THE FIRST EIGENVALUE OF A RIEMANN SURFACE

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ABSTRACT. We present a collection of results whose central theme is that the phenomenon of the first eigenvalue of the Laplacian being large is typical for Riemann surfaces. Our main analytic tool is a method for studying how the hyperbolic metric on a Riemann surface behaves under compactification of the surface. We make the notion of picking a Riemann surface at random by modeling this process on the process of picking a random 3-regular graph. With this model, we show that there are positive constants $C_1$ and $C_2$ independent of the genus, such that with probability at least $C_1$, a randomly picked surface has first eigenvalue at least $C_2$.

In this note, we announce a collection of results ([10, 11, 12]) connected to the behavior of the first eigenvalue $\lambda_1(S)$ of a compact Riemann surface of large genus, endowed with a metric of constant curvature $-1$. These results have as their common theme that the phenomenon of $\lambda_1$ large is in some sense typical. To make the notion of “typical” precise, we model the process of picking a Riemann surface at random on the process of picking a 3-regular graph at random.

The idea of studying the first eigenvalue of a Riemann surface via the study of eigenvalues of 3-regular graphs comes from the work of Buser [13, 14]. In effect, our approach here is a variation on his idea, where we first study the behavior of $\lambda_1$ on finite-area Riemann surfaces connected to 3-regular graphs, and then see how $\lambda_1$ changes when we compactify the surface.

Our main analytic tool is a method for studying how the hyperbolic metric of a finite-area Riemann surface behaves under such a compactification. This method was introduced in [6], and is based on the Ahlfors-Schwarz Lemma ([1]; see also [5]).

We then have:

**Theorem 1** ([10]). For all $\epsilon$, there exists $N$ such that, for $g \geq N$, there is a compact Riemann surface $S_g$ of genus $g$ satisfying

$$\lambda_1(S_g) \geq \frac{171}{784} - \epsilon.$$  

The number $171/784$ comes from the improvement by Luo, Rudnick, and Sarnak [17] of the Selberg $3/16$ Theorem [18]. If Selberg’s conjecture were true, we would be able to replace $3/16$ by $1/4$, the best possible value. More generally, Theorem 1...
contains a method to build from a collection of Riemann surfaces \( \{S_i\} \) of large first eigenvalue a larger collection of surfaces whose genera may include all but finitely many genera, whose first eigenvalues satisfy a similar bound.

To state our next results, let \( \mathcal{F}_{n,k} \) denote the set of \( k \)-regular graphs on \( n \) vertices, and \( \mathcal{F}^*_n \) the set of pairs \((\Gamma, \mathcal{O})\), where \( \Gamma \in \mathcal{F}_{n,k} \) and \( \mathcal{O} \) is an orientation on \( \Gamma \) —that is, for each vertex \( v \) of \( \Gamma \), \( \mathcal{O} \) prescribes a cyclic ordering of the edges emanating from \( \mathcal{O} \). Let \( \mathcal{F}^!_{n,k} \) denote the subset of \( \mathcal{F}_{n,k} \) consisting of graphs without loops, double edges, or 3-cycles. As a probability space, \( \mathcal{F}^!_{n,k} \) has positive measure in \( \mathcal{F}_{n,k} \) bounded away from 0 as \( n \to \infty \).

We describe a way of associating to the pair \((\Gamma, \mathcal{O})\) \( \in \mathcal{F}^*_{n,3} \) a compact Riemann surface \( S_{C}(\Gamma, \mathcal{O}) \). By a theorem of Belyi [2], the surfaces that arise in this way are dense in the space of all compact Riemann surfaces.

We then have:

**Theorem 2** ([11]). There exists a constant \( C_1 \) with the following property:

(a) If \( \Gamma \) is picked randomly from \( \mathcal{F}^!_{n,4} \) for \( n \) odd, then with probability \( \to 1 \) as \( n \to \infty \), there is an orientation \( \mathcal{O} \) on \( \Gamma \) so that for any splitting of \((\Gamma, \mathcal{O})\) to a 3-regular graph \((\Gamma', \mathcal{O}')\), the surface \( S_{C}(\Gamma', \mathcal{O}') \) satisfies:

(i) \( S_{C}(\Gamma', \mathcal{O}') \) has genus \( \frac{n+1}{2} \);

(ii) \( \lambda_1(S_{C}(\Gamma', \mathcal{O}')) \geq C_1 \).

(b) If \( \Gamma \) is picked randomly from \( \mathcal{F}^!_{n,3} \) with \( n \equiv 2 \pmod 4 \), then with probability \( \to 1 \) as \( n \to \infty \), there is an orientation \( \mathcal{O} \) on \( \Gamma \) such that the surface \( S_{C}(\Gamma, \mathcal{O}) \) satisfies:

(i) \( S_{C}(\Gamma, \mathcal{O}) \) has genus \( \frac{n+2}{2} \);

(ii) \( \lambda_1(S_{C}(\Gamma, \mathcal{O})) \geq C_1 \).

**Theorem 3** ([12]). There exist constants \( C_2, C_3, C_4, \) and \( C_5 \) with the following property: if \( (\Gamma, \mathcal{O}) \) is picked randomly from \( \mathcal{F}^*_{n,3} \), then, as \( n \to \infty \), the surface \( S_{C}(\Gamma, \mathcal{O}) \) will have the following properties, with probability at least \( C_2 \):

(a) \( \lambda_1(S_{C}(\Gamma, \mathcal{O})) \geq C_3 \).

(b) The length of the shortest geodesic \( \text{syst}(S_{C}(\Gamma, \mathcal{O})) \) of \( S_{C}(\Gamma, \mathcal{O}) \) satisfies

\[
\text{syst}(S_{C}(\Gamma, \mathcal{O})) \geq C_4.
\]

(c) The diameter \( \text{diam}(S_{C}(\Gamma, \mathcal{O})) \) satisfies

\[
\text{diam}(S_{C}(\Gamma, \mathcal{O})) \leq C_5 \log(\text{genus}(S_{C}(\Gamma, \mathcal{O}))).
\]

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1. **Open and closed Riemann surfaces**

Let \( S^O \) be an open Riemann surface, carrying a complete metric of constant curvature \(-1\) and finite area. Then there is a unique compactification \( S^C \) of \( S^O \) whose conformal structure is uniquely determined from \( S^O \). In general, \( S^C \) need not carry a hyperbolic metric, but under favorable circumstances it will, and indeed the hyperbolic metrics on \( S^O \) and \( S^C \) will be closely related.
Theorem 4 ([6]). For all $\epsilon$, there exists $L$ such that, if all the cusps of $S^O$ have length $\geq L$, then there are canonically defined cusp neighborhoods $\{U_i\}$ on $S^O$ and $S^C$ such that the hyperbolic metrics $ds^2_O$ and $ds^2_C$ satisfy
\[
\frac{1}{1+\epsilon} ds^2_O \leq ds^2_C \leq (1+\epsilon) ds^2_O
\]
outside the sets $\{U_i\}$.

See [6] for a precise statement. The length of a cusp is the length of the longest closed horocycle about the cusp.

The inverse process was described in [10]:

Theorem 5 ([10]). Given $L$, there exists a number $R$ with the following property: If $S$ is a compact Riemann surface, and $\{p_1,\ldots,p_k\}$ points on $S$ such that
(a) The injectivity radius about each point is at least $R$, and
(b) The balls $B(p_i,R)$ of radius $R$ about the points $p_i$ are pairwise disjoint,
then $S - \{p_1,\ldots,p_k\}$ has cusps of length $\geq L$.

Using this, one shows:

Theorem 6 ([6, 10]). For $L$ sufficiently large, there is a constant $C(L)$ such that:
(i) The Cheeger constants $h(S^O)$ and $h(S^C)$ satisfy:
\[
\frac{1}{C(L)} h(S^O) \leq h(S^C) \leq C(L) h(S^O).
\]
(ii) The first eigenvalues $\lambda_1(S^O)$ and $\lambda_1(S^C)$ satisfy
\[
\frac{1}{C(L)} \lambda_1(S^O) \leq \lambda_1(S^C) \leq C(L) \lambda_1(S^O).
\]
(iii) The shortest closed geodesics satisfy
\[
\frac{1}{C(L)} \text{syst}(S^O) \leq \text{syst}(S^C) \leq C(L) \text{syst}(S^O).
\]
Furthermore, $C(L) \to 1$ as $L \to \infty$.

By combining this result with the technique of [15] of closing off cusps by forming handles, we prove

Theorem 7 ([10]). Let $\{S_i\}$ be a collection of compact Riemann surfaces with the following properties:
(a) There exists $\lambda > 0$ such that $\lambda_1(S_i) > \lambda$ for all $i$.
(b) $\text{syst}(S_i) \to \infty$ as $i \to \infty$.
(c) For every $C > 1$ there exists an $N$ such that $\{x \in \mathbb{R} : x > N\} \subset \bigcup_i [1,C](\text{genus}(S_i))$.

Then, for all $\epsilon$, there exists $N$ such that for $g \geq N$ there exists a surface $S_g$ of genus $g$ with $\lambda_1(S_g) \geq \lambda - \epsilon$.

Applying this theorem to the compactifications of the modular surfaces then gives Theorem 1.
Figure 1. The marked ideal triangle $T$.

2. RIEMANN SURFACES AND 3-REGULAR GRAPHS

Let $\Gamma$ be a 3-regular graph. An orientation $O$ of $\Gamma$ is an assignment to each vertex $v$ of $\Gamma$ of a cyclic ordering of the edges emanating from $v$. It is clear that a 3-regular graph on $n$ vertices possesses $2^n$ orientations.

To the pair $(\Gamma, O)$ we will assign two surfaces $S^O(\Gamma, O)$ and $S^C(\Gamma, O)$. The surface $S^O(\Gamma, O)$ is obtained by gluing one copy of the ideal hyperbolic triangle $T$ shown in Figure 1 for each vertex of $\Gamma$, such that the natural orientation of the geodesic segments on $T$ matches up with the orientation about the vertex. Whenever two vertices are joined by an edge, we glue the corresponding triangles together, subject to the following conditions:

(i) The geodesic segments on the copies of $T$ are glued together.

(ii) The orientations on the copies of $T$ (as complex manifolds with boundary) are preserved.

The surfaces $S^C(\Gamma, O)$ are the compactifications of the surfaces $S^O(\Gamma, O)$.

As discussed in [8], the geometry and even the topology of the surfaces $S^O(\Gamma, O)$ and $S^C(\Gamma, O)$ depend very strongly on the orientation $O$. We will say that a path $\gamma$ on $(\Gamma, O)$ is a left-hand-turn path if, whenever it arrives at a vertex, it turns left according to the orientation $O$. Then each cusp of $S^O(\Gamma, O)$ is associated to a unique left-hand-turn path. If we denote by $\#(LHT)$ the number of these paths, then the genus of $S^C(\Gamma, O)$ is clearly

$$\text{genus}(S^C(\Gamma, O)) = 1 + \frac{n - 2\#(LHT)}{4}.$$ 

The length of the cusp corresponding to a given left-hand-turn path $\gamma$ is precisely the number of edges in $\gamma$. 


Many geometric properties of the surface $S^O(\Gamma, \mathcal{O})$ are reflected in the pair $(\Gamma, \mathcal{O})$. Some of these properties follow from general properties of covering manifolds ([7] and [9]). Other properties depend more delicately on this particular construction. When the graph has no short left-hand-turn paths, then these properties descend to properties on $S^C(\Gamma, \mathcal{O})$ via Theorem 6.

**Theorem 8.** For some $L$ sufficiently large, there are constants $C_1, \ldots, C_6$ with the following property: Suppose that $(\Gamma, \mathcal{O})$ has no left-hand-turn paths of length $\leq L$. Then

(i) $C_1 \lambda_1(\Gamma) \leq \lambda_1(S^C(\Gamma, \mathcal{O})) \leq C_2 \lambda_1(\Gamma)$.
(ii) $C_3 h(\Gamma) \leq S^C(\Gamma, \mathcal{O}) \leq C_4 h(\Gamma)$.
(iii) $\log(\text{syst}(\Gamma)) \leq \text{syst}(S^C(\Gamma, \mathcal{O})) \leq C_5 \text{syst}(\Gamma)$.
(iv) ([11]) $\text{diam}(S^C(\Gamma, \mathcal{O})) \leq C_6 \text{diam}(\Gamma)$.

With such strong control over the surface $S^C(\Gamma, \mathcal{O})$, one might be led to expect that the surfaces $S^C(\Gamma, \mathcal{O})$ are rather rare. It is therefore rather surprising that they in fact are quite common.

**Theorem 9 ([2, 12]).** Given any compact Riemann surface $S$, there are arbitrarily small deformations $S_\epsilon$ of $S$ such that $S_\epsilon = S^C(\Gamma, \mathcal{O})$ for some pair $(\Gamma, \mathcal{O})$.

### 3. Models of random graphs

Theorems 2 and 3 are now obtained by an analysis of the process of picking a random graph. To carry out this analysis, we make use of the model of random graphs considered by Bollobás [3, 4]. In this model, a $k$-regular graph on $n$ vertices is constructed at random by putting $nk$ balls into a hat, $k$ balls for each vertex. The balls are drawn out of the hat in pairs, and an edge drawn between $v_1$ and $v_2$ each time a pair of balls corresponding to $v_1$ and $v_2$ is drawn. An orientation on the graph may be determined by the order in which the corresponding pairs are drawn.

We will need the following results of [3] and [4]:

**Theorem 10.**

(i) ([4]) There is a constant $C_1$ such that, as $n \to \infty$, the probability that $H(\Gamma) \geq C_1$ tends to 1.

(ii) ([3]) Let $X_1, \ldots, X_L$ denote the random variable

$$X_j = \text{the number of closed paths of length } j \text{ in } \Gamma.$$ 

Then, for $L$ fixed and $n \to \infty$, the variables $X_1, \ldots, X_L$ tend to independent Poisson distributions.

To establish Theorem 2 (a), we seek the probability that a randomly chosen 4-regular graph will have an orientation with precisely one left-hand-turn path. Using ideas of [20, 21], it is shown in [11] that this will happen with probability $\to 1$ as long as $\Gamma$ has no closed loops of length 1. The proof of Theorem 2 (b) is similar, using [19] in place of [20, 21].

To establish Theorem 3, we estimate the probability that the pair $(\Gamma, \mathcal{O})$ has no left-hand-turn paths of length $\leq L$. This will certainly be the case if it has no closed paths of length $\leq L$ whatsoever, from which Theorem 3 follows. By refining this argument, we may get substantially better estimates for the constant $C_2$. 
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