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A ONE-BOX-SHIFT MORPHISM BETWEEN SPECHT MODULES

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Abstract. We give a formula for a morphism between Specht modules over $(\mathbf{Z}/m)\mathcal{S}_n$, where $n \geq 1$, and where the partition indexing the target Specht module arises from that indexing the source Specht module by a downwards shift of one box, m being the box shift length. Our morphism can be reinterpreted integrally as an extension of order m of the corresponding Specht lattices.

0. Notation

We write composition of maps on the right, $\xrightarrow{\alpha}\xrightarrow{\beta} = \xrightarrow{\alpha\beta}$. Intervals are to be read as subsets of $\mathbf{Z}$. Let $n \geq 1$, let $\mathcal{S}_n = \mathrm{Aut}_{\mathrm{Sets}}[1,n]$ denote the symmetric group on n letters and let ε_σ denote the sign of a permutation $\sigma \in \mathcal{S}_n$. Let

$$\begin{array}{ccc} \mathbf{N} & \xrightarrow{\lambda} & \mathbf{N}_0 \\ i & \longrightarrow & \lambda_i \end{array}$$

be a *partition* of n, i.e. assume $\sum_i \lambda_i = n$ and $\lambda_i \geq \lambda_{i+1}$ for $i \in \mathbf{N}$. Let

$$[\lambda] := \{i \times j \in \mathbf{N} \times \mathbf{N} \mid j \leq \lambda_i\}$$

denote the *diagram* of λ. We say that $i \times j \in [\lambda]$ lies in row i and in column j. A λ-*tableau* is a bijection

$$\begin{array}{ccc} [\lambda] & \xrightarrow[\sim]{[a]} & [1,n] \\ i \times j & \longrightarrow & a_{i,j}. \end{array}$$

The element $\sigma \in \mathcal{S}_n$ acts on the set T^λ of λ-tableaux via composition $[a] \xrightarrow{\sigma} [a]\sigma$. Let F^λ be the free $\mathbf{Z}$-module on T^λ with the induced operation of $\mathcal{S}_n$. Let

$$\begin{array}{ccc} [\lambda] & \xrightarrow{\rho} & \mathbf{N} \\ i \times j & \longrightarrow & i \end{array} \qquad\qquad \begin{array}{ccc} [\lambda] & \xrightarrow{\kappa} & \mathbf{N} \\ i \times j & \longrightarrow & j \end{array}$$

denote the projections. We denote by $\{a\} := [a]^{-1}\rho$ the λ-*tabloid* associated to the λ-tableaux $[a]$. The free $\mathbf{Z}$-module on the set of tabloids, equipped with the inherited $\mathcal{S}_n$-operation, is denoted by M^λ. Let

$$C_{[a]} := \{\sigma \in \mathcal{S}_n \mid [a]^{-1}\kappa = ([a]\sigma)^{-1}\kappa\}$$

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be the *column stabilizer* of $[a]$. Let the *Specht lattice* S^λ be the $\mathbf{Z}\mathcal{S}_n$-sublattice of M^λ generated over $\mathbf{Z}$ by the λ-*polytabloids*

$$\langle a\rangle := \sum_{\sigma\in C_{[a]}} \{a\}\sigma\varepsilon_\sigma .$$

Let λ' denote the *transposed partition* of λ, i.e. $j\le\lambda_i \iff i\times j\in[\lambda] \iff i\le\lambda'_j$.

1. CARTER-PAYNE

Let $d\in[1,n]$ be the number of shifted boxes. Let $1\le s<t\le n$, s being the row of $[\lambda]$ from which the boxes are shifted, and t being the row into which the boxes are shifted. Suppose

$$\mu_i := \begin{cases} \lambda_i - d & \text{for } i=s,\\ \lambda_i + d & \text{for } i=t,\\ \lambda_i & \text{else}\end{cases}$$

defines a partition of n. Let the *box shift length* be denoted by

$$m := (\lambda_s - s) - (\lambda_t - t) - d.$$

Let $m[p] := p^{v_p(m)}$ be the p-part of m. Using [1], CARTER and PAYNE proved the following

Theorem 1.1 ([2]). *Let K be an infinite field of characteristic p. Suppose $d < m[p]$. Then*

$$\operatorname{Hom}_{K\mathcal{S}_n}(K\otimes_{\mathbf{Z}} S^\lambda, K\otimes_{\mathbf{Z}} S^\mu)\neq 0.$$

2. INTEGRAL REINTERPRETATION

Assume $d=1$, i.e. $[\mu]$ arises from $[\lambda]$ by a one-box-shift. The condition $d<m[p]$ translates into $p|m$.

As we will see below, this particular case of the result of CARTER and PAYNE already holds over $K=\mathbf{F}_p$. So we obtain a nonzero element in

$$\operatorname{Hom}_{\mathbf{Z}\mathcal{S}_n}(S^\lambda/pS^\lambda, S^\mu/pS^\mu) \xleftarrow{\sim} \operatorname{Hom}_{\mathbf{Z}\mathcal{S}_n}(S^\lambda, S^\mu/pS^\mu).$$

We consider a part of the long exact $\operatorname{Ext}^*_{\mathbf{Z}\mathcal{S}_n}(S^\lambda,-)$-sequence on

$$0\longrightarrow S^\mu \xrightarrow{p} S^\mu \longrightarrow S^\mu/pS^\mu \longrightarrow 0,$$

viz.

$$\begin{array}{l} 0 \longrightarrow \underbrace{\operatorname{Hom}_{\mathbf{Z}\mathcal{S}_n}(S^\lambda,S^\mu)}_{=0} \xrightarrow{p} \underbrace{\operatorname{Hom}_{\mathbf{Z}\mathcal{S}_n}(S^\lambda,S^\mu)}_{=0} \longrightarrow \operatorname{Hom}_{\mathbf{Z}\mathcal{S}_n}(S^\lambda,S^\mu/pS^\mu)\\ \longrightarrow \operatorname{Ext}^1_{\mathbf{Z}\mathcal{S}_n}(S^\lambda,S^\mu) \xrightarrow{p} \operatorname{Ext}^1_{\mathbf{Z}\mathcal{S}_n}(S^\lambda,S^\mu).\end{array}$$

Mapping our morphism into Ext^1, we obtain a nonzero element of $\operatorname{Ext}^1_{\mathbf{Z}\mathcal{S}_n}(S^\lambda,S^\mu)$ which is annihilated by p. Conversely, the p-torsion elements of Ext^1 are given by morphisms modulo p.

Since $n!$ annihilates $\operatorname{Ext}^1_{\mathbf{Z}\mathcal{S}_n}(S^\lambda,S^\mu)$, replacement of p by $n!$ shows that any element in Ext^1 is given by a modular morphism modulo $n!$,

$$\operatorname{Hom}_{\mathbf{Z}\mathcal{S}_n}(S^\lambda, S^\mu/n!\,S^\mu) \xrightarrow{\sim} \operatorname{Ext}^1_{\mathbf{Z}\mathcal{S}_n}(S^\lambda,S^\mu).$$

Therefore, in order to get hold of the whole Ext^1, we need to calculate modulo prime powers in general.

3. One-box-shift formula

We keep the assumption $d = 1$. Let $s' := \lambda_s$ and let $t' := \lambda_t + 1$. A *path* of length $l \in [1, s' - t']$ is a map

$$\begin{array}{rcl} [0,l] & \xrightarrow{\gamma} & [\lambda] \cup [\mu] \\ k & \longrightarrow & \alpha_k \times \beta_k \end{array}$$

such that $k < k'$ implies $\beta_k < \beta_{k'}$, and such that $\alpha_0 \times \beta_0 = t \times t'$ and $\beta_l = s'$. For a λ-tableau $[a]$, we define the μ-tableau $[a^\gamma]$ by

$$\begin{array}{rcll} a^\gamma_{i,j} & := & a_{i,j} & \text{for } i \times j \in [\mu] \backslash (\gamma([1,l]) \cup \mathbf{N} \times \{s'\}), \\ a^\gamma_{\alpha_k,\beta_k} & := & a_{\alpha_{k+1},\beta_{k+1}} & \text{for } k \in [0, l-1], \\ a^\gamma_{i,s'} & := & a_{i,s'} & \text{for } i < \alpha_l, \\ a^\gamma_{i,s'} & := & a_{i+1,s'} & \text{for } i \geq \alpha_l. \end{array}$$

For $i \in [t'+1, s'-1]$, we denote

$$X_i := (s' - \lambda'_{s'}) - (i - \lambda'_i).$$

Let

$$x_\gamma := (-1)^{\alpha_l + 1} \frac{\prod_{i \in [t'+1, s'-1],\ \mu'_i > \mu'_{i+1}} X_i}{\prod_{k \in [1, l-1]} X_{\beta_k}}.$$

Let Γ be the set of paths of some length $l \in [1, s' - t']$.

Theorem 3.1 ([4], 4.3.31, cf. 0.7.1)**.** *The abelian group* $\mathrm{Hom}_{\mathbf{Z}\mathcal{S}_n}(S^\lambda, S^\mu/mS^\mu)$ *contains an element f of order $m = (\lambda_s - s) - (\lambda_t - t) - 1$ which is given by the commutative diagram of $\mathbf{Z}\mathcal{S}_n$-linear maps*

$$\begin{tikzcd}
 & {[a]} \arrow[r] & \sum_{\gamma \in \Gamma} x_\gamma \otimes \langle a^\gamma \rangle & \\
{[a]} \arrow[ddd] & F^\lambda \arrow[r] \arrow[ddd] \arrow[dr] & \mathbf{Q} \otimes_{\mathbf{Z}} S^\mu & 1 \otimes \langle b \rangle \\
 & & S^\mu \arrow[u] \arrow[dd] & \langle b \rangle \arrow[u] \arrow[dd] \\
 & & & \\
\langle a \rangle & S^\lambda \arrow[r, "f"] & S^\mu / mS^\mu & \langle b \rangle + mS^\mu .
\end{tikzcd}$$

Reducing modulo a prime dividing m, this recovers the case $d = 1$ of the result of Carter and Payne. By the long exact sequence as above, but with p replaced by m, we obtain a nonzero element in $\mathrm{Ext}^1_{\mathbf{Z}\mathcal{S}_n}(S^\lambda, S^\mu)$ of order m. [1]

The proof of this theorem proceeds by showing that a sufficient set of Garnir relations in F^λ is annihilated by $F^\lambda \longrightarrow S^\mu/mS^\mu$.

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[1] I do not know the structure of $\mathrm{Ext}^1_{\mathbf{Z}\mathcal{S}_n}(S^\lambda, S^\mu)$ as an abelian group. At least *in case* $n \leq 7$, direct computation yields that the projection of our element to its $2'$-part generates this $2'$-part. We have, however, for example $\mathrm{Ext}^1_{\mathbf{Z}\mathcal{S}_6}(S^{(4,1^2)}, S^{(3,1^3)})_{(2)} \simeq \mathbf{Z}/2 \oplus \mathbf{Z}/2$.

4. Example

Let $n = 9$, $\lambda = (4,3,2)$, $\mu = (3,3,2,1)$, $t' = 1$ and $s' = 4$, whence $m = 6$, $X_2 = 4$, $X_3 = 2$. We obtain a morphism of order 6 that maps

$$
\begin{array}{ccl}
S^{(4,3,2)} & \xrightarrow{f} & S^{(3,3,2,1)}/6\, S^{(3,3,2,1)} \\
\left\langle \begin{array}{cccc} 1 & 4 & 7 & 9 \\ 2 & 5 & 8 & \\ 3 & 6 & & \end{array} \right\rangle & \longrightarrow & 4^0 2^0 \left(\left\langle \begin{array}{ccc} 1 & \boxed{7} & \boxed{9} \\ 2 & 5 & 8 \\ 3 & 6 & \\ \boxed{4} & & \end{array} \right\rangle + \left\langle \begin{array}{ccc} 1 & 4 & \boxed{9} \\ 2 & \boxed{7} & 8 \\ 3 & 6 & \\ \boxed{5} & & \end{array} \right\rangle + \left\langle \begin{array}{ccc} 1 & 4 & \boxed{9} \\ 2 & 5 & 8 \\ 3 & \boxed{7} & \\ \boxed{6} & & \end{array} \right\rangle \right. \\
& & \left. + \left\langle \begin{array}{ccc} 1 & \boxed{8} & 7 \\ 2 & 5 & \boxed{9} \\ 3 & 6 & \\ \boxed{4} & & \end{array} \right\rangle + \left\langle \begin{array}{ccc} 1 & 4 & 7 \\ 2 & \boxed{8} & \boxed{9} \\ 3 & 6 & \\ \boxed{5} & & \end{array} \right\rangle + \left\langle \begin{array}{ccc} 1 & 4 & 7 \\ 2 & 5 & \boxed{9} \\ 3 & \boxed{8} & \\ \boxed{6} & & \end{array} \right\rangle \right) \\
& & +4^1 2^0 \left(\left\langle \begin{array}{ccc} 1 & 4 & \boxed{9} \\ 2 & 5 & 8 \\ 3 & 6 & \\ \boxed{7} & & \end{array} \right\rangle + \left\langle \begin{array}{ccc} 1 & 4 & 7 \\ 2 & 5 & \boxed{9} \\ 3 & 6 & \\ \boxed{8} & & \end{array} \right\rangle \right) \\
& & +4^0 2^1 \left(\left\langle \begin{array}{ccc} 1 & \boxed{9} & 7 \\ 2 & 5 & 8 \\ 3 & 6 & \\ \boxed{4} & & \end{array} \right\rangle + \left\langle \begin{array}{ccc} 1 & 4 & 7 \\ 2 & \boxed{9} & 8 \\ 3 & 6 & \\ \boxed{5} & & \end{array} \right\rangle + \left\langle \begin{array}{ccc} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & \boxed{9} & \\ \boxed{6} & & \end{array} \right\rangle \right) \\
& & +4^1 2^1 \left(\left\langle \begin{array}{ccc} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & \\ \boxed{9} & & \end{array} \right\rangle \right) .
\end{array}
$$

The $[0, l-1]$-part of the respective path is highlighted.

5. Motivation

We consider the rational Wedderburn isomorphism

$$
\begin{array}{ccc}
\mathbf{Q}\mathcal{S}_n & \xrightarrow{\sim} & \prod_\lambda (\mathbf{Q})_{n_\lambda \times n_\lambda} \\
\sigma & \longrightarrow & (\rho_\sigma^\lambda)_\lambda
\end{array}
$$

where λ runs over the partitions of n and where ρ_σ^λ denotes the matrix describing the operation of $\sigma \in \mathcal{S}_n$ on S^λ *with respect to a chosen tuple of integral bases.* The restriction

$$\mathbf{Z}\mathcal{S}_n \hookrightarrow \prod_\lambda (\mathbf{Z})_{n_\lambda \times n_\lambda}$$

of this isomorphism, viewed as an embedding of abelian groups, has index [2]

$$\prod_\lambda \left(\frac{n!}{n_\lambda} \right)^{n_\lambda^2/2} .$$

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[2] **Question.** *Given a central primitive idempotent* e^λ *of* $\Gamma := \prod_\lambda (\mathbf{Z})_{n_\lambda \times n_\lambda}$*, what is the index of* $e^\lambda \mathbf{Z}\mathcal{S}_n$ *in* $e^\lambda \Gamma$ *?* Cf. ([4], Section 1.1.3).

In particular, for $n \geq 2$ it is no longer an isomorphism.

Suppose, for partitions λ and μ of n and for some modulus $m \geq 2$, we are given a $\mathbf{Z}\mathcal{S}_n$-linear map

$$S^\lambda \xrightarrow{g} S^\mu / mS^\mu .$$

Let G be the matrix, with respect to the chosen integral bases of S^λ and S^μ, of a lifting of g to a $\mathbf{Z}$-linear map $S^\lambda \longrightarrow S^\mu$. The $\mathbf{Z}\mathcal{S}_n$-linearity of g reads

$$G\rho^\mu_\sigma - \rho^\lambda_\sigma G \in m(\mathbf{Z})_{n_\lambda \times n_\mu} \qquad \text{for all } \sigma \in \mathcal{S}_n .$$

Thus such a morphism yields a *necessary* condition for a tuple of matrices to lie in the image of the Wedderburn embedding.

For example, the evaluations of our one-box-shift morphism at hook partitions, i.e. at $\lambda = (k, 1^{n-k})$ and $\mu = (k-1, 1^{n-k+1})$, $k \in [2, n]$, furnish a long exact sequence. In the (simple) case of $n = p$ prime, and localized at (p), the set of necessary conditions imposed by these morphisms already turns out to be sufficient for a tuple of matrices over $\mathbf{Z}_{(p)}$ to lie in the image of the localized Wedderburn embedding ([4], Section 4.2.1). Therefore, it is advisable to chose a tuple of locally integral bases adapted to this long exact sequence. For instance, we obtain

$$\begin{aligned} \mathbf{Z}_{(3)}\mathcal{S}_3 \quad & \xrightarrow{\sim} \quad \left\{ a \times \begin{bmatrix} b & c \\ d & e \end{bmatrix} \times f \;\middle|\; a \equiv_3 b,\ d \equiv_3 0,\ e \equiv_3 f \right\} \\ & \subseteq \quad \mathbf{Z}_{(3)} \times \begin{bmatrix} \mathbf{Z}_{(3)} & \mathbf{Z}_{(3)} \\ \mathbf{Z}_{(3)} & \mathbf{Z}_{(3)} \end{bmatrix} \times \mathbf{Z}_{(3)} , \end{aligned}$$

the embedding *not* being written in the combinatorial standard polytabloid bases.

For an approach to the general case, see ([4], Chapters 3 and 5). Further examples may be found in ([4], Chapter 2).

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