COMPLETING LIE ALGEBRA ACTIONS TO LIE GROUP ACTIONS

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Abstract. For a finite-dimensional Lie algebra $\mathfrak{g}$ of vector fields on a manifold $M$ we show that $M$ can be completed to a $G$-space in a universal way, which however is neither Hausdorff nor $T_1$ in general. Here $G$ is a connected Lie group with Lie-algebra $\mathfrak{g}$. For a transitive $\mathfrak{g}$-action the completion is of the form $G/H$ for a Lie subgroup $H$ which need not be closed. In general the completion can be constructed by completing each $\mathfrak{g}$-orbit.

1. Introduction. In [7], Palais investigated when one could extend a local Lie group action to a global one. He did this in the realm of non-Hausdorff manifolds, since he showed that completing a vector field $X$ on a Hausdorff manifold $M$ may already lead to a non-Hausdorff manifold on which the additive group $\mathbb{R}$ acts. We reproved this result in [3], being unaware of Palais’ result. In [4] this result was extended to infinite dimensions and applied to partial differential equations like Burgers’ equation: solutions of the PDE were continued beyond the shocks and the universal completion was identified.

Here we give a detailed description of the universal completion of a Hausdorff $\mathfrak{g}$-manifold to a $G$-manifold. For a homogeneous $\mathfrak{g}$-manifold (where the finite-dimensional Lie algebra $\mathfrak{g}$ acts infinitesimally transitive) we show that the $G$-completion (for a Lie group $G$ with Lie algebra $\mathfrak{g}$) is a homogeneous space $G/H$ for a possibly non-closed Lie subgroup $H$ (Theorem 7). In Example 8 we show that each such situation can indeed be realized. For general $\mathfrak{g}$-manifolds we show that one can complete each $\mathfrak{g}$-orbit separately and replace the $\mathfrak{g}$-orbits in $M$ by the resulting $G$-orbits to obtain the universal completion $G\, M$ (Theorem 9). All $\mathfrak{g}$-invariant structures on $M$ ‘extend’ to $G$-invariant structures on $G\, M$. The relation between our results and those of Palais are described in Section 10.

2. $\mathfrak{g}$-manifolds. Let $\mathfrak{g}$ be a Lie algebra. A $\mathfrak{g}$-manifold is a (finite-dimensional Hausdorff) connected manifold $M$ together with a homomorphism of Lie algebras $\zeta = \zeta^M : \mathfrak{g} \rightarrow \mathfrak{X}(M)$ into the Lie algebra of vector fields on $M$. We may assume without loss of generality that it is injective; if not, replace $\mathfrak{g}$ by $\mathfrak{g}/\ker(\zeta)$. We shall also say that $\mathfrak{g}$ acts on $M$.

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The image of $\zeta$ spans an integrable distribution on $M$, which need not be of constant rank. So through each point of $M$ there is a unique maximal leaf of that distribution; we also call it the $g$-orbit through that point. It is an initial submanifold of $M$ in the sense that a mapping from a manifold into the orbit is smooth if and only if it is smooth into $M$; see [3, 2.14ff.].

Let $\ell : G \times M \to M$ be a left action of a Lie group with Lie algebra $\mathfrak{g}$. Let $\ell_a : M \to M$ and $\ell^x : G \to M$ be given by $\ell_a(x) = \ell^x(a) = \ell(a, x) = a.x$ for $a \in G$ and $x \in M$. For $X \in \mathfrak{g}$ the fundamental vector field $\zeta_X = \zeta^X \in \mathfrak{X}(M)$ is given by $\zeta_X(x) = -T_x(\ell^x).X = -T_{e;x}(X,e;x) = -\partial_x[0,exp(tX)].x$. The minus sign is necessary so that $\zeta : \mathfrak{g} \to \mathfrak{X}(M)$ becomes a Lie algebra homomorphism. For a right action the fundamental vector field mapping without minus would be a Lie algebra homomorphism. Since left actions are more common, we stick to them.

3. The graph of the pseudogroup. Let $M$ be a $g$-manifold, effective and connected, so that the action $\zeta = \zeta^M : \mathfrak{g} \to \mathfrak{X}(M)$ is injective. Recall from [1, 2.3] that the pseudogroup $\Gamma(\mathfrak{g})$ consists of all diffeomorphisms of the form

$$\text{Fl}^{\zeta_{X_i}}_{t_i} \circ \ldots \circ \text{Fl}^{\zeta_{X_2}}_{t_2} \circ \text{Fl}^{\zeta_{X_1}}_{t_1} |U,$$

where $X_i \in \mathfrak{g}$, $t_i \in \mathbb{R}$, and $U \subset M$ are such that $\text{Fl}^{\zeta_{X_1}}_{t_1}$ is defined on $U$, $\text{Fl}^{\zeta_{X_2}}_{t_2}$ is defined on $\text{Fl}^{\zeta_{X_1}}_{t_1}(U)$, and so on.

Now we choose a connected Lie group $G$ with Lie algebra $\mathfrak{g}$, and we consider the integrable distribution of constant rank $d = \dim(\mathfrak{g})$ on $G \times M$ which is given by

$$(L_X(g), \zeta^M_X(x)) : (g, x) \in G \times M, X \in \mathfrak{g} \subset TG \times TM,$$

where $L_X$ is the left invariant vector field on $G$ generated by $X \in \mathfrak{g}$. This gives rise to the foliation $\mathcal{F}^M_\zeta$ on $G \times M$, which we call the graph foliation of the $g$-manifold $M$.

Consider the following diagram, where $L(e, x)$ is the leaf through $(e, x)$ in $G \times M$, $O_g(x)$ is the $g$-orbit through $x$ in $M$, and $W_x \subset G$ is the image of the leaf $L(e, x)$ in $G$. Note that $\text{pr}_1 : L(e, x) \to W_x$ is a local diffeomorphism for the smooth structure of $L(e, x)$.

Moreover we consider a piecewise smooth curve $c : [0, 1] \to W_x$ with $c(0) = e$ and we assume that it is liftable to a smooth curve $\tilde{c} : [0, 1] \to L(e, x)$ with $\tilde{c}(0) = (e, x)$. Its endpoint $\tilde{c}(1) \in L(e, x)$ does not depend on small (i.e. liftable to $L(e, x)$) homotopies of $c$ which respect the ends. This lifting depends smoothly on the choice of the initial point $x$ and gives rise to a local diffeomorphism $\gamma_x(c) : U \to \{e\} \times U \to \{c(1)\} \times U' \to U'$, a typical element of the pseudogroup $\Gamma(\mathfrak{g})$ which is defined near $x$. See [1, 2.3] for more information and Example 4 below. Note that the leaf $L(g, x)$ through $(g, x)$ is given by

$$(3.3) \quad L(g, x) = \{(gh, y) : (h, y) \in L(e, x)\} = \{\mu_g \times \text{Id}\}(L(e, x)),$$
where $\mu : G \times G \to G$ is the multiplication and $\mu_g(h) = gh = \mu^h(g)$.

4. Examples. It is helpful to keep the following examples in mind, which elaborate upon [1, 5.3]. Let $G = g = \mathbb{R}^2$, let $W$ be an annulus in $\mathbb{R}^2$ containing 0, and let $M_1$ be a simply connected piece of finite or infinite length of the universal cover of $W$. Then the Lie algebra $g = \mathbb{R}^2$ acts on $M$ but not the group. Let $p : M_1 \to W$ be the restriction of the covering map, a local diffeomorphism.

Here $G \times_g M_1 \cong G = \mathbb{R}^2$. Namely, the graph distribution is then also transversal to the fiber of $pr_2 : G \times M_1 \to M_1$ (since the action is transitive and free on $M_1$), thus describes a principal $G$-connection on the bundle $pr_2 : G \times M_1 \to M_1$. Each leaf is a covering of $M_1$ and hence diffeomorphic to $M_1$ since $M_1$ is simply connected. For $g \in \mathbb{R}^2$ consider $j_g : M_1 \cong \{g\} \times M_1 \subset G \times M_1 \xrightarrow{\pi} G \times_g M_1$ and two points $x \neq y \in M_1$. We may choose a smooth curve $\gamma$ in $M_1$ from $x$ to $y$, lift it into the leaf $L(g, x)$ and project it to a curve $c$ in $g + W$ from $g$ to $c(1) = g + p(y) - p(x) \in g + W$. Then $(g, x)$ and $(c(1), y)$ are on the same leaf. So $j_g(x) = j_g(y)$ if and only if $p(x) = p(y)$. So we see that $j_g(x) = g + p(x)$, and thus $G \times_g M_1 = \mathbb{R}^2$. This will also follow from 7.

5. Enlarging to group actions. In the situation of Section 3 let us denote by $G \times g M = G \times M / F_\zeta$ the space of leaves of the foliation $F_\zeta$ on $G \times M$, with the quotient topology. For each $g \in G$ we consider the mapping

$$j_g : M \overset{\text{ins}}{\to} \{g\} \times M \subset G \times M \xrightarrow{\pi} G \times_g M = G \times_g M.$$ 

Note that the submanifolds $\{g\} \times M \subset G \times M$ are transversal to the graph foliation $F_\zeta$. The leaf space $G \times M$ of $G \times M$ admits a unique smooth structure, possibly singular and non-Hausdorff, such that a mapping $f : G \times M \to N$ into a smooth manifold $N$ is smooth if and only if the compositions $f \circ j_g : M \to N$ are smooth. For example
we may use the structure of a Frölicher space or smooth space induced by the mappings $j_g$ in the sense of [6] Section 23] on $G M = G \times_\sigma M$. The canonical open maps $j_g : M \to G M$ for $g \in G$ are called the charts of $G M$. By construction, for each $x \in M$ and for $g'g^{-1}$ near enough to $e$ in $G$ there exists a curve $c : [0,1] \to W_x$ with $c(0) = e$ and $c(1) = g'g^{-1}$ and an open neighborhood $U$ of $x$ in $M$ such that for the smooth transformation $\gamma_x(c)$ in the pseudogroup $\Gamma(g)$ we have

\[(5.2) \quad j_{g'}|U = j_g \circ \gamma_x(c).\]

Thus the mappings $j_g$ may serve as a replacement for charts in the description of the smooth structure on $G M$. Note that the mappings $j_g$ are not injective in general. Even if $g = g'$, there might be liftable smooth loops $c$ in $W_x$ such that (5.2) holds. Note also some similarity of the system of ‘charts’ $j_g$ with the notion of an orbifold where one uses finite groups instead of pseudogroup transformations.

The leaf space $G M = G \times_\sigma M$ is a smooth $G$-space where the $G$-action is induced by $(g',x) \mapsto (gg',x)$ in $G \times M$.

**Theorem.** The $G$-completion $G M$ has the following universal properties:

\[(5.3) \quad \text{Given any Hausdorff } G \text{-manifold } N \text{ and } g \text{-equivariant mapping } f : M \to N \text{ there exists a unique } G \text{-equivariant continuous mapping } \tilde{f} : G M \to N \text{ with } \tilde{f} \circ j_e = f. \text{ Namely, the mapping } \tilde{f} : G \times M \to N \text{ given by } \tilde{f}(g,x) = g \cdot f(x) \text{ is smooth and factors to } f : G M \to N.\]

\[(5.4) \quad \text{In the setting of (5.3), the universal property holds also for the } T_1 \text{-quotient of } G M, \text{ which is given as the quotient } G \times M/\mathcal{F}_\zeta \text{ of } G \times M \text{ by the equivalence relation generated by the closure of leaves.}\]

\[(5.5) \quad \text{If } M \text{ carries a symplectic or Poisson structure or a Riemannian metric such that the } g \text{-action preserves this structure or is even a Hamiltonian action, then the structure ‘can be extended to } G M \text{ so that the enlarged } G \text{-action preserves these structures or is even Hamiltonian’.}\

**Proof.** (5.3) Consider the mapping $\tilde{f} = \ell^N \circ (\text{Id}_G \times f) : G \times M \to N$ which is given by $\tilde{f}(g,x) = g \cdot f(x)$. Then by (3.1) and (3.2) we have for $X \in G$

\[T \tilde{f} \cdot (L_X(g), \zeta_X^M(x)) = T\ell \cdot (L_X(g), T_x f \cdot \zeta_X^M(x)) = T\ell \cdot (R_{Ad(g)}X(g), 0 f(x)) + T\ell \cdot (0, \zeta_X^M(f(x))) = -\zeta_{Ad(g)}X(g \cdot f(x)) + T\ell g \cdot \zeta_X^N(f(x)) = 0.\]

Thus $\tilde{f}$ is constant on the leaves of the graph foliation on $G \times M$ and thus factors to $\tilde{f} : G M \to N$. Since $\tilde{f}(g,g_1,x) = g \cdot g_1 \cdot f(x) = g \cdot f(g,x)$, the mapping $\tilde{f}$ is $G$-equivariant. Since $N$ is Hausdorff, $\tilde{f}$ is even constant on the closure of each leaf, thus (5.4) holds also.

(5.5) Let us treat Poisson structure $P$ on $M$. For symplectic structures or Riemannian metrics the argument is similar and simpler. Since the Lie derivative along fundamental vector fields of $P$ vanishes, the pseudogroup transformation $\gamma_x(c)$ in (5.2) preserves $P$. Since $G M$ is the quotient of the disjoint union of all spaces $(g) \times M$ for $g \in G$ under the equivalence relation described by (5.2), $P$ ‘passes down to this quotient’. Note that we refrain from putting too much meaning on this statement.

The universal property (5.3) holds also for smooth $G$-spaces $N$ which need not be Hausdorff, nor $T_1$, but should have tangent spaces and foliations so that it is
Therefore the leaves of the form $L_r$ where $r$ being independent of $e$ parametrize the space of leaves $G$ in (3.2) is a universal covering. This is visibly consistent with (3.3). In order to explicitly as follows. For any smooth curve $c(t) = (\xi(t), \eta(t)) \in G$ starting at $(\xi_0, \eta_0)$ we have $\dot{c}(t) = (\xi(t), \eta(t)) Y \in \mathfrak{g}$ and the lifted curve $(c(t), y(t))$ is in the leaf $L((\xi_0, \eta_0), y_0)$ if and only if it satisfies the first order ODE

$$y(t), y'(t) = \dot{\xi}(t) \zeta_\alpha^x (y(t)) + \dot{\eta}(t) \zeta_\alpha^y (y(t))$$

(6.2) with initial value $y(0) = y_0 = (x_0, y_0, u = z_0) \in M$. Substituting (6.1) into (6.2), we see that this ODE is linear, that is, $\dot{x} = \dot{\xi}$, $\dot{y} = \dot{\eta}$ and $\dot{z} = -\alpha z \frac{xy - y^2}{r^2} = -\alpha z \frac{xy - y^2}{r^2}$, where $r^2 = x^2 + y^2$. Thus the projection $x(t)$ of $y(t)$ to the $(x, y)$-plane is given by $x(t) = c(t) - ((\xi_0, \eta_0) - x_0) = c(t) - (\xi_0 - x_0, \eta_0 - y_0)$, whereas the third equation leads to

$$z(t) = u e^{-\alpha \int_0^t d\theta} = u e^{-\alpha (\theta(t) - \theta_0)} = u e^{\alpha \theta_0} e^{-\alpha \theta(t)}$$

(6.3) where $\theta$ is the angle function in the $(x, y)$-plane. This depends only on the endpoints $x_0, x(t)$ and the winding number of the curve $x$ and is otherwise independent of $x$. Incompleteness occurs whenever the curve $x$ goes to $(0, 0) \in \mathbb{R}^2$ in finite time $\bar{t} < \infty$, that is, $x(t) \to (0, 0)$, $t \to \bar{t}$, or equivalently $c(t) \to (\xi_0, \eta_0) - x_0, t \to \bar{t}$. It follows that the leaf $L((\xi_0, \eta_0), y_0)$ is parametrized by $(r, \theta) \in \mathbb{R}_+ \times \mathbb{R}$ with $z = z(\theta)$ being independent of $r > 0$ and that

$$\text{pr}_1 : L((\xi_0, \eta_0), y_0) \to W_{(\xi_0, \eta_0), y_0} = \mathbb{R}^2 \setminus \{(\xi_0, \eta_0) - x_0\}$$

(6.4) in (3.2) is a universal covering. This is visibly consistent with (3.3). In order to parametrize the space of leaves $G_M$, we observe that the parameter $x_0$ can be eliminated. In fact, from the previous formulas we see that

$$L((\xi_0', \eta_0'), (x_0', u')) = L((\xi_0, \eta_0), (x_0, u))$$

(6.5) if and only if $(\xi_0', \eta_0') - x_0 = (\xi_0, \eta_0) - x_0$ and $u' = u e^{\alpha (\theta_0 - \theta')}$, so that we have $z'(\theta) = u' e^{\alpha \theta'} e^{-\alpha \theta(\theta)} = u e^{\alpha \theta_0} e^{-\alpha \theta(t)} = z(\theta)$. In particular, it follows that

$$L((\xi_0, \eta_0), y_0) = L((\xi_0' + 1, \eta_0'), (1, 0, u'))$$

(6.6) where $(\xi_0', \eta_0') = (\xi_0, \eta_0) - x_0, u' = u e^{\alpha \theta_0}, \theta_0 = 0$, projecting to $\mathbb{R}^2 \setminus \{(\xi_0', \eta_0')\}$. Therefore the leaves of the form $L((\xi_0 + 1, \eta_0), (1, 0, u'))$ are distinct for different values of $(\xi_0, \eta_0)$ and fixed value of $u$ and from the relation (3.3) we conclude that

$$L((\xi_0 + 1, \eta_0), (1, 0, u')) = (\xi_0, \eta_0) + L((1, 0), (1, 0, u))$$

(6.7) that is, $G = \mathbb{R}^2$ acts without isotropy on $G_M$. We also need to determine the range for the parameter $u$. Obviously, we have $L((1, 0), (1, 0, u')) = L((1, 0), (1, 0, u))$
if and only if \( u' = e^{2\pi n} u \) for \( n \in \mathbb{Z} \). Thus these leaves are parametrized by \([u]\),

Taking values in the quotient of the additive group \( \mathbb{R} \) under the multiplicative group \( \{e^{2\pi n} : n \in \mathbb{Z}\} \), that is,

\[
\{0\} \cup S^1_+ \cup S^1_- \cong \{0\} \cup \mathbb{R}^2 / \{e^{2\pi n} : n \in \mathbb{Z}\} \cup \mathbb{R}^2 / \{e^{2\pi n} : n \in \mathbb{Z}\}.
\]

(6.8)

The topology on the above space is determined by the leaf closures, respectively the orbit closures. First we have \( \bar{L}((\xi_0 + 1, \eta_0), (1, 0, u)) = (\xi_0, \eta_0) + \bar{L}((1, 0), (1, 0, u)) \) in \( G \times M \), and it is sufficient to determine the closures of \( \bar{L}((1, 0), (1, 0, u)) \). For \( (1, 0, u) \in M \) with \( u \neq 0 \) we consider the curve \( c(\theta) = e^{i\theta} \in G = \mathbb{R}^2 \). It is liftable to \( G \times M \) and determines on \( M \) the curve \( y(t) = (\cos \theta, \sin \theta, u e^{-\alpha \theta}) \). Thus the curve \( (c(\theta), y(\theta)) \) in the leaf through \((1, 0; 1, 0, u) \in G \times M \subset \mathbb{R}^5 \) has a limit cycle for \( \theta \to \infty \) which lies in the different leaf through \((1, 0; 1, 0, 0) \), which is closed, given by the \((x, y)\)-plane \((\mathbb{R}^2 \times 0) \setminus 0 \) at level \((1, 0) \in G \). Thus we have

\[
\bar{L}((1, 0), (1, 0, u)) = \bar{L}((1, 0), (1, 0, u)) \cup \bar{L}((1, 0), (1, 0, 0)).
\]

(6.9)

Hence the leaf \( L((1, 0), (1, 0, u)) \) is not closed, and the topological space \( G \times M \) is not \( T^1 \) and not a manifold. The orbits of the \( g \)-action are determined by the leaf structure via \( \pi_2 \) in diagram (3.2), and they look here as follows. The \((x, y)\)-plane \((\mathbb{R}^2 \times 0) \setminus 0 \) is a closed orbit. Orbits above this plane are helicoidal staircases leading down and accumulating exponentially at the \((x, y)\)-plane. Orbits below this plane are helicoidal staircases leading up and again accumulating exponentially. Thus the orbit space \( M/\mathfrak{g} \) of the \( g \)-action is given by (6.8), with the point 0 being closed. By (6.9), the closure of any orbit represented by a point \([u]\) on one of the circles is given by \([u], 0\). From (6.6) and (6.7), we see that the \( G \)-completion \( G \times M \) has a section over the orbit space \( G \times M / G \cong M / \mathfrak{g} \) given by \([u] \mapsto L((1, 0), (1, 0, u)) \). Therefore \( G \times M \cong G \times M / \mathfrak{g} = \mathbb{R}^2 \times \{0\} \cup S^1_+ \cup S^1_- \} \).

The structure of the completion and the orbit spaces are independent of the deformation parameter \( \alpha > 0 \) in (6.1). However, for \( \alpha \downarrow 0 \), the completion just means adding in the \( z \)-axis, that is, we get \( G \times M \cong \mathbb{R}^3 \) with \( G = \mathbb{R}^2 \) acting by parallel translation on the affine planes \( z = c \), and \( M / \mathfrak{g} \cong G \times M / G \cong \mathbb{R} \) as it should be.

It was pointed out to us \([2]\) that one can make this example still more pathological. Consider the above example only in a cylinder over the annulus \( 0 < x^2 + y^2 < 1 \). Add an open handle to the disk and continue the \( \mathbb{R}^2 \)-action on the cylinder over the disk with an open handle added in such a way that there is a shift in the \( z \)-direction when one traverses the handle. Then one of the helicoidal staircases is connected to the disk itself, so it accumulates onto itself. This is called a ‘resilient leaf’ in foliation theory.

7. Theorem. Let \( M \) be a connected transitive effective \( \mathfrak{g} \)-manifold. Let \( G \) be a connected Lie group with Lie algebra \( \mathfrak{g} \). Then we have:

(7.1) There exists a subgroup \( H \subset G \) such that the \( G \)-completion \( G \times M \) is diffeomorphic to \( G / H \).

(7.2) The Hausdorff quotient of \( G \times M \) is the homogeneous manifold \( G / \overline{H} \). It has the following universal property: For each smooth \( \mathfrak{g} \)-equivariant mapping \( f : M \to N \) into a Hausdorff \( G \)-manifold \( N \) there exists a unique smooth \( G \)-equivariant mapping \( \bar{f} : G / \overline{H} \to N \) with \( f = \bar{f} o \pi o j : M \to G / \overline{H} \xrightarrow{\pi} G / H \to N \).
Since (7.1) we choose a base point \( x_0 \in M \). The \( G \)-completion is given by \( G \times \{ x_0 \} \), the orbit space of the \( g \)-action on \( G \times \) \( M \) which is given by \( g \ni X \mapsto L_X \times \xi_Y^M \), and the \( G \)-action on the completion is given by multiplication from the left. The submanifold \( G \times \{ x_0 \} \) meets each \( g \)-orbit in \( G \times M \) transversely, since
\[
T_{(g, x_0)} (G \times \{ x_0 \}) + T_{(g, x_0)} L(g, x_0) = \{ L_X (g) \times 0_{x_0} + L_Y (g) \times \xi_Y (x_0) : X, Y \in \mathfrak{g} \} = T_{(g, x_0)} (G \times M).
\]
By (3.3) we have \( L(g, x) = g.L(e, x) \) so that the isotropy Lie algebra \( \mathfrak{h} = \mathfrak{g}_{x_0} = \{ x \in \mathfrak{g} : \xi_X (x_0) = 0 \} \) is also given by
\[
X \in \mathfrak{h} \iff X \times 0_{x_0} \in T_{(e, x_0)} (G \times \{ x_0 \}) \cap T_{(e, x_0)} L(e, x_0)
\]
\[
\iff L_X (g) \times 0_{x_0} \in T_{(g, x_0)} (G \times \{ x_0 \}) \cap T_{(g, x_0)} L(g, x_0).
\]
Since \( G \times \{ x_0 \} \) is a leaf of a foliation and the \( L(e, x) \) also form a foliation, \( \mathfrak{h} \) is a Lie subalgebra of \( \mathfrak{g} \). Let \( H_0 \) be the connected Lie subgroup of \( G \) which corresponds to \( \mathfrak{h} \). Then clearly \( H_0 \times \{ x_0 \} \subset G \times \{ x_0 \} \cap L(e, x_0) \). Let the subgroup \( H \subset G \) be given by
\[
H = \{ g \in G : (g, x_0) \in L(e, x_0) \} = \{ g \in G : L(g, x_0) = L(e, x_0) \};
\]
then the \( C^\infty \)-curve component of \( H \) containing \( e \) is just \( H_0 \). So \( H \) consists of at most countably many \( H_0 \)-cosets. Thus \( H \) is a Lie subgroup of \( G \) (with a finer topology, perhaps). By construction the orbit space \( G \times \{ x_0 \} \) equals the quotient of the transversal \( G \times \{ x_0 \} \) by the relation induced by intersecting with each leaf \( L(g, x_0) \) separately, i.e., \( G \times \{ x_0 \} = G / H \).

(7.2) Obviously the \( T_1 \)-quotient of \( G / H \) equals the Hausdorff quotient \( G / \overline{\mathbb{P}} \), which is a smooth manifold. The universal property is easily seen.

(7.3) Let \( x \in M \) and \( (g, x) \in L(e, x_0) = L(g, x) = g.L(e, x) \). So it suffices to treat the leaf \( L(e, x) \). We choose \( X_1, \ldots, X_n \in \mathfrak{g} \) such that \( \xi_{X_1} (x), \ldots, \xi_{X_n} (x) \) form a basis of the tangent space \( T_x M \). Let \( u : U \to \mathbb{R}^n \) be a chart on \( M \) centered at \( x \) such that \( u(U) \) is an open ball in \( \mathbb{R}^n \) and such that \( \xi_{X_1} (y), \ldots, \xi_{X_n} (y) \) are still linearly independent for all \( y \in U \). For \( y \in U \) consider the smooth curve \( c_y : [0, 1] \to U \) given by \( c_y (t) = u^{-1} (t.u(y)) \). We consider
\[
\partial_t c_y (t) = c'_y (t) = \sum_{i=1}^n f^i_y (t) \xi_{X_i} (c_y (t)), \quad f^i_y \in C^\infty ([0, 1], \mathbb{R}),
\]
\[
X_y (t) = \sum_{i=1}^n f^i_y (t) X_i \in \mathfrak{g}, \quad X \in C^\infty ([0, 1], \mathfrak{g}),
\]
\[
g_y \in C^\infty ([0, 1], G), \quad T(\mu_{g_y (t)}) \partial_t g_y (t) = X_y (t), \quad g_y (0) = e,
\]
and everything is also smooth in \( y \in U \). Then for \( h \in H \) we have \( (h.g_y (t), c_y (t)) \in L(e, x) \) since
\[
\partial_t (h.g_y (t), c_y (t)) = (L_{X_y (t)} (h.g_y (t)), \xi_{X_y (t)} (c_y (t))).
\]
Thus \( U \times H \ni (y, h) \mapsto \text{pr}_2^{-1} (U) \cap L(e, x) \) is the required fiber bundle parameterization.
8. Example. Let $G$ be a simply connected Lie group and let $H$ be a connected Lie subgroup of $G$ which is not closed. For example, let $G = \text{Spin}(5)$, which is compact of rank 2, and let $H$ be a dense 1-parameter subgroup in its 2-dimensional maximal torus. Let $\text{Lie}(G) = \mathfrak{g}$ and $\text{Lie}(H) = \mathfrak{h}$. We consider the foliation of $G$ into right $H$-cosets $gH$ which is generated by $\{L_X : X \in \mathfrak{h}\}$ and is left invariant under $G$. Let $U$ be a chart centered at $e$ on $G$ which is adapted to this foliation, i.e. $u : U \rightarrow u(U) = V_1 \times V_2 \subset \mathbb{R}^k \times \mathbb{R}^{n-k}$ such that the sets $u^{-1}(V_1 \times \{x\})$ are the leaves intersected with $U$. We assume that $V_1$ and $V_2$ are open balls, and that $U$ is so small that $\exp : W \rightarrow U$ is a diffeomorphism for a suitable convex open set $W \subset \mathfrak{g}$. Of course, $\mathfrak{g}$ acts on $U$ and respects the foliation, so this $\mathfrak{g}$-action descends to the leaf space $M$ of the foliation on $U$ which is diffeomorphic to $V_2$.

Lemma. In this situation, for the $G$-completion we have $G \times \mathfrak{g} \times M = G/H$.

Proof. We use the method described in the end of the proof of Theorem 7: $\mathcal{G}M = G \times \mathfrak{g} \times M$ is the quotient of the transversal $G \times \{x_0\}$ by the relation induced by intersecting with each leaf $L(g, x_0)$ separately. Thus we have to determine the subgroup $H_1 = \{g \in G : (g, x_0) \in L(e, g)\}$.

Obviously any smooth curve $c_1 : [0, 1] \rightarrow H$ starting at $e$ is liftable to $L(e, x_0)$ since it does not move $x_0 \in M$. So $H \subseteq H_1$, and moreover $H$ is the $C^\infty$-path component of the identity in $H_1$.

Conversely, if $c = (c_1, c_2) : [0, 1] \rightarrow L(e, x_0) \subset G \times M$ is a smooth curve from $(e, x_0)$ to $(g, x_0)$, then $c_2$ is a smooth loop through $x_0$ in $M$ and there exists a smooth homotopy $h$ in $M$ which contracts $c_2$ to $x_0$, fixing the ends. Since $\text{pr}_2 : L(e, x_0) \rightarrow M$ is a fiber bundle by (7.3), we can lift the homotopy $h$ from $M$ to $L(e, x_0)$ with starting curve $c$, fixing the ends, and deforming $c$ to a curve $c'$ in $L(e, x_0) \cap \text{pr}_2^{-1}(x_0)$. Then $\text{pr}_1 \circ c'$ is a smooth curve in $H_1$ connecting $e$ and $g$.

Thus $H_1 = H$, and consequently $\mathcal{G}M = G/H$. \qed

9. Theorem. Let $M$ be a connected $\mathfrak{g}$-manifold. Let $G$ be a connected Lie group with Lie algebra $\mathfrak{g}$. Then the $G$-completion $\mathcal{G}M$ can be described in the following way:

9.1 Form the leaf space $M/\mathfrak{g}$, a quotient of $M$ which may be non-Hausdorff and not $T_1$, etc.

9.2 For each point $z \in M/\mathfrak{g}$, replace the orbit $\pi^{-1}(z) \subset M$ by the homogeneous space $G/H_x$ described in Theorem 7, where $x$ is some point in the orbit $\pi^{-1}(z) \subset M$. One can use transversals to the $\mathfrak{g}$-orbits in $M$ to describe this in more detail.

9.3 For each point $z \in M/\mathfrak{g}$, one can also replace the orbit $\pi^{-1}(z) \subset M$ by the homogeneous space $G/H_x$ described in Theorem 7, where $x$ is some point in the orbit $\pi^{-1}(z) \subset M$. The resulting $G$-space has then Hausdorff orbits which are smooth manifolds, but the same orbit space as $M/\mathfrak{g}$.

See Example 6 above.

Proof. Let $\mathcal{O}(x) \subset M$ be the $\mathfrak{g}$-orbit through $x$, i.e., the leaf through $x$ of the singular foliation (with non-constant leaf dimension) on $M$ which is induced by the $\mathfrak{g}$-action. Then the $G$-completion of the orbit $\mathcal{O}(x)$ is $\mathcal{G}\mathcal{O}(x) = G/H_x$ for the Lie subgroup $H_x \subset G$ described in Theorem (7.1). By the universal property of the $G$-completion we get a $G$-equivariant mapping $\mathcal{G}\mathcal{O}(x) \rightarrow \mathcal{G}M$ which is injective and a homeomorphism onto its image, since we can repeat the construction of Theorem
COMPLETING LIE ALGEBRA ACTIONS TO LIE GROUP ACTIONS

(7.1) on $M$. Clearly the mapping $j_e : M \to GM$ induces a homeomorphism between the orbit spaces $M/g \to GM/G$.

Now let $s : V \to M$ be an embedding of a submanifold which is a transversal to the $g$-foliation at $s(v_0)$. We have $Ts \cdot T_{v_0}V \oplus \zeta(s(v_0))(g) = T_{s(v_0)}M$. Then $s$ induces a mapping $V \to G \times M$ and $V \to GM$ and we may use the point $s(v)$ in replacing $O(s(v))$ by $G/H_{s(v)}$ for $v$ near $v_0$.

The following diagram summarizes the relation between the preceding constructions.

$$
\begin{array}{ccc}
M & \to & U_{[x] \in M/g} G/H_x \\
\downarrow & & \downarrow \\
M/g & \cong & G M = G \times _g M \\
\pi & \cong & \pi_G \\
M/G & \to & (G \times M/F_\zeta)/G \\
\end{array}
$$

Note that taking the $T_1$-quotient $G \times M/F_\zeta$ of the leaf space $gM$ may be a very severe reduction. In Example 6 the isotropy groups $H_x$ are trivial and we have $G \times M/F_\zeta = \mathbb{R}^2 \times \{0\}$ and $(G \times M/F_\zeta)/G = \{0\}$.

10. Palais’ treatment of $g$-manifolds. In [7], Palais considered $g$-actions on finite-dimensional manifolds $M$ in the following way. He assumed from the beginning that $M$ may be a non-Hausdorff manifold, since the completion may be non-Hausdorff. Then he introduced notions which we can express as follows in the terms introduced here:

(10.1) $(M, \zeta)$ is called generating if it generates a local $G$-transformation group. See [7, II, 2, Def. V and II, 7, Thm. XI]. This holds if and only if the leaves of the graph foliation on $G \times M$ described in Section 3 are Hausdorff. For Hausdorff $g$-manifolds this is always the case.

(10.2) $(M, \zeta)$ is called uniform if $pr_1 : L(e, x) \to G$ in (3.2) is a covering map for each $x \in M$. See [7, III, 6, Def. VIII and III, 6, Thm. XVII, Cor., Cor. 2]. In the Hausdorff case the $g$-action is then complete and it may be integrated to a $G$-action, where $G$ is a simply connected Lie group with Lie algebra $g$, so that $G M \cong M$.

(10.3) $(M, \zeta)$ is called univalent if $pr_1 : L(e, x) \to G$ in (3.2) is injective for all $x$. See [7, III, 2, Def. VI and III, 4, Thm. X].

(10.4) $(M, \zeta)$ is called globalizable if there exists a (non-Hausdorff) $G$-manifold $N$ which contains $M$ equivariantly as an open submanifold. See [7, III, 1, Def. II and III, 4, Thm. X]. This is a severe condition which is not satisfied in examples 4 and 6 above.

Palais’ main result on (non-Hausdorff) manifolds with a vector field says that (10.1), (10.3), and (10.4) are equivalent. See [7, III, 7, Thm. XX].

On (non-Hausdorff) $g$-manifolds his main result is that (10.3) and (10.4) are equivalent. See [7, III, 2, Def. VI and III, 4, Thm. X], and also [7, III, 1, Def. II and III, 4, Thm. X].
11. Concluding remarks. (11.1) A suitable setting for further development might be the class of discrete $g$-manifolds, that is, $g$-manifolds for which the $G$-space $\mathring{G}M$ is $T_1$, or equivalently the leaves of the graph foliation $F$ on $\mathring{G} \times M$ are closed. In this case, the charts $j_g : M \rightarrow \mathring{G}M$ in (5.1) are local diffeomorphisms with respect to the unique smooth structure on $\mathring{G}M$, and $\mathring{G}M$ is a smooth manifold, albeit not necessarily Hausdorff.

(11.2) In the context of (11.1), there are several definitions of proper $g$-actions, all of which are equivalent to saying that the $G$-action on $\mathring{G}M$ is proper. Many properties of proper actions will carry over to this case.

References