ENODSOPIC DECOMPOSITION OF CHARACTERS OF CERTAIN CUSPIDAL REPRESENTATIONS

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Abstract. We construct an endoscopic decomposition for local $L$-packets associated to irreducible cuspidal Deligne-Lusztig representations. Moreover, the obtained decomposition is compatible with inner twistings.

1. Introduction

Let $E$ be a local non-Archimedean field, with ring of integers $\mathcal{O}$ and residue field $\mathbb{F}_q$ of characteristic $p$. We denote by $\Gamma \supset W \supset I$ the absolute Galois, the Weil, and the inertia groups of $E$. Let $G$ be a reductive group over $E$, $L_G = bG$ its complex Langlands dual group, and $\mathcal{D}(G(E))$ the space of invariant distributions on $G(E)$.

Every admissible homomorphism $\lambda : W \to L_G$ (see [Ke1, \S 10]) gives rise to a finite group $S_\lambda := \pi_0(Z_G(\lambda)/Z(\hat{G})^I)$, where $Z_G(\lambda)$ is the centralizer of $\lambda(W)$ in $\hat{G}$. Every conjugacy class $\kappa$ of $S_\lambda$ defines an endoscopic subspace $\mathcal{D}_{\kappa, \lambda}(G(E)) \subset \mathcal{D}(G(E))$. For simplicity, we will restrict ourselves to the elliptic case, where $\lambda(W)$ does not lie in any proper Levi subgroup of $L_G$.

Langlands conjectured that every elliptic $\lambda$ corresponds to a finite set $\Pi_\lambda$, called an $L$-packet, of cuspidal irreducible representations of $G(E)$. Moreover, the subspace $\mathcal{D}_\lambda(G(E)) \subset \mathcal{D}(G(E))$, generated by the characters $\{\chi(\pi)\}_{\pi \in \Pi_\lambda}$, must have an endoscopic decomposition. More precisely, it is expected ([La1, IV, 2]) that there exists a basis $\{a_\pi\}_{\pi \in \Pi_\lambda}$ of the space of central functions on $S_\lambda$ such that $\chi_{\kappa, \lambda} := \sum_{\pi \in \Pi_\lambda} a_\pi(\kappa)\chi(\pi)$ belongs to $\mathcal{D}_{\kappa, \lambda}(G)$ for every conjugacy class $\kappa$ of $S_\lambda$.

The goal of this paper is to construct the endoscopic decomposition of $\mathcal{D}_{\lambda}(G(E))$ for tamely ramified $\lambda$'s such that $Z_G(\lambda(I))$ is a maximal torus. In this case, $G$ splits over an unramified extension of $E$, and $\lambda$ factors through $L^T \hookrightarrow L_G$ for an elliptic unramified maximal torus $T$ of $G$. By the local Langlands correspondence for tori ([La2]), a homomorphism $\lambda : W \to L_T$ defines a tamely ramified homomorphism $\theta : T(E) \to \mathbb{C}^\times$. Each $\kappa \in S_\lambda = \hat{T}^T/Z(\hat{G})^I$ gives rise to an elliptic endoscopic datum $E_{\kappa, \lambda}$ of $G$, while the characters of $S_\lambda$ are in bijection with the conjugacy classes of embeddings $T \hookrightarrow G$, stably conjugate to the inclusion. Therefore each
character $a$ of $S_{\lambda}$ gives rise to an irreducible cuspidal representation $\pi_{a,\lambda}$ of $G(E)$ (denoted by $\pi_{a,\delta}$ in Notation 2.3).

Our main result asserts, for fields $E$ of sufficiently large residual characteristic, that each $\chi_{\kappa,\lambda} := \sum a(\kappa)\chi(\pi_{a,\lambda})$ is $\mathcal{E}_{\kappa,\lambda}$-stable. Moreover, the resulting endoscopic decomposition of $\mathcal{D}_{\lambda}(G(E))$ is compatible with inner twistings. For simplicity, we restrict ourselves to local fields of characteristic zero, while the case of positive characteristic follows by approximation (see [Ka3], [De]).

Our argument goes as follows. First we prove the stability of the restriction of $\chi_{\kappa,\lambda}$ to the subset of topologically unipotent elements of $G(E)$. If $p$ is sufficiently large, this assertion reduces to the analogous assertion about distributions on the Lie algebra. Now the stability follows from a combination of a Springer hypothesis [Ka1] and a generalization of a theorem of Waldspurger [Wa]. To prove the result in general, we use the topological Jordan decomposition ([Ka2]).

When this work was in the process of writing, we have heard that S. DeBacker and M. Reeder obtained similar results.

**Notation and conventions.** In addition to the notation introduced above, we use the following conventions:

For a reductive group $G$, always assumed to be connected, we denote by $Z(G)$, $G^{\text{ad}}$, $G^{\text{der}}$, $G^{\text{sc}}$, $G_{\delta}$, and $G^{\text{ss}}$ the center of $G$, the adjoint group of $G$, the derived group of $G$, the simply connected covering of $G^{\text{der}}$, the centralizer of $\delta \in G$, and the set of strongly regular semisimple elements of $G$ (that is, the set of $\delta \in G$ such that $G_{\delta} \subset G$ is a maximal torus), respectively.

Denote by $G$, $T$, and $\mathcal{L}$ the Lie algebras of the algebraic groups $G$, $T$, and $L$.

Let $E$ be a local non-Archimedean field of characteristic zero, $\overline{E}$ a fixed algebraic closure of $E$, and $E^{nr}$ a maximal unramified extension of $E$ in $\overline{E}$.

For a reductive group $G$ (resp. its Lie algebra $\mathfrak{g}$) over $E$, we denote by $S(G(E))$ (resp. $S(\mathfrak{g}(E))$) the space of locally constant measures with compact support. We denote by $\mathcal{D}(G(E))$ (resp. $\mathcal{D}(\mathfrak{g}(E))$) the space of invariant distributions on $G(E)$ (resp. $\mathfrak{g}(E)$), namely $G(E)$-invariant linear functionals on $S(G(E))$ (resp. $S(\mathfrak{g}(E))$), where $G(E)$ acts by conjugation. Whenever necessary we equip $G(E)$ and $\mathfrak{g}(E)$ with invariant measures, denoted by $\mu$, defined by a translation-invariant top degree differential form on $G$. We denote by $G(E)_{\text{ua}}$ (resp. $\mathfrak{g}(E)_{\text{ua}}$) the set of topologically unipotent (resp. topologically nilpotent) elements of $G(E)$ (resp. $\mathfrak{g}(E)$).

Finally, we denote by $rk(G)$ the rank of $G$ over $E$, and put $e(G) := (-1)^{rk(G^{\text{ss}})}$. Note that our sign $e(G)$ differs from that defined by Kottwitz.

2. Formulation of the main result

2.1. Let $L$ be a connected reductive group over $\mathbb{F}_{q}$, and $\overline{T} : \overline{L} \hookrightarrow L$ an embedding of a maximal elliptic torus of $L$. Following Deligne and Lusztig, we associate to every character $\vartheta : \overline{T}(\mathbb{F}_{q}) \to \mathbb{C}^\times$ in general position an irreducible cuspidal representation $\rho_{\vartheta,\overline{T}}$ of $L(\mathbb{F}_{q})$ (see [DL] Prop. 7.4 and Thm. 8.3).

2.2. There is an equivalence of categories $T \mapsto \mathcal{T}$ between tori over $E$ splitting over $E^{nr}$ and tori over $\mathbb{F}_{q}$. Every such $T$ has a canonical $\mathcal{O}$-structure.

**Notation 2.3.** a) Let $G$ be a reductive group over $E$, $T$ a torus over $E$ splitting over $E^{nr}$, and $a : T \hookrightarrow G$ an embedding of a maximal elliptic torus of $G$. Then $G$ splits over $E^{nr}$, and $a(T(\mathcal{O}))$ lies in a unique parahoric subgroup $G_{a}$ of $G(E)$. Let
$G_a^+$ be the pro-unipotent radical of $G_a$. Then there exists a canonical reductive group $L_a$ over $\mathbb{F}_q$ with an identification $L_a(\mathbb{F}_q) = G_a/G_a^+$. Moreover, $a : T \twoheadrightarrow G$ induces an embedding $\pi : T \hookrightarrow L_a$ of a maximal elliptic torus of $L_a$.

b) Let $\theta : T(E) \rightarrow \mathbb{C}^\times$ be a character in general position, trivial on $\text{Ker}[T(O) \rightarrow T(\mathbb{F}_q)]$. Denote by $\vartheta : T(\mathbb{F}_q) \rightarrow \mathbb{C}^\times$ the character of $T(\mathbb{F}_q)$ defined by $\theta$. Then there exists a unique irreducible representation $\rho_{a, \theta}$ of $Z(G)(E)G_a$, whose central character is the restriction of $\theta$, extending the inflation to $G_a$ of the cuspidal Deligne-Lusztig representation $\rho_{\pi, \theta}$ of $L_a(\mathbb{F}_q)$. We denote by $\pi_{a, \theta}$ the induced cuspidal representation $\text{Ind}_{Z(G)(E)G_a}^{G(E)} \rho_{a, \theta}$ of $G(E)$.

**2.4.** Recall (see [Ko2 Thm 1.2]) that for every reductive group $G$ over $E$, $H^1(E, G)$ is canonically isomorphic to the group $\pi_0(Z(\hat{G})^\Gamma)$ of characters of $\pi_0(Z(\hat{G})^\Gamma)$. If $T$ is a maximal torus of $G$, we get a commutative diagram:

$$
\begin{array}{ccc}
H^1(E, T) & \overset{\sim}{\longrightarrow} & \pi_0(\hat{T}^\Gamma) \\
\downarrow & & \downarrow \\
H^1(E, G) & \overset{\sim}{\longrightarrow} & \pi_0(Z(\hat{G})^\Gamma) 
\end{array}
$$

In particular, we have a canonical surjection

$$
\hat{T}^\Gamma/Z(\hat{G})^\Gamma \twoheadrightarrow \text{Coker}[\pi_0(Z(\hat{G})^\Gamma) \rightarrow \pi_0(\hat{T}^\Gamma)] \cong (\text{Ker}[H^1(E, T) \rightarrow H^1(E, G)])^D.
$$

**Notation 2.5.** a) To every pair $a, a'$ of stably conjugate embeddings $T \hookrightarrow G$, one associates the class $\text{inv}(a', a) \in \text{Ker}[H^1(E, T) \rightarrow H^1(E, G)]$. This is the class of a cocycle $c_\sigma = g^{-1}\sigma(g)$, where $g \in G(T)$ is such that $a' = gag^{-1}$ (compare [Ko2 4.1]).

b) To each $\kappa \in \hat{T}^\Gamma/Z(\hat{G})^\Gamma$, an embedding $a_0 : T \hookrightarrow G$, and a character $\theta$ of $T(E)$ as in Notation 2.3, we associate the invariant distribution

$$
\chi_{a_0, \kappa, \theta} := e(G) \sum_a \langle \text{inv}(a, a_0), \kappa \rangle \chi(\pi_{a, \theta}).
$$

Here $a$ runs over a set of representatives of conjugacy classes of embeddings which are stably conjugate to $a_0$, and $\chi(\pi_{a, \theta})$ denotes the character of $\pi_{a, \theta}$.

**Notation 2.6.** Each pair $(a, \kappa)$, where $a : T \hookrightarrow G$ is an embedding of a maximal torus of $G$ and $\kappa$ is an element of $\hat{T}^\Gamma$, gives rise to an isomorphism class $E_{(a, \kappa, \eta)}$ of an endoscopic datum of $G$. Furthermore, $E_{(a, \kappa, \eta)}$ is elliptic if $a(T)$ is an elliptic torus of $G$ (see [Kol §7] for the definitions of endoscopic data, and compare [La1 II, 4]).

More precisely, each embedding $\eta : \hat{T} \hookrightarrow \hat{G}$, whose conjugacy class corresponds to the stable conjugacy class of $a$, defines an endoscopic datum $E_{(a, \kappa, \eta)} = (s, \rho)$, consisting of a semisimple element $s = \eta(\kappa)$ of $\hat{G}$ and a homomorphism $\rho : \Gamma_{\text{pr-}} \rightarrow \text{Norm}_{\text{Aut}_{\hat{G}}}^\kappa(\eta(\hat{T}))_{\mathfrak{s}}/\eta(\hat{T}) \rightarrow \text{Out}(\hat{G}_{\mathfrak{s}})$. Here $\rho_T$ is induced by the $E$-structure of $T$, and $\rho'$ is induced by the inclusion $\text{Norm}_{\text{Aut}_{\hat{G}}}^\kappa(\eta(\hat{T}))_{\mathfrak{s}} \subset \text{Norm}_{\text{Aut}_{\hat{G}}}^\kappa(\hat{G}_{\mathfrak{s}})$. Moreover, the isomorphism class of $E_{(a, \kappa, \eta)}$, denoted by $E_{(a, \kappa)}$, does not depend on $\eta$.

**Notation 2.7.** For each $\gamma \in G^\gamma(E)$ and $\xi \in \hat{G}_{\gamma}^\Gamma$,

(i) put $E_{(\gamma, \xi)} := E_{(a_\gamma, \xi)}$, where $a_\gamma : G_\gamma \hookrightarrow G$ is the inclusion map;
(ii) fix an invariant measure $dg_\gamma$ on $G_\gamma(E)$, and put

$$O_\gamma(\phi) := \int_{G(E)/G_\gamma(E)} f(\gamma^{-1}g) \, dg_\gamma$$

for each $\phi = f dg \in S(G(E))$;

(iii) denote by $\xi \in \pi_0(\hat{G}^\Gamma / \hat{G})$ the class of $\xi$;

(iv) denote by $O_{\gamma, \xi} \in \mathcal{D}(G(E))$ the sum $\sum_{\gamma'} (\text{inv}(\gamma', \gamma), \xi)O_{\gamma', \xi}$, taken over a set of representatives of the conjugacy classes stably conjugate to $\gamma$, where each $dg_{\gamma'}$ is compatible with $dg_\gamma$.

**Definition 2.8.** Let $\mathcal{E}$ be an endoscopic datum of $G$.

(i) A measure $\phi \in S(G(E))$ is called $\mathcal{E}$-unstable if $O_{\gamma, \xi}(\phi) = 0$ for all pairs $(\gamma, \xi)$ as in Notation 2.7, for which $E_{(\gamma, \xi)}$ is isomorphic to $\mathcal{E}$.

(ii) A distribution $F \in \mathcal{D}(G(E))$ is called $\mathcal{E}$-stable if $F(\phi) = 0$ for all $\mathcal{E}$-unstable $\phi \in S(G(E))$.

**Theorem 2.9.** Assume that $p > \dim G^{\der}$. Then for each triple $(a_0, \kappa, \theta)$, the distribution $\chi_{a_0, \kappa, \theta}$ is $\mathcal{E}_{(a_0, \kappa)}$-stable.

**Notation 2.10.** For an endoscopic datum $\mathcal{E} = (s, \rho)$, choose a representative $\tilde{s} \in \hat{G}^{\text{sc}}$ of $s$, and let $Z(\mathcal{E})$ be the set of $z \in Z(\hat{G}^{\text{sc}})^\Gamma$ for which there exists $g \in \tilde{G}$ commuting with $\rho : \Gamma \to \text{Out}(\tilde{G}_s^0)$ such that $g\tilde{s}g^{-1} = z\tilde{s}$. Then $Z(\mathcal{E})$ is a subgroup of $Z(\hat{G}^{\text{sc}})^\Gamma$, depending only on the isomorphism class of $\mathcal{E}$.

**Definition 2.11.** Let $\mathcal{E}$ be an endoscopic datum of $G$. An inner twisting $\varphi : G \to G'$ is called $\mathcal{E}$-admissible if the corresponding class $\text{inv}(G', G) \in H^1(E, G^{\ad}) \cong (Z(\hat{G}^{\text{sc}})^\Gamma)^\theta$ is orthogonal to $Z(\mathcal{E}) \subset Z(\hat{G}^{\text{sc}})^\Gamma$.

**Definition 2.12.** Let $G$ be a reductive group over $E$, $\mathcal{E} = (s, \rho)$ an elliptic endoscopic datum of $G$, and $\varphi : G \to G'$ an $\mathcal{E}$-admissible inner twisting. Fix a triple $(a, a'; \kappa)$, consisting of a pair $s : T \hookrightarrow G$ and $a' : T \hookrightarrow G'$ of stably conjugate embeddings of maximal tori, and an element $\kappa \in \hat{T}^\Gamma$ such that $E_{(a, \kappa)} \cong \mathcal{E}$.

a) Consider $\phi \in S(G(E))$ and $\phi' \in S(G'(E))$. They are called $(a, a'; \kappa)$-indistinguishable if they satisfy the following conditions.

(A) For every $\gamma \in G^{\text{sc}}(E)$ and $\xi \in \hat{G}_\gamma^\Gamma$ such that $E_{(\gamma, \xi)} \cong \mathcal{E}$ and $O_{\gamma, \xi}(\phi) \neq 0$,

(i) there exists $\gamma' \in G'(E)$ stably conjugate to $\gamma$;

(ii) we have $O_{\gamma'}(\phi') = (\gamma', \xi)O_{\gamma, \xi}(\phi)$.

Here $(\gamma', \xi) \in \mathbb{C}^\times$ is the invariant $(\gamma', \xi) = (a_{\gamma'}, \kappa) = (\gamma, \xi)$ defined in the Appendix for embeddings $a_\gamma : G_\gamma \hookrightarrow G$ and $a_{\gamma'} : G_{\gamma'} \hookrightarrow G'$ such that $a_{\gamma'}(\gamma) = \gamma$ and $a_{\gamma'}(\xi) = \gamma'$.

(B) Condition (A) holds if $G, a_0, \gamma, \phi$ are interchanged with $G', a'_0, \gamma', \phi'$.

b) The distributions $F \in \mathcal{D}(G(E))$ and $F' \in \mathcal{D}(G'(E))$ are called $(a, a'; \kappa)$-equivalent if $F(\phi) = F'(\phi')$ for every two $(a, a'; \kappa)$-indistinguishable measures $\phi$ and $\phi'$.

**Remark 2.13.** If $\phi$ is $\mathcal{E}_{(a_0, \kappa)}$-unstable, then $\phi$ and $\phi' = 0$ are $(a_0, a'_0, \kappa)$-indistinguishable. Therefore every two $(a_0, a'_0, \kappa)$-equivalent distributions $F$ and $F'$ are $\mathcal{E}_{(a_0, \kappa)}$-stable.
Main Theorem 2.14. Assume that $p > \dim G^{\text{der}}$. Let $\varphi : G \to G'$ be an $E(a_0, \kappa)$-admissible inner twisting. Let $a_0' : T \to G'$ be an embedding which is stably conjugate to $a_0$. Then the distributions $\chi_{a_0, \kappa, \theta}$ on $G(E)$ and $\chi_{a_0', \kappa, \theta}$ on $G'(E)$ are $(a_0, a_0'; \kappa)$-equivalent.

Remark 2.15. a) By Remark 2.13, Theorem 2.9 follows from the Main Theorem.

b) We believe that a much smaller bound on $p$ would suffice.

3. Basic ingredients of the argument

3.1. A generalization of a theorem of Waldspurger. Suppose that we are in the situation of Definition 2.12. Then $\varphi$ induces an inner twisting $G \to G'$. As in Definition 2.12, one can define $(a, a'; \kappa)$-equivalence of $F \in D(G(E))$ and $F' \in D(G'(E))$.

Fix a nontrivial character $\psi : E \to \mathbb{C}^\times$, a nondegenerate $G$-invariant pairing $\langle \cdot, \cdot \rangle$ on $G$, and $\varphi$-compatible invariant measures on $G(E)$ and $G'(E)$. Then $\varphi$ defines a nondegenerate $G'$-invariant pairing $\langle \cdot, \cdot \rangle'$ on $G'$. These data determine the Fourier transforms $F \mapsto F(F)$ on $G(E)$ and $G'(E)$.

Theorem 3.1. The distributions $F \in D(G(E))$ and $F' \in D(G'(E))$ are $(a, a'; \kappa)$-equivalent if and only if $\epsilon(G) F(F)$ and $\epsilon(G') F(F')$ are $(a, a'; \kappa)$-equivalent.

The proof is a generalization of that of Waldspurger [Wa], who treated the case $\phi' = 0$ (compare also [KP] Thm. 2.7.1, where the stable case is considered).

3.2. Springer hypothesis. In the notation of 2.1, assume that $\mathfrak{a}(T_0) \subset \mathcal{L}(F_q)$ contains an $L$-regular element $\overline{\mathfrak{t}}$ [and that $p$ is so large that the logarithm defines an isomorphism $\log : L_{\text{un}} \to L_{\text{nil}}$ between unipotent elements of $L$ and nilpotent elements of $\mathcal{L}$]. Let $\mathfrak{t}$ be the characteristic function of the Ad$(L(F_q))$-orbit of $\overline{\mathfrak{t}}$, and let $\mathcal{F}(\mathfrak{t})$ be its Fourier transform. We need the following result of [Ka1].

Theorem 3.2. For every $u \in L_{\text{un}}(F_q)$, we have

$$\Tr \rho_{\mathfrak{t}}(u) = q^{-(\dim L - \dim \mathfrak{t})/2} \mathcal{F}(\mathfrak{t})(\log(u)).$$

3.3. Topological Jordan decomposition. We will call an element $\gamma \in G(E)$ compact if it generates a relatively compact subgroup of $G(E)$. We will call an element $\gamma \in G(E)$ topologically unipotent if the sequence $\{\gamma^p\}$ converges to 1. Every topologically unipotent element is compact. The following result is a rather straightforward generalization of [Ka2, Lem. 2, p. 226].

Lemma 3.3. For every compact element $\gamma \in G(E)$ there exists a unique decomposition $\gamma = \delta u$ such that $\delta$ and $u$ commute, $\delta$ is of finite order prime to $p$, and $u$ is topologically unipotent. In particular, this decomposition is compatible with conjugation and field extensions.

4. A sketch of the proof of the main theorem

4.1. Reformulation of the problem.

Notation 4.1. To each $a : T \hookrightarrow G$ and $\theta : T(E) \to \mathbb{C}^\times$ as in Notation 2.3 we associate a function $t_{a, \theta}$ on $G(E)$ supported on $Z(G)(E)G_a$ and equal to $\Tr \rho_{a, \theta}$ there. Since $t_{a, \theta}$ is cuspidal, the integral

$$F_{a, \theta} : \gamma := \frac{1}{\mu((G^{\text{ad}})_a)} \int_{G(E)/Z(G)(E)} t_{a, \theta}(g^\gamma g^{-1})dg$$

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stabilizes for every $\gamma \in G^\text{st}(E)$ (see \cite[Lem. 23]{HC}), thus providing us with a locally constant invariant function $F_{a,\theta}$ on $G^\text{st}(E)$.

**Lemma 4.2.** For each $a$ and $\theta$, $F_{a,\theta}$ is a locally $L^1$-function on $G(E)$. Moreover, the corresponding distribution equals $\chi(\pi_{a,\theta})$.

**Proof.** The assertion follows from Harish-Chandra’s theorem [HC Thm. 16]. \hfill \Box

**Notation 4.3.** For every $\gamma_0 \in G^\text{st}$ and $\bar{\xi} \in \pi_0(G_{\gamma_0}^{-\Gamma} / Z(\hat{G})^{\Gamma})$ we define

\begin{equation}
\Sigma_{G;\gamma_0;\xi;\mathit{a}_0;\kappa} := e(G) \sum_a \sum_{\gamma} \{\text{inv}(a, a_0), \kappa\} \{\text{inv}(\gamma, \gamma_0), \bar{\xi}\}^{-1} F_{a,\theta}(\gamma),
\end{equation}

where $a$ and $\gamma$ run over sets of representatives of the conjugacy classes within the stable conjugacy classes of $a_0$ and $\gamma_0$, respectively.

**Theorem 4.4.** For all $\gamma_0 \in G^\text{st}$ and $\bar{\xi} \in \pi_0(G_{\gamma_0}^{-\Gamma} / Z(\hat{G})^{\Gamma})$ such that $\Sigma_{G;\gamma_0;\xi;\mathit{a}_0;\kappa} \neq 0$,

(i) there exists a representative $\xi \in G_{\gamma_0}^{-\Gamma}$ of $\bar{\xi}$ such that $\mathcal{E}_{\gamma_0, \xi} \simeq \mathcal{E}_{(a_0, \kappa)}$;

(ii) if $\varphi : G \to G'$ is $(\mathcal{E}, a_0, \bar{\xi})$-admissible (see Definition A.1), then for every $\xi$ as in (i) and every stably conjugate $\gamma_0' \in G'(E)$ of $\gamma_0$ we have

\[\Sigma_{G';\gamma_0';\xi;\mathit{a}_0';\kappa} = \left(\frac{\gamma_0'}{\gamma_0}\right) \Sigma_{G;\gamma_0;\xi;\mathit{a}_0;\kappa}.\]

**Notation 4.5.** It follows from Lemma 4.2 that Theorem 4.4 is equivalent to the Main Theorem. Moreover, by standard arguments, Theorem 4.4 reduces to the case when the derived group of $G$ is simply connected.

**4.6.** From now on we will assume that $G^\text{der} = G^\text{sc}$. In particular, the centralizer of each semisimple element of $G$ is connected, and each $G_a$ is a maximal compact subgroup of $G(E)$. We fix $(\gamma_0, \bar{\xi})$ such that $\Sigma_{G;\gamma_0;\xi;\mathit{a}_0;\kappa} \neq 0$. Since $\Sigma_{G;\gamma_0;\xi;\mathit{a}_0;\kappa} = \theta(z) \Sigma_{G;\gamma_0;\xi;\mathit{a}_0;\kappa}$ for each $z \in Z(G)(E)$ and since the support of each $t_{a,\theta}$ consists of elements compact modulo center, we can assume that $\gamma_0$ is compact with topological Jordan decomposition $\gamma_0 = \delta_0 u_0$. Moreover, we can assume that either $\gamma_0$ is topologically unipotent, or $\delta_0 \notin Z(G)(E)$.

**4.2. The topologically unipotent case.**

**4.7.** Since $p$ does not divide the order of $Z(G^\text{der})$, the canonical map $G^\text{der}(E)_{\text{tu}} \times Z(G)(E)_{\text{tu}} \to G(E)_{\text{tu}}$ is an isomorphism. Therefore to prove Theorem 4.4 for topologically unipotent $\gamma_0$, we can assume that $G$ is semisimple and simply connected.

**Notation 4.8.** Denote by $\Phi_G : G \to G$ the composition map

\[G \xrightarrow{\text{Ad}} GL(G) \xrightarrow{\text{log}(p)} \text{End}(G) \xrightarrow{\text{pr}} G,\]

where $\text{log}(p)(1-A) = -\sum_{i=1}^{p-1} A^i$, and $\text{pr}$ is the canonical projection, defined by the standard pairing $(A, B) \mapsto \text{Tr} AB$ on $\text{End}(G)$.

**Lemma 4.9.** The map $\Phi_G$ defines a $G(E)$-equivariant homeomorphism

\[G(E)_{\text{tu}} \xrightarrow{\sim} G(E)_{\text{tn}},\]

where $G(E)$ acts by conjugation. Moreover, for every parahoric subgroup $G_x$ of $G(E)$, $\Phi_G$ induces a bijection $(\Phi_G)_x : (G_x)_{\text{tu}} \xrightarrow{\sim} (G_x)_{\text{tn}}$, which in turn induces the logarithm map $\log : (L_x)_{\text{un}}(F_q) \xrightarrow{\sim} (L_x)_{\text{nil}}(F_q)$. 


Notation 4.10. a) By our assumption on \( p \), there exists \( t \in T(\mathcal{O}) \) whose reduction \( T(\mathbb{F}_q) \) is not fixed by any nontrivial element of the Weyl group of \( G \).

b) For every \( a : T \rightarrow G \) as in Notation 2.3 we denote by \( \Omega_{a,t} \subset \mathcal{L}_a(\mathbb{F}_q) \) the \( \text{Ad}(L_a(\mathbb{F}_q)) \)-orbit of \( \pi(t) \), by \( \overline{\Omega}_{a,t} \subset G_a \subset G \) the preimage of \( \Omega_{a,t} \), and let \( \delta_{a,t} \) be the characteristic function of \( \overline{\Omega}_{a,t} \).

c) As the centralizer \( G_y \) of each \( y \in \overline{\Omega}_{a,t} \) is \( G_a \)-conjugate to \( a(T) \), the integral

\[
\Delta_{a,t}(x) := \frac{1}{\mu(G_a)} \int_{G(E)} \delta_{a,t}(\text{Ad}(g)x)dg
\]

converges absolutely for each \( x \in G \). Thus it defines an element of \( \mathcal{D}(G(E)) \). Similarly to Notation 2.4 we consider \( \Delta_{a_0,\kappa,t} := e(G) \sum_a (\text{inv}(a, a_0), \kappa) \Delta_{a,t} \in \mathcal{D}(G(E)) \).

Lemma 4.11. Let \( T^+ \subset G(E) \) be a maximal topologically nilpotent subalgebra. Assume that \( \psi : E \rightarrow \mathbb{C}^\times \) is trivial on the maximal ideal \( M \subset \mathcal{O} \) and induces a nontrivial character of \( \mathbb{F}_q \). Then for each \( u \in G(E)_{tu} \) we have

\[
t_{a,\kappa}(u) = \mu(T^+)^{-1} \mathcal{F}(\delta_{a,t})(\Phi_G(u)).
\]

Proof. The assumption on \( \psi \) implies that \( G_{a^+} \) is the orthogonal complement of \( G_a \) with respect to the pairing \( (x, y) \mapsto \psi((x, y)) \). Therefore our lemma is an immediate consequence of the definition of the Fourier transform (over \( E \) and \( \mathbb{F}_q \)), Theorem 4.2, Lemma 4.9 and the equality \( q^{(\dim L_a - \dim T)/2} \mu(G_{a^+}) = \mu(T^+) \). \( \Box \)

4.12. Now we are ready to show that \( (\chi_{a_0,\kappa,\theta})|_{G(E)_{tu}} \) and \( (\chi_{a_0',\kappa,\theta})|_{G(E)_{tu}} \) are \( (a_0, a_0'; \kappa) \)-equivalent. First of all, by direct calculations, \( e(G) \Delta_{a_0,\kappa,t} \) is \( (a_0, a_0'; \kappa) \)-equivalent to \( e(G') \Delta_{a_0',\kappa,t} \). Hence, by Theorem 3.1 \( \mathcal{F}(\Delta_{a_0,\kappa,t}) \) is \( (a_0, a_0'; \kappa) \)-equivalent to \( \mathcal{F}(\Delta_{a_0',\kappa,t}) \). Using Lemmas 4.2, 4.9 and 4.11 we see that \( \chi_{a_0,\kappa,\theta} \) has the same restriction to \( G(E)_{tu} \) as \( \mu(T^+)^{-1} \Phi_G(\mathcal{F}(\Delta_{a_0,\kappa,t})) \), and similarly for \( \chi_{a_0',\kappa,\theta} \) and \( \mu(T^+)^{-1} \Phi_G(\mathcal{F}(\Delta_{a_0',\kappa,t})) \). Since \( \Phi_G \) is a \( G^\text{ad} \)-invariant algebraic morphism defined over \( E \), the assertion follows from the equality \( \mu(T^+) = \mu(T^+) \).

4.3. The general case. It remains to prove Theorem 4.1 for \( \delta_0 \notin Z(G)(E) \) (see 1.6). We are going to deduce the assertion from that for \( G_{b_0} \).

Proposition 4.13. For every embedding \( a : T \rightarrow G \) and a compact element \( \gamma \) in \( G(E) \) with topological Jordan decomposition \( \gamma = \delta u \), we have

\[
e(G)F_{a,\theta}(\gamma) = e(G_\delta) \sum_b \theta(b^{-1}(\delta))F_{b,\theta}(u).
\]

Here \( b \) runs over the set of conjugacy classes of embeddings \( b : T \rightarrow G_\delta \) whose composition with the inclusion \( G_\delta \subset G \) is conjugate to \( a \).

Proof. The proposition follows by direct calculation from the recursive formula ([DL, Thm. 4.2]) for characters of Deligne-Lusztig representations. \( \Box \)

Notation 4.14. a) We say that \( t \in T(E) \) is \( (G, a_0, \gamma_0) \)-relevant if there exists an embedding \( b_0 : T \rightarrow G_{b_0} \subset G \) stably conjugate to \( a_0 \) such that \( b(t) = \delta_0 \).

b) Assume that \( t \in T(E) \) is \( (G, a_0, \gamma_0) \)-relevant. Since \( a_0(T) \subset G \) is elliptic, for each \( \delta \in G(E) \) stably conjugate to \( \delta_0 \) there exists an embedding \( b_{\delta,\delta} : T \rightarrow G_\delta \subset G \) stably conjugate to \( a_0 \) such that \( b_{\delta,\delta}(t) = \delta \). Further, \( b_{\delta,\delta} \) is unique up to stable conjugacy, and the endoscopic datum \( \mathcal{E}_{t,\kappa} := \mathcal{E}_{t,\mu}(b_{\delta,\delta}, \kappa) \) of \( G_{b_0} \) is independent of \( \delta \).
Therefore for all \( \delta \in G(E) \) and \( \delta' \in G'(E) \) are stably conjugate to \( \delta_0 \), and \( G_{\delta_0} \) (resp. \( G'_{\delta_0'} \)) is an \( \mathcal{E}_{t,n} \)-admissible inner form of \( G_{\delta} \) (see Definition 2.11).

**4.15.** Using Proposition 4.13 we see that

\[
\sum_{G:G_{\delta_0}=0} \tau_{a,0} = \sum_t \theta(t) \sum_{\delta} I_{t,\delta},
\]

where

(i) \( t \) runs over the set of \((G, a_0, \gamma_0)\)-relevant elements of \( T(E) \);

(ii) \( \delta \) runs over a set of representatives of the conjugacy classes within the stable conjugacy class of \( \delta_0 \);

(iii) \( I_{t,\delta} \) vanishes unless there exists an element \( \gamma \in G(E) \) stably conjugate to \( \gamma_0 \) with topological Jordan decomposition \( \gamma = \delta u \), in which case we get

\[
I_{t,\delta} = \langle \text{inv}(\delta, \delta_0), a_0 \rangle \langle \text{inv}(\gamma, \gamma_0), \xi \rangle^{-1} \sum_{G:G_\delta=0} \tau_{b,0} \zeta_{b,0},\kappa.
\]

**4.16.** For simplicity of the exposition, we will restrict ourselves to the case when \( \gamma_0 \in G(E) \) is elliptic. Choose \( t \) which has a nonzero contribution to \( \text{(4.2)} \). Replacing \( \delta_0 \) by a stably conjugate element we can assume that \( \sum_{G:G_{\delta_0}=0} I_{t,\delta} \neq 0 \) and \( I_{t,\delta} \neq 0 \). Hence by Theorem 4.13 for \( G_{\delta_0} \) there exists a representative \( \xi \in \mathcal{G}_{\gamma_0}^{-1} \) such that the endoscopic datum \( \mathcal{E}(\gamma_0, \xi) \) of \( G_{\delta_0} \) is isomorphic to \( \mathcal{E}_{t,n} \). Therefore there exist embeddings \( \eta_1: \mathcal{G}_{\gamma_0} \hookrightarrow \mathcal{G}_{\delta_0} \) and \( \eta_2: \mathcal{T} \hookrightarrow \mathcal{G}_{\delta_0} \) such that \( \mathcal{E}(\gamma_0, \xi, \eta_1) = \mathcal{E}(\gamma_0, \xi, \eta_2) \) (compare Notation 2.10) and \( \text{inv}(\xi, \eta_1(\xi)) \) for a certain \( z \in Z(\mathcal{G}_{\delta_0})^\Gamma \). Moreover, \( z \) is defined up to multiplication by an element of \( Z(\mathcal{E}_{t,n}) \). Therefore for all \( \delta \sim_{\mathcal{E}_{t,n}} \delta_0 \), the expression \( \langle \text{inv}(\delta, \delta_0), z \rangle \) is independent of the choice of the \( \eta_1 \)’s.

**Claim 4.17.** For each \( \delta \sim_{\mathcal{E}_{t,n}} \delta_0 \) we have \( I_{t,\delta} = \langle \text{inv}(\delta, \delta_0), z \rangle I_{t,\delta_0} \).

**Proof.** Since \( \sum_{G:G_{\delta_0}=0} \tau_{b,0} \zeta_{b,0},\kappa \neq 0 \), Theorem 4.4 for inner forms \( G_{\delta} \) and \( G_{\delta_0} \) implies that for every stably conjugate \( u \in G_{\delta}(E) \) of \( u_0 \in G_{\delta_0}(E) \), we have

\[
\sum_{G:G_{\delta_0}=0} \tau_{b,0} \zeta_{b,0},\kappa = \langle u, u_0; \xi \rangle \sum_{G:G_{\delta_0}=0} \tau_{b,0} \zeta_{b,0},\kappa.
\]

Then \( \gamma := \delta u \in G(E) \) is stably conjugate to \( \gamma_0 \), and the assertion follows by direct calculation from \( \text{(4.3)} \).

**4.18.** Now we are ready to show the validity of (i), (ii) of Theorem 4.4

(i) As \( \sum_{G:G_{\delta_0}=0} I_{t,\delta} \neq 0 \), we get from Claim 4.17 that \( \sum_{G:G_{\delta_0}=0} \langle \text{inv}(\delta, \delta_0), z \rangle \neq 0 \). By the definition of \( \mathcal{E}_{t,n} \)-equivalence, this implies that \( z \) belongs to \( Z(\mathcal{E}_{t,n})Z(\mathcal{G})^\Gamma \). Thus changing \( \eta_1 \) (or \( \eta_2 \)), we can assume that \( z \in Z(\mathcal{G})^\Gamma \). Since \( \langle \gamma_0, \xi \rangle \) and \( \langle b, \gamma_0, \xi \rangle \) define isomorphic endoscopic data of \( G_{\delta_0} \), we therefore conclude that \( \mathcal{E}_{\eta_1} \equiv \mathcal{E}_{\eta_2} \).

(ii) Since \( T \) is elliptic, an element \( t \in T(E) \) is \((G, a_0, \gamma_0)\)-relevant if and only if it is \((G', a_0', \gamma_0')\)-relevant. Thus it will suffice to show that for every such \( t \) we have

\[
\sum_{G:G_{\delta_0}=0} I_{t,\delta} = \langle \gamma_0, \xi \rangle \sum_{G:G_{\delta_0}=0} I_{t,\delta}.
\]

For every stably conjugate \( \delta \in G(E) \) of \( \delta_0 \), there exists a stably conjugate \( \delta' \in G'(E) \) of \( \delta_0' \) such that \( \delta' \sim_{\mathcal{E}_{t,n}} \delta \). Therefore it will suffice to show that for every such pair \( \delta' \sim_{\mathcal{E}_{t,n}} \delta \), we have \( I_{t,\delta'} = \langle \gamma_0, \xi \rangle I_{t,\delta} \). The latter equality can be proved by the same arguments as Claim 4.17.
Let $G$ be a reductive group over $E$, $E = (s, \rho)$ an elliptic endoscopic datum of $G$, and $\varphi : G \to G'$ an $E$-admissible inner twisting. For every two triples $(a_i, a_i'; \kappa_i)$, $i = 1, 2$, where $a_i : T_i \to G$ and $a_i' : T_i \to G'$ are stably conjugate embeddings of maximal tori, and $\kappa_i$ is an element of $T_i^\Gamma$ such that $\mathcal{E}_{(a_i, \kappa_i)}$ is isomorphic to $\mathcal{E}$, we are going to define an invariant $(\frac{a_1';a_1}{a_2';a_2}) \in \mathbb{C}^\times$.

**Step 1.** Replacing $G$, $G'$, $T_i$, $\mathcal{E}$, and $\varphi$ by $G^{sc}$, $G'^{sc}$, $T_i^{sc} := a_i^{-1}(G^{sc}) = a_i'^{-1}(G'^{sc})$, the image of $\kappa_i$ in $T_{i}^{sc}$, and the corresponding endoscopic datum of $G^{sc}$, respectively, we can assume that $G$ is semisimple and simply connected. Let $T_{1,2}$ be the quotient of the product $T_1 \times T_2$ by the subgroup $\{(z, z^{-1}) | z \in Z(G) = Z(G')\}$.

**Step 2.** Choose elements $g_1, g_2$, and $\{\tilde{c}_\sigma\}_{\sigma \in \Gamma}$ of $G(\mathcal{E})$ such that $a_i' = \varphi(g_1 a_i g_1^{-1})$ and each $\tilde{c}_\sigma$ is a representative of $\varphi^{-1} \sigma \varphi \in G^{ad}(\mathcal{E})$. Then each $g_1^{-1} \tilde{c}_\sigma g_1 \in G(\mathcal{E})$ belongs to $a_i(T_i(\mathcal{E}))$, and the images of $(a_1^{-1}(g_1^{-1} \tilde{c}_\sigma g_1), a_2^{-1}((g_2^{-1} \tilde{c}_\sigma g_2)^{-1}))$ in $T_{1,2}(\mathcal{E})$ form a cocycle, whose cohomology class $\text{inv}(\frac{a_1';a_1}{a_2';a_2}) \in H^1(E, T_{1,2})$ is independent of the choices.

**Step 3.** Choose embeddings $\eta_i : \hat{T}_i \hookrightarrow \hat{G}$ such that $\mathcal{E}_{(a_i, \kappa_i, \eta_i)} = (s, \rho)$ and a representative $\tilde{s} \in \hat{G}^{sc} = \hat{G}^{ad}$ of $s$. Put $T_i^{ad} := T_i/a_i^{-1}(Z(G))$. Each $\eta_i$ defines an embedding $\tilde{\eta}_i : T_i^{ad} \hookrightarrow \hat{G}^{ad}$, hence an element $\tilde{\kappa}_i = \kappa(\tilde{s}, \eta_i) := \tilde{\eta}_i^{-1}(\tilde{s}) \in T_i^{ad}$. Then the image of $(\tilde{\kappa}_1, \tilde{\kappa}_2)$ in $T_1^{ad} \times T_2^{ad}/Z(\hat{G})^\Gamma \cong T_{1,2}$, denoted by $\kappa_{1,2}$, is $\Gamma$-invariant. Moreover, as $\varphi : G \to G'$ is $E$-admissible, the expression $(\frac{a_1';a_1}{a_2';a_2}) := \langle \text{inv}(\frac{a_1';a_1}{a_2';a_2}), \kappa_{1,2} \rangle \in \mathbb{C}^\times$ is independent of the choices.

**Definition A.1.** Let $\mathcal{E} = (s, \rho)$ be an endoscopic datum of $G$, $\varphi : G \to G'$ an inner twisting, $a : T \to G$ an embedding of a maximal torus, and $\kappa$ an element of $\hat{T}^\Gamma$ such that $\mathcal{E}_{(a, \kappa)} \cong \mathcal{E}$. We say that $\varphi : G \to G'$ is $(\mathcal{E}, a, \pi)$-admissible, if for all representatives $\kappa' \in \hat{T}^\Gamma$ of $\pi \in \pi_0(\hat{T}^\Gamma/Z(\hat{G})^\Gamma)$ satisfying $\mathcal{E}_{(a, \kappa')} \cong \mathcal{E}$, all embeddings $\eta, \eta' : \hat{T} \hookrightarrow \hat{G}$ such that $\mathcal{E}_{(a, \kappa, \eta)} = \mathcal{E}_{(a, \kappa, \eta')} = (s, \rho)$, and all representatives $\tilde{s} \in \hat{G}^{ad}$ of $s$, the difference $\kappa(\tilde{s}, \eta') - \kappa(\tilde{s}, \eta) \in Z(\hat{G})^{ad}$ is orthogonal to $\text{inv}(G', G) \in H^1(E, G^{ad})$.

**Remark A.2.** a) Every $(\mathcal{E}, a, \pi)$-admissible inner twisting is $\mathcal{E}$-admissible.

b) If $a(T) \subset G$ is elliptic, then every $E$-admissible inner twisting is $(\mathcal{E}, a, \pi)$-admissible.

c) $\varphi : G \to G'$ is $(\mathcal{E}, a_1, \pi_1)$-admissible if and only if $(\frac{a_1';a_1}{a_2';a_2}) = \langle \text{inv}(\frac{a_1';a_1}{a_2';a_2}), \kappa_{1,2} \rangle$ for all representatives $\kappa'_1 \in \hat{T}_1^\Gamma$ of $\pi_1 \in \pi_0(\hat{T}_1^\Gamma/Z(\hat{G})^\Gamma)$ satisfying $\mathcal{E}_{(a_1, \kappa'_1)} \cong \mathcal{E}$.

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