A NEW INEQUALITY FOR SUPERDIFFUSIONS AND ITS APPLICATIONS TO NONLINEAR DIFFERENTIAL EQUATIONS

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ABSTRACT. Our motivation is the following problem: to describe all positive solutions of a semilinear elliptic equation \( Lu = u^\alpha \) with \( \alpha > 1 \) in a bounded smooth domain \( E \subset \mathbb{R}^d \). In 1998 Dynkin and Kuznetsov solved this problem for a class of solutions which they called \( \sigma \)-moderate. The question if all solutions belong to this class remained open. In 2002 Mselati proved that this is true for the equation \( \Delta u = u^2 \) in a domain of class \( C^4 \). His principal tool—the Brownian snake—is not applicable to the case \( \alpha \neq 2 \). In 2003 Dynkin and Kuznetsov modified most of Mselati’s arguments by using superdiffusions instead of the snake. However a critical gap remained. A new inequality established in the present paper allows us to close this gap.

1. INTRODUCTION

1.1. Diffusions and superdiffusions. We denote by \( \mathcal{M}(S) \) the set of all finite measures, and by \( \mathcal{P}(S) \) the set of all probability measures on a measurable space \( S \). \( \mathcal{B}(E) \) stands for the set of all positive Borel functions on \( E \). We use notation \( \langle u, \mu \rangle \) for the integral of \( u \) with respect to a measure \( \mu \), and notation \( P\{A, Y\} \) for the integral \( \int_A Y \, dP \).

Let \( L \) be an elliptic differential operator of the second order in \( \mathbb{R}^d \). Under mild assumptions on the coefficients of \( L \), there exists a continuous Markov process \( \xi = (\xi_t, \Pi_x) \) in \( \mathbb{R}^d \) whose transition density is a fundamental solution of the parabolic equation \( \partial u / \partial t = Lu \). We call this process a diffusion. For every open set \( D \) we denote by \( \tau_D \) the first exit time of \( \xi \) from \( D \).

Let \( \psi \) be a positive Borel function on \( \mathbb{R}_+ = [0, \infty) \). Suppose that to every open set \( D \) and every \( \mu \in \mathcal{M}(\mathbb{R}^d) \) there corresponds a random measure \( (X_D, P_\mu) \) on \( \mathbb{R}^d \) such that, for every \( f \in \mathcal{B}(\mathbb{R}^d) \),

\[
P_\mu e^{-\langle f, X_D \rangle} = e^{-\langle u, \mu \rangle}
\]

where \( u \) satisfies the equation

\[
u(x) + \Pi_x \int_0^{\tau_D} \psi[u(\xi_t)] \, dt = \Pi_x f(\xi_{\tau_D}).
\]
A NEW INEQUALITY FOR SUPERDIFFUSIONS 69

We call the family \(X = (X_D, P_\mu)\) a superdiffusion. [Heuristically, we have here a model of a random evolution of the cloud of particles, and \(X_D\) is a mass distribution on \(\partial D\) if each particle is frozen at the first exit from \(D\).

The existence of a superdiffusion is proved for a convex class of positive convex functions which contains the functions

\[
\psi(u) = u^\alpha, \quad 0 < \alpha \leq 2.
\]

[See, e.g., Chapter 4 in [Dy02].]

By restricting the family \((X_D, P_\mu)\) to \(D \subset E\) and \(\mu \in \mathcal{M}(E)\) we define a superdiffusion in an open set \(E\).

A new tool—a family of measures \(N_x\), \(x \in E\) (defined on the same space \(\mathcal{O}\) as measures \(P_\mu\))—was introduced in [DK04]. Our inspiration was the role played by an analog of these measures in Le Gall’s theory of the Brownian snake.

The range \(\mathcal{R}_E\) of a superdiffusion in a domain \(E\) is a minimal closed set which supports, \(P_x\)-a.s. and \(N_x\)-a.s., an exit measure \(X_\mathcal{O}\) for an arbitrary open set \(\mathcal{O} \subset E\) and for every \(x \in E\).

1.2. Stochastic boundary values of harmonic functions. We say that a function \(h\) in \(E\) is harmonic in \(E\) if \(Lh = 0\) in \(E\) and we use the notation \(\mathcal{H}(E)\) for the set of all positive harmonic functions. If \(E\) is smooth, then there exists a 1-1 correspondence between \(\mathcal{H}(E)\) and \(\mathcal{M}(\partial E)\). The harmonic function \(h_\nu\) corresponding to \(\nu \in \mathcal{M}(\partial E)\) is given by the formula

\[
h_\nu(x) = \int_{\partial E} k_E(x, y) \nu(dy)
\]

where \(k_E(x, y)\) is the Poisson kernel for \(L\) in \(E\). For every \(\nu \in \mathcal{M}(\partial E)\), there exists a random variable \(Z_\nu\) such that

\[
Z_\nu = \lim_\nu \mathbf{E}[h_\nu, X_{D_n}]\quad \text{\(P_x\)-a.s. and \(N_x\)-a.s.}
\]

for every \(x \in E\) and for every sequence \(D_n\) exhausting \(E\) We call \(Z_\nu\) the stochastic boundary value of \(h_\nu\).

The energy function for \(\nu \in \mathcal{M}(\partial E)\) is defined by the formula

\[
\mathcal{E}_\nu = \Pi_x \int_0^{\mathcal{R}_E} \psi(h_\nu(\xi_t)) dt.
\]

1.3. Principal result.

**Theorem 1.1.** Suppose that \(D\) is a smooth open subset of a smooth domain \(E\). If \(\nu\) is a finite measure concentrated on \(\partial D \cap \partial E\) and if \(\mathcal{E}_\nu < \infty\), then

\[
\mathbb{N}_x (\mathcal{R}_E \subset D^*, Z_\nu \neq 0) \geq C(\alpha) [\mathbb{N}_x (\mathcal{R}_E \subset D^*, Z_\nu)]^{\alpha/(\alpha-1)} \mathcal{E}_\nu^{-1/(\alpha-1)}
\]

where \(C(\alpha) = (\alpha - 1)^{-1}\Gamma(\alpha - 1)\).

We prove Theorem 1.1 in Section 5 after the necessary tools have been prepared in Sections 2–4.

1Definitions of \(\mathbb{N}_x\) and other tools mentioned in the Introduction will be given in Section 2.

2We use the term smooth for open sets of class \(C^{2,\lambda}\) unless another class is indicated explicitly.

3Domains \(D_n\) exhaust \(E\) if \(D_n \subset D_{n+1}\) and if the union of \(D_n\) is equal to \(E\).

4Here \(\Gamma\) is Euler’s gamma-function.
1.4. Applications to differential equations. We denote by $U(E)$ the set of all positive solutions of the equation

$$Lu = \psi(u) \quad \text{in } E.$$  

We say that an element $u$ of $U(E)$ is moderate if $u \leq h$ for some $h \in \mathcal{H}(E)$. There exists a 1-1 correspondence between the set $\mathcal{U}_1(E)$ of all moderate solutions and a subset $\mathcal{H}_1(E)$ of $\mathcal{H}(E)$: $h \in \mathcal{H}_1(E)$ is the minimal harmonic function dominating $u \in \mathcal{U}_1(E)$, and $u$ is the maximal solution dominated by $h$. We put $\nu \in N^E_1$ if $h_\nu \in \mathcal{H}_1(E)$. We denote by $u_\nu$ the element of $\mathcal{U}_1(E)$ corresponding to $h_\nu$. These elements are related by the formula

$$u_\nu(x) + \mathcal{E}_x(\nu) = h_\nu(x).$$

If $\mathcal{E}_x(\nu) < \infty$ for some $x \in E$, then $\nu \in N^E_1$.

To every closed subset $K$ of $\partial E$ there correspond two elements of $U(E)$:

$$w_K(x) = -\log P_x\{\mathcal{R}_E \cap K = \emptyset\} = \mathcal{N}_x\{\mathcal{R}_E \cap K \neq \emptyset\}$$

and

$$u_K(x) = \sup u_\nu(x)$$

where the supremum is taken over all $\nu \in N^E_1$ concentrated on $K$.

We say that $u \in U(E)$ is $\sigma$-moderate if there exist moderate solutions $u_n$ such that $u_n \uparrow u$. All solutions $u_K$ are $\sigma$-moderate.

Theorem 1.1 in combination with the results presented in Chapter 11, Section 7.1 of \cite{Dy02} and in \cite{Dy04a, Dy04c, DK03, DK04, Ku04} makes it possible to prove the following two theorems:

**Theorem 1.2.** If $E$ is a domain of class $C^4$ and if $L$ is the Laplacian $\Delta$, then

$$u_K = w_K \quad \text{for all closed } K \subset \partial E.$$  

**Theorem 1.3.** Under the conditions of Theorem 1.2 all elements of $U(E)$ are $\sigma$-moderate.

[Marcus and Véron proved in \cite{MV04} that the equation (1.12) can be established by a purely analytical method applicable to all $\alpha > 1$.]

2. Tools

2.1. $h$-transform and conditional diffusion. Suppose $\xi$ is a diffusion in a domain $E$ with the transition function $p_t(x,dy)$, and let $h \in \mathcal{H}(E)$. Then

$$p_t^h(x,dy) = \frac{1}{h(x)}p_t(x,dy)h(y)$$

is the transition function of a continuous Markov process $(\xi_t, \Pi^h_x)$ in $E$ called the $h$-transform of $\xi$. We prefer to deal with measures $\Pi^h_x = h(x)\Pi^h_x$ which depend linearly on $h$. Put $\Pi^\nu_x = \Pi^\nu_x$ and $\Pi^\nu_y = \Pi^\nu_y$ where $\delta_y$ is the unit mass at a point $y$. The process $(\xi_t, \Pi^\nu_x)$ can be interpreted as a diffusion starting from $x \in E$ and conditioned to exit from $E$ at $y$.

The following lemma is proved, for instance, in \cite{Dy02}, page 103:

\footnote{This follows, for instance, from Theorem 3.2 of Chapter 8 in \cite{Dy02}.}

\footnote{$w_K$ can be characterized as the maximal element of $U(E)$ vanishing on $\partial E \setminus K$.}

\footnote{Proofs of these theorems are sketched in \cite{Dy04b}. The complete proofs are contained in the forthcoming book \cite{Dy04d}.}
Lemma 2.1. For every stopping time $\tau$ and every pre-$\tau$ positive $Y$,

\begin{equation}
\Pi^h_x Y_{1_{\tau<\tau_E}} = \Pi_x Y h(\xi_{\tau}) 1_{\tau<\tau_E}.
\end{equation}

2.2. Measures $N_x$. Denote by $Z_x$ the class of all functions of the form

\begin{equation}
Z = \sum_{i=1}^n \langle f_i, X_{O_i} \rangle
\end{equation}

where $O_1, \ldots, O_n$ is a finite family of neighborhoods of $x$ and $f_1, \ldots, f_n \in B(\mathbb{R}^d)$.

By Theorem 1.1 in [DK04], for every $x \in E$, there exists a unique measure $N_x$ with the properties:

(i) For every $Z \in Z_x$,

\begin{equation}
N_x(1 - e^{-Z}) = -\log P_x e^{-Z}.
\end{equation}

(ii) If $\bar{\Omega}$ is the intersection of $\{X_O = 0\}$ over all neighborhoods $O$ of $x$, then $N_x(\bar{\Omega}) = 0$.

2.3. Stochastic boundary values and range. Suppose that $u \in B(E)$. A random variable $Z_u$ is called a stochastic boundary value of $u$ [we write $Z_u = \text{SBV}(u)$] if

\begin{equation}
Z_u = \lim \langle u, X_{D_n} \rangle \quad \text{P}_x\text{-a.s. and } N_x\text{-a.s.}
\end{equation}

for every $x \in E$ and every sequence $D_n$ exhausting $E$.

For every $u \in B(\mathbb{R}^d)$, we put

\begin{equation}
V_D(u)(x) = -\log P_x e^{-\langle u, X_D \rangle}.
\end{equation}

Denote by $U^-(E)$ the set of such $u$ such that $V_D(u) \leq u$ for all $D \subset E$. This condition holds for all $u \geq 0$ such that $Lu \leq \psi(u)$ in $E$. In particular, it holds for $u \in \mathcal{H}(E)$.

Since $V_D(u_1 + u_2) \leq V_D(u_1) + V_D(u_2)$ (Theorem 2.1 of Chapter 8 in [Dy02]), the sum of two elements of $U^-(E)$ belongs to $U^-(E)$.

By Theorem 1.2 in [DK04], a stochastic boundary value $Z_u$ exist for every $u \in U^-(E)$ and

\begin{equation}
N_x(1 - e^{-Z_u}) = -\log P_x e^{-Z_u}.
\end{equation}

Formula (1.5) means that $Z_u = \text{SBV}(h_u)$. If $\nu \in N_E$, then $Z_\nu$ is also $\text{SBV}(u_\nu)$.

By Theorem 1.3 in [DK04], for every domain $E$, there exists a random closed set $R_E$ with the properties:

(a) For every open $O \subset E$ and every $x \in E$, the measure $X_O$ is concentrated, $P_x\text{-a.s. and } N_x\text{-a.s.}$, on $R_E$.

(b) If (a) holds for a random closed set $F$, then, for every $x \in E$, $R \subset F \ P_x\text{-a.s. and } N_x\text{-a.s.}$.

We call $R_E$ the range of $X$ in $E$. We denote by $R$ the range of $X$ in $\mathbb{R}^d$.

2.4. More relations between measures $P_x$ and $N_x$. By Theorem 1.4 in [DK04], for every $u \in U^-(E)$ and every Borel set $\Gamma \subset \partial E$,

\begin{equation}
-\log P_x \{ R_E \cap \Gamma = \emptyset, e^{-Z_u} \} = N_x \{ R_E \cap \Gamma \neq \emptyset \} + N_x \{ R_E \cap \Gamma = \emptyset, 1 - e^{-Z_u} \}.
\end{equation}

This function is the maximal element of $U(E)$ dominated by $w + u$.

By taking $Z = 0$, we get

\begin{equation}
-\log P_x \{ R_E \cap \Gamma = \emptyset \} = N_x \{ R_E \cap \Gamma \neq \emptyset \}.
\end{equation}
It follows from (2.11) and (2.8) that, if
\[ P_x \{ \mathcal{R}_E \cap \Gamma = \emptyset \} > 0, \]
then
\[ N_x \{ \mathcal{R}_E \cap \Gamma = \emptyset, 1 - e^{-Z} \} = -\log P_x \{ e^{-Z} \mid \mathcal{R}_E \cap \Gamma = \emptyset \}. \]

By applying (2.7) to $\lambda Z$ and passing to the limit as $\lambda \to +\infty$, we get
\[ -\log P_x \{ \mathcal{R}_E \cap \Gamma = \emptyset, Z = 0 \} = N_x \{ \mathcal{R}_E \cap \Gamma \neq \emptyset \} + N_x \{ \mathcal{R}_E \cap \Gamma = \emptyset, Z \neq 0 \}. \]

By Proposition 1.1 in [DK02],
\[ N_x Z_\nu = P_x Z_\nu \quad \text{if} \quad P_x Z_\nu < \infty. \]
On the other hand, for every $f \in \mathcal{B}(\bar{D})$,
\[ P_x (f, X_D) = \Pi_x f (\xi_D) \]
(see, e.g., [Dy02], Chapter 4, Lemma 4.1). It follows from (2.5), (2.11), (2.12), Fatou’s lemma and the mean value property of harmonic functions, that
\[ N_x Z_\nu = P_x Z_\nu \leq h_\nu(x) < \infty \quad \text{for every} \quad \nu \in \mathcal{M}(\partial E). \]

**Proposition 2.1.** Suppose $x \in D$, $\Lambda$ is a Borel subset of $\partial D$ and $\mathcal{A} = \{ \mathcal{R} \cap \Lambda = \emptyset \}$. We have $P_x \mathcal{A} > 0$ and, for all $Z', Z'' \in \mathcal{Z}_x$,
\[ N_x \{ \mathcal{A}, (e^{-Z'} - e^{-Z''})^2 \} = -2 \log P_x \{ e^{-Z'} \mid \mathcal{A} \} + 2 \log P_x \{ e^{-2Z'} \mid \mathcal{A} \} + \log P_x \{ e^{-2Z''} \mid \mathcal{A} \}. \]

If $Z' = Z''$ $P_x$-a.s. on $\mathcal{A}$ and if $P_x \{ \mathcal{A}, Z' < \infty \} > 0$, then $Z' = Z''$ $N_x$-a.s. on $\mathcal{A}$.

**Proof.** First, $P_x \mathcal{A} > 0$ because $P_x \mathcal{A} = e^{-w_\Lambda(x)}$. Next
\[ (e^{-Z'} - e^{-Z''})^2 = 2(1 - e^{-Z'} - Z'') - (1 - e^{-2Z'}) - (1 - e^{-2Z''}). \]
Therefore (2.14) follows from (2.9). The second part of the proposition is an obvious implication of (2.13). \hfill \Box

2.5. **Properties of superdiffusions.** The following properties are often used in the theory of superdiffusions. [They are a part of the definition of branching exit Markov systems, and superdiffusions are a special case of such systems (see [Dy02], Chapters 3 and 4).]

2.5.A. (Markov property) If $Y \geq 0$ is measurable with respect to the $\sigma$-algebra generated by $X_D$, $D' \subset D$ and $Z \geq 0$ is measurable with respect to the $\sigma$-algebra generated by $X_D$, $D'' \subset D$, then
\[ P_\mu(YZ) = P_\mu(YP_XZ). \]

2.5.B. If $\mu(E) = 0$, then $P_\mu \{ X_E = \mu \} = 1$.

We use 2.5.A, 2.5.B and Proposition 2.1 to prove the next proposition.

**Proposition 2.2.** Let $D \subset E$ be two open sets. Then, for every $x \in D$, $X_D$ and $X_E$ coincide $P_x$-a.s. and $N_x$-a.s. on the set $\mathcal{A} = \{ \mathcal{R}_D \subset D^* \}$.

[Note that
\[ D^* = \{ x \in \bar{D} : d(x, \Lambda) > 0 \} \]
where $\Lambda = \partial D \cap E$.]


3. Relations between superdiffusions and conditional diffusions
in two open sets

3.1. Now we consider two bounded smooth open sets $D \subset E$. We denote by $\tilde{Z}_\nu$ the stochastic boundary value of $\tilde{h}_\nu(x) = \int_{\partial D} k_D(x, y) \nu(dy)$ in $D$; $\tilde{\Pi}^E$ refers to the diffusion in $D$ conditioned to exit at $y \in \partial D$.

**Theorem 3.1.** Put $\mathcal{A} = \{\mathcal{R}_D \subset D^*\}$. For every $x \in D$, 
\[(3.1) \quad \mathcal{R}_E = \mathcal{R}_D \quad P_x$-a.s. and $\mathcal{N}_x$-a.s. \]
and 
\[(3.2) \quad Z_\nu = \tilde{Z}_\nu \quad P_x$-a.s. and $\mathcal{N}_x$-a.s. on $\mathcal{A}$
for all $\nu \in \mathcal{N}^E_\mathcal{F}$ concentrated on $\partial D \cap \partial E$.

**Proof.** 1°. First, we prove (3.1). Clearly, $\mathcal{R}_D \subset \mathcal{R}_E$ $P_x$-a.s. and $\mathcal{N}_x$-a.s. for all $x \in D$. We get (3.1) if we show that, if $O$ is an open subset of $E$, then, for every $x \in D$, $X_O = X_{O \cap D}$ $P_x$-a.s. on $\mathcal{A}$ and, for every $x \in O \cap D$, $X_O = X_{O \cap D}$ $P_x$-a.s. on $\mathcal{A}$. For $x \in O \cap D$ this follows from Proposition 2.2 applied to $O \cap D \subset O$ because $\{\mathcal{R}_D \subset D^*\}$. For $x \in D \setminus O$, $P_x \{X_O = X_{D \cap O} = \delta_x\} = 1$.

2°. Put 
\[(3.3) \quad D_m^* = \{x \in D : d(x, E \setminus D) > 1/m\}.
To prove (3.2), it is sufficient to prove that it holds on $\mathcal{A}_m = \{\mathcal{R}_D \subset D_m^*\}$ for all sufficiently large $m$. First we prove that, for all $x \in D$, 
\[(3.4) \quad Z_\nu = \tilde{Z}_\nu \quad P_x$-a.s. on $\mathcal{A}_m$.
We get (3.3) by proving that both $Z_\nu$ and $\tilde{Z}_\nu$ coincide $P_x$-a.s. on $\mathcal{A}_m$ with the stochastic boundary value $Z^*$ of $h_\nu$ in $D$.

Let 
\[E_n = \{x \in E : d(x, \partial E) > 1/n\}, \quad D_n = \{x \in D : d(x, \partial D) > 1/n\}.
If $n > m$, then 
\[\mathcal{A}_m \subset \mathcal{A}_n \subset \{\mathcal{R}_D \subset D_m^*\} \subset \{\mathcal{R}_{D_n} \subset D_n^*\}.
We apply Proposition 2.2 to $D_n \subset E_n$ and we get that, $P_x$-a.s. on $\{\mathcal{R}_{D_n} \subset D_n^*\}$ $\mathcal{A}_m$, $X_{D_n} = X_{E_n}$ for all $n > m$, which implies $Z^* = Z_\nu$.

3°. Now we prove that 
\[(3.5) \quad Z^* = \tilde{Z}_\nu \quad P_x$-a.s. on $\mathcal{A}_m$.
Consider $h^0 = h_\nu - \tilde{h}_\nu$ and $Z^0 = Z_\nu - \tilde{Z}_\nu$. If $y \in \partial D \cap \partial E$, then 
\[(3.6) \quad k_E(x, y) = k_D(x, y) + \Pi_x \tau_D < \tau_E, k_E(\xi_{\tau_D}, y)\}.
Therefore 
\[(3.7) \quad h^0(x) = \Pi_x \{\tau_D \in \partial D \cap E, h_\nu(\xi_{\tau_D})\}.
This is a harmonic function in $D$. It vanishes on $\Gamma_m = \partial D \cap D_m^* = \partial E \cap D_n^*$.

We claim that, for every $\varepsilon > 0$ and every $m$, $h^0 < \varepsilon$ on $\Gamma_{m,n} = \partial E_n \cap D_n^*$ for all sufficiently large $n$. [If this is not true, then there exists a sequence $n_i \to \infty$ such that $z_n \in \Gamma_{m,n}$, and $h^0(z_n) \geq \varepsilon$. If $z$ is a limit point of $z_n$, then $z \in \Gamma_m$ and $h^0(z) \geq \varepsilon$.]
All measures $X_{D_n}$ are concentrated, $P_x$-a.s., on $R_D$. Therefore $A_m$ implies that they are concentrated, $P_x$-a.s., on $D_m$. Since $\Gamma_{m,n} \subset D_m^*$, we conclude that, for all sufficiently large $n$, $\langle h^0, X_{D_n} \rangle < \epsilon(1, X_{D_n}) P_x$-a.s. on $A_m$. This implies (3.9).

4°. If $\nu \in M(\partial D)$ and $Z_{\nu} = SBV(h_{\nu})$, then

$$\Pi_x Z_{\nu} = P_x Z_{\nu} \leq h_{\nu}(x) < \infty.$$  

Note that $P_x A > 0$. It follows from (3.8) that $Z_{\nu} < \infty$ $P_x$-a.s. and therefore $P_x \{A, Z_{\nu} < \infty \} > 0$. By Proposition 2.1 (3.4) follows from (3.3).

3.2. We also need the following result (see [Dy04a], Lemma 3.2).

**Theorem 3.2.** Suppose that $D \subset E$ are smooth open sets. Denote by $\tilde{\mathcal{F}}$ the $\sigma$-algebra in $\Omega$ generated by the sets $\{s < \tau_D, \xi_s \in B\}$ where $s \geq 0, B \in \mathcal{B}(E)$. We have

$$\Pi_x Y = \Pi_x \{\tau_D = \tau_E, Y\}$$

for all $x \in D, y \in \partial D \cap \partial E$ and for all $Y \in \tilde{\mathcal{F}}$.

**Corollary 3.1.** If

$$F_t = \exp \left[ - \int_0^t a(\xi_s) \, ds \right]$$

where $a$ is a positive continuous function on $[0, \infty)$, then, for $y \in \partial D \cap \partial E$,

$$\Pi_x F_{\tau_D} = \Pi_x \{\tau_D = \tau_E, F_{\tau_E}\}.$$

Indeed, it is easy to see that $F_{\tau} \in \tilde{\mathcal{F}}$.

4. Equations connecting $P_x$ and $\Pi_x$ with $\Pi_x^c$

4.1.

**Theorem 4.1.** Let $Z_{\nu} = SBV(h_{\nu}), Z_u = SBV(u)$ where $\nu \in \Lambda_1^E$ and $u \in U(E)$. Then

$$P_x Z_{\nu} e^{-Z_u} = e^{-u(x)} \Pi_x^c e^{-\Phi(u)}$$

and

$$\Pi_x Z_{\nu} e^{-Z_u} = \Pi_x^c e^{-\Phi(u)}$$

where

$$\Phi(u) = \int_0^{\tau_E} \psi'[u(\xi_t)] dt.$$ 

**Proof.** Formula (4.1) follows from Theorem 3.1 in Chapter 9 of [Dy02]. To prove (4.2), we observe that, for every $\lambda \geq 0, \lambda Z_{\nu} + Z_u = SBV(v)$ where $v = \lambda h_{\nu} + u \in U^-(E)$ and therefore, by (2.4),

$$\Pi_x (1 - e^{-\lambda Z_{\nu} - Z_u}) = - \log P_x e^{-\lambda Z_{\nu} - Z_u}.$$ 

By taking the derivatives with respect to $\lambda$ at $\lambda = 0$ we get

$$\Pi_x Z_{\nu} e^{-Z_u} = P_x Z_{\nu} e^{-Z_u} / P_x e^{-Z_u}.$$ 

By Theorem 1.1 of Chapter 9 in [Dy02],

$$P_x e^{-Z_u} = e^{-u(x)}.$$ 

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8The differentiation under the integral signs is justified by (2.13).
Therefore (4.14) follows from (4.11), (4.3) and (4.6).

**Theorem 4.2.** Suppose that $D \subset E$ are bounded smooth open sets and $\Lambda, L, D^*$ are the sets introduced in Theorem 3.1. Let $\nu$ be a finite measure on $\partial D \cap \partial E$, $x \in E$ and $E_x(\nu) < \infty$. Put

\begin{equation}
    w_\Lambda(x) = N_x\{R_D \cap \Lambda \neq \emptyset\},
\end{equation}

\begin{equation}
    v_s(x) = w_\Lambda(x) + N_x\{R_D \cap \Lambda = \emptyset, 1 - e^{-sZ_\nu}\}
\end{equation}

for $x \in D$ and let $w_\Lambda(x) = v_s(x) = 0$ for $x \in E \setminus D$. For every $x \in E$, we have

\begin{equation}
    N_x\{R_E \subset D^*, Z_\nu \neq 0\} = \int_0^\infty \Pi^\nu_x\{A, e^{-\Phi(w_\Lambda)}\}ds
\end{equation}

where $\Phi$ is defined by (4.3) and

\begin{equation}
    A = \{\tau_E = \tau_D\} = \{\xi_t \in D \text{ for all } t < \tau_E\}.
\end{equation}

**Remark.** Since $E_x(\nu) < \infty$, $\nu$ belongs to $N_x^E$ and to $N_x^D$.

**Proof.** 1°. If $x \in E \setminus D$, then, $N_x$-a.s., $R_E$ is not a subset of $D^*$ because $R_E$ contains supports of $X_D$ for all neighborhoods $O$ of $x$ and we can choose $O$ such that $\bar{O} \cap D^* = \emptyset$. On the other hand, $\Pi^\nu_x(A) = 0$. Therefore (4.8) and (4.9) hold independently of values of $w_\Lambda$ and $v_s$.

2°. Now we assume that $x \in D$. Put $A = \{R_D \subset D^*\}$. We claim that

\begin{equation}
    A = \{R_E \subset D^*\} \text{ } N_x\text{-a.s.}
\end{equation}

Indeed, $\{R_E \subset D^*\} \subset A$ because $R_D \subset R_E$. By Theorem 3.1 $A \subset \{R_D = R_E\}$ $N_x$-a.s. Hence, $A \subset \{R_E \subset D^*\}$.

By Theorem 3.1 $R_D = R_E$ and $Z_\nu = \tilde{Z}_\nu$, $N_x$-a.s. on $A$. Therefore

\begin{equation}
    N_x\{R_E \subset D^*, Z_\nu \} = N_x\{A, Z_\nu \} = N_x\{A, \tilde{Z}_\nu \},
\end{equation}

\begin{equation}
    N_x\{R_E \subset D^*, Z_\nu e^{-s\tilde{Z}_\nu} \} = N_x\{A, Z_\nu e^{-s\tilde{Z}_\nu} \} = N_x\{A, \tilde{Z}_\nu e^{-s\tilde{Z}_\nu} \}.
\end{equation}

Formula (4.7) defines two elements of $U(D)$. The stochastic boundary value $Z_\Lambda$ of $w_\Lambda$ in $D$ is equal to $\infty 1_A$ (Remark 1.2 on p. 133 in [Dy02]) and therefore

\begin{equation}
    e^{-Z_\Lambda} = 1_A.
\end{equation}

By (2.7) and (2.8), $v_s(x) = -\log P_x\{R_D \cap \Lambda = \emptyset, e^{-s\tilde{Z}_\nu}\}$ and, by Remark 2.1 on p. 137 in [Dy02], the stochastic boundary value $Z^s$ of $v_s$ in $D$ is equal to $Z_\Lambda + s\tilde{Z}_\nu$. Hence,

\begin{equation}
    e^{-Z^s} = 1_A e^{-s\tilde{Z}_\nu}.
\end{equation}

By (4.11), (4.12) and (4.13),

\begin{equation}
    N_x\{A, Z_\nu \} = N_x\{\tilde{Z}_\nu e^{-Z_\Lambda} \}
\end{equation}

and

\begin{equation}
    N_x\{A, Z_\nu e^{-s\tilde{Z}_\nu} \} = N_x\{\tilde{Z}_\nu e^{-Z^s} \}.
\end{equation}
By applying formula (4.2) to $\tilde{Z}_\nu$ and to the restriction of $w_\Lambda$ to $D$, we conclude from (4.14) that

$$N_x A, Z_\nu = \Pi_x^\nu \exp \left[ - \int_0^{\tau_D} \psi'(w_\Lambda(\xi_s)) ds \right]$$

and, by Corollary 3.1,

$$N_x A, e^{-\Phi(v_s)} = \Pi_x^\nu A, e^{-\Phi(v_s)}.$$  \hspace{1cm} (4.16)

Analogously, (4.2) applied to the restriction of $v_s$ to $D$, in combination with (4.15) and (3.11), yields

$$N_x A, e^{-sZ_\nu} = \Pi_x^\nu A, e^{-\Phi(v_s)}.$$  \hspace{1cm} (4.17)

Formula (4.8) follows from (4.17) and formula (4.9) follows from (4.18) because

$$N_x A, Z_\nu \neq 0 \Rightarrow \lim_{t \to \infty} N_x A, 1 - e^{-tZ_\nu} = 0.$$  \hspace{1cm} (4.19)

and

$$1 - e^{-tZ_\nu} = \int_0^t Z_\nu e^{-sZ_\nu} ds.$$  \hspace{1cm} (4.20)

5. Proof of Theorem 4.1

We use the following two elementary inequalities:

5.A. For all $a, b \geq 0$ and $0 < \beta < 1$,

$$\alpha + b^\beta \leq a^\beta + b^\beta.$$  \hspace{1cm} (5.1)

Proof. It is sufficient to prove (5.1) for $a = 1$. Put $f(t) = (1 + t)^\beta - t^\beta$. Note that $f(0) = 1$ and $f'(t) \leq 0$ for $t > 0$. Hence $f(t) \leq 1$ for $t \geq 0$.

5.B. For every finite measure $M$, every positive measurable function $Y$ and every $\beta > 0$,

$$M(Y^{-\beta}) \geq M(1)^{1+\beta}(MY)^{-\beta}.$$  \hspace{1cm} (5.2)

Indeed $f(y) = y^{-\beta}$ is a convex function on $\mathbb{R}_+$, and we get (5.2) by applying Jensen’s inequality to the probability measure $M/M(1)$.

Proof of Theorem 4.1 1°. If $x \in E \setminus D$, then, $N_x$-a.s., $R_E$ is not a subset of $D^*$ (see the proof of Theorem 4.2). Hence, both sides of (4.7) vanish.

2°. Suppose $x \in D$. By (2.6), $N_x (1 - e^{-sZ_\nu}) = u_{sv}$. Thus (4.7) implies $v_s \leq w_\Lambda + u_{sv}$. Therefore, by (5.3) $u_s^{\alpha-1} \leq w_\Lambda^{\alpha-1} + u_{sv}^{\alpha-1}$ and, since $u_{sv} \leq h_{sv} = sh_{sv}$, $\Phi(v_s) \leq \Phi(w_\Lambda) + s^{\alpha-1} \Phi(h_{sv})$.

Put $A = \{ R_E \subset D^* \}$. It follows from (4.9) that

$$N_x A, Z_\nu \neq 0 \geq \Pi_x^\nu A, \int_0^\infty e^{-\Phi(w_\Lambda)} e^{-s^{\alpha-1} \Phi(h_{sv})} ds.$$  \hspace{1cm} (5.2)

Note that $\int_0^\infty e^{-as^\beta} ds = C a^{-1/\beta}$ where $C = \int_0^\infty e^{-t^\beta} dt$. Therefore (5.2) implies

$$N_x A, Z_\nu \neq 0 \geq C \Pi_x^\nu A, e^{-\Phi(w_\Lambda)} (\Phi(h_{sv})^{-1/(\alpha-1)})^{1-\beta}.$$  \hspace{1cm} (5.3)
The right side in (5.3) is equal to $CM(Y^{-\beta})$ where $\beta = 1/(\alpha - 1), Y = \Phi(h_\nu)$ and $M$ is the measure with the density $1_Ae^{-\Phi(w_\lambda)}$ with respect to $\Pi_x^{(\nu)}$. We get from (5.3) and 5.B that

$$N_x\{A, Z_\nu \neq 0\} \geq CM(1)^{1+\beta}(MY)^{-\beta}$$

$$= C[\Pi_x^{(\nu)}\{A, e^{-\Phi(w_\lambda)}\}]^{\alpha/(\alpha - 1)}[\Pi_x^{(\nu)}\{A, e^{-\Phi(w_\lambda)}\Phi(h_\nu)\}]^{-1/(\alpha - 1)}.$$

By (4.8), $\Pi_x^{(\nu)}\{A, e^{-\Phi(w_\lambda)}\} = N_x\{R_E \subset D^*, Z_\nu\}$ and, since $\Pi_x^{(\nu)}\{A, e^{-\Phi(w_\lambda)}\Phi(h_\nu)\} \leq \Pi_x^{(\nu)}\Phi(h_\nu)$, we have

$$N_x\{A, Z_\nu \neq 0\} \geq C[N_x\{R_E \subset D^*, Z_\nu\}]^{\alpha/(\alpha - 1)}[\Pi_x^{(\nu)}\Phi(h_\nu)]^{-1/(\alpha - 1)}.$$

3. By the definition of $h$-transform, for every $f \in B(E)$ and every $h \in H(E)$,

$$\Pi_x^{(\nu)}\int_0^{\tau_E} f(\xi_t)dt = \int_0^{\infty} \Pi_x^{(\nu)}[t < \tau_E, f(\xi_t)]dt = \int_0^{\infty} \Pi_x[t < \tau_E, f(\xi_t)]dt dt.$$

By taking $f = \alpha h_\nu^\alpha - 1$ and $h = h_\nu$ we get

(5.5) $\Pi_x^{(\nu)}\Phi(h_\nu) = \alpha E_x(\nu)$.

Formula (5.4) follows from (5.4) and (5.5). 

REFERENCES


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