

A NEW INEQUALITY FOR SUPERDIFFUSIONS AND ITS APPLICATIONS TO NONLINEAR DIFFERENTIAL EQUATIONS

E. B. DYNKIN

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ABSTRACT. Our motivation is the following problem: to describe all positive solutions of a semilinear elliptic equation $Lu = u^\alpha$ with $\alpha > 1$ in a bounded smooth domain $E \subset \mathbb{R}^d$. In 1998 Dynkin and Kuznetsov solved this problem for a class of solutions which they called σ -moderate. The question if all solutions belong to this class remained open. In 2002 Mselati proved that this is true for the equation $\Delta u = u^2$ in a domain of class C^4 . His principal tool—the Brownian snake—is not applicable to the case $\alpha \neq 2$. In 2003 Dynkin and Kuznetsov modified most of Mselati’s arguments by using superdiffusions instead of the snake. However a critical gap remained. A new inequality established in the present paper allows us to close this gap.

1. INTRODUCTION

1.1. Diffusions and superdiffusions. We denote by $\mathcal{M}(S)$ the set of all finite measures, and by $\mathcal{P}(S)$ the set of all probability measures on a measurable space S . $\mathcal{B}(E)$ stands for the set of all positive Borel functions on E . We use notation $\langle u, \mu \rangle$ for the integral of u with respect to a measure μ , and notation $P\{A, Y\}$ for the integral $\int_A Y dP$.

Let L be an elliptic differential operator of the second order in \mathbb{R}^d . Under mild assumptions on the coefficients of L , there exists a continuous Markov process $\xi = (\xi_t, \Pi_x)$ in \mathbb{R}^d whose transition density is a fundamental solution of the parabolic equation $\partial u / \partial t = Lu$. We call this process a diffusion. For every open set D we denote by τ_D the first exit time of ξ from D .

Let ψ be a positive Borel function on $\mathbb{R}_+ = [0, \infty)$. Suppose that to every open set D and every $\mu \in \mathcal{M}(\mathbb{R}^d)$ there corresponds a random measure (X_D, P_μ) on \mathbb{R}^d such that, for every $f \in \mathcal{B}(\mathbb{R}^d)$,

$$(1.1) \quad P_\mu e^{-\langle f, X_D \rangle} = e^{-\langle u, \mu \rangle}$$

where u satisfies the equation

$$(1.2) \quad u(x) + \Pi_x \int_0^{\tau_D} \psi[u(\xi_t)] dt = \Pi_x f(\xi_{\tau_D}).$$

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We call the family $X = (X_D, P_\mu)$ a superdiffusion. [Heuristically, we have here a model of a random evolution of the cloud of particles, and X_D is a mass distribution on ∂D if each particle is frozen at the first exit from D .]

The existence of a superdiffusion is proved for a convex class of positive convex functions which contains the functions

$$(1.3) \quad \psi(u) = u^\alpha, \quad 0 < \alpha \leq 2.$$

[See, e.g., Chapter 4 in [Dy02].]

By restricting the family (X_D, P_μ) to $D \subset E$ and $\mu \in \mathcal{M}(E)$ we define a superdiffusion in an open set E .

A new tool—a family of measures $\mathbb{N}_x, x \in E$ (defined on the same space \mathcal{O} as measures P_μ)—was introduced in [DK04]. Our inspiration was the role played by an analog of these measures in Le Gall's theory of the Brownian snake.¹

The range \mathcal{R}_E of a superdiffusion in a domain E is a minimal closed set which supports, P_x -a.s. and \mathbb{N}_x -a.s., an exit measure X_O for an arbitrary open set $O \subset E$ and for every $x \in E$.

1.2. Stochastic boundary values of harmonic functions. We say that a function h in E is harmonic in E if $Lh = 0$ in E and we use the notation $\mathcal{H}(E)$ for the set of all positive harmonic functions. If E is smooth,² then there exists a 1-1 correspondence between $\mathcal{H}(E)$ and $\mathcal{M}(\partial E)$. The harmonic function h_ν corresponding to $\nu \in \mathcal{M}(\partial E)$ is given by the formula

$$(1.4) \quad h_\nu(x) = \int_{\partial E} k_E(x, y) \nu(dy)$$

where $k_E(x, y)$ is the Poisson kernel for L in E . For every $\nu \in \mathcal{M}(\partial E)$, there exists a random variable Z_ν such that

$$(1.5) \quad Z_\nu = \lim \langle h_\nu, X_{D_n} \rangle \quad P_x\text{-a.s. and } \mathbb{N}_x\text{-a.s.}$$

for every $x \in E$ and for every sequence D_n exhausting E .³ We call Z_ν the stochastic boundary value of h_ν .

The energy function for $\nu \in \mathcal{M}(\partial E)$ is defined by the formula

$$(1.6) \quad \mathcal{E}_x(\nu) = \Pi_x \int_0^{\tau_E} \psi[h_\nu(\xi_t)] dt.$$

1.3. Principal result.

Theorem 1.1. *Suppose that D is a smooth open subset of a smooth domain E . If ν is a finite measure concentrated on $\partial D \cap \partial E$ and if $\mathcal{E}_x(\nu) < \infty$, then*

$$(1.7) \quad \mathbb{N}_x\{\mathcal{R}_E \subset D^*, Z_\nu \neq 0\} \geq C(\alpha) [\mathbb{N}_x\{\mathcal{R}_E \subset D^*, Z_\nu\}]^{\alpha/(\alpha-1)} \mathcal{E}_x(\nu)^{-1/(\alpha-1)}$$

where $C(\alpha) = (\alpha - 1)^{-1} \Gamma(\alpha - 1)$.⁴

We prove Theorem 1.1 in Section 5 after the necessary tools have been prepared in Sections 2–4.

¹Definitions of \mathbb{N}_x and other tools mentioned in the Introduction will be given in Section 2.

²We use the term smooth for open sets of class $C^{2,\lambda}$ unless another class is indicated explicitly.

³Domains D_n exhaust E if $\bar{D}_n \subset D_{n+1}$ and if the union of D_n is equal to E .

⁴Here Γ is Euler's gamma-function.

1.4. Applications to differential equations. We denote by $\mathcal{U}(E)$ the set of all positive solutions of the equation

$$(1.8) \quad Lu = \psi(u) \quad \text{in } E.$$

We say that an element u of $\mathcal{U}(E)$ is moderate if $u \leq h$ for some $h \in \mathcal{H}(E)$. There exists a 1-1 correspondence between the set $\mathcal{U}_1(E)$ of all moderate solutions and a subset $\mathcal{H}_1(E)$ of $\mathcal{H}(E)$: $h \in \mathcal{H}_1(E)$ is the minimal harmonic function dominating $u \in \mathcal{U}_1(E)$, and u is the maximal solution dominated by h . We put $\nu \in \mathcal{N}_1^E$ if $h_\nu \in \mathcal{H}_1(E)$. We denote by u_ν the element of $\mathcal{U}_1(E)$ corresponding to h_ν . These elements are related by the formula

$$(1.9) \quad u_\nu(x) + \mathcal{E}_x(\nu) = h_\nu(x).$$

If $\mathcal{E}_x(\nu) < \infty$ for some $x \in E$, then $\nu \in \mathcal{N}_1^E$.⁵

To every closed subset K of ∂E there correspond two elements of $\mathcal{U}(E)$:⁶

$$(1.10) \quad w_K(x) = -\log P_x\{\mathcal{R}_E \cap K = \emptyset\} = \mathbb{N}_x\{\mathcal{R}_E \cap K \neq \emptyset\}$$

and

$$(1.11) \quad u_K(x) = \sup u_\nu(x)$$

where the supremum is taken over all $\nu \in \mathcal{N}_1^E$ concentrated on K .

We say that $u \in \mathcal{U}(E)$ is σ -moderate if there exist moderate solutions u_n such that $u_n \uparrow u$. All solutions u_K are σ -moderate.

Theorem 1.1 in combination with the results presented in Chapter 11, Section 7.1 of [Dy02] and in [Dy04a], [Dy04c], [DK03], [DK04], [Ku04] makes it possible to prove the following two theorems:⁷

Theorem 1.2. *If E is a domain of class C^4 and if L is the Laplacian Δ , then*

$$(1.12) \quad u_K = w_K \quad \text{for all closed } K \subset \partial E.$$

Theorem 1.3. *Under the conditions of Theorem 1.2 all elements of $\mathcal{U}(E)$ are σ -moderate.*

[Marcus and Véron proved in [MV04] that the equation (1.12) can be established by a purely analytical method applicable to all $\alpha > 1$.]

2. TOOLS

2.1. h -transform and conditional diffusion. Suppose ξ is a diffusion in a domain E with the transition function $p_t(x, dy)$, and let $h \in \mathcal{H}(E)$. Then

$$(2.1) \quad p_t^h(x, dy) = \frac{1}{h(x)} p_t(x, dy) h(y)$$

is the transition function of a continuous Markov process $(\xi_t, \hat{\Pi}_x^h)$ in E called the h -transform of ξ . We prefer to deal with measures $\Pi_x^h = h(x) \hat{\Pi}_x^h$ which depend linearly on h . Put $\Pi_x^\nu = \Pi_x^{h_\nu}$ and $\hat{\Pi}_x^y = \hat{\Pi}_x^{\delta_y}$ where δ_y is the unit mass at a point y . The process $(\xi_t, \hat{\Pi}_x^y)$ can be interpreted as a diffusion starting from $x \in E$ and conditioned to exit from E at y .

The following lemma is proved, for instance, in [Dy02], page 103:

⁵This follows, for instance, from Theorem 3.2 of Chapter 8 in [Dy02].

⁶ w_K can be characterized as the maximal element of $\mathcal{U}(E)$ vanishing on $\partial E \setminus K$.

⁷Proofs of these theorems are sketched in [Dy04b]. The complete proofs are contained in the forthcoming book [Dy04d].

Lemma 2.1. *For every stopping time τ and every pre- τ positive Y ,*

$$(2.2) \quad \Pi_x^h Y 1_{\tau < \tau_E} = \Pi_x Y h(\xi_\tau) 1_{\tau < \tau_E}.$$

2.2. **Measures \mathbb{N}_x .** Denote by \mathcal{Z}_x the class of all functions of the form

$$(2.3) \quad Z = \sum_1^n \langle f_i, X_{O_i} \rangle$$

where O_1, \dots, O_n is a finite family of neighborhoods of x and $f_1, \dots, f_n \in \mathcal{B}(\mathbb{R}^d)$. By Theorem 1.1 in [DK04], for every $x \in E$, there exists a unique measure \mathbb{N}_x with the properties:

(i) For every $Z \in \mathcal{Z}_x$,

$$(2.4) \quad \mathbb{N}_x(1 - e^{-Z}) = -\log P_x e^{-Z}.$$

(ii) If $\bar{\Omega}$ is the intersection of $\{X_O = 0\}$ over all neighborhoods O of x , then $\mathbb{N}_x(\bar{\Omega}) = 0$.

2.3. **Stochastic boundary values and range.** Suppose that $u \in \mathcal{B}(E)$. A random variable Z_u is called a stochastic boundary value of u [we write $Z_u = \text{SBV}(u)$] if

$$(2.5) \quad Z_u = \lim \langle u, X_{D_n} \rangle \quad P_x\text{-a.s. and } \mathbb{N}_x\text{-a.s.}$$

for every $x \in E$ and every sequence D_n exhausting E .

For every $u \in \mathcal{B}(\mathbb{R}^d)$, we put

$$V_D(u)(x) = -\log P_x e^{-\langle u, X_D \rangle}.$$

Denote by $\mathcal{U}^-(E)$ the set of u such that $V_D(u) \leq u$ for all $D \subset E$. This condition holds for all $u \geq 0$ such that $Lu \leq \psi(u)$ in E . In particular, it holds for $u \in \mathcal{H}(E)$. Since $V_D(u_1 + u_2) \leq V_D(u_1) + V_D(u_2)$ (Theorem 2.1 of Chapter 8 in [Dy02]), the sum of two elements of $\mathcal{U}^-(E)$ belongs to $\mathcal{U}^-(E)$.

By Theorem 1.2 in [DK04], a stochastic boundary value Z_u exist for every $u \in \mathcal{U}^-(E)$ and

$$(2.6) \quad \mathbb{N}_x(1 - e^{-Z_u}) = -\log P_x e^{-Z_u}.$$

Formula (1.5) means that $Z_\nu = \text{SBV}(h_\nu)$. If $\nu \in \mathcal{N}_1^E$, then Z_ν is also $\text{SBV}(u_\nu)$.

By Theorem 1.3 in [DK04], for every domain E , there exists a random closed set \mathcal{R}_E with the properties:

(a) For every open $O \subset E$ and every $x \in E$, the measure X_O is concentrated, P_x -a.s. and \mathbb{N}_x -a.s., on \mathcal{R}_E .

(b) If (a) holds for a random closed set F , then, for every $x \in E$, $\mathcal{R} \subset F$ P_x -a.s. and \mathbb{N}_x -a.s.

We call \mathcal{R}_E the range of X in E . We denote by \mathcal{R} the range of X in \mathbb{R}^d .

2.4. **More relations between measures P_x and \mathbb{N}_x .** By Theorem 1.4 in [DK04], for every $u \in \mathcal{U}^-(E)$ and every Borel set $\Gamma \subset \partial E$,

$$(2.7) \quad -\log P_x \{\mathcal{R}_E \cap \Gamma = \emptyset, e^{-Z_u}\} = \mathbb{N}_x \{\mathcal{R}_E \cap \Gamma \neq \emptyset\} + \mathbb{N}_x \{\mathcal{R}_E \cap \Gamma = \emptyset, 1 - e^{-Z_u}\}.$$

This function is the maximal element of $\mathcal{U}(E)$ dominated by $w_\Gamma + u$.

By taking $Z = 0$, we get

$$(2.8) \quad -\log P_x \{\mathcal{R}_E \cap \Gamma = \emptyset\} = \mathbb{N}_x \{\mathcal{R}_E \cap \Gamma \neq \emptyset\}.$$

It follows from (2.7) and (2.8) that, if

$$P_x\{\mathcal{R}_E \cap \Gamma = \emptyset\} > 0,$$

then

$$(2.9) \quad \mathbb{N}_x\{\mathcal{R}_E \cap \Gamma = \emptyset, 1 - e^{-Z}\} = -\log P_x\{e^{-Z} \mid \mathcal{R}_E \cap \Gamma = \emptyset\}.$$

By applying (2.7) to λZ and passing to the limit as $\lambda \rightarrow +\infty$, we get

$$(2.10) \quad -\log P_x\{\mathcal{R}_E \cap \Gamma = \emptyset, Z = 0\} = \mathbb{N}_x\{\mathcal{R}_E \cap \Gamma \neq \emptyset\} + \mathbb{N}_x\{\mathcal{R}_E \cap \Gamma = \emptyset, Z \neq 0\}.$$

By Proposition 1.1 in [DK04],

$$(2.11) \quad \mathbb{N}_x Z_\nu = P_x Z_\nu \quad \text{if } P_x Z_\nu < \infty.$$

On the other hand, for every $f \in \mathcal{B}(\bar{D})$,

$$(2.12) \quad P_x\langle f, X_D \rangle = \Pi_x f(\xi_{\tau_D})$$

(see, e.g., [Dy02], Chapter 4, Lemma 4.1). It follows from (2.5), (2.11), (2.12), Fatou's lemma and the mean value property of harmonic functions, that

$$(2.13) \quad \mathbb{N}_x Z_\nu = P_x Z_\nu \leq h_\nu(x) < \infty \quad \text{for every } \nu \in \mathcal{M}(\partial E).$$

Proposition 2.1. *Suppose $x \in D$, Λ is a Borel subset of ∂D and $\mathcal{A} = \{\mathcal{R} \cap \Lambda = \emptyset\}$. We have $P_x \mathcal{A} > 0$ and, for all $Z', Z'' \in \mathcal{Z}_x$,*

$$(2.14) \quad \mathbb{N}_x\{\mathcal{A}, (e^{-Z'} - e^{-Z''})^2\} \\ = -2 \log P_x\{e^{-Z'-Z''} \mid \mathcal{A}\} + \log P_x\{e^{-2Z'} \mid \mathcal{A}\} + \log P_x\{e^{-2Z''} \mid \mathcal{A}\}.$$

If $Z' = Z''$ P_x -a.s. on \mathcal{A} and if $P_x\{\mathcal{A}, Z' < \infty\} > 0$, then $Z' = Z''$ \mathbb{N}_x -a.s. on \mathcal{A} .

Proof. First, $P_x \mathcal{A} > 0$ because $P_x \mathcal{A} = e^{-w_\Lambda(x)}$. Next

$$(e^{-Z'} - e^{-Z''})^2 = 2(1 - e^{-Z'-Z''}) - (1 - e^{-2Z'}) - (1 - e^{-2Z''}).$$

Therefore (2.14) follows from (2.9). The second part of the proposition is an obvious implication of (2.14). \square

2.5. Properties of superdiffusions. The following properties are often used in the theory of superdiffusions. [They are a part of the definition of branching exit Markov systems, and superdiffusions are a special case of such systems (see [Dy02], Chapters 3 and 4).]

2.5.A. (Markov property) If $Y \geq 0$ is measurable with respect to the σ -algebra generated by $X_{D'}$, $D' \subset D$ and $Z \geq 0$ is measurable with respect to the σ -algebra generated by $X_{D''}$, $D'' \supset D$, then

$$(2.15) \quad P_\mu(YZ) = P_\mu(Y P_{X_D} Z).$$

2.5.B. If $\mu(E) = 0$, then $P_\mu\{X_E = \mu\} = 1$.

We use 2.5.A, 2.5.B and Proposition 2.1 to prove the next proposition.

Proposition 2.2. *Let $D \subset E$ be two open sets. Then, for every $x \in D$, X_D and X_E coincide P_x -a.s. and \mathbb{N}_x -a.s. on the set $\mathcal{A} = \{\mathcal{R}_D \subset D^*\}$.*

[Note that

$$(2.16) \quad D^* = \{x \in \bar{D} : d(x, \Lambda) > 0\}$$

where $\Lambda = \partial D \cap E$.]

3. RELATIONS BETWEEN SUPERDIFFUSIONS AND CONDITIONAL DIFFUSIONS
IN TWO OPEN SETS

3.1. Now we consider two bounded smooth open sets $D \subset E$. We denote by \tilde{Z}_ν the stochastic boundary value of $\tilde{h}_\nu(x) = \int_{\partial D} k_D(x, y) \nu(dy)$ in D ; $\tilde{\Pi}_x^y$ refers to the diffusion in D conditioned to exit at $y \in \partial D$.

Theorem 3.1. *Put $\mathcal{A} = \{\mathcal{R}_D \subset D^*\}$. For every $x \in D$,*

$$(3.1) \quad \mathcal{R}_E = \mathcal{R}_D \quad P_x\text{-a.s. and } \mathbb{N}_x\text{-a.s.}$$

and

$$(3.2) \quad Z_\nu = \tilde{Z}_\nu \quad P_x\text{-a.s. and } \mathbb{N}_x\text{-a.s. on } \mathcal{A}$$

for all $\nu \in \mathcal{N}_1^E$ concentrated on $\partial D \cap \partial E$.

Proof. 1°. First, we prove (3.1). Clearly, $\mathcal{R}_D \subset \mathcal{R}_E$ P_x -a.s. and \mathbb{N}_x -a.s. for all $x \in D$. We get (3.1) if we show that, if O is an open subset of E , then, for every $x \in D$, $X_O = X_{O \cap D}$ P_x -a.s. on \mathcal{A} and, for every $x \in O \cap D$, $X_O = X_{O \cap D}$ \mathbb{N}_x -a.s. on \mathcal{A} . For $x \in O \cap D$ this follows from Proposition 2.2 applied to $O \cap D \subset O$ because $\{\mathcal{R}_D \subset D^*\} \subset \{\mathcal{R}_{O \cap D} \subset (O \cap D)^*\}$. For $x \in D \setminus O$, $P_x\{X_O = X_{D \cap O} = \delta_x\} = 1$.

2°. Put

$$(3.3) \quad D_m^* = \{x \in \bar{D} : d(x, E \setminus D) > 1/m\}.$$

To prove (3.2), it is sufficient to prove that it holds on $\mathcal{A}_m = \{\mathcal{R}_D \subset D_m^*\}$ for all sufficiently large m . First we prove that, for all $x \in D$,

$$(3.4) \quad Z_\nu = \tilde{Z}_\nu \quad P_x\text{-a.s. on } \mathcal{A}_m.$$

We get (3.4) by proving that both Z_ν and \tilde{Z}_ν coincide P_x -a.s. on \mathcal{A}_m with the stochastic boundary value Z^* of h_ν in D .

Let

$$E_n = \{x \in E : d(x, \partial E) > 1/n\}, \quad D_n = \{x \in D : d(x, \partial D) > 1/n\}.$$

If $n > m$, then

$$\mathcal{A}_m \subset \mathcal{A}_n \subset \{\mathcal{R}_D \subset D_n^*\} \subset \{\mathcal{R}_{D_n} \subset D_n^*\}.$$

We apply Proposition 2.2 to $D_n \subset E_n$ and we get that, P_x -a.s. on $\{\mathcal{R}_{D_n} \subset D_n^*\} \supset \mathcal{A}_m$, $X_{D_n} = X_{E_n}$ for all $n > m$, which implies $Z^* = Z_\nu$.

3°. Now we prove that

$$(3.5) \quad Z^* = \tilde{Z}_\nu \quad P_x\text{-a.s. on } \mathcal{A}_m.$$

Consider $h^0 = h_\nu - \tilde{h}_\nu$ and $Z^0 = Z^* - \tilde{Z}_\nu$. If $y \in \partial D \cap \partial E$, then

$$(3.6) \quad k_E(x, y) = k_D(x, y) + \Pi_x\{\tau_D < \tau_E, k_E(\xi_{\tau_D}, y)\}.$$

Therefore

$$(3.7) \quad h^0(x) = \Pi_x\{\xi_{\tau_D} \in \partial D \cap E, h_\nu(\xi_{\tau_D})\}.$$

This is a harmonic function in D . It vanishes on $\Gamma_m = \partial D \cap D_m^* = \partial E \cap D_m^*$.

We claim that, for every $\varepsilon > 0$ and every m , $h^0 < \varepsilon$ on $\Gamma_{m,n} = \partial E_n \cap D_m^*$ for all sufficiently large n . [If this is not true, then there exists a sequence $n_i \rightarrow \infty$ such that $z_{n_i} \in \Gamma_{m,n_i}$ and $h^0(z_{n_i}) \geq \varepsilon$. If z is a limit point of z_{n_i} , then $z \in \Gamma_m$ and $h^0(z) \geq \varepsilon$.]

All measures X_{D_n} are concentrated, P_x -a.s., on \mathcal{R}_D . Therefore \mathcal{A}_m implies that they are concentrated, P_x -a.s., on D_m^* . Since $\Gamma_{m,n} \subset D_m^*$, we conclude that, for all sufficiently large n , $\langle h^0, X_{D_n} \rangle < \varepsilon \langle 1, X_{D_n} \rangle$ P_x -a.s. on \mathcal{A}_m . This implies (3.5).

4°. If $\nu \in \mathcal{M}(\partial E)$ and $Z_\nu = \text{SBV}(h_\nu)$, then

$$(3.8) \quad \mathbb{N}_x Z_\nu = P_x Z_\nu \leq h_\nu(x) < \infty.$$

Note that $P_x \mathcal{A} > 0$. It follows from (3.8) that $Z_\nu < \infty$ P_x -a.s. and therefore $P_x \{\mathcal{A}, Z_\nu < \infty\} > 0$. By Proposition 2.1, (3.2) follows from (3.4). \square

3.2. We also need the following result (see [Dy04a], Lemma 3.2).

Theorem 3.2. *Suppose that $D \subset E$ are smooth open sets. Denote by $\tilde{\mathcal{F}}$ the σ -algebra in Ω generated by the sets $\{s < \tau_D, \xi_s \in B\}$ where $s \geq 0, B \in \mathcal{B}(E)$. We have*

$$(3.9) \quad \tilde{\Pi}_x^y Y = \Pi_x^y \{\tau_D = \tau_E, Y\}$$

for all $x \in D, y \in \partial E \cap \partial D$ and for all $Y \in \tilde{\mathcal{F}}$.

Corollary 3.1. *If*

$$(3.10) \quad F_t = \exp \left[- \int_0^t a(\xi_s) ds \right]$$

where a is a positive continuous function on $[0, \infty)$, then, for $y \in \partial D \cap \partial E$,

$$(3.11) \quad \tilde{\Pi}_x^y F_{\tau_D} = \Pi_x^y \{\tau_D = \tau_E, F_{\tau_E}\}.$$

Indeed, it is easy to see that $F_{\tilde{\tau}} \in \tilde{\mathcal{F}}$.

4. EQUATIONS CONNECTING P_x AND \mathbb{N}_x WITH Π_x^ν

4.1.

Theorem 4.1. *Let $Z_\nu = \text{SBV}(h_\nu), Z_u = \text{SBV}(u)$ where $\nu \in \mathcal{N}_1^E$ and $u \in \mathcal{U}(E)$. Then*

$$(4.1) \quad P_x Z_\nu e^{-Z_u} = e^{-u(x)} \Pi_x^\nu e^{-\Phi(u)}$$

and

$$(4.2) \quad \mathbb{N}_x Z_\nu e^{-Z_u} = \Pi_x^\nu e^{-\Phi(u)}$$

where

$$(4.3) \quad \Phi(u) = \int_0^{\tau_E} \psi'[u(\xi_t)] dt.$$

Proof. Formula (4.1) follows from Theorem 3.1 in Chapter 9 of [Dy02]. To prove (4.2), we observe that, for every $\lambda \geq 0$, $\lambda Z_\nu + Z_u = \text{SBV}(v)$ where $v = \lambda h_\nu + u \in \mathcal{U}^-(E)$ and therefore, by (2.6),

$$(4.4) \quad \mathbb{N}_x (1 - e^{-\lambda Z_\nu - Z_u}) = -\log P_x e^{-\lambda Z_\nu - Z_u}.$$

By taking the derivatives with respect to λ at $\lambda = 0$,⁸ we get

$$(4.5) \quad \mathbb{N}_x Z_\nu e^{-Z_u} = P_x Z_\nu e^{-Z_u} / P_x e^{-Z_u}.$$

By Theorem 1.1 of Chapter 9 in [Dy02],

$$(4.6) \quad P_x e^{-Z_u} = e^{-u(x)}.$$

⁸The differentiation under the integral signs is justified by (2.13).

Therefore (4.4) follows from (4.1), (4.5) and (4.6). \square

Theorem 4.2. *Suppose that $D \subset E$ are bounded smooth open sets and Λ, L, D^* are the sets introduced in Theorem 1.1. Let ν be a finite measure on $\partial D \cap \partial E$, $x \in E$ and $\mathcal{E}_x(\nu) < \infty$. Put*

$$(4.7) \quad \begin{aligned} w_\Lambda(x) &= \mathbb{N}_x\{\mathcal{R}_D \cap \Lambda \neq \emptyset\}, \\ v_s(x) &= w_\Lambda(x) + \mathbb{N}_x\{\mathcal{R}_D \cap \Lambda = \emptyset, 1 - e^{-sZ_\nu}\} \end{aligned}$$

for $x \in D$ and let $w_\Lambda(x) = v_s(x) = 0$ for $x \in E \setminus D$. For every $x \in E$, we have

$$(4.8) \quad \mathbb{N}_x\{\mathcal{R}_E \subset D^*, Z_\nu\} = \Pi_x^\nu\{A, e^{-\Phi(w_\Lambda)}\},$$

$$(4.9) \quad \mathbb{N}_x\{\mathcal{R}_E \subset D^*, Z_\nu \neq 0\} = \int_0^\infty \Pi_x^\nu\{A, e^{-\Phi(v_s)}\} ds$$

where Φ is defined by (4.3) and

$$(4.10) \quad A = \{\tau_E = \tau_D\} = \{\xi_t \in D \text{ for all } t < \tau_E\}.$$

Remark. Since $\mathcal{E}_x(\nu) < \infty$, ν belongs to \mathcal{N}_x^E and to \mathcal{N}_x^D .

Proof. 1°. If $x \in E \setminus D$, then, \mathbb{N}_x -a.s., \mathcal{R}_E is not a subset of D^* because \mathcal{R}_E contains supports of X_O for all neighborhoods O of x and we can choose O such that $\bar{O} \cap D^* = \emptyset$. On the other hand, $\Pi_x^\nu(A) = 0$. Therefore (4.8) and (4.9) hold independently of values of w_Λ and v_s .

2°. Now we assume that $x \in D$. Put $\mathcal{A} = \{\mathcal{R}_D \subset D^*\}$. We claim that

$$\mathcal{A} = \{\mathcal{R}_E \subset D^*\} \quad \mathbb{N}_x\text{-a.s.}$$

Indeed, $\{\mathcal{R}_E \subset D^*\} \subset \mathcal{A}$ because $\mathcal{R}_D \subset \mathcal{R}_E$. By Theorem 3.1, $\mathcal{A} \subset \{\mathcal{R}_D = \mathcal{R}_E\}$ \mathbb{N}_x -a.s. Hence, $\mathcal{A} \subset \{\mathcal{R}_E \subset D^*\}$.

By Theorem 3.1, $\mathcal{R}_D = \mathcal{R}_E$ and $Z_\nu = \tilde{Z}_\nu$ \mathbb{N}_x -a.s. on \mathcal{A} . Therefore

$$(4.11) \quad \begin{aligned} \mathbb{N}_x\{\mathcal{R}_E \subset D^*, Z_\nu\} &= \mathbb{N}_x\{\mathcal{A}, Z_\nu\} = \mathbb{N}_x\{\mathcal{A}, \tilde{Z}_\nu\}, \\ \mathbb{N}_x\{\mathcal{R}_E \subset D^*, Z_\nu e^{-sZ_\nu}\} &= \mathbb{N}_x\{\mathcal{A}, Z_\nu e^{-sZ_\nu}\} = \mathbb{N}_x\{\mathcal{A}, \tilde{Z}_\nu e^{-s\tilde{Z}_\nu}\}. \end{aligned}$$

Formula (4.7) defines two elements of $\mathcal{U}(D)$. The stochastic boundary value Z_Λ of w_Λ in D is equal to $\infty 1_{\mathcal{A}^c}$ (Remark 1.2 on p. 133 in [Dy02]) and therefore

$$(4.12) \quad e^{-Z_\Lambda} = 1_{\mathcal{A}}.$$

By (2.7) and (2.8), $v_s(x) = -\log P_x\{\mathcal{R}_D \cap \Lambda = \emptyset, e^{-sZ_\nu}\}$ and, by Remark 2.1 on p. 137 in [Dy02], the stochastic boundary value Z^s of v_s in D is equal to $Z_\Lambda + s\tilde{Z}_\nu$. Hence,

$$(4.13) \quad e^{-Z^s} = 1_{\mathcal{A}} e^{-s\tilde{Z}_\nu}.$$

By (4.11), (4.12) and (4.13),

$$(4.14) \quad \mathbb{N}_x\{\mathcal{A}, Z_\nu\} = \mathbb{N}_x\{\tilde{Z}_\nu e^{-Z_\Lambda}\}$$

and

$$(4.15) \quad \mathbb{N}_x\{\mathcal{A}, Z_\nu e^{-sZ_\nu}\} = \mathbb{N}_x\{\tilde{Z}_\nu e^{-Z^s}\}.$$

By applying formula (4.2) to \tilde{Z}_ν and to the restriction of w_Λ to D , we conclude from (4.14) that

$$(4.16) \quad \mathbb{N}_x\{\mathcal{A}, Z_\nu\} = \tilde{\Pi}_x^\nu \exp \left[- \int_0^{\tau_D} \psi'[w_\Lambda(\xi_s)] ds \right]$$

and, by Corollary 3.1,

$$(4.17) \quad \mathbb{N}_x\{\mathcal{A}, Z_\nu\} = \Pi_x^\nu\{A, e^{-\Phi(w_\Lambda)}\}.$$

Analogously, (4.2) applied to the restriction of v_s to D , in combination with (4.15) and (3.11), yields

$$(4.18) \quad \mathbb{N}_x\{\mathcal{A}, Z_\nu e^{-sZ_\nu}\} = \Pi_x^\nu\{A, e^{-\Phi(v_s)}\}.$$

Formula (4.8) follows from (4.17) and formula (4.9) follows from (4.18) because

$$(4.19) \quad \mathbb{N}_x\{\mathcal{A}, Z_\nu \neq 0\} = \lim_{t \rightarrow \infty} \mathbb{N}_x\{\mathcal{A}, 1 - e^{-tZ_\nu}\}$$

and

$$(4.20) \quad 1 - e^{-tZ_\nu} = \int_0^t Z_\nu e^{-sZ_\nu} ds.$$

□

5. PROOF OF THEOREM 1.1

We use the following two elementary inequalities:

5.A. For all $a, b \geq 0$ and $0 < \beta < 1$,

$$(5.1) \quad (a + b)^\beta \leq a^\beta + b^\beta.$$

Proof. It is sufficient to prove (5.1) for $a = 1$. Put $f(t) = (1 + t)^\beta - t^\beta$. Note that $f(0) = 1$ and $f'(t) \leq 0$ for $t > 0$. Hence $f(t) \leq 1$ for $t \geq 0$. □

5.B. For every finite measure M , every positive measurable function Y and every $\beta > 0$,

$$M(Y^{-\beta}) \geq M(1)^{1+\beta} (MY)^{-\beta}.$$

Indeed $f(y) = y^{-\beta}$ is a convex function on \mathbb{R}_+ , and we get 5.B by applying Jensen's inequality to the probability measure $M/M(1)$.

Proof of Theorem 1.1. 1°. If $x \in E \setminus D$, then, \mathbb{N}_x -a.s., \mathcal{R}_E is not a subset of D^* (see the proof of Theorem 4.2). Hence, both sides of (1.7) vanish.

2°. Suppose $x \in D$. By (2.6), $\mathbb{N}_x(1 - e^{-sZ_\nu}) = u_{s\nu}$. Thus (4.7) implies $v_s \leq w_\Lambda + u_{s\nu}$. Therefore, by 5.A, $v_s^{\alpha-1} \leq w_\Lambda^{\alpha-1} + u_{s\nu}^{\alpha-1}$ and, since $u_{s\nu} \leq h_{s\nu} = sh_\nu$, $\Phi(v_s) \leq \Phi(w_\Lambda) + s^{\alpha-1}\Phi(h_\nu)$.

Put $\mathcal{A} = \{\mathcal{R}_E \subset D^*\}$. It follows from (4.9) that

$$(5.2) \quad \mathbb{N}_x\{\mathcal{A}, Z_\nu \neq 0\} \geq \Pi_x^\nu\{A, \int_0^\infty e^{-\Phi(w_\Lambda) - s^{\alpha-1}\Phi(h_\nu)} ds\}.$$

Note that $\int_0^\infty e^{-as^\beta} ds = Ca^{-1/\beta}$ where $C = \int_0^\infty e^{-t^\beta} dt$. Therefore (5.2) implies

$$(5.3) \quad \mathbb{N}_x\{\mathcal{A}, Z_\nu \neq 0\} \geq C\Pi_x^\nu\{A, e^{-\Phi(w_\Lambda)}\Phi(h_\nu)^{-1/(\alpha-1)}\}.$$

The right side in (5.3) is equal to $CM(Y^{-\beta})$ where $\beta = 1/(\alpha - 1)$, $Y = \Phi(h_\nu)$ and M is the measure with the density $1_A e^{-\Phi(w_\Lambda)}$ with respect to Π_x^ν . We get from (5.3) and 5.B that

$$\begin{aligned} \mathbb{N}_x\{\mathcal{A}, Z_\nu \neq 0\} &\geq CM(1)^{1+\beta}(MY)^{-\beta} \\ &= C[\Pi_x^\nu\{A, e^{-\Phi(w_\Lambda)}\}]^{\alpha/(\alpha-1)}[\Pi_x^\nu\{A, e^{-\Phi(w_\Lambda)}\Phi(h_\nu)\}]^{-1/(\alpha-1)}. \end{aligned}$$

By (4.8), $\Pi_x^\nu\{A, e^{-\Phi(w_\Lambda)}\} = \mathbb{N}_x\{\mathcal{R}_E \subset D^*, Z_\nu\}$ and, since $\Pi_x^\nu\{A, e^{-\Phi(w_\Lambda)}\Phi(h_\nu)\} \leq \Pi_x^\nu\Phi(h_\nu)$, we have

$$(5.4) \quad \mathbb{N}_x\{\mathcal{A}, Z_\nu \neq 0\} \geq C[\mathbb{N}_x\{\mathcal{R}_E \subset D^*, Z_\nu\}]^{\alpha/(\alpha-1)}[\Pi_x^\nu\Phi(h_\nu)]^{-1/(\alpha-1)}.$$

3°. By the definition of h -transform, for every $f \in \mathcal{B}(E)$ and every $h \in \mathcal{H}(E)$,

$$\Pi_x^h \int_0^{\tau_E} f(\xi_t) dt = \int_0^\infty \Pi_x^h\{t < \tau_E, f(\xi_t)\} dt = \int_0^\infty \Pi_x\{t < \tau_E, f(\xi_t)h(\xi_t)\} dt.$$

By taking $f = \alpha h_\nu^{\alpha-1}$ and $h = h_\nu$ we get

$$(5.5) \quad \Pi_x^\nu\Phi(h_\nu) = \alpha \mathcal{E}_x(\nu).$$

Formula (1.7) follows from (5.4) and (5.5). \square

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DEPARTMENT OF MATHEMATICS, CORNELL UNIVERSITY, ITHACA, NY 14853

E-mail address: ebd1@cornell.edu