

LOWER BOUNDS FOR THE SPECTRAL FUNCTION AND FOR THE REMAINDER IN LOCAL WEYL'S LAW ON MANIFOLDS

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ABSTRACT. We announce asymptotic lower bounds for the spectral function of the Laplacian and for the remainder in the local Weyl's law on Riemannian manifolds. In the negatively curved case, methods of thermodynamic formalism are applied to improve the estimates. Our results develop and extend the unpublished thesis of A. Karnaukh. We discuss some ideas of the proofs; for complete proofs see our extended paper on the subject.

1. SPECTRAL FUNCTION AND THE WEYL'S LAW

Let X be a compact Riemannian manifold of dimension $n \geq 2$ with the metric $\{g_{ij}\}$ and the volume V . Let Δ be the Laplacian on X with eigenvalues $0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \dots$ and the corresponding orthonormal basis $\{\phi_i\}$ of eigenfunctions: $\Delta\phi_i = \lambda_i\phi_i$.

Given $x, y \in X$, let

$$N_{x,y}(\lambda) = \sum_{\sqrt{\lambda_i} \leq \lambda} \phi_i(x)\phi_i(y)$$

be the spectral function of the Laplacian. On the diagonal $x = y$ we denote it simply $N_x(\lambda)$. If $N(\lambda) = \#\{\sqrt{\lambda_i} \leq \lambda\}$ is the eigenvalue counting function, then $N(\lambda) = \int_X N_x(\lambda) dV$. Let

$$(1.1) \quad \sigma_n = \frac{2\pi^{n/2}}{n\Gamma(n/2)}$$

be the volume of the unit ball in \mathbf{R}^n . The asymptotic behavior of the spectral and the counting functions is given by ([13]; see also [28]):

$$(1.2) \quad \begin{aligned} N_{x,y}(\lambda) &= O(\lambda^{n-1}), & x \neq y; \\ N_x(\lambda) &= \frac{\sigma_n}{(2\pi)^n} \lambda^n + R_x(\lambda), & R_x(\lambda) = O(\lambda^{n-1}); \\ N(\lambda) &= \frac{V\sigma_n}{(2\pi)^n} \lambda^n + R(\lambda), & R(\lambda) = O(\lambda^{n-1}). \end{aligned}$$

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We refer to the asymptotics of $N_x(\lambda)$ as the *local Weyl's law*, the asymptotics for $N(\lambda)$ being the usual Weyl's law for the distribution of eigenvalues. The upper bounds for $R(\lambda)$ and $R_x(\lambda)$ are attained for round spheres and hence are sharp. Both local and integrated remainder estimates for the Weyl's law on manifolds under various geometric conditions have been actively studied in the last forty years (see [2], [8], [10], [14], [21], [24], [26], [29], [31], etc).

In the present paper we focus on asymptotic *lower* bounds for the spectral function and for the remainder in local Weyl's law.

We recall that $f_1(\lambda) = \Omega(f_2(\lambda))$ for an arbitrary function f_1 and a positive function f_2 means $\limsup_{\lambda \rightarrow \infty} |f_1(\lambda)|/f_2(\lambda) > 0$.

Theorem 1.3. *Let X be a compact n -dimensional Riemannian manifold, and let $x, y \in X$ be two points that are not conjugate along any shortest geodesic joining them. Then*

$$(1.4) \quad N_{x,y}(\lambda) = \Omega\left(\lambda^{\frac{n-1}{2}}\right).$$

Let us now formulate the on-diagonal counterpart of Theorem 1.3. Consider the heat trace asymptotics as $t \rightarrow 0^+$:

$$(1.5) \quad \sum_i e^{-\lambda_i t} \sim \frac{1}{(4\pi t)^{n/2}} \sum_{j=0}^{\infty} \left(\int_X a_j(x) d\text{vol} \right) t^n,$$

where $a_j(x)$ are the local heat invariants of X . Let $\kappa_x = \min\{j \geq 1 \mid a_j(x) \neq 0\}$. If $a_j(x) = 0$ for all $j \geq 1$, we set $\kappa_x = \infty$. We recall that $a_1(x) = \frac{\tau(x)}{6}$, where $\tau(x)$ is the scalar curvature of X at the point x .

Theorem 1.6. *Let X be an n -dimensional Riemannian manifold and $x \in X$ an arbitrary point. If $n - 2\kappa_x - 1 > 0$, then*

$$(1.7) \quad R_x(\lambda) = \Omega(\lambda^{n-2\kappa_x-1}).$$

If X has no conjugate points, then

$$(1.8) \quad R_x(\lambda) = \Omega(\lambda^{\frac{n-1}{2}}).$$

Remark 1.9. If $n - 4\kappa_x - 1 < 0$, then the bound (1.8) is better than the bound (1.7). If the scalar curvature $\tau(x) \neq 0$, then (1.7) becomes $R_x(\lambda) = \Omega(\lambda^{n-3})$.

Estimate (1.8) should be compared with the Hardy-Landau lower bound for the remainder in the Gauss circle problem or, equivalently, for the remainder in the Weyl's law on a 2-dimensional flat square torus. We note that (1.8) gives the same exponent as in the Hardy-Landau bound for any surface without conjugate points. In dimension 3 the exponent in (1.8) is also consistent with the lower bound due to Szegő for the error term in the sphere problem (see [30]). However, for a similar counting problem in a ball of dimension $n \geq 4$, the sharp error estimate is $\Omega(\lambda^{n-2})$ which is larger than (1.8).

2. ESTIMATES FOR NEGATIVELY CURVED MANIFOLDS

Asymptotic lower bounds (1.4), (1.7), and (1.8) can be improved for manifolds of negative curvature. We assume that for any pair of directions ξ, η the sectional curvature $K(\xi, \eta)$ satisfies

$$(2.1) \quad -K_1^2 \leq K(\xi, \eta) \leq -K_2^2.$$

Apart from the standard wave equation techniques (cf. [10], [2], [16]) our method uses thermodynamic formalism (see, for example, [6], [20]). Let M be the universal cover of X . Let G^t be the geodesic flow on the unit tangent bundle SM and let E_ξ^u be the unstable subspace for G^t , $\xi \in SM$. The *Sinai-Ruelle-Bowen potential* is a Hölder continuous function $\mathcal{H} : SM \rightarrow \mathbf{R}$ which for any $\xi \in SM$ is defined by the formula (see [7], [27])

$$(2.2) \quad \mathcal{H}(\xi) = \frac{d}{dt} \Big|_{t=0} \ln \det dG^t|_{E_\xi^u}.$$

For any continuous function $f : SM \rightarrow \mathbf{R}$ one can define the *topological pressure*

$$(2.3) \quad P(f) = \sup_{\mu} \left(h_{\mu} + \int f d\mu \right),$$

where the supremum is taken over all G^t -invariant measures μ , and h_{μ} denotes the *measure-theoretic entropy* of the geodesic flow (see [6]). In particular $P(0) = h$, where h is the *topological entropy* of the flow.

Theorem 2.4. *The remainder in the local Weyl's law on an n -dimensional compact negatively curved manifold satisfies*

$$(2.5) \quad R_x(\lambda) = \begin{cases} \Omega \left(\lambda^{\frac{n-1}{2}} (\log \lambda)^{\frac{P(-\mathcal{H}/2)}{h} - \delta} \right) & \forall \delta > 0, \quad n \leq 5; \\ \Omega(\lambda^{n-3}), & n \geq 6. \end{cases}$$

One can show that the $\frac{P(-\mathcal{H}/2)}{h}$ is estimated in terms of curvatures as follows:

$$\frac{P(-\mathcal{H}/2)}{h} \geq \frac{K_2}{2K_1} > 0.$$

We note that in dimensions $n \leq 5$ the main contribution to the remainder comes from the oscillating terms corresponding to geodesic loops. At the same time, in dimensions $n \geq 6$ the contribution of the singularity at $t = 0$ dominates the remainder. This explains different bounds in Theorem 2.4 for low and high dimensions.

Away from the diagonal, we get

Theorem 2.6. *On a compact n -dimensional negatively curved manifold the spectral function $N_{x,y}(\lambda)$ satisfies, for any $\delta > 0$ and $x \neq y$,*

$$(2.7) \quad N_{x,y}(\lambda) = \Omega \left(\lambda^{\frac{n-1}{2}} (\log \lambda)^{\frac{P(-\mathcal{H}/2)}{h} - \delta} \right).$$

2.1. Discussion: remainder estimates on negatively curved surfaces. Let us compare the results of this section with some known facts and conjectures regarding the error term in Weyl's law on negatively curved surfaces.

A lower bound for the local remainder $R_x(\lambda)$ on negatively curved surfaces was proved in an unpublished Princeton Ph.D. thesis [16]. On surfaces of *constant* negative curvature (2.5) coincides with its analogue in [16], but the techniques of thermodynamic formalism for hyperbolic flows allow us to improve the results of [16] when the curvature is variable. We do not need the hypothesis $K_1/K_2 < 2$ of [16] to get a logarithmic improvement in the estimates, and moreover, we get higher powers of the logarithm.

For surfaces of constant negative curvature it is proved in [23] that $R(\lambda) = \Omega((\log \lambda)^{\frac{1}{2} - \delta})$ for any $\delta > 0$. In a work in progress we aim to generalize this bound

for surfaces of *variable* negative curvature using the methods of thermodynamic formalism announced in the present paper.

For arithmetic hyperbolic surfaces Selberg proved a faster growth of the remainder: $R(\lambda) = \Omega(\frac{\sqrt{\lambda}}{\log \lambda})$ (see [12]). An improvement of this bound for the modular surface $SL(2, \mathbf{Z})/\mathbf{H}$ was recently obtained in [17]; see also [18] for some related estimates. It is conjectured in [24] that on any surface of constant negative curvature $R(\lambda) = O(\lambda^{\frac{1}{2}+\epsilon})$ for any $\epsilon > 0$. However, the best known upper bound is $R(\lambda) = O(\lambda/\log \lambda)$ ([2]).

On a *generic* negatively curved surface it is believed that $R(\lambda) = O(\lambda^\epsilon)$ for any $\epsilon > 0$. Such an estimate looks plausible in view of the results on spectral fluctuations, e.g. in [3], [5] and [1]. We note the difference between the predicted upper bound for the *global* error term, and the lower bound of Theorem 2.4 for the *local* remainder.

3. SOME IDEAS OF THE PROOFS

Let us focus on Theorem 2.6 and indicate some key ideas that are common for the proofs of all the main results (for complete proofs see [15]).

Consider the even part of the wave kernel $e(t, x, y)$ on X . It satisfies

$$(3.1) \quad e(t, x, y) = \sum_{i=0}^{\infty} \cos(\sqrt{\lambda_i}t) \phi_i(x) \phi_i(y).$$

Take a smooth function $\psi \in C_0^\infty(\mathbf{R})$ such that $\text{supp } \psi \subseteq [-1, 1]$, it is even and monotone decreasing on $[0, 1]$, $\psi \geq 0$, $\psi(0) = 1$. Fix two positive parameters λ, T and consider the function (cf. [16])

$$(3.2) \quad k_{\lambda, T}(x, y) = \int_{-\infty}^{\infty} \frac{\psi(t/T)}{T} \cos(\lambda t) e(t, x, y) dt.$$

We have

Lemma 3.3. *If $N_{x, y}(\lambda) = o(\lambda^a (\log \lambda)^b)$, $a, b > 0$, then $k_{\lambda, T}(x, y) = o(\lambda^a (\log \lambda)^b)$.*

Lemma 3.3 is used to prove Theorem 2.6 by contradiction. Assuming the contrary, we use a *pretrace formula* (3.6) to show that $k_{\lambda, T}(x, y)$ is large.

Consider the fundamental solution $E(t, x, y)$ of the wave equation on the universal cover M of X . Then given $x, y \in X$, we have

$$(3.4) \quad e(t, x, y) = \sum_{\omega \in \Gamma} E(t, x, \omega y),$$

where the sum is taken over $\Gamma = \pi_1(X)$. Let $K_{\lambda, T}(x, y)$ be the analogue of $k_{\lambda, T}(x, y)$ corresponding to the wave kernel $E(t, x, y)$ on M :

$$(3.5) \quad K_{\lambda, T}(x, y) = \int_{-\infty}^{\infty} \frac{\psi(t/T)}{T} \cos(\lambda t) E(t, x, y) dt.$$

Then

$$(3.6) \quad k_{\lambda, T}(x, y) = \sum_{\omega \in \Gamma} K_{\lambda, T}(x, \omega y).$$

We use the parametrix for $E(t, x, y)$ (see [2], [32]):

$$(3.7) \quad E(t, x, y) = \frac{1}{\pi^{\frac{n-1}{2}}} |t| \sum_{j=0}^{\infty} u_j(x, y) \frac{(r^2 - t^2)_-^{j - \frac{n-3}{2} - 2}}{4^j \Gamma(j - \frac{n-3}{2} - 1)} \pmod{C^\infty},$$

where $r = d(x, y)$. The expression (3.7) is understood in the sense of generalized functions [11]. The coefficients $u_j(x, y)$ are the solutions of the transport equations along the geodesic joining x and y (see [2]). We recall that $u_j(x, x) = a_j(x)$ as defined in (1.5).

Asymptotic analysis of the leading terms in (3.7) (cf. [2], [9]) yields

Proposition 3.8. *The integral $K_{\lambda, T}(x, y)$ defined by (3.5) satisfies, for any $x \neq y \in M$ as $\lambda \rightarrow \infty$,*

$$(3.9) \quad K_{\lambda, T}(x, y) = \frac{Q_0 \lambda^{\frac{n-1}{2}} \psi(r/T)}{T \sqrt{g(x, y)} r^{n-1}} \sin(\lambda r + \phi_n) + O(\lambda^{\frac{n-3}{2}}).$$

Here $g = \sqrt{\det g_{ij}}$ in geodesic normal coordinates, $\phi_n = \frac{\pi}{4}(3 - (n \bmod 8))$, Q_0 is a non-zero constant.

It follows from (3.6), (3.9), and the rate of growth for the number of lattice points on a negatively curved manifold, that

$$(3.10) \quad k_{\lambda, T}(x, y) = \sum_{\omega: r_\omega \leq T} \frac{Q_0 \lambda^{\frac{n-1}{2}} \psi(\frac{r_\omega}{T})}{T \sqrt{g(x, \omega y)} r_\omega^{n-1}} \sin(\lambda r_\omega + \phi_n) + O(\lambda^{\frac{n-3}{2}}) \exp(O(T)).$$

Let us now bound from below the following sum:

$$(3.11) \quad S_{x, y}(T) = \sum_{r_\omega \leq T, \omega \in \Gamma} \frac{1}{\sqrt{g(x, \omega y)} r_\omega^{n-1}},$$

where $y \notin \Gamma x$, $r_\omega = d(x, \omega y)$.

The sum (3.11) can be estimated by a sum over closed geodesics on X . This is possible due to the fact that near a geodesic segment $[x, \omega y]$ there exists a closed geodesic of comparable length (a similar idea was applied in [4]). Reduction to closed geodesics requires some careful analysis of the behavior of the geodesic flow.

We apply thermodynamic formalism to bound the sum over closed geodesics from below. Namely, using the results of [19], [20], we get

Theorem 3.12. *There exists a constant $C_0 > 0$ such that*

$$(3.13) \quad S_{x, y}(T) \geq C_0 e^{P(-\frac{\mathcal{H}}{2}) \cdot T}$$

as $T \rightarrow \infty$, where P is the topological pressure (2.3) and \mathcal{H} is the Sinai-Ruelle-Bowen potential (2.2).

One can show that $P(-\frac{\mathcal{H}}{2}) \geq \frac{(n-1)K_2}{2}$, hence $S_{x, y}(T)$ grows exponentially in T .

To get a contradiction with Lemma 3.3, we consider the expression $k_{\lambda, T}(x, y)$, and let T grow with λ . We then use Dirichlet box principle (cf. [22], [25]) to choose λ so that λr_ω be very close to an integer multiple of 2π for all ω with $r_\omega \leq T$. In order to make the Dirichlet box principle work we have to choose T of the size $\frac{1}{h} \ln \ln \lambda$. This together with (3.13) explains the logarithmic factor in Theorem 2.6.

Assume now that $n \not\equiv 3 \pmod{4}$. Then all the expressions $\sin(\lambda r_\omega + \phi_n)$ in (3.10) are of the same sign and bounded away from zero by a constant. Combining this

with the estimate (3.13), we obtain a lower bound for the right-hand side of (3.10), which leads to the desired contradiction with Lemma 3.3.

When $n \equiv 3 \pmod{4}$, $\phi_n = 0$ in (3.9) and the Dirichlet box principle cannot be used directly. We modify the argument in order to bound from below all the expressions $\sin(\lambda r_\omega)$ in (3.10) for $T/A \leq r_\omega \leq T$, where A is a suitably chosen constant. This proves Theorem 2.6 in all dimensions.

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