

## THE $SL(2, \mathbb{C})$ CHARACTER VARIETY OF A ONE-HOLED TORUS

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ABSTRACT. In this note we announce several results concerning the  $SL(2, \mathbb{C})$  character variety  $\mathcal{X}$  of a one-holed torus. We give a description of the largest open subset  $\mathcal{X}_{BQ}$  of  $\mathcal{X}$  on which the mapping class group  $\Gamma$  acts properly discontinuously, in terms of two very simple conditions, and show that a series identity generalizing McShane’s identity for the punctured torus holds for all characters in this subset. We also give variations of the McShane-Bowditch identities for characters fixed by an Anosov element of  $\Gamma$  with applications to closed hyperbolic three-manifolds. Finally we give a definition of end invariants for  $SL(2, \mathbb{C})$  characters and give a partial classification of the set of end invariants of a character in  $\mathcal{X}$ .

### 1. INTRODUCTION

Let  $T$  be a one-holed torus,  $\pi$  its fundamental group, and  $\Gamma := \pi_0(\text{Homeo}(T))$  the mapping class group of  $T$ . In this note we announce several results concerning the  $SL(2, \mathbb{C})$  character variety  $\mathcal{X}$  of  $T$ . Here is a brief description of our results. We first give a characterization of the largest open subset  $\mathcal{X}_{BQ}$  of  $\mathcal{X}$  on which the mapping class group  $\Gamma$  acts properly discontinuously, in terms of two very simple conditions, called the Bowditch Q-conditions. This generalizes results of Bowditch [5], who gave a similar description for the “type-preserving” characters, and also of Goldman [7] (see also [6]), who studied the dynamics of the action of  $\Gamma$  on the *real*  $SL(2)$  characters, and gave a (geometric) description of the set for the real characters. Note that for  $[\rho] \in \mathcal{X}$ , it is possible to verify algorithmically whether  $[\rho]$  satisfies these conditions.

We next show that a series identity generalizing McShane’s remarkable identity for the punctured torus holds for all characters in this subset (the original identity can be regarded as the formal derivative of the general identity evaluated at an appropriate parameter value). This generalizes results of McShane [11], [12], Bowditch [3], [5], Mirzakhani [13], and the authors [16], [17] for two-generator subgroups of  $SL(2, \mathbb{C})$ . We also give necessary and sufficient conditions for this identity to hold for characters  $[\rho] \in \mathcal{X}$ , thereby giving a complete answer to the question of when the identity holds for two-generator subgroups of  $SL(2, \mathbb{C})$ .

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The next set of results are for variations of the McShane-Bowditch identities to characters fixed by an Anosov element of  $\Gamma$ , and which satisfy a relative version of the Bowditch Q-conditions. These have applications to closed hyperbolic 3-manifolds, and generalize the results of Bowditch in [4].

Finally we give a definition of end invariants for  $\mathrm{SL}(2, \mathbb{C})$  characters, inspired by Bowditch's definition in [5], and give a partial classification of the set  $\mathcal{E}(\rho)$  of end invariants of a character  $[\rho] \in \mathcal{X}$ . When non-empty, this set gives information about the extent to which the action of  $\Gamma$  on  $[\rho]$  is not proper. In particular, we give classification results for real characters, imaginary characters, and discrete characters.

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## 2. PRELIMINARIES AND DEFINITIONS

**2.1. Basic Definitions.** Let  $T$  be a one-holed torus and  $\pi$  its fundamental group, which is freely generated by two elements  $X, Y$  corresponding to simple closed curves on  $T$  with geometric intersection number one. The  $\mathrm{SL}(2, \mathbb{C})$  character variety  $\mathcal{X} := \mathrm{Hom}(\pi, \mathrm{SL}(2, \mathbb{C})) // \mathrm{SL}(2, \mathbb{C})$  of  $T$  is the set of equivalence classes of representations  $\rho : \pi \mapsto \mathrm{SL}(2, \mathbb{C})$ , where the equivalence classes are obtained by taking the closure of the orbits under conjugation by  $\mathrm{SL}(2, \mathbb{C})$ . The character variety stratifies into relative character varieties: for  $\kappa \in \mathbb{C}$ , the  $\kappa$  relative character variety  $\mathcal{X}_\kappa$  is the set of equivalence classes  $[\rho]$  such that

$$\mathrm{tr} \rho(XYX^{-1}Y^{-1}) = \kappa$$

for one (and hence any) pair of generators  $X, Y$  of  $\pi$ . By the classical results of Fricke, we have the following identifications:

$$\begin{aligned} \mathcal{X} &\cong \mathbb{C}^3, \\ \mathcal{X}_\kappa &\cong \{(x, y, z) \in \mathbb{C}^3 \mid x^2 + y^2 + z^2 - xyz - 2 = \kappa\}; \end{aligned}$$

the identification is given by

$$\iota : [\rho] \mapsto (x, y, z) := (\mathrm{tr} \rho(X), \mathrm{tr} \rho(Y), \mathrm{tr} \rho(XY)),$$

where  $X, Y$  is a fixed pair of generators of  $\pi$ . The topology on  $\mathcal{X}$  and  $\mathcal{X}_\kappa$  will be that induced by the above identifications.

A character is *real* if  $\iota([\rho]) \in \mathbb{R}^3$ , *imaginary* if two of the entries of  $\iota([\rho])$  are purely imaginary and the third is real, and *dihedral* if two of the entries of  $\iota([\rho])$  are zero (so that the third entry is  $\pm\sqrt{\kappa+2}$ ).

The outer automorphism group of  $\pi$ ,  $\mathrm{Out}(\pi) := \mathrm{Aut}(\pi)/\mathrm{Inn}(\pi) \cong \mathrm{GL}(2, \mathbb{Z})$  is isomorphic to the mapping class group  $\Gamma := \pi_0(\mathrm{Homeo}(T))$  of  $T$  and acts on  $\mathcal{X}$  preserving the trace of the commutator of a pair of generators, hence it also acts on  $\mathcal{X}_\kappa$ , and the action is given by

$$\phi([\rho]) = [\rho \circ \phi^{-1}],$$

where  $\phi \in \mathrm{Out}(\pi)$  and  $[\rho] \in \mathcal{X}$  or  $\mathcal{X}_\kappa$  respectively. It is often convenient to consider only the subgroup  $\mathrm{Out}(\pi)^+$  of ‘‘orientation-preserving’’ automorphisms corresponding to the orientation-preserving homeomorphisms  $\Gamma^+$  of  $T$ , which is isomorphic to

$SL(2, \mathbb{Z})$ . The action of  $Out(\pi)^+$  (respectively,  $Out(\pi)$ ) on  $\mathcal{X}$  and  $\mathcal{X}_\kappa$  is not effective, the kernel is  $\{\pm I\}$ , generated by the elliptic involution of  $T$  so that the effective action is by  $PSL(2, \mathbb{Z})$  (respectively,  $PGL(2, \mathbb{Z})$ ).

**2.2. Simple curves; pants graph of  $T$ .** Let  $\mathcal{C}$  be the set of free homotopy classes of non-trivial, non-peripheral simple closed curves on  $T$ ; the elements of  $\mathcal{C}$  correspond to certain elements of  $\pi$ , up to conjugation and inverse. Let  $\mathcal{C}(T)$  be the ‘‘pants graph’’ of  $T$ ; the vertices of  $\mathcal{C}(T)$  are the elements of  $\mathcal{C}$ , where two vertices are joined by an edge if and only if the corresponding curves on  $T$  have geometric intersection number one.  $\Gamma$  and  $Out(\pi)$  act naturally on  $\mathcal{C}$  (respectively  $\mathcal{C}(T)$ ). We can realize  $\mathcal{C}(T)$  as the Farey graph/triangulation of the upper half-plane  $\mathbb{H}^2$  so that  $\mathcal{C}$  be identified with  $\hat{\mathbb{Q}}$ ; the action of  $\Gamma$  is realized by the action of  $PGL(2, \mathbb{Z})$  on the Farey graph. The projective lamination space  $\mathcal{PL}$  of  $T$  is then identified with  $\hat{\mathbb{R}}$  and contains  $\mathcal{C}$  as the (dense) subset of rational points.

**2.3. Bowditch Q-conditions (BQ-conditions).** For  $[\rho] \in \mathcal{X}$  and  $X \in \mathcal{C}$ ,  $\text{tr } \rho(X)$  is well defined. We define the *Bowditch space* as the subset  $\mathcal{X}_{BQ}$  of  $\mathcal{X}$  consisting of characters  $[\rho]$  satisfying the following conditions (the *Bowditch Q-conditions*):

- (1)  $\text{tr } \rho(X) \notin [-2, 2]$  for all  $X \in \mathcal{C}$ ;
- (2)  $|\text{tr } \rho(X)| \leq 2$  for only finitely many (possibly no)  $X \in \mathcal{C}$ .

3. STATEMENT OF RESULTS

**3.1. Quasi-convexity.** For fixed  $[\rho] \in \mathcal{X}$  and  $K > 0$ , define  $\mathcal{C}_K(T)$  to be the subgraph of  $\mathcal{C}(T)$  spanned by the set of  $X \in \mathcal{C}$  for which  $|\text{tr } \rho(X)| \leq K$ . Then we have the following simple but fundamental result of Bowditch, which plays a key role in the proofs of all subsequent results. The result was stated for type-preserving characters in [5] but the proof works for all characters.

**Theorem 3.1** (Bowditch [5]). *For any  $[\rho] \in \mathcal{X}$ ,  $\mathcal{C}_K(T)$  is connected for all  $K \geq 2$ .*

**3.2. Action of  $\Gamma$  and generalizations of McShane’s identity.**

**Theorem 3.2** ([19, Theorems 2.2, 2.3 and Proposition 2.4]).

- (a)  $\mathcal{X}_{BQ}$  is open in  $\mathcal{X}$ .
- (b)  $\Gamma$  acts properly discontinuously on  $\mathcal{X}_{BQ}$ . Furthermore,  $\mathcal{X}_{BQ}$  is the largest open subset of  $\mathcal{X}$  for which this holds.
- (c) For  $[\rho] \in \mathcal{X}_{BQ} \cap \mathcal{X}_\kappa$ ,

$$(1) \quad \sum_{X \in \mathcal{C}} \log \frac{e^\nu + e^{l(\rho(X))}}{e^{-\nu} + e^{l(\rho(X))}} = \nu \pmod{2\pi i},$$

and the sum converges absolutely; here  $\nu = \cosh^{-1}(-\kappa/2)$ , and the complex length  $l(\rho(X))$  is given by the formula

$$(2) \quad l(\rho(X)) = 2 \cosh^{-1} \left( \frac{\text{tr } \rho(X)}{2} \right),$$

where we adopt the convention that the function  $\cosh^{-1}$  has images with non-negative real part and imaginary part in  $(-\pi, \pi]$ .

*Remark 3.3.* If  $\kappa = -2$ , then  $\nu = 0$  and all the terms of (1) are identically zero. However, if we take the first order infinitesimal, or the formal derivative of (1) and evaluate at  $\nu = 0$ , we get

$$(3) \quad \sum_{X \in \mathcal{C}} \frac{1}{1 + e^{l(\rho(X))}} = \frac{1}{2},$$

which is McShane's original identity in [11] for real type-preserving characters, and also Bowditch's generalization in [3] and [5] for general type-preserving characters. Note also that when  $\kappa = 2$ , which corresponds to reducible characters, the Bowditch Q-conditions are never satisfied; see [20]. Parts (a) and (b) of Theorem 3.2 were originally stated in [19] in terms of relative character varieties.

**3.3. Necessary and sufficient conditions.** Replacing condition (1) of the BQ-conditions by (1')  $\text{tr } \rho(X) \notin (-2, 2)$  for all  $X \in \mathcal{C}$ , we get the *extended Bowditch space*  $\hat{\mathcal{X}}_{BQ}$ , and we have the following result:

**Theorem 3.4** ([18, Theorem 1.6]). *For  $[\rho] \in \mathcal{X}$ , the identity (1) of Theorem 3.2(c) holds (with absolute convergence of the sum) if and only if  $[\rho]$  lies in the extended Bowditch space  $\hat{\mathcal{X}}_{BQ}$ .*

This result gives a complete answer to the question of when the generalized McShane's identity holds for two-generator subgroups of  $\text{SL}(2, \mathbb{C})$ .

**3.4. McShane-Bowditch identities for punctured torus bundles.** We next consider further variations of the McShane-Bowditch identities. Recall that  $\theta \in \Gamma$  acts naturally on  $\pi$ , and hence on  $\mathcal{X}$ , where the action is given by

$$\theta([\rho]) = [\rho \circ \theta^{-1}].$$

Suppose that  $[\rho] \in \mathcal{X}$  is stabilized by an Anosov element  $\theta \in \Gamma^+$  (this corresponds to a hyperbolic element if we identify  $\Gamma^+$  with  $\text{SL}(2, \mathbb{Z})$ ). So there exists  $A \in \text{SL}(2, \mathbb{C})$  such that for  $\alpha \in \pi$ ,

$$\theta(\rho)(\alpha) = A \cdot \rho(\alpha) \cdot A^{-1}.$$

Note that  $\text{tr } \rho(X)$  is well defined on the classes  $[X] \in \mathcal{C}/\langle \theta \rangle$ . Suppose further that  $[\rho]$  satisfies the *relative* Bowditch Q-conditions on  $\mathcal{C}/\langle \theta \rangle$ , that is,

- (1)  $\text{tr } \rho(X) \notin [-2, 2]$  for all  $[X] \in \mathcal{C}/\langle \theta \rangle$ ;
- (2)  $|\text{tr } \rho(X)| \leq 2$  for only finitely many (possibly no)  $[X] \in \mathcal{C}/\langle \theta \rangle$ .

Using the identification of  $\Gamma^+$  with  $\text{SL}(2, \mathbb{Z})$  and  $\mathcal{C}$  with  $\hat{\mathbb{Q}} \subset \hat{\mathbb{R}} \cong \mathcal{P}\mathcal{L}$  in §2.2, we get that the repelling and attracting fixed points of  $\theta$ ,  $\mu_-, \mu_+ \in \mathcal{P}\mathcal{L}$ , partition  $\mathcal{C}$  into two subsets  $\mathcal{C}_L \sqcup \mathcal{C}_R$  which are invariant under the action of  $\theta$ . We have the following generalizations of the McShane-Bowditch identities:

**Theorem 3.5** ([19, Theorems 5.6 and 5.9]). *Suppose that  $[\rho]$  is stabilized by an Anosov element  $\theta \in \Gamma^+$  and satisfies the relative Bowditch Q-conditions as stated above. Then*

$$(4) \quad \sum_{[X] \in \mathcal{C}/\langle \theta \rangle} \log \frac{e^\nu + e^{l(\rho(X))}}{e^{-\nu} + e^{l(\rho(X))}} = 0 \pmod{2\pi i},$$

and

$$(5) \quad \sum_{[X] \in \mathcal{C}_L/\langle \theta \rangle} \log \frac{e^\nu + e^{l(\rho(X))}}{e^{-\nu} + e^{l(\rho(X))}} = \pm l(A) \pmod{2\pi i},$$

where the sums converge absolutely,  $l(A)$  is the complex length of the conjugating element  $A$  corresponding to  $\theta$  as described above, and the sign in (5) depends only on our choice of orientations.

*Remark 3.6.* For type-preserving characters ( $\kappa = -2$ ), the result is due to Bowditch [4], where the summands of (4) and (5) should be replaced appropriately as in Remark 3.3 by the summands of McShane's original identity, and  $l(A)$  in (5) should be replaced by  $\lambda$ , which has an interpretation as the modulus of the cusp of an associated complete hyperbolic three-manifold. There are also similar identities in the case where  $\theta$  is reducible, that is, corresponds to a parabolic element of  $SL(2, \mathbb{Z})$ ; see [18].

The above result has applications to hyperbolic 3-manifolds. Let  $M$  be an orientable 3-manifold which fibers over the circle, with the fiber a once-punctured torus,  $\mathbb{T}$ . Suppose that the monodromy  $\theta$  of  $M$  is Anosov. By results of Thurston (see [22] and [21]),  $M$  has a complete finite-volume hyperbolic structure with a single cusp, which can in turn be deformed to incomplete hyperbolic structures, on which hyperbolic Dehn surgery can be performed to obtain complete hyperbolic manifolds without cusps. Restricting the holonomy representation to the fiber gives us characters which are stabilized by  $\theta$ , and in the complete case, the relative Bowditch  $Q$ -conditions are satisfied, as for small deformations of the complete structure to incomplete structures (see [19]). The identities can be interpreted as series identities for these (in)complete structures, involving the complex lengths of certain geodesics corresponding to the homotopy classes of essential simple closed curves on the fiber. The quantity  $\nu$  can be interpreted as half the complex length of the meridian of the boundary torus, and  $l(A)$  as the complex length of a (suitably chosen) longitude of the boundary torus.

**3.5. End invariants.** In this subsection, we state some results concerning the end invariants of a character  $[\rho]$ ; these results can be found in [20].

**Definition 3.7** (End invariants). An element  $X \in \mathcal{P}\mathcal{L}$  is an end invariant of  $[\rho]$  if there exist  $K > 0$  and a sequence of distinct elements  $X_n \in \mathcal{C}$  such that  $X_n \rightarrow X$  and  $|\text{tr } \rho(X_n)| < K$  for all  $n$ .

This definition generalizes the notion of a geometrically infinite end for a discrete, faithful, type-preserving character. Denote by  $\mathcal{E}(\rho)$  the set of end invariants of  $[\rho]$ ; this is a closed subset of  $\mathcal{P}\mathcal{L}$  (see [20]).

**Theorem 3.8.** *The set of end invariants  $\mathcal{E}(\rho)$  is equal to  $\mathcal{P}\mathcal{L}$  if and only if (i)  $[\rho]$  is dihedral, or (ii)  $[\rho]$  corresponds to an  $SU(2)$  representation. Furthermore, if  $\mathcal{E}(\rho) \neq \mathcal{P}\mathcal{L}$ , then  $\mathcal{E}(\rho)$  has empty interior in  $\mathcal{P}\mathcal{L}$ .*

**Theorem 3.9.** *The set of end invariants  $\mathcal{E}(\rho)$  is empty if and only if  $[\rho]$  satisfies the extended Bowditch  $Q$ -conditions.*

The next set of results classify  $\mathcal{E}(\rho)$  for real characters, reducible characters ( $\kappa = 2$ ), imaginary characters, and discrete characters.

**Theorem 3.10** (End invariants for real characters). *Suppose  $[\rho] \in \mathcal{X}_\kappa$  is real, with  $\kappa \neq 2$ . Then exactly one of the following must hold:*

- (a)  $\mathcal{E}(\rho) = \emptyset$ , and  $\rho$  satisfies the extended BQ-conditions.

- (b)  $\mathcal{E}(\rho) = \{\hat{X}\}$ , where  $\hat{X} \in \mathcal{C}$ ,  $\rho$  is an  $\mathrm{SL}(2, \mathbb{R})$  representation,  $\mathrm{tr} \rho(\hat{X}) \in (-2, 2)$ , and  $\mathrm{tr} \rho(X) \notin (-2, 2)$  for all  $X \in \mathcal{C} \setminus \{\hat{X}\}$ .
- (c)  $\mathcal{E}(\rho)$  is a Cantor subset of  $\mathcal{P}\mathcal{L}$ ,  $\rho$  is an  $\mathrm{SL}(2, \mathbb{R})$  representation,  $\mathrm{tr} \rho(X) \in (-2, 2)$  for at least two distinct  $X \in \mathcal{C}$ , and  $\mathrm{tr} \rho(Y) \notin (-2, 2) \cup \{\pm\sqrt{\kappa+2}\}$  for some element  $Y \in \mathcal{C}$ .
- (d)  $\mathcal{E}(\rho) = \mathcal{P}\mathcal{L}$ , and  $\rho$  satisfies the conditions of Theorem 3.8, that is,  $\rho$  is the dihedral representation or an  $\mathrm{SU}(2)$  representation.

Furthermore, case (a) occurs only when  $\kappa \in (-\infty, 2) \cup [18, \infty)$ ; case (b) when  $\kappa \in [6, \infty)$ ; case (c) when  $\kappa \in (2, \infty)$ ; and case (d) when  $\kappa \in [-2, 2) \cup (2, \infty)$ .

**Theorem 3.11** (End invariants for reducible characters). *For  $[\rho] \in \mathcal{X}_2$ ,  $\mathcal{E}(\rho) = \{X_0\}$  or  $\mathcal{P}\mathcal{L}$ . Furthermore, in the first case, if  $X_0 \in \mathcal{C}$ , then  $\mathrm{tr} \rho(X_0) \in [-2, 2]$  and  $\mathrm{tr} \rho(X) \notin [-2, 2]$  for all  $X \in \mathcal{C} \setminus \{X_0\}$ ; and if  $X_0 \notin \mathcal{C}$ , then  $\mathrm{tr} \rho(X) \notin [-2, 2]$  for all  $X \in \mathcal{C}$ ; in the second case,  $\mathrm{tr} \rho(X) \in [-2, 2]$  for all  $X \in \mathcal{C}$ .*

The following theorem gives a partial classification for imaginary characters.

**Theorem 3.12** (End invariants for imaginary characters). *Suppose that  $[\rho]$  is imaginary.*

- (i)  $\kappa = -2$ : For  $[\rho] \in \mathcal{X}_{-2}$ ,  $\mathcal{E}(\rho)$  is either a Cantor subset of  $\mathcal{P}\mathcal{L}$ , or it consists of a single element  $X$  in  $\mathcal{C}$ . In the latter case,  $\mathrm{tr} \rho(X) = 0$  and  $[\rho]$  is equivalent under the action of  $\Gamma$  to a character corresponding to the triple  $(0, x, ix)$ , where  $x \in \mathbb{R}$  satisfies  $|x| \geq 2$ .
- (ii)  $-14 < \kappa < 2$ : For  $[\rho] \in \mathcal{X}_\kappa$ ,  $\mathcal{E}(\rho)$  is either a Cantor subset of  $\mathcal{P}\mathcal{L}$ , or it consists of a single element  $X$  in  $\mathcal{C}$ .
- (iii)  $\kappa \leq -14$ : For  $[\rho] \in \mathcal{X}_\kappa$ ,  $\mathcal{E}(\rho)$  is either a Cantor subset of  $\mathcal{P}\mathcal{L}$ , or it consists of a single element  $X$  in  $\mathcal{C}$ , or it is empty.

Finally, we say that a character is discrete if the set  $\{\mathrm{tr} \rho(X) \mid X \in \mathcal{C}\}$  is discrete in  $\mathbb{C}$ . We have the following result.

**Theorem 3.13.** *For a discrete  $[\rho] \in \mathcal{X}$ , if  $\mathcal{E}(\rho)$  has at least three elements, then  $\mathcal{E}(\rho)$  is either a Cantor subset of  $\mathcal{P}\mathcal{L}$  or all of  $\mathcal{P}\mathcal{L}$ .*

#### 4. FURTHER DIRECTIONS AND RELATED RESULTS

A natural question to pose is whether the results above extend to general surfaces, and especially to closed surfaces without boundary (where a suitable version of McShane's identity is still lacking). The identities (1) and (3) have been generalized by McShane himself to hyperbolic surfaces with cusps [12]; to hyperbolic surfaces with cusps and/or geodesic boundary components by Mirzakhani [13]; to hyperbolic surfaces with cusps, geodesic boundary and/or conical singularities, as well as to classical Schottky groups by the authors in [16], [17]. Further refinements and analogous results for punctured surface bundles over the circle have also been obtained by Akiyoshi, Miyachi, and Sakuma in [1], [2]. The methods used in the above-cited works closely follow those used by McShane in his original proof and differ markedly from those used to prove the results announced in this note, which are modeled on Bowditch's proof. In particular, a general set of conditions, equivalent to the Bowditch Q-conditions, for the identities to hold is still lacking. The Bowditch method does extend fairly naturally to the case of the four-holed sphere; see [8]. The one-holed torus and the four-holed sphere are natural building blocks for more complicated surfaces, according to Grothendieck's reconstruction principle

[9]; see also Luo's work on  $SL(2)$  character varieties of surfaces in [10]. It would be interesting to see if this method can be used to give an independent proof of the identities for characters of general surfaces, and to provide necessary and sufficient conditions for the identities to hold, and also to shed some light on the dynamics of the mapping class group action on the  $SL(2, \mathbb{C})$  character varieties for these general surfaces.

Mirzakhani found some beautiful applications of McShane's identity; in particular, she used it to compute the Weil-Petersson volume of the moduli space of bordered surfaces in [13]; see also [14] and [15] for other striking applications. Theorem 3.2 gives a natural definition of a moduli space for (relative)  $SL(2, \mathbb{C})$  characters and it would be interesting to know if the volume can be computed for this moduli space, with respect to the Poisson (complex-symplectic) structure, which is invariant under the mapping class group action.

There are also interesting generalizations concerning the end invariants of  $[\rho]$ . The results stated in §3.5 can be regarded as evidence towards the following conjecture, which is a refinement and generalization of the suggestion by Bowditch in [5] that for a generic  $[\rho] \in \mathcal{X}_{-2}$  not satisfying the BQ-conditions,  $\mathcal{E}(\rho)$  should be a Cantor set.

**Conjecture 4.1.** Suppose that  $\mathcal{E}(\rho)$  has more than two elements. Then either  $\mathcal{E}(\rho) = \mathcal{P}\mathcal{L}$  or  $\mathcal{E}(\rho)$  is a Cantor subset of  $\mathcal{P}\mathcal{L}$ .

There is also a generalization of the ending lamination conjecture for  $SL(2, \mathbb{C})$  characters (as Bowditch conjectured for the  $\kappa = -2$  case), which can be stated as follows:

**Conjecture 4.2.** Suppose that  $[\rho], [\rho'] \in \mathcal{X}_\kappa$  are such that  $\mathcal{E}(\rho) = \mathcal{E}(\rho')$ ,  $\mathcal{E}(\rho)$  has at least two elements, and  $\mathcal{E}(\rho) \neq \mathcal{P}\mathcal{L}$ . Then  $[\rho] = [\rho']$ , up to sign changes and conjugation.

Finally, we say a few words about the generalizations to arbitrary surfaces. The definition of  $\mathcal{E}(\rho)$  can be extended without much difficulty. The case of the four-holed sphere is similar and the techniques given here should lead to similar results in that case. In other cases,  $\mathcal{P}\mathcal{L}$  is homeomorphic to the sphere  $S^n$  for some  $n \geq 2$ , and a possible generalization of Theorem 3.8 is that  $\mathcal{E}(\rho)$  has either full measure or measure zero. A possible generalization of Conjecture 4.1 would be that  $\mathcal{E}(\rho)$  is perfect if it contains more than two elements. However, we do not have any insights into these more general cases.

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