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HIGH-ENERGY LIMITS OF LAPLACE-TYPE AND DIRAC-TYPE EIGENFUNCTIONS AND FRAME FLOWS

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Abstract. We relate high-energy limits of Laplace-type and Dirac-type operators to frame flows on the corresponding manifolds, and show that the ergodicity of frame flows implies quantum ergodicity in an appropriate sense for those operators. Observables for the corresponding quantum systems are matrix-valued pseudodifferential operators, and therefore the system remains noncommutative in the high-energy limit. We discuss to what extent the space of stationary high-energy states behaves classically.

1. Introduction and main results

If X is an oriented closed Riemannian manifold and Δ the Laplace operator on X, then a complete orthonormal sequence of eigenfunctions $\phi_j \in L^2(X)$ with eigenvalues $\lambda_j \nearrow \infty$ is known to converge in the mean to the Liouville measure, in the sense that

$$\lim_{N\to\infty} \frac{1}{N} \sum_{j\le N} \langle \phi_j, A\phi_j\rangle = \int_{T_1^*X} \sigma_A(\xi) dL(\xi),$$

for any zero-order pseudodifferential operator A, where integration is with respect to the normalized Liouville measure on the unit cotangent bundle T_1^*X, and σ_A is the principal symbol of A. In particular, A might be a smooth function on X and the above implies that the sequence

$$\frac{1}{N} \sum_{j\le N} |\phi_j(x)|^2$$

converges to the normalized Riemannian measure in the weak topology of measures. In case the geodesic flow on T_1X is ergodic, it is known that the following stronger result holds:

$$\lim_{N\to\infty} \frac{1}{N} \sum_{j\le N} \left| \langle \phi_j, A\phi_j\rangle - \int_{T_1^*X} \sigma_A(\xi) dL(\xi) \right| = 0.$$

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This property is commonly referred to as quantum ergodicity and it is equivalent to the existence of a density-one subsequence ϕ'_j such that

$$\lim_{j\to\infty}\langle \phi'_j, A\phi'_j\rangle = \int_{T_1^*X}\sigma_A(\xi)dL(\xi),$$

for any zero-order pseudodifferential operator A (see [Shn74, Shn93, CV85, Zel87]).

In this paper we show that the high-energy behavior of the Dirac operator D acting on spinors on a closed spin manifold X is determined by the frame flow in the same manner, as the geodesic flow determines the high-energy limit of the Laplace operator. If F_kX is the bundle of oriented orthonormal k-frames in T^*X, then projection to the first vector turns $F_kX \to T_1^*X$ into a fiber bundle. In particular, for $k = n$ this is the full frame bundle and $FX = F_nX$ is a principal fiber bundle over T_1^*X with structure group $SO(n-1)$. Transporting covectors parallel along geodesics extends the Hamiltonian flow on T_1^*X to a flow on F_kX. This is the so-called k-frame flow. In the case $k = n$ we will refer to it simply as the frame flow. Of course, ergodicity of the k-frame flow for any k implies ergodicity of the geodesic flow, whereas the conclusion in the other direction is not always true (cf. Section 2). Still there are many examples investigated in the literature where the frame flow is ergodic. Our first main result is, that quantum ergodicity holds for eigensections of the Dirac operator in case the frame flow is ergodic.

Theorem 1.1. *Let X be a closed Riemannian spin manifold of dimension $n \geq 3$ with Dirac operator D acting on sections of the spinor bundle. Suppose that $\phi_j \in L^2(X;S)$ is an orthonormal sequence of eigensections of D with eigenvalues $\lambda_k \nearrow \infty$ such that the ϕ_k span*[1] *the positive-energy subspace of D. Then, if the frame flow on FX is ergodic, we have*

$$\lim_{N\to\infty}\frac{1}{N}\sum_{j\leq N}\left|\langle\phi_j, A\phi_j\rangle - \frac{1}{2^{[\frac{n}{2}]}}\int_{T_1^*X}\mathrm{Tr}((1+\gamma(\xi))\,\sigma_A(\xi))dL(\xi)\right| = 0,$$

for all $A \in \Psi\mathrm{DO}^0_{cl}(X,S)$. Here $\gamma(\xi)$ denotes the operator of Clifford multiplication with ξ. In particular, there is a density-one subsequence ϕ'_j such that

$$\langle\phi'_j, A\phi'_j\rangle \to \frac{1}{2^{[\frac{n}{2}]}}\int_{T_1^*X}\mathrm{Tr}((1+\gamma(\xi))\,\sigma_A(\xi))dL(\xi).$$

A similar statement holds for the negative-energy subspace.

Another result is that the $(2\min(p, n-p))$-frame flow determines the high-energy behavior of the Laplace-Beltrami operator Δ_p acting on the space $C^\infty(X, \Lambda^p_{\mathbb{C}}X)$ of complex-valued p-forms. Note that the Hodge decomposition implies that there are three invariant subspaces for Δ_p, namely the closures of $dC^\infty(X, \Lambda^{p-1}_{\mathbb{C}}X)$, $\delta C^\infty(X, \Lambda^{p+1}_{\mathbb{C}}X)$, and the finite-dimensional space of harmonic forms. The latter subspace plays no role for the high-energy behavior. The eigenspaces of the first subspace consist of exterior derivatives of $(p-1)$-eigenforms; their high-energy behavior is therefore determined by the high-energy behavior of Δ_{p-1}. In particular, the high-energy behavior of Δ_1 restricted to $dC^\infty(X)$ is controlled by the geodesic flow. We therefore look at the second subspace only. Note that any co-closed form which is a nonzero eigenvalue to Δ_p is coexact. Hence, to investigate

[1] In the sense that the linear hull is dense.

the high-energy behavior of Δ_p we have to look at the system

$$\begin{aligned} \Delta_p \phi_j &= \lambda_j \phi_j, \\ \delta \phi_j &= 0. \end{aligned} \tag{1.2}$$

In the case $p = 1$ such systems appear in physics if one investigates Maxwell's equations or the Proca equation for spin-1 particles. The restriction to the coclosed forms corresponds to a gauge condition which restricts to the transversal subspaces only. Our result is, that this system is quantum ergodic if the $2\min(p, n-p)$-frame flow is ergodic.

Theorem 1.3. *Let X be an oriented closed Riemannian manifold of dimension $n \geq 3$ and let $0 < p < n$. Suppose that ϕ_k is an orthonormal sequence of p-eigenforms satisfying*

$$\begin{aligned} \Delta_p \phi_k &= \lambda_k \phi_k, \\ \delta \phi_k &= 0, \end{aligned}$$

such that the ϕ_k span $\ker(\delta)$ and with $\lambda_k \nearrow \infty$. Suppose that $p \neq \frac{n-1}{2}$. Then, if the $(2\min(p, n-p))$-frame flow is ergodic, the system is quantum ergodic in the sense that

$$\lim_{N \to \infty} \frac{1}{N} \sum_{k \leq N} |\langle \phi_k, A\phi_k \rangle - \omega_t(\sigma_A)| = 0,$$

for all $A \in \Psi\mathrm{DO}^0_{cl}(X; \Lambda^p_{\mathbb{C}} X)$. In particular, there is a density-one subsequence ϕ'_k such that

$$\lim_{k \to \infty} \langle \phi'_k, A\phi'_k \rangle = \omega_t(\sigma_A), \quad \text{for all } A \in \Psi\mathrm{DO}^0_{cl}(X; \Lambda^p_{\mathbb{C}} X).$$

Here ω_t is a state on the C^*-algebra of continuous $\mathrm{End}(\Lambda^p_{\mathbb{C}} X)$-valued functions on $T_1^* X$ which is defined by

$$\omega_t(a) := \binom{n-1}{p}^{-1} \int_{T_1^* X} \mathrm{Tr}\,(i(\xi) i^*(\xi) a(\xi))\, dL(\xi),$$

where $i(\xi)$ is the operator of interior multiplication with ξ, and the adjoint $i^*(\xi)$ is the operator of exterior multiplication with ξ.

Note that the system

$$\begin{aligned} \Delta_k \phi_j &= \lambda_j \phi_j, \\ d\phi_j &= 0 \end{aligned}$$

is equivalent to our system with $p = n - k$ via the Hodge star-operator.

The restriction $p \neq \frac{n-1}{2}$ is necessary, since if $p = \frac{n-1}{2}$, the operator $\mathrm{i}^{\,p+1} \delta *$ leaves the space $\overline{\mathrm{Rg}(\delta)}$ invariant and commutes with Δ_p ($*$ is the Hodge star-operator). In this case our result is the following.

Theorem 1.4. *Let X be an oriented closed Riemannian manifold of odd dimension $n \geq 3$. Let $p = \frac{n-1}{2}$ and suppose that ϕ_k is an orthonormal sequence of p-eigenforms satisfying*

$$\Delta_p \phi_k = \lambda_k \phi_k,$$
$$\delta \phi_k = 0,$$
$$\mathrm{i}^{\,p+1} \delta * \phi_k = \pm\sqrt{\lambda_k}\phi_k$$

such that the ϕ_k span $\overline{\mathrm{Ran}(\delta \pm \mathrm{i}^{\,p+1}\Delta_p^{-1/2}\delta * \delta)}$ and with $\lambda_k \nearrow \infty$. Then, if the $(n-1)$-frame flow is ergodic, the system is quantum ergodic in the sense that

$$\lim_{N\to\infty} \frac{1}{N} \sum_{k \leq N} |\langle \phi_k, A\phi_k\rangle - \omega_\pm(\sigma_A)| = 0,$$

for all $A \in \Psi\mathrm{DO}^0_{cl}(X; \Lambda^p_{\mathbb{C}} X)$. In particular, there is a density-one subsequence ϕ'_k such that

$$\lim_{k\to\infty} \langle \phi'_k, A\phi'_k\rangle = \omega_\pm(\sigma_A), \quad \textit{for all } A \in \Psi\mathrm{DO}^0_{cl}(X; \Lambda^p_{\mathbb{C}} X).$$

Here the states $\omega_\pm$ are defined by

$$\omega_\pm(a) := \frac{(p!)^2}{(2p)!} \int_{T_1^* X} \mathrm{Tr}\left((1 \pm \mathrm{i}^{\,p} i(\xi)*)\, i(\xi) i^*(\xi) a(\xi)\right) dL(\xi).$$

Our analysis is based on a version of Egorov's theorem for matrix-valued operators. A second-order differential operator P acting on sections of a vector bundle E is said to be of Laplace type if $\sigma_P(\xi) = g(\xi,\xi)\,\mathrm{id}_E$, i.e., if in local coordinates it is of the form $P = -\sum_{i,k} g^{ik}\partial_i\partial_k +$ lower-order terms. Examples are the Laplace-Beltrami operator Δ_p acting on p-forms or the square D^2 of the Dirac operator on a Riemannian spin manifold. For such operators the first-order term (the subprincipal symbol) defines a connection ∇^E on the bundle E. We will prove a Egorov theorem for matrix-valued pseudodifferential operators acting on sections of E. More precisely, for $A \in \Psi\mathrm{DO}^0_{cl}(X, E)$, a zero-order classical pseudodifferential operator, the principal symbol σ_A is an element in $C^\infty(T_1^* X, \mathrm{End}(\pi^*(E)))$, where $\pi^*(E)$ is the pull-back of the bundle $E \to X$ under the projection $\pi : T_1^* X \to X$. Note that the connection ∇^E determines a connection ∇ on $\mathrm{End}(\pi^*(E))$. Parallel transport along the Hamiltonian flow of σ_P then determines a flow β_t acting on $C^\infty(T_1^* X, \mathrm{End}(\pi^*(E)))$. Our version of Egorov's theorem specialized to Laplace-type operators reads as follows.

Theorem 1.5. *If $A \in \Psi\mathrm{DO}^0_{cl}(X, E)$ and if P is a positive second-order differential operator of Laplace type, then for all $t \in \mathbb{R}$ the operators $A_t := e^{-\mathrm{i}\,tP^{1/2}} A e^{+\mathrm{i}\,tP^{1/2}}$ are again in $\Psi\mathrm{DO}^0_{cl}(X, E)$ and $\sigma_{A_t} = \beta_t(\sigma_A)$.*

We actually prove a more general version of this theorem which applies to flows generated by first-order pseudodifferential operators with real scalar principal symbols. Note that, unlike in the scalar case, the first-order terms are needed to determine the flow. We show that for pseudodifferential operators with real scalar principal part, the subprincipal symbol is invariantly defined as a partial connection along the Hamiltonian vector field, thus allowing us to define all flows without referring to local coordinate systems.

1.1. **Discussion.** The Dirac equation on $\mathbf{R}^3$ (and, more generally, on $\mathbf{R}^d$) has been studied from the semiclassical point of view in the papers [BoK98, BoK99, Bol01, BoG04, BoG04.2] of Bolte, Glaser, and Keppeler. The authors would like to thank J. Bolte for bringing this problem on manifolds to their attention. Unlike in the works of Bolte, Glaser, and Keppeler, we investigate the high-energy limit rather than the semiclassical limit. Therefore, all nontrivial dynamical effects are due to the nontrivial curvature of the spin-connection. This is conceptually different from the stated previous results where quantum ergodicity is due to a spin precession in an external magnetic field. External fields are not seen in the high-energy limit, and therefore the strict analog of the result of Shnirelman, Colin de Verdière, and Zelditch [Shn74, Shn93, CV85, Zel87] cannot be expected to hold in $\mathbb{R}^n$ or on manifolds with integrable geodesic flow.

Apart from working on manifolds our methods also differ from those previously employed as we take the absolute value of the Dirac operator instead of the Dirac operator itself as the generator of the dynamics. This has the advantage of allowing for the full algebra of matrix-valued functions as the observable algebra rather than a subalgebra. One can also justify this from a physical point of view. Namely, in a fully quantized theory, the generator of the time evolution on the 1-particle Hilbert spaces is the absolute value of the Dirac operator. Furthermore, on the electron 1-particle subspace these two operators coincide.

Our Theorem 1.4 for $n = 3$ and $k = 1$ deals with the electromagnetic field on a 3-dimensional compact manifold. The statement of Theorem 1.4 means that quantum ergodicity holds for circular polarized photons if the 2-frame flow is ergodic.

We would also like to mention that the Egorov theorem as we state it is related to a work of Dencker ([D82]), who proved a propagation of singularity theorem for systems of real principal type. It follows from his work that the polarization set of solutions to the Dirac equation is invariant under a certain flow similar to ours. We also refer the reader to [EW96], [GMMP97] and references therein, and [San99] for discussion of semiclassical limits for matrix-valued operators, and relations to parallel transport.

S. Zelditch studied quantum ergodicity for general C^*-dynamical systems in [Zel96]. In our work we identify the classical flows corresponding to the Dirac operator and the Hodge Laplacian as frame flows, which allows us to use the results obtained by Brin, Arnold, Pesin, Gromov, Karcher, Burns, and Pollicott to exhibit many examples of manifolds where quantum ergodicity holds for the Dirac operator; see Corollary 2.1. The connection to their work has not been made before in the literature on quantum ergodicity. Finally, our results on quantum ergodicity for p-eigenforms for Hodge Laplacian, and the role played by $2\min(p, n-p)$-frame flow seem to be completely new. It is a hope of the authors that their results will stimulate further studies of relationship between ergodic theory of partially hyperbolic dynamical systems and high-energy behavior of eigenfunctions of matrix-valued operators.

2. Ergodic frame flows: known examples

The k-frame flow Φ^t, $k \geq 2$, is defined as follows: let $(v_1, \ldots, v_k)$ be an ordered orthonormal set of k unit vectors in T_pX. Then $\Phi^t v_1 = G^t v_1$, where G^t is the geodesic flow. $\Phi^t v_j, 2 \leq j \leq k$, translates v_j by the parallel translation at distance t along the geodesic determined by v_1. Here we summarize the cases where the frame flow is known to be ergodic. A k-frame flow is an $SO(k-1)$-extension of the geodesic flow; on an n-dimensional manifold, k-frame flow is a factor of the n-frame flow for $2 \leq k < n$, so ergodicity of the latter implies ergodicity of the former. Frame flow preserves orientation, so in dimension 2 its ergodicity (restricted to positively oriented frames, say) is equivalent to the ergodicity of the geodesic flow.

Frame flows were considered by Arnold in [Arn61]. In negative curvature, they were studied by Brin, together with Gromov, Karcher, and Pesin, in a series of papers [BrP74, Br75, Br76, BrG80, Br82, BrK84]. Recently, significant progress has been made in understanding ergodic behavior of general partially hyperbolic systems, including frame flows. In the current paper, the authors are primarily interested in the ergodicity of the flow; the most recent paper dealing with that question appears to be [BuP03] by Burns and Pollicott, where the authors establish ergodicity under certain pinched-curvature assumptions in the "exceptional" dimensions 7 and 8; see below.

In the sequel, we assume that M is negatively curved with sectional curvatures satisfying

$$-K_2^2 \leq K \leq -K_1^2.$$

The frame flow is known to be ergodic and have the K property

1) if M has constant curvature [Br76, BrP74];
2) for an open and dense set of negatively curved metrics (in the C^3 topology) [Br75];
3) if n is odd, but not equal to 7 [BrG80]; or if $n = 7$ and $K_1/K_2 > 0.99023...$ [BuP03];
4) if n is even, but not equal to 8, and $K_1/K_2 > 0.93$ [BrK84]; or if $n = 8$ and $K_1/K_2 > 0.99023...$ [BuP03].

Corollary 2.1. *Quantum ergodicity for the Dirac operator and Hodge Laplacian (conclusions of Theorems* 1.1*,* 1.3*, and* 1.4*) hold in each of the cases 1)–4).*

The frame is *not* ergodic on negatively curved Kähler manifolds, since the almost complex structure J is preserved. This is the only known example in negative curvature where the geodesic flow is ergodic, but the frame flow is not. In fact, given an orthonormal k-frame $(v_1, \ldots, v_k)$, the functions $(v_i, Jv_j), 1 \leq i, j \leq k$, are first integrals of the frame flow, and in some cases it is possible to describe the ergodic components [Br82, BrP74, BrG80]. Note that also the system (1.2) is not quantum ergodic in the Kähler case, because the decomposition into (p, q)-forms is a decomposition into invariant subspaces of the Laplace-Beltrami operator. The Kähler case is interesting in its own right and will be discussed in a forthcoming paper.

The frame flow is conjectured to be ergodic whenever the curvature satisfies $-1 < K < -1/4$; cf. [Br82]. That conjecture is still open.

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