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ON POLYHARMONIC OPERATORS WITH LIMIT-PERIODIC POTENTIAL IN DIMENSION TWO

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(Communicated by Svetlana Katok)

In memory of our colleague and friend Robert M. Kauffman.

Abstract. This is an announcement of the following results. We consider a polyharmonic operator $H = (-\Delta)^l + V(x)$ in dimension two with $l \geq 6$ and $V(x)$ being a limit-periodic potential. We prove that the spectrum of H contains a semiaxis and there is a family of generalized eigenfunctions at every point of this semiaxis with the following properties. First, the eigenfunctions are close to plane waves at the high-energy region. Second, the isoenergetic curves in the space of momenta corresponding to these eigenfunctions have the form of slightly distorted circles with holes (Cantor-type structure). Third, the spectrum corresponding to the eigenfunctions (the semiaxis) is absolutely continuous.

We study the operator

$$H = (-\Delta)^l + V(x) \tag{1}$$

in two dimensions, $l \geq 6$, $V(x)$ being a limit-periodic potential:

$$V(x) = \sum_{r=1}^{\infty} V_r(x), \tag{2}$$

where $\{V_r\}_{r=1}^{\infty}$ is a family of periodic potentials with doubling periods and decreasing L_∞-norms, namely, V_r has orthogonal periods $2^{r-1}\vec{b_1}$, $2^{r-1}\vec{b_2}$ and $\|V_r\|_\infty < \exp(-2^{\eta r})$ for some $\eta > \eta_0(l) > 0$.

The one-dimensional analog of (1), (2) with $l = 1$ has been already thoroughly investigated. It is proven in [1]–[7] that the spectrum of the operator $H_1 u = -u'' + Vu$ is a Cantor-type set. It has positive Lebesgue measure [1, 6]. The spectrum is absolutely continuous [1, 2], [5]–[9]. Generalized eigenfunctions can be represented in the form $e^{ikx}u(x)$, $u(x)$ being limit-periodic [5, 6, 7]. The case of a complex-valued potential is studied in [10]. Integrated density of states is investigated in [11]–[14]. Properties of eigenfunctions of a discrete multidimensional limit-periodic Schrödinger operator are studied in [15]. As to the continuum multidimensional case, it is known that the integrated density of states for (1) is the limit of densities of states for periodic operators [14].

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Here we concentrate on properties of the spectrum and eigenfunctions of (1), (2) in the high-energy region. We prove the following results for the case $d = 2$, $l \geq 6$.

(1) The spectrum of the operator (1), (2) contains a semiaxis. A proof of the analogous result by different means is to appear in the forthcoming paper [16]. In [16], the more general case, $8l > d+3$, $d \neq 1 \pmod 4$, is considered, however, under additional restriction on the potential: all the lattices of periods Q_r of periodic potentials V_r need to contain a nonzero common vector γ, i.e., V is periodic in one direction γ in [16].

(2) There are generalized eigenfunctions $\Psi_\infty(\vec{k}, \vec{x})$, corresponding to the semiaxis, which are close to plane waves: for every $\vec{k}$ in an extensive subset $\mathcal{G}_\infty$ of $\mathbb{R}^2$, there is a solution $\Psi_\infty(\vec{k}, \vec{x})$ of the equation $H\Psi_\infty = \lambda_\infty \Psi_\infty$ which can be described by the formula:

$$\Psi_\infty(\vec{k}, \vec{x}) = e^{i\langle \vec{k}, \vec{x}\rangle} \left(1 + u_\infty(\vec{k}, \vec{x})\right), \tag{3}$$

$$\|u_\infty\| =_{|\vec{k}|\to\infty} O(|\vec{k}|^{-\gamma_1}), \quad \gamma_1 > 0, \tag{4}$$

where $u_\infty(\vec{k}, \vec{x})$ is a limit-periodic function:

$$u_\infty(\vec{k}, \vec{x}) = \sum_{r=1}^{\infty} u_r(\vec{k}, \vec{x}), \tag{5}$$

$u_r(\vec{k}, \vec{x})$ being periodic with periods $2^{r-1}\vec{b_1},\ 2^{r-1}\vec{b_2}$. The eigenvalue $\lambda_\infty(\vec{k})$ corresponding to $\Psi_\infty(\vec{k}, \vec{x})$ is close to $|\vec{k}|^{2l}$:

$$\lambda_\infty(\vec{k}) =_{|\vec{k}|\to\infty} |\vec{k}|^{2l} + O(|\vec{k}|^{-\gamma_2}), \quad \gamma_2 > 0. \tag{6}$$

The "nonresonant" set $\mathcal{G}_\infty$ of the vectors $\vec{k}$ for which (3)–(6) hold is an extensive Cantor-type set: $\mathcal{G}_\infty = \bigcap_{n=1}^{\infty} \mathcal{G}_n$, where $\{\mathcal{G}_n\}_{n=1}^{\infty}$ is a decreasing sequence of sets with larger and larger number of holes in each bounded region, holes added on each step being of smaller and smaller size. The set $\mathcal{G}_\infty$ satisfies the estimate:

$$\frac{|(\mathcal{G}_\infty \cap \mathbf{B_R})|}{|\mathbf{B_R}|} =_{R\to\infty} 1 + O(R^{-\gamma_3}), \quad \gamma_3 > 0, \tag{7}$$

where $\mathbf{B_R}$ is the disk of radius R centered at the origin, and $|\cdot|$ is Lebesgue measure in $\mathbb{R}^2$.

(3) The set $\mathcal{D}_\infty(\lambda)$, defined as a level (isoenergetic) set for $\lambda_\infty(\vec{k})$,

$$\mathcal{D}_\infty(\lambda) = \left\{\vec{k} \in \mathcal{G}_\infty : \lambda_\infty(\vec{k}) = \lambda\right\},$$

is proven to be a slightly distorted circle with infinite number of holes. It can be described by the formula:

$$\mathcal{D}_\infty(\lambda) = \{\vec{k} : \vec{k} = \varkappa_\infty(\lambda, \vec{\nu})\vec{\nu},\ \vec{\nu} \in \mathcal{B}_\infty(\lambda)\}, \tag{8}$$

where $\mathcal{B}_\infty(\lambda)$ is a subset of the unit circle S_1. The set $\mathcal{B}_\infty(\lambda)$ can be interpreted as the set of possible directions of propagation for the almost plane waves (3). The set $\mathcal{B}_\infty(\lambda)$ has a Cantor-type structure and an asymptotically full measure on S_1 as $\lambda \to \infty$:

$$L\left(\mathcal{B}_n(\lambda)\right) =_{\lambda\to\infty} 2\pi + O\left(\lambda^{-\gamma_3/2l}\right); \tag{9}$$

here and below $L(\cdot)$ is the length of a curve. The value $\varkappa_\infty(\lambda,\vec{\nu})-\lambda^{1/2l}$ in (8) gives the deviation of $\mathcal{D}_\infty(\lambda)$ from the perfect circle of radius $\lambda^{1/2l}$ in the direction $\vec{\nu}$. It is proven that the deviation is asymptotically small,

$$\varkappa_\infty(\lambda,\vec{\nu}) =_{\lambda\to\infty} \lambda^{1/2l}+O\left(\lambda^{-\gamma_4}\right), \quad \gamma_4>0. \tag{10}$$

(4) Absolute continuity of the branch of the spectrum (the semiaxis) corresponding to $\Psi_\infty(\vec{k},\vec{x})$ is proven.

To prove the results listed above we develop a modification of the Kolmogorov-Arnold-Moser (KAM) method. This paper is inspired by [17, 18, 19], where the method is used for periodic problems. In [17], KAM method is applied to classical Hamiltonian systems. In [18, 19], the technique developed in [17] is applied to semiclassical approximation for multidimensional periodic Schrödinger operators at high energies.

We consider a sequence of operators

$$H_0=(-\Delta)^l, \qquad H^{(n)}=H_0+\sum_{r=1}^{M_n}V_r, \quad n\ge 1, \quad M_n\to\infty \text{ as } n\to\infty.$$

Obviously, $\|H-H^{(n)}\|\to 0$ as $n\to\infty$ and $H^{(n)}=H^{(n-1)}+W_n$, where $W_n=\sum_{r=M_{n-1}+1}^{M_n}V_r$. We regard each operator $H^{(n)}$, $n\ge 1$, as a perturbation of the previous operator $H^{(n-1)}$. Every operator $H^{(n)}$ is periodic; however, the periods go to infinity as $n\to\infty$. We show that there is a λ_0, $\lambda_0=\lambda_0(V)$, such that the semiaxis $[\lambda_0,\infty)$ is contained in the spectra of **all** operators $H^{(n)}$. For every operator $H^{(n)}$ there is a set of eigenfunctions (corresponding to the semiaxis) that are close to plane waves: for every $\vec{k}$ in an extensive subset $\mathcal{G}_n$ of $\mathbb{R}^2$, there is a solution $\Psi_n(\vec{k},\vec{x})$ of the differential equation $H^{(n)}\Psi_n=\lambda_n\Psi_n$, which can be described by the formula:

$$\Psi_n(\vec{k},\vec{x})=e^{i\langle\vec{k},\vec{x}\rangle}\left(1+\tilde{u}_n(\vec{k},\vec{x})\right), \quad \|\tilde{u}_n\|=_{|\vec{k}|\to\infty} O(|\vec{k}|^{-\gamma_1}), \quad \gamma_1>0, \tag{11}$$

where $\tilde{u}_n(\vec{k},\vec{x})$ is a periodic function. The corresponding eigenvalue $\lambda_n(\vec{k})$ is close to $|\vec{k}|^{2l}$:

$$\lambda_n(\vec{k})=_{|\vec{k}|\to\infty}|\vec{k}|^{2l}+O\left(|\vec{k}|^{-\gamma_2}\right), \quad \gamma_2>0.$$

The nonresonant set $\mathcal{G}_n$ is proven to be extensive in $\mathbb{R}^2$:

$$\frac{|(\mathcal{G}_\infty\cap\mathbf{B_R})|}{|\mathbf{B_R}|}=_{R\to\infty}1+O(R^{-\gamma_3}), \quad \gamma_3>0. \tag{12}$$

The set $\mathcal{D}_n(\lambda)$ is defined as the level (isoenergetic) set for nonresonant eigenvalue $\lambda_n(\vec{k})$:

$$\mathcal{D}_n(\lambda)=\left\{\vec{k}\in\mathcal{G}_n:\lambda_n(\vec{k})=\lambda\right\}.$$

This set is proven to be a slightly distorted circle with a finite number of holes; see Figures 1 and 2. It can be described by the formula:

$$\mathcal{D}_n(\lambda)=\{\vec{k}:\vec{k}=\varkappa_n(\lambda,\vec{\nu})\vec{\nu},\ \vec{\nu}\in\mathcal{B}_n(\lambda)\}, \tag{13}$$

where $\mathcal{B}_n(\lambda)$ is a subset of the unit circle S_1. The set $\mathcal{B}_n(\lambda)$ can be interpreted as the set of possible directions of propagation for almost plane waves (11). It has an

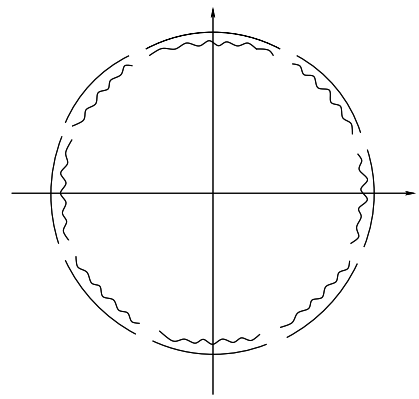

FIGURE 1. Distorted circle with holes, $\mathcal{D}_1(\lambda)$

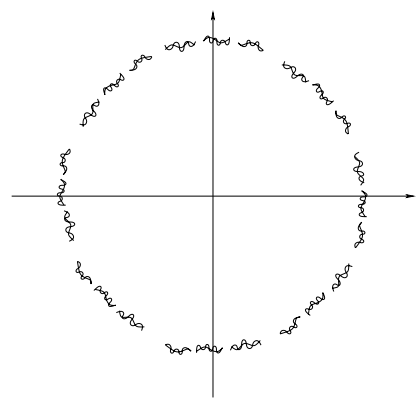

FIGURE 2. Distorted circle with holes, $\mathcal{D}_2(\lambda)$

asymptotically full measure on S_1 as $\lambda \to \infty$:

$$L\left(\mathcal{B}_n(\lambda)\right) =_{\lambda\to\infty} 2\pi + O\left(\lambda^{-\gamma_3/2l}\right). \tag{14}$$

The set $\mathcal{B}_n$ has only a finite number of holes; however, their number is growing with n. More and more holes of a smaller and smaller size are added on each step. The value $\varkappa_n(\lambda,\vec{\nu}) - \lambda^{1/2l}$ gives the deviation of $\mathcal{D}_n(\lambda)$ from the perfect circle of radius $\lambda^{1/2l}$ in the direction $\vec{\nu}$. It is proven that the deviation is asymptotically small:

$$\varkappa_n(\lambda,\vec{\nu}) = \lambda^{1/2l} + O\left(\lambda^{-\gamma_4}\right), \qquad \frac{\partial \varkappa_n(\lambda,\vec{\nu})}{\partial\varphi} = O\left(\lambda^{-\gamma_5}\right), \quad \gamma_4,\gamma_5 > 0, \tag{15}$$

φ being an angle variable $\vec{\nu} = (\cos\varphi, \sin\varphi)$. It is shown that $\mathcal{D}_1(\lambda)$ is strictly inside the perfect circle. Here and below we assume without loss of generality that $\int_{Q_r} V_r = 0$, for all r, the integral being taken over the elementary cell of periods Q_r.

On each step, more and more points are excluded from the nonresonant sets $\mathcal{G}_n$; thus $\{\mathcal{G}_n\}_{n=1}^{\infty}$ is a decreasing sequence of sets. The set $\mathcal{G}_\infty$ is defined as the limit set: $\mathcal{G}_\infty = \bigcap_{n=1}^{\infty} \mathcal{G}_n$. It has infinitely many holes in each bounded region, but

nevertheless satisfies the relation (7). For every $\vec{k} \in \mathcal{G}_\infty$ and every n, there is a generalized eigenfunction of $H^{(n)}$ of type (11). It is proven that the sequence $\Psi_n(\vec{k}, \vec{x})$ has a limit in $L_\infty(\mathbb{R}^2)$ when $\vec{k} \in \mathcal{G}_\infty$. The function $\Psi_\infty(\vec{k}, \vec{x}) = \lim_{n\to\infty} \Psi_n(\vec{k}, \vec{x})$ is a generalized eigenfunction of H. It can be written in the form (3) – (5), the functions u_n being related to $\tilde{u}_n$ in (11) by the formula $u_n = \tilde{u}_n - \tilde{u}_{n-1}$. Naturally, the corresponding eigenvalue $\lambda_\infty(\vec{k})$ is the limit of $\lambda_n(\vec{k})$ as $n \to \infty$.

Obviously, $\{\mathcal{B}_n(\lambda)\}_{n=1}^\infty$ is a decreasing sequence of sets, since on each step more and more directions are excluded. We consider the limit $\mathcal{B}_\infty(\lambda)$ of $\mathcal{B}_n(\lambda)$:

$$\mathcal{B}_\infty(\lambda) = \bigcap_{n=1}^{\infty} \mathcal{B}_n(\lambda).$$

This set has a Cantor-type structure on the unit circle. It is proven that $\mathcal{B}_\infty(\lambda)$ has an asymptotically full measure on the unit circle (see (9)). We prove that the sequence $\varkappa_n(\lambda, \vec{\nu})$, $n = 1, 2, \dots$, describing the isoenergetic curves $\mathcal{D}_n$, quickly converges as $n \to \infty$. Hence, $\mathcal{D}_\infty(\lambda)$ can be described as the limit of $\mathcal{D}_n(\lambda)$ in the sense of (8), where $\varkappa_\infty(\lambda, \vec{\nu}) = \lim_{n\to\infty} \varkappa_n(\lambda, \vec{\nu})$ for every $\vec{\nu} \in \mathcal{B}_\infty(\lambda)$. It is shown that the derivatives of the functions $\varkappa_n(\lambda, \vec{\nu})$ (with respect to the angle variable on the unit circle) have a limit as $n \to \infty$ for every $\vec{\nu} \in \mathcal{B}_\infty(\lambda)$. We denote this limit by $\frac{\partial \varkappa_\infty(\lambda, \vec{\nu})}{\partial \varphi}$. It follows from (15) that

$$\frac{\partial \varkappa_\infty(\lambda, \vec{\nu})}{\partial \varphi} = O\left(\lambda^{-\gamma_5}\right).$$

Thus, the limit curve $\mathcal{D}_\infty(\lambda)$ has a tangent vector in spite of its Cantor-type structure, the tangent vector being the limit of the corresponding tangent vectors for $\mathcal{D}_n(\lambda)$ as $n \to \infty$. The curve $\mathcal{D}_\infty(\lambda)$ looks like a distorted circle with infinitely many holes.

Absolute continuity of the branch of the spectrum (the semiaxis) corresponding to the functions $\Psi_\infty(\vec{k}, \vec{x})$, $\vec{k} \in \mathcal{G}_\infty$, follows from the convergence of the spectral projections corresponding to $\Psi_n(\vec{k}, \vec{x})$, $\vec{k} \in \mathcal{G}_\infty$, to spectral projections of H (in a strong sense uniformly in λ, $\lambda > \lambda_0(V)$) and properties of the level curves $\mathcal{D}_\infty(\lambda)$, $\lambda > \lambda_0(V)$.

The main technical difficulty to overcome is a construction of nonresonance sets $\mathcal{B}_n(\lambda)$ for every fixed sufficiently large λ, $\lambda > \lambda_0(V)$, where $\lambda_0(V)$ is the same for all n. The set $\mathcal{B}_n(\lambda)$ is obtained by deleting a "resonant" part from $\mathcal{B}_{n-1}(\lambda)$. The definition of $\mathcal{B}_{n-1} \setminus \mathcal{B}_n$ includes the Bloch eigenvalues of $H^{(n-1)}$. To describe $\mathcal{B}_{n-1} \setminus \mathcal{B}_n$, one has to use not only nonresonant eigenvalues of type (6), but also resonant eigenvalues, for which no suitable formulae are known. The absence of formulae causes difficulties in estimating the size of $\mathcal{B}_n \setminus \mathcal{B}_{n-1}$. To deal with this problem, we start with introducing an angle variable φ, $\varphi \in [0, 2\pi)$, $(\cos\varphi, \sin\varphi) \in S_1$, and considering sets $\mathcal{B}_n(\lambda)$ in terms of this variable. Next, we show that the resonant set $\mathcal{B}_{n-1} \setminus \mathcal{B}_n$ can be described as the set of zeros of determinants of the type $\mathrm{Det}(I + S_n)$, $S_n = S_n(\varphi)$ being a trace-type operator,

$$I + S_n(\varphi) = \left(H^{(n-1)}\left(\vec{\varkappa}_{n-1}(\varphi) + \vec{b}\right) - \lambda - \epsilon\right)\left(H_0\left(\vec{\varkappa}_{n-1}(\varphi) + \vec{b}\right) + \lambda\right)^{-1},$$

where $\vec{\varkappa}_{n-1}(\varphi)$ is a vector-function describing $\mathcal{D}_{n-1}(\lambda)$, $\vec{\varkappa}_{n-1}(\varphi) = \varkappa_{n-1}(\lambda, \vec{\nu})\vec{\nu}$. To obtain $\mathcal{B}_{n-1} \setminus \mathcal{B}_n$ we take all values of ϵ in a small interval and values of $\vec{b}$ in a finite set, $\vec{b} \neq 0$. Further, we extend our considerations to a complex neighborhood

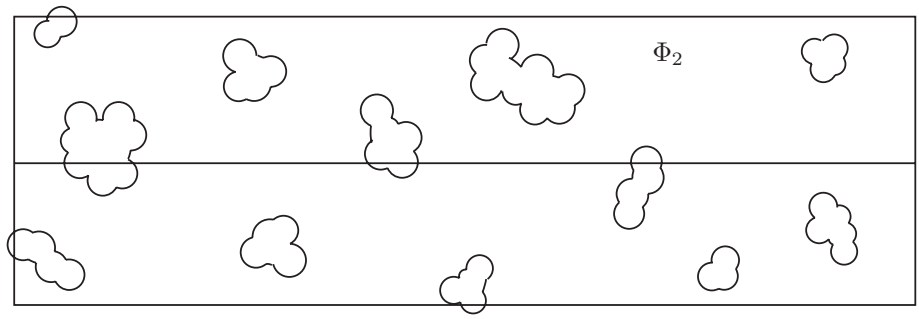

FIGURE 3. The set Φ_2.

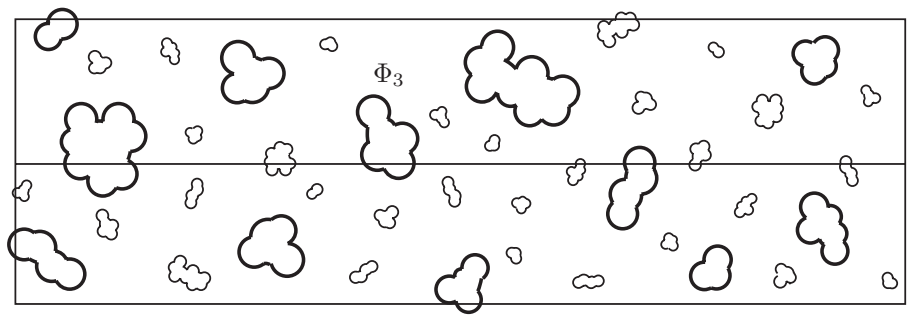

FIGURE 4. The set Φ_3.

Φ_0 of $[0, 2\pi)$. We show the determinants to be analytic functions of φ in Φ_0, and, by this, reduce the problem of estimating the size of the resonance set to a problem in complex analysis. We use theorems for analytic functions to count the zeros of the determinants and to investigate how far the zeros move when ε changes. This enables us to estimate the size of the zero set of the determinants, and, hence, the size of the nonresonance set $\Phi_n \subset \Phi_0$, which is defined as a nonzero set for the determinants. Having proven that the nonresonance set Φ_n is sufficiently large, we obtain estimates (12) for $\mathcal{G}_n$ and (14) for $\mathcal{B}_n$, the set $\mathcal{B}_n$ being the real part of Φ_n. To obtain Φ_n, we delete from Φ_0 more and more holes of smaller and smaller radii on each step. Thus, the nonresonance set $\Phi_n \subset \Phi_0$ has a Swiss Cheese structure (Figures 3 and 4).

Deleting the resonance set from Φ_0 on each step of the recurrence procedure is called a "Swiss Cheese Method". The essential difference of our method from those applied in similar situations before (see, e.g., [17, 18, 19]) is that we construct a nonresonance set not only in the whole space of a parameter ($\vec{k} \in \mathbb{R}^2$ here), but also on all isoenergetic curves $\mathcal{D}_n(\lambda)$ in the space of a parameter corresponding to sufficiently large energies. Estimates for the size of nonresonance sets in this case require more subtle technical considerations than those sufficient for a description of a nonresonant set in the whole space of the parameter.

The restriction $l \geq 6$ is technical, it is needed only for the first two steps of the recurrent procedure. The requirement for superexponential decay of $\|V_n\|$ as $n \to \infty$ is more essential, since it is needed to ensure convergence of the recurrence

procedure. It is not essential that the potential V_r has doubling periods; the periods of the type $q^{r-1}\vec{b}_1,\ q^{r-1}\vec{b}_2, q \in \mathbb{N}$, can be treated in the same way.

Proofs of the described results (except absolute continuity) are presented in [22]. A paper containing all the proofs is in preparation.

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