REPRESENTATION THEORY
An Electronic Journal of the American Mathematical Society
Volume 1, Pages 25–30 (November 4, 1996)
S 1088-4165(97)00003-4

NONLOCAL FINITENESS OF A W-GRAPH

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ABSTRACT. It is shown that the W-graph of an affine Weyl group of type B_2 (as defined by Kazhdan and Lusztig in *Representations of Coxeter groups and Hecke algebras*, Invent. Math. **53** (1979), 165–184) is not locally finite.

1. The purpose of this paper is to present an example of computation of the leading coefficients $\mu(y, w)$ of the polynomials $P_{y,w}$ of [KL], or equivalently, of the " inverse polynomials" $Q_{y,w}$ for an affine Weyl group.

We use the following method. We want to make use of the explicit formula [L] for the $P_{y,w}$ in the case where y, w have maximal length in their double coset with respect to the finite Weyl group. For such y, w we have $\mu(y, w) = 0$ except in trivial cases, so this by itself does not give interesting examples of $\mu(y, w)$. It would be much more useful to be able to compute, instead, the polynomials $Q_{y,w}$ where y, w have minimal length in their double coset with respect to the finite Weyl group. We show that these last polynomials can be directly related through a system of semilinear equations to the special polynomials $P_{y,w}$ above. These equations can sometimes be solved explicitly and we may hope to find in this way interesting examples of $\mu(y, w)$.

Consider the affine Weyl group of type $\tilde{B}_2$ with standard Coxeter generators a, b, c where a, c commute; set $p = aba, q = cba$. Using the method above, we shall obtain the following result:

(a) $$\mu(pq^n, pq^m) = 1 \text{ if } m > n > 0,\ m \text{ even},\ n \text{ odd}.$$

(Note that $\{pq^n | n = 1, 2, \dots\}$ are distinct involutions and the length of pq^n is $3n+3$.) We see that the W-graph (see [KL, §1]) of our Coxeter group is not locally finite. This example suggests that the W-graphs of most affine Weyl groups are not locally finite. (They are locally finite for type $\tilde{A}_1, \tilde{A}_2$.)

2. Let (W, S) be a Coxeter group (S is the set of simple reflections) and let $l : W \to \mathbf{N}$ be the corresponding length function. Let $\mathcal{A} = \mathbf{Z}[v, v^{-1}]$ where v is an indeterminate. For $f \in \mathcal{A}$, let $\mathrm{Res}_{v=0}(f) \in \mathbf{Z}$ denote the coefficient of v^{-1} in f. Let $^- : \mathcal{A} \to \mathcal{A}$ be the ring involution such that $\bar{v} = v^{-1}$. For $x, y \in W$, let $R_{x,y} \in \mathcal{A}$ be defined as in [KL, (2.0.a)] (where $q^{\frac{1}{2}}$ in *loc. cit.* is our v). We set $r_{x,y} = v^{l(x)-l(y)} R_{x,y}$. We have

(a) $$r_{x,x} = 1, \bar{r}_{x,y} = (-1)^{l(y)-l(x)} r_{x,y} \text{ for any } x, y \in W,$$

Received by the editors August 13, 1996 and, in revised form, August 21, 1996.
1991 *Mathematics Subject Classification.* Primary 20G99.
Supported in part by the National Science Foundation.

(see [KL, 2.1(i)]),

(b) $$r_{x,1} = \delta_{x,1},$$

(c) $$v^{-l(x)-l(y)} r_{x,y} + v^{-l(sx)-l(y)} r_{sx,y} = v^{-l(x)-l(sy)} r_{x,sy} + v^{-l(sx)-l(sy)} r_{sx,sy},$$

(d) $$v^{l(x)+l(y)} r_{x,y} + v^{l(x)+l(sy)} r_{x,sy} = v^{l(sx)+l(y)} r_{sx,y} + v^{l(sx)+l(sy)} r_{sx,sy},$$

(for any $x, y \in W$ and any $s \in S$). Note that (c) follows from [KL, (2.0.c)] and (d) follows by applying $\bar{}$ to (c) and using (a). Moreover,

(e) $$r_{y,z} \neq 0 \implies y \leq z \text{ in the Bruhat order of } W.$$

For any $x, z \in W$ we have (by [KL, 2.1(ii)])

(f) $$\sum_y r_{x,y} \bar{r}_{y,z} = \delta_{x,z}.$$

3. For $x \leq y$ in W, let $P_{x,y} \in \mathcal{A}$ be defined as in [KL, 1.1] (where $q^{\frac{1}{2}}$ in *loc. cit.* is our v). We set $p_{x,y} = v^{l(x)-l(y)} P_{x,y} \in \mathbf{Z}[v^{-1}]$ for $x \leq y$ and $p_{x,y} = 0$ for all other x, y. We have

(a) $$p_{x,x} = 1, \quad p_{x,y} \in v^{-1}\mathbf{Z}[v^{-1}] \text{ for } x < y,$$

(b) $$\sum_y r_{x,y} p_{y,z} = \bar{p}_{x,z} \text{ for any } x, z \text{ in } W,$$

(see [KL, (2.2.a)]).
We define elements $q_{x,y} \in \mathbf{Z}[v^{-1}]$ (for any x, y in W) by the system of equations

(c) $$\sum_z (-1)^{l(x)+l(y)} p_{x,y} q_{y,z} = \delta_{x,z}$$

(for any x, z in W). We have

(d) $$q_{x,x} = 1, \quad q_{x,y} \in v^{-1}\mathbf{Z}[v^{-1}] \text{ for } x < y \text{ and } q_{x,y} = 0 \text{ for all other } x, y.$$

For any $x, z \in W$ we have

(e) $$\sum_y (-1)^{l(x)-l(y)} q_{x,y} p_{y,z} = \delta_{x,z},$$

(f) $$\sum_y q_{x,y} r_{y,z} = \bar{q}_{x,z}.$$

As in [KL, 1.2], for $x, y \in W$, we set

$$\mu(x, y) = \text{Res}_{v=0}(p_{x,y}) = \text{Res}_{v=0}(q_{x,y}) \in \mathbf{Z}.$$

4. If I is a subset of S we denote by W_I the subgroup of W generated by I. If W_I is finite, we set

$$\xi_I = \sum_{w \in W_I} v^{2l(w)} \in \mathbf{Z}[v^2].$$

We now fix two subsets I, I' of S such that W_I and $W_{I'}$ are finite. Let ν (resp. ν') be the maximum length of an element of W_I (resp. $W_{I'}$). Let X^+ be the set of $W_I - W_{I'}$ double cosets in W. Each $\lambda \in X^+$ contains a unique element m_λ of minimal length and a unique element M_λ of maximal length; we define

(a) $$\xi_\lambda = \sum_{w \in \lambda} v^{2l(w)} \in \mathbf{Z}[v^2],$$

(b) $$\pi_\lambda = \frac{\xi_I}{v^\nu} \frac{\xi_{I'}}{v^{\nu'}} \frac{v^{l(M_\lambda)+l(m_\lambda)}}{\xi_\lambda} \in \mathcal{A};$$

then $\bar{\pi}_\lambda = \pi_\lambda$.

From 2(c) we see that the function $y \to \sum_{x \in \lambda} v^{-l(x)-l(y)} r_{x,y}$ is constant on left W_I-cosets; similarly, it is constant on right $W_{I'}$-cosets, hence it is constant on the double cosets in W. In particular, for $\lambda, \lambda' \in X^+$, we have

(c) $$\sum_{x \in \lambda; y \in \lambda'} v^{-l(x)+l(y)} r_{x,y} = \sum_{y \in \lambda'} v^{2l(y)} \sum_{x \in \lambda} v^{-l(x)-l(m_{\lambda'})} r_{x,m_{\lambda'}}.$$

From 2(d) we see that the function $x \to \sum_{y \in \lambda'} v^{l(x)+l(y)} r_{x,y}$ is constant on left W_I-cosets; similarly, it is constant on right $W_{I'}$-cosets, hence it is constant on the double cosets in W. In particular, for $\lambda, \lambda' \in X^+$, we have

(d) $$\sum_{x \in \lambda; y \in \lambda'} v^{-l(x)+l(y)} r_{x,y} = \sum_{x \in \lambda} v^{-2l(x)} \sum_{y \in \lambda'} v^{l(M_\lambda)+l(y)} r_{M_\lambda,y}.$$

Comparing (c),(d) we see that

(e) $$\xi_{\lambda'} \sum_{x \in \lambda} v^{-l(x)-l(m_{\lambda'})} r_{x,m_{\lambda'}} = \overline{\xi_\lambda} \sum_{y \in \lambda'} v^{l(M_\lambda)+l(y)} r_{M_\lambda,y}.$$

5. Let $\leq$ be the partial order on X^+ defined by $\lambda \leq \lambda'$ if $M_\lambda \leq M_{\lambda'}$. For $\lambda, \lambda' \in X^+$, we set

$$p_{\lambda,\lambda'} = p_{M_\lambda, M_{\lambda'}}, \quad q_{\lambda,\lambda'} = q_{m_\lambda, m_{\lambda'}}.$$

By [KL, 2.3.g], we have

$$v^{l(M_\lambda)-l(x)} p_{x,M_{\lambda'}} = p_{\lambda,\lambda'}$$

for all $x \in \lambda$ and similarly

$$v^{l(x')-l(m_{\lambda'})} q_{m_\lambda,x'} = q_{\lambda,\lambda'}$$

for all $x' \in \lambda'$. We set

(a) $$a_{\lambda,\lambda'} = \sum_{y \in \lambda'} (-v)^{-l(M_{\lambda'})+l(y)} q_{M_\lambda,y},$$

(b) $$b_{\lambda,\lambda'} = \sum_{z \in \lambda} (-v)^{l(m_\lambda)-l(z)} p_{z,m_{\lambda'}}.$$

Clearly, $p_{\lambda,\lambda'}, q_{\lambda,\lambda'}, a_{\lambda,\lambda'}, b_{\lambda,\lambda'}$ are zero unless $\lambda \le \lambda'$ and are equal to 1 when $\lambda = \lambda'$; when $\lambda < \lambda'$ they belong to $v^{-1}\mathbf{Z}[v^{-1}]$ and

(c) $$\mathrm{Res}_{v=0}(q_{\lambda,\lambda'}) = \mu(m_\lambda, m_{\lambda'}) = \mathrm{Res}_{v=0}(b_{\lambda,\lambda'}).$$

6. Using 3(c) and the definitions, we see that

(a) $$\sum_{\lambda'} (-1)^{l(M_\lambda)-l(M_{\lambda'})} a_{\lambda,\lambda'} p_{\lambda',\lambda''} = \delta_{\lambda,\lambda''},$$

(b) $$\sum_{\lambda'} (-1)^{l(m_\lambda)-l(m_{\lambda'})} q_{\lambda,\lambda'} b_{\lambda',\lambda''} = \delta_{\lambda,\lambda''},$$

for any $\lambda, \lambda'' \in X^+$.

Proposition 7. *For any $\lambda, \lambda'' \in X^+$, we have*
(a) $\sum_{\lambda'} q_{\lambda,\lambda'} \frac{1}{\pi_{\lambda'}} \bar{p}_{\lambda',\lambda''} = \sum_{\lambda'} \bar{q}_{\lambda,\lambda'} \frac{1}{\pi_{\lambda'}} p_{\lambda',\lambda''}$,
(b) $\sum_{\lambda'} a_{\lambda,\lambda'} (-1)^{l(m_{\lambda'})-l(M_{\lambda'})} \pi_{\lambda'} \bar{b}_{\lambda',\lambda''} = \sum_{\lambda'} \bar{a}_{\lambda,\lambda'} (-1)^{l(m_{\lambda'})-l(M_{\lambda'})} \pi_{\lambda'} b_{\lambda',\lambda''}$.

For any λ, λ' in X^+ we set

$$r_{\lambda,\lambda'} = \sum_{z \in \lambda'} v^{l(z)-l(M_{\lambda'})} r_{M_\lambda,z},$$

$$\tilde{r}_{\lambda,\lambda'} = \sum_{z \in \lambda} v^{-l(z)+l(m_\lambda)} r_{z,m_{\lambda'}}.$$

Using these definitions and 3(b), 3(f), we deduce, for any λ, λ'':

(c) $$\sum_{\lambda'} r_{\lambda,\lambda'} p_{\lambda',\lambda''} = \bar{p}_{\lambda,\lambda''},$$

(d) $$\sum_{\lambda'} q_{\lambda,\lambda'} \tilde{r}_{\lambda',\lambda''} = \bar{q}_{\lambda,\lambda''}.$$

We can rewrite 4(e) in the form

$$\xi(\lambda') v^{-l(m_{\lambda'})-l(m_\lambda)} \tilde{r}_{\lambda,\lambda'} = \overline{\xi(\lambda)} v^{l(M_\lambda)+l(M_{\lambda'})} r_{\lambda,\lambda'}$$

or in the form

(e) $$\frac{1}{\pi_{\lambda'}} \tilde{r}_{\lambda,\lambda'} = \frac{1}{\pi_\lambda} r_{\lambda,\lambda'}.$$

Using (c), we see that the left hand side of (a) is equal to

$$\sum_{\lambda'} q_{\lambda,\lambda'} \frac{1}{\pi_{\lambda'}} \sum_{\tilde{\lambda}} r_{\lambda',\tilde{\lambda}} p_{\tilde{\lambda},\lambda''}$$

and, using (e), to

$$\sum_{\lambda',\tilde{\lambda}} q_{\lambda,\lambda'} \frac{1}{\pi_{\tilde{\lambda}}} \tilde{r}_{\lambda',\tilde{\lambda}} p_{\tilde{\lambda},\lambda''}.$$

Using (d), we see that this equals $\sum_{\tilde{\lambda}} \bar{q}_{\lambda,\tilde{\lambda}} \frac{1}{\pi_{\tilde{\lambda}}} p_{\tilde{\lambda},\lambda''}$ which is the same as the right hand side of (a). Thus (a) is proved.

We prove (b). We can write (a) as an identity of matrices indexed by $X^+ \times X^+$:

$$QD^{-1}\bar{P} = \bar{Q}D^{-1}P \tag{f}$$

where $Q = (q_{\lambda,\lambda'}), P = (p_{\lambda,\lambda'})$, D is the diagonal matrix with entries π_λ; we agree that $^-$ applied to a matrix is obtained by applying $^-$ to each entry. Taking the inverse of both sides of (f), we obtain $P^{-1}D\bar{Q}^{-1} = \bar{P}^{-1}DQ^{-1}$. This is, up to signs, the same as (b) (we use 6(a),6(b)). The proposition is proved.

8. In the remainder of this paper we assume that (W,S) is an irreducible affine Weyl group regarded as a Coxeter group. The set X of elements of W which have only finitely many conjugates, is a normal subgroup of W, which is free abelian, finitely generated, of finite index. We shall take $I = I' = S - \{s_0\}$ where $s_0 \in S$ is such W is generated by X and by W_I. There is a unique $\mathbf{Z}$-basis $\{\alpha_s | s \in I\}$ of X and unique homomorphisms $\check{\alpha}_s : X \to \mathbf{Z}$ for $s \in I$ such that $s(x) = x - \check{\alpha}_s(x)\alpha_s$ for all $s \in I, x \in X$. (We write the group operation in X as addition and we write $w(x)$ instead of wxw^{-1} for $w \in W, x \in X$.) Let R be the finite set consisting of all elements of X of the form $w(\alpha_s)$ for various $s \in I$ and $w \in W_I$. Let $R^+ = R \cap \sum_{s\in I} \mathbf{N}\alpha_s$. Let $X^+ = \{x \in X | \check{\alpha}_s(x) \geq 0 \quad \forall s \in I\}$. For any $\lambda \in X^+$, we set

$$W_I^\lambda = \{w \in W_I | w(\lambda) = \lambda\}.$$

In our case, π_λ of 4(b) can be rewritten as follows:

$$\pi_\lambda = v^{-\nu_\lambda} \sum_{w \in W_I^\lambda} v^{2l(w)}$$

where ν_λ is the number of reflections of W_I^λ.

There is a 1-1 correspondence between X^+ and the set of $W_I - W_I$ double cosets in W (to an element of X^+ corresponds the unique double coset containing it). We identify in this way X^+ with the set of $W_I - W_I$ double cosets in W, thus reconciling the present notation with the notation X^+ in §4.

9. Let X' be the subgroup of $\mathbf{Q} \otimes X$ consisting of all x such that $\check{\alpha}_s(x) \in \mathbf{Z}$ for all $s \in I$. We have $X \subset X'$ and the action of W_I on X extends uniquely to a linear action of W_I on X'. For any subset $\mathbf{i} \subset R^+$, we set $\alpha_{\mathbf{i}} = \sum_{\alpha \in \mathbf{i}} \alpha \in X$. It is known that $\alpha_{R^+} = 2\rho$ where $\rho \in X'$ and that $w(\rho) - \rho \in X$ for any $w \in W_I$. Let Γ be the free $\mathcal{A}$-module with basis $Z_\lambda, (\lambda \in X^+)$. We define elements $Z_\lambda \in \Gamma$ for all $\lambda \in X$ (not just for $\lambda \in X^+$) by setting

$$Z_\lambda = (-1)^{l(w)} Z_{w(\lambda+\rho)-\rho},$$

if $w(\lambda + \rho) - \rho \in X^+$, for some $w \in W_I$ (necessarily unique),

$$Z_\lambda = 0,$$

otherwise.

Theorem 10.

$$v^{-\nu_{\lambda'}} \pi_{\lambda'} \sum_{\lambda \in X^+} a_{\lambda,\lambda'} Z_\lambda = \sum_{\mathbf{i};\mathbf{i} \subset R^+} (-v^2)^{-|\mathbf{i}|} Z_{\lambda' - \alpha_{\mathbf{i}}};$$

(*equality in* Γ) *holds for any* $\lambda' \in X^+$*; here,* $a_{\lambda,\lambda'}$ *are as in* §*5.*

This is obtained by assembling several identities in [L] in the affine Hecke algebra (notation of [L]):

$$J_\rho(v^{-l(p_{\lambda'})}K_{\lambda'}) = \frac{1}{\bar{\mathcal{P}}_{\lambda'}}\sum_{\mathbf{i}}(-v^2)^{-|\mathbf{i}|}J_{\lambda'+\rho-\alpha_{\mathbf{i}}}$$

([L, 6.7]);

$$J_\rho C'_\lambda = J_{\lambda+\rho}$$

([L, 6.9]);

$$C'_\lambda = \sum_{\lambda'\in X^+} p_{\lambda',\lambda}v^{-l(p_{\lambda'})}K_{\lambda'}$$

([L, 6.10, 6.13]) and using 6(a). (We take Γ to be the $\mathcal{A}$-submodule of the affine Hecke algebra spanned by $Z_\lambda = J_{\lambda+\rho}$ with $\lambda \in X^+$.) In our case, we have $(-1)^{l(M_\lambda)-l(M_{\lambda'})} = 1$ for λ, λ' in X^+.

Corollary 11. *For any $x \in X$ we set*

$$\Phi(x) = \sum_{\mathbf{i};\mathbf{i}\subset R^+;\alpha_{\mathbf{i}}=x}(-v^2)^{-|\mathbf{i}|}.$$

For any λ, λ' in X^+, we have

$$a_{\lambda,\lambda'} = \frac{v^{\nu_{\lambda'}}}{\pi_{\lambda'}}\sum_{w\in W_I}(-1)^{l(w)}\Phi(\lambda'+\rho-w(\lambda+\rho)).$$

12. It is likely that, in the case where $\check{\alpha}_s(\lambda') \geq 1$ for all $s \in I$ and $\lambda \in X^+$, we have

$$b_{\lambda,\lambda'} = (-1)^{l(m_\lambda - l(m_{\lambda'})}\frac{1}{\pi_\lambda}\sum_{w\in W_I}(-1)^{l(w)}\Phi(w(\lambda'-\rho)-(\lambda-\rho)).$$

13. Now let (W,S) be an affine Weyl group of type $\tilde{B}_2$. Write $I = \{1,2\}$ so that $\check{\alpha}_2(\alpha_1) = -1, \check{\alpha}_1(\alpha_2) = -2$. We solve the system of semilinear equations 7(b) with unknowns $b_{\lambda',\lambda''}$ for fixed $\lambda'' = t\alpha_1 + t\alpha_2$, where $t \geq 3$ and the quantities $a_{\lambda,\lambda'}$ are given by §11. (That system has a unique solution subject to the requirement that $b_{\lambda',\lambda''}$ is zero unless $\lambda' \leq \lambda''$, is equal to 1 when $\lambda' = \lambda''$ and belongs to $v^{-1}\mathbf{Z}[v^{-1}]$ when $\lambda' < \lambda''$.) We find

$$b_{t\alpha_1+(t-1)\alpha_2,t\alpha_1+t\alpha_2} = v^{-3}+v^{-1},$$
$$b_{s\alpha_1+(s-1)\alpha_2,t\alpha_1+t\alpha_2} = -v^{-5}+v^{-1},$$

for $s = t-1, t-2, \ldots, 3$. It follows that $\mathrm{Res}_{v=0}(b_{s\alpha_1+(s-1)\alpha_2,t\alpha_1+t\alpha_2}) = 1$, for $s = t, t-1, \ldots, 3$. Using 5(c), this yields formula 1(a).

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