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COHOMOLOGY OF CLASSIFYING SPACES AND HERMITIAN REPRESENTATIONS

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Abstract. It is shown that a large part of the cohomology of the classifying space of a Lie group satisfying certain hypotheses can be obtained by a difference construction from hermitian representations of that Lie group. This result is relevant to the study of Novikov's higher signatures.

1. Statement of the Theorem

1.1. Let $\mathcal{G}$ be a connected Lie group. We set $H^k_{\mathcal{G}} = H^k(B_{\mathcal{G}}, \mathbf{C})$ (cohomology with complex coefficients) where $B_{\mathcal{G}}$ is a classifying space of $\mathcal{G}$. We set $H^*_{\mathcal{G}} = H^0_{\mathcal{G}} \times H^1_{\mathcal{G}} \times H^2_{\mathcal{G}} \times \dots$; we regard this as a topological $\mathbf{C}$-algebra in which a fundamental system of neighbourhoods of 0 is provided by the subspaces $0 \times 0 \times \dots \times 0 \times H^l_{\mathcal{G}} \times H^{l+1}_{\mathcal{G}} \times \dots$ for various integers $l \geq 0$.

For any continuous finite dimensional complex representation V' of $\mathcal{G}$, we can form the associated complex vector bundle on $B_{\mathcal{G}}$ and consider its Chern character $\mathrm{ch}_{V'} \in H^*_{\mathcal{G}}$. It is well known that, in the case where $\mathcal{G}$ is compact, the elements $\mathrm{ch}_{V'}$ (for various V' as above) span over $\mathbf{C}$ a dense subspace of $H^*_{\mathcal{G}}$.

1.2. How to extend this result to not necessarily compact groups?

Let V be a finite dimensional $\mathbf{C}$-vector space with a given non-degenerate hermitian form on which $\mathcal{G}$ acts linearly and continuously, preserving the hermitian form. We associate to V an element $\tilde{\mathrm{ch}}_V \in H^*_{\mathcal{G}}$ as follows.

We choose a maximal compact subgroup $\mathcal{K}$ of $\mathcal{G}$. We can find a direct sum decomposition $V = V^+ \oplus V^-$ with V^+, V^- orthogonal to each other for the hermitian form such that V^+, V^- are $\mathcal{K}$-invariant subspaces and the hermitian form is positive definite on V^+ and negative definite on V^-. The Chern characters $\mathrm{ch}_{V^+} \in H^*_{\mathcal{K}}, \mathrm{ch}_{V^-} \in H^*_{\mathcal{K}}$ are then well defined since V^+, V^- are representations of $\mathcal{K}$. We may identify $H^*_{\mathcal{K}} = H^*_{\mathcal{G}}$ since the inclusion $\mathcal{K} \to \mathcal{G}$ induces a homotopy equivalence $B_{\mathcal{K}} \xrightarrow{\sim} B_{\mathcal{G}}$; we define

$$\tilde{\mathrm{ch}}_V = \mathrm{ch}_{V^+} - \mathrm{ch}_{V^-} \in H^*_{\mathcal{G}}.$$

Note that the element $\tilde{\mathrm{ch}}_V$ is independent of the choices of $\mathcal{K}$ and of the decomposition $V = V^+ \oplus V^-$, since the set of these choices is a contractible space.

Our main result is the following:

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Theorem 1.3. *Assume that $\mathcal{G}$ is the group of* **R***-rational points of a connected, simply connected semisimple algebraic group G over* **C** *with a given* **R***-structure. Assume also that $\mathcal{G}$ possesses some compact Cartan subgroup. Then the elements $\tilde{ch}_V$ for various V as above span over* **C** *a dense subspace of $H^*_{\mathcal{G}}$.*

In the case where $\mathcal{G}$ is a real symplectic group, this was proved in [L] where an application to the study of Novikov's higher signatures (for discrete cocompact subgroups of $\mathcal{G}$) was given. Gromov [G, p.139-140] realized that the result of [L] has also interesting differential-geometric applications, and asked the author (in January 1995) whether the more general statement of the Theorem above might be true. This provided the impetus for the present work.

2. Proof of the Theorem

2.1. Let G be as in 1.3 and let $\mathfrak{g}$ be the Lie algebra of G (over $\mathbf{C}$). Let Z_G be the centre of G. Let T be a maximal torus of G. We choose standard Chevalley generators $e_i, f_i, h_i (i \in I)$ for $\mathfrak{g}$ such that $\{h_i | i \in I\}$ is a $\mathbf{C}$-basis of the Lie algebra of T.

Let $\pi\colon G \to G$ be the involutive automorphism of G such that the tangent map $d\pi\colon \mathfrak{g} \to \mathfrak{g}$ is given by

$$d\pi(e_i) = -f_i, d\pi(f_i) = -e_i, d\pi(h_i) = -h_i$$

for all $i \in I$. Let $^- \colon G \to G$ be the antiholomorphic involution of G whose tangent map is the conjugate-linear map $^- \colon \mathfrak{g} \to \mathfrak{g}$ given by $\bar{e}_i = e_i, \bar{f}_i = f_i, \bar{h}_i = h_i$ for all $i \in I$. Clearly, $\pi\colon G \to G$ commutes with $^- \colon G \to G$. It is well known that

$$K = \{g \in G | \pi(\bar{g}) = g\}$$

is a maximal compact subgroup of G.

2.2. For any $\lambda = (\lambda_i)_{i \in I} \in \mathbf{N}^I$, let V_λ be a finite dimensional $\mathbf{C}$-vector space with a non-zero vector η on which G acts linearly as an algebraic group such that the corresponding representation of $\mathfrak{g}$ on V_λ is irreducible and satisfies $e_i\eta = 0, h_i\eta = \lambda_i\eta$ for all i. Note that (V_λ, η) is uniquely determined by λ up to unique isomorphism. It follows easily that there is a unique conjugate-linear involution $x \mapsto \bar{x}$ of V_λ such that $\overline{ax} = \bar{a}\bar{x}$ for all $a \in \mathfrak{g}, x \in V_\lambda$ and $\bar{\eta} = \eta$. It is well known and easy to prove that there is a unique hermitian form $\langle , \rangle \colon V \times V \to \mathbf{C}$ (linear in the first variable, conjugate-linear in the second variable) such that

$\langle \eta, \eta \rangle = 1$ and $\langle ax, y \rangle = \langle x, -d\pi(\bar{a})y \rangle$ for all $x, y \in V_\lambda$ and all $a \in \mathfrak{g}$.

It is well known that this hermitian form is positive definite.

From the definition we deduce the identity:

$$\langle gx, y \rangle = \langle x, \pi(\bar{g}^{-1})y \rangle$$

for all $x, y \in V_\lambda$ and all $g \in G$.

2.3. Let $\sigma \in T$ be such that $\sigma^2 \in Z_G$. We have

(a) $\bar{\sigma} = \sigma^{-1}$,

(b) $\pi(\sigma) = \sigma^{-1}$.

Indeed, it is easy to check that, if $t \in T$ is of finite order, then $\bar{t} = t^{-1}$. Since Z_G is a finite group, σ is of finite order. Hence (a) holds. Clearly, for any $t \in T$, we have $\pi(t) = t^{-1}$. Hence (b) holds.

2.4. We define an antiholomorphic map $\tau\colon G \to G$ by

$$g \mapsto \tau(g) = \sigma\pi(\bar{g})\sigma^{-1}.$$

It is easy to see that τ is an involution. Let

(a) $$\mathcal{G} = \{g \in G | \tau(g) = g\}.$$

Then $\mathcal{G}$ is the group of real points for an $\mathbf{R}$-rational structure on G. Note that

$$\mathcal{G} \cap T = \{t \in T | \tau(t) = t\} = \{t \in T | t\bar{t} = 1\}$$

is a compact Cartan subgroup of $\mathcal{G}$. Thus, $\mathcal{G}$ satisfies the assumptions of Theorem 1.3. Conversely, it is known that any $\mathcal{G}$ as in the assumptions of Theorem 1.3 can be obtained by the previous construction. (For example, $\sigma = 1$ gives rise to a compact $\mathcal{G}$.) Hence it suffices to prove Theorem 1.3 for $\mathcal{G}$ given by (a).

2.5. Let G^σ be the centralizer of σ in G. Let

$$\mathcal{K} = G^\sigma \cap \mathcal{G} = G^\sigma \cap K = \mathcal{G} \cap K.$$

We have

$$\sigma \in \mathcal{K}.$$

Indeed, from 2.3(a),(b) we see that $\sigma \in K$. Clearly, $\sigma \in G^\sigma$; our assertion follows.

The following result is well known.

Lemma 2.6. (a) *$\mathcal{K}$ is a maximal compact subgroup of G^σ.*
(b) *$\mathcal{K}$ is a maximal compact subgroup of $\mathcal{G}$.*

2.7. Since η is an eigenvector for the T-action on V_λ, we have

$$\sigma\eta = \delta\eta$$

for some $\delta \in \mathbf{C}^*$. Applying $^-$ to the last equality we obtain

$$\bar{\delta}\eta = \bar{\sigma}\eta = \sigma^{-1}\eta = \delta^{-1}\eta;$$

hence

$$\bar{\delta} = \delta^{-1}.$$

By Schur's lemma, σ^2 acts as a scalar on V_λ and this scalar is necessarily δ^2. Thus,

$$\sigma^2 = \delta^2$$

on V_λ.

For $x, y \in V_\lambda$ we set

$$\langle\langle x, y\rangle\rangle = \delta^{-1}\langle \sigma x, y\rangle.$$

Lemma 2.8. (a) *$\langle\langle , \rangle\rangle$ is a non-degenerate hermitian form on V_λ.*
(b) *For any $x, y \in V_\lambda$ and any $g \in \mathcal{G}$, we have $\langle\langle gx, gy\rangle\rangle = \langle\langle x, y\rangle\rangle$.*

Let $x, y \in V_\lambda$. We must show that

(c) $$\langle\langle x, y\rangle\rangle = \overline{\langle\langle y, x\rangle\rangle}.$$

The right hand side is

$$\overline{\delta^{-1}\langle \sigma y, x\rangle} = \bar{\delta}^{-1}\langle x, \sigma y\rangle = \delta\langle x, \sigma y\rangle$$

while the left hand side is

$$\delta^{-1}\langle \sigma x, y\rangle = \delta^{-1}\langle x, \pi(\bar\sigma^{-1})y\rangle = \delta^{-1}\langle x, \pi(\bar\sigma^{-1})y\rangle$$
$$= \delta^{-1}\langle x, \sigma^{-1}y\rangle = \langle x, \delta\sigma^{-1}y\rangle = \langle x, \delta^{-1}\sigma y\rangle = \delta\langle x, \sigma y\rangle.$$

Thus (c) is proved. The fact that $\langle\langle , \rangle\rangle$ is non-degenerate follows from the analogous property of $\langle , \rangle$. Next we show that, for g in G, we have

(d) $$\langle\langle gx, y\rangle\rangle = \langle\langle x, \tau(g^{-1})y\rangle\rangle.$$

The right hand side is

$$\langle\langle gx, y\rangle\rangle = \delta^{-1}\langle \sigma gx, y\rangle = \delta^{-1}\langle x, \pi(\bar g^{-1})\pi(\bar\sigma^{-1})y\rangle = \delta^{-1}\langle x, \pi(\bar g^{-1})\sigma^{-1}y\rangle$$
$$= \delta^{-1}\langle x, \sigma^{-1}\tau(g^{-1})y\rangle,$$

while the left hand side is

$$\langle\langle x, \tau(g^{-1})y\rangle\rangle = \delta^{-1}\langle \sigma x, \tau(g^{-1})y\rangle = \delta^{-1}\langle x, \pi(\bar\sigma^{-1})\tau(g^{-1})y\rangle = \delta^{-1}\langle x, \sigma^{-1}\tau(g^{-1})y\rangle.$$

Thus (d) is proved. In the case where $g \in \mathcal{G}$, identity (d) becomes $\langle\langle gx, y\rangle\rangle = \langle\langle x, g^{-1}y\rangle\rangle$. Replacing here y by gy we obtain (b). The lemma is proved.

Lemma 2.9. *Let* $V_\lambda^+ = \{x \in V_\lambda | \sigma x = \delta x\}, V_\lambda^- = \{x \in V_\lambda | \sigma x = -\delta x\}$. *Then*
(a) $V_\lambda = V_\lambda^+ \oplus V_\lambda^-$ *and* $\langle\langle V_\lambda^+, V_\lambda^-\rangle\rangle = 0$;
(b) $\langle\langle , \rangle\rangle|_{V_\lambda^+}$ *is positive definite and* $\langle\langle , \rangle\rangle|_{V_\lambda^-}$ *is negative definite.*

The first statement of (a) follows from the fact that $\sigma^2 = \delta^2$ on V_λ. We have $\sigma \in \mathcal{G}$ (see 2.5) hence, by 2.8(b), σ acts as an isometry of $\langle\langle , \rangle\rangle$. Hence the ζ-eigenspace of σ is orthogonal under $\langle\langle , \rangle\rangle$ to the ζ'-eigenspace of σ provided that $\zeta'\bar\zeta \neq 1$. The last condition is satisfied by $\zeta = \delta, \zeta' = -\delta$ since $-\delta\bar\delta = -1$. This proves (a).

We prove (b). If $x \in V_\lambda^+$ and $x \neq 0$, we have $\langle\langle x, x\rangle\rangle = \delta^{-1}\langle \sigma x, x\rangle = \langle x, x\rangle > 0$; if $x \in V_\lambda^-$ and $x \neq 0$, we have $\langle\langle x, x\rangle\rangle = \delta^{-1}\langle \sigma x, x\rangle = -\langle x, x\rangle < 0$. The lemma is proved.

2.10. Let W be the Weyl group of G with respect to T and let W' be the Weyl group of G^σ with respect to T. We regard W' naturally as a subgroup of W. Now W hence, by restriction, W' acts on T by conjugation. This induces an action of W, hence of W', through algebra automorphisms on $\mathcal{O}$, the algebra of regular functions $T \to \mathbf{C}$. We will denote the action of $w \in W$ on $\mathcal{O}$ by $f \mapsto w^*f$. Let $\mathcal{O}^W$ be the algebra of W-invariant elements of $\mathcal{O}$; let $\mathcal{O}^{W'}$ be the algebra of W'-invariant elements of $\mathcal{O}$. We have $\mathcal{O}^W \subset \mathcal{O}^{W'} \subset \mathcal{O}$.

For any $t \in T$, the set of elements of $\mathcal{O}$ that vanish at t is a maximal ideal I_t of $\mathcal{O}$; we denote the completion of $\mathcal{O}$ with respect to the maximal ideal I_t by $\hat{\mathcal{O}}_t$. If W_t is the stabilizer of t in W, then the W_t-action on $\mathcal{O}$ preserves I_t; hence it induces a W_t-action on $\hat{\mathcal{O}}_t$. Let $(\hat{\mathcal{O}}_t)^{W_t}$ be the space of invariants of this W_t-action. In particular, for $t = \sigma$ we have $W_\sigma = W'$; hence $(\hat{\mathcal{O}}_\sigma)^{W'}$ is well defined. For $t = 1$, we have $W_1 = W$; the W-action on $\hat{\mathcal{O}}_1$ may be restricted to W' and we denote by $(\hat{\mathcal{O}}_1)^{W'}$ the space of W'-invariants for this action.

For any continuous finite dimensional complex representation V' of $\mathcal{K}$, the function $t \to \mathrm{tr}(t, V)$ on $T \cap \mathcal{K} = T \cap K = \{t \in T | t\bar t = 1\}$ extends uniquely to a regular function $\chi_{V'} \colon T \to \mathbf{C}$ which belongs to $\mathcal{O}^{W'}$.

By classical results on cohomology of classifying spaces [B], we may identify

$$H^*_{\mathcal{K}} = (\hat{\mathcal{O}}_1)^{W'}$$

as topological algebras so that the following holds: For any continuous finite dimensional complex representation V' of $\mathcal{K}$, the element $\mathrm{ch}_{V'} \in H^*_{\mathcal{K}}$ corresponds to the image of $\chi_{V'}$ under the obvious imbedding $j\colon \mathcal{O}^{W'} \to (\hat{\mathcal{O}}_1)^{W'}$.

If V_λ is as in 2.2, we have for any $t \in T \cap K$

$$\mathrm{tr}(t, V_\lambda^+) - \mathrm{tr}(t, V_\lambda^-) = \delta^{-1}\mathrm{tr}(t\sigma, V_\lambda).$$

Let $\tilde{\chi}_{V_\lambda} \in \mathcal{O}^{W'}$ be the function on T given by $t \mapsto \delta^{-1}\mathrm{tr}(t\sigma, V_\lambda)$. (This function is W'-invariant since σ is W'-invariant; note also that δ depends on λ.) We see that the element $\tilde{\mathrm{ch}}_{V_\lambda} \in H^*_{\mathcal{G}} = H^*_{\mathcal{K}} = (\hat{\mathcal{O}}_1)^{W'}$ associated in 1.2 to the $\mathcal{G}$-module V_λ (restriction from G to $\mathcal{G}$) with its $\mathcal{G}$-invariant hermitian form $\langle\langle,\rangle\rangle$ is precisely the image of $\tilde{\chi}_{V_\lambda}$ under the obvious imbedding $j\colon \mathcal{O}^{W'} \to (\hat{\mathcal{O}}_1)^{W'}$. Now $\mathcal{O}^W$ is spanned as a vector space by the functions $t \mapsto \mathrm{tr}(t, V_\lambda)$ for various λ as in 2.2. It follows that the subspace of $H^*_{\mathcal{G}} = H^*_{\mathcal{K}} = (\hat{\mathcal{O}}_1)^{W'}$ spanned by the elements $\tilde{\mathrm{ch}}_{V_\lambda}$ for various λ as in 2.2 is precisely the image of the composition

(a) $$\mathcal{O}^W \xrightarrow{p_\sigma} \mathcal{O}^{W'} \xrightarrow{j} (\hat{\mathcal{O}}_1)^{W'}$$

where p_σ attaches to $f \in \mathcal{O}^W$ the function $t \mapsto f(t\sigma)$ in $\mathcal{O}^{W'}$.

Hence Theorem 1.3 is a consequence of the following result, in which σ may be taken to be an arbitrary element of T.

Proposition 2.11. *The image of the composition 2.10(a) is dense in* $(\hat{\mathcal{O}}_1)^{W'}$.

Consider the diagram

$$\mathcal{O}^W \xrightarrow{\alpha} (\hat{\mathcal{O}}_\sigma)^{W'} \xrightarrow{\beta} (\hat{\mathcal{O}}_1)^{W'}$$

where α is induced by the obvious imbedding $\mathcal{O} \to \hat{\mathcal{O}}_\sigma$ and β is the isomorphism induced by the isomorphism $\hat{\mathcal{O}}_\sigma \to \hat{\mathcal{O}}_1$ which comes from the translation by σ on T (an isomorphism of varieties which takes 1 to σ). It is clear that $\beta \circ \alpha$ is equal to the composition 2.10(a). Since β is an isomorphism of topological algebras, it is therefore enough to show that the image of α is dense in $(\hat{\mathcal{O}}_\sigma)^{W'}$. Let $f \in (\hat{\mathcal{O}}_\sigma)^{W'}$. Let X be the W-orbit of σ in T. For each $t \in X$, we define an element $f_t \in \hat{\mathcal{O}}_t$ as follows. We choose $w \in W$ such that $w(t) = \sigma$. Now $w^*\colon \mathcal{O} \to \mathcal{O}$ induces an isomorphism $w^*\colon \hat{\mathcal{O}}_\sigma \xrightarrow{\sim} \hat{\mathcal{O}}_t$ and we set $f_t = w^*(f)$. This element is independent of the choice of w since f is W'-invariant. Note that $f_\sigma = f$.

Let $\hat{I}_t$ be the maximal ideal of $\hat{\mathcal{O}}_t$. Let $n \geq 1$ be an integer. By the Chinese remainder theorem, we can find $\phi \in \mathcal{O}$ such that $\phi = f_t \mod \hat{I}_t^n$ for all $t \in X$. For any $w \in W$, the function $w^*\phi \in \mathcal{O}$ satisfies again $w^*\phi = f_t \mod \hat{I}_t^n$ for all $t \in X$ since, by definition, the family $(f_t)_{t\in X}$ is W-invariant in an obvious sense. Setting $\phi' = \sharp(W)^{-1}\sum_{w\in W} w^*\phi \in \mathcal{O}$, we deduce that $\phi' = f_t \mod \hat{I}_t^n$ for all $t \in X$. In particular, taking $t = \sigma$, we see that $\phi' = f \mod \hat{I}_\sigma^n$. Since $\phi' \in \mathcal{O}^W$, we see that $\mathcal{O}^W$ is dense in $(\hat{\mathcal{O}}_\sigma)^{W'}$.

This completes the proof of the Proposition hence, that of Theorem 1.3.

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