ORBITS IN THE FLAG VARIETY AND IMAGES OF THE MOMENT MAP FOR CLASSICAL GROUPS I

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Abstract. We propose algorithms to get representatives and the images of the moment map of conormal bundles of $GL(p, \mathbb{C}) \times GL(q, \mathbb{C})$-orbits in the flag variety of $GL(p+q, \mathbb{C})$, and $GL(p+q, \mathbb{C})$-orbits and $Sp(p, \mathbb{C}) \times Sp(q, \mathbb{C})$-orbits in the flag variety of $Sp(p+q, \mathbb{C})$ and their signed Young diagrams.

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1. Introduction

For an admissible representation of a real semisimple Lie group, the associated variety is one of the most important invariants of representations. Borho-Brylinski [1] clarify the relation between representations in a $D$-module picture and their associated varieties. The associated variety of a representation (Harish-Chandra module) is exactly the image of the moment map of the characteristic variety of the $D$-module corresponding to the representation. In particular, the associated variety of a discrete series representation is nothing but the image of the moment map of the conormal bundle of the corresponding closed orbit in the flag variety. Motivated by these facts, it seems to be useful to determine the image of the moment map of the conormal bundle of various orbits in the flag variety. In this paper, for indefinite unitary groups, real symplectic groups, and indefinite symplectic groups, we will determine the image as explicitly as possible. Garfinkle gave another algorithm in [3]. By her algorithm we can also get signed Young diagrams from parameters of $K$-orbits in the flag variety. It remains to examine how the two algorithms agree.

In order to describe the results in this paper, we introduce the following notation. Let $G_{\mathbb{R}}$ be a real classical group. Let $K$ be the complexification of a maximal compact subgroup of $G_{\mathbb{R}}$. The image of the moment map on a conormal bundle of a $K$-orbit in the flag variety is known to contain a (unique) dense nilpotent $K$-orbit in the orthogonal subspace $p$ of the Lie algebra $k$ of $K$ in the Lie algebra $g$ (see [1], [2]). What we want to write down is the map from a $K$-orbit in the flag variety to a nilpotent $K$-orbit in $p$. For a $K$-orbit $Q$ in the flag variety $X$, the image of the moment map $\mu$ of the conormal bundle $T^*_QX$ is a $K$-saturation of the image of a fiber $(T^*_QX)_x$ at a point $x \in Q$. Then the nilpotent $K$-orbit $O$ in $p$ is a $K$-saturation of the subspace $\mu((T^*_QX)_x)$ in $p$. The $K$-orbits in $p$ are parametrized by symbols called signed Young diagrams from which we can read off corresponding orbits (see [2]). Similarly, Matsuki and Rossman [5], [8], [9] classified $K$-orbits in the flag variety, and moreover, for classical groups, Matsuki and Oshima introduce symbols (called clans) parametrizing these orbits [6]. Hence the map of $K$-orbits in the flag variety into nilpotent $K$-orbits in $p$ defined above can be expressed in terms of the map of clans into signed Young diagrams.

In this paper, we give the subspaces $\mu((T^*_QX)_x) \subset p$ directly from the clan which parametrizes the corresponding $K$-orbit in the flag variety. We can determine the corresponding signed Young diagram from a generic point of this space.

In this paper we treat the cases $G_{\mathbb{R}} = U(p, q)$, $Sp(n, \mathbb{R})$, and $Sp(p, q)$. The same argument can be applied to groups which have a compact Cartan subgroup. The precise results will appear elsewhere.

We give a brief organization of this paper.

In Section 2, we treat the case of indefinite unitary groups $U(p, q)$. After we recall a standard notation of $U(p, q)$, we treat $K$-orbit ($GL(p, \mathbb{C}) \times GL(q, \mathbb{C})$-orbit) decomposition in the flag variety $X$ in Section 2.1.

In Section 2.2, we recall a symbolic parametrization (called clan) of $K$-orbits in the flag variety in terms of clans along [6]. Since no proofs are given in [6], we give proofs for completeness. We give representatives of $K$-orbits for clans also.

In Section 2.3, we give a dimension formula of $K$-orbits for clans, and in Section 2.4 a graph (called $M$-$O$ graph) which has clans for vertices. The graph shows closure relations between $K$-orbits.
In Section 2.5, after we recall the moment map, we give images of the moment map of the conormal bundles of $K$-orbits for clans.

In Section 2.6, we recall the parametrization of nilpotent $K$-orbits in $p$ by signed Young diagrams. Although we partly follow the treatment of [2], we do not use the Kostant-Sekiguchi correspondence. Our method stands on the following observation: for an element $X$ of a $K$-orbit in $p$, if we can compute the dimension of the kernel of $X^i$ for all $1 \leq i \leq n$, then we can obtain the signed Young diagram of the orbit using such information. We write down tables of signed Young diagrams for clans for $U(1,1)$, $U(2,1)$, $U(2,2)$, and $U(p,2)$ by way of examples.

In Section 3 (resp. in Section 4), we treat the case of real symplectic group $Sp(n,\mathbb{R})$ (resp. indefinite symplectic group $Sp(p,q)$). For applying the case of $U(2p,2q)$ to the case of $Sp(p,q)$, we use another realization of $U(2p,2q)$ in Section 4.

In Section 3.2-1 (resp. in Section 4.3-1), we define a symbolic parametrization (called generalized clan) of $K$-orbits ($GL(n,\mathbb{C})$-orbits (resp. $Sp(p,\mathbb{C}) \times Sp(q,\mathbb{C})$-orbits)) in the flag variety in the same way as in the case of $U(n,n)$ (resp. $U(2p,2q)$). Generalized clans for $Sp(n,\mathbb{R})$ (resp. $Sp(p,q)$) are given as elements of a subset of clans for $U(n,n)$ (resp. $U(2p,2q)$) satisfying some conditions. The notion of generalized clans is an improvement of the notion of clans in [6].

In Section 3.2-2 (resp. in Section 4.3-2), we give representatives of $K$-orbits for generalized clans.

In Section 3.3 (resp. in Section 4.4), we give a graph (called M-O graph) in terms of [6]. Instead of clans, the graph has generalized clans for vertices.

In Section 3.4 (resp. in Section 4.5), we give a dimension formula of $K$-orbits for generalized clans.

In Section 3.5 (resp. in Section 4.6), we give images of the moment map of the conormal bundle of $K$-orbits for generalized clans.

In Section 3.6 (resp. in Section 4.7), we recall the parametrization of nilpotent $K$-orbits in $p$ by signed Young diagrams. Lastly, we write down tables of signed Young diagrams for clans for $Sp(1,\mathbb{R})$, $Sp(2,\mathbb{R})$, and $Sp(3,\mathbb{R})$ (resp. $Sp(1,1)$, $Sp(2,1)$, and $Sp(2,2)$) by way of examples.

**Notation 1.0.1.** Let $\mathbb{N}$ denote the set of positive integers; $\mathbb{N} = \{1,2,\ldots\}$. For $n \in \mathbb{N}$, let an $n \times n$ matrix $E_{ij}$ ($1 \leq i,j \leq n$) denote the matrix unit which has $1$ for the $(i,j)$-entry and $0$ for other entries. Let an $n$-column vector $e_i$ be the vector which has $1$ for the $i$-th entry and $0$ for other entries. For a matrix $A$, let $A_{st}$ be the $(s,t)$-entry of $A$. Let Mat$(m,n)$ be the set of $m \times n$-matrices over $\mathbb{C}$. Let $I_n \in$ Mat$(n,n)$ be the identity matrix, $J_n \in$ Mat$(n,n)$ satisfy $(J_n)_{st} = \delta_{s+t,n+1}$:

$$J_n = \begin{pmatrix} 0 & \cdots & 0 & 1 \\ 0 & \cdots & 1 & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 1 & \cdots & 0 & 0 \end{pmatrix}.$$ 

Let diag$(a_1,\ldots,a_n) \in$ Mat$(n,n)$ be a diagonal matrix

$$\text{diag}(a_1,\ldots,a_n) = \begin{pmatrix} a_1 & 0 & \cdots & 0 \\ 0 & a_2 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & a_n \end{pmatrix}.$$
For a matrix $A \in \text{Mat}(n, n)$ and a subset $\{i(1), i(2), \ldots, i(m)\}$ of $\{1, \ldots, n\}$, we denote by $A_{i(1),i(2),\ldots,i(m)}$ an $m \times m$-matrix whose $(s, t)$-entry is $A_{i(s),i(t)}$. Let $\#S$ denote cardinality of a finite set $S$. For $m$ vectors $\{g_1, \ldots, g_m \mid g_i \in \mathbb{C}^n\}$ $(m < n)$ let $(g_1, \ldots, g_m)$ be vector space spanned by $\{g_1, \ldots, g_m\}$. Let $\mathcal{S}_n$ be the set of permutations of $\{1, \ldots, n\}$.

2. The case of $U(p, q)$

In this section we treat $G_{\mathbb{R}} = U(p, q)$, i.e., $GL(p, \mathbb{C}) \times GL(q, \mathbb{C})$-orbits in the flag variety of $GL(n, \mathbb{C})$ for $n = p + q$. Although we restrict ourselves to the case $G_{\mathbb{R}} = U(p, q)$, $Sp(n, \mathbb{R})$, and $Sp(p, q)$ in the main body of the paper, a preliminary discussion holds for an arbitrary linear connected reductive Lie group $G_{\mathbb{R}}$.

Let $G_{\mathbb{R}}$ be a real classical Lie group with Lie algebra $\mathfrak{g}_{\mathbb{R}}$, $G$ the complexification of $G_{\mathbb{R}}$, $\theta$ a Cartan involution of $\mathfrak{g}_{\mathbb{R}}$. Let $\mathfrak{g}_{\mathbb{R}} = \mathfrak{t}_{\mathbb{R}} + \mathfrak{p}_{\mathbb{R}}$ be the Cartan decomposition corresponding to $\theta$, $\mathfrak{t}$ the complexification of $\mathfrak{t}_{\mathbb{R}}$, $K$ the analytic subgroup of $G$ for $\mathfrak{t}$, and $B$ a Borel subgroup of $G$.

We realize the indefinite unitary group $G_{\mathbb{R}} = U(p, q)$ as a group of matrices $g$ in $GL(n, \mathbb{C})$ which leave invariant a Hermitian form of the signature $(p, q)$ in $G = GL(n, \mathbb{C})$

$$x_1\overline{x_1} + \cdots + x_p\overline{x_p} - x_{p+1}\overline{x_{p+1}} - \cdots - x_n\overline{x_n},$$

i.e.,

$$U(p, q) = \left\{ g \in GL(n, \mathbb{C}) \mid ^t g \left( \begin{array}{cc} I_p & 0 \\ 0 & -I_q \end{array} \right) g = \left( \begin{array}{cc} I_p & 0 \\ 0 & -I_q \end{array} \right) \right\}.$$ 

We fix a Cartan involution $\theta$ of $G_{\mathbb{R}}$ as follows:

$$\theta : g \mapsto \left( \begin{array}{cc} I_p & 0 \\ 0 & -I_q \end{array} \right) g \left( \begin{array}{cc} I_p & 0 \\ 0 & -I_q \end{array} \right).$$

Then we have

(1)

$$\mathfrak{t} = \left\{ \left( \begin{array}{cc} K_{11} & 0 \\ 0 & K_{22} \end{array} \right) \right\} K_{11} \in \text{Mat}(p, p) \land K_{22} \in \text{Mat}(q, q),$$

(2)

$$K = \left\{ \left( \begin{array}{cc} K_{11} & 0 \\ 0 & K_{22} \end{array} \right) \right\} K_{11} \in GL(p, \mathbb{C}) \land K_{22} \in GL(q, \mathbb{C}),$$

and

(3)

$$\mathfrak{p} = \left\{ \left( \begin{array}{cc} 0 & P_{12} \\ P_{21} & 0 \end{array} \right) \right\} P_{12} \in \text{Mat}(p, q) \land P_{21} \in \text{Mat}(q, p).$$

Let $\mathfrak{a}$ be a Cartan subalgebra of $\mathfrak{g}$ consisting of all diagonal matrices of $\mathfrak{g}$, $\mathfrak{b}$ a Borel subalgebra consisting of all upper triangular matrices. This choice corresponds to the choice of simple root system $\Psi$:

$$\Psi := \{ \alpha_1, \alpha_2, \ldots, \alpha_{n-1} \} \subset \mathfrak{a}^*$$

where $\alpha_i \in \mathfrak{a}^*$ satisfies $\alpha_i(E_{jj}) = \delta_{ij} - \delta_{i+1,j}$. 

2.1. \(GL(p, \mathbb{C}) \times GL(q, \mathbb{C})\)-orbits in the flag variety of \(GL(n, \mathbb{C})\). In this section we treat orbit decomposition in a flag variety.

**Definition 2.1.1.** A flag \(x\) of \(GL(n, \mathbb{C})\) is a sequence of \(n + 1\) vector spaces
\[
x = (V_0, V_1, V_2, \ldots, V_n),
\]
satisfying the following two conditions.
1. \(\dim V_i = i\) for all \(0 \leq i \leq n\).
2. \(\{0\} = V_0 \subset V_1 \subset V_2 \subset \cdots \subset V_n = \mathbb{C}^n\).

We denote the set of flags by \(X\). We call \(X\) a flag variety of \(GL(n, \mathbb{C})\).

**Remark 2.1.2.** We fix a \(G\)-equivariant natural isomorphism between \(X\) and \(G/B\). A left coset \(gB \in G/B\) of an element \(g = (g_1 \ldots g_n) \in G\) (here \(g_i (1 \leq i \leq n)\) are elements of \(\mathbb{C}^n\)) can be identified with a flag \(x = (V_0, V_1, \ldots, V_n)\) in the following manner:
\[
V_0 = \{0\}, \ V_i \text{ is spanned by } i\text{-vectors } g_1, g_2, \ldots, g_{i-1}, \text{ and } g_i, \text{ i.e., } V_i = \langle g_1, \ldots, g_i \rangle
\]
for all \(1 \leq i \leq n\).

**Notation 2.1.3.** Let \(V := \mathbb{C}^n\) and let \(\theta\) be an involution of \(V\) such that
\[
\theta : v \mapsto \begin{pmatrix} I_p & 0 \\ 0 & -I_q \end{pmatrix}v.
\]

Let \(V_+\) and \(V_-\) be the eigenspaces in \(V\) under \(\theta\) for eigenvalues +1 and -1, respectively:
\[
V_+ := \langle e_1, \ldots, e_p \rangle \quad \text{and} \quad V_- := \langle e_{p+1}, \ldots, e_n \rangle.
\]

**Remark 2.1.4.** This gives a decomposition into a direct sum
\[
(4) \quad V = V_+ \oplus V_-.
\]

**Notation 2.1.5.** Let \(\pi_+\) be the first projection and \(\pi_-\) the second projection with respect to the decomposition (4). Let \(V_{i,+}, V_{i,-}\) be subspaces of \(V_i\) such that
\[
V_{i,+} := V_i \cap V_+ \quad \text{and} \quad V_{i,-} := V_i \cap V_-.
\]
We have
\[
\pi_-(V_i) + V_j = \pi_+(V_i) + V_j = (V_i + V_-) \cap V_+ + V_j
\]
for \(i < j\). Because \(K = \{g \in GL(n, \mathbb{C}) \mid gV_+ = V_+, gV_- = V_-\}\), we have
\[
\dim(kV_i \cap V_+) = \dim(V_i \cap k^{-1}V_+) = \dim(V_i \cap V_+)
\]
for all \(k \in K\), and so on. Therefore we have the following proposition.

**Proposition 2.1.6.** For a flag \(x = (V_0, V_1, \ldots, V_n) \in X\), \(\dim V_{i,+}\), \(\dim V_{i,-}\), and \(\dim(\pi_+(V_i) + V_j)\) for \(0 \leq i \leq j \leq n\) are invariant under the actions of \(K\).

**Definition 2.1.7.** For a flag \(x = (V_0, V_1, \ldots, V_n) \in X\), \(0 \leq i \leq n\), and \(i < j\), we put
\[
(i;+) := \dim V_{i,+},
\]
\[
(i;-) := \dim V_{i,-},
\]
\[
(i;N) := i - (i;+) - (i;-), \text{ and}
\]
\[
(i;j) := \dim(\pi_+(V_i) + V_j).
\]
Remark 2.1.8. We have \( \dim \pi_{+}(V_i) = (i;+) + (i;N) \) and \( \dim \pi_{-}(V_i) = (i;-) + (i;N) \).

2.2. **A symbolic parametrization of \( GL(p, \mathbb{C}) \times GL(q, \mathbb{C}) \)-orbits.** In this section we recall a symbolic parametrization of \( K \)-orbits in \( X \) and give representatives of \( K \)-orbits for the parameters.

2.2-1. **Clans for \( U(p,q) \).** We give a set consisting of sequences of \( n \) symbols in the following definition. This set will parametrize \( K \)-orbits in the flag variety \( X \).

**Definition 2.2.1 (Clan).** (See [6].) An *indication* for \( U(p,q) \) is an ordered set \( (c_1 \cdots c_n) \) of \( n \) symbols satisfying the following four conditions.

1. A symbol \( c_i \) is +, −, or an element of \( \mathbb{N} \) for \( 1 \leq i \leq n \).
2. If \( c_i \in \mathbb{N} \), then there exists a unique \( j \neq i \) with \( c_j = c_i \), i.e.,
   \[
   \# \{i \mid c_i = a\} = 0 \text{ or } 2 \quad \text{for any } a \in \mathbb{N}.
   \]
3. The difference between numbers of + and − in indications \( (c_1 \cdots c_n) \) coincides with the difference of signatures of the Hermitian form defining the group \( \mathbb{R} \):
   \[
   \# \{i \mid c_i = +\} - \# \{i \mid c_i = -\} = p - q.
   \]
4. If \( c_i \in \mathbb{N}, a \in \mathbb{N}, \text{ and } a < c_i, \) then there exist some \( j \) such that \( c_j = a \).

We define an equivalence relation between two indications as follows. Two indications \( (c_1 \cdots c_n) \) and \( (c'_1 \cdots c'_n) \) are regarded as equivalent if and only if there exists a permutation \( \sigma \in \mathfrak{S}_m \) with \( m := \max \{c'_i \in \mathbb{N}\} \) such that

\[
c_i = \begin{cases} 
\sigma(c'_i) & \text{if } c'_i \in \mathbb{N}, \\
+ & \text{if } c'_i = +, \\
- & \text{if } c'_i = -,
\end{cases}
\]

for \( 1 \leq i \leq n \). A *clan* is an equivalence class of the indications with respect to the equivalence relation. For example, \((2, 2, 1 + 1 = 1 1 2 + 2 -)\) as a clan. We denote the set of clans for \( U(p,q) \) by \( \mathcal{C}(U(p,q)) \). By abuse of notation sometimes we represent a clan \( \gamma \) by an indication belonging to the clan \( \gamma \).

**Definition 2.2.2 (Standard indication).** If an indication \( (c_1 \cdots c_n) \) of a clan satisfies the following condition, we call it *standard*.

If \( c_i = c_j = a \in \mathbb{N} \) for \( i < j \), \( c_s = c_t = b \in \mathbb{N} \) for \( s < t \), and \( i < s \), then \( a < b \).

Obviously, every clan has a unique standard indication.

**Example 2.2.3.** The set \( \mathcal{C}(U(2,1)) \) consists of six clans:

\[
\mathcal{C}(U(2,1)) = \left\{ + + - , + - + , - + + , 1 + 1 , 1 1 + , 1 + 1 \right\}.
\]

Here \((+ + -)\), for example, is denoted by \(+ + -\), for simplicity.

**Example 2.2.4.** The set \( \mathcal{C}(U(2,2)) \) consists of 21 clans:
\[ C(U(2,2)) = \begin{cases} + + - -, + - + -, + - - +, - + + - \\ - + - +, - - + +, 1 1 + -, 1 1 - + \\ + 1 1 1 -, - 1 1 1 +, + - 1 1, - + 1 1 \\ 1 + 1 1 -, 1 - 1 +, + 1 1 - 1, - 1 1 + 1 \\ 1 + - 1, 1 - + 1, 1 1 2 2, 1 2 1 2 \\ 1 2 2 1 \end{cases} \]

Proposition 2.2.5. For a clan \( (c_1 \cdots c_n) \), suppose \( c_i = c_j \in \mathbb{N} \) and \( i < j \). If \( m = j \), then

\[ \# \{ a \in \mathbb{N} \mid c_s = c_t = a \text{ for } s \leq i < m < t \} = \# \{ a \in \mathbb{N} \mid c_s = c_t = a \text{ for } s \leq i-1 < m < t \}. \]

If \( m < j \), then

\[ \# \{ a \in \mathbb{N} \mid c_s = c_t = a \text{ for } s \leq i < m < t \} = \# \{ a \in \mathbb{N} \mid c_s = c_t = a \text{ for } s \leq i-1 < m < t \} + 1. \]

Proposition 2.2.6. For an \( x = (V_0, V_1, \ldots, V_n) \in X \), there exists a clan \( \gamma = (c_1 \cdots c_n) \in C(U(p,q)) \) satisfying the following conditions.

(5) \( (i; N) = \# \{ a \in \mathbb{N} \mid c_s = c_t = a \text{ for } s \leq i < t \} \),

(6) \( (i; +) = \# \{ l \mid c_l = + \text{ for } l \leq i \} + \# \{ a \in \mathbb{N} \mid c_s = c_t = a \text{ for } s < t \leq l \} \),

(7) \( (i; -) = \# \{ l \mid c_l = - \text{ for } l \leq i \} + \# \{ a \in \mathbb{N} \mid c_s = c_t = a \text{ for } s < t \leq l \} \),

(8) \( (i; j) = j + \# \{ a \in \mathbb{N} \mid c_s = c_t = a \text{ for } s \leq i < j < t \} \)

for all \( 1 \leq i \leq n \) and \( i < j \). Such a clan \( \gamma \) is determined uniquely by the flag \( x \).

We call \( \gamma \) the clan of \( x \).

Proof. For a flag \( x = (V_0, V_1, \ldots, V_n) \in X \), such a clan will be given inductively by the following procedure.

1. First of all, we remark that one of \( (1; +) \), \( (1; -) \), and \( (1; N) \) is 1, and others are 0.
   (a) If \( (1; +) = 1 \), then we set \( c_1 = + \).
   (b) If \( (1; -) = 1 \), then we set \( c_1 = - \).
   (c) If \( (1; N) = 1 \), then we set \( c_1 = 1 \).

2. Suppose we have already obtained \( (c_1 \cdots c_{i-1}) \). We remark that
   \( (i; N) - (i - 1; N) = 1, 0, \text{ or } -1 \).
   (a) If \( (i; N) - (i - 1; N) = 1 \), then we set \( c_i = a \). Here,
   \[ a = 1 + \max \{ b \in \mathbb{N} \mid c_j = b, j < i \} \]
   (b) If \( (i; N) - (i - 1; N) = 0 \), then we have either \( (i; +) - (i - 1; +) = 1 \) or \( (i; -) - (i - 1; -) = 1 \).
      (b-i) If \( (i; +) - (i - 1; +) = 1 \), then we set \( c_i = + \).
      (b-ii) If \( (i; -) - (i - 1; -) = 1 \), then we set \( c_i = - \).
   (c) If \( (i; N) - (i - 1; N) = -1 \), then we have
   \[ (i; +) - (i - 1; +) = 1 \text{ and} \]
   \[ (i; -) - (i - 1; -) = 1. \]
We will choose a $j < i$ and put $c_i = c_j \in \mathbb{N}$ as follows. It follows from (9) and (10), we have a nonzero $v_+ \in \pi_+(V_{i-1})$ and a nonzero $v_- \in \pi_-(V_{i-1})$ such that $v_+ \not\in V_{i-1}$, $v_- \not\in V_{i-1}$ and

$$V_i = V_{i-1} + \langle v_+, v_- \rangle.$$ 

We have a unique $j \leq i - 1$ such that

$$v_+ \in \pi_+(V_j) + V_{i-1} \quad \text{and} \quad v_+ \not\in \pi_+(V_{j-1}) + V_{i-1}.$$

We set $c_i = c_j$. We will show that $(c_1 c_2 \ldots c_n)$ is a clan and satisfies the conditions of Proposition 2.2.6. Let $v_+ = v_+^\prime + v_+^\prime\prime$, such that $v_+^\prime \in \pi_+(V_j)$ and $v_+^\prime\prime \in V_{i-1}$. Then $v_+^\prime \not\in \pi_+(V_{j-1})$ and $v_+^\prime\prime \in V_j$. Because of $v_+^\prime \not\in V_{i'}$ for all $j < i' < i$,

$$(j; i') = \dim(\pi_+(V_j) + V_{i'})$$

$$= \dim(\pi_+(V_{j-1}) + V_{i'} + \langle v_+^\prime \rangle)$$

$$= \dim(\pi_+(V_{j-1}) + V_{i'}) + 1$$

$$= (j - 1; i') + 1.$$

Thus, if $i' < i$, then $c_j = c_i = 1$ for some $i' < t$. So, $c_j = c_i$ for some $i \leq t$.

On the other hand, we have $V_j = V_{j-1} + \langle v_+^\prime + v_-^\prime \rangle$ for some nonzero $v_-^\prime \in V_-$. Then

$$(j; i) = \dim(\pi_+(V_j) + V_i)$$

$$= \dim(\pi_+(V_{j-1} + \langle v_+^\prime + v_-^\prime \rangle) + V_i)$$

$$= \dim(\pi_+(V_{j-1}) + V_i)$$

$$= \dim(\pi_+(V_{j-1}) + V_i)$$

$$= (j - 1; i).$$

Thus, $c_j \neq c_i$ for $i < t$. This means that $c_i = c_j$.

By induction, we have constructed a clan $\gamma = (c_1 \ldots c_n)$ of $x$. 

\begin{proposition}
Conversely, for any clan $\gamma \in C(U(p,q))$, there exists a flag $x \in X$ such that $\gamma$ is the clan of $x$.
\end{proposition}

\begin{proof}
For a clan $\gamma = (c_1 \ldots c_n) \in C(U(p,q))$, such a flag will be given inductively as follows. For $1 \leq i \leq n$ we will choose a nonzero $v_i \in V$ such that $V_i = \langle v_1, \ldots, v_i \rangle$.

1. Let $V_0 = \{0\}$.
2. Suppose we have already obtained $(V_0, V_1, \ldots, V_{i-1})$.
   a) If $c_i = +$, then we set $v_i = e_s \in V_+$ for

   $$(11) \quad s = \min \{ l \mid 1 \leq l \leq p \quad \text{and} \quad e_l \not\in \pi_+(V_{i-1}) \}.$$ 

   b) If $c_i = -$, then we set $v_i = e_t \in V_-$ for

   $$(12) \quad t = \min \{ l \mid p < l \leq n \quad \text{and} \quad e_l \not\in \pi_-(V_{i-1}) \}.$$ 

   c) If $c_i = c_j \in \mathbb{N}$ for some $i < j$, then we set $v_i = e_s + e_t$ for an $s$ satisfying

   $$(11) \quad \text{and a } t \text{ satisfying } (12).$$

   d) If $c_i = c_j \in \mathbb{N}$ for $j < i$, then we set $v_i = \theta(v_j)$.

3. Let $V_i = V_{i-1} \oplus \langle v_i \rangle$.

We have constructed a flag $x = (V_0, V_1, \ldots, V_n) \in X$ such that $\gamma$ is a clan of $x$. 
\end{proof}
For \( \{v_1, \ldots, v_n\} \) given in previous proof, let \( g = (v_1 \cdots v_n) \in G \). Since all flags in \( X \) corresponding to the clan \( \gamma \in \mathcal{C}(U(p, q)) \) belong to the \( K \)-orbit \( Kgb = \{ kgb \mid k \in K, b \in B \} \), we have the following theorem.

**Theorem 2.2.8.** Clans in \( \mathcal{C}(U(p, q)) \) parametrize \( K \)-orbits in the flag variety \( X \) via the correspondence in Proposition 2.2.6.

2.2-2. Representatives of \( GL(p, \mathbb{C}) \times GL(q, \mathbb{C}) \)-orbits in \( GL(n, \mathbb{C})/B \). For a clan \( \gamma \), we will give an element \( g \in G \) such that \( \gamma \) is a clan of the flag \( x = gB \).

**Definition 2.2.9** (Signed clan). A signed clan of a clan \( \gamma = (c_1 \cdots c_n) \) is an ordered set \( (d_1 \ldots d_n) \) of \( n \) symbols \( +, -, a_+ \) and \( a_- \) for some \( a \in \mathbb{N} \) satisfying the following two conditions.

1. If \( c_i = + \), then \( d_i = + \). If \( c_i = - \), then \( d_i = - \).
2. If \( c_i = c_j = a \) for some \( a \in \mathbb{N} \), then \( (d_i, d_j) = (a_+, a_-) \) or \( (a_-, a_+) \).

**Example 2.2.10.** There are four signed clans of \((1 1 2 2)\):

\[
(1_+ 1_+ 2_+ 2_-), (1_+ 1_- 2_+ 2_+), (1_- 1_+ 2_+ 2_-), \text{ and } (1_- 1_- 2_+ 2_+).
\]

**Remark 2.2.11.** As is seen above, there are several choices of signatures for a clan. We can always adapt the special choice such that \((d_i, d_j) = (a_+, a_-)\) for \( i < j \), which causes no problem for \( U(p, q) \). However, other choices of signature are appropriate for other classical groups, hence we admit an ambiguity of signatures.

**Definition 2.2.12.** For a signed clan \( \delta = (d_1 \ldots d_n) \), we say a signature of \( d_i \) is plus if \( d_i = + \) or \( a_+ \), minus if \( d_i = - \) or \( a_- \).

**Notation 2.2.13.** For each clan \( \gamma \), let \( Q_\gamma \) be a \( K \)-orbit in the flag variety \( X \) corresponding to \( \gamma \) via the parametrization of Theorem 2.2.8, i.e.,

\[
Q_\gamma = Kgb.
\]

Although we already have representatives \( g \in G \) of \( K \)-orbits for clans in the proof of Proposition 2.2.7, we give representatives in the following theorem.

**Theorem 2.2.14.** For a clan \( \gamma \in \mathcal{C}(U(p, q)) \) and a signed clan \( \delta = (d_1 \ldots d_n) \) of \( \gamma \), fix a permutation \( \sigma \in S_n \) satisfying the following condition.

\[
(13) \quad 1 \leq \sigma(i) \leq p \quad \text{if the signature of } d_i \text{ is plus}, \\
1 \leq \sigma(i) \leq n \quad \text{if the signature of } d_i \text{ is minus}.
\]

Then the representative \( g(\delta) := g(\delta, \sigma) = (g_1 \ldots g_n) \) is a representative of \( Q_\gamma \), i.e., \( Q_\gamma = Kg(\delta)B \). Here \( g_i \in V \) are the column vectors defined as follows:

- If \( d_i = \pm \), then \( g_i = e_{\sigma(i)} \).
- If \( d_i = a_+ \), \( d_j = a_- \), then
  \[
g_i = \frac{1}{\sqrt{2}} (e_{\sigma(i)} + e_{\sigma(j)}) \quad \text{and} \quad g_j = \frac{1}{\sqrt{2}} (-e_{\sigma(i)} + e_{\sigma(j)}) \).
\]

**Remark 2.2.15.** The representative \( g(\delta) \in GL(n, \mathbb{C}) \) given in Theorem 2.2.14 is real and orthogonal:

\[
g(\delta)^{-1} = \dagger g(\delta).
\]
Example 2.2.16. For a signed clan $\delta = ( + \ 1_+ - 2_+ 1_- 2_- )$, if we choose a permutation

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 3 & 5 & 2 & 6 & 4 \end{pmatrix},$$

we get a representative

$$g = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & 0 & 0 & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 0 & \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & 0 & 0 & \frac{1}{\sqrt{2}} & 0 \end{pmatrix}.$$  

Here $\sigma = \begin{pmatrix} 1 & 2 & \cdots \end{pmatrix}$ means $\sigma(1) = i_1$, $\sigma(2) = i_2$, and so on.

2.3. Dimensions of $GL(p,\mathbb{C}) \times GL(q,\mathbb{C})$-orbits in $GL(n,\mathbb{C})/B$. In this section we will give a dimension formula of $K$-orbits for clans. For calculation of a dimension, we need some preparations.

Notation 2.3.1. Suppose $x = (V_1, \ldots, V_n)$ be a flag. We take a complementary subspace $W_{i,+}$ of $\pi_+(V_i)$ in $V_+$ (resp. $W_{i,-}$ of $\pi_-(V_i)$ in $V_-$):

$$V_+ = \pi_+(V_i) \oplus W_{i,+} \quad (\text{resp. } V_- = \pi_-(V_i) \oplus W_{i,-}).$$

We also take a complementary subspace $W'_i$ of $V_i$ in $\pi_+(V_i) \oplus \pi_-(V_i)$:

$$\pi_+(V_i) \oplus \pi_-(V_i) = V_i \oplus W'_i.$$  

We have

$$V = V_i \oplus W_{i,+} \oplus W_{i,-} \oplus W'_i \quad \text{and} \quad \dim W'_i = (i; N).$$

Remark 2.3.2. If $v \in V_i$, then $\pi_+(v) \in \pi_+(V_i)$. If $v \in W_{i,+}$ and $v \neq 0$, then $\pi_+(v) = v \not\in \pi_+(V_i)$. If $v \in W'_i$, then $\pi_+(v) \in \pi_+(V'_i)$.

Proposition 2.3.3. Let $(c_1 \cdots c_n)$ be the clan of a flag $x = (V_0, \cdots, V_n)$. If

$$v = v' + w_+ + w_- + w',$$

$v' \in V_i$, $w_+ \in W_{i,+}$, $w_- \in W_{i,-}$, $w' \in W'_i$, and

$$V_{i+1} = V_i + \langle v \rangle,$$

then $c_{i+1}$ is as follows:

1. If $w_+ \neq 0$ and $w_- = 0$, then we have $c_{i+1} = +$.
2. If $w_+ = 0$ and $w_- \neq 0$, then we have $c_{i+1} = -$.
3. If $w_+ \neq 0$ and $w_- \neq 0$, then we have

$$c_{i+1} = 1 + \max\{c_s \mid c_s \in \mathbb{N} \text{ and } 1 \leq s \leq i \}.$$  

4. If $w_+ = 0$ and $w_- = 0$, then we have a unique $j$ such that

$$v \in \pi_+(V_j) + V_i \quad \text{and} \quad v \not\in \pi_+(V_{j-1}) + V_i.$$  

Then we have $c_{i+1} = c_j \in \mathbb{N}$.  

Proof. 1. If \( w_+ \neq 0 \) and \( w_- = 0 \), then
\[
\pi_+(V_{i+1}) = \pi_+(V_i) + (\pi_+(v') + w_+ + \pi_+(w')) = \pi_+(V_i) \oplus \langle w_+ \rangle
\]
and \( \pi_-(V_{i+1}) = \pi_-(V_i) \) by Remark 2.3.2. Then,
\[
\dim \pi_+(V_{i+1}) = \dim \pi_+(V_i) + 1 \quad \text{and} \quad \dim \pi_-(V_{i+1}) = \dim \pi_-(V_i).
\]
Thus,
\[
\dim W'_{i+1} = \dim \pi_+(V_{i+1}) + \dim \pi_-(V_{i+1}) - \dim V_{i+1}
= \dim \pi_+(V_i) + 1 + \dim \pi_-(V_i) - (i + 1)
= \dim \pi_+(V_i) + \dim \pi_-(V_i) - \dim V_i
= \dim W'_i
\]
by equation (14). So \((i + 1; N) = (i; N)\). On the other hand,
\[
(i + 1; +) = \dim \pi_+(V_{i+1}) - (i + 1; N)
= \dim \pi_+(V_i) + 1 - (i; N) = (i; +) + 1
\]
by Remark 2.1.8. Therefore \( c_{i+1} = + \).

2. If \( w_+ = 0 \) and \( w_- \neq 0 \), then \( c_{i+1} = - \) by the same argument of 1.

3. If \( w_+ \neq 0 \) and \( w_- \neq 0 \), then
\[
\dim \pi_+(V_{i+1}) = \dim \pi_+(V_i) + 1 \quad \text{and} \quad \dim \pi_-(V_{i+1}) = \dim \pi_-(V_i) + 1.
\]
Thus, \((i + 1; N) = (i; N) + 1\) by the same argument of 1. Therefore,
\[
c_{i+1} = 1 + \max \{ c_s \mid c_s \in N \quad \text{and} \quad 1 \leq s \leq i \}.
\]

4. By the end of the proof of Proposition 2.2.6, it is clear for \( w_+ = 0 \) and \( w_- = 0 \).
\( \square \)

Corollary 2.3.4. For a flag \( x = (V_0, \ldots, V_n) \) and a clan \((c_1 \cdots c_n)\) of \( x\), if \( c_i = c_j \in N \) for \( i < j \), then there exist \( v_+ \in V_+ \) and \( v_- \in V_- \) such that
\[
V_i = V_{i-1} + \langle v_+ + v_- \rangle \quad \text{and} \quad V_j = V_{j-1} + \langle v_+, v_- \rangle.
\]

Proof. By the proof of Proposition 2.3.3, there exist \( v_i \in V \) such that
\[
\pi_+(v_i) \in V_+ - \pi_+(V_{i-1}), \quad \pi_-(v_i) \in V_- - \pi_-(V_{i-1}),
\]
and \( v_j \in V_{j-1} \oplus W'_{j-1} \) such that
\[
v_j \in \pi_+(V_i) + V_{j-1} \quad \text{and} \quad v_j \notin \pi_+(V_{i-1}) + V_{j-1}.
\]
Since
\[
v_j \in \pi_+(V_i) + \pi_-(V_i) + V_{j-1} \quad \text{and} \quad v_j \notin \pi_+(V_{i-1}) + \pi_-(V_{i-1}) + V_{j-1},
\]
there exist \( v'_+ \in \pi_+(V_i), v'_- \in \pi_-(V_i) \) and \( v_{j-1} \in V_{j-1} \) such that
\[
v_j = v'_+ + v'_- + v_{j-1}.
\]
So, we have
\[
V_j = V_{j-1} + \langle v'_+ + v'_- \rangle.
\]
Since there exist \( v_s \in V_{i-1} \) (\( 1 \leq s \leq i - 1 \)) with \( v_i \) span \( V_i \),
\[
V_i = \langle v_s \mid 1 \leq s \leq i \rangle,
\]
We have
\[ \pi_+(V_i) = \langle \pi_+(v_s) \mid 1 \leq s \leq i \rangle, \quad \text{and} \quad \pi_-(V_i) = \langle \pi_-(v_s) \mid 1 \leq s \leq i \rangle. \]
Thus, there exist \( a_s, b_s \in \mathbb{C} \) for \( 1 \leq s \leq i \) such that
\[
v'_i = \sum_{s=1}^{i} a_s \pi_+(v_s) \quad \text{and} \quad v'_i = \sum_{s=1}^{i} b_s \pi_-(v_s).
\]
Here, \( a_i \neq b_i \). We have
\[
V_j = V_{j-1} + \left( \sum_{s=1}^{i} (a_s \pi_+(v_s) + b_s \pi_-(v_s)) \right).
\]
By simple calculations, we have
\[
V_i = V_{i-1} + \left( \sum_{s=1}^{i} (a_s - b_s)v_s \right)
\]
\[
V_j = V_{j-1} + \left( \sum_{s=1}^{i} (-a_s - b_s)v_s + 2 \sum_{s=1}^{i} (a_s \pi_+(v_s) + b_s \pi_-(v_s)) \right)
\]
\[
= V_{j-1} + \left( \sum_{s=1}^{i} (a_s - b_s)(\pi_+(v_s) - \pi_-(v_s)) \right)
\]
\[
= V_{j-1} + \left( \sum_{s=1}^{i} (a_s - b_s)\pi_+(v_s), \sum_{s=1}^{i} (a_s - b_s)\pi_-(v_s) \right)
\]
The two vectors
\[
v_+ = \sum_{s=1}^{i} (a_s - b_s)\pi_+(v_s) \quad \text{and} \quad v_- = \sum_{s=1}^{i} (a_s - b_s)\pi_-(v_s)
\]
satisfy the conditions of the corollary. \( \square \)

**Proposition 2.3.5.** For a clan \( \gamma \in \mathcal{C}(U(p,q)) \), we have the following equation.
\[
\sum_{c_l=+} (p - \# \{ l \mid c_l = + \text{ for } l \leq i \})
\]
\[
- \# \{ a \in \mathbb{N} \mid c_a = c_t = a \text{ for } s < t \leq i \}
\]
\[
+ \sum_{c_l=-} (q - \# \{ l \mid c_l = - \text{ for } l \leq i \})
\]
\[
- \# \{ a \in \mathbb{N} \mid c_a = c_t = a \text{ for } s < t \leq i \}
\]
\[
+ \sum_{c_j=c_l \in \mathbb{N}, j<i} (n - \# \{ l \mid c_l = + \text{ or } - \text{ for } l \leq i \})
\]
\[
- 2 \# \{ a \in \mathbb{N} \mid c_a = c_t = a \text{ for } s < t \leq i \}
\]
\[
= \frac{1}{2} (p(p-1) + q(q-1)).
\]
Proof. The left hand side of the equation is

\[
\sum_{c_i = +} (p - \# \{ l \mid c_i = + \text{ for } l \leq i \} - \# \{ a \in N \mid c_s = c_t = a \text{ for } s < t \leq i \})
\]

\[
\sum_{c_i = c_j \in N, j < i} (p - \# \{ l \mid c_i = + \text{ for } l \leq i \} - \# \{ a \in N \mid c_s = c_t = a \text{ for } s < t \leq i \})
\]

\[
\sum_{c_i = -} (q - \# \{ l \mid c_i = - \text{ for } l \leq i \} - \# \{ a \in N \mid c_s = c_t = a \text{ for } s < t \leq i \})
\]

\[
\sum_{c_i = c_j \in N, j < i} (q - \# \{ l \mid c_i = - \text{ for } l \leq i \} - \# \{ a \in N \mid c_s = c_t = a \text{ for } s < t \leq i \})
\]

= \sum_{i=1}^{p} (p - i) + \sum_{i=1}^{q} (q - i) = \frac{1}{2} (p(p - 1) + q(q - 1)). \qed

Proposition 2.3.6. For a clan \( \gamma \in C(U(p, q)) \), we have the following equation.

\[
\sum_{c_i = c_j \in N, i < j} (j - i - \# \{ \hat{c}_i \in N \mid i < t < j \text{ and } \hat{c}_t < \hat{c}_i \})
\]

\[= \sum_{c_i = c_j \in N, i < j} (j - i - \# \{ a \in N \mid c_s = c_t = a \text{ for } s < i < t < j \}),\]

where, \((\hat{c}_1 \cdots \hat{c}_n)\) is the standard indication.

Proof. A clan \((c_1 \cdots c_n)\) and its standard indication \((\hat{c}_1 \cdots \hat{c}_n)\) satisfy

\[
\sum_{c_i = c_j \in N, i < j} \# \{ \hat{c}_t \in N \mid i < t < j \text{ and } \hat{c}_t < \hat{c}_i \}
\]

\[= \sum_{c_i = c_j \in N, i < j} \# \{ a \in N \mid c_s = c_t = a \text{ for } s < i < t < j \}. \qed\]

Definition 2.3.7. We define a length \( \ell(\gamma) \) of a clan \( \gamma = (c_1 \cdots c_n) \in C(U(p, q)) \) by the value in the previous proposition:

\[
\ell(\gamma) := \sum_{c_i = c_j \in N, i < j} (j - i - \# \{ \hat{c}_t \in N \mid i < t < j \text{ and } \hat{c}_t > \hat{c}_i \})
\]

\[= \sum_{c_i = c_j \in N, i < j} (j - i - \# \{ a \in N \mid c_s = c_t = a \text{ for } s < i < t < j \}),\]

where, \((\hat{c}_1 \cdots \hat{c}_n)\) is the standard indication of \( \gamma \).

Proposition 2.3.8. For \( \gamma \in C(U(p, q)) \), we have the dimension and the codimension of \( Q_\gamma = Kg(\delta)B \):

\[
\dim Q_\gamma = \ell(\gamma) + \frac{1}{2} (p(p - 1) + q(q - 1)),
\]

\[
\text{codim } Q_\gamma = pq - \ell(\gamma).
\]
Proof. By Proposition 2.3.3, flags \((V_1, \ldots, V_n)\) in \(Q_\gamma\) satisfy the following conditions.

1. If \(c_i = +\), then
   \[
   V_i/V_{i-1} \subset (V_{i-1} + W_{i-1}^+, \oplus W_i^+)/V_{i-1}.
   \]

2. If \(c_i = -\), then
   \[
   V_i/V_{i-1} \subset (V_{i-1} + W_{i-1}^-, \oplus W_i^-)/V_{i-1}.
   \]

3. If \(c_i = c_j \in \mathbb{N}\) for \(i < j\), then
   \[
   V_i/V_{i-1} \subset V/V_{i-1},
   V_j/V_{j-1} \subset (V_{j-1} + \pi_+(V_i))/V_{j-1}.
   \]

The dimension of \(K\)-orbit \(Q_\gamma\) is the sum of the following four values.

\[
(17) \quad \sum_{c_i = +} \dim(V_{i-1} + W_{i-1}^+, \oplus W_i^+) - \dim V_{i-1} - 1),
\]
\[
(18) \quad \sum_{c_i = -} \dim(V_{i-1} + W_{i-1}^-, \oplus W_i^-) - \dim V_{i-1} - 1),
\]
\[
(19) \quad \sum_{c_i = c_j \in \mathbb{N}, i < j} \dim V - \dim V_{i-1} - 1),
\]
\[
(20) \quad \sum_{c_i = c_j \in \mathbb{N}, i < j} \dim(V_{j-1} + \pi_+(V_i)) - \dim V_{j-1} - 1).
\]

The dimension of \(K\)-orbit \(Q_\gamma\) is the sum of the following four values.

\[
(17) \quad \sum_{c_i = +} \dim(V_{i-1} + W_{i-1}^+, \oplus W_i^+) - \dim V_{i-1} - 1),
\]
\[
(18) \quad \sum_{c_i = -} \dim(V_{i-1} + W_{i-1}^-, \oplus W_i^-) - \dim V_{i-1} - 1),
\]
\[
(19) \quad \sum_{c_i = c_j \in \mathbb{N}, i < j} \dim V - \dim V_{i-1} - 1),
\]
\[
(20) \quad \sum_{c_i = c_j \in \mathbb{N}, i < j} \dim(V_{j-1} + \pi_+(V_i)) - \dim V_{j-1} - 1).
\]

Four equations

\[
(17) \quad \sum_{c_i = +} (p - \#\{l \mid c_l = + \text{ for } l \leq i\} - \#\{a \in \mathbb{N} \mid c_s = c_t = a \text{ for } s < t \leq i\}),
\]
\[
(18) \quad \sum_{c_i = -} (q - \#\{l \mid c_l = - \text{ for } l \leq i\} - \#\{a \in \mathbb{N} \mid c_s = c_t = a \text{ for } s < t \leq i\}),
\]
\[
(19) \quad \sum_{c_i = c_j \in \mathbb{N}, i < j} (n - i),
\]
and
\[
(20) \quad \sum_{c_i = c_j \in \mathbb{N}, i < j} \#\{a \in \mathbb{N} \mid c_s = c_t = a \text{ for } s < i < j < t\},
\]
and Proposition 2.3.5, lead

\[
(17) + (18) + \sum_{c_i = c_j \in \mathbb{N}, i < j} (n - \#\{l \mid c_l = + \text{ or } - \text{ for } l \leq j\})
\]
\[
- 2\#\{a \in \mathbb{N} \mid c_s = c_t = a \text{ for } s < t \leq j\})
\]
\[
= \frac{1}{2}(p(p - 1) + q(q - 1))
\]
and

\[ (19) + (20) - \sum_{c_i = c_j \in \mathbb{N}_{i<j}} (n - \# \{ l \mid c_l = + \text{ or } - \text{ for } l \leq j \}) \]
\[ - 2 \# \{ a \in \mathbb{N} \mid c_a = c_t = a \text{ for } s < t \leq j \} \]
\[ = \sum_{c_i = c_j \in \mathbb{N}_{i<j}} (-i + \# \{ l \mid c_l = + \text{ or } - \text{ for } l \leq j \}) + \# \{ l \mid c_l \in \mathbb{N} \text{ for } l \leq j \} \]
\[ - \# \{ s \mid c_s = c_t \in \mathbb{N} \text{ for } i < s < j \} \]
\[ = \sum_{c_i = c_j \in \mathbb{N}_{i<j}} (-i + j - \# \{ a \in \mathbb{N} \mid c_a = c_t \in \mathbb{N} \text{ for } i < s < j \}) \]
\[ = \sum_{c_i = c_j \in \mathbb{N}_{i<j}} (-i + j - \# \{ a \in \mathbb{N} \mid c_a = c_t \in \mathbb{N} \text{ for } s < i < t \}) \]
\[ = \ell(\gamma). \]

Therefore,

\[ \dim Q_\gamma = (17) + (18) + (19) + (20) \]
\[ = \frac{1}{2} (p(p-1) + q(q-1)) + \ell(\gamma). \]

Since \( \dim(G/B) = \frac{1}{2} n(n-1) \), the codimension of \( Q_\gamma \) is

\[ \operatorname{codim}(Q_\gamma) = pq - \ell(\gamma). \]

2.4. M-O graph of \( U(p,q) \). We define an oriented graph from which we can read off closure relations between \( K \)-orbits. The graph has clans as vertices in which an edge is labeled by \( i \) for some \( 1 \leq i \leq n - 1 \).

**Definition 2.4.1.** Let \( \Gamma'(U(p,q)) \) be the set of all triples \((\gamma, \gamma', i)\), with \( \gamma, \gamma' \in C(U(p,q)), 1 \leq i \leq n - 1 \), satisfying the following conditions. Let \( \gamma = (c_1 \cdots c_n) \) and \( \gamma' = (c_1' \cdots c_n') \).

1. \( c_i \neq c_{i+1} \).
2. If \((c_i, c_{i+1}) = (+, -) \text{ or } (-, +)\), then \( c_i' = c_{i+1}' \in \mathbb{N} \) and \( c_j' = c_j \) otherwise:
   \[ \gamma' = (c_1 \cdots c_{i-1} a a c_{i+2} \cdots c_n) \]
   for \( a = 1 + \max \{ b \in \mathbb{N} \mid c_j = b \text{ for } 1 \leq j \leq n \} \).
3. If \((c_i, c_{i+1}) \neq (+, -) \text{ or } (-, +)\), then \((c_i', c_{i+1}') = (c_{i+1}, c_{i}) \) and \( c_j' = c_j \) otherwise:
   \[ \gamma' = (c_1 \cdots c_{i-1} c_{i+1} c_i c_{i+2} \cdots c_n). \]
Example 2.4.2.

\[ \Gamma'(U(2,1)) \]

\[
\left\{
\begin{array}{l}
((+ - +), \ (1 \ 1 +), \ 1), \ ((1 \ 1 +), \ (1 \ 1 +), \ 2) \\
((+ - +), \ (+ \ 1 1), \ 2), \ ((1 \ 1 +), \ (1 \ 1 +), \ 2) \\
((+ + -), \ (+ \ 1 1), \ 2), \ ((+ 1 1), \ (1 \ 1 +), \ 1) \\
((+ + -), \ (+ \ 1 +), \ 1), \ ((1 \ 1 +), \ (+ \ 1 1), \ 1)
\end{array}
\right.
\]

Remark 2.4.3. If \((\gamma,\gamma',i)\in\Gamma'(U(p,q))\), \(\gamma = (c_1 \cdots c_n)\) and \((c_i,c_{i+1}) \neq (+,-)\) or \((-,+)\), then \((\gamma',\gamma,i)\) is also an element of \(\Gamma'(U(p,q))\).

Proposition 2.4.4. If \((\gamma,\gamma',i)\in\Gamma'(U(p,q))\), then

\[ \dim Q_\gamma - \dim Q_{\gamma'} = \pm 1. \]

Proof. We will prove \(\ell(\gamma') - \ell(\gamma) = \pm 1\). Suppose \((\gamma,\gamma',i)\in\Gamma'(U(p,q))\) and \(\gamma = (c_1 \cdots c_n)\). Then the clasps \(\gamma\) and \(\gamma'\) are

\[ \gamma' = (c_1 \cdots c_{i-1} c'_i c'_{i+1} c_{i+2} \cdots c_n) \]

and

\[ (c_i,c_{i+1};c'_i,c'_{i+1}) = (\pm,\mp: a,a), (a,\pm: \pm,a), (\pm,a: \pm,a), \text{or } (a;b,b,a) \]

for some \(a,b \in \mathbb{N}\). We can take \((c_1 \cdots c_n)\) to be the standard indication.

1. If \((c_i,c_{i+1}) = (\pm,\mp)\), then \(c'_i = c'_{i+1} = a\) for some \(a \in \mathbb{N}\). Then we have

\[
\sum_{c_i = c'_i \in \mathbb{N}, i < j} (j-i - \#\{a' \in \mathbb{N} | c'_s = c'_i = a' \text{ for } s < i < t < j\})
\]

and

\[
\sum_{c_i = c'_i \in \mathbb{N}, i < j} (j-i - \#\{a' \in \mathbb{N} | c_s = c_i = a' \text{ for } s < i < t < j\})
\]

Thus, \(\ell(\gamma') - \ell(\gamma) = 1\).

2. If \((c_i,c_{i+1}) = (a,\pm)\) or \((\pm,a)\) for some \(a \in \mathbb{N}\), the second case follows the first case by Remark 2.4.3. We prove that for the case of \((c_i,c_{i+1}) = (a,\pm)\).

If \((c_i,c_{i+1}) = (a,\pm)\), there exist \(j \neq i, i+1\) such that \(c_j = a\).

(a) If \(i+1 < j\), then

\[
(j-i+1) - \#\{a' \in \mathbb{N} | c'_s = c'_i = a' \text{ for } s < i+1 + 1 < j\}
\]

and

\[
(j-i - \#\{a' \in \mathbb{N} | c_s = c_i = a' \text{ for } s < i < t < j\})
\]

Thus, \(\ell(\gamma') - \ell(\gamma) = 1\).

(b) If \(j < i\), then

\[
(i+1) - j - \#\{a' \in \mathbb{N} | c'_s = c'_i = a' \text{ for } s < j < t < i+1\}
\]

and

\[
(i-j - \#\{a' \in \mathbb{N} | c_s = c_i = a' \text{ for } s < j < t < i\})
\]

There is no difference in other terms. Therefore, \(\ell(\gamma') - \ell(\gamma) = 1\).
3. If \((c_i, c_{i+1}) = (a, b)\) for some \(a, b \in \mathbb{N}\) there exist \(j, j' \notin \{i, i+1\}\) such that 
\(c_j = a\) and \(c_{j'} = b\).
(a) For \(i + 1 < j < j'\) or \(i + 1 < j' < j\), the second case follows the first case by Remark 2.4.3.
If \(i + 1 < j < j'\), then we have
\((j - (i + 1)) - \#\{a' \in \mathbb{N} | c'_s = c'_t = a', s < j < i + 1\} \geq 1\)
\(+j' - i - \#\{b' \in \mathbb{N} | c'_s = c'_t = b', s < i < j'\} \geq 1\)
\(-j - i - \#\{a' \in \mathbb{N} | c_a = c_t = a', s < j < t\} \geq 1\)
\(+j' - (i + 1) - \#\{b' \in \mathbb{N} | c_a = c_t = b', s < j < i + 1\} \geq 1\)
\(= (j - (i + 1)) + j' - i - (-1) - (j - i + j' - (i + 1)) = 1\).
There is no difference in other terms. Hence, \(\ell(\gamma) - \ell(\gamma') = 1\).
(b) For \(j < i + 1 < j'\) or \(j' < i + 1 < j\), the second case follows the first case by Remark 2.4.3.
If \(j < i + 1 < j'\), then
\(((i + 1) - j - \#\{a' \in \mathbb{N} | c'_s = c'_t = a', s < j < i + 1\} \geq 1\)
\(+j' - i - \#\{b' \in \mathbb{N} | c'_s = c'_t = b', s < i < j'\} \geq 1\)
\(-(i - j - \#\{a' \in \mathbb{N} | c_a = c_t = a', s < j < t\} \geq 1\)
\(+j' - (i + 1) - \#\{b' \in \mathbb{N} | c_a = c_t = b', s < i + 1 < j'\} \geq 1\)
\(= ((i + 1) - j + j' - i - 1) - (i - j + j' - (i + 1)) = 1\).
There is no difference in other terms. Thus, \(\ell(\gamma') - \ell(\gamma) = 1\).
(c) For \(j' < j < i\) or \(j' < j < i\), the second case follows the first case by Remark 2.4.3.
If \(j < j' < i\), then
\(((i + 1) - j - \#\{a' \in \mathbb{N} | c'_s = c'_t = a', s < j < i + 1\} \geq 1\)
\(+i - j' - \#\{b' \in \mathbb{N} | c'_s = c'_t = b', s < j' < i\} \geq 1\)
\(-(i - j - \#\{a' \in \mathbb{N} | c_a = c_t = a', s < j < t\} \geq 1\)
\+(i + 1) - j' - \#\{b' \in \mathbb{N} | c_a = c_t = b', s < j' < i + 1\} \geq 1\)
\(= ((i + 1) - j - j' - (-1)) - (i - j + (i + 1) - j') = 1\).
Therefore we have \(\ell(\gamma') - \ell(\gamma) = \pm 1\) for all \((\gamma, \gamma', i) \in \Gamma'(U(p, q))\).
We define subsets of $C(U(p, q))$ of which, elements have the same length.

**Definition 2.4.5.** Let $C_m(U(p, q))$ be the set of clans of the length $m$: $$C_m(U(p, q)) := \{ \gamma \in C(U(p, q)) \mid \ell(\gamma) = m \}.$$ Then we have

$$C(U(p, q)) = \bigsqcup_{m \geq 0} C_m(U(p, q)). \tag{21}$$

**Remark 2.4.6.** If $\gamma \in C_0(U(p, q))$, i.e., $\ell(\gamma) = 0$, then $Q_\gamma$ is a closed orbit.

**Proposition 2.4.7.** If $(c_1 \ldots c_n) \in C_0(U(p, q))$, then $c_i = + or -$ for all $i$, i.e.,

$$C_0(U(p, q)) = \{ (c_1 \ldots c_n) \in C(U(p, q)) \mid c_i \not\in N \text{ for all } 1 \leq i \leq n \}.$$ Proof. For $\gamma = (c_1 \ldots c_n) \in C(U(p, q))$, if $c_i = c_j \in N$ for some $i < j$,

$$j - i - \# \{ a \in N \mid c_s = c_t = a, \ s < i < t < j \} > 0.$$ Therefore if $c_i = c_j \in N$, then $\ell(\gamma) > 0$, and if $c_i \not\in N$ for all $1 \leq i \leq n$, then $\ell(\gamma) = 0$.

**Notation 2.4.8.** We denote by $B_i$ the parabolic subgroup of $G$ for the root $-\alpha_i$ and all positive roots:

$$B_i = \{ X \in GL(n, C) \mid X_{st} = 0 \text{ if } s > t \text{ and } (s, t) \neq (i + 1, i) \}.$$ Let

$$\pi_i : G/B \to G/B_i$$

be the canonical projection.

**Remark 2.4.9.** The projection $\pi_i$ sends $(V_0, \ldots, V_n) \in G/B$ to

$$(V_0, \ldots, V_{i-1}, V_{i+1}, \ldots, V_n).$$ For $(\gamma, \gamma', i) \in \Gamma'(U(p, q))$, there are two flags

$$(V_0, \ldots, V_{i-1}, V_i, V_{i+1}, \ldots, V_n)$$

having the clan $\gamma$ and

$$(V_0, \ldots, V_{i-1}, V'_i, V_{i+1}, \ldots, V_n)$$

having the clan $\gamma'$, by Theorem 2.2.14. Therefore we have the following proposition.

**Proposition 2.4.10.** If $(\gamma, \gamma', i) \in \Gamma'(U(p, q))$, then

$$\pi_i(Q_\gamma) = \pi_i(Q'_{\gamma'}).$$

By following [6], we introduce an oriented graph.

**Definition 2.4.11 (M-O graph of $U(p, q)$).** We give subsets $\Gamma_m(U(p, q))$ of $\Gamma'(U(p, q))$ for $m \geq 1$ as follows:

$$\Gamma_m(U(p, q)) := \{ (\gamma, \gamma', i) \in \Gamma'(U(p, q)) \mid \gamma \in C_{m-1}(U(p, q)), \gamma' \in C_m(U(p, q)) \}.$$ The M-O graph of $U(p, q)$ is a finite oriented graph whose vertices are $C(U(p, q))$ and whose oriented edges are $\Gamma(U(p, q)) := \bigcup_{m \in N} \Gamma_m(U(p, q))$. 

Example 2.4.12.

\[
\Gamma_1(U(2, 1)) = \left\{ \begin{array}{c}
((+-+), (11+), 1),
((+-+), (11+), 1),
((+-+), (+11), 2),
((+-+), (+11), 2)
\end{array} \right\} \\
\Gamma_2(U(2, 1)) = \left\{ ((11+), (1+1), 2), ((11+), (1+1), 1) \right\}
\]

\[
\Gamma(U(2, 1)) = \Gamma_1(U(2, 1)) \cup \Gamma_2(U(2, 1)).
\]

If we denote \((\gamma, \gamma', i)\) by \(\gamma \xrightarrow{i} \gamma'\), the M-O graph of \(U(2, 1)\) is as in Figure 1.

Remark 2.4.13. If \((\gamma, \gamma', i) \in \Gamma(U(p, q))\), then

\[
\dim Q_\gamma + 1 = \dim Q_{\gamma'} \quad \text{and} \quad \pi_i(Q_\gamma) = \pi_i(Q_{\gamma'}).
\]

2.5. Images of the moment map. In this section we give the image of the moment map of a fiber of the conormal bundle of the \(K\)-orbit for each clan.

For each point \(x \in X\) the morphism \(G \to X\) sending \(g \in G\) to \(gx\) gives rise to a linear map \(g \to T_x(X)\), the tangent-map at \(x\), and to a dual map \(T^*_x(X) \to g^*\) from the cotangent bundle into the dual of the Lie algebra, the cotangent-map for \(x \in X\) at \(x\). The collection of tangent-maps \(g \to T_x(X)\) at the various points \(x \in X\) gives rise to an algebraic map \(g \times X \to T(X)\), compatible with the projections to the base space \(X\). Composing the cotangent-map with the map forgetting the base point in \(X\), we obtain a canonical map \(T^*(X) \to g^*\) from the cotangent-bundle into the dual of the Lie algebra, denoted by \(\mu\). We call it the moment map of the \(G\)-space \(X\). (for example [1].)

Let \(\varphi\) be the morphism \(G \to X\) sending \(g \in G\) to \(gx\) for \(x \in X\). Then we obtain a linear map \(d\varphi : g = T_e G \to T_x(X)\) and a dual map \((d\varphi)^* : T^*_x(X) \to T^*_x G = g^*\).

Let \(G_x\) be the stabilizer of \(x\) in \(G\) and \(g_x\) its Lie algebra. Since the kernel of \(d\varphi\) is \(g_x\), the image of \((d\varphi)^*\) is \((g_x)^\perp\). Here

\[
(g_x)^\perp := \{ \xi \in g^* \mid \langle \xi, a \rangle = 0 \quad \text{for all} \quad a \in g_x \}
\]

and \(\langle \cdot, \cdot \rangle : g^* \times g \to \mathbb{C}\).

Let \(\psi\) be the morphism \(K \to X\) sending \(k \in K\) to \(kx\). Then we have \(\psi = \varphi|_K\) and \(d\psi = d\varphi|_t\). The image of \(\psi\) is the \(K\)-orbit \(Q\) of \(x\) in \(X\): \(Q = \{kx \mid k \in K\}\).
The fiber \((T^*_Q X)_x\) of conormal bundle \(T^*_Q X\) at \(x\) is
\[
(T^*_Q X)_x = \{ \eta \in (T^* X)_x \mid \langle \eta, u \rangle = 0 \text{ for all } u \in T_x Q \}
= \{ \eta \in (T^* X)_x \mid \langle \eta, d\psi(a) \rangle = 0 \text{ for all } a \in \mathfrak{t} \}
= \{ \eta \in (T^* X)_x \mid \langle \eta, d\varphi(a) \rangle = 0 \text{ for all } a \in \mathfrak{t} \}
= \{ \eta \in (T^* X)_x \mid \langle (d\varphi)^*(\eta), a \rangle = 0 \text{ for all } a \in \mathfrak{t} \}
= \{ \eta \in T^*_Q X \mid (d\varphi)^*(\eta) \in \mathfrak{t}^\perp \}.
\]
By (23) and (24) we have
\[
(d\varphi)^*((T^*_Q X)_x) = (\mathfrak{g}_x)^\perp \cap \mathfrak{t}^\perp.
\]
Hence, the image of the moment map \(\mu((T^*_Q X)_x)\) is \((\mathfrak{g}_x)^\perp \cap \mathfrak{t}^\perp\), where
\[
(\mathfrak{g}_x)^\perp = \{ \xi \in \mathfrak{g}^* \mid \langle \xi, a \rangle = 0 \text{ for all } a \in \mathfrak{g}_x \},
\mathfrak{t}^\perp = \{ \xi \in \mathfrak{g}^* \mid \langle \xi, a \rangle = 0 \text{ for all } a \in \mathfrak{t} \}.
\]
For a representative \(g' \in G\) with \(x = g'B\), we have \(G_x = g' \cdot B \cdot g'^{-1}\) and \(\mathfrak{g}_x = \text{Ad}(g')\mathfrak{b}\). Thus, we have
\[
(\mathfrak{g}_x)^\perp \cap \mathfrak{t}^\perp = (\text{Ad}^*(g')\mathfrak{b})^\perp \cap \mathfrak{t}^\perp.
\]
By identifying \(\mathfrak{g}^*\) with the dual of \(\mathfrak{g}\) by means of a nondegenerate symmetric invariant bilinear form on \(\mathfrak{g}\), the image of the moment map is identified with the vector subspace \((\text{Ad}(g')\mathfrak{b})^\perp \cap \mathfrak{p}\) of \(\mathfrak{g}\). Here \(\mathfrak{b}^\perp\) is a subalgebra of \(\mathfrak{g}\) that is orthogonal to \(\mathfrak{b}\):
\[
\mathfrak{b}^\perp = \{ b \in \mathfrak{g} \mid \beta(b, b') = 0 \text{ for all } b' \in \mathfrak{b} \},
\mathfrak{b}^\perp = \{ b \in \mathfrak{g} \mid b_{ij} = 0 \text{ if } i \leq j \}.
\]
where \(\beta(x, y)\) is the trace of \( ^txy\) on \(\mathfrak{g}\):
\[
\beta(x, y) = \text{tr} ^txy \text{ for } x, y \in \mathfrak{g}.
\]
For a clan \(\gamma = (c_1 \ldots c_n) \in \mathcal{C}(U(p, q))\), fix a signed clan \(\delta\) of \(\gamma\). Let \(g(\delta)\) be a representative given as in Theorem 2.2.14. Then \( ^tg(\delta)^{-1} = g(\sigma)\). We will describe the image of the moment map \(\mu((T^*_Q X)_x) = (\text{Ad}(g(\delta))\mathfrak{b}^\perp) \cap \mathfrak{p}\) for \(x = g(\delta)B\).

We regard \(\mathcal{S}_p \times \mathcal{S}_q \subset \mathcal{S}_n\) as a subgroup of \(\mathcal{S}_n\) as follows:
\[
(\sigma_1, \sigma_2) = \sigma' ; \sigma'(i) = \begin{cases} 
\sigma_1(i) & \text{if } 1 \leq i \leq p, \\
\sigma_2(i-p) + p & \text{if } p+1 \leq i \leq n.
\end{cases}
\]
We regard \(\sigma \in \mathcal{S}_n\) as an element of \(GL(n, \mathbb{C})\) such that \(\sigma(e_i) = e_{\sigma(i)}\) for all \(1 \leq i \leq n\), i.e.,
\[
\sigma = \begin{pmatrix} 
e_{\sigma(1)} & \cdots & e_{\sigma(n)} \end{pmatrix} \in GL(n, \mathbb{C}).
\]
Then \(\mathcal{S}_p \times \mathcal{S}_q \subset K\). The theorem following the next definition gives the image of the moment map.

**Definition 2.5.1** (Driving space for \(U(p, q)\)). For a signed clan \(\delta = (d_1 \ldots d_n)\) of a clan \(\gamma\), we give the following vector subspace \(\text{Dri}(\delta)\) of \(\mathfrak{g}\) and call it driving space of \(\delta\) (for \(U(p, q)\)).

\[
\text{Dri}(\delta) := \{ Y \in \mathfrak{g} \mid Y \text{ satisfies the following conditions } \}.
\]

1. If \(d_i\) and \(d_j\) have the same signature, then \(Y_{ij} = Y_{ji} = 0\).
Then $x$ and $\sigma$.

Proof of Proposition 2.5.2.

Let $d_i = a, d_j = b, d_s = b, d_t = a, d_m = -$. Then $Y_{ij} = Y_{ji} = 0$.

Proposition 2.5.2 means

Let $d_i = a, d_j = b, d_s = b, d_t = a, d_m = -$. Then $Y_{ij} = Y_{ji} = 0$.

Proposition 2.5.2

For a clan $\gamma = (c_1 \cdots c_n) \in \mathcal{C}(U(p, q))$, fix a signed clan $\delta = (d_1 \cdots d_n)$ of $\gamma$. Let the representative $g := g(\delta, \sigma)$ be given as in Theorem 2.2.14 and $x = gB \in Q_\gamma$. We can read off $\mu(T_{Q_\gamma}^\ast X)_x$ from $\text{Dri}(\delta)$ as follows:

\[ \mu \left( (T_{Q_\gamma}^\ast X)_x \right) = \{ Y_{(\sigma^{-1}(1), \ldots, \sigma^{-1}(n))} \mid Y \in \text{Dri}(\delta) \}. \]

Proposition 2.5.2 means

\[ \mu \left( (T_{Q_\gamma}^\ast X)_x \right) = (\text{Ad}(g) b^\perp) \cap p = \{ Y_{(\sigma^{-1}(1), \ldots, \sigma^{-1}(n))} \mid Y \in \text{Dri}(\delta) \} \]

\[ = \{ Y \in \text{Dri}(\delta) \} = \text{Ad}(\sigma) \text{Dri}(\delta). \]

Corollary 2.5.3.

For a representative $g = g(\delta, \sigma)$ of $Q_\gamma$ given in Theorem 2.2.14 and $x = gB$, the driving space is the following space.

\[ \text{Dri}(\delta) = \text{Ad}(\sigma^{-1}) \mu \left( (T_{Q_\gamma}^\ast X)_x \right) \]

\[ = \{ A_{(\sigma(1), \ldots, \sigma(n))} \mid A \in \mu \left( (T_{Q_\gamma}^\ast X)_x \right) \}. \]

Proof of Proposition 2.5.2. For two representatives $g(\delta, \sigma)$ and $g(\delta, \sigma')$ there exists $\sigma'' \in \mathfrak{S}_p \times \mathfrak{S}_q \subset \mathfrak{S}_n$ such that $\sigma' = \sigma'' \sigma$. Then

\[ g(\delta, \sigma') = \sigma'' g(\delta, \sigma). \]

Suppose Proposition 2.5.2 is true for $g(\delta, \sigma)$, i.e.,

\[ (\text{Ad}(g(\delta, \sigma)) b^\perp) \cap p = \text{Ad}(\sigma) \text{Dri}(\delta). \]

Then

\[ (\text{Ad}(g(\delta, \sigma')) b^\perp) \cap p = (\text{Ad}(\sigma'' \cdot g(\delta, \sigma)) \cdot b^\perp) \cap p \]

\[ = \text{Ad}(\sigma'')(\text{Ad}(g(\delta, \sigma)) \cdot b^\perp) \cap p \]

\[ = \text{Ad}(\sigma'') \text{Ad}(\sigma) \text{Dri}(\delta) \]

\[ = \text{Ad}(\sigma'' \sigma) \text{Dri}(\delta) \]

\[ = \text{Ad}(\sigma') \text{Dri}(\delta). \]
Thus, Proposition 2.5.2 is also true for $g(\delta, \sigma')$. So, we prove Proposition 2.5.2 for one case.

We calculate each element of $gb^+g^{-1} \cap p$.

Let $b \in b^+$. We have

$$
(gbg^{-1})_{st} = \sum_{i=1}^{n} \sum_{j=1}^{n} g_{st} b_{ij} g_{tj} = \sum_{i,j \in \{1, \ldots, n\}, i > j} g_{st} b_{ij} g_{tj}.
$$

1. If $(d_i, d_j) = (+, +)$ or $(-, -)$ for $i < j$, then we may assume $\sigma(i) < \sigma(j)$.

Since

$$
\{ A'_{(\sigma(i), \sigma(j))} \mid A' \in p \} = \left\{ \left( \begin{array}{cc} 0 & 0 \\ 0 & 0 \end{array} \right) \right\},
$$

we have

$$
\{ A_{(\sigma(i), \sigma(j))} \mid A \in (Ad(g)b^+) \cap p \} = \left\{ \left( \begin{array}{cc} 0 & 0 \\ 0 & 0 \end{array} \right) \right\} = \{ Y_{i,j} \mid Y \in \text{Dri}(\delta) \}.
$$

2. If $(d_i, d_j) = (a_+, a_-)$, then $\sigma(i) \leq p < \sigma(j)$. Let $i' = \min(i, j)$ and $j' = \max(i, j)$. We have

$$
\{ (Ad(g)b)_{(\sigma(i), \sigma(j))} \mid b \in b^+ \} = \left\{ g' \left( \begin{array}{cc} 0 & 0 \\ b_{j'i'} & 0 \end{array} \right) \right\} = \left\{ \left( \begin{array}{cc} y & \pm y \\ \mp y & -y \end{array} \right) \right\}
$$

and

$$
\{ A'_{(\sigma(i), \sigma(j))} \mid A' \in p \} = \left\{ \left( \begin{array}{cc} 0 & z_{\sigma(i)\sigma(j)} \\ z_{\sigma(i)\sigma(j)} & 0 \end{array} \right) \right\},
$$

where $g' = \tilde{g}$ or $k\tilde{g}$ for

$$
\tilde{g} := \left( \begin{array}{cc} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{array} \right) \quad \text{and} \quad k := \left( \begin{array}{cc} -1 & 0 \\ 0 & 1 \end{array} \right) \in GL(1, C) \times GL(1, C) \hookrightarrow K.
$$

So,

$$
\{ A_{(\sigma(i), \sigma(j))} \mid A \in (Ad(g)b^+) \cap p \} = \left\{ \left( \begin{array}{cc} 0 & 0 \\ 0 & 0 \end{array} \right) \right\} = \{ Y_{i,j} \mid Y \in \text{Dri}(\delta) \}.
$$

3. If $(d_i, d_m) = (+, -)$ for $l < m$, then $\sigma(l) \leq p < \sigma(m)$. We have

$$
\{ (Ad(g)b)_{(\sigma(l), \sigma(m))} \mid b \in b^+ \} = \left\{ \left( \begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & b_{ml} & 0 \end{array} \right) \left( \begin{array}{ccc} 1 & 0 & 0 \\ 0 & 0 & 1 \\ b_{ml} & 0 & 0 \end{array} \right) \right\}
$$

and

$$
\{ A'_{(\sigma(l), \sigma(m))} \mid A' \in p \} = \left\{ \left( \begin{array}{cc} 0 & z_{\sigma(l)\sigma(m)} \\ z_{\sigma(l)\sigma(m)} & 0 \end{array} \right) \right\}.
$$

Therefore,

$$
\{ A_{(\sigma(l), \sigma(m))} \mid A \in (Ad(g)b^+) \cap p \} = \left\{ \left( \begin{array}{cc} 0 & 0 \\ y_{ml} & 0 \end{array} \right) \right\} = \{ Y_{l,m} \mid Y \in \text{Dri}(\delta) \}.$$
4. If \((d_m, d_l) = (-, +)\) for \(m < l\), then \(\sigma(l) \leq p < \sigma(m)\). By the same argument of 3, we have
\[
\left\{ A_{(\sigma(l), \sigma(m))} \mid A \in (\text{Ad}(g) b^\perp) \cap p \right\} = \left\{ \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ b_{lm} & 0 & 0 \end{pmatrix} \right\} \cap \left\{ \begin{pmatrix} 0 & 0 & \sigma(l) \sigma(m) \\ \sigma(m) \sigma(l) & 0 & 0 \end{pmatrix} \right\} = \left\{ \begin{pmatrix} 0 & 0 & \gamma_{lm} \\ 0 & 0 & 0 \end{pmatrix} \right\} = \left\{ Y_{(l, m)} \mid Y \in \text{Dri}(\delta) \right\}.
\]

5. If \((d_i, d_j) = (a_+, a_-)\) and \(d_l = +\) for \(\min(i, j) < l < \max(i, j)\), then we may assume \(\sigma(i) < \sigma(l) \leq p < \sigma(j)\). Let \(i', j', l' \in \mathbb{N}\) satisfy
\[
\{ i', j', l' \} = \{ i, j, l \} \quad \text{and} \quad i' < j' < l'.
\]
We have
\[
\{ (gbg^{-1})_{(\sigma(i), \sigma(l), \sigma(j))} \mid b \in b^\perp \} = \left\{ g' \begin{pmatrix} 0 & 0 & 0 \\ b_{i'j'} & 0 & 0 \\ b_{i'j'} & 0 & 0 \end{pmatrix} \right\}
\]
and
\[
\left\{ A'_{(\sigma(i), \sigma(l), \sigma(j))} \mid A' \in p \right\} = \left\{ \begin{pmatrix} 0 & 0 & \sigma(i) \sigma(j) \\ \sigma(j) \sigma(i) & 0 & 0 \\ \sigma(j) \sigma(i) & 0 \end{pmatrix} \right\}
\]
for \(g' = \tilde{g}\) or \(k \tilde{g}\). Here,
\[
\tilde{g} := \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{pmatrix}
\]
and
\[
k := \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \in GL(2, \mathbb{C}) \times GL(1, \mathbb{C}) \hookrightarrow K.
\]
Therefore,
\[
\left\{ A_{(\sigma(i), \sigma(l), \sigma(j))} \mid A \in (\text{Ad}(g) b^\perp) \cap p \right\} = \left\{ \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \right\} = \left\{ Y_{(i, j, l)} \mid Y \in \text{Dri}(\delta) \right\}.
\]

6. If \((d_i, d_j) = (a_+, a_-)\) and \(d_m = -\) for \(\min(i, j) < m < \max(i, j)\), then we may assume \(\sigma(i) \leq p < \sigma(m) < \sigma(j)\). Let \(i', j', m' \in \mathbb{N}\) satisfy
\[
\{ i', j', m' \} = \{ i, j, m \} \quad \text{and} \quad i' < j' < m'.
\]
We have
\[
\{ (gbg^{-1})_{(\sigma(i), \sigma(m), \sigma(j))} \mid b \in b^\perp \} = \left\{ g' \begin{pmatrix} 0 & 0 & 0 \\ b_{i'j'} & 0 & 0 \\ b_{m'i'j'} & 0 & 0 \end{pmatrix} \right\}
\]
and
\[
\left\{ A'_{(\sigma(i), \sigma(m), \sigma(j))} \mid A' \in p \right\} = \left\{ \begin{pmatrix} 0 & \sigma(i) \sigma(m) & \sigma(i) \sigma(j) \\ \sigma(m) \sigma(i) & 0 & 0 \\ \sigma(j) \sigma(i) & 0 & 0 \end{pmatrix} \right\}.
for \( g' = \tilde{g} \) or \( k\tilde{g} \). Here,
\[
\tilde{g} := \begin{pmatrix}
\frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\
0 & 1 & 0 \\
\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}}
\end{pmatrix}
\]
and
\[
k := \begin{pmatrix}
-1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix} \in GL(1, \mathbb{C}) \times GL(2, \mathbb{C}) \hookrightarrow K.
\]

Hence,
\[
\{ A_{\sigma(i), \sigma(m), \sigma(j)} \mid A \in (\text{Ad}(g)b^\perp) \cap p \} = \{ Y \mid Y \in Dri(\delta) \}.
\]

7. If \((d_i, d_j) = (a_+, a_-)\) and \(d_l = +\) for \( l < \min(i, j) \), then we may assume \( \sigma(i) < \sigma(l) \leq p < \sigma(j) \). Let \( i', j', l' \in \mathbb{N} \) satisfy (26). We have (27) and (28), where \( g' = \tilde{g} \) or \( k\tilde{g} \) for
\[
\tilde{g} := \begin{pmatrix}
0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\
1 & 0 & 0 \\
0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}
\end{pmatrix}
\]
and \( k \) is as in (29). Hence,
\[
\{ A_{\sigma(i), \sigma(l), \sigma(j)} \mid (\text{Ad}(g)b^\perp) \cap p \} = \{ Y \mid Y \in Dri(\delta) \}.
\]

8. If \((d_i, d_j) = (a_+, a_-)\) and \(d_m = -\) for \( m < \min(i, j) \), then we may assume \( \sigma(i) \leq p < \sigma(m) < \sigma(j) \). Let \( i', j', m' \in \mathbb{N} \) satisfy (30). We have (31) and (32), where \( g' = \tilde{g} \) or \( k\tilde{g} \) for
\[
\tilde{g} := \begin{pmatrix}
0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\
1 & 0 & 0 \\
0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}
\end{pmatrix}
\]
and \( k \) is as in (33). Thus,
\[
\{ A_{\sigma(i), \sigma(m), \sigma(j)} \mid A \in (\text{Ad}(g)b^\perp) \cap p \} = \{ Y \mid Y \in Dri(\delta) \}.
\]

9. If \((d_i, d_j) = (a_+, a_-)\) and \(d_l = +\) for \( \max(i, j) < l \), then we may assume \( \sigma(i) < \sigma(l) \leq p < \sigma(j) \). Let \( i', j', l' \in \mathbb{N} \) satisfy (26). We have (27) and (28), where \( g' = \tilde{g} \) or \( k\tilde{g} \) for
\[
\tilde{g} := \begin{pmatrix}
\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\
0 & 0 & 1 \\
\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0
\end{pmatrix}
\]
and $k$ is as (29). So,
\[
\{ A(\sigma(i),\sigma(j)) \mid A \in (\text{Ad}(g)b^\perp) \cap p \} = \left\{ \left( \begin{array}{ccc} 0 & 0 & 0 \\ 0 & 0 & y_{ij} \\ 0 & 0 & 0 \end{array} \right) \right\} = \{ Y(i,j,i) \mid Y \in \text{Dri}(\delta) \}.
\]

10. If $(d_i, d_j) = (a_+, a_-)$ and $d_m = -$ for $\max(i, j) < m$, then we may assume $\sigma(i) \leq p < \sigma(m) < \sigma(j)$. Let $i', j', m' \in \mathbb{N}$ satisfy (30). We have (31) and (32), where $g' = \tilde{g}$ or $k\tilde{g}$ for
\[
\tilde{g} := \begin{pmatrix}
\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\
0 & 0 & 1 \\
\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0
\end{pmatrix}
\]
and $k$ is as in (33). Therefore,
\[
\{ A(\sigma(i),\sigma(m),\sigma(j)) \mid A \in (\text{Ad}(g)b^\perp) \cap p \} = \left\{ \left( \begin{array}{ccc} 0 & 0 & 0 \\ y_{mi} & 0 & 0 \\ 0 & 0 & 0 \end{array} \right) \right\} = \{ Y(i,m,j) \mid Y \in \text{Dri}(\delta) \}.
\]

11. If $(d_i, d_j) = (a_+, a_-)$ and $(d_s, d_i) = (b_+, b_-)$ for $\max(i, j) < \min(t, s)$, then we may assume $\sigma(i) < \sigma(s) \leq p < \sigma(j) < \sigma(t)$. Let $i', j', s', t' \in \mathbb{N}$ satisfy
\[
\{i', j', s', t'\} = \{i, j, s, t\} \quad \text{and} \quad i' < j' < s' < t'.
\]
We have
\[
\{ (gbg^{-1})_{(\sigma(i),\sigma(s),\sigma(j),\sigma(t))} \mid b \in b^\perp \} = \left\{ g \left( \begin{array}{cccc} 0 & 0 & 0 & 0 \\ b_{j'i'} & 0 & 0 & 0 \\ b_{s'i'} & b_{s'j'} & 0 & 0 \\ b_{t'i'} & b_{t'j'} & b_{t's'} & 0 \end{array} \right) t^\frac{1}{2}g \right\}
\]
and
\[
\{ A(\sigma(i),\sigma(s),\sigma(j),\sigma(t)) \mid A \in p \} = \left\{ \left( \begin{array}{cccc} 0 & 0 & z_{\sigma(i)\sigma(j)} & z_{\sigma(i)\sigma(t)} \\ 0 & 0 & z_{\sigma(j)\sigma(i)} & z_{\sigma(j)\sigma(t)} \\ z_{\sigma(s)\sigma(i)} & z_{\sigma(s)\sigma(j)} & 0 & 0 \\ z_{\sigma(t)\sigma(i)} & z_{\sigma(t)\sigma(s)} & z_{\sigma(t)\sigma(j)} & 0 \end{array} \right) \right\}
\]
for $g' = \tilde{g}$, $k_1\tilde{g}$, $k_2\tilde{g}$ or $k_1k_2\tilde{g}$. Here,
\[
\tilde{g} := \begin{pmatrix}
\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 & 0 \\
0 & 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\
\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & 0 \\
0 & 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}
\end{pmatrix}
\]
and $k_1, k_2 \in K$ are
\[
k_1 := \begin{pmatrix}
-1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix} \in GL(2, \mathbb{C}) \times GL(2, \mathbb{C}) \hookrightarrow K
\]
and

\[
(38) \quad k_2 := \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \in GL(2, \mathbb{C}) \times GL(2, \mathbb{C}) \hookrightarrow K.
\]

Hence,

\[
\{ A_{(\sigma(i),\sigma(s),\sigma(j),\sigma(t))} \mid A \in (\text{Ad}(g)b^+) \cap p \} = \begin{cases} 
\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ y_{ti} & 0 & 0 & 0 \end{pmatrix} & \text{if } g' = \tilde{g}, \\
\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -y_{ti} & 0 & 0 & 0 \end{pmatrix} & \text{if } g' = k_1\tilde{g},
\end{cases}
\]

\[
= \{ Y_{i,s,j,t} \mid Y \in \text{Dri}(\delta) \}.
\]

12. If \((d_i, d_j) = (a_+, a_-)\) and \((d_s, d_t) = (b_+, b_-)\) for \(\min(i, j) < \min(t, s) < \max(i, j) < \max(t, s)\), then we may assume \(\sigma(i) < \sigma(s) \leq p < \sigma(j) < \sigma(t)\). Let \(i', j', s', t' \in N\) satisfy (34). We have (36) and (35) for \(g' = \tilde{g}, k_1\tilde{g}, k_2\tilde{g}\) or \(k_1k_2\tilde{g}\). Here,

\[
\tilde{g} := \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & 0 & \frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & 0 & 1/\sqrt{2} \end{pmatrix}
\]

and \(k_1, k_2\) are as (37) and (38). Thus,

\[
\{ A_{(\sigma(i),\sigma(s),\sigma(j),\sigma(t))} \mid A \in (\text{Ad}(g)b^+) \cap p \}
\]

\[
= \begin{cases} 
\{ \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & y_{ti} & 0 \\ 0 & 0 & 0 & 0 \\ y_{ti} & 0 & 0 & 0 \end{pmatrix} & \text{if } g' = \tilde{g}, \\
\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ y_{ti} & 0 & 0 & 0 \end{pmatrix} & \text{if } g' = k_1\tilde{g},
\end{cases}
\]

\[
= \{ Y_{i,s,j,t} \mid Y \in \text{Dri}(\delta) \}.
\]

13. If \((d_i, d_j) = (a_+, a_-)\), \((d_s, d_t) = (b_+, b_-)\) for \(\min(i, j) < \min(s, t) < \max(s, t) < \max(i, j)\), then we may assume \(\sigma(i) < \sigma(s) \leq p < \sigma(j) < \sigma(t)\). Let \(i', j', s', t' \in N\) satisfy (34). We have (36) and (35) for \(g' = \hat{g}, k_1\hat{g}, k_2\hat{g},\) or \(k_1k_2\hat{g}\). Here,

\[
\hat{g} := \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & 0 & -\frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & 0 & 1/\sqrt{2} \\ 0 & \frac{1}{\sqrt{2}} & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{pmatrix}
\]
and \( k_1 \) and \( k_2 \) are as (37) and (38). Thus,

\[
\{ A_{(\sigma(i), \sigma(s), \sigma(j), \sigma(t))} \mid A \in (\text{Ad}(g)b^\perp) \cap p \} = \left\{ \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \right\} = \{ Y_{(i,s,j,t)} \mid Y \in \text{Dri}(\delta) \}.
\]

We have finished proving Proposition 2.5.2 for all the elements.

Since \( g(\delta) = (e_{\sigma(1)} \ldots e_{\sigma(n)}) = \sigma \), if a clan \( \gamma \) corresponds to a closed orbit, we have the following lemma.

Lemma 2.5.4. If a clan \( \gamma \) is an element of \( C_0(U(p,q)) \), then

\[
\text{Dri}(\delta) = b^\perp \cap (\sigma g(\delta)).
\]

Definition 2.5.5 (Driving matrix for \( U(p,q) \)). Let \( \delta = (d_1 \ldots d_n) \) be a signed clan of \( \gamma = (c_1 \ldots c_n) \) satisfying the following condition.

(39) If \( c_i = c_j = a \in N \) for \( i < j \), then \( (d_i, d_j) = (a_+, a_-) \).

Let \( F \) be a field \( C(\mathbb{C}^{s_t} \mid 1 \leq s, t \leq n) \) generated by algebraically independents \( y_{st} \) for \( 1 \leq s, t \leq n \). Let \( Y(\gamma) \) and \( Y(\gamma, m), 1 \leq m \leq 6 \) be elements of \( F \otimes C \text{Mat}(n, n) \) satisfying the following conditions.

1. \( Y(\gamma, 1)_{ij} = \begin{cases} 1 & \text{if } i > j, \\ 0 & \text{if } i \leq j. \end{cases} \)
2. \( Y(\gamma, 2)_{ij} = \begin{cases} 0 & \text{if } (d_i, d_j) = (+, +) \text{ or } (-, -), \\ Y(\gamma, 1)_{ij} & \text{otherwise}. \end{cases} \)
3. If \( (d_s, d_t) = (a_+, a_-) \) for some \( a \in N \), then
   - \( Y(\gamma, 3)_{s,k_1} = 0 \) if \( 1 \leq k_1 \leq s \),
   - \( Y(\gamma, 3)_{k_2,s} = Y(\gamma, 3)_{t,k_2} = 0 \) if \( s < k_2 \leq t \),
   - \( Y(\gamma, 3)_{k_3,t} = 0 \) if \( t < k_3 \leq n \),
   and \( Y(\gamma, 3)_{ij} = Y(\gamma, 2)_{ij} \) otherwise.

Remark 2.5.6. If \( Y(\gamma, 3)_{ij} = 1 \), then the following conditions are satisfied.

(a) \( j < i \).
(b) \( (d_i, d_j) = (a_-, b_+), (a_+, +), (a_-, -), (+, a_+), (-, a_-), (+, +), \text{ or } (-, +) \)
   for some \( a, b \in N, a \neq b \).
(c) If \( d_i = a_- \) for some \( a \in N \), then we have \( d_k = a_+ \) for some \( j < k < i \).
(d) If \( d_j = a_+ \) for some \( a \in N \), then we have \( d_k = a_- \) for some \( j < k < i \).

4. If \( Y(\gamma, 3)_{ij} = 1 \), \( (d_i, d_j) = (a_-, b_+) \) and \( (d_k, d_l) = (a_+, b_-) \) for some \( a, b \in N \), then

\[
Y(\gamma, 4)_{ij} = y_{ij},
\]

\[
Y(\gamma, 4)_{kl} = \begin{cases} y_{ij} & \text{if } j < k < l < i, \\ y_{kl} & \text{if } j < l < k < i, \end{cases}
\]

and \( Y(\gamma, 4)_{i', j'} = Y(\gamma, 3)_{i', j'} \) otherwise.

5. If \( Y(\gamma, 4)_{ij} = 1 \), \( (d_i, d_j) = (a_-, -) \), and \( d_k = a_+ \) for some \( a \in N \), then

\[
Y(\gamma, 5)_{ij} = 0, \quad Y(\gamma, 5)_{kj} = y_{kj},
\]

and \( Y(\gamma, 5)_{i', j'} = Y(\gamma, 4)_{i', j'} \) otherwise.
Example 2.5.7. For a clan $\gamma = (1 + 213 + 32)$, the signed clan of $\gamma$ is $\delta = (1_+ + 2_+1_--3_+ + 3_--2--)$. We give figures for $Y(\gamma, m)$, $1 \leq m \leq 6$ and $Y(\gamma)$ in Table 1. In these figures, the entry which is in the same row of $c_i$ and the same column of $c_j$ is the $(i, j)$-entry of the matrix. Therefore the driving matrix $Y(\gamma)$ is as follows:

$$ Y(\gamma) = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & y_{81} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & y_{54} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & y_{64} & 0 & 0 & 0 \\
y_{71} y_{72} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
y_{81} y_{82} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{pmatrix}. $$

We can read off $\text{Dri}(\delta)$ from $Y(\gamma)$. 

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</tbody>
</table>

**Table 1**

6. If $Y(\gamma, 5)_{ij} = 1$, $(d_i, d_j) = (+, a_+)$, and $d_k = a_-$ for some $a \in \mathbb{N}$, then $Y(\gamma, 6)_{ij} = 0, \quad Y(\gamma, 6)_{ik} = y_{ik},$

and $Y(\gamma, 6)_{i'j'} = Y(\gamma, 5)_{i'j'}$ otherwise.

7. $Y(\gamma)_{ij} = \begin{cases} y_{ij} & \text{if } Y(\gamma, 6)_{ij} = 1, \\
Y(\gamma, 6)_{ij} & \text{otherwise.} \end{cases}$

We call $Y(\gamma)$ the driving matrix of $\gamma$ (for $U(p, q)$).
Proposition 2.5.8. Let $\delta$ be the signed clan satisfying condition (39) of a clan $\gamma$, then

$$\text{Dri}(\delta) = \{ Y(\gamma) \mid y_{ij} \in C \text{ for } 1 \leq i, j \leq n \}.$$ 

Proof. Let $\delta$ be the signed clan of $\gamma$ satisfying condition (39). We compare $Y(\gamma)$ and $Y \in \text{Dri}(\delta)$.

1. If $(d_i, d_j) = (+, +)$ or $(-, -)$ for $i < j$, then
   $$Y(\gamma)_{(i,j)} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad Y_{(i,j)} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$ 

2. If $(d_i, d_j) = (a_+, a_-)$, then $i < j$ and
   $$Y(\gamma)_{(i,j)} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad Y_{(i,j)} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$ 

3. If $(d_i, d_j) = (+, -)$ or $(-, +)$, and $i < j$, then
   $$Y(\gamma)_{(i,j)} = \begin{pmatrix} 0 & y_{ji} \\ y_{ji} & 0 \end{pmatrix} \quad \text{and} \quad Y_{(i,j)} = \begin{pmatrix} 0 & 0 \\ x_{ji} & 0 \end{pmatrix}$$ 
   for some $x_{ji} \in C$.

4. If $(d_i, d_l, d_j) = (a_+, +, a_-)$ and $i < l < j$, then
   $$Y(\gamma)_{(l,i,j)} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ y_{lj} & 0 & 0 \end{pmatrix} \quad \text{and} \quad Y_{(l,i,j)} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ x_{lj} & 0 & 0 \end{pmatrix}$$ 
   for some $x_{lj} \in C$.

5. If $(d_i, d_m, d_j) = (a_+, -a_-)$ and $i < m < j$, then
   $$Y(\gamma)_{(i,m,j)} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad Y_{(i,m,j)} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$ 

6. If $(d_i, d_i, d_j) = (+, a_+, a_-)$ and $l < i < j$, then
   $$Y(\gamma)_{(l,i,j)} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ y_{jl} & 0 & 0 \end{pmatrix} \quad \text{and} \quad Y_{(l,i,j)} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ x_{jl} & 0 & 0 \end{pmatrix}$$ 
   for some $x_{jl} \in C$.

7. If $(d_m, d_i, d_j) = (- , a_+, a_-)$ and $m < i < j$, then
   $$Y(\gamma)_{(m,i,j)} = \begin{pmatrix} 0 & 0 & 0 \\ y_{im} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad Y_{(m,i,j)} = \begin{pmatrix} 0 & 0 & 0 \\ x_{im} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$ 
   for some $x_{im} \in C$.

8. If $(d_i, d_j, d_j) = (a_+, a_-, +)$ and $i < j < l$, then
   $$Y(\gamma)_{(i,j,l)} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ y_{ij} & 0 & 0 \end{pmatrix} \quad \text{and} \quad Y_{(i,j,l)} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & x_{ij} & 0 \end{pmatrix}$$ 
   for some $x_{ij} \in C$. 


9. If \((d_i, d_j, d_m) = (a_+, a_-, -)\) and \(i < j < m\), then

\[
Y(\gamma)_{i,j,m} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ y_{mi} & 0 & 0 \end{pmatrix} \quad \text{and} \quad Y_{i,j,m} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ x_{mi} & 0 & 0 \end{pmatrix}
\]

for some \(x_{mi} \in \mathbb{C}\).

10. If \((d_i, d_j, d_s, d_t) = (a_+, a_-, b_+, b_-)\) and \(i < j < s < t\), then

\[
Y(\gamma)_{i,j,s,t} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & y_{sj} & 0 \\ 0 & y_{ti} & 0 & 0 \\ y_{ti} & 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad Y_{i,j,s,t} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & x_{sj} & 0 \\ 0 & x_{ti} & 0 & 0 \\ x_{ti} & 0 & 0 & 0 \end{pmatrix}
\]

for some \(x_{sj}, x_{ti} \in \mathbb{C}\).

11. If \((d_i, d_s, d_j, d_t) = (a_+, b_+, a_-, b_-)\) and \(i < s < j < t\), then

\[
Y(\gamma)_{i,s,j,t} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & y_{ti} & 0 & 0 & 0 & 0 & 0 \\ 0 & y_{ti} & 0 & 0 & 0 & 0 & 0 & 0 \\ y_{ti} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad Y_{i,s,j,t} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & x_{ti} & 0 & 0 & 0 & 0 & 0 \\ 0 & x_{ti} & 0 & 0 & 0 & 0 & 0 & 0 \\ x_{ti} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}
\]

for some \(x_{ti} \in \mathbb{C}\).

12. If \((d_i, d_s, d_t, d_j) = (a_+, b_+, b_-, a_-)\) and \(i < s < t < j\), then

\[
Y(\gamma)_{i,s,t,j} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad Y_{i,s,t,j} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}
\]

We got \(\text{Dri}(\delta) = \{ Y(\gamma) \mid y_{ij} \in \mathbb{C} \text{ for } 1 \leq i, j \leq n \} \). \(\square\)

**Example 2.5.9.** For \(\gamma = (-++)\), we have

\[
\text{Dri}(\delta) = \left\{ \begin{pmatrix} 0 & 0 & 0 \\ y_{21} & 0 & 0 \\ y_{31} & 0 & 0 \end{pmatrix} \mid y_{ij} \in \mathbb{C} \right\}.
\]
By Proposition 2.5.2 and Proposition 2.5.8, we get the following theorem.

**Theorem 2.5.10.** For a clan \(\gamma \in C(U(p,q))\), fix a signed clan \(\delta\) of \(\gamma\). Let the representative \(g := g(\delta, \sigma)\) be given as in Theorem 2.2.14 and \(x = gB \in Q_{\gamma}\). We can read off \(\mu(T_{Q_{\gamma}}^* X_x)\) from the driving matrix \(Y(\gamma)\) for \(U(p,q)\) as follows:

\[
\left( \mu(T_{Q_{\gamma}}^* X_x) \right) = \{ Y(\gamma)_{(\sigma^{-1}(1), \ldots, \sigma^{-1}(n))} | y_{ij} \in C \text{ for } 1 \leq i, j \leq n \}.
\]

**Definition 2.5.11.** (Generic element) We call \(A \in \mu(T_{Q_{\gamma}}^* X)\) a generic element if \(A\) satisfies

\[
\dim(\ker(A^i)) = \min \{ \dim(\ker(A'^i)) | A' \in \mu(T_{Q_{\gamma}}^* X_x) \}
\]

for all \(1 \leq i \leq n\). Similarly, for a signed clan \(\delta\), we call \(Y\) in \(\text{Dri}(\delta)\) a generic element if \(Y\) satisfies

\[
\dim(\ker(Y^i)) = \min \{ \dim(\ker(Y'^i)) | Y' \in \text{Dri}(\delta) \}
\]

for all \(1 \leq i \leq n\).

### 2.6. Signed Young diagrams

In this section we give tables of signed Young diagrams for clans of \(U(1,1), U(2,1), U(2,2)\) and \(U(p,2)\) by way of examples.

A signed Young diagram is a Young diagram in which every box is labeled with a + or − sign in such a way that signs alternate across rows and they need not alternate down columns. Two signed Young diagrams are regarded as equal if and only if one can be obtained from the other by interchanging rows of equal length. The signature of a signed Young diagram is the ordered pair \((i, j)\) where \(i\) is the number of boxes labeled + and \(j\) is the number of boxes labeled −. For \(G_{\mathbb{R}} = U(p,q)\), nilpotent \(K\)-orbits in \(\mathfrak{p}\) are parametrized by signed Young diagrams of signature \((p, q)\) (see [2]).

An element \(A\) of \(\mathfrak{p}\) satisfies

\[
A \cdot V_+ \subset V_- \quad \text{and} \quad A \cdot V_- \subset V_+.
\]

**Definition 2.6.1.** The signed Young diagram of a nilpotent orbit is defined as follows. We remark that a signed Young diagram is determined by the number of boxes labeled + and the number of boxes labeled − in each column. Let \(D_{i,+}\) be the number of boxes labeled + in \(i\)-column and \(D_{i,-}\) be the number of boxes labeled − in \(i\)-column.

For an element \(A\) of an orbit, the signed Young diagram of the orbit satisfies

\[
\sum_{j=1}^{i} D_{j,+} = \dim(\ker(A^i|_{V_+})) \quad \text{and} \quad \sum_{j=1}^{i} D_{j,-} = \dim(\ker(A^i|_{V_-})).
\]

So, we have

\[
D_{i,+} = \dim(\ker(A^i|_{V_+})) - \dim(\ker(A^{i-1}|_{V_+}))
\]

and

\[
D_{i,-} = \dim(\ker(A^i|_{V_-})) - \dim(\ker(A^{i-1}|_{V_-})).
\]

**Proposition 2.6.2.** Under the conditions of Theorem 2.2.14, we put

\[
V'_+ := \sigma^{-1} V_+ = (e_{\sigma^{-1}(1)} \ldots e_{\sigma^{-1}(n)}) V_+ = (e_{\sigma^{-1}(1)}, \ldots, e_{\sigma^{-1}(p)}),
\]

\[
V'_- := \sigma^{-1} V_- = (e_{\sigma^{-1}(1)} \ldots e_{\sigma^{-1}(n)}) V_- = (e_{\sigma^{-1}(p+1)}, \ldots, e_{\sigma^{-1}(n)}).
\]
Then, we have
\[ \dim(\ker(Y^i|_{V'_i})) = \dim(\ker(A^i|_{V'_i})), \]
\[ \dim(\ker(Y^i|_{V'_i})) = \dim(\ker(A^i|_{V'_i})) \]
for a generic element \( Y \) in \( \text{Dri}(\delta) \) and \( A \) in \( \mu(T^*_Q,X) \).

We consider linear equations
\[ Y^i\vec{a} = \vec{0} \quad \text{for} \quad \vec{a} \in F \otimes_C V'_i. \]
(40)
The set of solutions of (40) is an \( F \)-subspace of \( F \otimes_C V'_i \). Because
\[ \dim_F \{ \vec{a} \in F \otimes_C V'_i \mid Y^i\vec{a} = \vec{0} \} = \dim_C(\ker(Y^i|_{V'_i})) \]
for a generic element \( Y \in \text{Dri}(\gamma) \), we have the following proposition

**Proposition 2.6.3.** Let \( Y^0 = I_n \). We have the following equations.
\[ D_{i,+} = \dim_F \{ \vec{a} \in F \otimes_C V'_i \mid Y^i\vec{a} = \vec{0} \} - \dim_F \{ \vec{a} \in F \otimes_C V'_i \mid Y^{i-1}\vec{a} = \vec{0} \}, \]
\[ D_{i,-} = \dim_F \{ \vec{b} \in F \otimes_C V'_i \mid Y^i\vec{b} = \vec{0} \} - \dim_F \{ \vec{b} \in F \otimes_C V'_i \mid Y^{i-1}\vec{b} = \vec{0} \}. \]

**Example 2.6.4.** For a clan \( \gamma = (+11-+) \), and a signed clan \( \delta = (+1+1-+-+) \).
We get a permutation, a driving matrix \( Y(\gamma) \), and two vectors \( \vec{a} \) and \( \vec{b} \) as follows:
\[ \sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 2 & 4 & 3 & 5 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}, \]
\[ Y(\gamma) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ y_{31} & y_{41} & y_{42} & 0 & 0 \\ y_{53} & y_{54} & 0 & 0 & 0 \end{pmatrix}, \]
\[ \vec{a} = \begin{pmatrix} a_1 \\ a_2 \\ 0 \\ 0 \\ a_5 \end{pmatrix}, \quad \text{and} \quad \vec{b} = \begin{pmatrix} 0 \\ 0 \\ b_3 \\ b_4 \\ 0 \end{pmatrix}. \]

We can calculate \( Y(\gamma)\vec{a} + Y(\gamma)\vec{b} \) as follows:
\[ Y(\gamma)\vec{a} + Y(\gamma)\vec{b} = Y(\gamma) \begin{pmatrix} a_1 \\ a_2 \\ b_3 \\ b_4 \\ a_5 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ y_{31}a_1 \\ y_{41}a_1 + y_{42}a_2 \\ y_{53}b_3 + y_{54}b_4 \end{pmatrix}. \]
The equations \( Y(\gamma)\vec{a} = \vec{0} \) and \( Y(\gamma)\vec{b} = \vec{0} \) lead \( a_1 = 0, a_2 = 0, x_{53}b_3 + x_{54}b_4 = 0. \) So, we have
\[ (D_{1,+}, D_{1,-}) = (1 - 0, 1 - 0) = (1, 1). \]
Thus, the first column of the signed Young diagram of \( \gamma \) is as Figure A.

We can calculate \( Y(\gamma)\vec{a} + Y(\gamma)\vec{b} \) as follows:

\[
Y(\gamma)\vec{a} + Y(\gamma)\vec{b} = Y(\gamma)(Y(\gamma)\vec{a} + Y(\gamma)\vec{b})
\]

\[
= Y(\gamma) \begin{pmatrix}
0 \\
0 \\
y_{31}a_1 \\
y_{41}a_1 + y_{42}a_2 \\
y_{53}b_3 + y_{54}b_4
\end{pmatrix} = \begin{pmatrix}
0 \\
0 \\
0 \\
0 \\
(y_{53}y_{31} + y_{54}y_{41})a_1 + y_{54}y_{42}a_2
\end{pmatrix}.
\]

The equations \( Y(\gamma)\vec{a} = \vec{0} \) and \( Y(\gamma)\vec{b} = \vec{0} \) lead \( (y_{53}y_{31} + y_{54}y_{41})a_1 + y_{54}y_{42}a_2 = 0 \). Therefore, we have

\[
( D_{2,+}, D_{2,-} ) = ( 2 - 1, 2 - 1 ) = ( 1, 1 ).
\]

Thus, the first and second columns of the signed Young diagram of \( \gamma \) is as Figure B.

We can calculate \( Y(\gamma)^3\vec{a} + Y(\gamma)^3\vec{b} \) as follows:

\[
Y(\gamma)^3\vec{a} + Y(\gamma)^3\vec{b} = Y(\gamma)(Y(\gamma)^2\vec{a} + Y(\gamma)^2\vec{b})
\]

\[
= Y(\gamma) \begin{pmatrix}
0 \\
0 \\
0 \\
0 \\
(y_{53}y_{31} + y_{54}y_{41})a_1 + y_{54}y_{42}a_2
\end{pmatrix} = \begin{pmatrix}
0 \\
0 \\
0 \\
0 \\
0
\end{pmatrix}.
\]

By the equations \( Y(\gamma)^3\vec{a} = \vec{0} \) and \( Y(\gamma)^3\vec{b} = \vec{0} \), we have

\[
( D_{3,+}, D_{3,-} ) = ( 3 - 2, 2 - 2 ) = ( 1, 0 ).
\]

Therefore, the signed Young diagram for the clan \( \gamma \) is as Figure C. We give tables of signed Young diagrams for clans by way of examples.
Example 2.6.5. This is the table of the case of $G_R = U(1, 1)$.

<table>
<thead>
<tr>
<th>clan $\gamma$</th>
<th>a representative $g(\gamma)$ of $K$-orbit</th>
<th>driving matrix $Y(\gamma)$</th>
<th>signed Young diagram</th>
</tr>
</thead>
<tbody>
<tr>
<td>$++$</td>
<td>$\begin{pmatrix} 1 &amp; 0 &amp; 0 \ 0 &amp; 0 &amp; 1 \ 0 &amp; 1 &amp; 0 \end{pmatrix}$</td>
<td>$\begin{pmatrix} 0 &amp; 0 &amp; 0 \ y_{21} &amp; 0 &amp; 0 \ 0 &amp; y_{32} &amp; 0 \end{pmatrix}$</td>
<td>$\begin{pmatrix} + &amp; + &amp; + \end{pmatrix}$</td>
</tr>
<tr>
<td>$+-$</td>
<td>$\begin{pmatrix} 0 &amp; 1 &amp; 0 \ 0 &amp; 1 &amp; 0 \end{pmatrix}$</td>
<td>$\begin{pmatrix} 0 &amp; 0 \ y_{21} &amp; 0 \end{pmatrix}$</td>
<td>$\begin{pmatrix} + &amp; + \end{pmatrix}$</td>
</tr>
<tr>
<td>$11$</td>
<td>$\begin{pmatrix} \frac{1}{\sqrt{2}} &amp; \frac{i}{\sqrt{2}} \ \frac{i}{\sqrt{2}} &amp; \frac{1}{\sqrt{2}} \end{pmatrix}$</td>
<td>$\begin{pmatrix} 0 &amp; 0 \ 0 &amp; 0 \end{pmatrix}$</td>
<td>$\begin{pmatrix} + \end{pmatrix}$</td>
</tr>
</tbody>
</table>

Example 2.6.6. This is the table of the case of $G_R = U(2, 1)$.

<table>
<thead>
<tr>
<th>clan $\gamma$</th>
<th>a representative $g(\gamma)$ of $K$-orbit</th>
<th>driving matrix $Y(\gamma)$</th>
<th>signed Young diagram</th>
</tr>
</thead>
<tbody>
<tr>
<td>$++-+1$</td>
<td>$\begin{pmatrix} 1 &amp; 0 &amp; 0 \ 0 &amp; 0 &amp; 1 \ 0 &amp; 1 &amp; 0 \end{pmatrix}$</td>
<td>$\begin{pmatrix} 0 &amp; 0 &amp; 0 \ y_{21} &amp; 0 &amp; 0 \ 0 &amp; y_{32} &amp; 0 \end{pmatrix}$</td>
<td>$\begin{pmatrix} + &amp; + &amp; + &amp; + \end{pmatrix}$</td>
</tr>
<tr>
<td>$+--$</td>
<td>$\begin{pmatrix} 1 &amp; 0 &amp; 0 \ 0 &amp; 1 &amp; 0 \ 0 &amp; 0 &amp; 1 \end{pmatrix}$</td>
<td>$\begin{pmatrix} 0 &amp; 0 &amp; 0 \ 0 &amp; 0 &amp; 0 \ y_{31} &amp; y_{32} &amp; 0 \end{pmatrix}$</td>
<td>$\begin{pmatrix} + &amp; + \end{pmatrix}$</td>
</tr>
<tr>
<td>$+11$</td>
<td>$\begin{pmatrix} 1 &amp; 0 &amp; 0 \ 0 &amp; \frac{1}{\sqrt{2}} &amp; \frac{-i}{\sqrt{2}} \ 0 &amp; \frac{i}{\sqrt{2}} &amp; \frac{1}{\sqrt{2}} \end{pmatrix}$</td>
<td>$\begin{pmatrix} 0 &amp; 0 &amp; 0 \ 0 &amp; 0 &amp; 0 \ y_{31} &amp; 0 &amp; 0 \end{pmatrix}$</td>
<td>$\begin{pmatrix} + \end{pmatrix}$</td>
</tr>
<tr>
<td>$-++$</td>
<td>$\begin{pmatrix} 0 &amp; 1 &amp; 0 \ 0 &amp; 0 &amp; 1 \ 1 &amp; 0 &amp; 0 \end{pmatrix}$</td>
<td>$\begin{pmatrix} 0 &amp; 0 &amp; 0 \ y_{21} &amp; 0 &amp; 0 \ y_{31} &amp; 0 &amp; 0 \end{pmatrix}$</td>
<td>$\begin{pmatrix} + &amp; + \end{pmatrix}$</td>
</tr>
<tr>
<td>$1-1$</td>
<td>$\begin{pmatrix} \frac{1}{\sqrt{2}} &amp; \frac{1}{\sqrt{2}} &amp; 0 \ \frac{-i}{\sqrt{2}} &amp; \frac{i}{\sqrt{2}} &amp; 0 \ 0 &amp; 0 &amp; 1 \end{pmatrix}$</td>
<td>$\begin{pmatrix} 0 &amp; 0 &amp; 0 \ 0 &amp; 0 &amp; 0 \ 0 &amp; y_{32} &amp; 0 \end{pmatrix}$</td>
<td>$\begin{pmatrix} + &amp; + \end{pmatrix}$</td>
</tr>
<tr>
<td>$1+1$</td>
<td>$\begin{pmatrix} \frac{1}{\sqrt{2}} &amp; 0 &amp; \frac{-i}{\sqrt{2}} \ 0 &amp; 1 &amp; 0 \ \frac{i}{\sqrt{2}} &amp; 0 &amp; \frac{1}{\sqrt{2}} \end{pmatrix}$</td>
<td>$\begin{pmatrix} 0 &amp; 0 &amp; 0 \ 0 &amp; 0 &amp; 0 \ 0 &amp; y_{32} &amp; 0 \end{pmatrix}$</td>
<td>$\begin{pmatrix} + \end{pmatrix}$</td>
</tr>
</tbody>
</table>
Example 2.6.7. This is the table of the case of $G_R = U(2, 2)$.

<table>
<thead>
<tr>
<th>clan $\gamma$</th>
<th>a representative</th>
<th>driving matrix $Y(\gamma)$</th>
<th>signed Young diagram</th>
</tr>
</thead>
<tbody>
<tr>
<td>$+ - + -$</td>
<td>$\begin{pmatrix} 1 &amp; 0 &amp; 0 &amp; 0 \ 0 &amp; 0 &amp; 1 &amp; 0 \ 0 &amp; 1 &amp; 0 &amp; 0 \ 0 &amp; 0 &amp; 0 &amp; 1 \end{pmatrix}$</td>
<td>$\begin{pmatrix} 0 &amp; 0 &amp; 0 &amp; 0 \ 0 &amp; 0 &amp; 0 &amp; 0 \ 0 &amp; 0 &amp; 0 &amp; 0 \ 0 &amp; 0 &amp; 0 &amp; 0 \end{pmatrix}$</td>
<td>$\begin{pmatrix} 0 &amp; 0 &amp; 0 &amp; 0 \ 0 &amp; 0 &amp; 0 &amp; 0 \ 0 &amp; 0 &amp; 0 &amp; 0 \ 0 &amp; 0 &amp; 0 &amp; 0 \end{pmatrix}$</td>
</tr>
<tr>
<td>$- + - +$</td>
<td>$\begin{pmatrix} 0 &amp; 1 &amp; 0 &amp; 0 \ 0 &amp; 0 &amp; 0 &amp; 1 \ 1 &amp; 0 &amp; 0 &amp; 0 \ 0 &amp; 0 &amp; 1 &amp; 0 \end{pmatrix}$</td>
<td>$\begin{pmatrix} 0 &amp; 0 &amp; 0 &amp; 0 \ 0 &amp; 0 &amp; 0 &amp; 0 \ 0 &amp; 0 &amp; 0 &amp; 0 \ 0 &amp; 0 &amp; 0 &amp; 0 \end{pmatrix}$</td>
<td>$\begin{pmatrix} 0 &amp; 0 &amp; 0 &amp; 0 \ 0 &amp; 0 &amp; 0 &amp; 0 \ 0 &amp; 0 &amp; 0 &amp; 0 \ 0 &amp; 0 &amp; 0 &amp; 0 \end{pmatrix}$</td>
</tr>
<tr>
<td>$+ - - +$</td>
<td>$\begin{pmatrix} 1 &amp; 0 &amp; 0 &amp; 0 \ 0 &amp; 0 &amp; 1 &amp; 0 \ 0 &amp; 1 &amp; 0 &amp; 0 \ 0 &amp; 0 &amp; 0 &amp; 1 \end{pmatrix}$</td>
<td>$\begin{pmatrix} 0 &amp; 0 &amp; 0 &amp; 0 \ 0 &amp; 0 &amp; 0 &amp; 0 \ 0 &amp; 0 &amp; 0 &amp; 0 \ 0 &amp; 0 &amp; 0 &amp; 0 \end{pmatrix}$</td>
<td>$\begin{pmatrix} 0 &amp; 0 &amp; 0 &amp; 0 \ 0 &amp; 0 &amp; 0 &amp; 0 \ 0 &amp; 0 &amp; 0 &amp; 0 \ 0 &amp; 0 &amp; 0 &amp; 0 \end{pmatrix}$</td>
</tr>
<tr>
<td>$+ - 1 1$</td>
<td>$\begin{pmatrix} 1 &amp; 0 &amp; 0 &amp; 0 \ 0 &amp; 0 &amp; \frac{\sqrt{2}}{2} &amp; \frac{\sqrt{2}}{2} \ 0 &amp; 1 &amp; 0 &amp; 0 \ 0 &amp; 0 &amp; 1 &amp; 0 \end{pmatrix}$</td>
<td>$\begin{pmatrix} 0 &amp; 0 &amp; 0 &amp; 0 \ 0 &amp; 0 &amp; 0 &amp; 0 \ 0 &amp; 0 &amp; 0 &amp; 0 \ 0 &amp; 0 &amp; 0 &amp; 0 \end{pmatrix}$</td>
<td>$\begin{pmatrix} 0 &amp; 0 &amp; 0 &amp; 0 \ 0 &amp; 0 &amp; 0 &amp; 0 \ 0 &amp; 0 &amp; 0 &amp; 0 \ 0 &amp; 0 &amp; 0 &amp; 0 \end{pmatrix}$</td>
</tr>
<tr>
<td>$1 1 - +$</td>
<td>$\begin{pmatrix} \frac{\sqrt{2}}{2} &amp; \frac{\sqrt{2}}{2} &amp; 0 &amp; 0 \ 0 &amp; 0 &amp; 1 &amp; 0 \ \frac{\sqrt{2}}{2} &amp; \frac{\sqrt{2}}{2} &amp; 0 &amp; 0 \ 0 &amp; 0 &amp; 1 &amp; 0 \end{pmatrix}$</td>
<td>$\begin{pmatrix} 0 &amp; 0 &amp; 0 &amp; 0 \ 0 &amp; 0 &amp; 0 &amp; 0 \ 0 &amp; 0 &amp; 0 &amp; 0 \ 0 &amp; 0 &amp; 0 &amp; 0 \end{pmatrix}$</td>
<td>$\begin{pmatrix} 0 &amp; 0 &amp; 0 &amp; 0 \ 0 &amp; 0 &amp; 0 &amp; 0 \ 0 &amp; 0 &amp; 0 &amp; 0 \ 0 &amp; 0 &amp; 0 &amp; 0 \end{pmatrix}$</td>
</tr>
<tr>
<td>$- + + -$</td>
<td>$\begin{pmatrix} 0 &amp; 1 &amp; 0 &amp; 0 \ 0 &amp; 0 &amp; 1 &amp; 0 \ 1 &amp; 0 &amp; 0 &amp; 0 \ 0 &amp; 0 &amp; 0 &amp; 1 \end{pmatrix}$</td>
<td>$\begin{pmatrix} 0 &amp; 0 &amp; 0 &amp; 0 \ 0 &amp; 0 &amp; 0 &amp; 0 \ 0 &amp; 0 &amp; 0 &amp; 0 \ 0 &amp; 0 &amp; 0 &amp; 0 \end{pmatrix}$</td>
<td>$\begin{pmatrix} 0 &amp; 0 &amp; 0 &amp; 0 \ 0 &amp; 0 &amp; 0 &amp; 0 \ 0 &amp; 0 &amp; 0 &amp; 0 \ 0 &amp; 0 &amp; 0 &amp; 0 \end{pmatrix}$</td>
</tr>
<tr>
<td>$- + 1 1$</td>
<td>$\begin{pmatrix} 0 &amp; 1 &amp; 0 &amp; 0 \ 0 &amp; 0 &amp; \frac{\sqrt{2}}{2} &amp; \frac{\sqrt{2}}{2} \ 1 &amp; 0 &amp; 0 &amp; 0 \ 0 &amp; 0 &amp; \frac{\sqrt{2}}{2} &amp; \frac{\sqrt{2}}{2} \end{pmatrix}$</td>
<td>$\begin{pmatrix} 0 &amp; 0 &amp; 0 &amp; 0 \ 0 &amp; 0 &amp; 0 &amp; 0 \ 0 &amp; 0 &amp; 0 &amp; 0 \ 0 &amp; 0 &amp; 0 &amp; 0 \end{pmatrix}$</td>
<td>$\begin{pmatrix} 0 &amp; 0 &amp; 0 &amp; 0 \ 0 &amp; 0 &amp; 0 &amp; 0 \ 0 &amp; 0 &amp; 0 &amp; 0 \ 0 &amp; 0 &amp; 0 &amp; 0 \end{pmatrix}$</td>
</tr>
<tr>
<td>$1 1 + -$</td>
<td>$\begin{pmatrix} \frac{\sqrt{2}}{2} &amp; \frac{\sqrt{2}}{2} &amp; 0 &amp; 0 \ 0 &amp; 0 &amp; 1 &amp; 0 \ \frac{\sqrt{2}}{2} &amp; \frac{\sqrt{2}}{2} &amp; 0 &amp; 0 \ 0 &amp; 0 &amp; 0 &amp; 1 \end{pmatrix}$</td>
<td>$\begin{pmatrix} 0 &amp; 0 &amp; 0 &amp; 0 \ 0 &amp; 0 &amp; 0 &amp; 0 \ 0 &amp; 0 &amp; 0 &amp; 0 \ 0 &amp; 0 &amp; 0 &amp; 0 \end{pmatrix}$</td>
<td>$\begin{pmatrix} 0 &amp; 0 &amp; 0 &amp; 0 \ 0 &amp; 0 &amp; 0 &amp; 0 \ 0 &amp; 0 &amp; 0 &amp; 0 \ 0 &amp; 0 &amp; 0 &amp; 0 \end{pmatrix}$</td>
</tr>
<tr>
<td>$++ - -$</td>
<td>$\begin{pmatrix} 1 &amp; 0 &amp; 0 &amp; 0 \ 0 &amp; 1 &amp; 0 &amp; 0 \ 0 &amp; 0 &amp; 1 &amp; 0 \ 0 &amp; 0 &amp; 0 &amp; 1 \end{pmatrix}$</td>
<td>$\begin{pmatrix} 0 &amp; 0 &amp; 0 &amp; 0 \ 0 &amp; 0 &amp; 0 &amp; 0 \ 0 &amp; 0 &amp; 0 &amp; 0 \ 0 &amp; 0 &amp; 0 &amp; 0 \end{pmatrix}$</td>
<td>$\begin{pmatrix} 0 &amp; 0 &amp; 0 &amp; 0 \ 0 &amp; 0 &amp; 0 &amp; 0 \ 0 &amp; 0 &amp; 0 &amp; 0 \ 0 &amp; 0 &amp; 0 &amp; 0 \end{pmatrix}$</td>
</tr>
<tr>
<td>$+ 1 1 -$</td>
<td>$\begin{pmatrix} 1 &amp; 0 &amp; 0 &amp; 0 \ \frac{\sqrt{2}}{2} &amp; \frac{\sqrt{2}}{2} &amp; 0 &amp; 0 \ \frac{\sqrt{2}}{2} &amp; \frac{\sqrt{2}}{2} &amp; 0 &amp; 0 \ 0 &amp; 0 &amp; 0 &amp; 1 \end{pmatrix}$</td>
<td>$\begin{pmatrix} 0 &amp; 0 &amp; 0 &amp; 0 \ 0 &amp; 0 &amp; 0 &amp; 0 \ 0 &amp; 0 &amp; 0 &amp; 0 \ 0 &amp; 0 &amp; 0 &amp; 0 \end{pmatrix}$</td>
<td>$\begin{pmatrix} 0 &amp; 0 &amp; 0 &amp; 0 \ 0 &amp; 0 &amp; 0 &amp; 0 \ 0 &amp; 0 &amp; 0 &amp; 0 \ 0 &amp; 0 &amp; 0 &amp; 0 \end{pmatrix}$</td>
</tr>
<tr>
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<td>$\begin{pmatrix} 0 &amp; 0 &amp; 0 &amp; 0 \ 0 &amp; 0 &amp; 0 &amp; 0 \ 0 &amp; y_{31} &amp; 0 &amp; 0 \ y_{42} &amp; y_{43} &amp; 0 &amp; 0 \end{pmatrix}$</td>
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</table>
Example 2.6.8. This is the table for the case of $G_R = U(p, 2)$.

<table>
<thead>
<tr>
<th>Signed Young diagram</th>
<th>Clan $(i, j, k, l \geq 1, \ a, b, c \geq 0)$</th>
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<th>Clan $(i, j, k \geq 1, \ a, b, c \geq 0)$</th>
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3. The case of $Sp(n,\mathbb{R})$

In this section, we treat the case of $G_\mathbb{R} = Sp(n,\mathbb{R})$, i.e., we treat $GL(n,\mathbb{C})$-orbits in the flag variety of $Sp(n,\mathbb{C})$. We will apply the case of $U(n,n)$ to the case of $Sp(n,\mathbb{R})$.

Definition 3.0.9. For a clan $\gamma = (c_1 \cdots c_n) \in \mathcal{C}(U(p,q))$, transpose $t_\gamma$, of $\gamma$ is a clan in $\mathcal{C}(U(p,q))$ such that

$$t_\gamma = (c_n \cdots c_1)$$
and minus $-\gamma = (c'_1 \cdots c'_n)$ is a clan in $\mathcal{C}(U(q,p))$ such that

$$
c'_i = \begin{cases} 
- & \text{if } c_i = +, \\
+ & \text{if } c_i = -, \\
c_i & \text{if } c_i \in \mathbb{N}.
\end{cases}
$$

We call a clan $\gamma$ symmetric if the transpose of $\gamma$ is equal to $\gamma$ as a clan, i.e.,

$$t_\gamma = \gamma.$$

We call a clan $\gamma$ skew symmetric if the transpose of $\gamma$ is equal to minus of $\gamma$ as a clan, i.e.,

$$t_\gamma = -\gamma.$$

**Example 3.0.10.** A clan $\gamma_1 = ( + 1 - 2 1 - 2 + )$ is symmetric because

$$t_{\gamma_1} = ( + 2 - 1 2 - 1 + ) = ( + 1 - 2 1 - 2 + ) = \gamma_1.$$

A clan $\gamma_2 = ( + 1 - 2 1 + 2 - )$ is skew symmetric because

$$t_{\gamma_2} = ( - 2 + 1 2 - 1 + ) = - ( + 1 - 2 1 - 2 + ) = -\gamma_2.$$

From now on, we denote $G$, $B$, $K$, $g$, $b$, $k$, and $p$ for $G_{\text{AH}}$, $B_{\text{AH}}$, $K_{\text{AH}}$, $g_{\text{AH}}$, $b_{\text{AH}}$, $k_{\text{AH}}$, and $p_{\text{AH}}$. Let $U(p',q') = U(n,n)$. We denote $U(n,n)$ by $G_{\text{AH}}$.

3.1. **Flags of $Sp(n,C)$**. In this section, after realizing a real form $Sp(n,R)$ of $Sp(n,C)$, we recall flags of $Sp(n,C)$. We realize a complex symplectic group $Sp(n,C)$ as a group of matrices $g$ in $GL(2n,C)$ which leave invariant an exterior form

$$x_1 \wedge x_2 + x_2 \wedge x_3 + \cdots + x_n \wedge x_{n+1},$$

i.e.,

$$(41) \quad Sp(n,C) = \left\{ g \in GL(2n,C) \left| t^g \begin{pmatrix} 0 & J_n \\ -J_n & 0 \end{pmatrix} g = \begin{pmatrix} 0 & J_n \\ -J_n & 0 \end{pmatrix} \right. \right\}.$$

By this realization, a Borel subgroup can be upper triangular.

We realize a real symplectic group $G_{\text{R}} = Sp(n,R)$ as a group of matrices $g$ in $Sp(n,C)$ which leave invariant a Hermitian form of the signature $(n,n)$

$$x_1 \overline{x_1} + \cdots + x_n \overline{x_n} - x_{n+1} \overline{x_{n+1}} - \cdots - x_{2n} \overline{x_{2n}},$$

i.e.,

$$(42) \quad Sp(n,R) = \left\{ g \in Sp(n,C) \left| t^g \begin{pmatrix} I_n & 0 \\ 0 & -I_n \end{pmatrix} g = \begin{pmatrix} I_n & 0 \\ 0 & -I_n \end{pmatrix} \right. \right\}.$$

By this realization of the real form, a compact Cartan subgroup can be diagonal. This realization is not only the author’s taste but also corresponds to $U(n,n)$. (See (43)).
Remark 3.1.1. The group (42) and a group
\[ \text{Sp}(n, \mathbb{C}) \cap \text{GL}(2n, \mathbb{R}) \]
are conjugate by a matrix
\[ \frac{1}{\sqrt{2}} \begin{pmatrix} I_n & J_n \\ -iJ_n & iI_n \end{pmatrix}. \]

We fix a Cartan involution \( \theta \) of \( G_\mathbb{R} \):
\[ \theta : g \mapsto \begin{pmatrix} I_n & 0 \\ 0 & -I_n \end{pmatrix} g \begin{pmatrix} I_n & 0 \\ 0 & -I_n \end{pmatrix}. \]

Then we have
\[ k = g \cap k_{\text{III}}, \quad p = g \cap p_{\text{III}}, \quad b = g \cap b_{\text{III}}. \]
and
\[ K = \left\{ \begin{pmatrix} K' & 0 \\ 0 & J_n \end{pmatrix} \middle| K' \in \text{GL}(n, \mathbb{C}) \right\} \simeq \text{GL}(n, \mathbb{C}). \]

Notation 3.1.2. Let \( V = \mathbb{C}^{2n} \) and \( \theta \) be an involution of \( V \) such that
\[ \theta : v \mapsto \begin{pmatrix} I_n & 0 \\ 0 & -I_n \end{pmatrix} v. \]

Let \( V_+ \) and \( V_- \) be the eigenspaces in \( V \) under \( \theta \) for eigenvalues +1 and -1, respectively:
\[ V_+ := \langle e_1, \ldots, e_n \rangle, \quad \text{and} \quad V_- := \langle e_{n+1}, \ldots, e_{2n} \rangle. \]

Let \( \mathfrak{a} \) be a Cartan subalgebra of \( \mathfrak{g} \) consisting of all diagonal matrices of \( \mathfrak{g} = \text{sp}(n, \mathbb{C}) \), \( \mathfrak{b} \) a Borel subalgebra consisting of all upper triangular matrices of \( \mathfrak{g} \). This choice corresponds to the choice of simple root system \( \Psi \):
\[ \Psi := \{ \alpha_1, \alpha_2, \ldots, \alpha_n \} \subset \mathfrak{a}^*. \]
where \( \alpha_i \in \mathfrak{a}^* \) satisfies
\[ \alpha_i(E_{jj} - E_{2n+1-j,2n+1-j}) = \begin{cases} \delta_{ij} - \delta_{i+1,j} & \text{if } 1 \leq i \leq n - 1, \\ 2\delta_{nj} & \text{if } i = n. \end{cases} \]

Let \( \omega \) be a nondegenerate skew-symmetric bilinear form on \( V = \mathbb{C}^{2n} \) such that
\[ \omega(v, w) = t_v \begin{pmatrix} 0 & J_n \\ -J_n & 0 \end{pmatrix} w. \]

If \( W \) is a subspace of \( V \), we write
\[ W^\perp = \{ v \in V \mid \omega(v, w) = 0 \text{ for all } w \in W \}. \]
This is the orthogonal subspace of \( W \), relative to \( \omega \). We say that \( W \) is isotropic if \( W \subset W^\perp \).

Remark 3.1.3. The orthogonal subspace of \( V_+ \) and \( V_- \) are \( V_+ \) and \( V_- \) respectively:
\[ (V_+)^\perp = V_- \quad \text{and} \quad (V_-)^\perp = V_. \]

Remark 3.1.4. A flag \( x \) of \( \text{Sp}(n, \mathbb{C}) \) is a sequence of \( 2n + 1 \) vector spaces
\[ x = (V_0, V_1, V_2, \ldots, V_{2n}), \]
satisfying the following three conditions.
1. \( \dim V_i = i \) for all \( 0 \leq i \leq 2n \).
2. \( \{0\} = V_0 \subset V_1 \subset V_2 \subset \ldots \subset V_{2n} = \mathbb{C}^{2n} \).
3. For \( 0 \leq i \leq 2n \), \( V_{2n-i} \) is the orthogonal subspace of \( V_i \):

\[
V_{2n-i} = V_i^\perp,
\]

i.e., for \( 1 \leq i \leq n \), \( V_i \) are isotropic subspaces.

We denote the set of flags by \( X \).

**Remark 3.1.5.** We fix a \( G \)-equivariant natural isomorphism between \( X \) and \( G/B \) via the manner in Remark 2.1.2.

### 3.2. A symbolic parametrization of \( GL(n, \mathbb{C}) \)-orbits in \( Sp(n, \mathbb{C})/B \)

In this section we give a symbolic parametrization of \( K \)-orbits in the flag variety \( X \) and give representatives of \( K \)-orbits for the parameters. Our parameters are an improvement of the Matsuki-Oshima’s parameters [6]. The set of parameters of the \( K \)-orbits in the flag variety \( X \) is a subset of \( C(U(n, n)) \). The parameter we give for a \( K \)-orbit \( Q \) in the flag variety for \( G_\mathbb{R} = Sp(n, \mathbb{R}) \) coincides with a clan in \( C(U(n, n)) \) which includes \( Q \).

#### 3.2-1. Generalized clans for \( Sp(n, \mathbb{R}) \)

We give a subset of \( C(U(n, n)) \). The subset parametrizes \( K \)-orbits in the flag variety \( X \).

**Proposition 3.2.1.** For a flag \( x = (V_0, V_1, \ldots , V_{2n}) \in X \), there exists a clan \( \gamma = (c_1 \cdots c_{2n}) \in C(U(n, n)) \) via the conditions in Proposition 2.2.6.

**Proposition 3.2.2.** Let \( x \in X \) and \( \gamma = (c_1 \cdots c_{2n}) \in C(U(n, n)) \) be the clan of \( x \). Then the clan is skew symmetric, i.e., \( b_\gamma = -\gamma \).

**Proof.** We remark that

\[
V_{2n-i,\pm} = V_{2n-i} \cap V_\pm = (V_i)^\perp \cap (V_\pm)^\perp = (V_i + V_\pm)^\perp = (\pi_\pm(V_i) \oplus V_\pm)^\perp.
\]

1. If \( c_i = + \), then there exists a vector \( v_+ \) in \( V_+ \) such that \( v_+ \notin \pi_+(V_{i-1}) \) and \( V_i = V_{i-1} \oplus \langle v_+ \rangle \). Since

\[
V_{2n-i+1,+} = (\pi_-(V_{i-1}) \oplus V_+) \perp, \quad \text{and} \quad V_{2n-i,+} = (\pi_-(V_i) \oplus V_+)^\perp = (\pi_-(V_{i-1}) \oplus V_+)^\perp,
\]

we have

\[
(2n - i + 1; +) - (2n - i; +) = 0.
\]

Since

\[
V_{2n-i+1,-} = (\pi_+(V_{i-1}) \oplus V_-)^\perp \quad \text{and} \quad V_{2n-i,-} = (\pi_+(V_i) \oplus V_-)^\perp = (\pi_+(V_{i-1}) \oplus \langle v_+ \rangle \oplus V_-)^\perp,
\]

we have

\[
(2n - i + 1; -) - (2n - i; -) = 1.
\]

Thus, we have \( c_{2n+1-i} = - \).

2. If \( c_i = - \), then we have \( c_{2n+1-i} = + \) by the same argument of 1.
3. We will prove that $c_{2n+1-i} = c_i' \in \mathbb{N}$ for some $i' < 2n+1 - i$ if $c_i \in \mathbb{N}$ and $c_i \neq c_j$ for all $j < i$. If $c_i \in \mathbb{N}$ and $c_i \neq c_j$ for all $j < i$, then there exists a $v_i \in \mathbb{C}^{2n}$ such that $\pi_+(v_i) \notin \pi_+(V_{i-1})$, $\pi_-(v_i) \notin \pi_-(V_{i-1})$, and

$$V_i = V_{i-1} \oplus \langle v_i \rangle.$$ 

Since

$$V_{2n-i+1,\pm} = (\pi_+(V_{i-1}) \oplus V_\pm)^\perp$$

and

$$V_{2n-i,\pm} = (\pi_+(V_i) \oplus V_\pm)^\perp = (\pi_+(V_{i-1}) \oplus (\pi_+(v_i)) \oplus V_\pm)^\perp,$$

we have

$$(2n - i + 1, \pm) - (2n - i, \pm) = 1.$$

Therefore, $c_{2n+1-i} = c_i' \in \mathbb{N}$ for some $i' < 2n+1 - i$.

4. We will prove that $c_{2n+1-j} \in \mathbb{N}$ and $c_{2n+1-j} \neq c_{2n+1-j}'$ for all $j' < 2n+1 - j$ if $c_i = c_j \in \mathbb{N}$ for $i < j$. Then there exists a $v_i \in \mathbb{C}^{2n}$ such that $\pi_+(v_i) \notin \pi_+(V_{i-1})$, $\pi_-(v_i) \notin \pi_-(V_{i-1})$, and

$$V_i = V_{i-1} \oplus \langle v_i \rangle \subset V_j = V_{j-1} \oplus \langle \pi_+(v_i) \rangle.$$ 

Since $\pi_\pm(v_i) \in \pi_\pm(V_i) \subset \pi_\pm(V_{j-1})$, we have

$$V_{2n-j,\pm} = (\pi_+(V_j) \oplus V_\pm)^\perp$$

and

$$V_{2n-j,\pm} = ((\pi_+(V_{j-1}) + (\pi_+(v_i)) \oplus V_\pm)^\perp$$

Thus,

$$(2n - j + 1, \pm) - (2n - j, \pm) = 0.$$

Therefore $c_{2n+1-j} \in \mathbb{N}$ and $c_{2n+1-j} \neq c_{2n+1-j}'$ for all $j' < 2n+1 - j$.

5. It is enough to prove that $c_i = c_j \in \mathbb{N}$ if $c_{2n+1-i} = c_{2n+1-j} \in \mathbb{N}$ for $2n+1 - i < j < i$. Let $2n+1 - i < j < i$.

We prove that by induction on $i$. Suppose that if $c_k = c_l \in \mathbb{N}$ for $2n+1 - i < k < l < i$, then $c_{2n+1-k} = c_{2n+1-l} \in \mathbb{N}$.

(a) We assume now $c_i = c_s \in \mathbb{N}$ for $s < j$ hoping to show a contradiction.

Then there exist $v_i, v_s \in \mathbb{C}^{2n}$, such that

$$V_{2n+1-i} = V_{2n-i} \oplus \langle v_i \rangle \subset V_{2n+1-j} = V_{2n-j} \oplus \langle \pi_+(v_i) \rangle$$

and

$$V_s = V_{s-1} \oplus \langle v_s \rangle \subset V_{j-1} \subset V_i = V_{i-1} \oplus \langle \pi_-(v_s) \rangle.$$ 

Thus,

$$V_{2n-i} = V_i^\perp \subset \langle \pi_-(v_s) \rangle^\perp.$$ 

Since

$$v_s \in V_s \subset V_{j-1} = V_{2n-j+1}^\perp = V_{2n-j}^\perp \cap (\pi_+(v_i))^\perp \subset (\pi_+(v_i))^\perp,$$

we have

$$\omega(\pi_-(v_s), v_i) = \omega(\pi_-(v_s), \pi_+(v_i)) = \omega(v_s, \pi_+(v_i)) = 0.$$ 

So

$$\langle v_i \rangle \subset (\pi_-(v_s))^\perp.$$
On the other hand,
\[ V_{2n-i} = \frac{v_i}{v_i} = V_{2n-i} \cap \langle \pi_-(v_s) \rangle \]
\[ = V_{2n-i+1} \cap \langle \pi_-(v_s) \rangle = (V_{2n-i} \oplus \langle v_i \rangle) \cap \langle \pi_-(v_s) \rangle \]
\[ = V_{2n-i} \oplus \langle v_i \rangle = V_{2n-i+1} \]
by (44) and (45). That contradicts \( V_{2n-i} \neq V_{2n-i+1} \). We have proved \( c_i \neq c_s \) for all \( s < j \).

(b) We will assume that \( c_i = c_s \) for \( j < s < i \) and see what happens. Then \( c_{2n+1-s} \in \mathbb{N} \) and there exists \( 1 \leq t < 2n+1-s \) such that \( c_t = c_{2n+1-s} \).

By assumptions, \( t \) satisfies \( 1 \leq t < 2n+1 - i \). Then there exist \( v_t, v_s \in \mathbb{C}^n \) satisfying
\[ V_{t-1} \oplus \langle v_t \rangle \subseteq V_{2n-i} \subseteq V_{2n-s+1} = V_{2n-s} \oplus \langle \pi_+(v_t) \rangle \]
and
\[ V_s = V_{s-1} \oplus \langle v_s \rangle \subseteq V_t = V_{i-1} \oplus \langle \pi_-(v_s) \rangle. \]
Thus,
\[ V_{2n-s} = V_s \oplus \langle v_s \rangle \subseteq \langle v_s \rangle. \]

Since \( v_t \in V_{2n-i} = V_s \oplus \langle v_s \rangle \subseteq \langle \pi_-(v_s) \rangle \subset \langle \pi_-(v_s) \rangle \),
we have
\[ \omega(\pi_+(v_t), v_s) = \omega(\pi_+(v_t), \pi_-(v_s)) = 0. \]
So
\[ \langle \pi_+(v_t) \rangle \subseteq \langle v_s \rangle \]
and the other hand
\[ V_{2n-s} = V_s \oplus \langle v_s \rangle \subseteq \langle \pi_+(v_t) \rangle \]
\[ = V_{2n-s+1} \cap \langle v_s \rangle \]
\[ = (V_{2n-s} \oplus \langle \pi_+(v_t) \rangle) \cap \langle v_s \rangle \]
\[ = V_{2n-s} \oplus \langle \pi_+(v_t) \rangle = V_{2n-s+1} \]
by (46) and (47). That contradicts \( V_{2n-s} \neq V_{2n-s+1} \).
Therefore \( c_{2n+1-j} = c_{2n+1-i} \in \mathbb{N} \) if \( c_i = c_j \in \mathbb{N} \) by induction. \( \square \)

**Definition 3.2.3** (Generalized clan). We call a clan satisfying the condition in Proposition 3.2.2 a *generalized clan* for \( Sp(n, \mathbb{R}) \). We denote the set of generalized clans for \( Sp(n, \mathbb{R}) \) by \( \mathcal{G}(Sp(n, \mathbb{R})) \):
\[ \mathcal{G}(Sp(n, \mathbb{R})) = \{ \gamma \in \mathcal{G}(U(n, n)) \mid \gamma = -\gamma \}. \]

**Example 3.2.4.** The set \( \mathcal{G}(Sp(2, \mathbb{R})) \) consists of 11 generalized clans:
\[
\begin{align*}
\{ & + - - - , + - + - , - + - + , - - + + , + 1 1 - , - 1 1 + , 1 + - 1 , - - 1 + , 1 1 2 2 , 1 2 1 2 , 1 2 2 1 \}.
\end{align*}
\]
We recall the notion of clans in [6] in order to compare our notion of generalized clans.

**Definition 3.2.5.** [6] A *clan* for $Sp(n, \mathbb{R})$ is an equivalence class of ordered sets $(c_1 \cdots c_n)$ of $n$ symbols satisfying the following five conditions.

1. For $1 \leq i \leq n$, a symbol $c_i$ is $+, -, 0, a,$ or $\overline{a}$ for $a \in \mathbb{N}$.
2. If $c_i = a \in \mathbb{N}$, then there exists a unique $j \neq i$ with $c_j = a$ and $c_t \neq a$ for all $t$.
3. If $c_i = +$ or $\overline{a}$ for $a \in \mathbb{N}$, then there exists a unique $j \neq i$ with $c_j = +$ and $c_t \neq a$ for all $t$.
4. If $c_i = a$ or $\overline{a}$ for $a \in \mathbb{N}$ and $b < a$, then there exists some $j$ such that $c_j = b$ or $\overline{b}$.
5. Two ordered sets $(c_1 \cdots c_n)$ and $(c'_1 \cdots c'_n)$ are regarded as equivalent if and only if there exists a permutation $\sigma \in \mathcal{S}_n$ for

$m := \max\{b \in \mathbb{N} \mid b \text{ or } \overline{b} \text{ for some } i\}$

such that

$$c_i = \begin{cases} 
\sigma(a) & \text{if } c'_i = a \in \mathbb{N}, \\
+ & \text{if } c'_i = +, \\
- & \text{if } c'_i = -, \\
0 & \text{if } c'_i = 0, \\
\sigma(a) & \text{if } c'_i = \overline{a} (a \in \mathbb{N})
\end{cases}$$

for $1 \leq i \leq n$.

For example, $(\overline{2} 1 1 + \overline{2} -) = (\overline{1} 2 2 + \overline{1} -)$ as a clan.

The following one-to-one correspondence exists between clans for $Sp(n, \mathbb{R})$ and generalized clans for $Sp(n, \mathbb{R})$.

A clan $(c'_1 \cdots c'_n)$ corresponds to a generalized clan $(c_1 \cdots c_{2n})$ satisfying the following five conditions for $1 \leq i, j \leq n$:

1. If $c'_i = +$, then $(c_i, c_{2n+1-i}) = (+, -)$.
2. If $c'_i = -$, then $(c_i, c_{2n+1-i}) = (-, +)$.
3. If $c'_i = c'_j \in \mathbb{N}$, then $c_i = c_j \in \mathbb{N}$ and $c_{2n+1-i} = c_{2n+1-j} \in \mathbb{N}$.
4. If $c'_i = c'_j = \overline{a}$, then $c_i = c_{2n+1-j} \in \mathbb{N}$ and $c_{2n+1-i} = c_j \in \mathbb{N}$.
5. If $c_i = 0$, then $c_i = c_{2n+1-i} \in \mathbb{N}$.

For example, a clan $(\overline{1} 2 2 + 0 \overline{1})$ corresponds to a generalized clan

$$(1 2 2 + 3 4 1 3 - 5 5 4).$$

Therefore $C(Sp(n, \mathbb{R}))$ is a parametrization of $K$-orbits in $X$ by [6, Theorem 4.1].

**Corollary 3.2.6.** Generalized clans in $C(Sp(n, \mathbb{R}))$ parametrize $K$-orbits in the flag variety $X$ via the correspondence in Proposition 2.2.6.

**Notation 3.2.7.** For each generalized clan $\gamma$, let $Q_\gamma$ be the $K$-orbit in the flag variety $X$ corresponding to $\gamma$ via the parametrization of Corollary 3.2.6.

3.2-2. *Representatives of $GL(n, \mathbb{C})$-orbits in $Sp(n, \mathbb{C})/B$.* We will give an element $g \in G$ such that the flag $x = gB$ in $X = Sp(n, \mathbb{C})/B$ corresponds to a generalized clan $\gamma \in C(Sp(n, \mathbb{R}))$ and the flag $x = gB_{AIII}$ in $GL(2n, \mathbb{C})/B_{AIII}$ corresponds to the clan $\gamma \in C(U(n, n))$. 

Definition 3.2.8 (Skew symmetric signed clan). A skew symmetric signed clan \( \gamma \) = \( (d_1 \cdots d_m) \) of a generalized clan \( \gamma = (c_1 \cdots c_m) \) is a signed clan (in Definition 2.2.9) of the clan \( \gamma \) satisfying the following condition.

\[
(48) \quad \text{If } c_i = c_j = a \in \mathbb{N} \text{ for } i < j, \text{ then } (d_i, d_j) = (a_+, a_-).
\]

Example 3.2.9. The skew symmetric signed clan of \( \gamma = (121 - 32 + 434) \) is

\[
\gamma_- = \begin{pmatrix} 1_+ & 2_+ & 1_- & 3_+ & 2_- & 4_+ & 3_- & 4_-
\end{pmatrix}.
\]

Corollary 3.2.10. In a skew symmetric signed clan \( (d_1 \cdots d_m) \), the signature of \( d_i \) is not equal to the signature of \( d_{m+1-i} \) for \( 1 \leq i \leq m \).

In this section, we call a skew symmetric signed clan just a signed clan.

We will give a representative \( g \in G \) of \( Q_\gamma = K \cdot g \cdot B \).

Theorem 3.2.11. For \( \gamma \in C(\text{Sp}(n, \mathbb{R})) \) and the signed clan \( \gamma_- = (d_1 \cdots d_{2n}) \) of \( \gamma \), fix a permutation \( \sigma'' \in \mathfrak{S}_n \) and a permutation \( \sigma' \in \mathfrak{S}_{2n} \) satisfying the following condition.

For \( i \leq n, \)

\[
\sigma'(i) = \sigma''(i) \quad \text{and} \quad \sigma'(2n+1-i) = 2n+1-\sigma''(i)
\]

if the signature of \( d_i \) is plus,

\[
\sigma'(i) = 2n+1-\sigma''(i) \quad \text{and} \quad \sigma'(2n+1-i) = \sigma''(i)
\]

if the signature of \( d_i \) is minus.

The matrix \( g(\gamma) = (g_1 \ g_2 \ \cdots \ g_{2n}) \) in \( \text{Mat}(2n, 2n) \) is a representative of \( Q_\gamma \), i.e., \( Q_\gamma = K g(\gamma) B \). Here \( g_i \in V \) are column vectors defined as follows:

- If \( d_i = - \), then \( g_i = e_{\sigma''(i)} \).
- If \( d_i = + \), then

\[
g_i = \begin{cases} 
  e_{\sigma''(i)} & \text{if } i \leq n, \\
  -e_{\sigma''(i)} & \text{if } n < i.
\end{cases}
\]

- If \( (d_i, d_j) = (a_+, a_-) \) for some \( a \in \mathbb{N} \), then

\[
g_i = \begin{cases} 
  \frac{1}{\sqrt{2}}(e_{\sigma''(i)} + e_{\sigma''(j)}) & \text{if } i \leq n, \\
  -\frac{1}{\sqrt{2}}(e_{\sigma''(i)} + e_{\sigma''(j)}) & \text{if } n < i,
\end{cases}
\]

and

\[
g_j = \frac{1}{\sqrt{2}}(-e_{\sigma''(i)} + e_{\sigma''(j)}).
\]

Proof. The permutation \( \sigma' \) satisfies the conditions (13) of \( \sigma \) in Theorem 2.2.14 for \( U(n, n) \). Thus, \( g(\gamma) \) is a representative of \( K\mathfrak{A}_{2n} \cdot Q_\gamma \cdot B \mathfrak{A}_{2n} \), i.e., \( Q_\gamma = K \cdot g(\gamma) \cdot B \mathfrak{A}_{2n} \), and \( g(\gamma) \in G \). Therefore, we have

\[ Q_\gamma = K g(\gamma) B. \]

Example 3.2.12. For a signed clan \( \gamma_- = (1_+ 2_+ 1_- 3_+ 2_- 4_+ 3_- 4_-) \), for example, if we choose an identity permutation as \( \sigma'' \),

\[
\sigma'' = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\
 1 & 2 & 3 & 4 & 5 & 6
\end{pmatrix},
\]
then $\sigma'$ is as follows:

$$
\sigma' = \begin{pmatrix}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 \\
1 & 2 & 10 & 9 & 5 & 6 & 7 & 8 & 4 & 3 & 11 & 12
\end{pmatrix}.
$$

**Remark 3.2.13.** The representative $g(\gamma)$ is real and orthogonal:

$$
g(\gamma)^{-1} = hg(\gamma).
$$

### 3.3. M-O graph for $Sp(n, \mathbb{R})$

We define an oriented graph which has generalized clans as vertices and in which an edge is labeled by $i$ for some $1 \leq i \leq n$. We can read off closure relations in the graph.

**Definition 3.3.1.** Let $\Gamma(Sp(n, \mathbb{R}))$ be the set of all triples $(\gamma, \gamma', i)$ with $\gamma, \gamma' \in C(Sp(n, \mathbb{R}))$ and $1 \leq i \leq n$, satisfying one of the following two conditions.

1. $(\gamma, \gamma', i) \in \Gamma(U(n, n))$. (This case happens only if $i = n$ or $i \neq n$ and $(c_i, c_{i+1}) = (c_{2n-i}, c_{2n+1-i}) = (a, b)$ for $a, b \in \mathbb{N}$.)

2. There exists a clan $\gamma'' \in C(U(n, n))$ such that

$$
\{(\gamma, \gamma'', i), (\gamma'', \gamma', 2n - i)\} \in \Gamma(U(n, n)).
$$

**Example 3.3.2.** Since $(+ - + -), (+ 1 1 -) \in C(Sp(2, \mathbb{R}))$, and

$$
((+ - + -), (+ 1 1 -), 2) \in \Gamma(U(2, 2)),
$$

$$(+ - + -), (+ 1 1 -), 2 \in \Gamma(Sp(2, \mathbb{R})).
$$

Since $(1212), (1221) \in C(Sp(2, \mathbb{R}))$, and $((1212), (1221), 1) \in \Gamma(U(2, 2))$

$$(1212), (1221), 1 \in \Gamma(Sp(2, \mathbb{R})).
$$

Since $(+ - + -), (1222) \in C(Sp(2, \mathbb{R}))$, $(11 + -) \in C(U(2, 2))$, and

$$
\{((+ - + -), (11 + -), 1), ((11 + -), (1222), 3)\} \in \Gamma(U(2, 2)),
$$

$$(+ - + -), (1222), 1 \in \Gamma(Sp(2, \mathbb{R})).
$$

**Notation 3.3.3.** We denote by $B_{i\mathbb{A}H}$ the parabolic subgroup of $G_{i\mathbb{A}H}$ in Notation 2.4.8. We denote by $B_i$ the parabolic subgroup of $G$ for the root $-\alpha_i$ and all positive roots:

$$
B_i = (B_{i\mathbb{A}H}B_{2n-i\mathbb{A}H}) \cap G.
$$

Let

$$
\pi_i : G/B \to G/B_i
$$

be the canonical projection.

**Remark 3.3.4.** The projection $\pi_i$ sends $(V_0, \ldots, V_{2n}) \in G/B$ to

$$
\begin{cases}
(V_0, \ldots, V_{i-1}, V_{i+1}, \ldots, V_{2n-i-1}, V_{2n-i+1}, \ldots, V_{2n}) & \text{if } 1 \leq i < n, \\
(V_0, \ldots, V_{n-1}, V_{n+1}, \ldots, V_{2n}) & \text{if } i = n.
\end{cases}
$$

By Theorem 2.2.14 and Remark 3.3.4, we have the following proposition.

**Proposition 3.3.5.** If $(\gamma, \gamma', i) \in \Gamma(Sp(n, \mathbb{R}))$, then

$$
\pi_i(Q_\gamma) = \pi_i(Q_{\gamma'}). 
$$
Proposition 3.3.6. If \((\gamma, \gamma', i) \in \Gamma(Sp(n, R))\), then
\[ \dim Q_\gamma = \dim Q_{\gamma'} - 1 \]

and
\[ Kg(\gamma)B \subset Kg(\gamma')B. \]

Proof. By [6], if \((\gamma, \gamma', i) \in \Gamma(Sp(n, R))\), then we have
\[ \dim Q_\gamma = \dim Q_{\gamma'} + 1 \]

and
\[ Kg(\gamma)B \subset Kg(\gamma')B, \quad \text{or} \quad Kg(\gamma)B \subset Kg(\gamma)B. \]

If \(i = n\), then \((\gamma, \gamma', n) \in \Gamma(U(n, n))\). Therefore
\[ K_{\text{AHG}}(\gamma)B_{\text{AH}} \subset K_{\text{AHG}}(\gamma')B_{\text{AH}}. \]

Although the definition of the following graph follows [6], we use generalized clans instead of clans as vertices.

Definition 3.3.7 (M-O graph of \(Sp(n, R)\)). Let \(C_0(Sp(n, R)) = C(Sp(n, R)) \cap C_0(U(n, n))\):
\[ C_0(Sp(n, R)) = \{(c_1 \cdots c_n) \in C(Sp(n, R)) | c_i \notin N \text{ for all } 1 \leq i \leq n\}. \]

We give subsets \(\Gamma_m(Sp(n, R))\) of \(\Gamma(Sp(n, R))\) and subsets \(C_m(Sp(n, R))\) of \(C(Sp(n, R))\) for \(m \geq 1\) by induction as follows:
\[ \Gamma_m(Sp(n, R)) := \{(\gamma', \gamma, i) \in \Gamma'(Sp(n, R)) | \gamma' \in C_{m-1}(Sp(n, R))\}, \]
\[ C_m(Sp(n, R)) := \{\gamma \in C(Sp(n, R)) | (\gamma', \gamma, i) \in \Gamma_m(Sp(n, R))\}. \]

The M-O graph with generalized clans of \(Sp(n, R)\) is a finite oriented graph whose vertices are generalized clans in \(C(Sp(n, R))\) and whose oriented edges are \(\Gamma(Sp(n, R)) = \bigcup_{m \in N} \Gamma_m(Sp(n, R))\).

Example 3.3.8. The \(\Gamma(Sp(2, R))\) is as Figure 2.

Remark 3.3.9. If \((\gamma, \gamma', i) \in \Gamma(Sp(n, R))\), then
\[ \pi_i(Q_\gamma) = \pi_i(Q_{\gamma'}) \quad \text{and} \quad \dim Q_\gamma + 1 = \dim Q_{\gamma'}. \]

3.4. Dimensions of \(GL(n, C)\)-orbits in \(Sp(n, C)/B\). In this section we will give a dimension formula of \(K\)-orbits for generalized clans.

Definition 3.4.1. We define the length \(\ell_C(\gamma)\) of a generalized clan \(\gamma = (c_1 \cdots c_{2n}) \in C(Sp(n, R))\) such that
\[ (49) \quad \ell_C(\gamma) = \frac{1}{2} (\ell(\gamma) + \#\{t \in N | c_s = c_t \in N \text{ and } s \leq n < t \leq 2n + 1 - s\}). \]

Where \(\ell(\gamma)\) is the length of the clan \(\gamma\) which is given in Definition 2.3.7.

Proposition 3.4.2. If \(\gamma \in C_m(Sp(n, R))\), then \(\ell_C(\gamma) = m\).

Proof. We prove the proposition by induction on \(m\). Let \(\gamma = (c_1 \cdots c_{2n})\).
1. If $\gamma \in C_0(Sp(n, \mathbb{R}))$, then $\ell_C(\gamma) = 0$.
2. Suppose $\ell_C(\gamma') = j$, if $\gamma' \in C_j(Sp(n, \mathbb{R}))$. For $\gamma \in C_{j+1}(Sp(n, \mathbb{R}))$, there exists $\gamma'' = (c_1 \cdots c_{2n}) \in C_j(Sp(n, \mathbb{R}))$ such that $(\gamma'', \gamma, i) \in \Gamma(Sp(n, \mathbb{R}))$.
   
(a) If $i = n$, then $\ell(\gamma) = \ell(\gamma') + 1$ and
   $$\# \{t \in \mathbb{N} | c_s = c_t \in \mathbb{N} \text{ and } s \leq t \leq 2n + 1 - s\} - \# \{n + 1\} = \# \{t \in \mathbb{N} | c'_s = c'_t \in \mathbb{N} \text{ and } s \leq t \leq 2n + 1 - s\}.$$

So, $\ell_C(\gamma') = j + 1$.

(b) If $i \neq n$ and $(c_i, c_{i+1}) = (c_{2n-i}, c_{2n-i+i}) = (a, b)$ for some $a \in \mathbb{N}$ and $b \in \mathbb{N}$, then
   $$\ell(\gamma) = \ell(\gamma') + 1$$

and
   $$\# \{t \in \mathbb{N} | c_s = c_t \in \mathbb{N} \text{ and } s \leq t \leq 2n + 1 - s\} - \# \{2n + 1 - i\} = \# \{t \in \mathbb{N} | c'_s = c'_t \in \mathbb{N} \text{ and } s \leq t \leq 2n + 1 - s\}$$

(c) If $i \neq n$ and not $(c_i, c_{i+1}) = (c_{2n-i}, c_{2n-i+i}) = (a, b)$, then there exists a clan $\gamma'' \in \mathcal{C}(U(n, n))$ such that $(\gamma'', \gamma', i), (\gamma'', \gamma, 2n - i) \in \Gamma(U(n, n))$.

So, we have
   $$\ell(\gamma) = \ell(\gamma'') + 1 = \ell(\gamma') + 2$$

and
   $$\# \{t \in \mathbb{N} | c_s = c_t \in \mathbb{N} \text{ and } s \leq t \leq 2n + 1 - s\} = \# \{t \in \mathbb{N} | c'_s = c'_t \in \mathbb{N} \text{ and } s \leq t \leq 2n + 1 - s\}.$$

Therefore $\ell_C(\gamma) = j + 1$.

So, $\ell_C(\gamma) = m$ if $\gamma \in C_m(Sp(n, \mathbb{R}))$ for all $m \in \mathbb{N}$.

A generalized clan
   $$\gamma = (1 \ 2 \ \cdots \ n \ \cdots \ 2 \ 1)$$
corresponds to the open orbit and
\[ \ell_C(\gamma) = \frac{1}{2} n(n + 1). \]
Since the dimension of a closed orbit is \( \frac{1}{2} n(n - 1) \), we have the following dimension formulas.

**Proposition 3.4.3.** For \( \gamma \in C(Sp(n, \mathbb{R})) \), we have the dimension and the codimension of \( Q_\gamma = K g(\delta) B \subset X \):

\[
\begin{align*}
\dim Q_\gamma &= \ell_C(\gamma) + \frac{1}{2} n(n - 1), \\
\text{codim } Q_\gamma &= \frac{1}{2} n(n + 1) - \ell_C(\gamma).
\end{align*}
\]

3.5. **Images of the moment map.** We denote \( (\mathfrak{b}_{\mathfrak{AII}})_{\perp} \) by \( \mathfrak{b}_{\perp}^\perp \). We have

\[
(\operatorname{Ad}(g) \mathfrak{b}_{\perp}^\perp) \cap \mathfrak{p} = (\operatorname{Ad}(g) \cdot \mathfrak{b}_{\perp}^\perp \cap \mathfrak{g}) \cap (\mathfrak{p}_{\mathfrak{AII}} \cap \mathfrak{g}) = ((\operatorname{Ad}(g) \cdot \mathfrak{b}_{\perp}^\perp) \cap \mathfrak{p}_{\mathfrak{AII}}) \cap \mathfrak{g},
\]
and elements \( X \) in \( \mathfrak{p} \) satisfy

\[ X_{ij} = X_{2n+1-j,2n+1-i}. \]
Since we gave the image \( (\operatorname{Ad}(g) \cdot \mathfrak{b}_{\perp}^\perp) \cap \mathfrak{p}_{\mathfrak{AII}} \) of the moment map \( \mu \) in Proposition 2.5.2 and \( \sigma := \sigma' \) in Theorem 3.2.11, we get the following proposition.

**Proposition 3.5.1.** For a generalized clan \( \gamma = (c_1 \cdots c_{2n}) \in C(Sp(n, \mathbb{R})) \), fix a permutation \( \sigma' \in S_{2n} \) as in Theorem 3.2.11. Let the representative \( g := g(\gamma) \) be given as in Theorem 3.2.11, and \( x = gB \in Q_\gamma \). We can read off \( \mu(T_{Q_\gamma}X)_x \) from a vector subspace \( \text{Dri}(\gamma) \) of \( \mathfrak{g} \).

\[
\mu(T_{Q_\gamma}X)_x = \{ Y(\sigma'^{-1}1, \ldots, \sigma'^{-1}2n) \mid Y \in \text{Dri}(\gamma) \}.
\]
Here, \( \text{Dri}(\gamma) \) is the subspace of \( \text{Dri}(\gamma_-) \), the driving space of the signed clan \( \gamma_- \) for \( U(n, n) \) in Definition 2.5.1, is defined as follows:

\[
\text{Dri}(\gamma) := \text{Dri}(\gamma_-) \cap \operatorname{Ad}(\sigma'^{-1}) \cdot \mathfrak{g} = \{ Y \in \text{Dri}(\gamma_-) \mid Y_{ij} = Y_{2n+1-j,2n+1-i} \}.
\]
We call the space \( \text{Dri}(\gamma) \) the driving space of the generalized clan \( \gamma \) (for \( Sp(n, \mathbb{R}) \)).

**Proof.** For the signed clan \( \gamma_- = (d_1 \cdots d_{2n}) \) and the permutation \( \sigma' \) in Theorem 3.2.11 and a representative \( g_{\mathfrak{AII}}(\gamma_-, \sigma') \) in Theorem 2.2.14, we have

\[ g = g(\gamma_, \sigma') \operatorname{diag}(\varepsilon_1, \ldots, \varepsilon_{2n}). \]
Here,

\[ \varepsilon_i = \begin{cases} 
-1 & \text{if } n < i \text{ and the signature of } d_i \text{ is plus,} \\
1 & \text{otherwise.}
\end{cases} \]

So,

\[
(\operatorname{Ad}(g) \mathfrak{b}_{\perp}^\perp) \cap \mathfrak{p}_{\mathfrak{AII}} = (\operatorname{Ad}(g(\gamma_-, \sigma'))(\operatorname{Ad}(\operatorname{diag}(\varepsilon_1, \ldots, \varepsilon_{2n})) \cdot \mathfrak{b}_{\perp}^\perp) \cap \mathfrak{p}_{\mathfrak{AII}} = (\operatorname{Ad}(g(\gamma_-, \sigma')) \cdot \mathfrak{b}_{\perp}^\perp) \cap \mathfrak{p}_{\mathfrak{AII}} = \operatorname{Ad}(\sigma') \cdot \text{Dri}(\gamma_-)
\]
Therefore,
\[
\mu(T_Q^* X)_x = (\text{Ad}(\sigma') \cdot \text{Dri}(\gamma)) \cap g
\]
\[
= \text{Ad}(\sigma') \cdot (\text{Dri}(\gamma) \cap \text{Ad}(\sigma'^{-1})g)
\]
\[
= \text{Ad}(\sigma') \cdot \text{Dri}(\gamma).
\]
is as stated in the proposition. \(\square\)

**Corollary 3.5.2.** For the representative \(g := g(\gamma)\) of \(Q_\gamma\) given as in Theorem 3.2.11, we have the following equation.

\[
\text{Dri}(\gamma) = \text{Ad}(\sigma'^{-1})((\text{Ad}(g)b^\perp) \cap p)
\]
\[
= \{ A(\sigma'(1),\ldots,\sigma'(2n)) | A \in (\text{Ad}(g)b^\perp) \cap p \}. 
\]

Since \(g(\gamma) = (e_{\sigma'(1)} \cdots e_{\sigma'(2n)}) = \sigma'\) if a generalized clan \(\gamma\) corresponds to a closed orbit, we have the following lemma.

**Lemma 3.5.3.** If a generalized clan \(\gamma\) is an element of \(C_0(\text{Sp}(n, \mathbb{R}))\), then

\[
\text{Dri}(\gamma) = b^\perp \cap (\text{Ad}(g(\gamma)^{-1}) \cdot p).
\]

**Definition 3.5.4** (Driving matrix). For a generalized clan \(\gamma \in C(\text{Sp}(n, \mathbb{R}))\), let \(Y(\gamma)\) and \(Y(\gamma, m), 1 \leq m \leq 8\) be elements of \(F \otimes_{\mathbb{C}} \text{Mat}(2n, 2n)\) satisfying the following conditions. Let \(\gamma = (d_1 \cdots d_{2n})\).

1. \(Y(\gamma, 1)_{ij} = \begin{cases} 1 & \text{if } j < i \leq 2n + 1 - j, \\ 0 & \text{otherwise}. \end{cases}\)
2. \(Y(\gamma, 2)_{ij} = \begin{cases} 0 & \text{if } (d_i, d_j) = (+, +) \text{ or } (-, -), \\ Y(\gamma, 1)_{ij} & \text{otherwise}. \end{cases}\)
3. If \((d_i, d_j) = (a_+, a_-),\) then
   - \(Y(\gamma, 3)_{i,k} = 0\) if \(1 \leq k_1 \leq i,\)
   - \(Y(\gamma, 3)_{k,i} = Y(\gamma, 3)_{i,k} = 0\) if \(i < k_2 \leq j,\)
   - \(Y(\gamma, 3)_{k,j} = 0\) if \(j < k_3 \leq 2n,\)

for all \(a \in \mathbb{N}\) and \(Y(\gamma, 3)_{i,j'} = Y(\gamma, 2)_{i,j'}\) otherwise.

**Remark 3.5.5.** If \(Y(\gamma, 3)_{ij} = 1\), then the following conditions are satisfied.

(a) \(j < i \leq 2n + 1 - j\) (so \(j \leq n\)).
(b) \((d_i, d_j) = (a_-, b_+), (a_-, +), (a_-, -), (+, a_+), (-, a_+), (+, -), \text{ or } (-, +)\) for some \(a, b \in \mathbb{N}, a \neq b.\)
(c) If \(d_i = a_-\) for some \(a \in \mathbb{N},\) then we have \(d_k = a_+\) for some \(j < k < i.\)
(d) If \(d_j = a_+\) for some \(a \in \mathbb{N},\) then we have \(d_k = a_-\) for some \(j < k < i.\)

4. If \(Y(\gamma, 3)_{ij} = 1, (d_i, d_j) = (a_-, b_+)\) and \((d_k, d_l) = (a_+, b_-)\) for some \(a, b \in \mathbb{N},\) then
   - \(Y(\gamma, 4)_{ij} = Y(\gamma, 4)_{2n+1-j, 2n+1-i} = y_{ij},\)
   - \(Y(\gamma, 4)_{kl} = Y(\gamma, 4)_{2n+1-l, 2n+1-k} = \begin{cases} y_{ij} & \text{if } j < k < l < i, \\ y_{kl} & \text{if } j < l < k < i, \end{cases}\)

for all \(a, b \in \mathbb{N},\) and \(Y(\gamma, 4)_{i,j'} = Y(\gamma, 3)_{i,j'}\) otherwise.
5. If \( Y(\gamma, 4)_{ij} = 1 \), \((d_i, d_j) = (a_-, -)\), and \( d_k = a_+ \) for \( a \in \mathbb{N} \), then
\[
Y(\gamma, 5)_{ij} = 0 \quad \text{and} \quad Y(\gamma, 5)_{kj} = Y(\gamma, 5)_{2n+1-j, 2n+1-k} = y_{kj},
\]
and \( Y(\gamma, 5)_{ij'} = Y(\gamma, 4)_{ij'} \) otherwise.

6. If \( Y(\gamma, 5)_{ij} = 1 \), \((d_i, d_j) = (+, a_+)\), and \( d_k = a_- \) for \( a \in \mathbb{N} \), then
\[
Y(\gamma, 6)_{ij} = 0 \quad \text{and} \quad Y(\gamma, 6)_{ik} = Y(\gamma, 6)_{2n+1-k, 2n+1-i} = y_{ik},
\]
and \( Y(\gamma, 6)_{ij'} = Y(\gamma, 5)_{ij'} \) otherwise.

7. \( Y(\gamma)_{ij} = Y(\gamma)_{2n+1-j, 2n+1-i} = \begin{cases} y_{ij} & \text{if } Y(\gamma, 6)_{ij} = 1, \\ Y(\gamma, 6)_{ij} & \text{otherwise.} \end{cases} \)

We call \( Y(\gamma) \) the \textit{driving matrix} of \( \gamma \) (for \( Sp(n, \mathbb{R}) \)).

By the same arguments of Proposition 2.5.8, we get the following proposition.

**Proposition 3.5.6.** For a generalized clan \( \gamma \in C(Sp(n, \mathbb{R})) \), the Driving space \( \text{Dri}(\gamma) \) for \( Sp(n, \mathbb{R}) \) and the Driving matrix \( Y(\gamma) \) of \( \gamma \) for \( Sp(n, \mathbb{R}) \) satisfy the following condition.

\[
\text{Dri}(\gamma) = \{ Y(\gamma) \mid y_{ij} \in \mathbb{C}, \text{ for } 1 \leq i, j \leq 2n \}.
\]

By Proposition 3.5.1 and Proposition 3.5.6, we got the following theorem which means we can read off \( \mu(T_Q^n, X)_x \) from the driving matrix \( Y(\gamma) \).

**Theorem 3.5.7.** For a generalized clan \( \gamma \in C(Sp(n, \mathbb{R})) \), fix a permutation \( \sigma' \in \mathfrak{S}_{2n} \) in Theorem 3.2.11. Let the representative \( q := g(\gamma) \) be given as in Theorem 3.2.11, \( x = gB \in Q_\gamma \). We can read off \( \mu(T_Q^n, X)_x \) from the driving matrix \( Y(\gamma) \) for \( Sp(n, \mathbb{R}) \) as follows:

\[
\mu(T_Q^n, X)_x = \{ Y(\gamma)_{(\sigma'^{-1}(1), \ldots, \sigma'^{-1}(2n))} \mid y_{ij} \in \mathbb{C}, \text{ for } 1 \leq i, j \leq 2n \}.
\]

### 3.6. Signed Young diagrams.

In this section, after we recall a parametrization of nilpotent \( K \)-orbits in \( p \), we give tables of signed Young diagrams for the clans of \( Sp(1, \mathbb{R}), \ Sp(2, \mathbb{R}), \) and \( Sp(3, \mathbb{R}) \) by way of examples.

Nilpotent \( K \)-orbits in \( p \) are parametrized by signed Young diagrams of signature \((n, n)\) satisfying the following conditions.

If \( m \in \mathbb{N} \) is odd, then the number of rows of which the length are \( m \) and which are labeled + in the first column is equal to the number of rows of which the length are \( m \) and which are labeled — in the first column.

For an element \( A \) of an orbit, the signed Young diagram of the orbit satisfies

\[
\sum_{j=1}^{i} D_{j,+} = \dim(\ker(A^i|V_+)) \quad \text{and} \quad \sum_{j=1}^{i} D_{j,-} = \dim(\ker(A^i|V_-)).
\]

**Proposition 3.6.1.** Under the conditions of Theorem 3.2.11, we put

\[
V'_+ := \sigma'^{-1}V_+ = (e_{\sigma'^{-1}(1)} \cdots e_{\sigma'^{-1}(2n)})V_+ \\
= (e_{\sigma'^{-1}(1)}, \ldots, e_{\sigma'^{-1}(n)}),
\]

\[
V'_- := \sigma'^{-1}V_- = (e_{\sigma'^{-1}(1)} \cdots e_{\sigma'^{-1}(2n)})V_- \\
= (e_{\sigma'^{-1}(n+1)}, \ldots, e_{\sigma'^{-1}(2n)}).
\]
By the same argument of the case of $U(n, n)$, we have the following proposition.

**Proposition 3.6.2.** Let $Y(\gamma)^0 = I_n$. We have the following equations:

$D_{i,+} = \dim_F \{ \vec{a} \in F \otimes_C V'_+ | Y(\gamma)^i \vec{a} = \vec{0} \} - \dim_F \{ \vec{a} \in F \otimes_C V'_+ | Y(\gamma)^{i-1} \vec{a} = \vec{0} \}$,

$D_{i,-} = \dim_F \{ \vec{b} \in F \otimes_C V'_- | Y(\gamma)^i \vec{b} = \vec{0} \} - \dim_F \{ \vec{b} \in F \otimes_C V'_- | Y(\gamma)^{i-1} \vec{b} = \vec{0} \}$.

At last we give tables of signed Young diagrams for the clans of $Sp(1, \mathbb{R})$ in Example 3.6.3, $Sp(2, \mathbb{R})$ in Example 3.6.4, and $Sp(2, \mathbb{R})$ in Example 3.6.5.

**Example 3.6.3.** This is the table of the case of $G_{\mathbb{R}} = Sp(1, \mathbb{R})$.

<table>
<thead>
<tr>
<th>clan $\gamma$</th>
<th>a representative $g(\gamma)$ of $K$-orbit</th>
<th>driving matrix $Y(\gamma)$</th>
<th>signed Young diagram</th>
</tr>
</thead>
<tbody>
<tr>
<td>$+-$</td>
<td>$\begin{pmatrix} 1 &amp; 0 \ 0 &amp; 1 \end{pmatrix}$</td>
<td>$\begin{pmatrix} 0 &amp; 0 \ y_{21} &amp; 0 \end{pmatrix}$</td>
<td>![Diagram]</td>
</tr>
<tr>
<td>$-+$</td>
<td>$\begin{pmatrix} 0 &amp; -1 \ 1 &amp; 0 \end{pmatrix}$</td>
<td>$\begin{pmatrix} 0 &amp; 0 \ y_{21} &amp; 0 \end{pmatrix}$</td>
<td>![Diagram]</td>
</tr>
<tr>
<td>$1\ 1$</td>
<td>$\begin{pmatrix} \frac{1}{\sqrt{2}} &amp; \frac{1}{\sqrt{2}} \ \frac{1}{\sqrt{2}} &amp; -\frac{1}{\sqrt{2}} \end{pmatrix}$</td>
<td>$\begin{pmatrix} 0 &amp; 0 \ 0 &amp; 0 \end{pmatrix}$</td>
<td>![Diagram]</td>
</tr>
</tbody>
</table>
Example 3.6.4. This is the table of the case of $G_{\mathbb{R}} = Sp(2, \mathbb{R})$.

<table>
<thead>
<tr>
<th>clan $\gamma$</th>
<th>a representative $g(\gamma)$ of $K$-orbit</th>
<th>driving matrix $Y(\gamma)$</th>
<th>signed Young diagram</th>
</tr>
</thead>
<tbody>
<tr>
<td>$+++$</td>
<td>$\begin{pmatrix} 1 &amp; 0 &amp; 0 &amp; 0 \ 0 &amp; 0 &amp; -1 &amp; 0 \ 0 &amp; 1 &amp; 0 &amp; 0 \ 0 &amp; 0 &amp; 0 &amp; 1 \end{pmatrix}$</td>
<td>$\begin{pmatrix} 0 &amp; 0 &amp; 0 &amp; 0 \ y_{21} &amp; 0 &amp; 0 &amp; 0 \ 0 &amp; y_{32} &amp; 0 &amp; 0 \ y_{41} &amp; 0 &amp; y_{21} &amp; 0 \end{pmatrix}$</td>
<td>$\begin{array}{c} \square \square \square \square \square \square \end{array}$</td>
</tr>
<tr>
<td>$--+$</td>
<td>$\begin{pmatrix} 0 &amp; 1 &amp; 0 &amp; 0 \ 0 &amp; 0 &amp; 0 &amp; -1 \ 1 &amp; 0 &amp; 0 &amp; 0 \ 0 &amp; 0 &amp; 1 &amp; 0 \end{pmatrix}$</td>
<td>$\begin{pmatrix} 0 &amp; 0 &amp; 0 &amp; 0 \ y_{21} &amp; 0 &amp; 0 &amp; 0 \ 0 &amp; y_{32} &amp; 0 &amp; 0 \ y_{41} &amp; 0 &amp; y_{21} &amp; 0 \end{pmatrix}$</td>
<td>$\begin{array}{c} \square \square \square \square \square \square \end{array}$</td>
</tr>
<tr>
<td>$++-$</td>
<td>$\begin{pmatrix} 1 &amp; 0 &amp; 0 &amp; 0 \ 0 &amp; 1 &amp; 0 &amp; 0 \ 0 &amp; 0 &amp; 1 &amp; 0 \ 0 &amp; 0 &amp; 0 &amp; 1 \end{pmatrix}$</td>
<td>$\begin{pmatrix} 0 &amp; 0 &amp; 0 &amp; 0 \ 0 &amp; 0 &amp; 0 &amp; 0 \ y_{31} &amp; y_{32} &amp; 0 &amp; 0 \ y_{41} &amp; y_{31} &amp; 0 &amp; 0 \end{pmatrix}$</td>
<td>$\begin{array}{c} \square \square \square \square \square \square \end{array}$</td>
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<tr>
<td>$+11-$</td>
<td>$\begin{pmatrix} 1 &amp; 0 &amp; 0 &amp; 0 \ 0 &amp; \frac{1}{\sqrt{2}} &amp; \frac{1}{\sqrt{2}} &amp; 0 \ 0 &amp; \frac{1}{\sqrt{2}} &amp; \frac{1}{\sqrt{2}} &amp; 0 \ 0 &amp; 0 &amp; 0 &amp; 1 \end{pmatrix}$</td>
<td>$\begin{pmatrix} 0 &amp; 0 &amp; 0 &amp; 0 \ 0 &amp; 0 &amp; 0 &amp; 0 \ y_{31} &amp; 0 &amp; 0 &amp; 0 \ y_{41} &amp; y_{31} &amp; 0 &amp; 0 \end{pmatrix}$</td>
<td>$\begin{array}{c} \square \square \square \square \square \square \end{array}$</td>
</tr>
<tr>
<td>$-++$</td>
<td>$\begin{pmatrix} 0 &amp; 0 &amp; 1 &amp; 0 \ 0 &amp; 0 &amp; 0 &amp; 1 \ 1 &amp; 0 &amp; 0 &amp; 0 \ 0 &amp; 1 &amp; 0 &amp; 0 \end{pmatrix}$</td>
<td>$\begin{pmatrix} 0 &amp; 0 &amp; 0 &amp; 0 \ 0 &amp; 0 &amp; 0 &amp; 0 \ y_{31} &amp; y_{32} &amp; 0 &amp; 0 \ y_{41} &amp; y_{31} &amp; 0 &amp; 0 \end{pmatrix}$</td>
<td>$\begin{array}{c} \square \square \square \square \square \square \end{array}$</td>
</tr>
<tr>
<td>$-11+$</td>
<td>$\begin{pmatrix} 0 &amp; 0 &amp; 0 &amp; 1 \ 0 &amp; \frac{1}{\sqrt{2}} &amp; \frac{1}{\sqrt{2}} &amp; 0 \ 0 &amp; \frac{1}{\sqrt{2}} &amp; \frac{1}{\sqrt{2}} &amp; 0 \ 1 &amp; 0 &amp; 0 &amp; 0 \end{pmatrix}$</td>
<td>$\begin{pmatrix} 0 &amp; 0 &amp; 0 &amp; 0 \ 0 &amp; 0 &amp; 0 &amp; 0 \ y_{31} &amp; 0 &amp; 0 &amp; 0 \ y_{41} &amp; y_{31} &amp; 0 &amp; 0 \end{pmatrix}$</td>
<td>$\begin{array}{c} \square \square \square \square \square \square \end{array}$</td>
</tr>
<tr>
<td>$1122$</td>
<td>$\begin{pmatrix} \frac{1}{\sqrt{2}} &amp; \frac{1}{\sqrt{2}} &amp; 0 &amp; 0 \ 0 &amp; 0 &amp; \frac{1}{\sqrt{2}} &amp; \frac{1}{\sqrt{2}} \ \frac{1}{\sqrt{2}} &amp; \frac{1}{\sqrt{2}} &amp; 0 &amp; 0 \ 0 &amp; 0 &amp; \frac{1}{\sqrt{2}} &amp; \frac{1}{\sqrt{2}} \end{pmatrix}$</td>
<td>$\begin{pmatrix} 0 &amp; 0 &amp; 0 &amp; 0 \ 0 &amp; 0 &amp; 0 &amp; 0 \ 0 &amp; y_{32} &amp; 0 &amp; 0 \ y_{41} &amp; 0 &amp; 0 &amp; 0 \end{pmatrix}$</td>
<td>$\begin{array}{c} \square \square \square \square \square \square \end{array}$</td>
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<tr>
<td>$1212$</td>
<td>$\begin{pmatrix} \frac{1}{\sqrt{2}} &amp; 0 &amp; \frac{1}{\sqrt{2}} &amp; 0 \ 0 &amp; \frac{1}{\sqrt{2}} &amp; 0 &amp; \frac{1}{\sqrt{2}} \ \frac{1}{\sqrt{2}} &amp; 0 &amp; \frac{1}{\sqrt{2}} &amp; 0 \ 0 &amp; \frac{1}{\sqrt{2}} &amp; 0 &amp; \frac{1}{\sqrt{2}} \end{pmatrix}$</td>
<td>$\begin{pmatrix} 0 &amp; 0 &amp; 0 &amp; 0 \ 0 &amp; 0 &amp; y_{41} &amp; 0 \ 0 &amp; 0 &amp; 0 &amp; 0 \ y_{41} &amp; 0 &amp; 0 &amp; 0 \end{pmatrix}$</td>
<td>$\begin{array}{c} \square \square \square \square \square \square \end{array}$</td>
</tr>
<tr>
<td>clan $\gamma$</td>
<td>a representative $g(\gamma)$ of $K$-orbit</td>
<td>driving matrix $Y(\gamma)$</td>
<td>signed Young diagram</td>
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<tr>
<td>$1 + - 1$</td>
<td>$\begin{pmatrix} \frac{1}{2} &amp; 0 &amp; 0 &amp; \frac{1}{2} \ 0 &amp; 1 &amp; 0 &amp; 0 \ \frac{1}{2} &amp; 0 &amp; 0 &amp; \frac{1}{2} \ 0 &amp; 0 &amp; 1 &amp; 0 \end{pmatrix}$</td>
<td>$\begin{pmatrix} 0 &amp; 0 &amp; 0 &amp; 0 \ 0 &amp; 0 &amp; 0 &amp; 0 \ 0 &amp; 0 &amp; 0 &amp; 0 \ 0 &amp; 0 &amp; 0 &amp; 0 \end{pmatrix}$</td>
<td>$\begin{array}{c} \pm \ \pm \ \pm \end{array}$</td>
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<tr>
<td>$1 - + 1$</td>
<td>$\begin{pmatrix} \frac{1}{2} &amp; 0 &amp; 0 &amp; \frac{1}{2} \ 0 &amp; 0 &amp; 1 &amp; 0 \ \frac{1}{2} &amp; 0 &amp; 0 &amp; \frac{1}{2} \ 0 &amp; 1 &amp; 0 &amp; 0 \end{pmatrix}$</td>
<td>$\begin{pmatrix} 0 &amp; 0 &amp; 0 &amp; 0 \ 0 &amp; 0 &amp; 0 &amp; 0 \ 0 &amp; 0 &amp; 0 &amp; 0 \ 0 &amp; 0 &amp; 0 &amp; 0 \end{pmatrix}$</td>
<td>$\begin{array}{c} \pm \ \pm \ \pm \end{array}$</td>
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<tr>
<td>$1 2 2 1$</td>
<td>$\begin{pmatrix} \frac{1}{2} &amp; 0 &amp; 0 &amp; \frac{1}{2} \ 0 &amp; \frac{1}{2} &amp; 0 &amp; \frac{1}{2} \ \frac{1}{2} &amp; 0 &amp; 0 &amp; \frac{1}{2} \ 0 &amp; \frac{1}{2} &amp; \frac{1}{2} &amp; 0 \end{pmatrix}$</td>
<td>$\begin{pmatrix} 0 &amp; 0 &amp; 0 &amp; 0 \ 0 &amp; 0 &amp; 0 &amp; 0 \ 0 &amp; 0 &amp; 0 &amp; 0 \ 0 &amp; 0 &amp; 0 &amp; 0 \end{pmatrix}$</td>
<td>$\begin{array}{c} \pm \ \pm \ \pm \end{array}$</td>
</tr>
</tbody>
</table>
Example 3.6.5. This is the table of the case of $G_R = Sp(3, R)$.

<table>
<thead>
<tr>
<th>clan $\gamma$</th>
<th>signed Young diagram</th>
<th>clan $\gamma$</th>
<th>signed Young diagram</th>
</tr>
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<tbody>
<tr>
<td>++-+-+</td>
<td>+-----+</td>
<td>1+1 2 -2</td>
<td></td>
</tr>
<tr>
<td>-+-++-</td>
<td>-++++</td>
<td>1+2 1-2</td>
<td></td>
</tr>
<tr>
<td>+-11++</td>
<td></td>
<td>1-1 2+2</td>
<td></td>
</tr>
<tr>
<td>11+-22</td>
<td></td>
<td>1-2 1+2</td>
<td></td>
</tr>
<tr>
<td>-++++-</td>
<td></td>
<td>12+12</td>
<td></td>
</tr>
<tr>
<td>-+11--</td>
<td>+++++</td>
<td>1+-1+1</td>
<td></td>
</tr>
<tr>
<td>11+-22</td>
<td></td>
<td>+1221-</td>
<td></td>
</tr>
<tr>
<td>++-++-</td>
<td></td>
<td>1+22-1</td>
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<tr>
<td>+1122-</td>
<td></td>
<td>1-+-+1</td>
<td></td>
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<tr>
<td>+1212-</td>
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<td>1-221+</td>
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<tr>
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<td>+++++</td>
<td>122331</td>
<td></td>
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<td>+++++</td>
<td>12-21</td>
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<td>-1+-1+</td>
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<tr>
<td>1-++1+</td>
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<td></td>
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<tr>
<td>+++-+-</td>
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<td>12-21</td>
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<td>++11--</td>
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<tr>
<td>+1+-1-</td>
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<td></td>
<td>123321</td>
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</tr>
<tr>
<td>--11++</td>
<td></td>
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<td>-1-+1+</td>
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In this section, we treat the case of $G_{\mathbb{R}} = Sp(p, q)$, i.e., $Sp(p, C) \times Sp(q, C)$-orbits in the flag variety of $Sp(n, C)$. Here, $n = p + q$. We will apply the case of $U(2p, 2q)$ to the case of $Sp(p, q)$. For this, we change the realization of $U(2p, 2q)$. We denote $U(2p, 2q)$ by $G_{\text{RHI}}$.

4. Changing the realization of $U(2p, 2q)$. From now on, we use the following realization of $U(2p, 2q)$.

We realize $G_{\text{RHI}} = U(2p, 2q)$ as the group of matrices $g$ in $GL(2n, C)$ which leave invariant the Hermitian form of the signature $(2p, 2q)$

$$x_1 \bar{x}_1 + \cdots + x_p \bar{x}_p - x_{p+1} \bar{x}_{p+1} - \cdots - x_{p+2q} \bar{x}_{p+2q} + x_{p+2q+1} \bar{x}_{p+2q+1} + \cdots + x_{2n} \bar{x}_{2n},$$

i.e.,

$$U(2p, 2q) = \left\{ g \in GL(2n, C) \mid ^{t}g \begin{pmatrix} I_p & 0 & 0 \\ 0 & -I_{2q} & 0 \\ 0 & 0 & I_p \end{pmatrix} g = \begin{pmatrix} I_p & 0 & 0 \\ 0 & -I_{2q} & 0 \\ 0 & 0 & I_p \end{pmatrix} \right\}.$$

We fix a Cartan involution $\theta$ of $G_{\text{RHI}}$ as follows:

$$\theta : g \mapsto \begin{pmatrix} I_p & 0 & 0 \\ 0 & -I_{2q} & 0 \\ 0 & 0 & I_p \end{pmatrix} \cdot g \cdot \begin{pmatrix} I_p & 0 & 0 \\ 0 & -I_{2q} & 0 \\ 0 & 0 & I_p \end{pmatrix}$$

for $g \in G_{\text{RHI}}$. Then we have

$$\mathfrak{t}_{\text{RHI}} = \left\{ \begin{pmatrix} K_{11} & 0 & K_{13} \\ 0 & K_{22} & 0 \\ K_{31} & 0 & K_{33} \end{pmatrix} \mid K_{11}, K_{13}, K_{31}, K_{33} \in \text{Mat}(p,p), K_{22} \in \text{Mat}(2q, 2q) \right\},$$

$$\mathfrak{K}_{\text{RHI}} = \left\{ \begin{pmatrix} K_{11} & 0 & K_{13} \\ 0 & K_{22} & 0 \\ K_{31} & 0 & K_{33} \end{pmatrix} \mid K_{11}, K_{13}, K_{31}, K_{33} \in \text{Mat}(p,p), K_{22} \in \text{GL}(2q, 2q) \right\},$$

and

$$\mathfrak{p}_{\text{RHI}} = \left\{ \begin{pmatrix} 0 & P_{12} & 0 \\ P_{21} & 0 & P_{23} \\ 0 & P_{32} & 0 \end{pmatrix} \mid P_{12}, P_{32} \in \text{Mat}(p, 2q), P_{21}, P_{23} \in \text{Mat}(2q, p) \right\}.$$

Notation 4.1.1. Let $V = \mathbb{C}^{2n}$ and $\theta$ be an involution of $V$ such that

$$\theta : v \mapsto \begin{pmatrix} I_p & 0 & 0 \\ 0 & -I_{2q} & 0 \\ 0 & 0 & I_p \end{pmatrix} v.$$

Let $V_+$ and $V_-$ be the eigenspaces in $V$ under $\theta$ for eigenvalues $+1$ and $-1$ respectively:

$$V_+ := \langle e_1, \ldots, e_p, e_{p+2q+1}, \ldots, e_{2n} \rangle,$$

$$V_- := \langle e_{p+1}, \ldots, e_{p+2q} \rangle.$$

Under this realization, Theorem 2.2.14 changes into the following proposition.
Proposition 4.1.2. For $\gamma \in \mathbb{C}(U(2p, 2q))$ and a signed clan $\delta = (d_1 \ldots d_{2n})$ of $\gamma$, fix a permutation $\sigma \in S_{2n}$ satisfying the following condition.

\begin{equation}
\begin{aligned}
1 \leq \sigma(i) \leq p & \quad \text{or} \quad p + 2q + 1 \leq \sigma(i) \leq 2n & \quad \text{if the signature of } d_i \text{ is plus}, \\
\quad p + 1 \leq \sigma(i) \leq p + 2q & \quad \text{if the signature of } d_i \text{ is minus}.
\end{aligned}
\end{equation}

Then the matrix $g_{\mathbb{A}_H}(\delta) := g_{\mathbb{A}_H}(\delta, \sigma) = (g_1, g_2 \ldots g_n)$ is a representative of $Q_\gamma$. Here $g_i \in V$ are the column vectors defined as follows:

- If $c_i = \pm$, then $g_i = e_{\sigma(i)}$.
- If $c_i = a_+, c_j = a_-$, then

$$
g_i = \frac{1}{\sqrt{2}}(e_{\sigma(i)} + e_{\sigma(j)}) \quad \text{and} \quad g_j = \frac{1}{\sqrt{2}}(-e_{\sigma(i)} + e_{\sigma(j)}).
$$

4.2. The case of $Sp(p, q)$. We use the realization of $Sp(n, \mathbb{C})$ as (41) in Section 3. We realize an indefinite symplectic group $G_\mathbb{R} = Sp(p, q)$ as a group of matrices $g$ in $Sp(n, \mathbb{C})$ which leave invariant the Hermitian form (50) of the signature $(2p, 2q)$, i.e.,

$$
Sp(p, q) = \left\{ g \in Sp(n, \mathbb{C}) \mid \bar{g} \begin{pmatrix} I_p & 0 & 0 \\ 0 & -I_{2q} & 0 \\ 0 & 0 & I_p \end{pmatrix} g = \begin{pmatrix} I_p & 0 & 0 \\ 0 & -I_{2q} & 0 \\ 0 & 0 & I_p \end{pmatrix} \right\}.
$$

We fix a Cartan involution $\theta$ of $G_\mathbb{R}$ as follows:

$$
\theta : g \mapsto \begin{pmatrix} I_p & 0 & 0 \\ 0 & -I_{2q} & 0 \\ 0 & 0 & I_p \end{pmatrix} g \begin{pmatrix} I_p & 0 & 0 \\ 0 & -I_{2q} & 0 \\ 0 & 0 & I_p \end{pmatrix}.
$$

Then we have

$$
\mathfrak{k} = g \cap \mathfrak{a}_H, \quad \mathfrak{p} = g \cap \mathfrak{p}_H, \quad \mathfrak{b} = g \cap \mathfrak{b}_H,
$$

and

$$
K = \left\{ \begin{pmatrix} K_{11} & 0 & K_{13} \\ 0 & K_{22} & 0 \\ K_{31} & 0 & K_{33} \end{pmatrix} \mid K_{11}, K_{13}, K_{31}, K_{33} \in \text{Mat}(p, p), K_{11}, K_{13}, K_{31}, K_{33} \in \text{Mat}(p, p) \right\}
\approx Sp(p, \mathbb{C}) \times Sp(q, \mathbb{C}).
$$

Remark 4.2.1. We have $(V_+)^\perp = V_-$ and $(V_-)^\perp = V_+.$

Remark 4.2.2. We fix a $G$-equivariant natural isomorphism between $X$ and $G/B$ via the manner in Remark 2.1.2.

4.3. A symbolic parametrization of $Sp(p, \mathbb{C}) \times Sp(q, \mathbb{C})$-orbits. In this section we give a symbolic parametrization of $K$-orbits in $X$ and give representatives of $K$-orbits for the parameters. Our parameters are an improvement of the Matsuki-Oshima’s parameters [6]. The set of parameters for $G_\mathbb{R} = Sp(p, q)$ is a subset of $\mathbb{C}(U(2p, 2q))$. The parameter we give for a $K$-orbit $Q$ in the flag variety for $G_\mathbb{R} = Sp(p, q)$ coincides with a clan in $\mathbb{C}(U(2p, 2q))$ which includes $Q$. 

4.3-1. Generalized clans for $Sp(p,q)$. We give a subset of $C(U(2p,2q))$. The subset parametrizes $K$-orbits in the flag variety $X$.

**Proposition 4.3.1.** For a flag $x = (V_0,V_1,\ldots,V_{2n}) \in X$, there exists a clan $\gamma = (c_1 \cdots c_{2n}) \in C(U(2p,2q))$ via the conditions in Proposition 2.2.6.

**Proposition 4.3.2.** Let $x \in X$ and $\gamma = (c_1 \cdots c_{2n}) \in C(U(2p,2q))$ be the clan of $x$. Then the clan $\gamma$ is symmetric, i.e., $\gamma^t = \gamma$ and $i + j \neq 2n + 1$ if $c_i = c_j \in \mathbb{N}$.

**Proof.** We remark that

$$V_{2n-i,\pm} = V_{2n-i} \cap V_{\pm} = (V_i)^\perp \cap (V_{\mp})^\perp = (V_i + V_{\mp})^\perp = (\pi_{\pm}(V_i) \oplus V_{\mp})^\perp.$$  

1. If $c_i = +$, then there exists a vector $v_+$ in $V_+$ such that $v_+ \not\in \pi_+(V_{i-1})$ and $V_i = V_{i-1} \oplus (v_+)$. Since

$$V_{2n-i+1,+} = (\pi_+(V_{i-1}) \oplus V_-)^\perp,$$

and

$$V_{2n-i,+} = (\pi_+(V_i) \oplus V_-)^\perp,$$

we have

$$(2n - i + 1; +) - (2n - i; +) = 1.$$  

Since

$$V_{2n-i+1,-} = (\pi_-(V_{i-1}) \oplus V_+)^\perp$$

and

$$V_{2n-i,-} = (\pi_-(V_i) \oplus V_+)^\perp = (\pi_-(V_{i-1}) \oplus V_+)^\perp,$$

we have

$$(2n - i + 1; -) - (2n - i; -) = 0.$$  

Thus, we have $c_{2n+1-i} = +$.

2. If $c_i = -$, then we have $c_{2n+1-i} = -$ by the same argument of 1.

3. We prove $(c_i,c_{2n+1-i}) \neq (a,a)$ for any $a \in \mathbb{N}$. Suppose $c_i = c_{2n+1-i} \in \mathbb{N}$ and $1 \leq i \leq n$, then there exists a vector $v_i$ in $C^{2n}$ such that

$$V_i = V_{i-1} \oplus \langle v_i \rangle \subset V_{2n-i+1} = V_{2n-i} \oplus \langle \pi_+(v_i) \rangle.$$  

Then we have

$$v_i \not\in V_{i-1} = V_{2n-i+1}$$

$$= (V_{2n-i} \oplus \langle \pi_+(v_i) \rangle)^\perp$$

$$= V_{2n-i} \cap \langle \pi_+(v_i) \rangle^\perp.$$  

That contradicts

$$v_i \in V_i = V_{2n-i}^\perp \quad \text{and} \quad v_i \not\in \langle \pi_+(v_i) \rangle^\perp.$$  

So, we got $(c_i, c_j) \neq (a,a)$ for any $a \in \mathbb{N}$.
4. We will prove that $c_{2n+1-i} = c_i' \in \mathbb{N}$ for some $i' < 2n + 1 - i$ if $c_i \in \mathbb{N}$ and $c_i \neq c_j$ for all $j < i$. If $c_i \in \mathbb{N}$ and $c_i \neq c_j$ for all $j < i$, then there exists a $v_i \in C^{2n}$ such that $\pi_+(v_i) \notin \pi_+(V_{i-1})$, $\pi_-(v_i) \notin \pi_-(V_{i-1})$, and

$$V_i = V_{i-1} \oplus \langle v_i \rangle.$$ 

Since

$$V_{2n-i+1,\pm} = (\pi_+(V_{i-1}) \oplus V_\mp)^\perp$$

and

$$V_{2n-i,\pm} = (\pi_+(V_i) \oplus V_\mp)^\perp = (\pi_+(V_{i-1}) \oplus (\pi_+(v_i) \oplus V_\mp)^\perp,$$

we have

$$(2n - i + 1, \pm) - (2n - i, \pm) = 1.$$ 

Therefore, $c_{2n+1-i} = c_i' \in \mathbb{N}$ for some $i' < 2n + 1 - i$.

5. We will prove that $c_{2n+1-j} \in \mathbb{N}$ and $c_{2n+1-j}' \neq c_{j'}$ for all $j' < 2n + 1 - j$ if $c_i = c_j \in \mathbb{N}$ for $i < j$. Then there exists a $v_i \in C^{2n}$ such that $\pi_+(v_i) \notin \pi_+(V_{i-1})$, $\pi_-(v_i) \notin \pi_-(V_{i-1})$, and

$$V_i = V_{i-1} \oplus \langle v_i \rangle \subset V_j = V_{j-1} \oplus \langle \pi_+(v_i) \rangle.$$ 

Since $\pi_+(v_i) \in \pi_+(V_i) \subset \pi_+(V_{j-1})$, we have

$$V_{2n-i,\pm} = (\pi_+(V_j) \oplus V_\mp)^\perp = ((\pi_+(V_{j-1}) + (\pi_+(v_i))) \oplus V_\mp)^\perp = (\pi_+(V_{j-1}) \oplus \pi_+(v_i))^\perp = V_{2n-j+1,\pm}.$$ 

Thus,

$$(2n - j + 1, \pm) - (2n - j, \pm) = 0.$$ 

Therefore $c_{2n+1-j} \in \mathbb{N}$ and $c_{2n+1-j}' \neq c_{j'}$ for all $j' < 2n + 1 - j$.

6. It is enough to prove that $c_i = c_j \in \mathbb{N}$ if $c_{2n+1-i} = c_{2n+1-j} \in \mathbb{N}$ for $2n + 1 - i \leq j < i$. Let $2n + 1 - i \leq j < i$.

We prove that by induction on $i$. We suppose that if $c_k = c_l \in \mathbb{N}$ for $2n + 1 - i < k < l < i$, then $c_{2n+1-k} = c_{2n+1-l} \in \mathbb{N}$.

(a) We assume now $c_i = c_s \in \mathbb{N}$ for $s < j$ hoping to show a contradiction.

Then there exist $v_i, v_s \in C^{2n}$, such that

$$V_{2n+1-i} = V_{2n-i} \oplus \langle v_i \rangle \subset V_{2n+1-j} = V_{2n-j} \oplus \langle \pi_+(v_i) \rangle$$

and

$$V_s = V_{s-1} \oplus \langle v_s \rangle \subset V_{j-1} \subset V_i = V_{i-1} \oplus \langle \pi_+(v_s) \rangle.$$ 

Thus,

$$V_{2n-i} = V_i^\perp \subset \langle \pi_+(v_s) \rangle^\perp.$$ 

Since

$$v_s \in V_s \subset V_{j-1} = V_{2n-j+1} = V_{2n-j} \cap \langle \pi_+(v_i) \rangle^\perp \subset \langle \pi_+(v_i) \rangle^\perp,$$

we have

$$\omega(\pi_+(v_s), v_i) = \omega(\pi_+(v_s), \pi_+(v_i)) = \omega(v_s, \pi_+(v_i)) = 0.$$ 

So

$$\langle v_i \rangle \subset \langle \pi_+(v_s) \rangle^\perp.$$
On the other hand,
\[ V_{2n-i} = V_i^⊥ = V_{i-1}^⊥ \cap \langle \pi_+(v_s) \rangle ∥ \\
= V_{2n-i+1} \cap \langle \pi_+(v_s) \rangle ∥ = (V_{2n-i} \oplus \langle v_i \rangle) \cap \langle \pi_+(v_s) \rangle ∥ \\
= V_{2n-i} \oplus \langle v_i \rangle = V_{2n-i+1} \]
by (44) and (45). That contradicts \( V_{2n-i} \neq V_{2n-i+1} \). We have proved \( c_i \neq c_s \) for all \( s < j \).

(b) We will assume that \( c_i = c_s \) for \( j < s < i \) and see what happens. Then \( c_{2n+1-s} \in N \) and there exists \( 1 \leq t < 2n+1-s \) such that \( c_t = c_{2n+1-s} \). By assumptions, \( t \) satisfies \( 1 \leq t < 2n+1 - i \). Then there exist \( v_t, v_s \in C^n \) satisfying
\[ V_{t-1} \oplus \langle v_t \rangle \subset V_{2n-i} \subset V_{2n-s+1} = V_{2n-s} \oplus \langle \pi_+(v_t) \rangle \]
and
\[ V_s = V_{s-1} \oplus \langle v_s \rangle \subset V_i = V_{i-1} \oplus \langle \pi_+(v_s) \rangle. \]

Thus,
\[ V_{2n-s} = V_s^⊥ \subset \langle v_s \rangle ∥. \]

Since \( v_t \in V_{2n-i} = V_i^⊥ = V_{i-1}^⊥ \cap \langle \pi_+(v_s) \rangle ∥ \subset \langle \pi_+(v_s) \rangle ∥, \)
we have
\[ \omega(\pi_+(v_t), v_s) = \omega(\pi_+(v_t), \pi_+(v_s)) = \omega(v_t, \pi_+(v_s)) = 0. \]

So
\[ \langle \pi_+(v_t) \rangle \subset \langle v_s \rangle ∥. \]

On the other hand
\[ V_{2n-s} = V_s^⊥ = V_{s-1}^⊥ \cap \langle v_s \rangle ∥ \\
= V_{2n-s+1} \cap \langle v_s \rangle ∥ \\
= (V_{2n-s} \oplus \langle \pi_+(v_t) \rangle) \cap \langle v_s \rangle ∥ \\
= V_{2n-s} \oplus \langle \pi_+(v_t) \rangle = V_{2n-s+1} \]
by (46) and (47). That contradicts \( V_{2n-s} \neq V_{2n-s+1} \).

Therefore \( c_{2n+1-j} = c_{2n+1-i} \in N \) if \( c_i = c_j \in N \) by induction.

**Definition 4.3.3** (Generalized clan). We call a clan satisfying the condition in Proposition 4.3.2 a *generalized clan* for \( Sp(p, q) \). We denote the set of generalized clans for \( Sp(p, q) \) by \( C(Sp(p, q)) \):
\[ C(Sp(p, q)) = \left\{ \gamma \in C(U(2p, 2q)) \bigg| \begin{array}{l} \iota \gamma = \gamma \text{ and } i + j \neq 2n + 1 \\ \text{if } c_i = c_j \not\in N \end{array} \right\}. \]

**Example 4.3.4.** The set \( C(Sp(1, 1)) \) consists of four generalized clans:
\[ \left\{ \begin{array}{ccccccc} + & - & - & + & - & + & - \\ 1 & 1 & 2 & 2 & 1 & 2 & 1 \end{array} \right\}. \]
Example 4.3.5. The set $C(Sp(2,1))$ consists of nine generalized clans:

$$
\begin{align*}
\{ &++--+++, +--+--+, --++--+, ++--+++, \\
&+1122++, 1+122++, 11++22, 11++22, \\
&+1212++, 1+212++, 12++12, 12++12 \}
\end{align*}
$$

We recall the definition of clans in [6] in order to compare our notion of generalized clans.

Definition 4.3.6. [6] A clan for $Sp(p,q)$ is an equivalence class of ordered sets $(c_1 \cdots c_n)$ of $n$ symbols satisfying the following six conditions.

1. For $1 \leq i \leq n$, a symbol $c_i$ is $+$, $-$, $a$, or $\overline{a}$ for $a \in \mathbb{N}$.
2. If $c_i = a \in \mathbb{N}$, then there exists a unique $j \neq i$ with $c_j = a$ and $c_t \neq \overline{a}$ for all $t$.
3. If $c_i = \overline{a}$ for $a \in \mathbb{N}$, then there exists a unique $j \neq i$ with $c_j = a$ and $c_t \neq a$ for all $t$.
4. The difference between numbers of $+$ and $-$ in clans $(c_1 \cdots c_n)$ coincides with half of the difference of signatures of the Hermitian form defining the group $Gr$:

$$
\#\{ i \mid c_i = + \} - \#\{ i \mid c_i = - \} = p - q.
$$

5. If $c_i = a$ or $\overline{a}$ for $a > 1$, then there exist some $j$ such that $c_j = a-1$ or $\overline{a}-1$.

6. Two ordered sets $(c_1 \cdots c_n)$ and $(c'_1 \cdots c'_n)$ are regarded as equivalent if and only if there exists a permutation $\sigma \in S_n$ with $m := \max\{ c'_i \in \mathbb{N} \}$ such that

$$
c_i = \begin{cases} 
\sigma(a) & \text{if } c'_i = a \\
+ & \text{if } c'_i = + \\
- & \text{if } c'_i = - \\
\overline{\sigma(a)} & \text{if } c'_i = \overline{a} (a \in \mathbb{N})
\end{cases}
$$

for $1 \leq i \leq n$.

For example, $(211+\overline{2}-) = (122+\overline{1}-)$ as a clan.

The following one-to-one correspondence exists between clans for $Sp(p,q)$ and generalized clans for $Sp(p,q)$. A clan $(c'_1 \cdots c'_n)$ corresponds to a generalized clan $(c_1 \cdots c_{2n})$ satisfying the following four conditions for $1 \leq i, j \leq n$:

1. If $c'_i = +$, then $(c_i, c_{2n+1-i}) = (+, +)$.
2. If $c'_i = -$, then $(c_i, c_{2n+1-i}) = (-, -)$.
3. If $c'_i = c'_j \in \mathbb{N}$, then $c_i = c_j \in \mathbb{N}$ and $c_{2n+1-i} = c_{2n+1-j} \in \mathbb{N}$.
4. If $c'_i = c'_j = \overline{a}$, then $c_i = c_{2n+1-j} \in \mathbb{N}$ and $c_j = c_{2n+1-i} \in \mathbb{N}$.

For example, a clan $(122+\overline{1})$ corresponds to a generalized clan $(122+31+443)$.

Therefore $C(Sp(p,q))$ is a parametrization of $K$-orbits in $X$ by [6, Theorem 4.1].

Corollary 4.3.7. Generalized clans in $C(Sp(p,q))$ parametrize $K$-orbits in the flag variety $X$ via the correspondence in Proposition 2.2.6.

Notation 4.3.8. For each generalized clan $\gamma$, let $Q_\gamma$ be the $K$-orbit in the flag variety $X$ corresponding to $\gamma$ via the parametrization of Corollary 4.3.7.
4.3-2. Representatives of \( Sp(p, C) \times Sp(q, C) \)-orbits in \( Sp(n, C)/B \). We will give an element \( g \in G \) such that the flag \( x = gB \) corresponds to a generalized clan \( \gamma \in C(Sp(p, q)) \) and the flag \( x = gB_{AIII} \) in \( X \) corresponds to the clan \( \gamma \in C(U(2p, 2q)) \).

**Definition 4.3.9** (Symmetric signed clan). A symmetric signed clan \( \gamma_+ = (d_1 \cdots d_{2n}) \) of a generalized clan \( \gamma = (c_1 \cdots c_{2n}) \) is a signed clan (in Definition 2.2.9) of the clan \( \gamma \) satisfying the following condition. If \( c_i = c_j = a \in N \), then we have \( c_{2n+1-j} = c_{2n+1-i} = b \in N \). We can assume \( \min(i, 2n+1-i) < \min(j, 2n+1-j) \). Then

\[
(d_i, d_j) = (a_+, a_-) \quad \text{and} \quad (d_{2n+1-j}, d_{2n+1-i}) = (b_-, b_+).
\]

**Example 4.3.10.** The symmetric signed clan of \( \gamma = (1 2 1 + 3 2 + 4 3 4) \) is

\[
\gamma_+ = (1_+ 2_+ 1_- 3_- 2_- 4_- 3_+ 4_-).
\]

**Corollary 4.3.11.** In a symmetric signed clan \( \gamma_+ = (d_1 \cdots d_{2n}) \), the signature of \( d_i \) is equal to the signature of \( d_{2n+1-i} \) for \( 1 \leq i \leq 2n \).

In this section, we call a skew symmetric signed clan just a clan.

We give a representative \( g \in G \) of \( Q_\gamma \).

**Theorem 4.3.12.** For \( \gamma \in C(Sp(p, q)) \) and the signed clan \( \gamma_+ = (d_1 \cdots d_{2n}) \) of \( \gamma \), fix two permutations \( \sigma'' \in S_n \) such that

\[
1 \leq \sigma''(i) \leq p \quad \text{if the signature of} \ d_i \ \text{is plus},
\]

\[
p+1 \leq \sigma''(i) \leq n \quad \text{if the signature of} \ d_i \ \text{is minus},
\]

and \( \sigma' \in S_{2n} \) such that

- if the signature of \( d_i \) is plus or \( d_i = - \), then
  \[
  \sigma'(i) = \begin{cases} 
  \sigma''(i) & \text{if } i \leq n, \\
  2n+1 - \sigma''(i) & \text{if } n < i,
  \end{cases}
  \]

- and if \( d_i = a_- \) and \( d_j = a_+ \) for some \( a \in N \), then
  \[
  \sigma'(i) = \begin{cases} 
  \sigma''(i) & \text{if } j \leq n, \\
  2n+1 - \sigma''(i) & \text{if } n < j.
  \end{cases}
  \]

The matrix \( g(\gamma) = (g_1, g_2 \cdots g_{2n}) \in \text{Mat}(2n, 2n) \) is a representative of \( Q_\gamma \), i.e.,

\[
Q_\gamma = Kg(\gamma)B.
\]

Here \( g_i \in V \) are column vectors defined as follows:

- If \( c_i = \pm, \) then \( g_i = e_{\sigma'(i)} \).
- If \( (c_i, c_j) = (a_+, a_-) \) for some \( a \in N \), then
  \[
  g_i = \frac{1}{\sqrt{2}}(e_{\sigma'(i)} + e_{\sigma'(j)}) \quad \text{and} \quad g_j = \begin{cases} 
  -\frac{1}{\sqrt{2}}(e_{\sigma'(i)} + e_{\sigma'(j)}) & \text{if } j \leq n < i, \\
  \frac{1}{\sqrt{2}}(e_{\sigma'(i)} + e_{\sigma'(j)}) & \text{otherwise}.
  \end{cases}
  \]

**Proof.** The permutation \( \sigma' \) satisfies the conditions (52) of \( \sigma \) in Proposition 4.1.2.

Thus, \( g(\gamma) \) is a representative of \( K_{AIII} \)-orbit \( Q_{\gamma AIII} \), i.e., \( Q_{\gamma AIII} = K_{AIII}g(\gamma)B_{AIII} \), and \( g(\gamma) \in G \). Therefore, we have

\[
Q_\gamma = Kg(\gamma)B. \quad \square
\]
Example 4.3.13. For a signed clan
\[ \gamma_+ = (1_+ - 1_- + 3_+ - 3_+ - 3_+ - 3_+), \]
for example, if we choose \( \sigma'' = (\begin{array}{cccccc}
1 & 2 & 3 & 4 & 5 & 6 \\
1 & 2 & 6 & 4 & 3 & 5 
\end{array}) \),
then \( \sigma' \) is
\[ \sigma' = (\begin{array}{ccccccc}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 \\
1 & 2 & 6 & 4 & 3 & 8 & 5 & 10 & 9 & 7 & 11 & 12 
\end{array}) \).

Remark 4.3.14. The representation \( g(\gamma) \) is real and orthogonal:
\[ g(\gamma)^{-1} = \iota g(\gamma). \]

4.4. M-O graph of \( Sp(p,q) \). We define an oriented graph which has generalized clans as vertices and in which an edge is labeled by \( i \) for some \( 1 \leq i \leq n \). We can read off closure relations from the graph.

Definition 4.4.1. Let \( \Gamma(Sp(p,q)) \) be the set of all triples \( (\gamma, \gamma', i) \) with \( \gamma, \gamma' \in C(Sp(p,q)) \) and \( 1 \leq i \leq n \), satisfying one of the following two conditions.
1. \( (\gamma, \gamma', i) \in \Gamma(U(2p, 2q)) \). (This case happens only if \( i = n \).)
2. There exists a clan \( \gamma'' \in C(U(2p, 2q)) \) such that
   \[ \{ (\gamma, \gamma'', i), (\gamma'', \gamma', 2n - i) \} \subset \Gamma(U(2p, 2q)). \]

By Theorem 2.2.14 and Remark 3.3.4, we have the following proposition.

Proposition 4.4.2. If \( (\gamma, \gamma', i) \in \Gamma(Sp(p,q)) \), then
\[ \pi_i(Q_\gamma) = \pi_i(Q_{\gamma'}). \]

Proposition 4.4.3. If \( (\gamma, \gamma', i) \in \Gamma(Sp(p,q)) \), then
\[ \dim Q_\gamma = \dim Q_{\gamma'} - 1 \]
and
\[ Kg(\gamma)B \subset Kg(\gamma')B. \]

Proof. By [6], if \( (\gamma, \gamma', i) \in \Gamma(Sp(p,q)) \), then we have
\[ \dim Q_\gamma = \dim Q_{\gamma'} \pm 1 \]
and
\[ Kg(\gamma)B \subset Kg(\gamma')B. \]
If \( i = n \), then \( (\gamma, \gamma', n) \in \Gamma(U(2p, 2q)) \). Therefore
\[ K_{AH}g(\gamma)B_{AH} \subset K_{AH}g(\gamma')B_{AH} \quad \text{and} \quad K_{AH}g(\gamma')B_{AH} \not\subset K_{AH}g(\gamma)B_{AH}. \]
Thus, we have \( Kg(\gamma)B \subset Kg(\gamma')B. \)

If \( i \neq n \), then both of \( (\gamma, \gamma'', i) \) and \( (\gamma'', \gamma', 2n - i) \) are elements of \( \Gamma(U(2p, 2q)) \) for some \( \gamma'' \in C(U(2p, 2q)) \). By the same argument of the case of \( i = n \), \( Kg(\gamma)B \subset Kg(\gamma')B \). Therefore \( \dim Q_\gamma = \dim Q_{\gamma'} - 1 \) for all \( (\gamma, \gamma', i) \in \Gamma(Sp(p,q)). \)

Although the definition of the following graph follows [6], we use generalized clans instead of clans as vertices.
**Definition 4.4.4** (M-O graph of $Sp(p, q)$). Let

$C_0(Sp(p, q)) = C(Sp(p, q)) \cap C_0(U(2p, 2q)) :$

$C_0(Sp(p, q)) = \{ (c_1 \ldots c_n) \in C(Sp(p, q)) \mid c_i \notin \mathbb{N} \text{ for all } 1 \leq i \leq n \}.$

We give subsets $\Gamma_m(Sp(p, q))$ of $\Gamma(Sp(p, q))$ and subsets $C_m(Sp(p, q))$ of $C(Sp(p, q))$ for $m \geq 1$ by induction as follows:

$\Gamma_m(Sp(p, q)) := \{ (\gamma', \gamma, i) \in \Gamma'(Sp(p, q)) \mid \gamma \in C_{m-1}(Sp(p, q)) \}$,

$C_m(Sp(p, q)) := \{ \gamma \in C(Sp(p, q)) \mid (\gamma', \gamma, i) \in \Gamma_m(Sp(p, q)) \}.$

The **M-O graph** with generalized clans of $Sp(p, q)$ is a finite oriented graph whose vertices are generalized clans in $C(Sp(p, q))$ and whose oriented edges are

$\Gamma(Sp(p, q)) = \bigcup_{m \in \mathbb{N}} \Gamma_m(Sp(p, q)).$

**Example 4.4.5.** The $\Gamma(Sp(2, 1))$ is as Figure 3.

**Remark 4.4.6.** If $(\gamma, \gamma', i) \in \Gamma(Sp(p, q))$, then

$\pi_i(Q_\gamma) = \pi_i(Q_{\gamma'})$ and $\dim Q_\gamma + 1 = \dim Q_{\gamma'}$.

**4.5. Dimensions of $Sp(p, C) \times Sp(q, C)$-orbits in $Sp(n, C)$**. In this section we will give a dimension formula of $K$-orbits for generalized clans. We defined the **length** $\ell_C(\gamma)$ of a generalized clan in Definition 3.4.1

$\ell_C(\gamma) = \frac{1}{2} ( \ell(\gamma) + \# \{ t \in \mathbb{N} \mid c_s = c_t \in \mathbb{N} \text{ and } s \leq n < t \leq 2n + 1 - s \}).$

**Proposition 4.5.1.** If $\gamma \in C_m(Sp(p, q))$, then $\ell_C(\gamma) = m$.

**Proof.** We prove the proposition by induction on $m$. Let $\gamma = (c_1 \ldots c_{2n})$.

1. If $\gamma \in C_0(Sp(p, q))$, then $\ell_C(\gamma) = 0$. 

2. Suppose $\ell_C(\gamma') = j$, if $\gamma' \in C_j(Sp(p, q))$. For $\gamma \in C_{j+1}(Sp(p, q))$, there exists $\gamma' = (c'_1 \ldots c'_{2n}) \in C_j(Sp(p, q))$ such that $(\gamma', \gamma, i) \in \Gamma(Sp(p, q))$ for some $i$.

(a) If $i = n$, then $\ell(\gamma) = \ell(\gamma') + 1$, and

$$\# \{ t \in \mathbb{N} | c_s = c_t \in \mathbb{N} \text{ and } s \leq n < t \leq 2n + 1 - s \} - \# \{ n + 1 \}
= \# \{ t \in \mathbb{N} | c'_s = c'_t \in \mathbb{N} \text{ and } s \leq n < t \leq 2n + 1 - s \}.$$ 

So, $\ell_C(\gamma') = j + 1$.

(b) If $i \neq n$, then there exists a clan $\gamma'' \in C(U(2p, 2q))$ such that $(\gamma', \gamma'', i), (\gamma'', \gamma, 2n - i) \in \Gamma(U(2p, 2q))$.

So, we have

$$\ell(\gamma) = \ell(\gamma'') + 1 = \ell(\gamma') + 2$$

and

$$\# \{ t \in \mathbb{N} | c_s = c_t \in \mathbb{N} \text{ and } s \leq n < t \leq 2n + 1 - s \} = \# \{ t \in \mathbb{N} | c'_s = c'_t \in \mathbb{N} \text{ and } s \leq n < t \leq 2n + 1 - s \}.$$ 

Therefore $\ell_C(\gamma) = j + 1$.

So, $\ell_C(\gamma) = m$ if $\gamma \in C_m(Sp(p, q))$ for all $m \in \mathbb{N}$.

A generalized clan

$$\gamma = (1 \ 2 \ \cdots \ (2p) \ - \cdots - \ (2p - 1) \ (2p) \ \cdots \ 5 \ 6 \ 3 \ 4 \ 2)$$

corresponds to the open orbit and

$$\ell_C(\gamma) = 2pq.$$ 

Since the dimension of a closed orbit is $p^2 + q^2$, we have the following dimension formulas.

**Proposition 4.5.2.** For $\gamma \in C(Sp(p, q))$, we have the dimension and the codimension of $Q_\gamma = Kg(\delta)B \subset X$:

$$\dim Q_\gamma = \ell_C(\gamma) + p^2 + q^2,$$

$$\operatorname{codim} Q_\gamma = 2pq - \ell_C(\gamma).$$

4.6. **Images of the moment map.** We have

$$(Ad(g(\gamma))b^\perp) \cap p = (Ad(g(\gamma)) \cdot (b^\perp_{A I I I} \cap g)) \cap (p_{A I I I} \cap g) = (Ad(g(\gamma)) \cdot b^\perp_{A I I I}) \cap p_{A I I I} \cap g,$$

and elements $X$ in $g$ satisfy

$$X_{ij} = \begin{cases} 
-X_{2n+1-j, 2n+1-i} & \text{if } 1 \leq i, j \leq n \text{ or } n + 1 \leq i, j \leq 2n, \\
X_{2n+1-j, 2n+1-i} & \text{if } i \leq n < j \text{ or } j \leq n < i.
\end{cases}$$

Since we gave the image $(Ad(g) \cdot b^\perp_{A I I I}) \cap p_{A I I I}$ of the moment map $\mu$ in Proposition 2.5.2 with $g := g_{A I I I}(\gamma)$ in Proposition 4.1.2 and $\sigma := \sigma'$ in Theorem 4.3.12, we get the following proposition.
Proposition 4.6.1. For a generalized clan $\gamma = (c_1 \cdots c_{2n}) \in C(\text{Sp}(p,q))$, fix a $\sigma' \in S_{2n}$ given as in Theorem 4.3.12. Let the representative $g := g(\gamma)$ be given as in Theorem 4.3.12, and $x = gB \in Q_\gamma$. We can read off $\mu(T_{Q_\gamma}^\ast X)_x$ from a vector subspace $\text{Dri}(\delta)$ of $\mathfrak{g}$.

$$\mu(T_{Q_\gamma}^\ast X)_x = \text{Ad}(\sigma') \cdot \text{Dri}(\gamma)$$

$$= \{ \sigma'Y \sigma' | Y \in \text{Dri}(\gamma) \}$$

$$= \{ Y_{(\sigma'^{i-1}(1), \ldots, \sigma'^{i-1}(n))} | Y \in \text{Dri}(\gamma) \}.$$ 

Here, $\text{Dri}(\gamma)$ is the subspace of $\text{Dri}(\gamma_+)$, the driving space of the signed clan $\gamma_+$ for $U(2p,2q)$ in Definition 2.5.1, defined as follows:

$$\text{Dri}(\gamma) := \text{Dri}(\gamma_+) \cap \text{Ad}(\sigma'^{-1}) \cdot \mathfrak{g}$$

$$= \{ Y \in \text{Dri}(\gamma_+) | Y \text{ satisfies the following two conditions } \}.$$ 

1. If $1 \leq \sigma'(i), \sigma'(j) \leq n$ or $n + 1 \leq \sigma'(i), \sigma'(j) \leq 2n$, then

$$Y_{i,j} = -Y_{2n+1-j,2n+1-i}.$$ 

2. If $\sigma'(i) \leq n < \sigma'(j)$ or $\sigma'(j) \leq n < \sigma'(i)$, then

$$Y_{i,j} = Y_{2n+1-j,2n+1-i}.$$ 

We call the space $\text{Dri}(\gamma)$ the driving space of the generalized clan $\gamma \in C(\text{Sp}(p,q))$.

Proof. For the signed clan $\gamma_+ = (d_1 \cdots d_{2n})$ and the permutation $\sigma'$ in Theorem 4.3.12 and a representative $g_{\text{III}}(\gamma_+, \sigma')$ in Proposition 4.1.2, we have

$$g = g(\gamma_+, \sigma') \cdot \text{diag}(\varepsilon_1 \cdots \varepsilon_{2n}).$$

Here,

$$\varepsilon_i = \begin{cases} 
-1 & \text{if } (d_i, d_j) = (a_-, a_+) \text{ for } i \leq j, \\
1 & \text{otherwise}. 
\end{cases}$$

So,

$$(\text{Ad}(g)b_{\text{III}} \perp) \cap \mathfrak{p}_{\text{III}}$$

$$= (\text{Ad}(g(\gamma_+, \sigma'))(\text{Ad}(\text{diag}(\varepsilon_1 \cdots \varepsilon_{2n})) \cdot b_{\text{III}} \perp) \cap \mathfrak{p}_{\text{III}}$$

$$= (\text{Ad}(g(\gamma_+, \sigma')) \cdot b_{\text{III}} \perp) \cap \mathfrak{p}_{\text{III}}.$$ 

Therefore, $\mu(T_{Q_\gamma}^\ast X)_x = \text{Ad}(\sigma') \cdot \text{Dri}(\gamma)$ is as above. \hfill \Box

Corollary 4.6.2. For the representative $g(\gamma)$ of $Q_\gamma$ given as in Theorem 4.3.12, we have the following equation.

$$\text{Dri}(\gamma) = \text{Ad}(\sigma'^{-1})(\text{Ad}(g(\gamma)) \cdot b_{\perp}) \cap \mathfrak{p}$$

$$= \{ A_{(\sigma'(1), \ldots, \sigma'(2n))} | A \in (\text{Ad}(g(\gamma)) \cdot b_{\perp}) \cap \mathfrak{p} \}.$$ 

Since $g(\gamma) = (e_{\sigma'(1)} \cdots e_{\sigma'(2n)}) = \sigma'$ if a generalized clan $\gamma$ corresponds to a closed orbit, we have the following lemma.

Lemma 4.6.3. If a generalized clan $\gamma$ is an element of $C_0(\text{Sp}(p,q))$, then

$$\text{Dri}(\gamma) = b_{\perp} \cap (\text{Ad}(g(\gamma)^{-1}) \cdot \mathfrak{p}).$$

Definition 4.6.4 (Driving matrix). For a generalized clan $\gamma \in C(\text{Sp}(p,q))$, let $Y(\gamma)$ and $Y(\gamma,m), 1 \leq m \leq 8$ be elements of $F \otimes \mathbb{C} \text{Mat}(2n,2n)$ satisfying the following conditions. Let $\gamma_+ = (d_1 \cdots d_{2n})$. 

\[
\text{Dri}(\delta) \in \text{Mat}(2n,2n)
\]
1. \( Y(\gamma, 1)_{ij} = \begin{cases} 1 & \text{if } j < i \leq 2n + 1 - j \\ 0 & \text{otherwise.} \end{cases} \)

2. \( Y(\gamma, 2)_{ij} = \begin{cases} 0 & \text{if } (d_i, d_j) = (+, +) \text{ or } (-, -), \\ Y(\gamma, 1)_{ij} & \text{otherwise.} \end{cases} \)

3. If \((d_s, d_t) = (a_+, a_-) \text{ or } (a_-, a_+)\) for \(s < t\), then
   \begin{align*}
   &\quad Y(\gamma, 3)_{i,k} = 0 \quad \text{if } 1 \leq k \leq s, \\
   &\quad Y(\gamma, 3)_{k,s} = Y(\gamma, 3)_{t,k} = 0 \quad \text{if } s \leq k \leq t, \\
   &\quad Y(\gamma, 3)_{k,t} = 0 \quad \text{if } t \leq k \leq 2n,
   \end{align*}
   for all \(a \in \mathbb{N}\) and \(Y(\gamma, 3)_{ij} = Y(\gamma, 2)_{ij}\) otherwise.

**Remark 4.6.5.** If \( Y(\gamma, 3)_{ij} = 1\), then the following conditions are satisfied.

(a) \(j < i \leq 2n + 1 - j\) (so \(j \leq n\)).

(b) \((d_i, d_j) = (a_+, b_+), (a_-, b_+), (a_+, +), (a_-, +), (a_-, -), (+, a_+), (-, a_+), (+, -), \text{ or } (-, +)\) for some \(a, b \in \mathbb{N}\), \(a \neq b\).

(c) If \(i + j = 2n + 1\), then \((d_i, d_j) = (a_+, b_+)\) for some \(a, b \in \mathbb{N}\), \(a \neq b\).

(d) If \(d_i = + \text{ or } -, \text{ then } \sigma'(i) < n \text{ if } i \leq n, \sigma'(i) > n \text{ if } i > n\).

(e) If \(d_i = a_+\), then \(d_k = a_-\) for some \(j < k < i\). Then \(n < \sigma'(k) < \sigma'(i)\).

(f) If \(d_i = a_-\), then \(d_k = a_+\) for some \(j < k < i\). Then \(\sigma'(k) < \sigma'(i) \leq n\).

(g) If \(d_i = + \text{ or } -, \text{ then } \sigma'(j) < n\).

(h) If \(d_i = a_+\), then \(d_k = a_-\) for \(j < k < i\). Then \(\sigma'(j), \sigma'(k) \leq n\).

4. If \(Y(\gamma, 3)_{i,2n+1-i} = 1\), \((d_i, d_{2n+1-i}) = (a_+, b_+),\) and \((d_j, d_{2n+1-j}) = (a_-, b_-)\), for some \(a, b \in \mathbb{N}\), then \(Y(\gamma, 4)_{i,2n+1-i} = 0\),
   \[ (Y(\gamma, 4)_{i,2n+1-j}, Y(\gamma, 4)_{j,2n+1-i}) = \begin{cases} (y_{i,2n+1-j}, y_{i,2n+1-j}) & \text{if } 2n + 1 - j < j, \\
(0,0) & \text{if } j < 2n + 1 - j, \end{cases} \]
   and \(Y(\gamma, 4)_{i',j'} = Y(\gamma, 3)_{i',j'}\) otherwise \((i', j')\).

5. If \((Y(\gamma, 4)_{ij} = 1, i + j \neq 2n + 1, (d_i, d_j) = (a_+, b_+),\) and \((d_k, d_l) = (a_-, b_-)\) for some \(a, b \in \mathbb{N}\), then \(Y(\gamma, 5)_{ij} = 0\),
   \[ (Y(\gamma, 5)_{il}, Y(\gamma, 5)_{2n+1-l,2n+1-i}) = (y_{il}, y_{it}), \]
   and
   \[ (Y(\gamma, 5)_{kj}, Y(\gamma, 5)_{2n+1-j,2n+1-j}) = \begin{cases} (-y_{il}, -y_{il}) & \text{if } k < l, \\
(y_{kj}, y_{kj}) & \text{if } l < k, \end{cases} \]
   and \(Y(\gamma, 5)_{i',j'} = Y(\gamma, 4)_{i',j'}\) otherwise.

6. If \((Y(\gamma, 5)_{ij} = 1, (d_i, d_j) = (a_-, b_+)\) and \((d_k, d_l) = (a_+, b_-)\) for some \(a, b \in \mathbb{N}\), then
   \[ (Y(\gamma, 6)_{ij}, Y(\gamma, 6)_{2n+1-j,2n+1-i}) = (y_{ij}, -y_{ij}), \]
   \[ (Y(\gamma, 6)_{kl}, Y(\gamma, 6)_{2n+1-l,2n+1-k}) = \begin{cases} (-y_{ij}, y_{ij}) & \text{if } k < l, \\
(y_{kl}, -y_{kl}) & \text{if } l < k, \end{cases} \]
   and \(Y(\gamma, 6)_{i',j'} = Y(\gamma, 5)_{i',j'}\) otherwise.
7. If $Y(\gamma, 6)_{ij} = 1$, $(d_i, d_j) = (a_+, +)$ or $(d_i, d_j) = (a_-, -)$, $d_k = a_-$ or $d_k = a_+$, for some $a \in \mathbb{N}$, then $Y(\gamma, 7)_{ij} = 0$

\[
( Y(\gamma, 7)_{kj}, Y(\gamma, 7)_{2n+1-j, 2n+1-k} ) = \begin{cases} (y_{kj}, -y_{kj}) & \text{if } \sigma'(k) \leq n, \\ (y_{kj}, y_{kj}) & \text{if } \sigma'(k) > n, \end{cases}
\]

and $Y(\gamma, 7)_{i'j'} = Y(\gamma, 6)_{i'j'}$ otherwise.

8. If $Y(\gamma, 7)_{ij} = 1$, $(d_i, d_j) = (+, a_+)$, and $d_k = a_-$ for some $a \in \mathbb{N}$, then $Y(\gamma, 8)_{ij} = 0$

\[
( Y(\gamma, 8)_{ik}, Y(\gamma, 8)_{2n+1-k, 2n+1-i} ) = \begin{cases} (y_{ik}, -y_{ik}) & \text{if } \sigma'(i) \leq n, \\ (y_{ik}, y_{ik}) & \text{if } \sigma'(i) > n, \end{cases}
\]

and $Y(\gamma, 8)_{i'j'} = Y(\gamma, 7)_{i'j'}$ otherwise.

9. If $Y(\gamma, 8)_{ij} = 1$, then

\[
( Y(\gamma)_{ij}, Y(\gamma)_{2n+1-j, 2n+1-i} ) = \begin{cases} (y_{ij}, -y_{ij}) & \text{if } \sigma'(i) \leq n, \\ (y_{ij}, y_{ij}) & \text{if } \sigma'(i) > n, \end{cases}
\]

and $Y(\gamma)_{i'j'} = Y(\gamma, 8)_{i'j'}$ for other $(i', j')$.

We call $Y(\gamma)$ the driving matrix of $\gamma$ (for $Sp(p, q)$).

By the same arguments of Proposition 2.5.8, we get the following proposition.

**Proposition 4.6.6.** For a generalized clan $\gamma \in \mathcal{C}(Sp(p, q))$, the driving space $\text{Dri}(\gamma)$ for $Sp(p, q)$ and the driving matrix $Y(\gamma)$ of $\gamma$ for $Sp(p, q)$ satisfy the following condition.

\[
\text{Dri}(\gamma) = \{ Y(\gamma) \mid y_{ij} \in \mathbb{C} \text{ for } 1 \leq i, j \leq 2n \}.
\]

By Proposition 4.6.1 and Proposition 4.6.6, we got the following theorem which means we can read off $\mu(T^*_Q, X)_x$ from the driving matrix $Y(\gamma)$.

**Theorem 4.6.7.** For a generalized clan $\gamma \in \mathcal{C}(Sp(p, q))$, fix a $\sigma' \in \mathfrak{S}_{2n}$ given as in Theorem 4.3.12. Let the representative $g := g(\gamma)$ be given as in Theorem 4.3.12, and $x = gB \in Q_\gamma$. We can read off $\mu(T^*_Q, X)_x$ from the driving matrix $Y(\gamma)$ for $Sp(p, q)$ as follows:

\[
\mu(T^*_Q, X)_x = \{ Y(\gamma)_{(\sigma'-1(1), \ldots, \sigma'-1(2n))} \mid y_{ij} \in \mathbb{C} \text{ for } 1 \leq i, j \leq 2n \}.
\]

### 4.7 Signed Young diagrams

In this section, after we recall a parametrization of nilpotent $K$-orbits in $p$, we give tables of signed Young diagrams for the clans of $Sp(1, 1)$, $Sp(2, 1)$, and $Sp(2, 2)$ by way of examples.

Nilpotent $K$-orbits in $p$ are parametrized by signed Young diagrams of signature $(2p, 2q)$ satisfying the following conditions.

1. If $m \in \mathbb{N}$ is even, then the number of rows of which the length are $m$ and which are labeled + in the first column is equal to the number of rows of which the length are $m$ and which are labeled − in the first column.
2. If \( m \in \mathbb{N} \) is odd, the number of rows of which the length are \( m \) and which are labeled + in the first column is even.
3. If \( m \in \mathbb{N} \) is odd, the number of rows of which the length are \( m \) and which are labeled − in the first column is even.

For an element \( A \) of an orbit, the signed Young diagram of the orbit satisfies

\[
\sum_{j=1}^{i} D_{j,+} = \dim(\ker(A^i|_{V_+})) \quad \text{and} \quad \sum_{j=1}^{i} D_{j,-} = \dim(\ker(A^i|_{V_-})).
\]

**Proposition 4.7.1.** Under the conditions of Theorem 4.3.12, we put

\[
V'_+ := \sigma'^{-1} V_+ = (e_{\sigma'^{-1}(1)} \cdots e_{\sigma'^{-1}(2n)}) V_+ \quad = \langle e_{\sigma'^{-1}(1)}, \ldots, e_{\sigma'^{-1}(p+2q+1)} \rangle \oplus \langle e_{\sigma'^{-1}(p+2q+1)}, \ldots, e_{\sigma'^{-1}(2n)} \rangle,
\]

\[
V'_- := \sigma'^{-1} V_- = (e_{\sigma'^{-1}(1)} \cdots e_{\sigma'^{-1}(2n)}) V_- \quad = \langle e_{\sigma'^{-1}(p+1)}, \ldots, e_{\sigma'^{-1}(p+2q)} \rangle.
\]

By the same argument of the case of \( U(2p, 2q) \), we have the following proposition.

**Proposition 4.7.2.** Let \( Y(\gamma)^0 = I_n \). We have the following equations:

\[
D_{i,+} = \dim_F \{ \bar{a} \in F \otimes \mathbb{C} V'_+ \mid Y(\gamma)^i \bar{a} = \bar{0} \} - \dim_F \{ \bar{a} \in F \otimes \mathbb{C} V'_+ \mid Y(\gamma)^{i-1} \bar{a} = \bar{0} \},
\]

\[
D_{i,-} = \dim_F \{ \bar{b} \in F \otimes \mathbb{C} V'_- \mid Y(\gamma)^i \bar{b} = \bar{0} \} - \dim_F \{ \bar{b} \in F \otimes \mathbb{C} V'_- \mid Y(\gamma)^{i-1} \bar{b} = \bar{0} \}.
\]

At last we give tables of signed Young diagrams for the clans of \( Sp(1, 1) \) in Example 4.7.3, \( Sp(2, 1) \) in Example 4.7.4, \( Sp(2, 2) \) in Example 4.7.5, and a subgraph of \( \Gamma(Sp(2, 2)) \) in Figure 4 (which has edges \((\gamma, \gamma', i)\) such that the first signature of the clan \( \gamma = (c_1 \ldots c_8) \) is plus, i.e., \( c_m = + \) for

\[
m = \min \{ j \mid c_j = + \ \text{or} \ - \}
\]

and the corresponding signed Young diagrams). In Figure 4, clans and signed Young diagrams are separated by dots. A clan and its signed Young diagram are in the same component. In Figure 4,

\[
( + - - + + + - - ) \quad \text{shaded} \quad (1 1 + - - + 2 2)
\]

instead of

\[
( + - - + + - - ) \quad \text{shaded} \quad (1 1 - + - + 2 2) \quad \in \Gamma(Sp(2, 2)).
\]
Figure 4
**Example 4.7.3.** This is the table of the case of $G_R = Sp(1,1)$.

<table>
<thead>
<tr>
<th>clan $\gamma$</th>
<th>a representative $g(\gamma)$ of $K$-orbit</th>
<th>driving matrix $Y(\gamma)$</th>
<th>signed Young diagram</th>
</tr>
</thead>
</table>
| $+--+$        | $\begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}$ | $\begin{pmatrix}
0 & 0 & 0 & 0 \\
y_{21} & 0 & 0 & 0 \\
y_{31} & 0 & 0 & 0 \\
0 & y_{31} - y_{21} & 0
\end{pmatrix}$ | 

| $-++-$        | $\begin{pmatrix}
0 & 1 & 0 & 0 \\
1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 1 & 0
\end{pmatrix}$ | $\begin{pmatrix}
0 & 0 & 0 & 0 \\
y_{21} & 0 & 0 & 0 \\
y_{31} & 0 & 0 & 0 \\
0 & y_{31} - y_{21} & 0
\end{pmatrix}$ | 

| $1122$        | $\begin{pmatrix}
\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 & 0 \\
\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & 0 \\
0 & 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\
0 & 0 & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}
\end{pmatrix}$ | $\begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
y_{42} & 0 & 0 & 0 \\
0 & y_{42} & 0 & 0
\end{pmatrix}$ | 

| $1212$        | $\begin{pmatrix}
\frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} & 0 \\
\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} & 0 \\
0 & \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\
0 & \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}}
\end{pmatrix}$ | $\begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}$ | 


Example 4.7.4. This is the table of the case of $G_R = Sp(2,1)$.

<table>
<thead>
<tr>
<th>clan $\gamma$</th>
<th>a representative $g(\gamma)$ of $K$-orbit</th>
<th>driving matrix $Y(\gamma)$</th>
<th>signed Young diagram</th>
</tr>
</thead>
<tbody>
<tr>
<td>$++-++$</td>
<td>1 0 0 0 0 0 0</td>
<td>0 0 0 0 0 0 0</td>
<td>$\begin{pmatrix}+\end{pmatrix}$</td>
</tr>
<tr>
<td></td>
<td>0 1 0 0 0 0</td>
<td>0 0 0 0 0 0 0</td>
<td></td>
</tr>
<tr>
<td></td>
<td>0 0 1 0 0 0</td>
<td>$y_{31}$ $y_{32}$ 0 0 0 0</td>
<td></td>
</tr>
<tr>
<td></td>
<td>0 0 0 1 0 0</td>
<td>$y_{41}$ $y_{42}$ 0 0 0 0</td>
<td></td>
</tr>
<tr>
<td></td>
<td>0 0 0 0 1 0</td>
<td>0 0 $y_{42} - y_{32}$ 0 0</td>
<td></td>
</tr>
<tr>
<td></td>
<td>0 0 0 0 0 1</td>
<td>0 0 $y_{41} - y_{31}$ 0 0</td>
<td></td>
</tr>
<tr>
<td>$+-+-+-+$</td>
<td>1 0 0 0 0 0 0</td>
<td>0 0 0 0 0 0 0</td>
<td>$\begin{pmatrix}+\end{pmatrix}$</td>
</tr>
<tr>
<td></td>
<td>0 0 1 0 0 0</td>
<td>$y_{21}$ 0 0 0 0 0 0</td>
<td></td>
</tr>
<tr>
<td></td>
<td>0 1 0 0 0 0</td>
<td>$y_{32}$ 0 0 0 0 0 0</td>
<td></td>
</tr>
<tr>
<td></td>
<td>0 0 0 0 1 0</td>
<td>$y_{42}$ 0 0 0 0 0 0</td>
<td></td>
</tr>
<tr>
<td></td>
<td>0 0 0 0 1 0</td>
<td>$y_{51}$ $y_{42} - y_{32}$ 0 0</td>
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</tr>
<tr>
<td></td>
<td>0 0 0 0 0 1</td>
<td>0 $y_{51}$ 0 0 $-y_{21}$ 0</td>
<td></td>
</tr>
<tr>
<td>$+1122+$</td>
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<td>0 0 0 0 0 0 0</td>
<td>$\begin{pmatrix}+\end{pmatrix}$</td>
</tr>
<tr>
<td></td>
<td>$\frac{1}{\sqrt{2}}$ $\frac{1}{\sqrt{2}}$ 0 0 0 0</td>
<td>$y_{31}$ 0 0 0 0 0 0</td>
<td></td>
</tr>
<tr>
<td></td>
<td>0 0 0 $\frac{1}{\sqrt{2}}$ $\frac{1}{\sqrt{2}}$ 0 0</td>
<td>$y_{41}$ $y_{53}$ 0 0 0 0</td>
<td></td>
</tr>
<tr>
<td></td>
<td>0 0 0 0 $\frac{1}{\sqrt{2}}$ $\frac{1}{\sqrt{2}}$ 0 0</td>
<td>0 0 $y_{53}$ 0 0 0 0</td>
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</tr>
<tr>
<td></td>
<td>0 0 0 0 0 1</td>
<td>0 0 $y_{41} - y_{53}$ 0 0</td>
<td></td>
</tr>
<tr>
<td>$--+++$</td>
<td>0 1 0 0 0 0 0</td>
<td>0 0 0 0 0 0 0</td>
<td>$\begin{pmatrix}+\end{pmatrix}$</td>
</tr>
<tr>
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<td>0 0 1 0 0 0</td>
<td>$y_{21}$ 0 0 0 0 0 0</td>
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<td></td>
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<td>$y_{31}$ 0 0 0 0 0 0</td>
<td></td>
</tr>
<tr>
<td></td>
<td>0 0 0 0 0 1</td>
<td>$y_{41}$ 0 0 0 0 0 0</td>
<td></td>
</tr>
<tr>
<td></td>
<td>0 0 0 1 0 0</td>
<td>$y_{51}$ 0 0 0 0 0 0</td>
<td></td>
</tr>
<tr>
<td></td>
<td>0 0 0 0 1 0</td>
<td>0 $y_{51}$ $y_{41} - y_{31} - y_{21}$ 0</td>
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</tr>
<tr>
<td>$11++2+$</td>
<td>$\frac{1}{\sqrt{2}}$ $\frac{1}{\sqrt{2}}$ 0 0 0 0</td>
<td>0 0 0 0 0 0 0</td>
<td>$\begin{pmatrix}+\end{pmatrix}$</td>
</tr>
<tr>
<td></td>
<td>0 0 1 0 0 0</td>
<td>$y_{32}$ 0 0 0 0 0 0</td>
<td></td>
</tr>
<tr>
<td></td>
<td>$\frac{1}{\sqrt{2}}$ $\frac{1}{\sqrt{2}}$ 0 0 0 0</td>
<td>$y_{42}$ 0 0 0 0 0 0</td>
<td></td>
</tr>
<tr>
<td></td>
<td>0 0 0 0 $\frac{1}{\sqrt{2}}$ $\frac{1}{\sqrt{2}}$ 0 0</td>
<td>$y_{62}$ 0 $y_{42} - y_{32}$ 0 0</td>
<td></td>
</tr>
<tr>
<td></td>
<td>0 0 0 0 0 1</td>
<td>0 $y_{62}$ 0 0 0 0 0</td>
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</tr>
<tr>
<td>clan $\gamma$</td>
<td>a representative $g(\gamma)$ of $K$-orbit</td>
<td>driving matrix $Y(\gamma)$</td>
<td>signed Young diagram</td>
</tr>
<tr>
<td>---------------</td>
<td>---------------------------------</td>
<td>------------------------</td>
<td>-------------------</td>
</tr>
<tr>
<td>$+1\ 2\ 1\ 2\ +$</td>
<td>$\begin{pmatrix} 1 &amp; 0 &amp; 0 &amp; 0 &amp; 0 &amp; 0 \ 0 &amp; \frac{1}{\sqrt{2}} &amp; 0 &amp; \frac{1}{\sqrt{2}} &amp; 0 &amp; 0 \ 0 &amp; \frac{1}{\sqrt{2}} &amp; 0 &amp; \frac{1}{\sqrt{2}} &amp; 0 &amp; 0 \ 0 &amp; 0 &amp; \frac{1}{\sqrt{2}} &amp; 0 &amp; \frac{1}{\sqrt{2}} &amp; 0 \ 0 &amp; 0 &amp; \frac{1}{\sqrt{2}} &amp; 0 &amp; \frac{1}{\sqrt{2}} &amp; 0 \ 0 &amp; 0 &amp; 0 &amp; 0 &amp; 0 &amp; 1 \end{pmatrix}$</td>
<td>$\begin{pmatrix} 0 &amp; 0 &amp; 0 &amp; 0 &amp; 0 &amp; 0 \ 0 &amp; 0 &amp; 0 &amp; 0 &amp; 0 &amp; 0 \ 0 &amp; 0 &amp; 0 &amp; 0 &amp; 0 &amp; 0 \ 0 &amp; 0 &amp; 0 &amp; 0 &amp; 0 &amp; 0 \ 0 &amp; 0 &amp; 0 &amp; 0 &amp; 0 &amp; 0 \ 0 &amp; 0 &amp; 0 &amp; 0 &amp; 0 &amp; 0 \end{pmatrix}$</td>
<td>$\begin{pmatrix} + \end{pmatrix}$</td>
</tr>
<tr>
<td>$1 + 1\ 2\ + 2$</td>
<td>$\begin{pmatrix} \frac{1}{\sqrt{2}} &amp; 0 &amp; \frac{1}{\sqrt{2}} &amp; 0 &amp; 0 &amp; 0 \ 0 &amp; 1 &amp; 0 &amp; 0 &amp; 0 &amp; 0 \ \frac{1}{\sqrt{2}} &amp; 0 &amp; \frac{1}{\sqrt{2}} &amp; 0 &amp; 0 &amp; 0 \ 0 &amp; 0 &amp; \frac{1}{\sqrt{2}} &amp; 0 &amp; \frac{1}{\sqrt{2}} &amp; 0 \ 0 &amp; 0 &amp; 0 &amp; 0 &amp; 1 &amp; 0 \ 0 &amp; 0 &amp; \frac{1}{\sqrt{2}} &amp; 0 &amp; \frac{1}{\sqrt{2}} &amp; 0 \end{pmatrix}$</td>
<td>$\begin{pmatrix} 0 &amp; 0 &amp; 0 &amp; 0 &amp; 0 &amp; 0 \ 0 &amp; 0 &amp; 0 &amp; 0 &amp; 0 &amp; 0 \ 0 &amp; 0 &amp; 0 &amp; 0 &amp; 0 &amp; 0 \ 0 &amp; 0 &amp; 0 &amp; 0 &amp; 0 &amp; 0 \ 0 &amp; 0 &amp; 0 &amp; 0 &amp; 0 &amp; 0 \ 0 &amp; 0 &amp; 0 &amp; 0 &amp; 0 &amp; 0 \end{pmatrix}$</td>
<td>$\begin{pmatrix} + \end{pmatrix}$</td>
</tr>
<tr>
<td>$1 + 2\ 1\ + 2$</td>
<td>$\begin{pmatrix} \frac{1}{\sqrt{2}} &amp; 0 &amp; 0 &amp; \frac{1}{\sqrt{2}} &amp; 0 &amp; 0 \ 0 &amp; 1 &amp; 0 &amp; 0 &amp; 0 &amp; 0 \ \frac{1}{\sqrt{2}} &amp; 0 &amp; 0 &amp; \frac{1}{\sqrt{2}} &amp; 0 &amp; 0 \ 0 &amp; 0 &amp; \frac{1}{\sqrt{2}} &amp; 0 &amp; \frac{1}{\sqrt{2}} &amp; 0 \ 0 &amp; 0 &amp; 0 &amp; 0 &amp; 1 &amp; 0 \ 0 &amp; 0 &amp; \frac{1}{\sqrt{2}} &amp; 0 &amp; \frac{1}{\sqrt{2}} &amp; 0 \end{pmatrix}$</td>
<td>$\begin{pmatrix} 0 &amp; 0 &amp; 0 &amp; 0 &amp; 0 &amp; 0 \ 0 &amp; 0 &amp; 0 &amp; 0 &amp; 0 &amp; 0 \ 0 &amp; 0 &amp; 0 &amp; 0 &amp; 0 &amp; 0 \ 0 &amp; 0 &amp; 0 &amp; 0 &amp; 0 &amp; 0 \ 0 &amp; 0 &amp; 0 &amp; 0 &amp; 0 &amp; 0 \ 0 &amp; 0 &amp; 0 &amp; 0 &amp; 0 &amp; 0 \end{pmatrix}$</td>
<td>$\begin{pmatrix} + \end{pmatrix}$</td>
</tr>
<tr>
<td>$1 \ 2\ + + 1\ 2$</td>
<td>$\begin{pmatrix} \frac{1}{\sqrt{2}} &amp; 0 &amp; 0 &amp; 0 &amp; \frac{1}{\sqrt{2}} &amp; 0 \ 0 &amp; 0 &amp; 1 &amp; 0 &amp; 0 &amp; 0 \ \frac{1}{\sqrt{2}} &amp; 0 &amp; 0 &amp; 0 &amp; \frac{1}{\sqrt{2}} &amp; 0 \ 0 &amp; \frac{1}{\sqrt{2}} &amp; 0 &amp; 0 &amp; \frac{1}{\sqrt{2}} &amp; 0 \ 0 &amp; 0 &amp; 0 &amp; 1 &amp; 0 &amp; 0 \ 0 &amp; \frac{1}{\sqrt{2}} &amp; 0 &amp; 0 &amp; \frac{1}{\sqrt{2}} &amp; 0 \end{pmatrix}$</td>
<td>$\begin{pmatrix} 0 &amp; 0 &amp; 0 &amp; 0 &amp; 0 &amp; 0 \ 0 &amp; 0 &amp; 0 &amp; 0 &amp; 0 &amp; 0 \ 0 &amp; 0 &amp; 0 &amp; 0 &amp; 0 &amp; 0 \ 0 &amp; 0 &amp; 0 &amp; 0 &amp; 0 &amp; 0 \ 0 &amp; 0 &amp; 0 &amp; 0 &amp; 0 &amp; 0 \ 0 &amp; 0 &amp; 0 &amp; 0 &amp; 0 &amp; 0 \end{pmatrix}$</td>
<td>$\begin{pmatrix} + \end{pmatrix}$</td>
</tr>
</tbody>
</table>
Example 4.7.5. This is the table of $G_R = Sp(2, 2)$.

<table>
<thead>
<tr>
<th>clan $\gamma$</th>
<th>signed Young diagram</th>
<th>clan $\gamma$</th>
<th>signed Young diagram</th>
</tr>
</thead>
<tbody>
<tr>
<td>$+ - + - - + -$</td>
<td>$1 1 2 3 2 3 4 4$</td>
<td>$1 1 2 3 2 3 4 4$</td>
<td>$+$</td>
</tr>
<tr>
<td>$- + - + - + -$</td>
<td>$1 2 2 1 3 4 4 3$</td>
<td>$1 2 2 1 3 4 4 3$</td>
<td>$+$</td>
</tr>
<tr>
<td>$+ - + - + - + -$</td>
<td>$1 2 3 2 4 3 4$</td>
<td>$1 2 3 2 4 3 4$</td>
<td>$+$</td>
</tr>
<tr>
<td>$- + - - + - + -$</td>
<td>$1 2 2 3 1 4 4 3$</td>
<td>$1 2 2 3 1 4 4 3$</td>
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<tr>
<td>$1 + - - + 2 2$</td>
<td>$1 2 3 1 4 2 3 4$</td>
<td>$1 2 3 1 4 2 3 4$</td>
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<tr>
<td>$1 1 + - + - 2 2$</td>
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<td>$1 2 3 2 4 1 4 3$</td>
<td>$+$</td>
</tr>
<tr>
<td>$++ - 1 1 2 2 - +$</td>
<td>$1 2 3 4 1 2 3 4$</td>
<td>$1 2 3 4 1 2 3 4$</td>
<td>$+$</td>
</tr>
<tr>
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<td>$1 2 3 4 2 1 4 3$</td>
<td>$1 2 3 4 2 1 4 3$</td>
<td>$+$</td>
</tr>
<tr>
<td>$1 1 2 2 - 3 3 4 4$</td>
<td>$+ 1 2 - - 1 2 +$</td>
<td>$+ 1 2 - - 1 2 +$</td>
<td>$+$</td>
</tr>
<tr>
<td>$1 2 1 2 3 4 3 4$</td>
<td>$- 1 2 + + 1 2 -$</td>
<td>$- 1 2 + + 1 2 -$</td>
<td>$+$</td>
</tr>
<tr>
<td>$++ - - - - - +$</td>
<td>$1 - 2 + + 1 - 2$</td>
<td>$1 - 2 + + 1 - 2$</td>
<td>$+$</td>
</tr>
<tr>
<td>$+ 1 1 - - 2 2 +$</td>
<td>$1 + 2 - - 1 + 2$</td>
<td>$1 + 2 - - 1 + 2$</td>
<td>$+$</td>
</tr>
<tr>
<td>$1 + 1 - - 2 + 2$</td>
<td>$1 2 + - - + 1 2$</td>
<td>$1 2 + - - + 1 2$</td>
<td>$+$</td>
</tr>
<tr>
<td>$+ 1 - 1 2 - 2 +$</td>
<td>$1 2 - + + - 1 2$</td>
<td>$1 2 - + + - 1 2$</td>
<td>$+$</td>
</tr>
<tr>
<td>$++ - - 1 2 1 2 - -$</td>
<td>$1 2 3 3 4 4 1 2$</td>
<td>$1 2 3 3 4 4 1 2$</td>
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<tr>
<td>$1 + - 1 2 - + 2$</td>
<td>$1 + - 1 2 - + 2$</td>
<td>$1 + - 1 2 - + 2$</td>
<td>$+$</td>
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<tr>
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<td>$1 + - 2 1 - + 2$</td>
<td>$1 + - 2 1 - + 2$</td>
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</tr>
<tr>
<td>$- 1 + 2 1 + + 2 -$</td>
<td>$1 - + 2 1 + + 2 -$</td>
<td>$1 - + 2 1 + + 2 -$</td>
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<tr>
<td>$1 - + 2 1 + - 2$</td>
<td>$1 - + 2 1 + - 2$</td>
<td>$1 - + 2 1 + - 2$</td>
<td>$+$</td>
</tr>
</tbody>
</table>

Acknowledgement

I would like to thank Professor Toshio Oshima, Professor Hisayosi Matumoto, and Professor David A. Vogan Jr. for stimulating discussions. I also thank the referees for carefully checking the paper and J.S.P.S. for a research fellowship during which most of this work was done.
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