REPRESENTATION THEORY
An Electronic Journal of the American Mathematical Society
Volume 1, Pages 424–461 (December 10, 1997)
S 1088-4165(97)00035-6

SPHERICAL REPRESENTATIONS AND MIXED SYMMETRIC SPACES

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Abstract. Let G/H be a symmetric space admitting a G-invariant hyperbolic cone field. For each such cone field we construct a local tube domain Ξ containing G/H as a boundary component. The domain Ξ is an orbit of an Ol'shanskii type semi group Γ. We describe the structure of the group G and the domain Ξ. Furthermore we explore the correspondence between Γ-modules of holomorphic sections of line bundles over Ξ and spherical highest weight modules.

Introduction

Let $(\mathfrak{g}, \tau)$ be a real symmetric Lie algebra and $\mathfrak{g} = \mathfrak{h} + \mathfrak{q}$ the corresponding eigenspace decomposition for τ. We call an element $X \in \mathfrak{q}$ hyperbolic if the operator $\operatorname{ad} X$ is diagonalizable over $\mathbb{R}$. The existence of "enough" hyperbolic elements in $\mathfrak{q}$ is important in many contexts. For Cartan decompositions it is crucial for the restricted root decomposition of semisimple real Lie algebras, and hence for the whole structure theory of these algebras. If $(\mathfrak{g}, \tau)$ is a non-compactly causal symmetric (NCC) Lie algebra in the sense of [HÓ96], then $\mathfrak{q}$ contains open convex cones which are invariant under the group $\operatorname{Inn}_{\mathfrak{g}}(\mathfrak{h})$ of inner automorphisms of $\mathfrak{g}$ generated by $e^{\operatorname{ad} \mathfrak{h}}$ and which consist entirely of hyperbolic elements. In the last years this class of reductive symmetric Lie algebras and the associated symmetric spaces have become a topic of very active research spreading in more and more areas. For a survey of the state of the art we refer to [HÓ96] and the literature cited there.

On the other hand there have been attempts to push this theory further to symmetric Lie algebras which are not necessarily semisimple or reductive. The simplest type (called the complex type) is where $\mathfrak{g} = \mathfrak{h}_{\mathbb{C}}$ is a complexification and τ is a complex conjugation. Among these symmetric Lie algebras those for which $\mathfrak{h}$ contains an open invariant convex cone W consisting of elliptic elements play a crucial role (cf. [Ne94a], [Ne96a], [Ne96b]). Then $iW \subseteq \mathfrak{q} = i\mathfrak{h}$ is an open cone consisting of hyperbolic elements so that, in the special case of reductive Lie algebras, we obtain on the one hand the non-compactly causal spaces of complex type and, if we allow $W = \mathfrak{h}$, also the Riemannian symmetric spaces coming from Cartan involutions of complex semisimple Lie algebras. For the associated symmetric spaces of complex type and the reductive spaces mentioned above one nowadays has a well developed picture of the harmonic analysis (holomorphic representations: [Ne94b], [Ne95];

Received by the editors June 24, 1997 and, in revised form, September 25, 1997.
1991 *Mathematics Subject Classification.* Primary 22E47, 22E15, 53C35, 54H15.

spherical functions [FHÓ94], [HiNe96]; Hardy spaces [HÓØ91], [Kr97]) and the invariant complex analysis (invariant Stein domains and plurisubharmonic functions [Ne96b]).

The first step in this program, i.e., the description of an appropriate class of not necessarily reductive symmetric Lie algebras which is general enough to incorporate all the cases mentioned above such as the mixed complex type case, the non-compactly causal spaces, and also the Riemannian symmetric spaces has been carried out in [KN96], which we will use as a reference for the basic structure theory and convex geometry of mixed, i.e., non-reductive, symmetric Lie algebras.

The next step that we carry out in this paper is the description of the structure of the associated global objects such as complex domains which are curved analogs of tube domains over convex cones. Furthermore we investigate the general representation theory and explain how certain representations can be realized in spaces of holomorphic functions on the aforementioned domains.

In Section I we collect the notation and the facts from [KN96] that we shall need throughout this paper. In Section II we then turn to product decompositions of the corresponding groups. These decompositions have various applications such as integration formulas and trivializations of certain holomorphic vector bundles. In this section many proofs consist in reducing everything to the case of reductive or simple Lie algebras which is completely discussed in [HÓ96]. As a first application of the decomposition theorems we explain how one can construct on certain domains acted on by H holomorphic vector valued functions which are H-eigenfunctions for a prescribed character. These functions play a key role in the realization theory of spherical representations.

Section III contains the description of various semigroups related to symmetric spaces. Basically these semigroups are of the type $\Gamma_H(C) = H\exp(C)$, where G/H is a symmetric space, and $C \subseteq \mathfrak{q}$ is a weakly hyperbolic H-invariant cone. Such semigroups arise naturally if G/H carries an invariant non-compactly causal structure ([HÓ96]). On the other hand the polar decomposition of $\Gamma_H(C)$ is quite similar to the Cartan decomposition of a real semisimple Lie group. We also describe a complex version of these semigroups and how they fit into the product decompositions discussed in Section II.

To each semigroup $\Gamma_H(C)$ we can associate the domain $\operatorname{Exp}(C) \subseteq G/H$ which should be thought of as the future of the base point. In Section IV we construct a complex domain $\Xi(C)$ which is the curved analog of a tube domain over the cone C. It contains the dual symmetric space G^c/H in its boundary (it is sort of a Shilov boundary for $\Xi(C)$). On the other hand it has the structure of an associated bundle of the type $G^c \times_H C$. The key point in Section IV is to clarify the interplay between the complex analysis of $\Xi(C)$ and its bundle structure which is not in any obvious way related to its complex geometry.

In Section V we explain which Hilbert spaces carrying representation of $\Gamma_H(C)$ can be realized in an equivariant way as spaces of holomorphic functions on $\Xi(C)$ and that the corresponding representations can be characterized as spherical representations of some sort. An important tool for these realizations is a result describing holomorphic functions on $\Xi(C)$ as the set of all holomorphic functions on a complex semigroup which are invariant under the group H.

In the last section we turn to irreducible spherical representations. We explain in which sense they are highest weight representations of the dual Lie algebra

$\mathfrak{g}^c = \mathfrak{h} + i\mathfrak{q}$ and give a necessary condition for a unitary highest weight representation to correspond to a spherical representation.

One of the next steps in the investigation of spherical representations of mixed symmetric spaces is to classify all irreducible spherical unitary highest weight representations. Since this involves quite a detailed analysis of certain singular highest weight representations and it is not even carried out for irreducible spaces, the solution to this problem seems to be quite complicated.

Another project building on this paper will be the construction of Hardy spaces on the domains $\Xi(C)$ for general groups. For non-reductive groups this construction is more involved because the symmetric space G^c/H does not always possess a G^c-invariant measure. Nevertheless it seems possible to construct analogs of Hardy spaces on $\Xi(C)$.

We would like to express our gratitude to the Mittag-Leffler Institute in Djursholm, Sweden for the hospitality and pleasant working atmosphere during our stay in spring 1996 where we started working on the present paper.

I. Basic facts and definitions

In this first section we collect some material concerning symmetric Lie algebras. Let $\mathfrak{g}$ denote a finite dimensional real Lie algebra. An element $X \in \mathfrak{g}$ is called *hyperbolic* if $\operatorname{ad} X$ is diagonalizable over $\mathbb{R}$. A convex subset $C \subseteq \mathfrak{g}$ is called *hyperbolic* if all its relative interior points are hyperbolic.

Definition I.1. (a) A *symmetric Lie algebra* $(\mathfrak{g}, \tau)$ is a pair consisting of a finite dimensional Lie algebra $\mathfrak{g}$ and an involutive automorphism τ of $\mathfrak{g}$. We put $\mathfrak{h} := \{X \in \mathfrak{g} \colon \tau.X = X\}$ and $\mathfrak{q} := \{X \in \mathfrak{g} \colon \tau.X = -X\}$, and note that $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{q}$.

(b) An abelian subspace $\mathfrak{a} \subseteq \mathfrak{q}$ is called *abelian maximal hyperbolic* if $\mathfrak{a}$ consists of hyperbolic elements and is maximal w.r.t. this property.

A subspace $\mathfrak{l} \subseteq \mathfrak{q}$ is called a *Lie triple system* if $[\mathfrak{l}, [\mathfrak{l}, \mathfrak{l}]] \subseteq \mathfrak{l}$. This means that the space $\mathfrak{l}_L := \mathfrak{l} \oplus [\mathfrak{l}, \mathfrak{l}]$ is a subalgebra of $\mathfrak{g}$. Recall that all abelian maximal hyperbolic subspaces as well as all maximal hyperbolic Lie triple systems in $\mathfrak{q}$ are conjugate under $\operatorname{Inn}_{\mathfrak{g}}(\mathfrak{h})$ (cf. [KN96, Cor. II.9, Th. III.3]).

(c) Let $\mathfrak{r}$ denote the radical of $\mathfrak{g}$ and let $\mathfrak{r} = \mathfrak{r}_{\mathfrak{h}} + \mathfrak{r}_{\mathfrak{q}}$ be its τ-eigenspace decomposition. In the following subscripts indicate intersections, for example $\mathfrak{r}_{\mathfrak{h}} := \mathfrak{r} \cap \mathfrak{h}$ etc. According to [KN96, Prop. III.5], there exists a τ-invariant Levi complement $\mathfrak{s} \subseteq \mathfrak{g}$ with the following properties. There exists a maximal hyperbolic Lie triple system $\mathfrak{p} \subseteq \mathfrak{q}$ such that $\mathfrak{p} = \mathfrak{p}_{\mathfrak{r}} \oplus \mathfrak{p}_{\mathfrak{s}}$, where $\mathfrak{p}_{\mathfrak{s}} \subseteq \mathfrak{s}_{\mathfrak{q}}$ is a maximal hyperbolic Lie triple system in $\mathfrak{s}_{\mathfrak{q}}$ and $[\mathfrak{p}_{\mathfrak{r}}, \mathfrak{s}] = \{0\}$. Then each maximal abelian subspace $\mathfrak{a} \subseteq \mathfrak{p}$ is of the form $\mathfrak{a} = \mathfrak{p}_{\mathfrak{r}} \oplus \mathfrak{a}_{\mathfrak{s}}$, and $[\mathfrak{p}_L, \mathfrak{s}] \subseteq \mathfrak{s}$, where $\mathfrak{p}_L = \mathfrak{p} \oplus [\mathfrak{p}, \mathfrak{p}]$ (cf. part (b)). Furthermore there exists a Cartan involution θ on $\mathfrak{s}$ commuting with $\operatorname{ad}[\mathfrak{p}, \mathfrak{p}]$ and $\tau|_{\mathfrak{s}}$ (cf. [KN96, Prop. I.5]). The corresponding Cartan decomposition is denoted by $\mathfrak{s} = \mathfrak{s}_{\mathfrak{k}} \oplus \mathfrak{s}_{\mathfrak{p}}$. The largest ideal of $\mathfrak{s}$ contained in $\mathfrak{s}_{\mathfrak{h}}$ is denoted by $\mathfrak{s}_{\mathrm{iso}}$. So the semisimple symmetric Lie algebra $(\mathfrak{s}, \tau|_{\mathfrak{s}})$ decomposes as

$$(\mathfrak{s}, \tau|_{\mathfrak{s}}) = (\mathfrak{s}_{\mathrm{iso}}, \tau|_{\mathfrak{s}_{\mathrm{iso}}}) \oplus \bigoplus_{i=1}^{n} (\mathfrak{s}_i, \tau|_{\mathfrak{s}_i})$$

with $(\mathfrak{s}_i, \tau|_{\mathfrak{s}_i})$ irreducible and effective.

(d) Let $\mathfrak{a} \subseteq \mathfrak{q}$ be an abelian maximal hyperbolic subspace. For every subalgebra $\mathfrak{b} \subseteq \mathfrak{g}$ we set $\mathfrak{b}^0 := \mathfrak{z}_{\mathfrak{b}}(\mathfrak{a})$. For $\alpha \in \mathfrak{a}^*$ we define

$$\mathfrak{g}^\alpha := \{X \in \mathfrak{g} \colon (\forall Y \in \mathfrak{a})[Y, X] = \alpha(Y)X\}$$

and write $\Delta := \{\alpha \in \mathfrak{a}^* \backslash \{0\} \colon \mathfrak{g}^\alpha \neq \{0\}\}$ for the set of roots. Then we get the root space decomposition $\mathfrak{g} = \mathfrak{g}^0 \oplus \bigoplus_{\alpha\in\Delta} \mathfrak{g}^\alpha$. For each $\alpha \in \Delta$ we put $m_\alpha := \dim \mathfrak{g}^\alpha$.

We call a root $\alpha \in \Delta$ *semisimple*, resp. *solvable*, if $\mathfrak{s}^\alpha := \mathfrak{g}^\alpha \cap \mathfrak{s} \neq \{0\}$, resp. $\mathfrak{g}^\alpha \subseteq \mathfrak{r}$. The set of all semisimple, resp. solvable, roots is denoted by Δ_s, resp. Δ_r. Note that $\Delta = \Delta_r \dot\cup \Delta_s$ (cf. [KN96, Lemma IV.5(i)]).

A root $\alpha \in \Delta$ is called *compact* if $\mathfrak{p}_L^\alpha \neq \{0\}$ and *non-compact* otherwise. We write Δ_k, Δ_n, resp. Δ_p, for the set of all compact, non-compact, resp. non-compact semisimple roots. Note that Δ_k is independent of the choice of $\mathfrak{p} \supseteq \mathfrak{a}$ (cf. [KN96, Def. V.1]) and that $\Delta = \Delta_k \dot\cup \Delta_n$ holds by definition.

(e) The *Weyl group* $\mathcal{W}_\mathfrak{a}$ of $(\mathfrak{g}, \tau)$ w.r.t. $\mathfrak{a}$ is defined by

$$\mathcal{W}_\mathfrak{a} := N_{\mathrm{Inn}_\mathfrak{g}(\mathfrak{h})}(\mathfrak{a}) / Z_{\mathrm{Inn}_\mathfrak{g}(\mathfrak{h})}(\mathfrak{a}).$$

Call $X_0 \in \mathfrak{a}$ *regular* if $\alpha(X_0) \neq 0$ for all $\alpha \in \Delta$. We call $\Delta^+ \subseteq \Delta$ a *positive system* if there exists $X_0 \in \mathfrak{a}$ regular such that $\Delta^+ = \{\alpha \in \Delta \colon \alpha(X_0) > 0\}$. A positive system is called $\mathfrak{p}$-*adapted* if the set $\Delta_n^+ := \Delta_n \cap \Delta^+$ of positive non-compact roots is invariant under the Weyl group.

(f) The symmetric Lie algebra $(\mathfrak{g}, \tau)$ is called *quasihermitian* if $\mathfrak{z}_\mathfrak{q}(\mathfrak{z}(\mathfrak{p})) = \mathfrak{p}$. In this case $\mathfrak{a}$ is maximal abelian in $\mathfrak{q}$ and there exists a $\mathfrak{p}$-adapted positive system Δ^+ (cf. [KN96, Prop. V. 10]). An irreducible effective quasihermitian symmetric Lie algebra $(\mathfrak{g}, \tau)$ is called *non-compactly Riemannian* (NCR), resp. *non-compactly causal* (NCC), if $\mathfrak{z}(\mathfrak{p}) = \{0\}$, resp. $\mathfrak{z}(\mathfrak{p}) \neq \{0\}$. The property of being quasihermitian is inherited by $\mathfrak{s}$. This means that the irreducible constituents $(\mathfrak{s}_i, \tau|_{\mathfrak{s}_i})$ of $\mathfrak{s}$ are either (NCR) or (NCC) (cf. [KN96, Prop. V.9(v)]).

(g) Let V be a finite dimensional real vector space and V^* its dual. For a subset $E \subseteq V$ the *dual cone* is defined by $E^\star := \{\omega \in V^* \colon (\forall x \in E)\omega(x) \geq 0\}$. A cone $C \subseteq V$ is called *generating* if $V = C - C$ and *pointed* if $C \cap -C = \{0\}$.

We associate to a positive system of non-compact roots Δ_n^+ the convex cones

$$C_{\min} := \operatorname{cone}\left(\{[X_\alpha, \tau(X_\alpha)] \colon X_\alpha \in \mathfrak{g}^\alpha, \alpha \in \Delta_n^+\}\right),$$

$C_{\max} := (\Delta_n^+)^\star = \{X \in \mathfrak{a} \colon (\forall \alpha \in \Delta_n^+)\alpha(X) \geq 0\}$ and $C_{\max,s} := (\Delta_p^+)^\star \cap \mathfrak{a}_\mathfrak{s}$. ■

Definition I.2. (a) Let $(\mathfrak{g}, \tau)$ be a symmetric Lie algebra and $\mathfrak{g}_\mathbb{C}$ be the complexification of $\mathfrak{g}$. We extend τ to a complex linear involution τ of $\mathfrak{g}_\mathbb{C}$. The *c-dual* $(\mathfrak{g}^c, \tau^c)$ of $(\mathfrak{g}, \tau)$ is defined by $\mathfrak{g}^c := \mathfrak{h} \oplus i\mathfrak{q}$ and $\tau^c := \tau|_{\mathfrak{g}^c}$. The complex conjugation in $\mathfrak{g}_\mathbb{C}$ w.r.t. the real form $\mathfrak{g}^c$ is called $\widehat{\tau}$. Thus the inclusion map $(\mathfrak{g}, \tau) \hookrightarrow (\mathfrak{g}_\mathbb{C}, \widehat{\tau})$ is an embedding of symmetric Lie algebras. We call $(\mathfrak{g}_\mathbb{C}, \widehat{\tau})$ the *canonical extension* of $(\mathfrak{g}, \tau)$ and write $\widehat{\mathfrak{h}} := \mathfrak{g}^c$ and $\widehat{\mathfrak{q}} := i\mathfrak{g}^c$ for the eigenspaces of $\widehat{\tau}$.

(b) A finite dimensional real Lie algebra $\mathfrak{g}^c$ is called *quasihermitian* if $(\mathfrak{g}^c_\mathbb{C}, \sigma)$ is quasihermitian, where σ denotes the complex conjugation on $\mathfrak{g}_\mathbb{C}$ w.r.t. $\mathfrak{g}^c$. We note that the symmetric Lie algebra $(\mathfrak{g}_\mathbb{C}, \widehat{\tau})$ considered in (a) is quasihermitian if and only if the Lie algebra $\mathfrak{g}^c$ is quasihermitian. ■

II. Product decompositions

In this section we describe certain product decompositions of open domains in groups associated to a symmetric Lie algebra. Most of these results generalize decompositions which are known for the reductive case (cf. [HÓ96]). We begin with some useful lemmas which give an idea of how to obtain certain product decompositions of open subsets of groups.

Lemma II.1. *Let $\mathfrak{n}$ be a nilpotent Lie algebra and N be a corresponding connected Lie group. Suppose that $\mathfrak{n} = \mathfrak{a} + \mathfrak{b}$ for subalgebras $\mathfrak{a}$ and $\mathfrak{b}$.*

(i) *If A and B are the analytic subgroups of N corresponding to $\mathfrak{a}$ and $\mathfrak{b}$, then $N = AB$.*
(ii) *If, in addition, $\mathfrak{b} = \mathfrak{c} \ltimes \mathfrak{d}$ and C, D are the corresponding analytic subgroups of N, then $N = ACD$.*

Proof. (i) We prove the assertion by induction. For $\dim \mathfrak{n} = 0$ there is nothing to prove. Suppose that $\mathfrak{n} \neq \{0\}$. Then $\mathfrak{z} := \mathfrak{z}(\mathfrak{n}) \neq \{0\}$, we can apply induction to $\widetilde{N} := N/Z$ and obtain $\widetilde{N} = \widetilde{A}\widetilde{B}$, where $\widetilde{A} := AZ/Z$ and $\widetilde{B} := BZ/Z$. From this we get $N = AZBZ = AZB$. Since A and B are subgroups, the lemma will follow if we can show that $Z \subseteq AB$. To see this, let $z \in Z$ and choose $X \in \mathfrak{z}$ with $z = \exp(X)$. Write $X = X_1 + X_2$, where $X_1 \in \mathfrak{a}$ and $X_2 \in \mathfrak{b}$. Then $[X_1, X_2] = [X_1, X] = 0$ implies that $z = \exp(X) = \exp(X_1)\exp(X_2) \in AB$.

(ii) As $B = CD$, this follows from (i). ∎

Lemma II.2. *Let G be a connected Lie group with Lie algebra $\mathfrak{g}$ and $\mathfrak{n} \subseteq \mathfrak{g}$ be a subalgebra for which $\mathfrak{g}$ is a nilpotent module under the adjoint representation. Assume further that $\mathfrak{n} \cap \mathfrak{z}(\mathfrak{g}) = \{0\}$ and put $N = \exp(\mathfrak{n})$. Then N is closed, simply connected and isomorphic to $\mathfrak{n}$ under the exponential mapping.*

Proof. According to our assumptions, the mapping $\varphi\colon \mathfrak{n} \to \mathrm{Ad}(N), X \mapsto e^{\mathrm{ad}\, X}$ is a homeomorphism. As $\varphi = \mathrm{Ad} \circ \exp|_{\mathfrak{n}}$ and $\mathfrak{n}$ is nilpotent, $\exp|_{\mathfrak{n}}\colon \mathfrak{n} \to N$ is a homeomorphism onto a locally compact, hence closed subgroup of G. This proves the lemma. ∎

Lemma II.3. *Let G be a Lie group whose Lie algebra $\mathfrak{g}$ can be written as the sum of subalgebras $\mathfrak{g} = \mathfrak{a} + \mathfrak{b} + \mathfrak{c}$, where $\mathfrak{b}$ normalizes $\mathfrak{c}$. Denote by A, B and C the analytic subgroups corresponding to $\mathfrak{a}$, $\mathfrak{b}$ and $\mathfrak{c}$ and consider the mapping*

$$\Phi\colon A \times B \times C \to G, \quad (a, b, c) \mapsto abc.$$

(i) *The mapping Φ is a submersion; in particular, the image ABC of Φ is open in G.*
(ii) *If $\mathfrak{g} = \mathfrak{a} \oplus \mathfrak{b} \oplus \mathfrak{c}$ is a direct sum, then $d\Phi$ is everywhere bijective.*

Proof. (i) Since Φ is left A and right C-equivariant, it suffices to prove that $d\Phi(\mathbf{1}, b, \mathbf{1})$ is surjective. For every $g \in G$ we denote by λ_g, resp. ρ_g, the left, resp. right, translation by g, i.e., $\lambda_g(x) = gx$, resp. $\rho_g(x) = xg$, for all $x \in G$. Then the differential is computed to

$$d\Phi(\mathbf{1}, b, \mathbf{1})(X, d\lambda_b(\mathbf{1}).Y, Z) = d\rho_b(\mathbf{1}).\big(X + \mathrm{Ad}(b).Y + \mathrm{Ad}(b).Z\big).$$

As $\mathrm{Ad}(b).\mathfrak{b} = \mathfrak{b}$ and $\mathrm{Ad}(b).\mathfrak{c} = \mathfrak{c}$, we conclude from $\mathfrak{g} = \mathfrak{a} + \mathfrak{b} + \mathfrak{c}$ that $d\Phi(\mathbf{1}, b, \mathbf{1})$ is surjective.

(ii) This follows from (i) by dimension counting. ∎

The HAN-decomposition. Let $(\mathfrak{g},\tau)$ be a symmetric Lie algebra. For the remainder of this section we assume that the abelian maximal hyperbolic subspace $\mathfrak{a}\subseteq\mathfrak{q}$ is also maximal abelian in $\mathfrak{q}$ and not only in $\mathfrak{p}$. For a positive system Δ^+ we define the subalgebras

$$\mathfrak{n}:=\sum_{\alpha\in\Delta^+}\mathfrak{g}^\alpha \qquad\text{and}\qquad \overline{\mathfrak{n}}:=\sum_{\alpha\in\Delta^-}\mathfrak{g}^\alpha.$$

Then $\mathfrak{g}^0=\mathfrak{z}_\mathfrak{g}(\mathfrak{a})=\mathfrak{a}\oplus\mathfrak{h}^0$ (cf. Definition I.1(d)), so that $\mathfrak{g}=\mathfrak{h}\oplus\mathfrak{a}\oplus\mathfrak{n}$ follows from the observation that each $X\in\mathfrak{g}^{-\alpha}$, $\alpha\in\Delta^+$ can be written as $X=\big(X+\tau(X)\big)-\tau(X)\in\mathfrak{h}+\mathfrak{n}$.

If not otherwise stated G denotes a simply connected Lie group with Lie algebra $\mathfrak{g}$. Then τ integrates to an involution on G which is also denoted by τ. We write H for the fixed point group of τ and recall from [Lo69, Th. 3.4] that H is connected. Further we define A, H, H^0, N, $\overline{N}$, R and S as the analytic subgroups of G corresponding to $\mathfrak{a}$, $\mathfrak{h}$, $\mathfrak{h}^0$, $\mathfrak{n}$, $\overline{\mathfrak{n}}$, $\mathfrak{r}$ and $\mathfrak{s}$. By subscripts we indicate intersections, for instance $H_R=H\cap R$, $N_S=N\cap S$ etc.

Proposition II.4 (The HAN-decomposition)**.** *For a simply connected symmetric Lie group (G,τ) associated to $(\mathfrak{g},\tau)$ the following assertions hold:*

(i) *The groups A, resp. N, are closed, simply connected and diffeomorphic to $\mathfrak{a}$, resp. $\mathfrak{n}$, under the exponential mapping. Moreover $A\cap N=\{\mathbf{1}\}$.*

(ii) *The map*

$$\Phi\colon H\times A\times N\to G,\quad (h,a,n)\mapsto han$$

is a diffeomorphism onto its open image.

(iii) *The multiplication mapping $\Phi_R\colon H_R\times A_R\times N_R\to R$ is a diffeomorphism.*

(iv) *The set HAN is R-saturated, i.e., left and right R-invariant.*

Proof. (i) Since $\mathfrak{g}$ is a nilpotent $\mathfrak{n}$-module and $\mathfrak{n}\cap\mathfrak{z}(\mathfrak{g})=\{0\}$, the statement concerning N follows from Lemma II.2. For the assertion about A we choose a vector space complement $\mathfrak{a}_1$ in $\mathfrak{a}$ to $\mathfrak{a}_0:=\mathfrak{a}\cap\mathfrak{z}(\mathfrak{g})$. Since G is simply connected, $Z(G)_0$ is a vector group and hence $\mathfrak{a}_0\cong A_0$. Moreover the hyperbolicity of $\mathfrak{a}$ together with $\mathfrak{a}_1\cap\mathfrak{z}(\mathfrak{g})=\{0\}$ imply that $\mathfrak{a}_1\cong\mathrm{Ad}(A_1)$ and hence $\mathfrak{a}_1\cong A_1$. As $A=A_0A_1$ and $A_0\cap A_1\subseteq A_1\cap\ker\mathrm{Ad}=\{\mathbf{1}\}$ we see that $A=A_0\times A_1$. Hence $A\cong\mathfrak{a}$ proving our assertion about A.

It remains to show that $A\cap N=\{\mathbf{1}\}$. Let $x\in A\cap N$ and choose an appropriate basis adapted to the root decomposition of $\mathfrak{g}$ w.r.t. $\mathfrak{a}$ and therefore to $\mathfrak{g}=\mathfrak{n}\oplus\mathfrak{a}\oplus\mathfrak{h}^0\oplus\overline{\mathfrak{n}}$. Then, since $x\in A$, $\mathrm{Ad}(x)$ is represented by a diagonal matrix and, since $x\in N$, also by a unipotent matrix. Hence $\mathrm{Ad}(x)=\mathbf{1}$ and $\mathrm{Ad}(N)\cong\mathfrak{n}$ implies that $x=\mathbf{1}$.

(ii) As the assumptions of Lemma II.3 are satisfied, $d\Phi$ is everywhere bijective and HAN is open in G. So it remains to check that Φ is injective, which, according to (i), will follow from $H\cap AN=\{\mathbf{1}\}$.

Let $x\in H\cap AN$ and write $x=an$ with $a\in A$ and $n\in N$. Then $x\in H$ entails that

$$\tau(n)=\tau(a^{-1})\tau(x)=ax=a^2n.$$

Take an element $X\in\mathfrak{a}$ in the interior of the positive Weyl chamber and set $a_t=\exp(tX)$ for all $t\in\mathbb{R}$. Note that $\lim_{t\to\infty}a_t^{-1}na_t=\mathbf{1}$ and, moreover, since N is closed, if $\tau(n) \neq \mathbf{1}$, then $a_t^{-1}\tau(n)a_t$ leaves every compact subset of G if t tends to infinity. So

$$a^2 = \lim_{t\to\infty} a_t^{-1}a^2 n a_t = \lim_{t\to\infty} a_t^{-1}\tau(n)a_t$$

entails $a^2 = \tau(n) = \mathbf{1}$. Finally, in view of (i), we have $a = 1$ and hence the assertion.

(iii) In accordance with Lemma II.3 and (ii), we only have to prove that Φ_R is onto. Let $\mathfrak{u}$ be the nilradical of $\mathfrak{r}$ and note that $\mathfrak{n}_\mathfrak{r} := \mathfrak{n} \cap \mathfrak{r} \subseteq [\mathfrak{a}, \mathfrak{r}] \subseteq \mathfrak{u}$. Since $\mathfrak{u}$ is τ-invariant, we get a decomposition $\mathfrak{u} = \mathfrak{h}_\mathfrak{u} \oplus \mathfrak{a}_\mathfrak{u} \oplus \mathfrak{n}_\mathfrak{r}$, where $\mathfrak{h}_\mathfrak{u} = \mathfrak{h} \cap \mathfrak{u}$ and $\mathfrak{a}_\mathfrak{u} = \mathfrak{a} \cap \mathfrak{u}$. Thus the assumptions of Lemma II.1(ii) are satisfied, so $U = H_U A_U N_R$ and finally $\mathfrak{r} = \mathfrak{r}^0 \oplus [\mathfrak{a}, \mathfrak{r}] = \mathfrak{a}_\mathfrak{r} + \mathfrak{h}_\mathfrak{r}^0 + \mathfrak{u}$ gives

$$R = UA_R H_R^0 = H_U A_U N_R A_R H_R^0 \subseteq H_U H_R^0 A_R N_R \subseteq H_R A_R N_R.$$

(iv) We only have to prove the right R-invariance of HAN because then the left R-invariance follows from the normality of R.

In view of (iii), it suffices to show that $ANH_R \subseteq HAN$. Let $b \in AN$ and $r \in H_R$. Since $brb^{-1} \in R$ it follows from (iii) that $brb^{-1} = h'b'$ with $h' \in H_R$ and $b' \in A_R N_R$. Now the assertion follows from $br = brb^{-1}b = h'b'b \in HAN$. ∎

The $P^+K_{1,\mathbb{C}}^cP^-$-decomposition. Let $(\mathfrak{g}, \tau)$ be a quasihermitian symmetric Lie algebra, $\mathfrak{p}$ a maximal hyperbolic Lie triple system in $\mathfrak{q}$, $\mathfrak{a} \subseteq \mathfrak{p}$ a maximal abelian subspace, and $\Delta^+ \subseteq \mathfrak{a}^*$ a $\mathfrak{p}$-adapted positive system of roots. We assume that $\mathfrak{h}^0 = \mathfrak{z}_\mathfrak{h}(\mathfrak{a})$ is compactly embedded in $\mathfrak{g}$ and recall from [KN96, Th. VIII.1(ii)] that this implies that $\mathfrak{k}^c := i\mathfrak{p} + [\mathfrak{p}, \mathfrak{p}] + \mathfrak{h}^0$ is a maximal compactly embedded subalgebra of $\mathfrak{g}^c$. Further we define subalgebras of $\mathfrak{g}_\mathbb{C}$ by

$$\mathfrak{p}^\pm := \sum_{\alpha\in\Delta_n^\pm} \mathfrak{g}_\mathbb{C}^\alpha, \qquad \mathfrak{p}_\mathfrak{r}^\pm := \sum_{\alpha\in\Delta_r^\pm} \mathfrak{g}_\mathbb{C}^\alpha \qquad \text{and} \qquad \mathfrak{p}_s^\pm := \sum_{\alpha\in\Delta_p^\pm} \mathfrak{g}_\mathbb{C}^\alpha,$$

and note that $[\mathfrak{k}^c, \mathfrak{p}^\pm] \subseteq \mathfrak{p}^\pm$ (cf. [KN96, Prop. V.4]). Thus $\mathfrak{p}_0 := \mathfrak{k}_\mathbb{C}^c \ltimes \mathfrak{p}^-$ is a subalgebra of $\mathfrak{g}_\mathbb{C}$ and we obtain a triangular decomposition $\mathfrak{g}_\mathbb{C} = \mathfrak{p}^+ \oplus \mathfrak{k}_\mathbb{C}^c \oplus \mathfrak{p}^-$.

Let G^c be a simply connected group with Lie algebra $\mathfrak{g}^c$, $H := (G^c)^\tau$ be the connected group of τ-fixed points in G^c, and $G_\mathbb{C} \cong (G^c)_\mathbb{C}$ be a simply connected complex group with Lie algebra $\mathfrak{g}_\mathbb{C}$. Further we have the maximal compactly embedded subgroup $K^c \subseteq G^c$, and the corresponding universal complexifications $c_{K^c} \colon K^c \to K_\mathbb{C}^c$ and $c_H \colon H \to H_\mathbb{C}$. Note that since K^c has a compact Lie algebra, we can identify K^c with a subgroup of its complexification $K_\mathbb{C}^c$ and thus surpress c_{K^c}. Moreover the simple connectedness of G^c implies that K^c and therefore also $K_\mathbb{C}^c$ are simply connected. Further we have the corresponding subgroups $G_1^c, K_1^c, H_1, K_{1,\mathbb{C}}^c, H_{1,\mathbb{C}} \subseteq G_\mathbb{C}$ which are the images of the canonical morphisms $\eta_H \colon H \to G_\mathbb{C}$, $\eta_{H_\mathbb{C}} \colon H_\mathbb{C} \to G_\mathbb{C}$, $\eta_{G^c} \colon G^c \to G_\mathbb{C}$, and $\eta_K \colon K_\mathbb{C}^c \to G_\mathbb{C}$.

The involutions $\widehat{\tau}$ and τ integrate to involutions on $G_\mathbb{C}$ denoted by the same letters. As $G_\mathbb{C}$ is simply connected, the fixed point groups $G_1^c = (G_\mathbb{C})^{\widehat{\tau}}$, $H_{1,\mathbb{C}} = (G_\mathbb{C})^\tau$, and $G_1 := (G_\mathbb{C})^{\widehat{\tau}\tau}$ are connected (cf. [Lo69, Th. 3.4]). The corresponding Lie algebras are given by $\mathfrak{g}^c$, $\mathfrak{h}_\mathbb{C}$ and $\mathfrak{g}$. We write P_0, $P^\pm$, $P_R^\pm$, $R_\mathbb{C}$ and $S_\mathbb{C}$ for the analytic subgroups of $G_\mathbb{C}$ associated to $\mathfrak{p}_0$, $\mathfrak{p}^\pm$, $\mathfrak{p}_\mathfrak{r}^\pm$, $\mathfrak{r}_\mathbb{C}$ and $\mathfrak{s}_\mathbb{C}$.

Proposition II.5 (The $P^+K_{1,\mathbb{C}}^cP^-$-decomposition)**.**

(i) *The groups $K_{1,\mathbb{C}}^c$ and $P^\pm$ are closed subgroups of $G_\mathbb{C}$ and $P^\pm$ is diffeomorphic to $\mathfrak{p}^\pm$ via the exponential mapping.*

(ii) *The mapping*

$$\Omega\colon P^+ \times K^c_{1,\mathbb{C}} \times P^- \to G_{\mathbb{C}}, \quad (p_+, k, p_-) \mapsto p_+ k p_-$$

is a biholomorphic diffeomorphism onto its open image.

(iii) *The multiplication mapping* $\Omega_R\colon P^+_R \times (K^c_{1,\mathbb{C}} \cap R_{\mathbb{C}}) \times P^-_R \to R_{\mathbb{C}}$ *is a biholomorphism.*

(iv) *The image* $P^+ K^c_{1,\mathbb{C}} P^-$ *of* Ω *is* $R_{\mathbb{C}}$*-saturated.*

(v) $G^c_1 \subseteq P^+ K^c_{1,\mathbb{C}} P^-$.

Proof. (i) First we prove that $K^c_{1,\mathbb{C}}$ is closed. Let $\mathfrak{t}_{\mathbb{C}} \subseteq \mathfrak{g}^0_{\mathbb{C}} \subseteq \mathfrak{k}^c_{\mathbb{C}}$ be a Cartan subalgebra of $\mathfrak{g}_{\mathbb{C}}$ containing $\mathfrak{a}_{\mathbb{C}}$. Then [Bou90, Ch. 7, §2, no. 1, Cor. 4] implies that the corresponding analytic subgroup $K^c_{1,\mathbb{C}}$ is closed.

Since $\mathfrak{g}_{\mathbb{C}}$ is a nilpotent $\mathfrak{p}^{\pm}$-module and $\mathfrak{p}^{\pm} \cap \mathfrak{z}(\mathfrak{g}_{\mathbb{C}}) = \{0\}$, the assertion about $P^{\pm}$ follows from Lemma II.2.

(ii) As the assumptions of Lemma II.3 are satisfied, the differential of Ω is everywhere bijective and the image $P^+ K^c_{1,\mathbb{C}} P^-$ is open. So it remains to establish the injectivity of Ω. This means that $P^+ \cap P_0 = \{\mathbf{1}\}$, and that $K^c_{1,\mathbb{C}} \cap P^- = \{\mathbf{1}\}$. Let $X_0 \in \mathfrak{z}(\mathfrak{p})$ such that $\Delta^+_n = \{\alpha \in \Delta_n \colon \alpha(X_0) > 0\}$ (cf. [KN96, Prop. V.4]) and define $a_t := \exp(tX_0)$ for $t \in \mathbb{R}$. Since $X_0 \in \mathfrak{z}(\mathfrak{p}) \subseteq \mathfrak{a}$, it centralizes $\mathfrak{k}^c_{\mathbb{C}}$. Furthermore we have $\lim_{t\to\infty} a_t^{-1} p_+ a_t = \mathbf{1}$ for all $p_+ \in P^+$ and $a_t^{-1} p_- a_t$ eventually leaves every compact set of G if $p_- \in P^- \setminus \{\mathbf{1}\}$ and t tends to infinity. From that we see that $P^+ \cap P_0 = \{\mathbf{1}\}$ and $K^c_{1,\mathbb{C}} \cap P^- = \{\mathbf{1}\}$.

(iii) In view of Lemma II.3 and (ii), it remains to check surjectivity. Let $\mathfrak{u}_{\mathbb{C}}$ be the nilradical of $\mathfrak{r}_{\mathbb{C}}$. Then $\mathfrak{u}_{\mathbb{C}} = \mathfrak{p}^+_{\mathfrak{r}} \oplus (\mathfrak{k}^c_{\mathbb{C}} \cap \mathfrak{u}_{\mathbb{C}}) \oplus \mathfrak{p}^-_{\mathfrak{r}}$ and Lemma II.1(ii) applies. We obtain that $U_{\mathbb{C}} = P^+_R (K^c_{\mathbb{C}} \cap U_{\mathbb{C}}) P^-_R$ and finally $\mathfrak{r}_{\mathbb{C}} = \mathfrak{r}^0_{\mathbb{C}} \oplus [\mathfrak{a}, \mathfrak{r}_{\mathbb{C}}] = (\mathfrak{k}^c_{\mathbb{C}} \cap \mathfrak{r}_{\mathbb{C}}) + \mathfrak{u}_{\mathbb{C}}$ yields

$$R_{\mathbb{C}} = (K^c_{1,\mathbb{C}} \cap R_{\mathbb{C}}) U_{\mathbb{C}} = (K^c_{1,\mathbb{C}} \cap R_{\mathbb{C}}) P^+_R (K^c_{1,\mathbb{C}} \cap U_{\mathbb{C}}) P^-_R = P^+_R (K^c_{1,\mathbb{C}} \cap R_{\mathbb{C}}) P^-_R.$$

(iv) This follows from (iii) as in the proof of Proposition II.4(iv).

(v) In view of (iv), we may assume that $(\mathfrak{g}, \tau)$ is simple and quasihermitian. In the case where $(\mathfrak{g}, \tau)$ is (NCR), we have $P^+ K^c_{1,\mathbb{C}} P^- = K^c_{1,\mathbb{C}} = G_{\mathbb{C}}$ and we are done. If $(\mathfrak{g}, \tau)$ is (NCC), then the assertions follow from [Hel78, p.399]. ∎

The $H_{1,\mathbb{C}} K^c_{1,\mathbb{C}} P^-$-decomposition. We keep the setup of the preceding subsection. The direct product group $H_{\mathbb{C}} \times (P^- \rtimes K^c_{\mathbb{C}})$ acts naturally on $G_{\mathbb{C}}$ by

$$(h, p, k).x := \eta_H(h) x \eta_K(k)^{-1} p^{-1}$$

and the orbit of $\mathbf{1}$ coincides with the set $H_{1,\mathbb{C}} K^c_{1,\mathbb{C}} P^-$. We write

$$K^{\mathbf{1}} := \{(h, p, k) \colon \eta_H(h) = p \eta_K(k)\},$$

for the stabilizer of $\mathbf{1}$.

Proposition II.6 (The $H_{1,\mathbb{C}} K^c_{1,\mathbb{C}} P^-$-decomposition)**.**

(i) *The orbit map* $H_{\mathbb{C}} \times (P^- \rtimes K^c_{\mathbb{C}}) \to G_{\mathbb{C}}$ *of* $\mathbf{1}$ *factors to a biholomorphic map*

$$\Psi\colon \big(H_{\mathbb{C}} \times (P^- \rtimes K^c_{\mathbb{C},1})\big)/K^{\mathbf{1}} \to H_{1,\mathbb{C}} K^c_{1,\mathbb{C}} P^- \subseteq G_{\mathbb{C}}$$

with open and dense range $H_{1,\mathbb{C}} K^c_{1,\mathbb{C}} P^-$.

(ii) $K^{\mathbf{1}} = \{(h, \mathbf{1}, k) \colon \eta_H(h) = \eta_K(k)\}$ *and its Lie algebra is given by*

$$\{(X, X) \colon X \in \mathfrak{h}_{\mathbb{C}} \cap \mathfrak{k}^c_{\mathbb{C}}\} \cong (\mathfrak{h} \cap \mathfrak{k}^c)_{\mathbb{C}}.$$

(iii) *The multiplication mapping* $\Psi_R\colon (H_{1,\mathbb{C}} \cap R_\mathbb{C}) \times (P_0 \cap R_\mathbb{C}) \to R_\mathbb{C}$ *is surjective.*
(iv) *The set* $H_{1,\mathbb{C}} K^c_{1,\mathbb{C}} P^+$ *is* $R_\mathbb{C}$*-saturated.*

Proof. (i) The mapping Ψ is obtained by factorization of the orbit map $H_\mathbb{C} \times (P^- \rtimes K^c_{\mathbb{C},1}) \to G_\mathbb{C}$. Thus it follows from $\mathfrak{g}_\mathbb{C} = \mathfrak{h}_\mathbb{C} + \mathfrak{k}_\mathbb{C} + \mathfrak{p}^-$ and Lemma II.3(i) that Ψ is everywhere submersive, hence a biholomorphic map of the complex homogeneous space $\big(H_\mathbb{C} \times (P^- \rtimes K^c_{\mathbb{C},1})\big)/K^{\mathbf{1}}$ onto an open subset of $G_\mathbb{C}$. The density of the image will be proved below.

(ii) From $\mathfrak{h}_\mathbb{C} \cap (\mathfrak{k}_\mathbb{C} + \mathfrak{p}^-) = \mathfrak{h}_\mathbb{C} \cap \mathfrak{k}^c_\mathbb{C}$ we further conclude that the Lie algebra of $K^{\mathbf{1}}$ is given by $\{(X,X)\colon X \in \mathfrak{h}_\mathbb{C} \cap \mathfrak{k}^c_\mathbb{C}\} \cong (\mathfrak{h} \cap \mathfrak{k}^c)_\mathbb{C}$. We claim that $H_{1,\mathbb{C}} \cap (K^c_{1,\mathbb{C}} P^-) \subseteq K^c_{1,\mathbb{C}}$. Let $x \in H_{1,\mathbb{C}} \cap (K^c_{1,\mathbb{C}} P^-)$ and write $x = kp$ with $k \in K^c_{1,\mathbb{C}}$ and $p \in P^-$. Then $x \in H_{1,\mathbb{C}}$ implies

$$\mathbf{1} = \tau(x)^{-1} x = \tau(p)^{-1} \tau(k)^{-1} kp.$$

Now, as $\tau(p)^{-1} \in P^+$, the uniqueness of the $P^+ K^c_{1,\mathbb{C}} P^-$-decomposition entails that $p = \mathbf{1}$ and so $x = k \in K^c_{1,\mathbb{C}}$. This proves that $(h,p,k) \in K^{\mathbf{1}}$ implies that $p = \mathbf{1}$ and completes the proof of (ii).

(iii) Since $\mathfrak{u}_\mathbb{C} = (\mathfrak{h}_\mathbb{C} \cap \mathfrak{u}_\mathbb{C}) + (\mathfrak{p}_0 \cap \mathfrak{u}_\mathbb{C})$, Lemma II.1(ii) implies that $U_\mathbb{C} = (H_{1,\mathbb{C}} \cap U_\mathbb{C})(P_0 \cap U_\mathbb{C})$. Hence the assertion follows from

$$\begin{aligned} R_\mathbb{C} = (H^0_{1,\mathbb{C}} \cap R_\mathbb{C}) U_\mathbb{C} (A_\mathbb{C} \cap R_\mathbb{C}) &\subseteq (H^0_{1,\mathbb{C}} \cap R_\mathbb{C})(H_{1,\mathbb{C}} \cap U_\mathbb{C})(P_0 \cap U_\mathbb{C})(A_\mathbb{C} \cap R_\mathbb{C}) \\ &\subseteq (H_{1,\mathbb{C}} \cap R_\mathbb{C})(P_0 \cap R_\mathbb{C}). \end{aligned}$$

(iv) This is done in the same way as the proof of Proposition II.4(iv).

It remains to prove the density of $\operatorname{im}\Psi$. In view of (iv), we may assume that $(\mathfrak{g}, \tau)$ is simple and quasihermitian. In the case where $(\mathfrak{g}, \tau)$ is (NCR), we have $H_{1,\mathbb{C}} K^c_{1,\mathbb{C}} P^+ = K^c_{1,\mathbb{C}} = G_\mathbb{C}$ and we are done. If $(\mathfrak{g}, \tau)$ is (NCC), then the assertions follow from [ÓØ88, Th. 2.4]. ∎

Constructing holomorphic H-eigenfunctions. We recall from Proposition II.5(v) that $G^c_1 \subseteq P^+ K^c_{1,\mathbb{C}} P^-$. We define the domain $\mathcal{D} \subseteq \mathfrak{p}^+$ by $\exp(\mathcal{D}) K^c_{1,\mathbb{C}} P^- = G^c_1 K^c_{1,\mathbb{C}} P^-$. Then we obtain an action of G^c on $\mathcal{D}$ which is given by $\exp(g.z) = \zeta\big(\eta_G(g) \exp z\big)$, where $\zeta\colon P^+ K^c_{1,\mathbb{C}} P^- \to P^+$ denotes the projection onto the first factor. We write $\kappa\colon P^+ K^c_{1,\mathbb{C}} P^- \to K^c_{1,\mathbb{C}}$ for the projection onto the middle factor.

According to [Ne98, Prop. VIII.1.9], the stabilizer of 0 in G^c coincides with K^c, so that $\mathcal{D} = G^c.0 \cong G^c/K^c$. This realization of G^c/K^c as an open domain in $\mathfrak{p}^+$ is called the *generalized Harish-Chandra embedding*. It clearly exhibits the complex manifold structure on G^c/K^c.

Let us briefly recall how this is related to the symmetric Lie algebra $(\mathfrak{g}, \tau)$. We assume that $(\mathfrak{g}, \tau)$ is a quasihermitian symmetric Lie algebra and that $\mathfrak{h}^0$ is compactly embedded. Then [KN96, Th. VIII.1] implies the existence of an element $X_0 \in \mathfrak{z}(\mathfrak{p}) \subseteq \mathfrak{z}(\widehat{\mathfrak{p}})$ such that $\alpha(X_0) > 0$ for all $\alpha \in \widehat{\Delta}^+_n$. Let $\sigma := \tau\widehat{\tau}$ denote the antilinear involution of $\mathfrak{g}_\mathbb{C}$ whose fixed point set coincides with the real subalgebra $\mathfrak{g}$. Then $\sigma(X_0) = X_0$, $\mathfrak{k}^c = \mathfrak{z}_{\mathfrak{g}^c}(X_0)$ and $\mathfrak{p}^+ = \sum_{\alpha \in \widehat{\Delta}^+_n} \mathfrak{g}^\alpha_\mathbb{C}$ imply that

$$\sigma(\mathfrak{k}^c_\mathbb{C}) = \mathfrak{k}^c_\mathbb{C} \quad \text{and} \quad \sigma(\mathfrak{p}^\pm) = \mathfrak{p}^\pm.$$

We conclude in particular that

$$\mathfrak{g} = (\mathfrak{p}^+ \cap \mathfrak{g}) \oplus (\mathfrak{k}^c_\mathbb{C} \cap \mathfrak{g}) \oplus (\mathfrak{p}^- \cap \mathfrak{g}),$$

where these three subspaces are real forms of $\mathfrak{p}^+$, $\mathfrak{k}^c_\mathbb{C}$, resp. $\mathfrak{p}^-$.

Our next objective is to show that the subgroup $H \subseteq G^c$ acts transitively on $\mathcal{D}_{\mathbb{R}} := \mathcal{D} \cap \mathfrak{g}$. The following lemma prepares the proof of this fact. If $\rho\colon \mathfrak{h} \to \mathfrak{gl}(V)$ is a representation of the Lie algebra $\mathfrak{h}$, then a subalgebra $\mathfrak{e} \subseteq \mathfrak{h}$ is said to be *ρ-compactly embedded* if the group $\overline{\langle e^{\rho(\mathfrak{e})}\rangle} \subseteq \mathrm{GL}(V)$ is compact.

Lemma II.7. *Let $\rho\colon \mathfrak{h} \to \mathfrak{gl}(\mathfrak{g}^c), X \mapsto \mathrm{ad}_{\mathfrak{g}^c} X$. If $(\mathfrak{g}, \tau)$ is quasihermitian and $\mathfrak{h}^0$ is compactly embedded, then the subalgebra $\mathfrak{k}^c \cap \mathfrak{h}$ is maximal ρ-compactly embedded in $\mathfrak{h}$.*

Proof. First we note that the lemma is equivalent to the statement that $\mathfrak{h} \cap \mathfrak{k}^c$ is maximal as a compactly embedded subalgebra of $\mathfrak{g}$ which is contained in $\mathfrak{h}$.

To prove this statement we first observe that if $\kappa(X,Y) = \mathrm{tr}(\mathrm{ad}\, X\, \mathrm{ad}\, Y)$ denotes the Cartan-Killing form of $\mathfrak{g}$, then κ is negative semidefinite on each compactly embedded subalgebra $\mathfrak{b}$, and that the isotropic part of $\mathfrak{b}$ is central in $\mathfrak{g}$. In fact, if $X \in \mathfrak{g}$ is elliptic, then the operator $(\mathrm{ad}\, X)^2$ has non-positive real eigenvalues, hence $\mathrm{tr}(\mathrm{ad}\, X)^2 \leq 0$ and $\mathrm{tr}(\mathrm{ad}\, X)^2 = 0$ is equivalent to $\mathrm{ad}\, X = 0$.

In terms of the root decomposition with respect to $\mathfrak{a} \subseteq \mathfrak{p}$, we have

$$\mathfrak{h} = \mathfrak{h}^0 + \sum_{\alpha \in \Delta^+} (\mathbf{1} + \tau).\mathfrak{g}^\alpha \quad \text{and} \quad \mathfrak{p}_L = \mathfrak{p}_L^0 \oplus \sum_{\alpha \in \Delta_k} \mathfrak{g}^\alpha.$$

Further our special construction of $\mathfrak{k}^c$ gives $\mathfrak{k}^c = \mathfrak{h}^0 + [\mathfrak{p}, \mathfrak{p}] + i\mathfrak{p}$, and therefore $\mathfrak{h} \cap \mathfrak{k}^c = \mathfrak{h}^0 + [\mathfrak{p}, \mathfrak{p}]$ (cf. [KN96, Prop. V.9(iii)]). From that we obtain

$$\mathfrak{h} \cap \mathfrak{k}^c = \mathfrak{h}^0 + [\mathfrak{p}, \mathfrak{p}] = \mathfrak{h}^0 + \sum_{\alpha \in \Delta_k^+} (\mathbf{1} + \tau).\mathfrak{g}^\alpha.$$

The fact that $\mathfrak{h} \cap \mathfrak{k}^c$ is compactly embedded in $\mathfrak{g}$ implies that κ is negative semidefinite on this subspace. We will show that κ is positive semidefinite on the complementary space

$$\mathfrak{h}_1 := \sum_{\alpha \in \Delta_n^+} (\mathbf{1} + \tau).\mathfrak{g}^\alpha.$$

This implies the lemma because it shows that each $X \in \mathfrak{h}_1$ which is a compact element of $\mathfrak{g}$ must be central, hence contained in $\mathfrak{h}^0$, and therefore $X = 0$.

From $\tau(\mathfrak{g}^\alpha) = \mathfrak{g}^{-\alpha}$ and the fact that τ preserves the Killing form, it follows that $\kappa(\mathfrak{g}^\alpha, \mathfrak{g}^\beta)$ vanishes if $\alpha + \beta \neq 0$. Hence it suffices to show that

$$\kappa\big((\mathbf{1} + \tau).X, (\mathbf{1} + \tau).X\big) \geq 0$$

for $X \in \mathfrak{g}^\alpha$, $\alpha \in \Delta_n^+$. Using the fact that the spaces $\mathfrak{g}^{\pm\alpha}$ are isotropic for κ, this expression can be evaluated to

$$\kappa\big((\mathbf{1} + \tau).X, (\mathbf{1} + \tau).X\big) = 2\kappa(\tau.X, X)$$

which, in view of [KN96, Prop. IV.7], is non-negative. This proves the lemma. ∎

Lemma II.8. *For $g \in G^c$ and $z \in \mathcal{D}$ we have $\sigma(g.z) = \tau(g).\sigma(z)$. In particular σ induces an antiholomorphic involution of the domain $\mathcal{D} \cong G^c/K^c$.*

Proof. Since σ preserves all factors in the $P^+K^c_{1,\mathbb{C}}P^-$-decomposition, it follows that

$$\sigma(g.z) = \log\zeta\Big(\sigma(\eta_G(g))\exp\sigma(z)\Big) = \log\zeta\Big(\eta_G(\tau.g)\exp\sigma(z)\Big) = \tau(g).\sigma(z). \quad ∎$$

Theorem II.9. $\mathcal{D}_{\mathbb{R}} = H.0$.

Proof. From Lemma II.8 we conclude in particular that

$$H.0 \subseteq \mathcal{D}_{\mathbb{R}} = \{z \in \mathcal{D} \colon \sigma(z) = z\}.$$

To see that we even have equality, let $z \in \mathcal{D}_{\mathbb{R}}$. Then $z \in \mathcal{D} \cong G^c/K^c$ shows that the isotropy group $G^c_z \subseteq G^c$ is conjugate to K^c, hence that $\mathfrak{g}^c_z$ is compactly embedded. Moreover, the fact that $\sigma(z) = z$ and Lemma II.8 imply that $\mathfrak{g}^c_z$ is τ-invariant. Therefore $\mathfrak{h}_z = \mathfrak{g}^c_z \cap \mathfrak{h}$ is compactly embedded in $\mathfrak{g}^c$. In view of Lemma II.7 and [Ne98, Th. V.1.4], there exists $\gamma \in \mathrm{Inn}_{\mathfrak{g}^c}(\mathfrak{h})$ with $\gamma.(\mathfrak{g}^c \cap \mathfrak{h}) \subseteq \mathfrak{k}^c \cap \mathfrak{h} = \mathfrak{h}_0$. We conclude in particular that $\dim \mathfrak{h}_z \leq \dim \mathfrak{h}_0$. Hence $\dim(H.z) \geq \dim(H.0) = \dim(\mathfrak{p}^+ \cap \mathfrak{g})$ implies that $H.z$ is open in $\mathcal{D}_{\mathbb{R}}$. On the other hand the convexity of $\mathcal{D}$ ([Ne98, Lemma VIII.1.10]) implies that $\mathcal{D}_{\mathbb{R}}$ is convex, hence that $\mathcal{D}_{\mathbb{R}}$ is connected. Therefore the partition of $\mathcal{D}_{\mathbb{R}}$ into H-orbits must be trivial, i.e., H acts transitively on $\mathcal{D}_{\mathbb{R}}$. ∎

Let

$$J \colon G^c \times \mathcal{D} \to K^c_{1,\mathbb{C}} \quad (g, z) \mapsto \kappa\big(\eta_{G^c}(g)\exp(z)\big)$$

and recall that J is a $(G^c, \mathcal{D}, K^c_{1,\mathbb{C}})$-cocycle (cf. [Ne98, Lemma VIII.1.7]). Since $G^c \times \mathcal{D}$ is simply connected, we obtain a lift

$$\widetilde{J} \colon G^c \times \mathcal{D} \to K^c_{\mathbb{C}}$$

which is uniquely determined by $\eta_K \circ \widetilde{J} = J$ and $\widetilde{J}(\mathbf{1}, 0) = \mathbf{1}$. Further the uniqueness of lifts implies that $\widetilde{J}$ is a $(G^c, \mathcal{D}, K^c_{\mathbb{C}})$-cocycle (cf. [Ne98, Lemma VIII.1.7]).

The direct product group $H_{\mathbb{C}} \times (P^- \rtimes K^c_{\mathbb{C}})$ acts naturally on $G_{\mathbb{C}}$ by

$$(h, p, k).x := \eta_{H_{\mathbb{C}}}(h)x\eta_K(k)^{-1}p^{-1}$$

and the open orbit of $\mathbf{1}$ coincides with the set $H_{1,\mathbb{C}}K^c_{1,\mathbb{C}}P^-$. According to Proposition II.6(ii), the stabilizer of $\mathbf{1}$ is given by the complex group

$$K^{\mathbf{1}} = \{(h, \mathbf{1}, k) \colon \eta_{H_{\mathbb{C}}}(h) = \eta_K(k)\},$$

whose Lie algebra is isomorphic to $(\mathfrak{h} \cap \mathfrak{k}^c)_{\mathbb{C}}$. Thus

$$H_{1,\mathbb{C}}K^c_{1,\mathbb{C}}P^- \cong \big(H_{\mathbb{C}} \times (P^- \rtimes K^c_{\mathbb{C}})\big)/K^{\mathbf{1}} \cong \big((H_{\mathbb{C}} \times K^c_{\mathbb{C}})/K^{\mathbf{1}}\big) \times P^-.$$

Since $K^{\mathbf{1}}$ is a complex subgroup of $H_{\mathbb{C}} \times K^c_{\mathbb{C}}$, we obtain complex homogeneous spaces $L := (H_{\mathbb{C}} \times K^c_{\mathbb{C}})/K^{\mathbf{1}}$, $\widetilde{L} := (H_{\mathbb{C}} \times K^c_{\mathbb{C}})/K^{\mathbf{1}}_0$, and $H_{1,\mathbb{C}}K^c_{1,\mathbb{C}}P^- \cong L \times P^-$ (cf. Proposition II.6(i)), where $\widetilde{L} \to L, xK^{\mathbf{1}}_0 \mapsto xK^{\mathbf{1}}$ is a covering of complex homogeneous spaces. We have a natural map

$$q \colon \widetilde{L} \to H_{1,\mathbb{C}}K^c_{1,\mathbb{C}} \subseteq G_{\mathbb{C}}, \quad (h, k)K^{\mathbf{1}}_0 \mapsto \eta_{H_{\mathbb{C}}}(h)\eta_K(k)^{-1}.$$

According to Proposition II.5.(v), $G^c_1 \subseteq H_{1,\mathbb{C}}K^c_{1,\mathbb{C}}P^-$, and therefore

$$\exp(\mathcal{D}) \subseteq G^c_1 K^c_{1,\mathbb{C}}P^- \subseteq H_{1,\mathbb{C}}K^c_{1,\mathbb{C}}P^- \cong L \times P^-.$$

If we assign to $z \in \mathcal{D}$ with $\exp(z) = \eta_{H_{\mathbb{C}}}(h)\eta_K(k)^{-1}p$ and $h \in H_{\mathbb{C}}$, $k \in K^c_{\mathbb{C}}$ and $p \in P^-$ the element $(h, k)K^{\mathbf{1}} \in L$, we thus obtain a holomorphic map $\gamma \colon \mathcal{D} \to L$. Since $\mathcal{D}$ is convex and therefore simply connected, this map lifts to a unique holomorphic map $\widetilde{\gamma} \colon \mathcal{D} \to \widetilde{L}$ with $\widetilde{\gamma}(0) = [\mathbf{1}, \mathbf{1}]$.

For the following lemma we recall the homomorphism $c_H \colon H \to H_{\mathbb{C}}$.

Lemma II.10. *For $h \in H$ and $z \in \mathcal{D}$ we have $\widetilde{\gamma}(h.z) = \big(c_H(h), \widetilde{J}(h, z)\big).\widetilde{\gamma}(z)$.*

Proof. Write $\gamma(z) = (h_1, k_1)K^{\mathbf{1}}$ with $h_1 \in H_{\mathbb{C}}$ and $k_1 \in K^c_{\mathbb{C}}$. We further put $h_0 := \eta_H(h) = \eta_{H_{\mathbb{C}}}\big(c_H(h)\big)$. Then $\exp z \in \eta_{H_{\mathbb{C}}}(h_1)\eta_K(k_1)^{-1}P^-$ shows that

$$\begin{aligned}\exp(h.z)J(h_0, z)P^- &= \exp(h.z)\kappa(h_0 \exp z)P^- = \exp(h_0.z)\kappa(h_0 \exp z)P^- \\ &= \zeta(h_0 \exp z)\kappa(h_0 \exp z)P^- = h_0 \exp zP^- \\ &= \eta_{H_{\mathbb{C}}}\big(c_H(h)h_1\big)\eta_K(k_1)^{-1}P^-\end{aligned}$$

and hence

$$\begin{aligned}\exp(h.z)P^- &= \eta_{H_{\mathbb{C}}}\big(c_H(h)h_1\big)\eta_K(k_1)^{-1}J(h_0, z)^{-1}P^- \\ &= \eta_{H_{\mathbb{C}}}\big(c_H(h)h_1\big)\eta_K\big(k_1^{-1}\tilde{J}(h, z)^{-1}\big)P^-.\end{aligned}$$

Thus

$$\begin{aligned}\gamma(h.z) &= \big(c_H(h)h_1, \tilde{J}(h, z)k_1\big)K^{\mathbf{1}} = \big(c_H(h), \tilde{J}(h, z)\big)(h_1, k_1)K^{\mathbf{1}} \\ &= \big(c_H(h), \tilde{J}(h, z)\big).\gamma(z).\end{aligned}$$

Since the lemma holds trivially for $h = \mathbf{1}$, the connectedness of H and the uniqueness of liftings eventually proves the assertion of the lemma. ∎

Let (ρ, V) be a finite dimensional unitary representation of the group K^c which we extend to a holomorphic representation $\rho\colon K^c_{\mathbb{C}} \to \mathrm{GL}(V)$. We consider the action of the simply connected group G^c on $\mathrm{Hol}(\mathcal{D}, V)$ given by

$$(g.f)(z) = J_\rho(g^{-1}, z)^{-1}.f(g^{-1}.z), \qquad \text{where} \qquad J_\rho := \rho \circ \widetilde{J}. \tag{2.1}$$

We are interested in a description of the space of H-eigenfunctions with respect to this action. So we consider a continuous homomorphism $\chi\colon H \to \mathbb{C}^*$ and its holomorphic "extension" $\chi_{\mathbb{C}}\colon H_{\mathbb{C}} \to \mathbb{C}^*$ which is uniquely determined by $\chi_{\mathbb{C}} \circ c_H = \chi$. In the following theorem we will obtain a description of the eigenspace

$$\mathrm{Hol}(\mathcal{D}, V)^{H,\chi} := \{f \in \mathrm{Hol}(\mathcal{D}, V) \colon (\forall h \in H) h.f = \chi(h)f\}$$

corresponding to the character χ of H. We put $H_K := \langle \exp_H \mathfrak{h} \cap \mathfrak{k}^c\rangle$.

Theorem II.11. *The evaluation map*

$$\Phi\colon \mathrm{Hol}(\mathcal{D}, V)^{H,\chi} \to V^{H_K, \chi|_{H_K}}, \quad f \mapsto f(0)$$

is a bijection.

Proof. We consider the evaluation map

$$\Phi\colon \mathrm{Hol}(\mathcal{D}, V)^{H,\chi} \to V, \quad f \mapsto f(0).$$

The semi-invariance of f under H with respect to the character χ means that

$$J_\rho(h^{-1}, z)^{-1}.f(h^{-1}.z) = \chi(h)f(z)$$

for all $z \in \mathcal{D}$ and $h \in H$ which can also be written as

$$f(h.z) = \chi(h)^{-1}J_\rho(h, z).f(z). \tag{2.2}$$

For $h \in H_K$ this implies in particular that

$$f(h.z) = \chi(h)^{-1}\rho(h).f(z), \tag{2.3}$$

i.e., that $f\colon \mathcal{D} \to V$ is an equivariant map with respect to the action of H_K on V given by $h * v := \chi(h)^{-1}\rho(h).v$. Thus the fact that H_K fixes the origin implies that $\rho(h).f(0) = \chi(h)f(0)$ for all $h \in H_K$, hence that $\mathrm{im}\,\Phi \subseteq V_\chi := V^{H_K,\chi|_{H_K}}$.

We claim that Φ is injective. If $\Phi(f) = f(0) = 0$, then (2.2) implies that f vanishes on $H.0 = \mathcal{D} \cap \mathfrak{g}$ (cf. Theorem II.9) which, in view of $\mathfrak{p}^+ \cong (\mathfrak{p}^+ \cap \mathfrak{g})_{\mathbb{C}}$, is the intersection of $\mathcal{D}$ with a real form of $\mathfrak{p}^+$. This shows that f vanishes, hence that Φ is injective.

We define a holomorphic representation (ρ_e, V) of the direct product group $H_{\mathbb{C}} \times K_{\mathbb{C}}^c$ on V by $\rho_e(h,k) := \chi_{\mathbb{C}}(h)^{-1}\rho(k)$. We consider the holomorphic orbit map

$$\delta\colon H_{\mathbb{C}} \times K_{\mathbb{C}}^c \to V, \quad (h,k) \mapsto \chi_{\mathbb{C}}(h)^{-1}\rho(k).v_0$$

of an element $v_0 \in V_\chi$. Then the stabilizer of v_0 is a closed complex subgroup of $H_{\mathbb{C}} \times K_{\mathbb{C}}^c$ whose Lie algebra is given by

$$\mathfrak{d} := \{(X,Y) \in \mathfrak{h}_{\mathbb{C}} \times \mathfrak{k}_{\mathbb{C}}^c \colon d\rho(Y).v_0 = d\chi_{\mathbb{C}}(X)\cdot v_0\}.$$

If $X \in \mathfrak{h}_{\mathbb{C}} \cap \mathfrak{k}_{\mathbb{C}}^c$, then the definition of the space V_χ implies that $d\rho(X).v_0 = d\chi(X)\cdot v_0$, hence that $(X,X) \in \mathfrak{d}$. This shows that $\mathfrak{d}$ contains the Lie algebra of $K^{\mathbf{1}}$, so that $K_0^{\mathbf{1}}$ is contained in the stabilizer of v_0. Hence the orbit map of v_0 factors to a holomorphic $(H_{\mathbb{C}} \times K_{\mathbb{C}}^c)$-equivariant map

$$\widetilde{\delta}\colon \widetilde{L} \to V, \quad (h,k)K_0^{\mathbf{1}} \mapsto \chi_{\mathbb{C}}(h)^{-1}\rho(k).v_0.$$

Now we obtain a holomorphic map

$$f\colon \mathcal{D} \to V, \quad z \mapsto \widetilde{\delta}\big(\widetilde{\gamma}(z)\big) \quad \text{with} \quad f(0) = v_0.$$

It remains to show that $f \in \mathrm{Hol}(D,V)^{H,\chi}$. In view of Lemma II.10, we have

$$\begin{aligned} f(h.z) = \widetilde{\delta}\big(\widetilde{\gamma}(h.z)\big) &= \widetilde{\delta}\Big(\big(c_H(h), \widetilde{J}(h,z)\big).\widetilde{\gamma}(z)\Big) \\ &= \chi(h)^{-1}J_\rho(h,z).\widetilde{\delta}\big(\widetilde{\gamma}(z)\big) = \chi(h)^{-1}J_\rho(h,z).f(z). \end{aligned}$$

This completes the proof. ∎

III. Ol'shanskiĭ semigroups

In this section we describe certain semigroups which are naturally associated to quasihermitian symmetric Lie algebras.

To understand the construction of these semigroups, we first have to make some remarks on symmetric spaces locally isomorphic to a given one.

Remark III.1. (a) Let $M = G/H$ be a symmetric space, where (G,τ) is a symmetric Lie group and $H \subseteq G^\tau$ an open subgroup. Further let $q\colon \widetilde{G} \to G$ denote the universal covering homomorphism and note that there exists a unique involution $\widetilde{\tau}$ on $\widetilde{G}$ with $q \circ \widetilde{\tau} = \tau \circ q$.

Let $\widetilde{H} := q^{-1}(H) \subseteq \widetilde{G}$. The action of $\widetilde{G}$ on M is transitive and therefore $M \cong \widetilde{G}/\widetilde{H}$. For $g \in \widetilde{G}$ we put $g^\sharp := \widetilde{\tau}(g)^{-1}$ and similarly $g^\sharp = \tau(g)^{-1}$ for $g \in G$. Let $\Gamma := \{g \in \widetilde{G}\colon gg^\sharp \in \ker q\}$. Then $G^\tau = \{g \in G\colon gg^\sharp = \mathbf{1}\}$ and $q(g^\sharp) = q(g)^\sharp$ implies that $\Gamma = q^{-1}(G^\tau)$ is a subgroup of $\widetilde{G}$ containing $\widetilde{H}$, but in general $\widetilde{G}^\tau = \Gamma_0 \neq \Gamma$ (cf. [Lo69, Th. 3.4]). So the only symmetric space of $\widetilde{G}$ associated to $\widetilde{\tau}$ is the simply connected covering space $\widetilde{M} := \widetilde{G}/\widetilde{G}^\tau$ of M.

(b) Suppose, conversely, that (G,τ) is a simply connected symmetric Lie group. We want to determine all those symmetric spaces which are locally isomorphic to G/G^τ, i.e., which are symmetric spaces of groups locally isomorphic to G.

Put $\Gamma := \{g \in \widetilde{G}\colon gg^\sharp \in Z(G)\}$. Then one easily checks that Γ is a closed subgroup of G and that $\gamma\colon \Gamma \to Z(G), g \mapsto gg^\sharp$ is a group homomorphism whose kernel coincides with $G^\tau = \Gamma_0$. In fact, we have

$$\gamma(ab) = abb^\sharp a^\sharp = aa^\sharp bb^\sharp = \gamma(a)\gamma(b)$$

for $a, b \in \Gamma$. If $D \subseteq G$ is a discrete central subgroup which is τ-invariant, then G/D is a symmetric Lie group and for any open subgroup $H \subseteq (G/D)^\tau$ we have seen in (a) that $\widetilde{H} := q^{-1}(H) \subseteq \gamma^{-1}(D) \subseteq \Gamma$.

If, conversely, $\widetilde{H} \subseteq \Gamma$ is a τ-invariant subgroup for which $D := \gamma(\widetilde{H}) \subseteq \widetilde{H}$ is discrete in $Z(G)$, then G/D is a symmetric Lie group, $H := \widetilde{H}/D \subseteq (G/D)^\tau$ is an open subgroup, and $(G/D)/H \cong G/\widetilde{H}$ is a symmetric space of G/D locally isomorphic to the simply connected symmetric space G/H of G. ∎

Definition III.2. An *involutive semigroup* $(S,^*)$ is a semigroup S together with an involutive antiautomorphism $^*\colon S \to S$, i.e., $(s^*)^* = s$ and $(st)^* = t^*s^*$ holds for $s, t \in S$. If, in addition, there is a second involutive antiautomorphism $^\sharp\colon S \to S$ commuting with *, then we call $(S,^*,^\sharp)$ *bi-involutive.* ∎

Definition III.3. (a) Let $\mathfrak{g}$ be a real Lie algebra. A subset $W \subseteq \mathfrak{g}$ is called *weakly hyperbolic* if $\operatorname{Spec}(\operatorname{ad} X) \subseteq \mathbb{R}$ holds for all $X \in W$. Note that closures of weakly hyperbolic subsets are also weakly hyperbolic.

(b) Let $(\mathfrak{g}, \tau)$ be a symmetric Lie algebra and $W \subseteq \mathfrak{q}$ be a weakly hyperbolic $\operatorname{Inn}(\mathfrak{h})$-invariant closed convex cone. Further let H, resp. G, be simply connected Lie groups with Lie algebra $\mathfrak{h}$, resp. $\mathfrak{g}$. We write τ for the involution on G obtained by integrating τ. Let $\eta\colon H \to G$ be the natural homomorphism for which $d\eta(\mathbf{1})\colon \mathfrak{h} \to \mathfrak{g}$ is the canonical embedding and put $H_1 := \eta(H)$. Then the subset $\Gamma_{H_1}(W) := H_1 \exp(W) \subseteq G$ is a closed subsemigroup of G and the *polar map*

$$H_1 \times W \to \Gamma_{H_1}(W), \quad (h, X) \mapsto h \exp X$$

is a homeomorphism. All of that follows from Lawson's Theorem ([La94, Cor. 3.2]). The assumptions of Lawson's Theorem are satisfied because $\exp \mathfrak{z}(\mathfrak{g}) \subseteq Z(G)_0$, and the latter group is a vector group because G is simply connected.

Now the universal covering semigroup $\Gamma_H(W) := \widetilde{\Gamma}_{H_1}(W)$ has a similar structure. We can lift the exponential function $\exp\colon W \to \Gamma_{H_1}(W)$ to an exponential function $\operatorname{Exp}\colon W \to \Gamma_H(W)$ with $\operatorname{Exp}(0) = \mathbf{1}$ and thus obtain a polar map

$$H \times W \to \Gamma_H(W), \quad (h, X) \mapsto h \operatorname{Exp} X$$

which is a homeomorphism.

If $D \subseteq H$ is a discrete central subgroup acting trivially on $\mathfrak{q}$, then $D \subseteq \Gamma_H(W)$ is a discrete central subgroup, and we obtain a covering homomorphism

$$\Gamma_H(W) \to \Gamma_{H/D}(W) := \Gamma_H(W)/D, \quad s \mapsto sD$$

(cf. [HiNe93, Ch. 3]). It is easy to see that $\Gamma_{H/D}(W)$ also has a polar map which is a homeomorphism. Note that for $D = \ker \eta$ we have $H/D \cong H_1$ and $\Gamma_{H_1}(W) \cong \Gamma_{H/D}(W)$.

The semigroups of the type $\Gamma_{H/D}(W)$ are called *real Ol'shanskiĭ semigroups.* If $W^0 \neq \emptyset$ then the subset $\Gamma^0_{H/D}(W) := (H/D) \operatorname{Exp}(W^0)$ is an open subsemigroup which is a manifold with an analytic semigroup multiplication. If W is an open cone, then we also write $\Gamma_{H/D}(W)$ for $H \operatorname{Exp}(W)$ and put $\overline{\Gamma}_{H/D}(W) := \Gamma_{H/D}(\overline{W})$. We note that the involution

$$s = h \operatorname{Exp} X \mapsto s^\sharp := (\operatorname{Exp} X) h^{-1} = h^{-1} \operatorname{Exp}\big(\operatorname{Ad}(h).X\big)$$

turns $\Gamma_{H/D}(W)$ into an involutive semigroup. ∎

From now on we assume that $(\mathfrak{g}, \tau)$ is quasihermitian and that Δ^+ is $\mathfrak{p}$-adapted (cf. Definition I.1(f)).

Definition III.4. Let $W_{\max,s} \subseteq \mathfrak{s}_\mathfrak{q}$ denote the uniquely determined hyperbolic closed convex cone with $W_{\max,s} \cap \mathfrak{a}_\mathfrak{s} = (\Delta^+_{n,s})^\star$ (cf. [KN96, Cor. IX.7]). Assume that G is simply connected and that $H = G^\tau$. Then

$$S_{\max,s} := R \rtimes \Gamma_{S_H}(W_{\max,s}) \subseteq G$$

is a closed subsemigroup of G which is invariant under the involution $g \mapsto g^\sharp := \tau(g)^{-1}$ turning $(S_{\max,s}, \sharp)$ into an involutive semigroup. ∎

Lemma III.5. *With the notation of Definition III.4 we have* $S_{\max,s} \subseteq HAN \cap \overline{N}AH$.

Proof. Since $S_{\max,s}$ is $\sharp$-invariant and $(HAN)^\sharp = \overline{N}AH$, it suffices to show that $S_{\max,s} \subseteq HAN$. As both HAN and $S_{\max,s}$ are R-saturated (Proposition II.4(iv)), left H-invariant and adapted to the product decomposition $G = R \times S_{\rm iso} \times \prod_{j=1}^n S_j$ (cf. Definition I.1(c)), we may assume that G is quasihermitian and simple. If $(\mathfrak{g}, \tau)$ is (NCR), then $HAN = G$ is the Iwasawa decomposition and we are done. If $(\mathfrak{g}, \tau)$ is (NCC), then the assertion follows from [HÓ96, Th. 5.4.7]. ∎

Definition III.6. Let $G_\mathbb{C}$ be the simply connected Lie group with Lie algebra $\mathfrak{g}_\mathbb{C}$ and $\widehat{S}_{\max,s} \subseteq G_\mathbb{C}$ denote the subsemigroup defined in Definition III.4 for the quasihermitian symmetric Lie algebra $(\mathfrak{g}_\mathbb{C}, \widehat{\tau})$ (cf. [KN96, Th. VIII.1]). Then $\widehat{S}_{\max,s} \cong R_\mathbb{C} \rtimes \Gamma_{S_1^c}(\widehat{W}_{\max,s})$. Note that the interior of $\widehat{S}_{\max,s}$ is a complex manifold on which the semigroup multiplication is holomorphic. Moreover it is invariant under the holomorphic involution $g^\sharp := \tau(g)^{-1}$ and under the antiholomorphic involution $g^* := \widehat{\tau}(g)^{-1}$. Thus $(\widehat{S}_{\max,s}, ^*, ^\sharp)$ is a bi-involutive semigroup in the sense of Definition III.2. ∎

Lemma III.7. *With the notation of Definition III.6 we have*

$$\widehat{S}_{\max,s} \subseteq P^+ K^c_{1,\mathbb{C}} P^- \quad \textit{and} \quad \widehat{S}_{\max,s} \subseteq H_{1,\mathbb{C}} K^c_{1,\mathbb{C}} P^+.$$

Proof. To prove these inclusions, we first note that since the sets $\widehat{S}_{\max,s}$, $P^+ K^c_{1,\mathbb{C}} P^-$, and $H_{1,\mathbb{C}} K^c_{1,\mathbb{C}} P^-$ are $R_\mathbb{C}$-invariant (Propositions II.5(iv) and II.6(iv)), we may w.l.o.g. assume that $(\mathfrak{g}, \tau)$ is simple and quasihermitian. If $(\mathfrak{g}, \tau)$ is (NCR), we are done, since $G_\mathbb{C} = (K_1^c)_\mathbb{C}$ in this case. If $(\mathfrak{g}, \tau)$ is (NCC), then the first assertion follows from [HiNe95, Prop. II.7] and the second inclusion follows from [ÓØ88, Th. 2.4]. ∎

IV. The domains $\Xi(C)$

Let $(\mathfrak{g}, \tau)$ be a symmetric Lie algebra and $(\mathfrak{g}_\mathbb{C}, \widehat{\tau})$ its canonical extension. We write $G_\mathbb{C}$ for a simply connected Lie group with Lie algebra $\mathfrak{g}_\mathbb{C}$ and, as before, denote by $H_{1,\mathbb{C}}$, G_1 and G_1^c the fixed point groups corresponding to the involutions τ, $\tau\widehat{\tau}$ and $\widehat{\tau}$. Further we define $H_1 := (G_1^c)^\tau = G_1 \cap G_1^c = (G_\mathbb{C})^\tau \cap (G_\mathbb{C})^{\widehat{\tau}}$, denote by G^c a simply connected Lie group associated to $\mathfrak{g}^c$, and set $H := (G^c)^\tau$. Note that in general H_1 is not connected.

We fix an $\{\mathrm{Inn}_{\mathfrak{g}_{\mathbb{C}}}(\widehat{\mathfrak{h}}), -\tau\}$-invariant weakly hyperbolic convex cone $\widehat{C} \subseteq \widehat{\mathfrak{q}}$ such that $C := \widehat{C} \cap \mathfrak{q}$ is an open cone in $\mathfrak{q}$. Note that we do not assume that $\widehat{C}$ is generating in $\widehat{\mathfrak{q}} = i\mathfrak{g}^c$.

Lemma IV.1. *The cone C is H_1-invariant.*

Proof. Since H_1 commutes with τ, the group H_1 leaves $\mathfrak{q}$ invariant. As $\mathrm{Ad}(G_1^c).\widehat{C} = \widehat{C}$, we have $\mathrm{Ad}(H_1).\widehat{C} = \widehat{C}$. Hence $\mathrm{Ad}(H_1).C = \mathrm{Ad}(H_1).(\mathfrak{q} \cap \widehat{C}) = \mathfrak{q} \cap \widehat{C} = C$. ∎

Definition IV.2. (a) In view of Lemma IV.1, the subsemigroup $\Gamma_{G_0^\tau}(C)$ of G_1 is invariant under multiplication with elements of H_1, and so we obtain an open subsemigroup $\Gamma_{H_1}(C) := H_1\Gamma_{G_0^\tau}(C) = H_1 \exp(C)$. Moreover, Lawson's Theorem ([La94, Th. 3.1]) even shows that the polar map $H_1 \times C \to \Gamma_{H_1}(C)$ is a homeomorphism (cf. Definition III.3, where we have discussed the case of a connected group H_1).

(b) If $G_2^c = G^c/D$ is a connected Lie group locally isomorphic to G^c, then $D \subseteq Z(G^c)$, so that D acts trivially on $\mathfrak{g}^c$ and hence on $\widehat{\mathfrak{q}}$. Thus we obtain the *complex Ol'shanskiĭ semigroup* $\Gamma_{G_1^c}(\widehat{C})$ (cf. Definition III.3(b)). The semigroup $\Gamma_{G^c}(\widehat{C})$ is the universal covering semigroup of $\Gamma_{G_2^c}(\widehat{C})$. If, in addition, $\widehat{C}$ is open in $\widehat{\mathfrak{q}}$, then the semigroup $\Gamma_{G_2^c}(\widehat{C})$ carries a natural complex structure such that the semigroup multiplication is holomorphic (cf. [HiNe93, Th. 9.15]). In addition to the natural involution $s \mapsto s^*$ which is given by $s = g\,\mathrm{Exp}\,X \mapsto (\mathrm{Exp}\,X)g^{-1}$ we have a second involution $s = g\,\mathrm{Exp}\,X \mapsto s^\sharp := \mathrm{Exp}(-\tau.X)g^\sharp$. These involutions yield on $\Gamma_{G_2^c}(\widehat{C})$ the structure of a bi-involutive semigroup. If, in addition, $\widehat{C}$ is open in $\widehat{\mathfrak{q}}$, then $^\sharp$ is holomorphic and * is antiholomorphic w.r.t. the underlying complex structures.

(c) If $H_2 := (G_2^c)_0$, then we have a natural embedding $\Gamma_{H_2}(C) \hookrightarrow \Gamma_{G_2^c}(\widehat{C})$ and it follows from the polar decomposition that $\Gamma_{H_2}(C)$ is a connected component of the fixed point set of the antiholomorphic involution $s \mapsto (s^*)^\sharp$ of $\Gamma_{G^c}(\widehat{C})$. If $\widehat{C}$ is open, then we may consider the real Ol'shanskiĭ semigroup $\Gamma_{H_2}(C)$ as a "real form" of the complex one. Note that both involutions $s \mapsto s^*$ and $s \mapsto s^\sharp$ coincide on $\Gamma_{H_2}(C)$. ∎

We consider the connected component $Q := \{g \in G_{\mathbb{C}} \colon g^\sharp = g\}_0$ of the set of *symmetric elements* of $G_{\mathbb{C}}$ containing the identity and recall from [Lo69, Prop. 4.4] that the *quadratic representation*

$$q\colon G_{\mathbb{C}}/H_{1,\mathbb{C}} \to Q, \quad gH_{1,\mathbb{C}} \mapsto gg^\sharp$$

which is equivariant with respect to the $G_{\mathbb{C}}$ action on $G_{\mathbb{C}}$ given by $g.x := gxg^\sharp$ is a homeomorphism. It follows in particular that Q carries a complex manifold structure inherited from the complex symmetric space $G_{\mathbb{C}}/H_{1,\mathbb{C}}$. We define

$$Q(\widehat{C}) := \{ss^\sharp \colon s \in \Gamma_{G_1^c}(\widehat{C})\} \subseteq Q \cap \Gamma_{G_1^c}(\widehat{C})$$

and note that under the quadratic representation this set is the orbit of the base point in $G_{\mathbb{C}}/H_{1,\mathbb{C}}$ under the action of the complex semigroup $\Gamma_{G^c}(\widehat{C})$.

To analyze the structure of the domains $Q(\widehat{C})$, the following point of view will be quite enlightening. We consider the left action of H_1 on $G_1^c \times C$ given by

$h.(g,X) := (gh^{-1}, \mathrm{Ad}(h).X)$ and write

$$\Xi_1(C) := G_1^c \times_{H_1} C\colon = (G_1^c \times C)/H_1$$

for the quotient space. We write $[g,X] := H_1.(g,X)$. Similarly one defines $\Xi(C) := G^c \times_H C$ and $\overline{\Xi}_1(C) := G_1^c \times_{H_1} \overline{C}$. Then $\Xi_1(C)$ is an open dense subset of $\overline{\Xi}_1(C)$.

Theorem IV.3. *If C is open, then the following assertions hold:*

(i) *The set $Q(\widehat{C})$ coincides with the connected component of the set*

$$\{s \in \Gamma_{G_1^c}(\widehat{C})\colon s^\sharp = s\}$$

containing $\exp(C)$ *and it can also be written as*

$$Q(\widehat{C}) = G_1^c.\exp(C) = \{g\exp(X)g^\sharp\colon g \in G_1^c, X \in C\}.$$

(ii) *The set $Q(\widehat{C})$ is an open submanifold of Q and the mapping*

$$\psi\colon \Xi_1(C) \to Q(\widehat{C}), \quad [g,X] \mapsto g\exp(2X)g^\sharp$$

is a diffeomorphism. In particular $\Xi_1(C)$ carries the structure of a complex manifold.

(iii) *There is a natural left action of $\overline{\Gamma}_{G_1^c}(\widehat{C})$ on $Q(\widehat{C}) \cong \Xi_1(C)$ given by $s.x = sxs^\sharp$. If $\widehat{C} \subseteq \widehat{\mathfrak{q}}$ is open, then the corresponding action map σ_1 is holomorphic on the open semigroup.*

Proof. (i) The chain of inclusions

$$\{g\exp(X)g^\sharp\colon g \in G_1^c, X \in C\} \subseteq Q(\widehat{C}) \subseteq \{s \in \Gamma_{G_1^c}(\widehat{C})\colon s^\sharp = s\}_0 \tag{4.1}$$

is clear. Thus we have to show that the right hand side of (4.1) is contained in the left hand side. Take an arbitary element from the right hand side and write $s = g\exp(X)$, where $g \in G_1^c$, $X \in \widehat{C}$. Then $s^\sharp = s$ entails that

$$g^\sharp \exp(\mathrm{Ad}(g^\sharp)^{-1}.X^\sharp) = \exp(X^\sharp)g^\sharp = s^\sharp = s = g\exp(X),$$

and so $g = g^\sharp$ by the uniqueness of the polar decomposition of $\Gamma_{G_1^c}(\widehat{C})$. Since the projection of the right hand side of (4.1) onto G_1^c obtained by the polar map is connected, we see that $g \in \{g_1 \in G_1^c\colon g_1^\sharp = g_1\}_0$. Therefore [Lo69, Prop. 4.4] implies the existence of an element $g_1 \in G_1^c$ such that $g = g_1 g_1^\sharp$ and thus

$$s = g_1 g_1^\sharp \exp(X) = g_1\exp(\mathrm{Ad}(g_1^\sharp).X)g_1^\sharp = g_1\exp(Y)g_1^\sharp, \tag{4.2}$$

where $Y := \mathrm{Ad}(g_1^\sharp).X$. So $s^\sharp = s$ together with (4.2) yields

$$\exp(Y) = g_1^{-1}s(g_1^{-1})^\sharp = g_1^{-1}s^\sharp(g_1^{-1})^\sharp = (g_1^{-1}s(g_1^{-1})^\sharp)^\sharp = \exp(Y)^\sharp = \exp(-\tau(Y)),$$

so $Y = -\tau(Y)$ and thus $Y \in \widehat{C} \cap \mathfrak{q} = C$. Now the assertion follows from (4.2).

(ii) First we show that ψ is bijective. According to (i), it suffices to check injectivity. Assume that $\psi(g_1, X_1) = \psi(g_2, X_2)$ for $g_1, g_2 \in G_1^c$ and $X_1, X_2 \in C$. This means that

$$g_1 g_1^\sharp \exp(\mathrm{Ad}(g_1^\sharp)^{-1}.2X_1) = g_2 g_2^\sharp \exp(\mathrm{Ad}(g_2^\sharp)^{-1}.2X_2).$$

Hence the uniqueness of the polar decomposition entails $g_1 g_1^\sharp = g_2 g_2^\sharp$, i.e., $h := g_1^{-1}g_2 \in H_1 = (G_1^c)^\tau$. Thus $X_2 = \mathrm{Ad}(g_2^\sharp g_1^{-\sharp}).X_1 = \mathrm{Ad}(h^\sharp).X_1 = \mathrm{Ad}(h^{-1}).X_1$ and so $[g_1, X_1] = [g_2, X_2]$.

Next we show that $d\psi$ is everywhere regular. We realize $Q(\widehat{C})$ inside of $G_{\mathbb{C}}/H_{1,\mathbb{C}}$ via the inverse of the quadratic representation. Then the mapping ψ is given by

$\psi([g,X]) = g\exp(X)H_{1,\mathbb{C}}$. Denote by m the map $G_1^c \times C \to G_{\mathbb{C}}, (g,X) \mapsto g\exp(X)$ and by p and π the projections $p\colon G_1^c \times C \to \Xi_1(C)$ and $\pi\colon G_{\mathbb{C}} \to G_{\mathbb{C}}/H_{1,\mathbb{C}}$. We obtain a commutative diagram

$$\begin{array}{ccc} G_1^c \times C & \xrightarrow{m} & G_{\mathbb{C}} \\ \big\downarrow p & & \big\downarrow \pi \\ \Xi_1(C) & \xrightarrow{\psi} & G_{\mathbb{C}}/H_{1,\mathbb{C}} \end{array}$$

Since $\psi \circ p = \pi \circ m$ and all maps involved are G_1^c-equivariant, we only have to check that $d(\pi \circ m)(\mathbf{1}, X)$ is onto for all $X \in C$. Denote by λ_g, resp. μ_g, the left translation by $g \in G_{\mathbb{C}}$ in $G_{\mathbb{C}}$, resp. $G_{\mathbb{C}}/H_{1,\mathbb{C}}$. Then the differential is given by

$$\begin{aligned} d(\pi\circ m)(\mathbf{1},X)(Y,Z) &= \frac{d}{dt}\Big|_{t=0}\pi\big(\exp(tY)\exp(X+tZ)\big) \\ &= d\pi(\exp(X)).d\lambda_{\exp(X)}(\mathbf{1}).\Big(e^{-\operatorname{ad}(X)}.Y + \frac{\mathbf{1}-e^{-\operatorname{ad}(X)}}{\operatorname{ad}(X)}.Z\Big) \\ &= d\mu_{\exp(X)}(\pi(\mathbf{1})).d\pi(\mathbf{1}).\Big(e^{-\operatorname{ad}(X)}.Y + \frac{\mathbf{1}-e^{-\operatorname{ad}(X)}}{\operatorname{ad}(X)}.Z\Big). \end{aligned}$$

We claim that

$$p_{\mathfrak{q}_{\mathbb{C}}}\Big(e^{-\operatorname{ad}(X)}.\mathfrak{g}^c + \frac{\mathbf{1}-e^{-\operatorname{ad}(X)}}{\operatorname{ad}(X)}.\mathfrak{q}\Big) = \mathfrak{q}_{\mathbb{C}}, \tag{4.3}$$

where $p_{\mathfrak{q}_{\mathbb{C}}}\colon \mathfrak{g}_{\mathbb{C}} \to \mathfrak{q}_{\mathbb{C}}$ denotes the projection of $\mathfrak{g}_{\mathbb{C}}$ onto $\mathfrak{q}_{\mathbb{C}}$ along $\mathfrak{h}_{\mathbb{C}}$. We establish (4.3) in two steps.

Step 1. $p_{\mathfrak{q}_{\mathbb{C}}}\Big(e^{-\operatorname{ad}(X)}.i\mathfrak{q}\Big) \supseteq i\mathfrak{q}$. Since $\operatorname{Spec}(\operatorname{ad}(X)) \subseteq \mathbb{R}$, the Spectral Mapping Theorem implies that $\cosh(\operatorname{ad}(X))\colon \mathfrak{q} \to \mathfrak{q}$ is an isomorphism. Thus Step 1 follows from

$$p_{\mathfrak{q}_{\mathbb{C}}}(e^{-\operatorname{ad}(X)}.i\mathfrak{q}) = \cosh(\operatorname{ad}(X)).i\mathfrak{q} = i\mathfrak{q}.$$

Step 2. $p_{\mathfrak{q}_{\mathbb{C}}}\Big(\frac{\mathbf{1}-e^{-\operatorname{ad}(X)}}{\operatorname{ad}(X)}.\mathfrak{q}\Big) \supseteq \mathfrak{q}$. Again by the Spectral Mapping Theorem, the map $\frac{\sinh(\operatorname{ad}(X))}{\operatorname{ad}(X)}\colon \mathfrak{q} \to \mathfrak{q}$ is an isomorphism. So Step 2 follows from

$$p_{\mathfrak{q}_{\mathbb{C}}}\Big(\frac{\mathbf{1}-e^{-\operatorname{ad}(X)}}{\operatorname{ad}(X)}.\mathfrak{q}\Big) = \frac{\sinh(\operatorname{ad}(X))}{\operatorname{ad} X}.\mathfrak{q} = \mathfrak{q}.$$

Step 1 and 2 imply (4.3), i.e., ψ is a submersion. Finally, $\dim \Xi_1(C) = \dim G_{\mathbb{C}}/H_{1,\mathbb{C}}$ implies that $d\psi$ is everywhere regular.

(iii) Using ψ, we realize $\Xi_1(C)$ as $Q(\widehat{C}) \subseteq \Gamma_{G_1^c}(\widehat{C})$ and by the quadratic representation as the orbit of the base point in $G_{\mathbb{C}}/H_{1,\mathbb{C}}$ under the action of $\Gamma_{G_1^c}(\widehat{C})$. Then it is clear how $\overline{\Gamma}_{G_1^c}(\widehat{C})$ acts on this space and that this action is holomorphic whenever $\widehat{C}$ is open in $\widehat{\mathfrak{q}}$. ∎

Since, according to Theorem IV.3, the manifold $Q(\widehat{C})$ only depends on $C = \widehat{C} \cap \mathfrak{q}$, we also write $Q(C)$ instead of $Q(\widehat{C})$.

Proposition IV.4. *Let* (G_2^c, τ) *be a connected symmetric Lie group with Lie algebra* $(\mathfrak{g}^c, \tau)$ *and* $H_2 \subseteq (G_2^c)^\tau$ *an open subgroup such that* H_2 *leaves the open cone* C *invariant. Put* $\Xi_2(C) := G_2^c \times_{H_2} C$.

(i) *The inclusion mappings* $G_2^c/H_2 \to \overline{\Xi}_2(C)$ *and* $\Xi_2(C) \to \overline{\Xi}_2(C)$ *induce isomorphisms of homotopy groups*

$$\pi_1(\Xi_2(C)) \cong \pi_1(\overline{\Xi}_2(C)) \cong \pi_1(G_2^c/H_2).$$

(ii) *If* $H_2 = (G_2^c)^\tau$, *then the mapping*

$$\Xi_2(C) \to Q_2(C) := G_2^c.\operatorname{Exp}(C) \subseteq \Gamma_{G_2^c}(\widehat{C}), \quad [g, X] \mapsto g.\operatorname{Exp} X$$

is a G_2^c*-equivariant diffeomorphism and thus* $\Xi_2(C)$ *inherits the structure of a complex manifold. Furthermore* $Q_2(C)$ *is invariant under the action of* $\overline{\Gamma}_{G_2^c}(\widehat{C})$ *on* $Q_2(C)$ *given by* $s.x = sxs^\sharp$ *which therefore gives rise to an action of* $\Gamma_{G_2^c}(\widehat{C})$ *on* $\Xi_2(C)$ *by holomorphic mappings. If* $\widehat{C}$ *is open, then this action is holomorphic. Moreover it extends to a continuous action on the closures.*

(iii) *If* $H_2 \subseteq (G_2^c)^\tau$ *is an open subgroup, then the domain* $\Xi_2(C) = G_2^c \times_{H_2} C$ *inherits a complex manifold structure via the natural covering map* $\Xi_2(C) \to \Xi_2'(C) := G_2^c \times_{(G_2^c)^\tau} C$. *In this case the action of the semigroup* $\Gamma_{G_2^c}(\widehat{C})$ *on* $\Xi_2'(C)$ *lifts to an action on* $\Xi_2(C)$ *with similar properties.*

Proof. (i) The homogeneous space G_2^c/H_2 is a deformation retract of $\overline{\Xi}_2(C)$ via the homotopy

$$f\colon [0,1] \times \overline{\Xi}_2(C) \to \overline{\Xi}_2(C), \quad f(t, [g, X]) := [g, (1-t)X].$$

Hence the inclusion $G_2^c/H_2 \to \overline{\Xi}(C)$ induces an isomorphism $\pi_1(\overline{\Xi}_2(C)) \cong \pi_1(G_2^c/H_2)$.

It remains to prove the corresponding statement for the inclusion $\Xi_2(C) \to \overline{\Xi}_2(C)$. Let $x_0 \in \Xi_2(C)$ and denote by $\Omega(\Xi_2(C), x_0)$, resp. $\Omega(\overline{\Xi}_2(C), x_0)$, the homotopy classes of paths $\gamma\colon [0,1] \to \Xi_2(C)$, resp. $\overline{\gamma}\colon [0,1] \to \overline{\Xi}_2(C)$, with $\gamma(0) = \overline{\gamma}(0) = \gamma(1) = \overline{\gamma}(1) = x_0$. Identify $\pi_1(\Xi_2(C))$, resp. $\pi_1(\overline{\Xi}_2(C))$, with $\Omega(\Xi_2(C), x_0)$, resp. $\Omega(\overline{\Xi}_2(C), x_0)$. Denote by $[\gamma]$, resp. $\overline{[\gamma]}$ the homotopy class of γ, resp. $[\overline{\gamma}]$, and let $X \in C$. We claim that the mapping

$$\Omega(\Xi_2(C), x_0) \to \Omega(\overline{\Xi}_2(C), x_0), \quad [\gamma] \mapsto \overline{[\gamma]}$$

is an isomorphism. To establish surjectivity, we have to show that each path $\overline{\gamma}$ in $\overline{\Xi}_2(C)$ is homotopic to a path in $\Xi_2(C)$. The map $\overline{H}(s,t) := \operatorname{Exp}(s(1-t)X).\overline{\gamma}(t)$ provides a homotopy transforming $\overline{\gamma}$ into a path in $\Xi_2(C)$.

Finally injectivity will follow if we can show that $\overline{[\gamma]} = \overline{[\gamma_{x_0}]}$ implies $[\gamma] = [\gamma_{x_0}]$, where γ_{x_0} is the constant path in x_0. Let $\overline{H}$ be a homotopy in $\overline{\Xi}_2(C)$ transforming γ into γ_{x_0}. Then $H(s,t) := \operatorname{Exp}(s(1-t)X).\overline{H}(t,s)$ is a homotopy in $\Xi_2(C)$ transforming γ into γ_{x_0}.

(ii) Let $q\colon \Gamma_{G^c}(\widehat{C}) \to \Gamma_{G_1^c}(\widehat{C}) \subseteq G_{\mathbb{C}}$ denote the universal covering homomorphism and put $\widetilde{Q}(C) := G^c.\operatorname{Exp}(C) \subseteq \Gamma_{G^c}(\widehat{C})$. Then $q\big(\widetilde{Q}(C)\big) = G_1^c.\operatorname{Exp}(C) = Q(C)$ is a submanifold of $G_{\mathbb{C}}$. If $D := \ker q \cong \pi_1(G_1^c)$ is the kernel of q, then this implies that $q^{-1}\big(Q(C)\big) = D\widetilde{Q}(C)$. If $d_1 g_1.\operatorname{Exp} X_1 = d_1 g_1 \operatorname{Exp} X_1 g_1^\sharp = d_2 g_2 \operatorname{Exp} X_2 g_2^\sharp$ with $X_j \in C$, $d_j \in D$, and $g_j \in G^c$, then the uniqueness of the polar decomposition shows that $d_1 g_1 g_1^\sharp = d_2 g_2 g_2^\sharp$, i.e., $d_1^{-1} d_2 = g g^\sharp$ for $g = g_2^{-1} g_1$.

From $gg^\sharp \in D$ we further conclude that $\operatorname{Ad}_{G^c}(g) \in \operatorname{Ad}_{G_1^c}(H_1)$, hence that $\operatorname{Ad}(g)$ commutes with τ and leaves $C \subseteq \mathfrak{q}$ invariant (Lemma IV.1). This means that

$$\begin{aligned} d_2 g_2.\operatorname{Exp} X_2 &= d_1 g g^\sharp g_2 (\operatorname{Exp} X_2) g_2^\sharp = d_1 g_2 g g^\sharp (\operatorname{Exp} X_2) g_2^\sharp \\ &= d_1 g_2 g \big(\operatorname{Exp} \operatorname{Ad}(g^\sharp).X_2\big) g^\sharp g_2^\sharp \in d_1 G^c.\operatorname{Exp}(C). \end{aligned}$$

So $d_1G^c.\operatorname{Exp}(C)$ intersects $d_2G^c.\operatorname{Exp}(C)$ if and only if the two sets coincide. This shows that the sets $d\widetilde{Q}(C)$, $d \in D$, are the connected components of the submanifold $q^{-1}\big(Q(C)\big)$ of $\Gamma_{G^c}(\widehat{C})$ and in particular complex manifolds.

We consider the natural map

$$\widetilde{\psi}\colon \Xi(C) = G^c \times_H C \to \widetilde{Q}(C), \quad [g, X] \mapsto g \operatorname{Exp} 2Xg^\sharp.$$

We note that it is surjective and well defined because $h \in H$ implies that $h^\sharp = h^{-1}$ so that $h^{-1}\big(\operatorname{Exp} 2\operatorname{Ad}(h).X\big)(h^{-1})^\sharp = h^{-1}\big(\operatorname{Exp} 2\operatorname{Ad}(h).X\big)h = \operatorname{Exp} 2X$. Furthermore, (ii) implies that $q \circ \widetilde{\psi}\colon \Xi(C) \to Q(C)$ is a submersion because it factors over the covering map $\Xi(C) \to \Xi_1(C)$. To see that it is injective, we note that the injectivity of ψ shows that $\widetilde{\psi}([g_1, X_1]) = \widetilde{\psi}([g_2, X_2])$ implies that $X_1 = X_2$ and $q(g_1) = q(g_2)$, i.e., $g_2 \in g_1 \ker q$. From that we conclude $\operatorname{Ad}(g_1) = \operatorname{Ad}(g_2)$ and thus

$$\begin{aligned}\widetilde{\psi}([g_1, X_1]) &= g_1(\operatorname{Exp} 2X_1)g_1^\sharp = g_1g_1^\sharp \operatorname{Exp}\big(2\operatorname{Ad}(g_1^\sharp)^{-1}.X_1\big)\\ &= g_1g_1^\sharp \operatorname{Exp}\big(2\operatorname{Ad}(g_2^\sharp)^{-1}.X_2\big)\end{aligned}$$

implies that $g_1g_1^\sharp = g_2g_2^\sharp$. Hence $d := g_1^{-1}g_2$ satisfies $dd^\sharp = \mathbf{1}$, i.e., $d \in H$. So $\widetilde{\psi}$ is injective and hence a diffeomorphism.

Let $D \subseteq G^c$ be a discrete central τ-invariant subgroup with $G_2^c \cong G^c/D$ and $H_2 = (G_2^c)^\tau$. Further let $\Gamma_{G_2^c}(\widehat{C})$ be the corresponding complex Ol'shanskiĭ semigroup and put $Q_2(C) := G_2^c.\operatorname{Exp}(C) \subseteq \Gamma_{G_2^c}(\widehat{C})$. Then we have the universal covering map $q_2\colon \Gamma_{G^c}(\widehat{C}) \to \Gamma_{G_2^c}(\widehat{C})$ satisfying $q_2\big(\widetilde{Q}(C)\big) = G_2^c.\operatorname{Exp} C = Q_2(C)$. Furthermore the following diagram commutes:

$$\begin{array}{ccc}\Xi(C) & \xrightarrow{\widetilde{\psi}} & \widetilde{Q}(C)\\ \big\downarrow & & \big\downarrow{\scriptstyle q_2}\\ \Xi_2(C) & \xrightarrow{\psi_2} & Q_2(C)\end{array}$$

This shows that ψ_2 is a surjective submersion. If $\psi_2([q_2(g_1), X_1]) = \psi_2([q(g_2), X_2])$, then $q_2(g_1.\operatorname{Exp} X_1) = q_2(g_2.\operatorname{Exp} X_2)$ implies that there exists $d \in D$ with $dg_1.\operatorname{Exp} X_1 = g_2.\operatorname{Exp} X_2$. Then the uniqueness of the polar decomposition yields $\operatorname{Ad}(g_1^\sharp)^{-1}.X_1 = \operatorname{Ad}(g_2^\sharp)^{-1}.X_2$ and $dg_1g_1^\sharp = g_2g_2^\sharp$. This means that $g := g_1^{-1}g_2$ satisfies $gg^\sharp \in D$, i.e., $q_2(g) \in (G_2^c)^\tau = H_2$, so that $[q(g_1), X_2] = [q(g_2), X_2]$ holds in $\Xi_2(C)$.

(iii) Let $\Xi_2'(C) := G_2^c \times_{(G_2^c)^\tau} C$ and note that the natural map $\Xi_2(C) \to \Xi_2'(C)$ given by $[g, X] \mapsto [g, X]$ is a covering map whose group of deck transformations is given by $(G_2^c)^\tau/H_2$. Thus $\Xi_2(C)$ carries a unique complex structure for which this covering map is holomorphic.

We write $\widehat{q}_2\colon \Gamma_{G^c}(\widehat{C}) \to \Gamma_{G_2^c}(\widehat{C})$ for the universal covering map. In (ii) we have seen that we have a natural action of $\Gamma_{G^c}(\widehat{C})$ on $\widetilde{Q}(C) \cong \Xi(C)$ by holomorphic mappings. Furthermore we have the covering map $q_2\colon \Xi(C) \to \Xi_2(C)$ whose group of deck transformations is given by $\widetilde{H}_2/H \cong \pi_0(\widetilde{H}_2)$ for $\widetilde{H}_2 := \widehat{q}_2^{-1}(H_2)$. So it suffices to show that the action of $\Gamma_{G^c}(\widehat{C})$ on $\Xi(C)$ factors over $\widehat{q}_2$ to an action of

$\Gamma_{G_2^c}(\widehat{C})$ on $\Xi_2(C)$, i.e., we want the following commutative diagram:

$$\begin{array}{ccc} \overline{\Gamma}_{G^c}(\widehat{C}) \times \overline{\Xi}(C) & \xrightarrow{\sigma} & \overline{\Xi}(C) \\ \Big\downarrow{\scriptstyle \widehat{q}_2 \times q_2} & & \Big\downarrow{\scriptstyle q_2} \\ \overline{\Gamma}_{G_2^c}(\widehat{C}) \times \overline{\Xi}_2(C) & \longrightarrow & \overline{\Xi}_2(C) \end{array}$$

So let $g_0 \in \ker \widehat{q}_2$. Then $g_0 \in \widetilde{H}_2 \cap Z(G^c)$ and for $[g, X] \in \Xi(C)$ we then have $g_0.[g, X] = [g_0 g, X] = [g g_0, X]$, i.e., g_0 acts on $\Xi(C)$ as an element of $\pi_1\big(\Xi_2(C)\big) \cong \widetilde{H}_2/H$. This shows that g_0 induces the identity on $\Xi_2(C)$, hence that the action of $\Gamma_{G^c}(\widehat{C})$ on $\Xi(C)$ factors to an action of $\Gamma_{G_2^c}(\widehat{C})$ on $\Xi_2(C)$. ■

V. Spherical representations of real Ol'shanskiĭ semigroups

Throughout this section (G^c, τ) denotes a simply connected Lie group with Lie algebra $(\mathfrak{g}^c, \tau)$ and $C \subseteq \mathfrak{q}$ an open convex hyperbolic $\mathrm{Inn}_{\mathfrak{g}}(\mathfrak{h})$-invariant convex cone. Note that the existence of such a cone implies that $(\mathfrak{g}, \tau)$ is quasihermitian, that $C_{\min} \subseteq C_{\max}$ holds for a positive $\mathfrak{p}$-adapted system Δ^+ (cf. [KN96, Th. VI.6]), and that $C \subseteq W_{\max}$ (cf. [KN96, Th. X.2]).

In this section we investigate general properties of spherical representations of the group G^c which are related to representations of the semigroup $\Gamma_H(C)$. In particular we will see that the existence of such representations with discrete kernel has consequences for the structure of the group G^c and that such representations can always be realized in certain spaces of holomorphic functions on the domain $\Xi(C)$. The main result describing this realization is Theorem V.8.

Definition V.1. Let $\mathcal{H}$ be a Hilbert space.

(a) A *representation* $(\pi, \mathcal{H})$ of $\Gamma_H(C)$ is a weakly continuous semigroup homomorphism $\pi\colon \Gamma_H(C) \to B(\mathcal{H})$ satisfying $\pi(s^\sharp) = \pi(s)^*$ for all $s \in \Gamma_H(C)$. We always assume that $(\pi, \mathcal{H})$ is *non-degenerate*, i.e., $\mathrm{span}\{\pi(\Gamma_H(C)).\mathcal{H}\}$ is dense in $\mathcal{H}$. If $\widehat{C} \subseteq \widehat{\mathfrak{q}}$ is an open convex hyperbolic $\mathrm{Inn}_{\mathfrak{g}_{\mathbb{C}}}(\widehat{\mathfrak{h}})$-invariant cone, then a *holomorphic representation* $(\widehat{\pi}, \mathcal{H})$ of $\Gamma_{G^c}(\widehat{C})$ is a holomorphic semigroup homomorphism $\widehat{\pi}\colon \Gamma_{G^c}(\widehat{C}) \to B(\mathcal{H})$ satisfying $\pi(s^*) = \pi(s)^*$ for all $s \in \Gamma_{G^c}(\widehat{C})$.

(b) Let $(\pi, \mathcal{H})$ be a representation of $\Gamma_H(C)$. The Lüscher-Mack Theorem (cf. [HiNe96, Th. 3.1]) provides a unique unitary representation $\pi^c\colon G^c \to U(\mathcal{H})$ satisfying $d\pi^c|_{\mathfrak{h}} = d\pi|_{\mathfrak{h}}$ and $d\pi^c(iX) = id\pi(X)$ for all $X \in C$. We say that π^c *extends* to π if π^c is obtained from π via the Lüscher-Mack Theorem. The *analytic vectors* $\mathcal{H}^\omega$ of $(\pi, \mathcal{H})$ are defined as the analytic vectors of $(\pi^c, \mathcal{H})$.

If $\widehat{C}$ is open and $\Gamma_{G^c}(\widehat{C})$ also acts on $\mathcal{H}$, then $\pi(s).\mathcal{H} \subseteq \mathcal{H}^\omega$ holds for all $s \in \Gamma_H(C)$ (cf. Proposition A.5). We endow $\mathcal{H}^\omega$ with the finest locally convex topology, making all the maps $\pi(s)\colon \mathcal{H} \to \mathcal{H}^\omega$, $s \in \Gamma_H(C)$ continuous. If this condition is not satisfied, then one can also define a natural topology on $\mathcal{H}^\omega$ which in general is more complicated to describe (cf. Appendix). We write $\mathcal{H}^{-\omega}$ for the space of continuous antilinear functionals $\mathcal{H}^\omega \to \mathbb{C}$ and endow $\mathcal{H}^{-\omega}$ with the strong dual topology. Then there is a natural chain of continuous inclusions

$$\mathcal{H}^\omega \hookrightarrow \mathcal{H} \hookrightarrow \mathcal{H}^{-\omega}. \tag{5.1}$$

An element of $\mathcal{H}^{-\omega}$ is called a *hyperfunction vector* of $(\pi, \mathcal{H})$. The representation $(\pi, \mathcal{H})$ of $\Gamma_H(C)$ naturally extends to a representation $(\pi^{-\omega}, \mathcal{H}^{-\omega})$ of $\Gamma_H(C)$ by setting $(\pi^{-\omega}(s).\nu)(v) := \nu(\pi(s^\sharp).v)$ for all $s \in \Gamma_H(C)$, $\nu \in \mathcal{H}^{-\omega}$ and $v \in \mathcal{H}$.

(c) Let $(\pi, \mathcal{H})$ be a representation of $\Gamma_H(C)$ and $\chi\colon H \to \mathbb{C}^*$ a character of H. We write $(\mathcal{H}^{-\omega})^{(H,\chi)}$ for the set of all $\nu \in \mathcal{H}^{-\omega}$ satisfying $\pi^{-\omega}(h).\nu = \chi(h).\nu$ for all $h \in H$. A representation $(\pi, \mathcal{H})$ is called (H, χ)-*spherical* if there exists a cyclic vector in $\nu \in (\mathcal{H}^{-\omega})^{(H,\chi)}$. If $\chi = \mathbf{1}$ is the trivial character, then $(\pi, \mathcal{H})$ is called *spherical*. Since the orbit maps $H \to \mathcal{H}^\omega$ are continuous with respect to the topology on $\mathcal{H}^\omega$ (cf. Lemma A.4), each character χ of H for which $(\mathcal{H}^{-\omega})^{(H,\chi)}$ is non-zero is continuous. ■

For the proof of the following lemma we recall the definition of the following cones, resp. vector spaces, associated to a convex set C in a real vector space:

$$\lim(C) := \{v \in V \colon v + C \subseteq C\}$$

and

$$H(C) := \{v \in V \colon v + C = C\} = \lim(C) \cap -\lim(C).$$

We call a symmetric Lie algebra $(\mathfrak{g}, \tau)$ *admissible* if $\mathfrak{q}$ contains an open convex hyperbolic $\mathrm{Inn}_{\mathfrak{g}}(\mathfrak{h})$-invariant subset which does not contain any affine subspace.

Lemma V.2. *Let $(\pi, \mathcal{H})$ be an (H, χ)-spherical representation of $\Gamma_H(C)$. If we define an action of H on the function space $\mathbb{C}^C$ by $(h.f)(X) := \chi^*(h) f\big(\mathrm{Ad}(h)^{-1}.X\big)$ for $X \in C$, $h \in H$, where $\chi^*(h) := \overline{\chi(h^{-1})}$, then the following assertions hold:*

(i) *To each cyclic $\nu \in (\mathcal{H}^{-\omega})^{(H,\chi)}$ corresponds an H-equivariant injective linear map*

$$l_\nu \colon \mathcal{H}^\omega \to \mathbb{C}^C, \quad v \mapsto \big(X \mapsto \overline{\langle \nu, \pi(\mathrm{Exp}\, X).v\rangle}\big).$$

(ii) *Denote by $\mathfrak{h}_{\mathrm{iso}} = \mathfrak{z}_{\mathfrak{h}}(\mathfrak{q})$ the maximal ideal of $\mathfrak{g}$ contained in $\mathfrak{h}$. If, in addition, $(\pi, \mathcal{H})$ has discrete kernel, then*
   (a) *$(\mathfrak{g}, \tau)$ is admissible, and*
   (b) *if, moreover, π is irreducible, then the isotropy algebra $\mathfrak{h}_{\mathrm{iso}}$ is a central, at most one-dimensional direct factor of $\mathfrak{g}$.*

Proof. (i) The linearity of l_ν is obvious and injectivity follows from the cyclicity of ν. Let $X \in C$, $h \in H$ and $v \in \mathcal{H}$. Then the H-equivariance follows from

$$\begin{aligned} l_\nu(\pi(h).v) &= \overline{\nu(\pi(\mathrm{Exp}(X))\pi(h).v)} \\ &= \overline{\nu(\pi(h)\pi(\mathrm{Exp}(\mathrm{Ad}(h)^{-1}.X)).v)} = \overline{(\pi^{-\omega}(h^{-1}).\nu)(\pi(\mathrm{Exp}(\mathrm{Ad}(h)^{-1}.X).v))} \\ &= \chi^*(h)\overline{\nu(\pi(\mathrm{Exp}(\mathrm{Ad}(h)^{-1}.X).v))} = (h.l_\nu(v))(X). \end{aligned}$$

(ii) Let $I_\pi^{\mathfrak{q}} \subseteq \mathfrak{q}^*$ denote the convex moment set of the representation π, i.e., the closed convex hull of the set of all linear functionals of the form $X \mapsto \langle d\pi(X).v, v\rangle$, where $v \in \mathcal{H}^\infty$ is a unit vector. Now the boundedness of the operator $\pi(\mathrm{Exp}\, X)$, $X \in C$, implies that $C \subseteq -B(I_\pi^{\mathfrak{q}}) := \{X \in \mathfrak{q} \colon \sup\langle I_\pi^{\mathfrak{q}}, X\rangle < \infty\}$. If $I_\pi \subseteq \widehat{\mathfrak{q}}^*$ is the corresponding moment set of $\mathfrak{g}^c$ and $p_{\mathfrak{q}} \colon \widehat{\mathfrak{q}} \to \mathfrak{q}$ the projection along $i\mathfrak{h}$, then $I_\pi^{\mathfrak{q}} = \overline{p_{\mathfrak{q}}(I_\pi)}$, and so

$$(I_\pi^{\mathfrak{q}})^\perp \cap \mathfrak{q} = I_\pi^\perp \cap \mathfrak{q} = \ker d\pi \cap \mathfrak{q} = \{0\}$$

([Ne98, Prop. VI.1.8]).

Hence $I_\pi^{\mathfrak{q}} \subseteq \mathfrak{q}^*$ is a generating convex subset. Furthermore the assumption that C has interior points implies that $B(I_\pi^{\mathfrak{q}})$ is generating, i.e., that $I_\pi^{\mathfrak{q}}$ contains no affine lines.

For $X \in C$ let $s(X) := \sup\langle I_\pi^{\mathfrak q}, X\rangle$. Then [Ne98, Prop. III.5.10(ii)] shows that

$$C = \bigcup_{m\in\mathbb{R}} C_m, \qquad \text{where} \qquad C_m = \{X \in C\colon s(X) \leq m\}^0.$$

It is clear that $\lim C_m = \lim \overline{C_m} \subseteq (I_\pi^{\mathfrak q})^\star$, hence that $H(C_m) \subseteq (I_\pi^{\mathfrak q})^\perp = \{0\}$. Thus $\overline{C_m}$ is an $\mathrm{Inn}_{\mathfrak g}(\mathfrak h)$-invariant open convex subset which in addition satisfies $H(\overline{C_m}) = \{0\}$, and so [KN96, Th. VI.6(iii)] implies that the symmetric Lie algebra $(\mathfrak g, \tau)$ has strong cone potential and is admissible (cf. [KN96, Def. VI.4(b)]).

Denote by $H_{\rm iso}$ the analytic subgroup of G^c corresponding to $\mathfrak h_{\rm iso}$. Then (i) implies that $\pi^c|_{H_{\rm iso}} = \chi|_{H_{\rm iso}} \cdot \mathrm{id}_{\mathcal H}$. Since $\ker\pi$ is discrete, it follows that $\mathfrak h_{\rm iso}$ is at most one-dimensional and $\mathfrak h_{\rm iso} \subseteq \mathfrak z(\mathfrak h)$. Thus $\mathfrak h_{\rm iso} \subseteq \mathfrak z(\mathfrak g)$ and it remains to show that $\mathfrak h_{\rm iso}$ is a direct factor. If $\mathfrak g$ is reductive, then this is obvious. Otherwise the nilradical $\mathfrak u$ of $\mathfrak g$ is non-central. From the fact that $(\mathfrak g, \tau)$ has strong cone potential, we get that $\mathfrak z(\mathfrak g) \cap \mathfrak q = \mathfrak z(\mathfrak g) \cap \mathfrak a \neq \{0\}$ (cf. [KN96, Def. V.1]). If, in addition, $(\pi, \mathcal H)$ is irreducible with discrete kernel, then Schur's Lemma implies that $\mathfrak z(\mathfrak g)$ is at most one-dimensional, hence contained in $\mathfrak q$. Therefore $\mathfrak h_{\rm iso} = \{0\}$ proves (b). ■

Remark V.3. In the setting of Lemma V.2(ii) we have seen that $(\mathfrak g, \tau)$ is admissible, so that [KN96, Th. VI.6] implies in particular that $(\mathfrak g, \tau)$ is quasihermitian. Since it is also effective up to a direct central factor in $\mathfrak h$, it follows from [KN96, Th. VI.6, Prop. VII.2] that $\mathfrak h^0$ is compactly embedded in $\mathfrak g$. Now the assumptions of [KN96, Th. X.7] are satisfied, thus $(\mathfrak g_{\mathbb C}, \widehat\tau)$ is quasihermitian and C extends to a convex hyperbolic $\{\mathrm{Inn}_{\mathfrak g_{\mathbb C}}(\widehat{\mathfrak h}), -\tau\}$-invariant hyperbolic open cone $\widehat C$ in $\widehat{\mathfrak q}$ satisfying $\widehat C \cap \mathfrak q = C$.

In particular there exists a maximal hyperbolic Lie triple system $\widehat{\mathfrak p} \subseteq \widehat{\mathfrak q}$ extending $\mathfrak p$, a maximal hyperbolic subspace $\widehat{\mathfrak a} \subseteq \widehat{\mathfrak p}$ extending $\mathfrak a$, and a $\widehat{\mathfrak p}$-adapted positive system $\widehat\Delta_n^+$ such that $\widehat\Delta_n^+|_{\mathfrak a} = \Delta_n^+$ (cf. [KN96, Prop. VII.12, Th. VIII.1]) and $C_{\min} \subseteq \overline{C} \cap \mathfrak q \subseteq C_{\max}$. Furthermore, there exists a unique closed convex hyperbolic invariant cone $\widehat W_{\max} \subseteq \widehat{\mathfrak q}$ which is $(-\tau)$-invariant and satisfies $\widehat W_{\max} \cap \mathfrak q = W_{\max}$. In particular we have $\widehat C \subseteq \widehat W_{\max}$. ■

Proposition V.4. *Let $(\pi, \mathcal H)$ be an irreducible representation of $\Gamma_H(C)$. We assume that either π is (H, χ)-spherical with discrete kernel or that $C \subseteq \widehat W^0_{\max}$. Then the following assertions hold:*

(i) *The representation $(\pi, \mathcal H)$ extends to a holomorphic representation $(\widehat\pi, \mathcal H)$ of $\Gamma_{G^c}(\widehat W^0_{\max})$.*
(ii) *The analytic vectors are given by $\mathcal H^\omega = \widehat\pi(\Gamma_{G^c}(\widehat W^0_{\max})).\mathcal H$.*

Proof. We have seen in Remark V.3 the assumption that π is (H, χ)-spherical with discrete kernel implies that $\mathfrak g^c$ is quasihermitian and that $C \subseteq \widehat W^0_{\max}$ holds for a suitable $\mathfrak k$-adapted positive system $\widehat\Delta^+$.

(i) The inclusion $C \subseteq \widehat W^0_{\max}$ means in particular that iC lies in the interior of the set of elliptic elements of $\mathfrak g^c$. In view of this and Lemma V.2(ii)(b), the assumptions of [HiNe96, Th. 3.6] are satisfied and (i) follows from [HiNe96, Th. 3.6, Th. B] because the discrete kernel assumption made in [HiNe96, Th. B] is only used to ensure that the cone $B(I_\pi)$ is weakly elliptic.

(ii) This follows from Proposition A.5. ■

According to Proposition IV.4, there exists a holomorphic $\Gamma_{G_1^c}(\widehat{W}^0_{\max})$-equivariant submersive map $p_1\colon \Gamma_{G_1^c}(\widehat{W}^0_{\max}) \to \Xi_1(W^0_{\max})$. By the simple connectedness of $\Gamma_{G^c}(\widehat{W}^0_{\max})$ and $\Xi(W^0_{\max})$ this mapping lifts to a map $p\colon \Gamma_{G^c}(\widehat{W}_{\max}) \to \Xi(W_{\max})$ with the same properties. Thus we obtain a commutative diagram:

$$\begin{array}{ccc} \Gamma_{G^c}(\widehat{W}_{\max}) & \xrightarrow{\ p\ } & \Xi(W_{\max}) \\ \big\downarrow \widehat{q} & & \big\downarrow q \\ \Gamma_{G_1^c}(\widehat{W}_{\max}) & \xrightarrow{\ p_1\ } & \Xi_1(W_{\max}) \end{array} \tag{5.2}$$

Note that, since p is submersive, the fibers $p^{-1}([g,X])$ are complex submanifolds of $\Gamma_{G^c}(\widehat{W}^0_{\max})$.

Let M, N be complex manifolds. Denote by $\mathrm{Hol}(N,M)$ the space of all holomorphic maps $f\colon N \to M$. Suppose that a group G acts by holomorphic mappings on N and M. Then G acts on $\mathrm{Hol}(N,M)$ via $(g.f)(n) := g.f(g^{-1}.n)$ for $g \in G$, $n \in N$. We write $\mathrm{Hol}(N,M)^G$ for the set of G-invariant elements in $\mathrm{Hol}(M,N)$, i.e., for the G-equivariant maps $N \to M$.

Lemma V.5. *Let M be a complex manifold, $U \subseteq \Xi(W^0_{\max})$ be an open subset, and let $V \subseteq \Gamma_{G^c}(\widehat{W}^0_{\max})$ be an open set invariant under the right H-action, such that $p(V) = U$. Let H act trivially on U. Realize U inside $Q(W^0_{\max})$* (cf. Proposition IV.4(ii))*. Then the push down mapping*

$$(p|_V)_*\colon \mathrm{Hol}(V,M)^H \to \mathrm{Hol}(U,M)^H, \quad f \mapsto \widetilde{f}\colon g\,\mathrm{Exp}(2X)g^\sharp \mapsto f(g\,\mathrm{Exp}(X))$$

is well defined and bijective. Its inverse is given by the pull back $(p|_V)^*$ of $p|_V : V \to U$. In particular, we have $\mathrm{Hol}(\Gamma_{G^c}(\widehat{W}^0_{\max}), M)^H \cong \mathrm{Hol}(\Xi(W^0_{\max}), M)$.

Proof. Since f is H-fixed, the map $\widetilde{f}$ is well defined. We claim that $\widetilde{f}$ is holomorphic (cf. Theorem IV.3(ii)). As the assertion is purely local, we may assume that $\Gamma_{G^c}(\widehat{W}^0_{\max}) \subseteq G_{\mathbb{C}}$. Fix $g_0\,\mathrm{Exp}(2X_0)g_0^\sharp \in U$ and $g_0\,\mathrm{Exp}(X_0) \in V$. As $p|V\colon V \to U$ is a holomorphic submersion, $p|V$ admits local trivializations. In particular, we find open connected neighborhoods $U_1 \subseteq U$ and $V_1 := (p|_V)^{-1}(U_1) \subseteq V$ of $g_0\,\mathrm{Exp}(2X_0)g_0^\sharp$, resp. $g_0\,\mathrm{Exp}(X_0)$, such that V_1 is biholomorphically equivalent to $U_1 \times W$, where $W \subseteq \mathbb{C}^m$ is an open set, and $p|V_1\colon U_1 \times W \to U_1$ is given by the projection onto the first factor. We define a holomorphic map f_1 on V_1 by $f_1(x,w) := f(x,w_0)$, where $w_0 \in W$ is defined by $g_0\,\mathrm{Exp}(X_0) = (g_0\,\mathrm{Exp}(2X)g_0^\sharp, w_0)$. We claim that $f|_{V_1} = f_1$. Let B be a convex open neighborhood of 0 in $\mathfrak{h}$. Then we find open connected neighborhoods $U_2 \subseteq U_1$, resp. $B_1 \subseteq B$, of $g_0\,\mathrm{Exp}(2X_0)g_0^\sharp$, resp. 0, such that

$$V_2 := (U_1 \times \{w_0\}).\,\mathrm{Exp}_{H_{1,\mathbb{C}}}(B+iB) = U_1 \times \big(w_0.\,\mathrm{Exp}_{H_{1,\mathbb{C}}}(B+iB)\big) \subseteq V_1.$$

Note that V_2 is connected, has non-empty interior and contains $g_0\,\mathrm{Exp}(X_0)$. Let $x \in U_2$ and $h \in \mathrm{Exp}_{H_{1,\mathbb{C}}}(B) \subseteq H$. Then the right H-invariance of f and $(x,w_0).h = (x,w')$ imply that

$$f((x,w_0).h) = f(x,w_0) = f_1(x,w_0) = f_1((x,w_0).h).$$

Thus f_1 and f coincide on the neighborhood $(U_2 \times \{w_0\}).\,\mathrm{Exp}_{H_{1,\mathbb{C}}}(B) \subseteq V_1$ of $g_0\,\mathrm{Exp}(X_0)$. As f and f_1 are holomorphic on V_1, the Identity Theorem for Holomorphic Functions implies that $f|V_1 = f_1$. Thus $f|V_1$ is independent of the second variable.

By construction, we have $g_0 \operatorname{Exp} X_0 \in V_1 \cap G^c \operatorname{Exp}(W^0_{\max})$. As V_1 is open, we find an open neighborhood $U_3 \subseteq U_1$ of $g_0 \operatorname{Exp}(2X_0)g_0^\sharp$ and a mapping $\gamma\colon U_3 \to W$ such that $(x, \gamma(x)) \in G^c \operatorname{Exp}(W^0_{\max})$ for all $x \in U_3$. By the definition of $\widetilde{f}$, we have

$$\widetilde{f}(x) = f(x, \gamma(x)) = f(x, w_0) \tag{5.3}$$

for $x \in U_3$. Hence $\widetilde{f}$ is holomorphic on $U_3 \subseteq U_1$, and therefore holomorphic as $g_0 \operatorname{Exp}(2X_0)g_0^\sharp$ was arbitrary.

It follows from (5.3) that $f = 0$ if and only if $\widetilde{f} = 0$, hence $(p|_V)_*$ is injective. As the pull back of a holomorphic function on U is clearly holomorphic on V, the lemma follows. ∎

Lemma V.6. *Let $(\widehat{\pi}, \mathcal{H})$ be a holomorphic representation of $\Gamma_{G^c}(\widehat{W}^0_{\max})$ and consider the $\Gamma_{G^c}(\widehat{W}^0_{\max}) \times \Gamma_{G^c}(\widehat{W}^0_{\max})$-action on $\operatorname{Hol}(\Gamma_{G^c}(\widehat{W}^0_{\max}))$ given by $\big((s_1, s_2).f\big)(s) := f(s_2^* s s_1)$. Then all hyperfunction matrix coefficients are holomorphic and the sesquilinear map*

$$\mathcal{H}^{-\omega} \times \mathcal{H}^{-\omega} \to \operatorname{Hol}(\Gamma_{G^c}(\widehat{W}^0_{\max})), \quad (\lambda, \mu) \mapsto \pi_{\lambda,\mu}\colon s \mapsto \langle \lambda, \widehat{\pi}^{-\omega}(s^*).\mu\rangle$$

is $\Gamma_{G^c}(\widehat{W}^0_{\max}) \times \Gamma_{G^c}(\widehat{W}^0_{\max})$-equivariant.

Proof. Since $\widehat{\pi}^{-\omega}(\Gamma_{G^c}(\widehat{W}^0_{\max})).\mathcal{H}^{-\omega} \subseteq \mathcal{H}^{\omega}$ (cf. Definition V.1(b)), the functions $\widehat{\pi}_{\lambda,\mu}$ are defined. Next we show the $\Gamma_{G^c}(\widehat{W}^0_{\max}) \times \Gamma_{G^c}(\widehat{W}^0_{\max})$-equivariance. Let $s, s_1, s_2 \in \Gamma_{G^c}(\widehat{W}^0_{\max})$ and $\lambda, \mu \in \mathcal{H}^{-\omega}$. Then the equivariance follows from

$$\begin{aligned}\pi_{\hat{\pi}^{-\omega}(s_1).\lambda, \hat{\pi}^{-\omega}(s_2).\mu}(s) &= \langle \hat{\pi}^{-\omega}(s_1).\lambda, \hat{\pi}^{-\omega}(s^*)\hat{\pi}^{-\omega}(s_2).\mu\rangle \\ &= \langle \lambda, \hat{\pi}(s_1^*)\hat{\pi}^{-\omega}(s^*)\hat{\pi}^{-\omega}(s_2).\mu\rangle \\ &= \langle \lambda, \hat{\pi}^{-\omega}(s_1^* s^* s_2).\mu\rangle = \big((s_1, s_2).\pi_{\lambda,\mu}\big)(s).\end{aligned}$$

It remains to show that all $\pi_{\lambda,\mu}$, $\lambda, \mu \in \mathcal{H}^{-\omega}$, are holomorphic. Fix $s_0 \in \Gamma_{G^c}(\widehat{W}^0_{\max})$. According to [HiNe93, Th. 3.20], we find an open neighborhood $U \subseteq \Gamma_{G^c}(\widehat{W}^0_{\max})$ of s_0, an open set $V \subseteq \Gamma_{G^c}(\widehat{W}^0_{\max})$, and elements $s_1, s_2 \in U$ such that $\varphi\colon U \to V$, $s \mapsto s_1 s s_2$ is biholomorphic. Then for all $s \in U$ the $\Gamma_{G^c}(\widehat{W}^0_{\max}) \times \Gamma_{G^c}(\widehat{W}^0_{\max})$-equivariance implies that

$$\pi_{\lambda,\mu}\big(\varphi(s)\big) = \pi_{\lambda,\mu}(s_1 s s_2) = \pi_{\widehat{\pi}^{-\omega}(s_2).\lambda, \widehat{\pi}^{-\omega}(s_1^*).\mu}(s).$$

Now $\widehat{\pi}^{-\omega}(s_2).\lambda, \widehat{\pi}^{-\omega}(s_1^*).\mu \in \mathcal{H}$, the holomorphy of $(\widehat{\pi}, \mathcal{H})$, and the holomorphy of φ^{-1} imply the assertion. ∎

Fix a continuous character $\chi\colon H \to \mathbb{C}^*$ and consider the left action of H on $(G^c \times W^0_{\max}) \times \mathbb{C}$, resp. $\Gamma_{G^c}(\widehat{W}^0_{\max}) \times \mathbb{C}$, given by $h.(g, X, z) := (gh^{-1}, \operatorname{Ad}(h).X, \chi(h)^{-1}.z)$, resp. $h.(s, z) = (sh^{-1}, \chi(h)^{-1}.z)$. Denote by $(G \times W^0_{\max}) \times_H \mathbb{C}$, resp. $\Gamma_{G^c}(\widehat{W}^0_{\max}) \times_H$

$\mathbb{C}$, the corresponding quotient spaces. Then we obtain a commutative diagram

$$\begin{array}{ccc} (G^c \times W^0_{\max}) \times_H \mathbb{C} & \longrightarrow & \Gamma_{G^c}(\widehat{W}^0_{\max}) \times_H \mathbb{C} \\ \downarrow \alpha & & \downarrow \widehat{\alpha} \\ \Xi(W^0_{\max}) & \longrightarrow & \Gamma_{G^c}(\widehat{W}^0_{\max})/H, \\ [g, X, z] & \longrightarrow & [g \operatorname{Exp}(X), z] \\ \downarrow & & \downarrow \\ [g, X] & \longrightarrow & g \operatorname{Exp}(X) H \end{array} \tag{5.4}$$

Lemma V.7. *Let $(\pi, \mathcal{H})$ be an irreducible (H, χ)-spherical representation of $\Gamma_H(C)$ with discrete kernel.*

(i) *To each $0 \neq \nu \in (\mathcal{H}^{-\omega})^{(H,\chi)}$ corresponds an analytic line bundle structure for $\widehat{\alpha}\colon \Gamma_{G^c}(\widehat{W}^0_{\max}) \times_H \mathbb{C} \to \Gamma_{G^c}(\widehat{W}^0_{\max})/H$ such that the restriction $\alpha :$ $(G^c \times W^0_{\max}) \times_H \mathbb{C} \to \Xi(W^0_{\max})$ is a holomorphic line bundle with respect to an appropriate complex structure. Here H acts on $\Gamma_{G^c}(\widehat{W}^0_{\max}) \times \mathbb{C}$ by $h.(s, z) = \big(sh^{-1}, \chi(h)^{-1}.z\big)$.*

(ii) *The holomorphic sections*

$$\Gamma^h(\Xi(W^0_{\max}), \chi) := \{s\colon \Xi(W^0_{\max}) \to (G^c \times W^0_{\max}) \times_H \mathbb{C}\colon s \ \ \text{holomorphic},\ \alpha \circ s = \mathrm{id}_{\Xi(W^0_{\max})}\}$$

of α are in one-to-one correspondence with the (H, χ)-semi-invariant holomorphic functions

$$\operatorname{Hol}(\Gamma_{G^c}(\hat{W}^0_{\max}))^{(H,\chi)} := \{f \in \operatorname{Hol}(\Gamma_{G^c}(\hat{W}^0_{\max}))\colon (\forall s \in \Gamma_{G^c}(\hat{W}^0_{\max}))(\forall h \in H) f(sh) = \chi(h) f(s)\}.$$

The correspondence is given by

$$\operatorname{Hol}(\Gamma_{G^c}(\widehat{W}^0_{\max}))^{(H,\chi)} \to \Gamma^h(\Xi(W^0_{\max}), \chi), \quad f \mapsto s_f\colon [g, X] \mapsto [g, X, f(g \operatorname{Exp}(X))].$$

Proof. (i) According to Proposition V.4(i), $(\pi, \mathcal{H})$ extends to a holomorphic representation $(\widehat{\pi}, \mathcal{H})$ of $\Gamma_{G^c}(\widehat{W}^0_{\max})$. Fix $0 \neq \nu \in (\mathcal{H}^{-\omega})^{(H,\chi)}$. For each $v \in \mathcal{H}$ we define an element $\sigma_v \in \operatorname{Hol}(\Gamma_{G^c}(\widehat{W}^0_{\max}))^{(H,\chi)}$ by $\sigma_v(s) := \nu(\widehat{\pi}(s)^*.v)$. As every operator $\widehat{\pi}(s)$, $s \in \Gamma_{G^c}(\widehat{W}^0_{\max})$, is injective and has dense range, the family $\{\sigma_v\colon v \in \mathcal{H}\}$ has no common zeros on on $\Gamma_{G^c}(\widehat{W}^0_{\max})$.

First we show that $\widehat{\alpha}$ is an analytic line bundle. Fix $s_0 \in \Gamma_{G^c}(\widehat{W}^0_{\max})/H$. Since $\{\sigma_v\colon v \in \mathcal{H}\}$ has no common zeros and consists of χ-semi-invariant functions, we find $v \in \mathcal{H}$ and an open neighborhood V of s_0 such that $\sigma_v|_{\widetilde{V}}$ has no zeros, where $\widetilde{V}$ denotes the inverse image of V under the quotient map $\Gamma_{G^c}(\widehat{W}^0_{\max}) \to \Gamma_{G^c}(\widehat{W}^0_{\max})/H$. The mapping

$$\varphi_V\colon \widehat{\alpha}^{-1}(V) \to V \times \mathbb{C}, \quad [s, z] \mapsto (sH, \sigma_v(s)^{-1}.z)$$

yields a trivialization of $\widehat{\alpha}^{-1}(V)$. We claim that the φ_V give rise to an analytic line bundle structure. Let $U, V \subseteq \Gamma_{G^c}(\widehat{W}^0_{\max})/H$ be open subsets with $U \cap V \neq \emptyset$ and

$u, v \in \mathcal{H}$ such that $\sigma_u|_{\widetilde{U}}$, $\sigma_v|_{\widetilde{V}}$ have no zeros. Then the corresponding transition function is given by

$$\varphi_U \circ \varphi_V^{-1} \colon (U \cap V) \times \mathbb{C} \to (U \cap V) \times \mathbb{C}, \quad (sH, z) \mapsto \Big(sH, \frac{\sigma_v(s)}{\sigma_u(s)}.z\Big).$$

Since

$$U \cap V \to \mathrm{GL}(1, \mathbb{C}) \cong \mathbb{C}^*, \ sH \mapsto \frac{\sigma_v(s)}{\sigma_u(s)} \tag{5.5}$$

is analytic, the claim follows.

Finally, Lemma V.5 and (5.5) show that the restricted line bundle α is holomorphic.

(ii) This follows from Lemma V.5 and the line bundle structure given in (i). ∎

According to Lemma V.7(ii), we can define a $\Gamma_{G^c}(\widehat{W}_{\max}^0)$-left action on $\Gamma^h(\Xi(W_{\max}^0), \chi)$ by

$$(s.s_f)([g, X]) := [g, X, f(s^* g \operatorname{Exp}(X))].$$

We equip $\mathrm{Hol}(\Gamma_{G^c}(\widehat{W}_{\max}^0))^{(H,\chi)}$ with the topology of compact convergence and transfer this topology via the correspondence of Lemma V.7(ii) to $\Gamma^h(\Xi(W_{\max}^0), \chi)$.

Theorem V.8. *Let $(\pi, \mathcal{H})$ be an irreducible representation of $\Gamma_H(C)$ and suppose that $C \subseteq \widehat{W}_{\max}^0$. Then the following statements are equivalent:*

1. *$(\pi, \mathcal{H})$ is (H, χ)-spherical.*
2. *The representation $(\pi, \mathcal{H})$ extends to a holomorphic representation of $\Gamma_{G^c}(\widehat{W}_{\max}^0)$ and there exists a continuous $\Gamma_{G^c}(\widehat{W}_{\max}^0)$-equivariant injective linear map $r \colon \mathcal{H} \to \Gamma^h(\Xi(W_{\max}^0), \chi)$.*

Proof. (1) $\Rightarrow$ (2): First we note that, in view of our assumptions, Proposition V.4(i) implies that $(\pi, \mathcal{H})$ extends to a holomorphic representation $(\widehat{\pi}, \mathcal{H})$ of $\Gamma_{G^c}(\widehat{W}_{\max}^0)$. Let $0 \neq \nu \in (\mathcal{H}^{-\omega})^{(H,\chi)}$ be a cyclic element and consider the mapping

$$r : \mathcal{H} \to \mathrm{Hol}(\Gamma_{G^c}(\widehat{W}_{\max}^0))^{(H,\chi)}, \quad v \mapsto \big(s \mapsto \nu(\widehat{\pi}(s)^*.v)\big).$$

In view of Lemma V.6, this map is well defined. It is clear that r is injective, $\Gamma_{G^c}(\widehat{W}_{\max}^0)$-equivariant and linear. We show that it is continuous. Let $v_n \to v$ in $\mathcal{H}$ and $r(v_n) \to f$ in $\mathrm{Hol}(\Gamma_{G^c}(\widehat{W}_{\max}^0))^{(H,\chi)}$. As $r(v_n) \to r(v)$ pointwise, we have $r(v) = f$. Hence continuity follows from the Closed Graph Theorem. In view of Lemma V.7(ii), this proves the assertion.

(2) $\Rightarrow$ (1): Let $r \colon \mathcal{H} \to \Gamma^h(\Xi(W_{\max}^0), \chi)$ be as stated in (2), and $v \in \mathcal{H}^\omega$. In view of Lemma V.7(ii), we may identify $\Gamma^h(\Xi(W_{\max}^0), \chi)$ with $\mathrm{Hol}(\Gamma_{G^c}(\widehat{W}_{\max}^0))^{(H,\chi)}$. According to Proposition V.4(ii), there are elements $s_0 \in \Gamma_{G^c}(\widehat{W}_{\max}^0)$ and $w \in \mathcal{H}$ with $v = \widehat{\pi}(s_0).w$. By [HiNe93, Th. 3.20] we find $s_1, s_2 \in \Gamma_{G^c}(\widehat{W}_{\max}^0)$ such that $s_0 = s_1 s_2$. Then

$$\begin{aligned} r(v)(s) &= r(\hat{\pi}(s_0).w)(s) = r(\hat{\pi}(s_1).(\hat{\pi}(s_2).w))(s) \\ &= \big(s_1.r(\hat{\pi}(s_2).w)\big)(s) = r(\hat{\pi}(s_2).w)(s_1^* s) \end{aligned}$$

entails that $r(v)$ extends to a continuous function $r(v)$ on $\overline{\Gamma}_{G^c}(\widehat{W}_{\max}^0)$ satisfying $r(v)(\mathbf{1}) = r(\widehat{\pi}(s_2).w)(s_1^*)$. Now the prescription $\nu(v) := r(v)(\mathbf{1})$ defines an element of $\mathcal{H}^{-\omega}$. Moreover the equality

$$\begin{aligned}\big(\pi^{-\omega}(h).\nu\big)(v) &= \nu(\pi(h)^{-1}.v) = r(\pi(h)^{-1}.v)(\mathbf{1})\\ &= \big(h^{-1}.r(v)\big)(\mathbf{1}) = r(v)(h) = \chi(h)r(v)(\mathbf{1}) = (\chi(h).\nu)(v)\end{aligned}$$

for $v \in \mathcal{H}^{\omega}$, $h \in H$, shows that $\nu \in (\mathcal{H}^{-\omega})^{(H,\chi)}$. As ν is the evaluation in $\mathbf{1}$, injectivity and equivariance of r show that ν is cyclic. ■

VI. Spherical highest weight representations

Definitions and elementary properties. The following lemma is crucial. The assumption on $(\mathfrak{g},\tau)$ stated there will be assumed for the whole section. Also we will assume from now on that $(\mathfrak{g},\tau)$ is effective. The following lemma should be compared with Remark V.3 which discusses the structural consequences of the existence of an irreducible spherical representation with discrete kernel.

Lemma VI.1. *Let $(\mathfrak{g},\tau)$ be a symmetric Lie algebra and $\mathfrak{p} \subseteq \mathfrak{q}$ a maximal hyperbolic Lie triple system. Let $\mathfrak{k}^c \supseteq i\mathfrak{p} + [\mathfrak{p},\mathfrak{p}]$ be a maximal compactly embedded subalgebra of $\mathfrak{g}^c$. Assume that there exists an element $X_0 \in i\mathfrak{z}(\mathfrak{k}^c) \cap \mathfrak{q}$ such that $\mathfrak{z}_{\mathfrak{g}^c}(iX_0) = \mathfrak{k}^c$. Then the following assertions hold:*

(i) *$(\mathfrak{g}_{\mathbb{C}},\widehat{\tau})$ is quasihermitian.*
(ii) *$(\mathfrak{g},\tau)$ is quasihermitian.*
(iii) *Let $\mathfrak{a} \subseteq \mathfrak{p}$ be a maximal abelian subspace containing X_0. Then there exists a compactly embedded Cartan subalgebra $\mathfrak{t}^c$ of $\mathfrak{g}^c$ such that $i\mathfrak{a} \subseteq \mathfrak{t}^c \subseteq \mathfrak{k}^c$.*
(iv) *Let $\widehat{\Delta} := \widehat{\Delta}(\mathfrak{t}^c_{\mathbb{C}},\mathfrak{g}_{\mathbb{C}})$, $\Delta := \Delta(\mathfrak{a},\mathfrak{g})$, and $\widehat{\Delta}^+$ be any positive system such that*

$$\widehat{\Delta}^+_n := \{\alpha \in \widehat{\Delta}_n \colon \alpha(X_0) > 0\}.$$

Then $\widehat{\Delta}^+$ is $\mathfrak{k}^c$-adapted and $\Delta^+ := (\widehat{\Delta}^+|\mathfrak{a})\backslash\{0\}$ is a $\mathfrak{p}$-adapted positive system of roots.

Proof. (i) This follows from $\mathfrak{k}^c \subseteq \mathfrak{z}_{\mathfrak{g}^c}(\mathfrak{z}(\mathfrak{k}^c)) \subseteq \mathfrak{z}_{\mathfrak{g}^c}(iX_0) = \mathfrak{k}^c$, in view of Definition I.2(b).

(ii) First we note that $\mathfrak{k}^c$ is τ-invariant because $\tau(X_0) = -X_0$, and hence that $\mathfrak{k}^c$ can be written as a direct Lie algebra sum $\mathfrak{k}^c = \mathfrak{k}^c_1 \oplus (i\mathfrak{p} + [\mathfrak{p},\mathfrak{p}])$, where $\mathfrak{k}^c_1 \subseteq \mathfrak{h}$ because $i\mathfrak{k}^c \cap \mathfrak{q}$ is a hyperbolic Lie triple system in $\mathfrak{q}$ containing $\mathfrak{p}$, and hence coincides with $\mathfrak{p}$. From the maximality of $\mathfrak{p}$ we conclude in particular that $X_0 \in \mathfrak{p}$ and that $i\mathfrak{z}(\mathfrak{p}) \subseteq \mathfrak{z}(\mathfrak{k}^c)$. Now the assertion follows from

$$i\mathfrak{p} \subseteq \mathfrak{z}_{i\mathfrak{q}}(i\mathfrak{z}(\mathfrak{p})) \subseteq \mathfrak{z}_{\mathfrak{g}^c}(X_0) \cap i\mathfrak{q} = \mathfrak{k}^c \cap i\mathfrak{q} = i\mathfrak{p}.$$

(iii), (iv) This is an immediate consequence of (i) and (ii) and Definition I.1(d)-(f). ■

Remark VI.2. Even though it might seem to be quite restrictive, the assumption of Lemma V.1 is quite natural. For instance let $(\mathfrak{g},\tau)$ be a symmetric Lie algebra, $C \subseteq \mathfrak{q}$ an open hyperbolic cone, and $\Gamma_H(C)$ the corresponding real Ol'shanskiĭ semigroup. If $\Gamma_H(C)$ admits an irreducible (H,χ)-spherical unitary representation with discrete kernel, then Lemma V.2(ii) together with [KN96, Th. VIII.1] implies that the condition of Lemma VI.1 holds for any $X_0 \in C^0_{\max} \cap \mathfrak{z}(\mathfrak{p})$. ■

Definition VI.3. Let $(\mathfrak{g},\tau)$ be a symmetric Lie algebra and $\mathfrak{k}^c$, $\mathfrak{a}$, $\mathfrak{t}^c$, $\widehat{\Delta}^+$ and Δ^+ as in Lemma VI.1.

(a) For a $\mathfrak{g}_{\mathbb{C}}$-module V and $\beta \in (\mathfrak{t}^c_{\mathbb{C}})^*$ we write

$$V^\beta := \{v \in V : (\forall X \in \mathfrak{t}^c_{\mathbb{C}}) X.v = \beta(X)v\}$$

for the *weight space of weight* β and $\mathcal{P}_V = \{\beta\colon V^\beta \neq \{0\}\}$ for the set of weights of V.

(b) Let V be a $\mathfrak{g}_{\mathbb{C}}$-module and $v \in V^\lambda$ a $\mathfrak{t}^c_{\mathbb{C}}$-weight vector. We say that v is a *primitive element of* V (with respect to $\widehat{\Delta}^+$) if $\mathfrak{g}^\alpha_{\mathbb{C}}.v = \{0\}$ holds for all $\alpha \in \widehat{\Delta}^+$.

(c) A $\mathfrak{g}_{\mathbb{C}}$-module V is called a *highest weight module* with highest weight λ (with respect to $\widehat{\Delta}^+$) if it is generated by a primitive element of weight λ.

(d) Let $\lambda \in i\mathfrak{t}^{c*}$ be dominant integral w.r.t. $\widehat{\Delta}^+_k$ and $F(\lambda)$ the corresponding irreducible $\mathfrak{k}^c_{\mathbb{C}}$-module of highest weight λ. We define the *generalized Verma module* by

$$N(\lambda) := \mathcal{U}(\mathfrak{g}_{\mathbb{C}}) \otimes_{\mathcal{U}(\mathfrak{k}^c_{\mathbb{C}}+\mathfrak{p}^+)} F(\lambda).$$

Note that $N(\lambda)$ is a highest weight module for $\mathcal{U}(\mathfrak{g}_{\mathbb{C}})$ with highest weight λ. We denote by $L(\lambda)$ the unique irreducible quotient of $N(\lambda)$.

(e) Let G^c be a connected Lie group with Lie algebra $\mathfrak{g}^c$. We write K^c for the analytic subgroup of G^c corresponding to $\mathfrak{k}^c$. Let $(\pi^c, \mathcal{H})$ be a unitary representation of G^c. A vector $v \in \mathcal{H}$ is called K^c*-finite* if it is contained in a finite dimensional K^c-invariant subspace. We write $\mathcal{H}^{K^c,\omega}$ for the space of analytic K^c-finite vectors.

(f) An irreducible unitary representation $(\pi^c, \mathcal{H})$ of G^c is called a *highest weight representation* w.r.t. $\widehat{\Delta}^+$ with highest weight $\lambda \in i\mathfrak{t}^{c*}$ if $\mathcal{H}^{K^c,\omega}$ is a highest weight module for $\mathfrak{g}_{\mathbb{C}}$ w.r.t. $\widehat{\Delta}^+$ and highest weight λ. We write $F(\lambda)$ for the *lowest* K^c*-type*, i.e., the K^c-submodule generated by a highest weight vector. Assume that G^c is simply connected. We then say that the irreducible highest weight module $L(\lambda)$ is *unitarizable* if there exists a unitary highest weight representation $(\pi^c_\lambda, \mathcal{H}_\lambda)$ of G^c with $\mathcal{H}_\lambda^{K^c,\omega} \cong L(\lambda)$ as $\mathfrak{g}_{\mathbb{C}}$-modules.

(g) If $L(\lambda)$ is unitarizable, then the corresponding unitary representation of $(\pi^c_\lambda, \mathcal{H}_\lambda)$ of G^c is called *singular* if the natural map $N(\lambda) \to L(\lambda)$ has a non-trivial kernel and *non-singular* otherwise. ∎

As before G^c denotes a simply connected Lie group with Lie algebra $\mathfrak{g}^c$ and $H = (G^c)^\tau$ is the fixed point group of τ in G^c.

The following lemma furnishes a bridge between semigroup representations and highest weight representations. We recall the Lüscher-Mack correspondence $\pi \mapsto \pi^c$ from Definition V.1.

Proposition VI.4. *Let $\emptyset \neq C \subseteq \mathfrak{q}$ be an $\mathrm{Inn}_{\mathfrak{g}}(\mathfrak{h})$-invariant open convex cone and $\widehat{\Delta}^+$ be a positive $\mathfrak{k}^c$-adapted system such that $C \subseteq \widehat{W}^0_{\max}$. Then the mapping $\pi \mapsto \pi^c$ provides a bijection between the irreducible (H,χ)-spherical representations of $\Gamma_H(C)$ and the (H,χ)-spherical highest weight representations of G^c w.r.t. $\widehat{\Delta}^+$.*

Proof. Let $(\pi, \mathcal{H})$ be an irreducible (H,χ)-spherical representation of $\Gamma_H(C)$. According to Proposition V.4(i), the representation π extends to a holomorphic repesentation $\widehat{\pi}$ of the complex Ol'shanskiĭ semigroup $\Gamma_{G^c}(\widehat{W}^0_{\max})$. Since $\widehat{W}_{\max} \subseteq \widehat{\mathfrak{q}}$ is a hyperbolic cone and all the operators $d\pi(X)$, $X \in \widehat{W}_{\max}$, are bounded from above, [HiNe96, Th. 3.4] shows that $(\pi^c, \mathcal{H})$ is a highest weight representation of G^c w.r.t. $\widehat{\Delta}^+$. Obviously, $(\pi^c, \mathcal{H})$ is (H,χ)-spherical.

Conversely, let $(\pi^c, \mathcal{H})$ be a unitary highest weight representation of G^c w.r.t. $\widehat{\Delta}^+$ and note that

$$\widehat{W}^0_{\max} \subseteq \{X \in \widehat{\mathfrak{q}}\colon d\pi^c(X) \ \text{ bounded from above}\}$$

(cf. [HiNe96, Th. 3.6(ii)]). According to [HiNe96, Th. B], $(\pi^c, \mathcal{H})$ extends to a holomorphic representation $(\widehat{\pi}, \mathcal{H})$ of $\Gamma_{G^c}(\widehat{W}^0_{\max})$. Hence $\pi := \widehat{\pi}|_{\Gamma_H(C)}$ is an (H, χ)-spherical representation of $\Gamma_H(C)$. ∎

For the following proposition we recall the definition of the group $H_K := \exp(\mathfrak{h} \cap \mathfrak{k}^c) \subseteq G^c$.

Proposition VI.5. *Let $(\pi^c_\lambda, \mathcal{H}_\lambda)$ be an (H, χ)-spherical highest weight representation of G^c. Then the following assertions hold:*

(i) $\dim(\mathcal{H}^{-\omega}_\lambda)^{(H,\chi)} = \dim F(\lambda)^{(H_K, \chi^*|_{H_K})} = 1$.

(ii) *The restriction mapping*

$$(\mathcal{H}^{-\omega}_\lambda)^{(H,\chi)} \to F(\lambda)^{(H_K, \chi^*|_{H_K})}, \quad \nu \mapsto \nu|_{F(\lambda)}$$

is a linear bijection.

Proof. (i) In view of Lemma III.5, we have $\Gamma_{H_1}(W_{\max}) \subseteq H_1 AN \subseteq G_1$. Hence the same argument as in the proof of [HiNe96, Lemma 4.6] implies that each primitive element $v_\lambda \in \mathcal{H}_\lambda$ is cyclic with respect to the action of the group H. From that it follows directly that the map

$$(\mathcal{H}^{-\omega}_\lambda)^{(H,\chi)} \to \mathbb{C}, \quad \nu \mapsto \nu(v_\lambda)$$

is injective. Using that $\big(\pi^c_\lambda|_{K^c}, F(\lambda)\big)$ is a highest weight representation of K^c, one similarly shows that the mapping

$$F(\lambda)^{(H\cap K^c, \chi^*|_{H\cap K^c})} \to \mathbb{C}, \quad \nu \mapsto \nu(v_\lambda)$$

is injective. This proves (i).

(ii) This is a direct consequence of the proof of (i). ∎

It is interesting to see how the conclusion of the preceding proposition is related to Theorem II.11. To explain this connection, let $\rho = \pi^K_\lambda\colon K^c_{\mathbb{C}} \to B\big(F(\lambda)\big)$ be the irreducible holomorphic representation of $K^c_{\mathbb{C}}$ defined by the dominant integral weight λ and consider the corresponding smooth action of G^c on $\operatorname{Hol}\big(\mathcal{D}, F(\lambda)\big)$ given by (2.1):

$$\big(\pi(g).f\big)(z) = J_\rho(g^{-1}, z)^{-1}.f(g^{-1}.z).$$

Differentiating this action leads to an action of the Lie algebra $\mathfrak{g}_{\mathbb{C}}$ on the same space. It turns out that the subspace of the constant functions is isomorphic to $F(\lambda)$ as a $\mathfrak{k}^c_{\mathbb{C}}$-module, and that it generates a $\mathfrak{g}_{\mathbb{C}}$-submodule isomorphic to $L(\lambda)$, the irreducible highest weight module with highest weight λ. If, in addition, $L(\lambda)$ is unitarizable, then we even obtain a G^c-equivariant embedding $\mathcal{H}_\lambda \hookrightarrow \operatorname{Hol}\big(\mathcal{D}, F(\lambda)\big)$.

We have already seen in Theorem II.11 that the H-eigenfunctions in $\operatorname{Hol}\big(\mathcal{D}, F(\lambda)\big)$ can be classified by the H_K-eigenvectors in $F(\lambda)$ which reduces the classification problem on the level of holomorphic functions to the representation theory of the compact Lie algebra $\mathfrak{k}^c$.

Let $K_z\colon \mathcal{H}_\lambda \to F(\lambda), f \mapsto f(z)$ denote the point evaluation in $z \in \mathcal{D}$. Then the mapping $\Phi\colon \mathcal{D} \to B\big(\mathcal{H}_\lambda, F(\lambda)\big), z \mapsto K^*_z$ is holomorphic (cf. [Ne98, Lemma A.III.11]) and therefore for each $v \in F(\lambda)$ the mapping $\mathcal{D} \to \mathcal{H}_\lambda$, $z \mapsto K^*_z.v$ is antiholomorphic. From that we conclude that $K_z^*.v \in \mathcal{H}_\lambda^\omega$ for all $z \in \mathcal{D}$, $v \in F(\lambda)$. Now let $\nu \in \mathcal{H}_\lambda^{-\omega}$. Then we define

$$f_\nu \colon \mathcal{D} \to F(\lambda) \quad \text{by} \quad \langle f_\nu(z), v\rangle = \langle \nu, K_z^*.v\rangle$$

for all $v \in F(\lambda)$. For each $s \in S := \Gamma_{G^c}(\widehat{W}^0_{\max})$ we then have

$$\langle f_\nu(s.z), v\rangle = \langle \nu, K_{s.z}^*.v\rangle.$$

Since the semigroup S acts also on $\mathcal{D}$ by

$$s.z := \log\big(\kappa(\eta(s)\exp z)\big),$$

where $\eta\colon S \to G_{\mathbb{C}}$ is the natural homomorphism, we obtain a holomorphic extension of the cocycle $\widetilde{J}\colon G^c \times \mathcal{D} \to K_{\mathbb{C}}^c$ to $\widetilde{J}\colon S \times \mathcal{D} \to K_{\mathbb{C}}^c$, and hence an action of S on $\mathrm{Hol}\big(\mathcal{D}, F(\lambda)\big)$ is given by

$$(s.f)(z) = J_\rho(s^*, z)^{-1}.f(s^*.z).$$

Furthermore the semigroup S acts by bounded operators $\widehat{\pi}_\lambda(s)$ on the Hilbert space $\mathcal{H}_\lambda$ and we have

$$K_z \circ \widehat{\pi}_\lambda(s) = J_\rho(s^*, z)^{-1} \circ K_{s^*.z},$$

i.e., $K_{s.z} = J_\rho(s,z)K_z \circ \widehat{\pi}_\lambda(s^*)$. Therefore

$$\begin{aligned}\langle f_\nu(s.z), v\rangle &= \langle \nu, K_{s.z}^*.v\rangle = \langle \nu, \big(J_\rho(s,z)K_z \circ \hat{\pi}_\lambda(s^*)\big)^*.v\rangle \\ &= \langle \nu, \hat{\pi}_\lambda(s)K_z^*J_\rho(s,z)^*.v\rangle = \langle \nu \circ \hat{\pi}_\lambda(s), K_z^*J_\rho(s,z)^*.v\rangle \\ &= \langle J_\rho(s,z)K_z.\big(\nu\circ\hat{\pi}_\lambda(s)\big), v\rangle.\end{aligned}$$

Hence the holomorphy of the mappings $z \mapsto J_\rho(s,z)$ and $z \mapsto K_z$ implies that this expression depends holomorphically on z, hence that f_ν is holomorphic. Furthermore

$$\begin{aligned}\langle f_{s.\nu}(z), v\rangle &= \langle s.\nu, K_z^*.v\rangle = \langle \nu, \widehat{\pi}_\lambda(s^*)K_z^*.v\rangle \\ &= \langle \nu, K_{s^*.z}^*J_\rho(s^*,z)^{-*}.v\rangle = \langle J_\rho(s^*,z)^{-1}.f_\nu(s^*.z), v\rangle \\ &= \langle (s.f_\nu)(z), v\rangle,\end{aligned}$$

i.e., $\nu \mapsto f_\nu$ is S-equivariant. So we obtain an S-equivariant embedding

$$\mathcal{H}_\lambda^{-\omega} \to \mathrm{Hol}\big(\mathcal{D}, F(\lambda)\big), \quad \nu \mapsto f_\nu.$$

Using this embedding, it is clear that the conclusion of Proposition VI.5 follows from Theorem II.11. The interesting point in the preceding construction is that Theorem II.11 always guarantees the existence of H-invariant holomorphic functions whenever $F(\lambda)$ is a spherical K^c-module. For singular highest weight representations $(\pi_\lambda, \mathcal{H}_\lambda)$ it is not true in general that $\mathcal{H}_\lambda$ is spherical whenever $F(\lambda)$ is. In terms of the picture explained above, the corresponding H-invariant holomorphic function is not contained in the closure of the Hilbert space when it is realized by holomorphic functions on $\mathcal{D}$. For more details on these problems for the special type of Cayley type spaces we refer to [HiNe97].

Proposition VI.6. *Let $(\pi_\lambda^c, \mathcal{H}_\lambda)$ be an (H,χ)-spherical highest weight representation of G^c. Then $(\pi_\lambda^c|_{K^c}, F(\lambda))$ is $(H_K, \chi^*|_{H_K})$-spherical and $\lambda|_{\mathfrak{t}_{\mathfrak{h}}^c} = d\chi^*|_{\mathfrak{t}_{\mathfrak{h}}^c}$. Moreover $\chi^*|_{H_K}$ is unitary, hence coincides with $\chi|_{H_K}$.*

Proof. The first assertion follows from Lemma VI.5(i). For the second one let $0 \neq \nu \in F(\lambda)^{(H_K, \chi^*|_{H_K})}$. Then we obtain for all $X \in \mathfrak{t}_{\mathfrak{h}}^c$ that

$$\begin{aligned}\langle \chi(\exp(X)).\nu, v_\lambda\rangle &= \langle (\pi_\lambda^c)^{-\omega}(\exp X).\nu, v_\lambda\rangle = \langle \nu, \pi_\lambda^c(\exp -X).v_\lambda\rangle \\ &= \nu(e^{-\lambda(X)}.v_\lambda) = e^{\lambda(X)}.\nu(v_\lambda).\end{aligned}$$

In view of $\nu(v_\lambda) \neq 0$, this proves the assertion. Finally the unitarity of $\chi^*|_{H_K}$ follows from the fact that K^c acts unitarily on $F(\lambda)$ and that $F(\lambda)$ is $\chi^*|_{H_K}$-spherical. ■

Appendix. Analytic vectors and semigroups

In this appendix $(\pi, \mathcal{H})$ denotes a continuous unitary representation of the Lie group G with Lie algebra $\mathfrak{g}$ on the Hilbert space $\mathcal{H}$. We write $\mathcal{H}^\omega \subseteq \mathcal{H}$ for the space of *analytic vectors*, i.e., $v \in \mathcal{H}^\omega$ means that the orbit map $G \to \mathcal{H}, g \mapsto \pi(g).v$ is real analytic. It is clear that $\mathcal{H}^\omega \subseteq \mathcal{H}^\infty$.

In this section we discuss a natural topology on the space $\mathcal{H}^\omega$ and some of its properties.

If $v \in \mathcal{H}^\omega$, then there exists an open connected 0-neighborhood $U \subseteq \mathfrak{g}_\mathbb{C}$ and a holomorphic map $\gamma_{v,U} \colon U \to \mathcal{H}$ with $\gamma_{v,U}(0) = v$ and $\gamma_{v,U}(X) = \pi(\exp X).v$ for $X \in U \cap \mathfrak{g}$. Let $\mathcal{H}_U \subseteq \mathcal{H}$ denote the subspace of all elements v for which $\gamma_{v,U}$ exists. Then we have a natural linear embedding

$$\eta_U \colon \mathcal{H}_U \to \mathrm{Hol}(U, \mathcal{H}), \quad v \mapsto \gamma_{v,U}.$$

Lemma A.1. *If we endow the space* $\mathrm{Hol}(U, \mathcal{H})$ *with the topology of uniform convergence on compact subsets of* U*, then the image* $\eta_U(\mathcal{H}_U) \subseteq \mathrm{Hol}(U, \mathcal{H})$ *is a closed subspace of the Fréchet space* $\mathrm{Hol}(U, \mathcal{H})$*, hence inherits the structure of a Fréchet space.*

Proof. First we note that the topology of uniform convergence turns $\mathrm{Hol}(U, \mathcal{H})$ into a Fréchet space. Since U can be covered by countably many compact subsets, it is clear that the topology on $\mathrm{Hol}(U, \mathcal{H})$ is defined by a countable family of seminorms, hence locally convex and metrizable. To see that it is complete, we first observe that every Cauchy sequence $(f_n)_{n \in \mathbb{N}}$ in $\mathrm{Hol}(U, \mathcal{H})$ converges uniformly on compact subsets of U to a continuous function $f \colon U \to \mathcal{H}$. Then f is weakly holomorphic and locally bounded, hence holomorphic. This proves that $\mathrm{Hol}(U, \mathcal{H})$ is complete, i.e., a Fréchet space.

To see that $\eta_U(\mathcal{H}_U)$ is a closed subspace, let $\gamma_{v_n,U} \to f$ hold in $\mathrm{Hol}(U, \mathcal{H})$. Then $\gamma_{v_n,U}(0) = v_n \to v := f(0)$ and hence

$$f(X) = \lim_{n \to \infty} \gamma_{v_n,U}(X) = \lim_{n \to \infty} \pi(\exp X).v_n = \pi(\exp X).v$$

for all $X \in U \cap \mathfrak{g}$. This means that $f = \gamma_{v,U} = \eta_U(v)$. Thus η_U has a closed image. ■

In view of Lemma A.1, we obtain on each of the spaces $\mathcal{H}_U$ a Fréchet space structure. If $U_1 \subseteq U_2$, then $\mathcal{H}_{U_2} \hookrightarrow \mathcal{H}_{U_1}$. That this inclusion is continuous follows from the fact that locally uniform convergence on U_2 implies locally uniform convergence on the smaller domain U_1.

Since $\mathcal{H}^\omega$ is the union of the subspaces $\mathcal{H}_U$, we can endow it with the finest locally convex topology which makes all the maps $\mathcal{H}_U \to \mathcal{H}^\omega$ continuous.

Lemma A.2. *A linear mapping $A\colon \mathcal{H}^\omega \to V$ into a locally convex space V is continuous if and only if it is continuous on all the subspaces $\mathcal{H}_U$ with respect to their Fréchet space topology.*

Proof. This is an immediate consequence of the definition of the topology on $\mathcal{H}^\omega$. ∎

We write $\mathcal{H}^{-\omega}$ for the space of antilinear continuous functionals on $\mathcal{H}^\omega$, i.e., those functions on $\mathcal{H}^\omega$ which are continuous on each space $\mathcal{H}_U$ with respect to its Fréchet topology. Since all the inclusions $\mathcal{H}_U \hookrightarrow \mathcal{H}$ are continuous, Lemma A.2 shows that the inclusion $\mathcal{H}^\omega \to \mathcal{H}$ is continuous. Its adjoint yields a continuous map $\mathcal{H} \hookrightarrow \mathcal{H}^{-\omega}$ which is injective because $\mathcal{H}^\omega$ is dense in $\mathcal{H}$. Thus we have natural inclusions

$$\mathcal{H}^\omega \hookrightarrow \mathcal{H} \hookrightarrow \mathcal{H}^{-\omega}$$

which are equivariant with respect to the natural actions of G and $\mathfrak{g}_\mathbb{C}$.

Proposition A.3. *Let $X \in \mathfrak{g}$ be such that* $\operatorname{Spec}\big(i \cdot d\pi(X)\big)$ *is bounded from above. Then the subspaces $\mathcal{H}^\infty$ and $\mathcal{H}^\omega$ are invariant under the hermitian operator $e^{id\pi(X)}$ which induces a continuous mapping on these spaces.*

Proof. First we prove the invariance of $\mathcal{H}^\infty$. In view of [Ne98, Th. II.4.38], we only have to show that if we realize $\mathcal{H}$ as a reproducing kernel Hilbert space $\mathcal{H}_K$ in $\mathbb{C}^M$ for $M = \mathcal{H}^\infty$, then

$$D.\big(\pi(\operatorname{Exp} X).v\big) \in \mathcal{H}$$

holds for all $D \in \mathcal{U}(\mathfrak{g}_\mathbb{C})$ and $v \in \mathcal{H}^\infty$.

Since for $X \in \mathfrak{g}_\mathbb{C}$ the operator $\operatorname{ad} X\colon D \mapsto [X, D]$ on $\mathcal{U}(\mathfrak{g}_\mathbb{C})$ is locally finite, it makes sense to consider $e^{i \operatorname{ad} X}$ as an operator on $\mathcal{U}(\mathfrak{g}_\mathbb{C})$. We claim that

$$D.\big(e^{i\pi(X)}.v\big) = e^{id\pi(X)}\big(e^{-i \operatorname{ad} X}.D\big).v \tag{A.1}$$

holds for $v \in \mathcal{H}^\infty$ and $D \in \mathcal{U}(\mathfrak{g}_\mathbb{C})$. It is clear that (A.1) implies the invariance of $\mathcal{H}^\infty$ because the right hand side is contained in $e^{id\pi(X)}.\mathcal{H}^\infty \subseteq \mathcal{H}$.

We write $D \mapsto D^*$ for the unique antilinear antiautomorphism of $\mathcal{U}(\mathfrak{g}_\mathbb{C})$ with $X^* = -\overline{X}$ for $X \in \mathfrak{g}_\mathbb{C}$. Let $x \in \mathcal{H}^\infty$, $\mathbb{C}^+ := \{z \in \mathbb{C}\colon \operatorname{Im} z \geq 0\}$ and consider the function

$$f\colon \mathbb{C}^+ \to \mathbb{C}, z \mapsto \langle v, e^{zd\pi(X)} D^*.x\rangle.$$

Then the strong continuity of the representation $\mathbb{C}^+ \to B(\mathcal{H}), z \mapsto e^{zd\pi(X)}$ (cf. [Ne98, Lemma IV.5.2]) implies that f is continuous and antiholomorphic on the open upper half plane $\operatorname{int}(\mathbb{C}^+)$. The same holds for the function

$$h\colon \mathbb{C}^+ \to \mathbb{C}, z \mapsto \langle v, (e^{z \operatorname{ad} X}.D^*)e^{zd\pi(X)}.x\rangle$$

because $[X, A]^* = [A^*, X^*] = -[A^*, X] = [X, A^*]$ for $A \in \mathcal{U}(\mathfrak{g}_\mathbb{C})$ yields

$$(e^{z \operatorname{ad} X}.D^*)^* = e^{\overline{z} \operatorname{ad} X}.D$$

and therefore

$$h(z) = \langle (e^{\overline{z} \operatorname{ad} X}.D).v, e^{zd\pi(X)}.x\rangle$$

which immediately shows that h is antiholomorphic.

For $z \in \mathbb{R}$ we have

$$f(z) = \langle v, e^{zd\pi(X)} D^*.x\rangle = \langle v, \pi(\exp zX) D^*.x\rangle$$
$$= \langle v, (e^{z\,\mathrm{ad}(X)}.D^*)\pi(\exp zX).x\rangle = \langle v, (e^{z\,\mathrm{ad}\,X}.D^*)e^{zd\pi(X)}.x\rangle = h(z).$$

Therefore the Identity Theorem for Antiholomorphic Functions on half planes implies that $f = h$. We conclude with

$$\big(D.\big(e^{id\pi(X)}.v\big)\big)(x) = \langle D.\big(e^{id\pi(X)}.v\big), x\rangle = \langle e^{id\pi(X)}.v, D^*.x\rangle = \langle v, e^{id\pi(X)} D^*.x\rangle$$

that

$$\langle D.\big(e^{id\pi(X)}.v\big), x\rangle = f(i) = h(i) = \langle e^{id\pi(X)}.(e^{-i\,\mathrm{ad}\,X}.D).v, x\rangle$$

for all $x \in M$, and thus

$$D.\big(e^{id\pi(X)}.v\big) = e^{id\pi(X)}\big(e^{-i\,\mathrm{ad}\,X}.D\big).v. \tag{A.2}$$

This completes the proof of the invariance of $\mathcal{H}^\infty$. Moreover, it shows that $e^{id\pi(X)}|_{\mathcal{H}^\infty}$ is a continuous map.

Now we turn to the space $\mathcal{H}^\omega$. Let $v \in \mathcal{H}^\omega$ and fix $U \subseteq \mathfrak{g}_\mathbb{C}$ as above with $v \in \mathcal{H}_U$. Then $\gamma_{v,U}(Y) = \pi(\exp Y).v$ for all $Y \in U \cap \mathfrak{g}$. Let $V \subseteq U$ be an open convex 0-neighborhood in $\mathfrak{g}_\mathbb{C}$ with $e^{z\,\mathrm{ad}\,X}.\overline{V} \subseteq U$ whenever $|z| \leq 1$. We claim that

$$\pi(\exp Y)e^{id\pi(X)}.v = e^{id\pi(X)}\gamma_{v,U}(e^{-i\,\mathrm{ad}\,X}.Y) \tag{A.3}$$

holds for all $Y \in V \cap \mathfrak{g}$. In fact, the mappings

$$\mathbb{C}^+ \to \mathcal{H}, \quad z \mapsto \pi(\exp Y)e^{zd\pi(X)}.v \quad \text{and} \quad z \mapsto e^{zd\pi(X)}\gamma_{v,U}(e^{-z\,\mathrm{ad}\,X}.Y)$$

are both continuous and holomorphic on the open upper half plane. Moreover, for $z \in \mathbb{R}$, they coincide. Hence a similar argument as in the first part of the proof shows that both functions coincide, and for $z = i$ this proves the proposition. Now the analyticity of $\gamma_{v,U}$ implies that the mapping $G \to \mathcal{H}, g \mapsto \pi(g)e^{id\pi(X)}.v$ is analytic on V and thus by equivariance analytic on G. This means that $e^{id\pi(X)}.v \in \mathcal{H}^\omega$.

To see that $e^{id\pi(X)}$ induces a continuous map $\mathcal{H}^\omega \to \mathcal{H}^\omega$, we first note that we only have to show that for each U the corresponding map $\mathcal{H}_U \to \mathcal{H}^\omega$ is continuous (Lemma A.2).

In view of the Identity Theorem for Holomorphic Functions, (A.3) shows that

$$\gamma_{e^{id\pi(X)}.v,V}(Y) = e^{id\pi(X)}\gamma_{v,U}(e^{-i\,\mathrm{ad}\,X}.Y)$$

for all $Y \in V$. This proves that $e^{id\pi(X)}.\mathcal{H}_U \subseteq \mathcal{H}_V$ and that $v_n \to v$ in $\mathcal{H}_U$ implies $e^{id\pi(X)}.v_n \to e^{id\pi(X)}.v$ in $\mathcal{H}_V$. Since the mapping $\mathcal{H}_V \to \mathcal{H}^\omega$ is continuous by definition of the topology on $\mathcal{H}^\omega$, this completes the proof. ∎

Lemma A.4. (a) *For each $g \in G$ the map $\pi(g)\colon \mathcal{H}^\omega \to \mathcal{H}^\omega$ is continuous and the orbit maps $G \to \mathcal{H}^\omega, g \mapsto g.v$ for $v \in \mathcal{H}^\omega$ are continuous.*

(b) *For each $v \in \mathcal{H}^\omega$ the limit*

$$d\pi(X).v = \lim_{t\to 0} \frac{1}{t}\big(\pi(\exp tX).v - v\big)$$

exists in $\mathcal{H}^\omega$.

(c) *For each $X \in \mathfrak{g}_\mathbb{C}$ the operator $d\pi(X)\colon \mathcal{H}^\omega \to \mathcal{H}^\omega$ is continuous.*

Proof. (a) We have to show that for each U the map $\pi(g)\colon \mathcal{H}_U \to \mathcal{H}^\omega$ is continuous whenever U is sufficiently small. Let $U \subseteq \mathfrak{g}_{\mathbb{C}}$ be a convex 0-neighborhood on which the Campbell-Hausdorff series $X * Y = X + Y + \frac{1}{2}[X,Y] + \dots$ defines a holomorphic multiplication $U \times U \to \mathfrak{g}_{\mathbb{C}}$. For $g = \exp(Y)$, $Y \in U \cap \mathfrak{g}$, $v \in \mathcal{H}_{U*U}$ we then have

$$\begin{aligned}\gamma_{\pi(g).v,U}(X) &= \pi(\exp X)\pi(g).v = \pi(\exp X)\pi(\exp Y).v\\ &= \pi(\exp X * Y).v = \gamma_{v,U*U}(X * Y)\end{aligned}$$

for all $X \in U$. This shows that $\pi(g)\colon \mathcal{H}_{U*U} \subseteq \mathcal{H}_U$ and that the corresponding linear map $\mathcal{H}_{U*U} \to \mathcal{H}_U$ is continuous. Thus $\pi(g)\colon \mathcal{H}_{U*U} \to \mathcal{H}^\omega$ is continuous, and hence $\pi(g)\colon \mathcal{H}^\omega \to \mathcal{H}^\omega$ is continuous because the 0-neighborhoods of the form $U * U$ form a 0-neighborhood basis. Since $G = \langle \exp(U \cap \mathfrak{g})\rangle$, we see that for each $g \in G$ the corresponding map $\mathcal{H}^\omega \to \mathcal{H}^\omega$ is continuous.

To show that the orbit maps $G \to \mathcal{H}^\omega$ are continuous, in view of the first part, it suffices to prove that they are continuous in $\mathbf{1}$. For U as above, $v \in \mathcal{H}_{U*U}$, and $g = \exp Y \in \exp(U \cap \mathfrak{g})$ we have just seen that

$$\gamma_{\pi(g).v,U}(X) = \gamma_{v,U*U}(X * Y),$$

and this implies that $g_n \to \mathbf{1}$, i.e., $Y \to 0$ entails that $\pi(g_n).v \to v$ in $\mathcal{H}_{U*U}$, and hence in $\mathcal{H}^\omega$.

(b) Since the inclusions $\mathcal{H}_U \to \mathcal{H}^\omega$ are continuous, it suffices to show that for each $v \in \mathcal{H}_{U*U}$ the relation

$$d\pi(X).v = \lim_{t\to 0} \frac{1}{t}\big(\pi(\exp tX).v - v\big) \tag{A.4}$$

holds with respect to the topology in $\mathcal{H}_U$. We have seen in (a) that for $tX \in U$ we have $\pi(\exp tX).\mathcal{H}_{U*U} \subseteq \mathcal{H}_U$. Furthermore

$$\gamma_{\pi(\exp tX).v,U}(Y) = \gamma_{v,U*U}(Y * tX)$$

gives

$$\begin{aligned}&\frac{1}{t}\big(\gamma_{\pi(\exp tX).v,U}(Y) - \gamma_{v,U}(Y)\big)\\ &\qquad = \frac{1}{t}\big(\gamma_{v,U}(Y * tX) - \gamma_{v,U}(Y)\big) \to d\gamma_{v,U}(Y)d\lambda_Y^*(0).X,\end{aligned}$$

where $\lambda_Y^*(X) = Y * X$ and the limit exists uniformly on compact subsets of U. This proves that the limit exists in $\mathcal{H}_U$. Finally the continuity and the injectivity of the mapping $\mathcal{H}^\omega \to \mathcal{H}$ shows that (A.4) holds in $\mathcal{H}_U$, hence in $\mathcal{H}^\omega$.

(c) First we show that for each open connected 0-neighborhood $U \subseteq \mathfrak{g}_{\mathbb{C}}$ we have $d\pi(X).\mathcal{H}_U \subseteq \mathcal{H}_U$. Let $v \in \mathcal{H}_U$ and $\gamma_{v,U} : U \to \mathcal{H}$ be the corresponding holomorphic map. We put

$$\eta_{v,U}(Y) := d\pi(e^{\operatorname{ad} Y}.X)\gamma_{v,U}(Y)$$

and note that since $\gamma_{v,U}(U) \subseteq \mathcal{H}^\infty$, the right hand side is well defined. Moreover, for each $Z \in \mathfrak{g}_{\mathbb{C}}$ the mapping $Y \mapsto d\pi(Z).\gamma_{v,U}(Y)$ is holomorphic because it arises by applying a right invariant holomorphic vector field on U, endowed with the canonical local group structure, to a holomorphic function. Now the fact that $\mathfrak{g}$ is finite dimensional easily implies that $\eta_{v,U}$ is a holomorphic map.

For $Y \in \mathfrak{g} \cap U$ we have

$$\begin{aligned}\eta_{v,U}(Y) &= d\pi(e^{\operatorname{ad} Y}.X)\gamma_{v,U}(Y) = \pi(\exp Y)d\pi(X)\pi(\exp -Y)\pi(\exp Y).v\\ &= \pi(\exp Y)d\pi(X).v = \gamma_{d\pi(X).v,U}(Y).\end{aligned}$$

This formula shows that $d\pi(X).v \in \mathcal{H}_U$ and that

$$\gamma_{d\pi(X).v,U}(Y) = d\pi(e^{\operatorname{ad} Y}.X)\gamma_{v,U}(Y)$$

for all $Y \in U$. Therefore $v_n \to v$ in $\mathcal{H}_U$ implies locally uniform convergence of $\gamma_{v_n,U}$ to $\gamma_{v,U}$, and hence locally uniform convergence of $\gamma_{d\pi(X).v_n,U}$ to $\gamma_{d\pi(X).v,U}$ because holomorphic vector fields yield continuous operators on spaces of holomorphic functions. ■

Proposition A.5. *Suppose that $S = \Gamma_G(W)$ is an open complex Ol'shanskiĭ semigroup, where $W \subseteq i\mathfrak{g}$ is a hyperbolic cone and that $(\pi, \mathcal{H})$ is a holomorphic representation of S corresponding to the unitary representation $(\pi, \mathcal{H})$ of G. Then the following assertions hold:*

(i) $\mathcal{H}^\omega = \operatorname{span}\big(\pi(S).\mathcal{H}\big)$.
(ii) *If $X \in W^0$, then $\mathcal{H}^\omega = \bigcup_{t>0} \pi(\operatorname{Exp} tX).\mathcal{H}$.*
(iii) *The topology on $\mathcal{H}^\omega$ coincides with the finest locally convex topology for which all the maps $\pi(s)\colon \mathcal{H} \to \mathcal{H}^\omega$ are continuous.*

Proof. (i), (ii) Since S is an open Ol'shanskiĭ semigroup and π is a holomorphic representation, the maps $S \to \mathcal{H}, s \mapsto \pi(s).v$ are holomorphic. So it is clear that $\pi(S).\mathcal{H} \subseteq \mathcal{H}^\omega$. Therefore it suffices to prove (ii).

So let $X \in W^0$ and suppose that $v \in \mathcal{H}^\omega$. Then there exists an open convex 0-neighborhood $U \subseteq \mathfrak{g}_\mathbb{C}$ and a complex analytic map

$$\gamma\colon U \to \mathcal{H} \quad \text{with} \quad \gamma(Y) = \sum_{n=0}^\infty \frac{1}{n!} d\pi(Y)^n.v$$

for all $Y \in U$. In view of the Identity Theorem for Holomorphic Functions, the relation $\gamma(Y) = \pi(\operatorname{Exp} Y).v$ for $Y \in U \cap (\mathfrak{g} + iW^0)$ follows from the holomorphy of the representation π.

We choose $t_0 > 0$ such that $-t_0X \in U$ and an open convex 0-neighborhood $V \subseteq U$ containing $-t_0X$ with $t_0X + V \subseteq U$. We claim that

$$\pi(\operatorname{Exp} t_0X)\gamma(zX) = \gamma\big((z + t_0)X\big)$$

whenever $zX \in V$. In fact, this holds for $\operatorname{Re} z > 0$ because $\gamma\big((z + t_0)X\big) = \pi\big(\operatorname{Exp}(z + t_0)X\big).v$ in this case, and π is a representation. Hence the Identity Theorem for Holomorphic Functions proves the claim. For $z = -t_0$ we obtain in particular

$$v = \gamma(0) = \pi(\operatorname{Exp} t_0X)\gamma(-t_0X) \in \pi(\operatorname{Exp} t_0X).\mathcal{H}.$$

This completes the proof of (i) and (ii).

(iii) We write $\mathcal{H}^\omega_a$ for the space $\mathcal{H}^\omega$ endowed with the topology defined under (iii). First we show that the map $\mathcal{H}^\omega \to \mathcal{H}^\omega_a$ is continuous, i.e., that for each open connected 0-neighborhood U the map $\mathcal{H}_U \to \mathcal{H}^\omega_a$ is continuous. Pick $X \in W^0$ such that $-X \in U$. Then we have seen in the proof of (ii) that $\pi(\operatorname{Exp} X).\gamma_{v,U}(-X) = v$. Hence $\mathcal{H}_U \subseteq \pi(\operatorname{Exp} X).\mathcal{H}$ and therefore $\gamma_{v,U}(-X) = \pi(\operatorname{Exp} X)^{-1}.v$. If $v_n \to v$ in $\mathcal{H}_U$, then $\gamma_{v_n,U}(-X) \to \gamma_{v,U}(-X)$, and hence $\pi(\operatorname{Exp} X)^{-1}.v_n \to \pi(\operatorname{Exp} X)^{-1}.v$. This proves continuity of the map $\mathcal{H}^\omega \to \mathcal{H}^\omega_a$.

To see that the map in the other direction is also continuous, fix $s \in S^0$. We have to show that the map $\pi(s)\colon \mathcal{H} \to \mathcal{H}^\omega$ is continuous. Let $U \subseteq \mathfrak{g}_\mathbb{C}$ be an open connected 0-neighborhood such that $U \to S^0, Y \mapsto \mathrm{Exp}(Y)s$ is biholomorphic onto an open subset of S^0. For $v \in \mathcal{H}$ we consider the map

$$\gamma_{\pi(s).v,U} \colon U \to \mathcal{H}, \quad Y \mapsto \pi(\mathrm{Exp}\, Ys).v \tag{A.5}$$

which is holomorphic and satisfies

$$\gamma_{\pi(s).v,U}(Y) = \pi(\exp Y)\pi(s).v$$

for all $Y \in U \cap \mathfrak{g}$. The existence of this map proves that $\pi(s).\mathcal{H} \subseteq \mathcal{H}_U$. Now the local boundedness of the representation of S further shows that if $v_n \to v$, then, in view of (A.5), $\gamma_{\pi(s).v_n,U}$ converges uniformly on each compact subset of U to $\gamma_{\pi(s).v,U}$. This means that the map $\pi(s) \colon \mathcal{H} \to \mathcal{H}_U$ is continuous. ■

We note that in the situation of Proposition A.5 the family of subspaces $\pi(s).\mathcal{H}$, $s \in S$, is directed. In fact, for $s_1, s_2 \in S$ there exists $s_3 \in S$ with $s_1, s_2 \in s_3S$. Then clearly $\pi(s_1).\mathcal{H} \cup \pi(s_2).\mathcal{H} \subseteq \pi(s_3).\mathcal{H}$.

Recall the space $\mathcal{H}^{-\omega}$ of antilinear continuous functionals on $\mathcal{H}^\omega$. In the situation of Proposition A.5 a linear functional f on $\mathcal{H}^\omega$ lies in $\mathcal{H}^{-\omega}$ if and only if $s.f = f \circ \pi(s^*) \colon \mathcal{H} \to \mathbb{C}$ is continuous for all $s \in S$. This means that $s.f \in \mathcal{H}$ for all $s \in S$. Thus we can describe $\mathcal{H}^{-\omega}$ as the space

$$\mathcal{H}^{-\omega} = \bigcap_{s \in S} \pi(s)^{-1}.\mathcal{H}.$$

In this sense the operators in $\pi(S)$ act as regularizing operators on $\mathcal{H}^{-\omega}$.

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