

INTERTWINING OPERATORS INTO COHOMOLOGY REPRESENTATIONS FOR SEMISIMPLE LIE GROUPS II

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ABSTRACT. One approach to constructing unitary representations for semisimple Lie groups utilizes analytic cohomology on open orbits of generalized flag manifolds. This work gives explicit formulas for harmonic cocycles associated to certain holomorphic homogeneous vector bundles, extending previous results of the author (*Intertwining operators into cohomology representations for semisimple Lie groups*, J. Funct. Anal. **151** (1997), 138–165). The key step shows that holomorphic discrete series representations and their limits are well-behaved with respect to restriction to certain submanifolds.

1. INTRODUCTION

Two goals of the unitary dual problem for semisimple Lie groups are a classification and a unified construction for irreducible unitary representations. This problem is still open, but if one instead considers irreducible admissible representations, several solutions to the problem occur.

Here we are concerned with the Langlands classification and the Vogan-Zuckerman classification. The former relies on methods of real analysis, and the latter is a cohomological construction that produces representations using algebraic techniques. It is of interest to compare these two classifications.

Placement of Vogan-Zuckerman modules in the Langlands classification is a consequence of results in [Vo2] (see also Theorem 11.216 in [KV]), which are proved using homological methods. Results of Wong ([Wo]) allow for an analytic interpretation of this result. One can consider using an intertwining operator from a given nonunitary principal series representation into the Dolbeault cohomology representation associated to the corresponding Vogan-Zuckerman module. Many $A^q(\lambda)$ cases are handled in this manner in [BKZ] and [Ba], and [Do] generalizes the former in some $\mathcal{R}^S(W)$ cases where W is a nonunitary principal series representation.

One consequence of constructing such operators is the production of strongly harmonic forms. Harmonic forms have been useful for constructing unitary representations; notable instances occur in [S2], [RSW], [Zi], and [BZ].

In this paper we construct intertwining operators as above. The domain for such operators will be nonunitary principal series representations induced from irreducible unitary highest weight representations.

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Constructions deviate slightly from [Do]. To use the usual formulations of the Borel-Weil-Bott Theorem and the construction of holomorphic discrete series representations, antiholomorphic tangent spaces are constructed with respect to negative roots; [Do] uses the set of positive roots.

Let G be a linear connected semisimple Lie group with maximal compact subgroup K . Associated to $Lie(K) = \mathfrak{k}_0$ is the Cartan involution θ and Cartan decomposition

$$(1.1) \quad \mathfrak{g}_0 = \mathfrak{k}_0 \oplus \mathfrak{p}_0.$$

Real Lie algebras are denoted by the German letter with the subscript 0. Their complexifications are denoted by dropping the subscript. Representations for real Lie algebras are extended to the complexification complex linearly. Conjugation in \mathfrak{g} with respect to \mathfrak{g}_0 will be denoted by bar.

Let $\mathfrak{h}_0 = \mathfrak{t}_0 \oplus \mathfrak{a}_0$ be a θ -stable Cartan subalgebra of \mathfrak{g}_0 with associated decomposition according to (1.1). This subalgebra will vary in later sections. Let $H = TA$ be the associated analytic subgroup.

Linearity assumptions are necessary for invoking results from [Wo] and for constructing holomorphic discrete series representations. Otherwise one need only assume G be semisimple with finite center.

2. DOLBEAULT COHOMOLOGY AND HARMONIC FORMS

Let X be a nonzero element in \mathfrak{t}_0 , and let $L = Z_G(X)$. Then L is connected. We assume also that \mathfrak{a}_0 is nonzero unless otherwise specified.

Roots in $\Delta(\mathfrak{g}, \mathfrak{h})$ take real values on $\mathfrak{h}_\mathbb{R} = i\mathfrak{t}_0 \oplus \mathfrak{a}_0$. Positivity is determined lexicographically from a choice of ordered basis for $\mathfrak{h}_\mathbb{R}$. The ordering is determined by choosing iX as the first element, choosing the following elements such that they form a basis with iX for $i\mathfrak{t}_0$, and choosing the remaining basis elements from \mathfrak{a}_0 .

Form the subalgebra \mathfrak{u} constructed from the root spaces associated to roots α such that $\alpha(iX) < 0$ and let \mathfrak{q} be the θ -stable parabolic subalgebra of \mathfrak{g} with Levi decomposition

$$(2.1) \quad \mathfrak{q} = \mathfrak{l} \oplus \mathfrak{u}.$$

Let $G^\mathbb{C}$ be the complexification of the adjoint group of G and let Q be the analytic subgroup of $G^\mathbb{C}$ with Lie algebra \mathfrak{q} . Then the natural map

$$(2.2) \quad G/L \hookrightarrow G^\mathbb{C}/Q$$

is an open inclusion. The space of antiholomorphic tangent vectors at the identity coset eQ in $G^\mathbb{C}/Q$ can be naturally identified with \mathfrak{u} , and the above inclusion gives a choice of complex structure for G/L .

For more details on holomorphic vector bundles, the reader is referred to [K5]. Our primary interest lies in defining sections and cohomology in terms of functions of G .

Let (π_L, W) be a Frechét representation of L . As per [K5] and [TW], we form the associated homogeneous holomorphic vector bundle \mathcal{L}_W over G/L .

Smooth sections of this bundle can be interpreted as smooth functions

$$f : G \rightarrow W$$

such that

$$(2.3) \quad f(gl) = \pi_L(l)^{-1}f(g);$$

the space of all such functions is denoted $C^\infty(G/L, W)$.

Smooth $(0, k)$ -forms can also be interpreted as smooth functions

$$(2.4) \quad \omega : G \rightarrow W \otimes \wedge^k \mathfrak{u}^*$$

such that

$$\omega(gl) = (\pi_L \otimes Ad)^{-1}(l)\omega(g);$$

the space of all such forms is denoted $C^{0,k}(G/L, W)$.

Associated to \mathcal{L}_W is the Cauchy-Riemann operator in degree $(0, k)$

$$(2.5) \quad \bar{\partial}_k : C^{0,k}(G/L, W) \rightarrow C^{0,k+1}(G/L, W),$$

which has the property that

$$(2.6) \quad \bar{\partial}_k \circ \bar{\partial}_{k-1} = 0.$$

Define the $(0, k)$ -th Dolbeault cohomology group as

$$(2.7) \quad H_{\bar{\partial}}^{0,k}(G/L, W) = \text{Ker } \bar{\partial}_k / \text{Im } \bar{\partial}_{k-1}.$$

Let $\{X_\alpha\}$ be a basis for \mathfrak{u} in terms of root vectors, and let $\{\omega_\alpha\}$ be the corresponding dual basis for \mathfrak{u}^* . A formula for $\bar{\partial}$ (analogous to formula (1.6b) in [GS]) is given by

$$(2.8) \quad \bar{\partial}(f \cdot \omega) = \sum_{\alpha \in \Delta(\mathfrak{u})} X_\alpha f \cdot \omega_\alpha \wedge \omega + \frac{1}{2} \sum_{\alpha \in \Delta(\mathfrak{u})} f \cdot \omega_\alpha \wedge Ad(X_\alpha)\omega,$$

where $f \in C^\infty(G/L, W)$ and $\omega \in \wedge^k \mathfrak{u}^*$.

There exists a corresponding formal adjoint; the operator which acts on the space $C^{0,k}(G/L, W)$ is not the formal adjoint to $\bar{\partial}$ as above. We refer the reader to section 9 of [Do] for more details. A formula for the operator

$$(2.9) \quad \bar{\partial}^* : C^{0,k}(G/L, W) \rightarrow C^{0,k-1}(G/L, W)$$

is given by

$$(2.10) \quad \bar{\partial}^*(f \cdot \omega) = \sum_{\alpha \in \Delta(\mathfrak{u})} c_\alpha \bar{X}_\alpha f \cdot i(\theta X_\alpha)\omega + \frac{1}{2} \sum_{\alpha \in \Delta(\mathfrak{u})} f \cdot c'_\alpha i(\theta X_\alpha)Ad(X_\alpha)^*\omega,$$

where c_α, c'_α are constants that depend on the choice of basis for \mathfrak{u} and $i(\cdot)$ denotes interior product.

Define the space of strongly harmonic $(0, k)$ -forms

$$(2.11) \quad \mathcal{H}^{0,k}(G/L, W) = \text{Ker } \bar{\partial}_k \cap \text{Ker } \bar{\partial}_k^*.$$

Because every strongly harmonic form is a cocycle, there is a natural map

$$(2.12) \quad Q : \mathcal{H}^{0,k}(G/L, W) \rightarrow H_{\bar{\partial}}^{0,k}(G/L, W).$$

When G is compact and W has finite dimension, the Kodaira-Hodge theorem (Theorem 3.19 in [Kd]) states that Q is an isomorphism. In the situation for constructing discrete series representations (see [S1],[S2], and [AR]) and with the domain restricted to square-integrable harmonic forms, Q is a continuous inclusion with dense image and is an isomorphism at the K -finite level. In general (as in [Zi] or [BZ]), one hopes for at least surjectivity of Q at the K -finite level.

One serious issue concerning Dolbeault cohomology representations is a well-defined topology. This problem, known as the Maximal Globalization Conjecture, was solved by Wong in [Wo]. Among other important facts, this theorem states

that if W is the maximal globalization of an $(\mathfrak{l}, L \cap K)$ -module of finite length, then $H_{\bar{\delta}}^{0,k}(G/L, W)$ is the maximal globalization of the cohomologically induced (\mathfrak{g}, K) -module $\mathcal{R}^k(W)$. The difficult part of the problem lies in showing that $\text{Im } \bar{\partial}_{k-1}$ is closed. The notion of a maximal globalization was introduced in [S3].

3. THE BOREL-WEIL-BOTT THEOREM

We recall how to construct intertwining operators in the compact case; the relevant cohomological theory is the Borel-Weil-Bott Theorem. Here one can construct harmonic forms using irreducible representations obtained through the theory of Cartan and Weyl.

Let K be a connected compact Lie group with complexification $K^{\mathbb{C}}$ and T a maximal torus. Let

$$(3.1) \quad \mathfrak{k} = \mathfrak{t} \oplus \sum_{\alpha \in \Delta} \mathfrak{k}_{\alpha}$$

be the root space decomposition and let Δ^+ be a choice of positive roots. Let

$$(3.2) \quad \mathfrak{b} = \mathfrak{t} \oplus \sum_{\alpha < 0} \mathfrak{k}_{\alpha}$$

be the root space decomposition of the Borel subalgebra associated to the negative roots and with nilradical \mathfrak{n}' . Associated to \mathfrak{b} is the subgroup $B \subset K^{\mathbb{C}}$. Here

$$(3.3) \quad K/T \cong K^{\mathbb{C}}/B$$

as smooth manifolds. Let $W_K = N_K(T)/T$ be the Weyl group associated to K and T . Suppose $(\chi_{\lambda}, \mathbb{C}_{\lambda})$ is a character of T with differential λ , and let

$$\delta = \frac{1}{2} \sum_{\alpha \in \Delta^+} \alpha.$$

Borel-Weil-Bott Theorem.

- (1) If $\langle \lambda + \delta, \alpha \rangle = 0$ for some $\alpha \in \Delta$, then $H_{\bar{\delta}}^{0,k}(K/T, \mathbb{C}_{\lambda}) = 0$ for all k .
- (2) If not, there exists a unique $w \in W_K$ such that

$$(3.4) \quad \mu = w(\lambda + \delta) - \delta$$

is dominant with respect to Δ^+ . Let

$$(3.5) \quad l(w) = \#\{\alpha \in \Delta^+ \mid w^{-1}\alpha < 0\}.$$

Then

$$(3.6) \quad H_{\bar{\delta}}^{0,k}(K/T, \mathbb{C}_{\lambda}) = \begin{cases} 0, & k \neq l(w), \\ V^{\mu}, & k = l(w), \end{cases}$$

where V^{μ} is the K -representation with highest weight μ .

Different proofs can be found in [Bo], [Ks], or [GS].

To construct the harmonic forms associated to the above cohomology groups, we work backwards, following section 6 of [GS]. The construction assigns a harmonic representative of a given \mathfrak{n} -cohomology class (explicitly realized in Corollary 5.15 of [Ks]) to a space of harmonic forms. A harmonic representative for $H^{l(w)}(\mathfrak{n}, (V^{\mu})^*)_{-\lambda}$ is evident when one deletes the term $\sigma^{-1}(k)v$ from (3.11) below.

Fix a dominant weight μ and associated irreducible K -representation (σ, V^μ) . Also fix $w \in W_K$. If $\mu = w(\lambda + \delta) - \delta$, then $\lambda = w^{-1}\mu + w^{-1}\delta - \delta$. Set

$$(3.7) \quad \Delta^+(w^{-1}) = \{\alpha \in \Delta^+ \mid w\alpha < 0\}$$

and

$$(3.8) \quad \omega_{w^{-1}} = \bigwedge_{\alpha \in \Delta^+(w^{-1})} \omega_{-\alpha}.$$

Define the intertwining operator

$$(3.9) \quad S : V^\mu \rightarrow \mathcal{H}^{0,l(w)}(K/T, \mathbb{C}_\lambda)$$

by

$$(3.10) \quad v \mapsto f_v$$

where

$$(3.11) \quad f_v(k) = P_\mu(\sigma(\tilde{w})\sigma^{-1}(k)v) \otimes \omega_{w^{-1}}.$$

Here P_μ is the T -equivariant projection onto the μ -weight space, and \tilde{w} is a representative in $N_K(T)$ for w .

Several facts need to be checked. First one must verify that f_v actually lies in the space of cochains. The nontrivial step in showing this is

$$(3.12) \quad Ad(t)\omega_{w^{-1}} = \chi_{w^{-1}\delta-\delta}(t)^{-1}\omega_{w^{-1}},$$

where χ_β denotes the character of T with differential β ; this fact occurs as Proposition 3.19 in [K3].

Next one needs to verify the strongly harmonic property. The cocycle property follows from

$$(3.13) \quad P_\mu(\sigma(X_{-w\alpha})V^\mu) = 0 \text{ if } \alpha > 0 \text{ and } w\alpha > 0, \text{ and}$$

$$(3.14) \quad Ad(X_\alpha)\omega_\beta = c_{\alpha\beta}\omega_{\beta-\alpha} \text{ if } \beta - \alpha \text{ is a root for } \mathfrak{n}'.$$

All other terms vanish from wedge products.

Vanishing under $\bar{\partial}^*$ follows from

$$(3.15) \quad P_\mu(\sigma(X_{w\alpha})V^\mu) = 0 \text{ if } \alpha > 0 \text{ and } w\alpha < 0, \text{ and}$$

$$(3.16) \quad Ad(X_\alpha)^*\omega_\beta = c'_{\alpha\beta}\omega_{\beta+\alpha} \text{ if } \beta + \alpha \text{ is a root for } \mathfrak{n}'.$$

All other terms vanish from interior products.

Finally to see that S is an isomorphism, the Kodaira-Hodge theorem implies that nonzero harmonic forms are nonzero in cohomology. If ϕ is a nonzero μ -weight vector, evaluate $f_{\sigma(\tilde{w})^{-1}\phi}$ at the identity. The Borel-Weil-Bott Theorem now implies that S is an isomorphism.

We consider the case where T is replaced with L as in section 2. The Borel-Weil-Bott Theorem in this case is formulated in section 6 of [GS] or Corollary 4.160 in [KV]. We note that the fiber of the bundle is allowed to be finite-dimensional irreducible.

We choose the parabolic subgroup $\mathfrak{q} = \mathfrak{l} \oplus \mathfrak{u}$ such that

$$(3.17) \quad \mathfrak{b} \subset \mathfrak{q}.$$

The cases of interest are those in the bottom and top degrees of cohomology; in the latter case, the element ω_s (given by any nonzero element in $\wedge^s \mathfrak{u}^*$, where $s = \dim \mathfrak{u}$) spans a one-dimensional L -representation.

We consider the former case, the Borel-Weil Theorem for K/L . Let V^μ be as before, let (τ, V^τ) be the irreducible L -representation generated by the highest weight

space, and let P_τ be the L -equivariant projection from V^μ to V^τ . Holomorphic sections (elements in $H_{\bar{\partial}}^{0,0}(K/L, V^\tau)$) are given by the formula

$$(3.18) \quad f_v(k) = P_\tau(\sigma(k)^{-1}v).$$

Proofs are given as before.

Consider the case of top degree, $H_{\bar{\partial}}^{0,s}(K/L, V^{\tau^*})$. Let (τ, V^τ) be the irreducible L -representation determined by the \mathfrak{u} -invariants in V^μ . Denote the associated L -equivariant projection by

$$(3.19) \quad P_\tau : V^\mu \rightarrow V^\tau.$$

The analog of (3.18) is given as

$$(3.20) \quad f_v(k) = P_\tau(\sigma^{-1}(k)v) \otimes \omega_s.$$

The cocycle property holds immediately, and the analogs of (3.15) and (3.16) yield vanishing under the adjoint. The fiber V^{τ^*} of the vector bundle has L -type

$$(3.21) \quad \tau^* = \tau \otimes \chi_{-2\delta(\bar{\mathfrak{u}})}.$$

4. NONUNITARY PRINCIPAL SERIES REPRESENTATIONS

The goal of this work is essentially a gross generalization of the Borel-Weil-Bott Theorem. The substitute for the Cartan-Weyl representation is a nonunitary principal series representation. These representations and their generalizations are enough to classify all irreducible admissible representations of G . The question of classifying irreducible unitary representations is still an open problem.

We outline the general construction. Let $\mathfrak{h}_0 = \mathfrak{t}_0 \oplus \mathfrak{a}_{\mathfrak{p},0}$ be a Cartan subalgebra of \mathfrak{g}_0 with the property that $\mathfrak{a}_{\mathfrak{p},0}$ is maximal abelian in \mathfrak{p}_0 . Let $M_{\mathfrak{p}} = Z_K(\mathfrak{a}_{\mathfrak{p},0})$; this subgroup is compact but not necessarily connected. We assume $M_{\mathfrak{p}}$ is connected for simplicity.

Form the set of restricted roots $\Sigma(\mathfrak{g}_0, \mathfrak{a}_{\mathfrak{p},0})$. A choice of positive system Σ^+ yields subalgebras

$$(4.1) \quad \mathfrak{n}_{\mathfrak{p},0} = \sum_{\alpha \in \Sigma^+} \mathfrak{g}_\alpha$$

and

$$(4.2) \quad \mathfrak{n}_{\mathfrak{p},0}^- = \theta \mathfrak{n}_{\mathfrak{p},0}$$

with associated subgroups $N_{\mathfrak{p}}$ and $N_{\mathfrak{p}}^-$. Now $M_{\mathfrak{p}}A_{\mathfrak{p}}N_{\mathfrak{p}}$ is a real parabolic subgroup of G (given in terms of its Langlands decomposition).

We construct representations of G induced from $M_{\mathfrak{p}}A_{\mathfrak{p}}N_{\mathfrak{p}}$. Let (σ, V^σ) be an irreducible unitary representation of $M_{\mathfrak{p}}$, $\nu \in \mathfrak{a}_{\mathfrak{p}}^*$, and

$$(4.3) \quad \rho_G = \frac{1}{2} \sum_{\alpha \in \Sigma^+} \alpha.$$

Form the space $C^\infty(G/M_{\mathfrak{p}}A_{\mathfrak{p}}N_{\mathfrak{p}}, \sigma \otimes e^\nu \otimes 1)$ defined by

$$(4.4) \quad \{\text{smooth } f : G \rightarrow V^\sigma \mid f(gman) = e^{-(\nu+\rho_G) \log a} \sigma^{-1}(m)f(g)\}$$

where $m \in M_{\mathfrak{p}}, a \in A_{\mathfrak{p}}$ and $n \in N_{\mathfrak{p}}$. The G -action is given by left translation. This space can be completed with respect to the norm

$$(4.5) \quad \|f\|_G^2 = \int_K |f(k)|_{\sigma}^2 dk.$$

Two important theorems pertaining to such representations are the Subrepresentation Theorem ([CM]) and the Langlands classification ([La], [KZ]; see also Theorem 14.92 of [K2]) as revised by Knapp and Zuckerman. The subrepresentation theorem states that every irreducible admissible representation of G is infinitesimally equivalent to a subrepresentation of some nonunitary principal series representation. To get a parameterization of such representations, one needs the revised Langlands classification. Here one considers a construction analogous to the one above by replacing $M_{\mathfrak{p}}A_{\mathfrak{p}}N_{\mathfrak{p}}$ with a cuspidal real parabolic subgroup MAN that contains $M_{\mathfrak{p}}A_{\mathfrak{p}}N_{\mathfrak{p}}$ (possibly G itself) and letting σ be a discrete series representation or limit of such. When ν is suitably restricted, this representation will have a unique irreducible quotient, and these account for all irreducible admissible representations in a precise manner.

5. THE INTERTWINING OPERATOR IN THE CASE OF COMPACT M

We recall the operator constructed in [Do] and note some important points. Let (σ, V^{σ}) be an irreducible unitary representation of $M_{\mathfrak{p}}$. The fiber W of the bundle \mathcal{L}_W is a nonunitary principal series representation of L induced from the real parabolic subgroup $(L \cap M_{\mathfrak{p}})A_{\mathfrak{p}}(L \cap N_{\mathfrak{p}})$; the parameter τ' for the $L \cap M_{\mathfrak{p}}$ -representation comes from (3.21) and (5.4). We construct an operator

$$(5.1) \quad S : C^{\infty}(G/M_{\mathfrak{p}}A_{\mathfrak{p}}N_{\mathfrak{p}}, \sigma \otimes e^{\nu} \otimes 1) \rightarrow C^{0,s}(G/L, W)$$

where $s = \dim_{\mathbb{C}} \mathfrak{u} \cap \mathfrak{k}$. Here

$$W = C^{\infty}(L/(L \cap M_{\mathfrak{p}})A_{\mathfrak{p}}(L \cap N_{\mathfrak{p}}), \tau' \otimes e^{\nu} \otimes 1);$$

we leave it to the reader to compute W in sections 8 and 9.

The intertwining operator S in (3.9) works because the projection operator and choice of form are chosen compatibly. Such a philosophy persists throughout the remainder.

The choice of form comes from [BKZ]. Let

$$(5.2) \quad \Delta_s = \Delta(\mathfrak{u} \cap \mathfrak{m}_{\mathfrak{p}} \cap \mathfrak{k}) \cup \Delta(\mathfrak{u} \cap \mathfrak{n}_{\mathfrak{p}}^{-}).$$

(The “ \mathfrak{k} ” is redundant here; it becomes necessary later.) Define

$$(5.3) \quad \omega_s = \bigwedge_{\alpha \in \Delta_s} \omega_{\alpha}.$$

The construction depends on the following two properties of ω_s :

$$(5.4) \quad \omega_s \text{ is a one-dimensional } (L \cap M_{\mathfrak{p}})A_{\mathfrak{p}}(L \cap N_{\mathfrak{p}})\text{-representation with } \mathfrak{a}_{\mathfrak{p}}\text{-weight } \rho_G - \rho_L \text{ and } \mathfrak{k}\text{-weight } -\delta(\mathfrak{u}) - \delta(\mathfrak{u} \cap \mathfrak{m}), \text{ and}$$

$$(5.5) \quad \omega_s \text{ vanishes under } Ad(X) \text{ when } X \in \mathfrak{u} \cap \mathfrak{m}_{\mathfrak{p}} \oplus \mathfrak{u} \cap \mathfrak{n}_{\mathfrak{p}}.$$

We define the associated projection. Let (τ, V^{τ}) be the irreducible $L \cap M_{\mathfrak{p}}$ -representation of $\mathfrak{u} \cap \mathfrak{m}_{\mathfrak{p}}$ -invariants in V^{σ} . Let

$$(5.6) \quad P_{\tau} : V^{\sigma} \rightarrow V^{\tau}$$

be the $L \cap M_{\mathfrak{p}}$ -equivariant projection.

With these definitions in place, define

$$(5.7) \quad [\mathcal{S}f(x)](l) = P_\tau(f(xl)) \otimes Ad(l)\omega_s.$$

This operator is constructed from elementary operations in representation theory; essentially we are combining the Borel-Weil-Bott construction with induction techniques. An explicit decomposition of \mathcal{S} can be found in [Do]. The operator \mathcal{S} has its origins in the “heuristic principle” of [K4].

When ν is sufficiently dominant, we can further apply the Langlands quotient operator to elements of W ; we note that in general these elements are not $L \cap M_{\mathfrak{p}} \cap K$ -finite, but they are always smooth. Details for such operators are in [VW].

6. AN EXAMPLE

Let $G = SO(4, 1)_e$, the identity component of $SO(4, 1)$. The latter group is the subgroup of $SL(5, \mathbb{R})$ whose elements preserve the form

$$(6.1) \quad \langle x, y \rangle_{4,1} = x_1 \cdot y_1 + \cdots + x_4 \cdot y_4 - x_5 \cdot y_5.$$

Let E_{ij} be the matrix with all entries zero except for a 1 in the (i, j) -th entry.

The Cartan subalgebra \mathfrak{h}_0 is spanned by the elements $X = E_{12} - E_{21}$ and $Y = E_{45} + E_{54}$. Let $\{e_1, \alpha\}$ be the dual basis for \mathfrak{h}^* corresponding to the basis $\{iX, Y\}$ for \mathfrak{h} . The set of roots is

$$\Delta(\mathfrak{g}, \mathfrak{h}) = \{\pm e_1 \pm \alpha, \pm e_1, \pm \alpha\}$$

and we choose for a positive system

$$\Delta^+(\mathfrak{g}, \mathfrak{h}) = \{e_1 \pm \alpha, e_1, \alpha\}.$$

Let L be the centralizer of X ; then

$$(6.2) \quad L = S^1 \times SO(2, 1)_e.$$

With the above choices, $\Delta(\mathfrak{l}) = \{\pm \alpha\}$ and $\Delta(\mathfrak{u}) = \{-e_1 \pm \alpha, -e_1\}$. The subgroup Q in $SO(5, \mathbb{C})$ is the stabilizer of the line spanned by $(1, i, 0, 0, 0)$. In fact, $G^{\mathbb{C}}/Q$ is the set of null lines in \mathbb{C}^5 with respect to the complexification of $\langle \cdot, \cdot \rangle_{4,1}$. Furthermore

$$(6.3) \quad M_{\mathfrak{p}} = SO(3),$$

$$(6.4) \quad L \cap M_{\mathfrak{p}} = S^1,$$

$$(6.5) \quad s = \dim \mathfrak{u} \cap \mathfrak{k} = 2,$$

$$(6.6) \quad N_{\mathfrak{p}} \text{ has dimension } 3, \text{ and}$$

$$(6.7) \quad L \cap N_{\mathfrak{p}} \text{ has dimension } 1.$$

Ignoring the S^1 factor, W is a nonunitary principal series representation for $PSL(2, \mathbb{R})$, which is well understood ([K1], [Wa]). In this example, many computations from [Do] are simplified since \mathfrak{u} is abelian.

7. PROPERTIES OF \mathcal{S}

In this section we summarize the main points of the proofs in [Do]. Many facts about the operator \mathcal{S} in (5.7) need to be verified.

The right $(L \cap M_{\mathfrak{p}})A_{\mathfrak{p}}(L \cap N_{\mathfrak{p}})$ -translation property at the L -variable and the right L -translation property at the G -variable are handled by Theorem 7.2 of [Do].

The proof of the harmonic property has much in common with the harmonic property in section 3. The proof occurs in Sections 8 and 9 of [Do]. We reproduce the main ideas here.

Let $f \in C^\infty(G/M_{\mathfrak{p}}A_{\mathfrak{p}}N_{\mathfrak{p}}, \sigma \otimes e^\nu \otimes 1)$. The cocycle property relies on two lemmas (with analogs for $\bar{\partial}^*$).

Lemma 7.1. *For $x \in G$ and $l \in L$,*

$$[\bar{\partial}|_{\tilde{x}=x}\mathcal{S}f(\tilde{x})](l) = (1 \otimes Ad(l))[\bar{\partial}|_{\tilde{x}=xl}\mathcal{S}f(\tilde{x})](e).$$

Lemma 7.2. *For $x \in G$, $[\bar{\partial}|_{\tilde{x}=x}\mathcal{S}f(\tilde{x})](e) = 0$.*

The first lemma follows by conjugating the \mathfrak{u} basis by l ; this works since $\bar{\partial}$ is invariant under complex changes of basis. The second lemma is calculus with root systems; terms not killed by the wedge product vanish by the right invariance of f under $N_{\mathfrak{p}}$ and by (5.5). The arguments for $\bar{\partial}^*$ follow from two similar lemmas, the second of which uses the analog of (3.15).

Here the harmonic property does not imply nonvanishing as in the Borel-Weil-Bott Theorem. Instead one uses a nonholomorphic Penrose transform, defined in [BKZ]; a general account of Penrose transforms in representation theory is given in [BE]. With V^μ a given minimal K -type of the right-hand side of (5.1), this map

$$(7.3) \quad \mathcal{P} : \ker \bar{\partial}_s \rightarrow C^\infty(G/K, V^\mu)$$

has the property that $\mathcal{P} \circ \bar{\partial}_{s-1} = 0$. Thus \mathcal{P} descends to $H_{\bar{\partial}}^{0,s}(G/L, W)$. One needs to find f such that

$$(7.4) \quad (\mathcal{P}\mathcal{S}f)(e) \neq 0;$$

such functions are given in [BKZ] and [Do]. The composition

$$(7.5) \quad C^\infty(G/M_{\mathfrak{p}}A_{\mathfrak{p}}N_{\mathfrak{p}}, \sigma \otimes e^\nu \otimes 1) \rightarrow H_{\bar{\partial}}^{0,s}(G/L, W) \rightarrow C^\infty(G/K, V^\mu)$$

is given by the formula

$$(7.6) \quad (\mathcal{P}\mathcal{S}f)(x) = \int_K \mu(k)P_\tau(f(xk)) dk.$$

8. A SECOND CASE - $L \cap M$ COMPACT

We consider the case where σ is an irreducible unitary highest weight module and $L \cap M$ is compact. General details for such representations occur in [EH]. Essentially all constructions from before work with changes as below.

Suppose \mathfrak{h}_0 is chosen such that \mathfrak{a}_0 is no longer maximal abelian in \mathfrak{p}_0 but \mathfrak{a}_0 lies in $\mathfrak{a}_{\mathfrak{p},0}$. Let

$$\mathfrak{m}_0 \oplus \mathfrak{a}_0 = Z_{\mathfrak{g}_0}(\mathfrak{a}_0)$$

be the orthogonal decomposition with respect to the Killing form. Since we still want M connected, we define M to be the analytic subgroup of G with Lie algebra \mathfrak{m}_0 . M is no longer compact, but it contains a compact Cartan subgroup. In fact T is such a subgroup. Let MAN be the analytic subgroup of G corresponding to the real parabolic subalgebra $\mathfrak{m}_0 \oplus \mathfrak{a}_0 \oplus \mathfrak{n}_0$ containing $\mathfrak{m}_{\mathfrak{p},0} \oplus \mathfrak{a}_{\mathfrak{p},0} \oplus \mathfrak{n}_{\mathfrak{p},0}$. We are interested in adjusting the definition of (4.4) for MAN . Reusing notation, let ρ_G stand for half the sum of roots of \mathfrak{n}_0 .

Form the set of roots $\Delta(\mathfrak{m}, \mathfrak{t})$. The choice of Cartan subalgebra \mathfrak{t} implies that every root is compact or noncompact; that is, the root space \mathfrak{m}_α satisfies either

$$(8.1) \quad \mathfrak{m}_\alpha \subset \mathfrak{m} \cap \mathfrak{k} \text{ or } \mathfrak{m}_\alpha \subset \mathfrak{m} \cap \mathfrak{p}.$$

There are several equivalent conditions necessary for the existence of irreducible unitary highest weight representations. We impose the condition that $\Delta(\mathfrak{m}, \mathfrak{t})$ have a good ordering; this means that there exists a choice of positive roots such that every positive noncompact root is greater than every compact root. Note that \mathfrak{u} is chosen such that

$$(8.2) \quad \Delta(\mathfrak{u} \cap \mathfrak{m}) \subset \Delta^-(\mathfrak{m}, \mathfrak{t}).$$

Another consequence of the good ordering is

Proposition 8.3 ([HC3]). *Fix a simple ideal \mathfrak{m}_i in \mathfrak{m} . Then $\mathfrak{m}_i \cap \mathfrak{p}$ decomposes into two irreducible $\mathfrak{m}_i \cap \mathfrak{k}$ -representations, each associated to either the set of positive or negative roots in $\Delta(\mathfrak{m}_i \cap \mathfrak{p})$. These representations are abelian subalgebras. Thus adjoining any choice of positive system for $\Delta(\mathfrak{m} \cap \mathfrak{k})$ to $\Delta^+(\mathfrak{m} \cap \mathfrak{p})$ yields a positive system for $\Delta(\mathfrak{m})$.*

The formula for \mathcal{S} in (5.7) is valid if we make the following changes:

- (8.4) (σ, V^σ) is an irreducible unitary highest weight representation for M , and
- (8.5) let (μ, V^μ) be the $M \cap K$ -representation generated by the highest weight vector with respect to $\Delta^+(\mathfrak{m})$, let (τ, V^τ) be the irreducible $L \cap M$ -subrepresentation in V^σ consisting of $\mathfrak{u} \cap \mathfrak{m} \cap \mathfrak{k}$ -invariants in V^μ , and as before, denote the $L \cap M$ -equivariant projection onto this space by P_τ .

In (8.5), V^τ is nonzero.

The proof of the harmonic property is essentially unchanged. Vanishing under $\bar{\delta}^*$ requires no changes; the “ \mathfrak{k} ” in (5.2) is no longer a redundancy.

Terms from $\mathfrak{u} \cap \mathfrak{m} \cap \mathfrak{p}$ enter into the $\bar{\delta}$ -computation in Lemma 7.2. Vanishing of these terms follows from two facts:

- (8.6) $P_\tau(\sigma(X_\alpha) \cdot V^\sigma) = 0$ for $\alpha \in \Delta(\mathfrak{u} \cap \mathfrak{m} \cap \mathfrak{p})$, and
- (8.7) $Ad(X)\omega_S = 0$ for $X \in \mathfrak{u} \cap \mathfrak{m} \cap \mathfrak{p}$.

The first statement is analogous to (3.13); the second follows since

$$(8.8) \quad Ad(X_\alpha)\omega_\beta = c_{\alpha\beta}\omega_{\beta-\alpha}$$

if $\beta - \alpha$ is a root. Since $\mathfrak{u} \cap \mathfrak{m}$ normalizes $\mathfrak{u} \cap \mathfrak{n}^-$, we need only consider when $\alpha \in \Delta(\mathfrak{u} \cap \mathfrak{m} \cap \mathfrak{p})$ and $\beta \in \Delta(\mathfrak{u} \cap \mathfrak{m} \cap \mathfrak{k})$. In this case the good ordering implies that when $\beta - \alpha$ is a root, it is not a root in $\Delta(\mathfrak{u} \cap \mathfrak{m})$.

9. A THIRD CASE - $L \cap M$ NONCOMPACT

We generalize the situation of section 8 to include noncompact $L \cap M$. Since our arguments rely on a concrete realization for σ , we restrict to the case of holomorphic discrete series representations. Similar arguments apply for limits of holomorphic discrete series representations [KO].

When $L \cap M$ is noncompact, the $L \cap M$ -cyclic span of ω_s is no longer one-dimensional. This difficulty forces the use of an explicit construction; one needs to give analytic formulas for $\bar{\mathfrak{u}}$ -cohomology.

Define

$$(9.1) \quad \omega_n = \bigwedge_{\alpha \in \Delta(\mathfrak{u} \cap \mathfrak{n}^-)} \omega_\alpha$$

and

$$(9.2) \quad \omega_c = \bigwedge_{\alpha \in \Delta(\mathfrak{u} \cap \mathfrak{m} \cap \mathfrak{k})} \omega_\alpha.$$

The $L \cap M$ -cyclic span of ω_c is not one-dimensional, but $\mathbb{C}\omega_n$ is a one-dimensional $(L \cap M)A(L \cap N)$ -representation with \mathfrak{t} -parameter $\delta(\mathfrak{u} \cap \mathfrak{m}) - \delta(\mathfrak{u})$ and \mathfrak{a} -parameter $\rho_G - \rho_L$.

Let w^{-1} be the unique element in $W_{M \cap K}$ such that

$$\Delta(\bar{\mathfrak{u}} \cap \mathfrak{m} \cap \mathfrak{k}) = \Delta^+(w^{-1});$$

we use the same notation for a fixed representative in $N_{M \cap K}(T)$. If w_0, w_L denote the longest elements in $W_{M \cap K}, W_{L \cap M \cap K}$ respectively, then $w^{-1} = w_L \cdot w_0$. While w preserves $\Delta^+(\mathfrak{m} \cap \mathfrak{p})$, sends $\Delta^+(\mathfrak{l} \cap \mathfrak{m} \cap \mathfrak{k})$ into $\Delta^+(\mathfrak{m} \cap \mathfrak{k})$, and sends $\Delta^+(\bar{\mathfrak{u}} \cap \mathfrak{m} \cap \mathfrak{k})$ into $\Delta^-(\mathfrak{m} \cap \mathfrak{k})$, it need not preserve $L \cap M$.

Define an intertwining operator

$$(9.3) \quad \mathcal{S} : C^\infty(G/MAN, \sigma \otimes e^\nu \otimes 1) \rightarrow C^{0,s}(G/L, W)$$

by

$$(9.4) \quad [\mathcal{S}f(x)(l)](m) = r(\sigma(w) \cdot f(xl))(Ad(w)m) \otimes Ad(lm)(\omega_c \wedge \omega_n),$$

where $x \in G, l \in L$ and $m \in L \cap M$. The operator r denotes restriction of an element in the holomorphic discrete series representation for M to an element of the holomorphic discrete series representation for $w(L \cap M)w^{-1}$. This operation will be made explicit in the next section.

Several items need to be checked concerning parameters. The right L -translation property at the G -variable is as before; the right $(L \cap M)A(L \cap N)$ -translation property at the L -variable uses properties of (9.1) and (9.2). We remark on the $L \cap M$ -representation used to define W in the next section.

The only changes needed in the proof of the harmonic property in section 8 are removal of the $L \cap M$ -variable, which is essentially the idea in Lemma 7.1, and the r -analog of (3.13), which occurs as (10.18).

10. RESTRICTING HOLOMORPHIC DISCRETE SERIES REPRESENTATIONS

We recall the construction of the holomorphic discrete series and produce an operator r which behaves the same as P_τ . The construction follows [K2, VI.4]. In (9.4), we implicitly consider elements in these representation spaces as sections over M/T . Results in this section are related to the general problem of restricting representations to subgroups, as found in [JV] or [Ko]. Arguments for holomorphic limits are essentially the same; we refer the reader to [KO] for definitions.

Suppose M is linear connected, $(\mathfrak{m} \cap \mathfrak{k}) \cap i(\mathfrak{m} \cap \mathfrak{p}) = 0$, and the roots of \mathfrak{m} are given a good ordering. Let

$$(10.1) \quad \mathfrak{b} = \mathfrak{t} \oplus \bar{\mathfrak{n}}'$$

be the Borel subalgebra of \mathfrak{m} associated to the negative roots. Let B and \bar{N}' be the analytic subgroups in $M^\mathbb{C}$ associated to \mathfrak{b} and $\bar{\mathfrak{n}}'$, respectively. Let χ_λ be a holomorphic one-dimensional representation of $T^\mathbb{C}$ with differential λ . Extend χ_λ to a holomorphic one-dimensional representation of B by defining χ_λ to be trivial on \bar{N}' . Let δ_M be half the sum of the positive roots for \mathfrak{m} .

Let $V_\lambda(M)$ be the space of functions $F : MB \rightarrow \mathbb{C}$ such that

- (10.2) F is holomorphic,
- (10.3) $F(xb) = \chi_\lambda(b)^{-1}F(x)$ for $x \in MB$ and $b \in B$, and
- (10.4) $\|F\|_M^2 = \int_M |F(m)|^2 dm < \infty$.

Denote the space of functions satisfying (10.2) and (10.3) by $\Gamma(MB, \lambda)$. This space is Fréchet with respect to the topology of uniform convergence on compact sets.

Let M act on $V_\lambda(M)$ and $\Gamma(MB, \lambda)$ by left translation:

$$(10.5) \quad (\sigma(m)f)(x) = f(m^{-1}x).$$

Theorem 10.6 ([HC2], [HC3]). *Suppose λ is dominant with respect to $\Delta^+(\mathfrak{m} \cap \mathfrak{k})$. Then $V_\lambda(M)$ is a Hilbert space and σ is a continuous unitary representation on it. If*

$$(10.7) \quad \langle \lambda + \delta_M, \alpha \rangle < 0 \text{ for each } \alpha \in \Delta^+(\mathfrak{m} \cap \mathfrak{p}),$$

then $V_\lambda(M)$ is nonzero, σ is irreducible, the irreducible representation of $M \cap K$ with highest weight λ occurs in $\sigma|_{M \cap K}$ and, the matrix coefficients of σ are square-integrable.

We can also construct such representations for $L \cap M$. In the above theorem, we replace M with $L \cap M$ and B with $L^\mathbb{C} \cap B$. The latter space is the analytic subgroup in $M^\mathbb{C}$ with Lie subalgebra $\mathfrak{l} \cap \mathfrak{b}$. Define $\delta_{L \cap M}$ analogously, and let $\delta_{\bar{\mathfrak{u}} \cap \mathfrak{m}}$ be half the sum of the roots for $\bar{\mathfrak{u}} \cap \mathfrak{m}$.

If λ yields a nonzero holomorphic discrete series representation for M , it does so for $L \cap M$. We need only check that (10.7) holds. If $\alpha \in \Delta^+(\mathfrak{l} \cap \mathfrak{m} \cap \mathfrak{p})$, then

$$(10.8) \quad \begin{aligned} \langle \lambda + \delta_M, \alpha \rangle &= \langle \lambda + \delta_{L \cap M} + \delta_{\bar{\mathfrak{u}} \cap \mathfrak{m}}, \alpha \rangle \\ &= \langle \lambda + \delta_{L \cap M}, \alpha \rangle. \end{aligned}$$

The last line follows because $\wedge^{top} \bar{\mathfrak{u}} \cap \mathfrak{m}$ is a one-dimensional $L \cap M$ -representation with differential $2\delta_{\bar{\mathfrak{u}} \cap \mathfrak{m}}$, and the assertion holds.

Theorem 10.9. *Restricting the domain of elements in $V_\lambda(M)$ to $(L \cap M)(L^\mathbb{C} \cap B)$ gives an $L \cap M$ -equivariant operator*

$$(10.10) \quad r : V_\lambda(M) \rightarrow V_\lambda(L \cap M)$$

with the property that

$$(10.11) \quad r(\sigma(X) \cdot f) = 0 \text{ for } X \in \mathfrak{u} \cap \mathfrak{m}.$$

Proof. Instead of using the explicit formula for elements of $V_\lambda(M)$ (as found in Lemma 6.7 of [K2]), we represent these functions as extended matrix coefficients. Let $(\pi, V, \langle \cdot, \cdot \rangle)$ be an abstract irreducible unitary representation of M that is unitarily equivalent to $V_\lambda(M)$; let ϕ be a nonzero highest weight vector for V . For $v \in V$, define

$$(10.12) \quad \Phi : V \rightarrow V_\lambda(M)$$

by

$$(10.13) \quad (\Phi v)(m) = \langle \pi^{-1}(m)v, \phi \rangle.$$

That Φv is well-defined on MB follows since $B = T^\mathbb{C} \bar{N}'$, $\bar{N}' = \exp \bar{\mathfrak{n}}'$, and ϕ is a highest weight vector. Properties (10.2)–(10.3) are easily checked; (10.4) follows

from the Schur orthogonality relations for discrete series. Theorem 10.6 and the orthogonality relations imply that Φ is a unitary equivalence up to scalar.

Let $X \in \mathfrak{u} \cap \mathfrak{m}$ and $l \in L \cap M$. Equation (10.11) follows from

$$\begin{aligned}
 (10.14) \quad r(\sigma(X)f)(l) &= \frac{d}{dt} f(\exp(-tX) \cdot l)|_{t=0} \\
 &= \frac{d}{dt} f(l \cdot \exp(-tAd(l)^{-1}X))|_{t=0} \\
 &= 0 \text{ by (10.3) and because } L \cap M \text{ normalizes } \mathfrak{u} \cap \mathfrak{m}.
 \end{aligned}$$

Note that $r \circ \sigma(X)$ is well-defined when f is not associated to a smooth vector of V .

Elements in the image of r satisfy the $L \cap M$ -analogs of (10.2) and (10.3) evidently. Square integrability of the image of r occupies the remainder of this section.

To see that r is continuous as a map from $V_\lambda(M)$ into $\Gamma((L \cap M)(L^\mathbb{C} \cap B), \lambda)$, we note two facts. It is well-known [S3] that the inclusion of $V_\lambda(M)$ into $\Gamma(MB, \lambda)$ is continuous with dense image. Since $(L \cap M)/T$ is closed in M/T , r is continuous as a map between the corresponding spaces of all holomorphic sections equipped with the topology of uniform convergence on compact supports. (Continuity with respect to the Hilbert space topology of $V_\lambda(L \cap M)$ occurs at the end of the proof.)

The following decompositions of $L \cap M$ -modules correspond:

$$\begin{aligned}
 (10.15) \quad V_\lambda(M) &= (\ker r) \oplus (\ker r)^\perp \\
 &= \sigma(\mathfrak{u} \cap \mathfrak{m}) \cdot V_\lambda(M) \oplus V_\lambda(M)^{\bar{\mathfrak{u}} \cap \mathfrak{m}}.
 \end{aligned}$$

The terms in the second line correspond to the closures of the decomposition at the $M \cap K$ -finite level, where the second term is the closure of the space of $\bar{\mathfrak{u}} \cap \mathfrak{m}$ -invariant $M \cap K$ -finite vectors. The $M \cap K$ -finite vectors are smooth and dense in $V_\lambda(M)$. This decomposition is the irreducible highest weight module analog of the well-known decomposition for finite dimensional representations.

Using the unitary structure and (10.11),

$$(10.16) \quad (\ker r)^\perp \subseteq V_\lambda(M)^{\bar{\mathfrak{u}} \cap \mathfrak{m}};$$

the reverse inclusion for the first terms follows from (10.11). Algebraic considerations show that both spaces in (10.16) are infinitesimally equivalent to $V_\lambda(L \cap M)$. For the former, one uses (10.13) to characterize its Harish-Chandra module as the highest weight module (for $\mathfrak{l} \cap \mathfrak{m}$) generated by ϕ . Setting $v = \phi$ in (10.13) gives an element in both sides of (10.16); the inclusion is a continuous nonzero $L \cap M$ -equivariant map between irreducible unitary representations. Thus (10.16) is an equality.

Now $r(\Phi v)$ is automatically square integrable for any $v \in V$. Let $v = v_0 + v_1$ be the decomposition of $v \in V$ corresponding to (10.15). Then

$$\begin{aligned}
 r(\Phi v) &= r(\Phi(v_0 + v_1)) \\
 &= r(\Phi v_1).
 \end{aligned}$$

Restricted to $L \cap M$, the last term is a matrix coefficient for an irreducible unitary representation in the discrete series of $L \cap M$. This matrix coefficient is square integrable by definition.

We give an argument without using the orthogonality relations. A theorem of Harish-Chandra [HC1] states that irreducible unitary representations that are

infinitesimally equivalent are unitarily equivalent. By remarks following (10.16), there exists a unitary equivalence

$$(10.17) \quad E : (\ker r)^\perp \rightarrow V_\lambda(L \cap M).$$

As above the inclusion

$$i : V_\lambda(L \cap M) \rightarrow \Gamma((L \cap M)(L^\mathbb{C} \cap B), \lambda)$$

is continuous with dense image; on the level of $L \cap M \cap K$ -finite vectors, one has

$$(im\ r)_{L \cap M \cap K} \subseteq \Gamma((L \cap M)(L^\mathbb{C} \cap B), \lambda)_{L \cap M \cap K} = (V_\lambda(L \cap M))_{L \cap M \cap K}.$$

Let P_r be the $L \cap M$ -equivariant projection from $V_\lambda(M)$ to $(\ker r)^\perp$. On the dense subset of $M \cap K$ -finite vectors, r and $i \circ E \circ P_r$ agree up to a scalar. Continuity (with respect to uniform convergence on compact subsets) implies that r and $i \circ E \circ P_r$ agree on all of $V_\lambda(M)$ after rescaling. Thus the image of r consists of square-integrable functions. Note that the factorization implies that r is continuous with respect to the topology on $V_\lambda(L \cap M)$. QED

Remark. The above proof can be modified for the case of limits of holomorphic discrete series. Without loss of generality, assume M is simple. The main results needed are Theorems 4.1 and 4.2 of [KO]. The replacement for condition (10.4) occurs there as (4.1) and one also has an explicit highest weight vector ψ_Λ . For existence of such representations, one adds to (10.7) the condition

$$\langle \lambda + \delta_M, \alpha_0 \rangle = 0$$

where α_0 is the highest root in $\Delta^+(\mathfrak{m} \cap \mathfrak{p})$. Since α_0 is not in $\Delta^+(\mathfrak{l} \cap \mathfrak{m} \cap \mathfrak{p})$, the target space is always a holomorphic discrete series representation of $L \cap M$.

Continuity of r still holds since the space of holomorphic sections is the maximal globalization of its underlying Harish-Chandra module [S3], which is equivalent to an irreducible Verma module. The correspondence of decompositions in (10.15) is valid without reference to matrix coefficients. One finishes the proof by the second method.

Results on orthogonality relations for matrix coefficients of nonsquare-integrable representations can be found in [Mi].

We describe the $L \cap M$ -representation that defines W in (9.3). Let r be the operator defined in Theorem 10.9 with respect to $w(L \cap M)w^{-1}$.

Vanishing occurs if we take left invariant derivatives in (9.3) at the $L \cap M$ -variable with respect to $\mathfrak{l} \cap \mathfrak{b}$. The terms acting on the form component vanish by (8.7) and the fact that ω_s spans a one-dimensional $L \cap M \cap K$ -module. For the terms in the function part, note that w preserves $\Delta^-(\mathfrak{m} \cap \mathfrak{p})$ and sends $\Delta^-(\mathfrak{l} \cap \mathfrak{m} \cap \mathfrak{k})$ into $\Delta^-(\mathfrak{m} \cap \mathfrak{k})$. By a similar argument, one has

$$(10.18) \quad r(\sigma(w)\sigma(X) \cdot V_\lambda(M))(e) = 0 \quad \text{for} \quad X \in \bar{\mathfrak{u}} \cap \mathfrak{m} \cap \mathfrak{k} \oplus \mathfrak{u} \cap \mathfrak{m} \cap \mathfrak{p}.$$

Ignoring the character associated to ω_n , the right translation action at the $L \cap M$ -variable with respect to T has differential $\lambda' = w^{-1}\lambda + 2\delta(\mathfrak{u} \cap \mathfrak{m} \cap \mathfrak{k})$. Now

$$\langle \delta(\mathfrak{u} \cap \mathfrak{m} \cap \mathfrak{k}), \alpha \rangle = 0$$

for $\alpha \in \Delta(\mathfrak{l} \cap \mathfrak{m} \cap \mathfrak{k})$. Hence λ' is dominant with respect to $\Delta^+(\mathfrak{l} \cap \mathfrak{m} \cap \mathfrak{k})$. From Theorem 10.9, we have

$$(10.19) \quad \langle w^{-1}\lambda + \delta_{L \cap M}, \alpha \rangle = \langle \lambda + w\delta_{L \cap M}, w\alpha \rangle < 0$$

when $\alpha \in \Delta^+(\mathfrak{l} \cap \mathfrak{m} \cap \mathfrak{p})$. To satisfy (10.7), we impose a restriction on λ via the following lemma.

Lemma 10.20. $\langle 2\delta(\mathfrak{u} \cap \mathfrak{m} \cap \mathfrak{k}), \alpha \rangle \geq 0$ for $\alpha \in \Delta^+(\mathfrak{l} \cap \mathfrak{m} \cap \mathfrak{p})$.

Proof. Without loss of generality, assume M is simple noncompact and $L \cap M$ noncompact. All α of interest are of the form

$$\alpha = \alpha_s + \sum n_\beta \beta$$

where α_s is the simple positive noncompact root, $n_\beta \geq 0$, and β ranges over the compact simple roots in $\Delta^+(\mathfrak{l} \cap \mathfrak{m})$. By the paragraph preceding the lemma, it is enough to consider $\alpha = \alpha_s$.

The good ordering implies that all compact roots in $\Delta^+(\mathfrak{m})$ lie in the integer span of the simple positive compact roots. Thus we can expand

$$2\delta(\mathfrak{u} \cap \mathfrak{m} \cap \mathfrak{k}) = - \sum m_\beta \beta,$$

where the β range over the simple compact roots in $\Delta^+(\mathfrak{m})$ and $m_\beta \geq 0$. Since the inner product of two simple roots is always nonpositive, the lemma follows.

The $L \cap M$ -representation occurring in W has space

$$(10.21) \quad \Gamma((L \cap M)(L^{\mathbb{C}} \cap B), \lambda') \otimes \mathbb{C}_\chi,$$

where χ is the one-dimensional $L \cap M$ -representation with differential $\delta(\mathfrak{u}) - \delta(\mathfrak{u} \cap \mathfrak{m})$. We leave square integrability of these sections as an open problem. QED

This last difficulty can be overcome by lowering the degree of cohomology. In (9.4), delete ω_c and set $w = 1$. Further details are left to the reader.

11. ANOTHER EXAMPLE

One example handles both sections 8 and 9. We retain the matrix notation from section 6. Let $G = SO(3, 4)_e$, the identity component of the subgroup of $SL(7, \mathbb{R})$ whose elements preserve the form

$$(11.1) \quad \langle x, y \rangle_{3,4} = \sum_{i=1}^3 x_i \cdot y_i - \sum_{j=4}^7 x_j \cdot y_j.$$

Let $X = E_{12} - E_{21}$, $Y = E_{34} + E_{43}$, and $Z = E_{56} - E_{65}$. Let the basis for \mathfrak{h}^* dual to $\{iX, Y, iZ\}$ be given by $\{e_1, \alpha, e_3\}$. Here

$$(11.2) \quad \mathfrak{t}_0 = \mathbb{R}X \oplus \mathbb{R}Z$$

and

$$(11.3) \quad \mathfrak{a}_0 = \mathbb{R}Y.$$

The set of positive roots is given by

$$\Delta^+(\mathfrak{g}, \mathfrak{h}) = \{e_1 \pm \alpha, e_3 \pm \alpha, e_1 \pm e_3, e_1, e_3, \alpha\}.$$

Now M_0 is isomorphic to $SO(2, 3)_e$. The good ordering for $\Delta^+(\mathfrak{m}, \mathfrak{t})$ has simple roots given by $e_1 - e_3$ (noncompact) and e_3 (compact). There are two choices of L which apply to our situation.

First consider the case $L = Z_G(X)$. Here $L \cap M_0$ is compact. Then we have (or choose)

- (11.4) $L = S^1 \times SO(4, 1)_e$,
- (11.5) $\Delta(\mathfrak{l}, \mathfrak{h}) = \{\pm\alpha \pm e_3, \pm\alpha, \pm e_3\}$,
- (11.6) $\Delta(\mathfrak{u}, \mathfrak{h}) = \{-e_1 \pm \alpha, -e_1 \pm e_3, -e_1\}$,
- (11.7) $L \cap M_0 \cong S^1 \times SO(3)$,
- (11.8) $\Delta(\mathfrak{l} \cap \mathfrak{m}, \mathfrak{t}) = \{\pm e_3\}$, and
- (11.9) $s = \dim \mathfrak{u} \cap \mathfrak{k} = 4$.

Note that $L \cap M_0 = K \cap M_0$ so τ is the minimal $K \cap M_0$ -type of σ .

For the noncompact $L \cap M_0$ case, let $L = Z_G(X + Z)$. We have

- (11.10) $\Delta(\mathfrak{l}, \mathfrak{h}) = \{\pm(e_1 - e_3), \pm\alpha\}$,
- (11.11) $\Delta(\mathfrak{u}, \mathfrak{h}) = \{-e_1 \pm \alpha, -e_3 \pm \alpha, -e_1, -e_3, -e_1 - e_3\}$,
- (11.12) $L \cap M_0 \cong U(1, 1)$; specifically (with zero blocks deleted)

$$\mathfrak{l}_0 \cap \mathfrak{m}_0 = \begin{pmatrix} 0 & a & c & d \\ -a & 0 & -d & c \\ c & -d & 0 & b \\ d & c & -b & 0 \end{pmatrix} \quad (a, b, c, d \in \mathbb{R});$$

exponentiation gives the center of the semisimple part.

- (11.13) $\Delta(\mathfrak{l} \cap \mathfrak{m}, \mathfrak{t}) = \{\pm(e_1 - e_3)\}$, and
- (11.14) $s = \dim \mathfrak{u} \cap \mathfrak{k} = 3$.

τ is a discrete series representation for $U(1, 1)$; see the references in section 6.

12. THE CASE OF DISCONNECTED M

To handle disconnected M , we augment the construction in Theorem 10.6 with the Cartan-Weyl theory for disconnected compact groups. We refer the reader to chapter 4, section 2 of [KV] for the latter.

Disconnectedness of M is captured in a way that is compatible with the geometry of discrete series representations. We construct an irreducible M -subrepresentation of $\text{ind}_{M_0}^M(V_\lambda(M_0))$. To avoid using another variable, we consider a simultaneous action of a large Cartan subgroup T on the fibers and complex structure associated to a holomorphic bundle over M_0/T_0 . A finer analysis of the disconnectedness of M occurs in [Vo1].

In general, we define

$$(12.1) \quad M = Z_K(\mathfrak{a}_0)M_0$$

where M_0 is the analytic subgroup of G with Lie algebra \mathfrak{m}_0 . Since $M = (M \cap K)M_0$, the disconnectedness of M is captured by a large Cartan subgroup of $M \cap K$. Let

$$(12.2) \quad T = N_{M \cap K}(\mathfrak{b} \cap \mathfrak{k})$$

be such a subgroup with identity component T_0 . Proposition 4.22 of [KV] states

- (12.3) T_0 has finite index in T ,
- (12.4) every element of $M \cap K$ lies in $T(M_0 \cap K)$, and
- (12.5) $T_0 = T \cap (M_0 \cap K)$.

Furthermore we note that the Ad -action of T on M_0 preserves

- (12.6) the Cartan decomposition of M_0 ,
- (12.7) the sets $\Delta(\mathfrak{m}, \mathfrak{t})$, $\Delta(\mathfrak{m} \cap \mathfrak{k})$, and $\Delta(\mathfrak{m} \cap \mathfrak{p})$,
- (12.8) the set of elements in \mathfrak{t}^* which are analytically integral with respect to $M_0 \cap K$, and
- (12.9) $\Delta^+(\mathfrak{m} \cap \mathfrak{k})$ and the subset of $\Delta^+(\mathfrak{m} \cap \mathfrak{k})$ -dominant elements in (12.8) .

In Theorem 10.6, assume λ is associated to an irreducible dominant representation (π_T, V^λ) ; that is, we assume that the representation space for the T_0 -character χ_λ is a fixed one-dimensional \mathfrak{t} -weight space \mathbb{C}_λ in V^λ .

Theorem 4.25 of [KV] states that such representations are in one-to-one correspondence with the irreducible representations of $M \cap K$. Let Π denote the set of \mathfrak{t} -weights of π_T , counted with multiplicity. If v is a \mathfrak{t} -weight vector of weight λ and $t \in T$, then $\pi_T(t)v$ is a \mathfrak{t} -weight vector of weight $Ad(t)\lambda$. The action of T is transitive on Π , and each weight for π_T has the same multiplicity.

Fix $t \in T$. We define a map (which is the usual T_0 -action when $t \in T_0$)

$$(12.10) \quad \sigma(t) : V_\lambda(M_0) \rightarrow V_\lambda^t(M_0)$$

by

$$(12.11) \quad (\sigma(t)F)(m) = \pi_T(t)(F(Ad(t^{-1})m)),$$

where $m \in M_0$. The space $V_\lambda^t(M_0)$ is a holomorphic discrete series representation space for M_0 ; the domain for functions in this space is $M_0(Ad(t)B)$. We verify that all assumptions in Theorem 10.6 hold with respect to the parameters $Ad(t)\lambda$ and $Ad(t)B$. When $t \in T_0$, we recover the original space $V_\lambda(M_0)$.

First note that for $h \in T_0$ and $m \in M_0$

$$(12.12) \quad \begin{aligned} (\sigma(t)F)(mh) &= \pi_T(t)F(Ad(t^{-1})(mh)) \\ &= \pi_T(t)F((Ad(t^{-1})m)(Ad(t^{-1})h)) \\ &= \chi_{Ad(t)\lambda}(h^{-1})(\sigma(t)F)(m). \end{aligned}$$

Thus T_0 translates from the right with respect to $\chi_{Ad(t)\lambda}$. A similar computation shows that $\sigma(t)F$ is right-invariant under $Ad(t)\bar{N}'$.

By (12.7) and T -invariance of the Killing form, we have

$$(12.13) \quad \langle Ad(t)\lambda + Ad(t)\delta_{M_0}, Ad(t)\alpha \rangle = \langle \lambda + \delta_{M_0}, \alpha \rangle < 0$$

for $\alpha \in \Delta^+(\mathfrak{m} \cap \mathfrak{p})$. Hence (10.7) holds for the positive system associated to t , and (12.11) is surjective onto the holomorphic discrete series representation of type $\sigma \circ Ad(t^{-1})$.

Let \bar{t} run over a set of representatives for each element of T/T_0 , and define V_M to be the external direct sum of the distinct $V_\lambda^{\bar{t}}(M_0)$. The sum is invariant under the T and M_0 -actions. Note that (12.11) extends the usual T_0 -action (with $T_0 = T \cap M_0$) and thus gives an M -action on V_M , which we denote by σ_M .

We give V_M the Hilbert space structure as a finite direct sum of Hilbert spaces. Since T preserves the left-invariant measure on M_0 , (σ_M, V_M) is a unitary representation with respect to this inner product.

As an M_0 -representation, V_M is infinitesimally equivalent to a finite direct sum of irreducible discrete series representations for M_0 . As an M -representation, V_M contains the irreducible $M \cap K$ -representation associated to V^λ ; this representation intersects nontrivially with each irreducible constituent of the underlying $(\mathfrak{m}, M_0 \cap K)$ -module. Hence V_M is irreducible under M .

13. THE OPERATOR \mathcal{S} FOR DISCONNECTED M

We combine the methods of sections 10 and 12 to construct the analog of (9.3) when M is disconnected.

Note that $T_{L \cap M} = L \cap T$ is a large Cartan subgroup of $L \cap M \cap K$. To see that

$$(13.1) \quad T_{L \cap M} = N_{L \cap M \cap K}(\mathfrak{l} \cap \mathfrak{b} \cap \mathfrak{k}),$$

note that elements in either $T \cap L$ or $L \cap M \cap K$ normalize $\bar{\mathfrak{u}} \cap \mathfrak{m} \cap \mathfrak{k} (\subset \mathfrak{b} \cap \mathfrak{k})$.

Since $T_{L \cap M}$ is compact, the space V^λ is fully reducible under $T_{L \cap M}$ -action. We choose an irreducible constituent in the decomposition, say $V^{L \cap M}$, and define the $T_{L \cap M}$ -equivariant projection

$$(13.2) \quad P_{L \cap M} : V^\lambda \rightarrow V^{L \cap M}.$$

We assume that the \mathfrak{k} -weight space \mathbb{C}_λ is contained in $V^{L \cap M}$. This constituent is an irreducible dominant representation for $T_{L \cap M}$ and hence is associated to some irreducible $L \cap M \cap K$ -representation. Thus the construction in section 12 allows one to construct a holomorphic discrete series representation for $L \cap M$, say $(\tau, V_{L \cap M})$, and the procedure of section 10 can be applied to construct an $L \cap M$ -equivariant projection

$$P_\tau : V_M \rightarrow V_{L \cap M}.$$

To adjust (9.4) to handle disconnectedness, replace r with the appropriate P_τ for $w(L \cap M)w^{-1}$. Note that $wT_{L \cap M}w^{-1} = T_{w(L \cap M)w^{-1}}$. By (12.9), the $T_{L \cap M}$ -span of $\omega_c \wedge \omega_n$ is one-dimensional. One sorts out the parameters of the $L \cap M$ -representation as in section 10.

14. THE NONHOLOMORPHIC PENROSE TRANSFORM

The proof for nonvanishing in cohomology is almost verbatim from sections 10 and 11 of [Do]. The main facts concerning the relationships between the minimal K - and $L \cap K$ -types still hold, as does the ability to express the elements of such types explicitly. In this case one uses the $KMAN$ decomposition for general real parabolic subgroup MAN .

We indicate the adjustment needed for representing elements in the minimal K -type of the domain of (9.3). Let (μ, V^μ) be an abstract irreducible unitary copy of the given minimal K -type, and let (μ_M, V^{μ_M}) be the unique minimal $M \cap K$ -type of σ . By inspection of the proof of Proposition 7.9 in [Vo1], a component of type μ_M occurs in the decomposition of μ into $M \cap K$ -types exactly once. Let

$$P_\mu : V^\mu \rightarrow V^{\mu_M}$$

be the $M \cap K$ -equivariant composition of the projection from V^μ to the μ_M -isotypic component followed by the equivalence sending this component onto $V^{\mu_M} \subset V_\lambda(M)$. With respect to the $KMAN$ decomposition, elements of the minimal K -type μ are given by

$$f_v(kman) = e^{-(\nu + \rho_G)(\log a)} \sigma(m)^{-1} P_\mu(\mu^{-1}(k)v)$$

where $v \in V_\mu$.

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