

WALL-CROSSING FUNCTORS AND \mathcal{D} -MODULES

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ABSTRACT. We study Translation functors and Wall-Crossing functors on infinite dimensional representations of a complex semisimple Lie algebra using \mathcal{D} -modules. This functorial machinery is then used to prove the Endomorphism-theorem and the Structure-theorem; two important results were established earlier by W. Soergel in a totally different way. Other applications to the category \mathcal{O} of Bernstein-Gelfand-Gelfand are given, and some conjectural relationships between Koszul duality, Verdier duality and convolution functors are discussed. A geometric interpretation of tilting modules is given.

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0. INTRODUCTION

We will be concerned here with infinite dimensional representations of a complex semisimple Lie algebra \mathfrak{g} .

In more detail, let $U\mathfrak{g}$ be the universal enveloping algebra of \mathfrak{g} , and let $Z(\mathfrak{g})$ be the center of $U\mathfrak{g}$. We consider the category of $U\mathfrak{g}$ -modules annihilated by a great enough (unspecified) power of a maximal ideal in $Z(\mathfrak{g})$. It has been observed by many people during the 1970's (see e.g., [W], [LW], [Ja], and [Z]) that various results can be usefully transferred between the categories corresponding to two different maximal ideals, using tensor products with finite-dimensional representations. The most relevant for us is the work of Jantzen [Ja], who introduced certain functors between the two categories, called *translation functors*. Jantzen showed that if both maximal ideals satisfy certain regularity and integrality conditions, then the translation functor establishes an equivalence of the two categories. If one of the two ideals is regular while the other is not, the corresponding translation functor is

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no longer an equivalence. The composition of the translation functor that sends the category at a regular maximal ideal to the category at a non-regular maximal ideal with the translation functor acting in the opposite direction is called a *wall-crossing* functor. The terminology stems from the identification of (integral) maximal ideals of $Z(\mathfrak{g})$ with Weyl group orbits in the weight lattice of the maximal torus. Non-regular ideals correspond to the orbits contained in the union of walls of the Weyl chambers.

In this paper we will be mainly interested in the “most singular” case, where the non-regular maximal ideal corresponds to the fixed point of the Weyl group, that is the point contained in all the walls. Our study was partly motivated by trying to understand two important results, the “Endomorphism-theorem” and the “Structure-theorem,” proved by W. Soergel [S1] in the course of the proof of the Koszul duality conjecture; see [BG], [BGS]. Soergel’s argument was very clever, but rather technical. A shorter proof of the Endomorphism-theorem was found by J. Bernstein [Be]. An alternative entirely geometric approach to the Structure-theorem, based on perverse sheaves and leaving almost all of the representation theory aside was given in [Gi2]. We propose below new proofs of both theorems in the framework of representation theory, and based on the technique of \mathcal{D} -modules. An advantage of our approach is that the bulk of the argument goes through in a quite general setting, while the proofs in [S1] and [Be] relied heavily on some special features of the category \mathcal{O} right from the beginning. One of our goals is to convince the reader (or at least ourselves) that the results in question are not so “hard,” and that they follow quite naturally from the \mathfrak{g} -module \longleftrightarrow \mathcal{D} -module correspondence combined with some basic functoriality properties.

It would be very interesting to apply the general results obtained in Section 3 below to the category of Harish-Chandra modules instead of the category \mathcal{O} . It is also tempting to extend our results to representations of affine Lie algebras (cf. [FrMa]). We remark that the \mathcal{D} -module approach of the present paper provides natural and simple proofs of most of the results concerning *tilting* modules (over the *finite-dimensional* semisimple Lie algebra \mathfrak{g}), proved by Soergel in [S2] for tilting modules over an *affine* Lie algebra. The main problem of extending our results to the affine case is that the category at the regular *positive* level is known ([KT1]) to be related to \mathcal{D} -modules on one affine flag manifold (which is the union of cells of finite *codimension*); the category at the regular *negative* level is known ([KT2]) to be related to \mathcal{D} -modules on another affine flag manifold (which is the union of cells of finite dimension), and the category at the “most singular” = *critical* level is expected [FM] to be related to \mathcal{D} -modules on yet another, so-called *periodic* flag manifold, (which is the union of cells of *semi-infinite* dimension). It is therefore especially intriguing how to find a geometric construction of translation functors to and from the “most singular” maximal ideal.

We now briefly outline the contents of the paper. In Section 1 we introduce a Galois extension of $U\mathfrak{g}$ with the Galois group W , the Weyl group. We compare modules over $U\mathfrak{g}$ with those over the Galois extension. Section 2 provides a self-contained exposition of translation functors from the geometric view-point. Although some of the results of this section are undoubtedly known to experts, we could not find a relevant reference in the literature. Section 3 is the heart of the paper containing our main results about wall-crossing functors. In Section 4 we

prove the *Endomorphism-theorem* and the *Structure-theorem* of Soergel and give some applications to the category \mathcal{O} of Bernstein-Gelfand-Gelfand [BGG].

Section 5 is devoted to various convolution functors. A convolution functor was first introduced in the geometric setting of perverse sheaves back in 1980, independently by Beilinson-Bernstein, Brylinski, Lusztig, MacPherson and others, in the course of the proof of the Kazhdan-Lusztig conjecture. An algebraic counterpart of that convolution in terms of Harish-Chandra bi-modules was considered in [Gi1]. It was claimed in [Gi1] (without proof) that the algebraic convolution of Harish-Chandra bi-modules goes into the geometric convolution under the localization functor of [BB1]. This claim turns out to be not quite correct, and in Section 5 we establish a precise relationship between the algebraic and the geometric convolution functors, respectively. We also propose conjectural relationships (see Conjecture 5.18 and Theorem 5.24) between the various convolutions, Koszul duality of [BGS], and the Verdier duality functors.

In Section 6 we express projective functors defined in [BeGe] in terms of the convolution functors studied in the previous section. We establish a direct connection between projective functors and projective Harish-Chandra bi-modules (somewhat analogous to a result of [BeGe] that motivated the name “projective functor”). We discuss some applications. In particular, we introduce *tilting* Harish-Chandra modules, establish their relation to the tilting objects of the category \mathcal{O} , and derive their basic properties using convolution functors.

The present paper grew out of an unpublished chapter of one of the (many) preliminary versions of [BGS], written in 1992. That chapter was not directly related to the subject of [BGS], and we decided to publish it separately to keep the size of [BGS] to a minimum.

1. TRANSLATION FUNCTORS FOR THE EXTENDED ENVELOPING ALGEBRA

Throughout the paper we fix \mathfrak{g} , a complex semisimple Lie algebra. We write \mathfrak{h} for the corresponding *abstract* Cartan subalgebra. The reader should be warned that \mathfrak{h} is not a subalgebra of \mathfrak{g} ; it is defined (see e.g. [CG, p. 137]) as the quotient of a Borel subalgebra modulo its nil-radical, and this quotient is independent of any choices. Recall further (see *loc. cit.*) that \mathfrak{h}^* , the dual space, comes equipped with a root system $R \subset \mathfrak{h}^*$ which has a preferred choice of simple roots. Thus there is a well defined element $\rho \in \mathfrak{h}^* = \text{half-sum of positive roots}$. An element $\lambda \in \mathfrak{h}^*$ is called ρ -*dominant* if $\langle \lambda + \rho, \check{\alpha} \rangle \geq 0$, for every positive coroot $\check{\alpha}$.

Let W be the Weyl group of \mathfrak{g} , the Coxeter group generated by reflections with respect to roots. The group W acts naturally on \mathfrak{h} and on \mathfrak{h}^* . In this paper we will always use the so-called *dot-action* of W on \mathfrak{h}^* . The dot-action of $w \in W$ is obtained by “twisting” the standard w -action $\mathfrak{h}^* \ni \lambda \mapsto w(\lambda)$ as follows: $\lambda \mapsto w \cdot \lambda := w(\lambda + \rho) - \rho$. Thus, the point $(-\rho)$ is the unique W -fixed point of the dot-action. Given $\lambda \in \mathfrak{h}^*$, let $W_\lambda \subset W$ denote the isotropy group, and $|\lambda| \simeq W/W_\lambda$ the W -orbit of λ with respect to the dot-action.

The dot-action on \mathfrak{h}^* induces a W -action on the polynomial algebra $\mathbb{C}[\mathfrak{h}^*]$. Let $\mathbb{C}[\mathfrak{h}^*]^W \subset \mathbb{C}[\mathfrak{h}^*]$ be the subalgebra of W -invariant polynomials on \mathfrak{h}^* . We have $\text{Specm } \mathbb{C}[\mathfrak{h}^*]^W = \mathfrak{h}^*/W$, where Specm stands for the maximal spectrum of a commutative \mathbb{C} -algebra. Given $\lambda \in \mathfrak{h}^*$, we let $\mathcal{I}_\lambda \subset \mathbb{C}[\mathfrak{h}^*]$ denote the maximal ideal of all polynomials vanishing at λ . We will often identify $\mathbb{C}[\mathfrak{h}^*]$ with $S\mathfrak{h}$, the symmetric algebra on \mathfrak{h} .

Let $Z(\mathfrak{g})$ be the center of the universal enveloping algebra $U\mathfrak{g}$. The Harish-Chandra isomorphism $Z(\mathfrak{g}) \simeq S\mathfrak{h}^W$ gives rise to the following composition of algebra morphisms:

$$\chi : Z(\mathfrak{g}) \xrightarrow{\sim} S\mathfrak{h}^W \hookrightarrow S\mathfrak{h} \xrightarrow{\sim} \mathbb{C}[\mathfrak{h}^*].$$

We will identify $Z(\mathfrak{g})$ with the image of χ , the subalgebra $\mathbb{C}[\mathfrak{h}^*]^W$ of dot-invariant polynomials. Thus, the map $\text{Specm } \mathbb{C}[\mathfrak{h}^*] \rightarrow \text{Specm } Z(\mathfrak{g})$ induced by the imbedding χ can (and will) be identified with the projection $\pi : \mathfrak{h}^* \rightarrow \mathfrak{h}^*/W$. Given $\lambda \in \mathfrak{h}^*$, we will often view the W -orbit $|\lambda|$ as a point of the orbispace \mathfrak{h}^*/W , and we let $I_{|\lambda|} = \pi^{-1}(\mathcal{J}_\lambda) \in \text{Specm } Z(\mathfrak{g})$ denote the corresponding maximal ideal in $Z(\mathfrak{g})$.

Definition. The extended enveloping algebra is defined by $\tilde{U} := U\mathfrak{g} \otimes_{Z(\mathfrak{g})} \mathbb{C}[\mathfrak{h}^*]$.

Thus \tilde{U} contains both $U\mathfrak{g}$ and $\mathbb{C}[\mathfrak{h}^*]$ as subalgebras and the subalgebra $\mathbb{C}[\mathfrak{h}^*]$ coincides with the center of \tilde{U} . Furthermore, the W -action on $\mathbb{C}[\mathfrak{h}^*]$ gives rise to a W -action on \tilde{U} such that $\tilde{U}^W = U\mathfrak{g}$.

Given $\lambda \in \mathfrak{h}^*$, let $\text{Mod}_{|\lambda|}(U\mathfrak{g})$, resp. $\text{Mod}_\lambda(\tilde{U})$, denote the category of finitely-generated $U\mathfrak{g}$ -modules (resp. \tilde{U} -modules) M such that $I_{|\lambda|}^n \cdot M = 0$, resp. $\mathcal{J}_\lambda^n \cdot M = 0$, for great enough $n = n(M) \gg 0$. Restricting \tilde{U} -modules to $U\mathfrak{g}$ -modules yields an exact functor $\text{Res}_\lambda : \text{Mod}_\lambda(\tilde{U}) \rightarrow \text{Mod}_{|\lambda|}(U\mathfrak{g})$.

Fix $\lambda \in \mathfrak{h}^*$. We introduce the intermediate algebra of W_λ -invariants: $\mathbb{C}[\mathfrak{h}^*]^W \subset \mathbb{C}[\mathfrak{h}^*]^{W_\lambda} \subset \mathbb{C}[\mathfrak{h}^*]$. We have

$$(1.1) \quad \begin{aligned} \tilde{U} &\simeq U\mathfrak{g} \otimes_{\mathbb{C}[\mathfrak{h}^*]^W} \mathbb{C}[\mathfrak{h}^*] \simeq U\mathfrak{g} \otimes_{\mathbb{C}[\mathfrak{h}^*]^W} \mathbb{C}[\mathfrak{h}^*]^{W_\lambda} \otimes_{\mathbb{C}[\mathfrak{h}^*]^{W_\lambda}} \mathbb{C}[\mathfrak{h}^*] \\ &\simeq \tilde{U}^{W_\lambda} \otimes_{\mathbb{C}[\mathfrak{h}^*]^{W_\lambda}} \mathbb{C}[\mathfrak{h}^*]. \end{aligned}$$

Geometrically, introducing the intermediate algebra $\mathbb{C}[\mathfrak{h}^*]^W \hookrightarrow \mathbb{C}[\mathfrak{h}^*]^{W_\lambda} \hookrightarrow \mathbb{C}[\mathfrak{h}^*]$ corresponds to the factorisation of the projection $\mathfrak{h}^* \rightarrow \mathfrak{h}^*/W$ as the composition $\mathfrak{h}^* \rightarrow \mathfrak{h}^*/W_\lambda \rightarrow \mathfrak{h}^*/W$. Let $\mathcal{J}_\lambda^{W_\lambda} := \mathbb{C}[\mathfrak{h}^*]^{W_\lambda} \cap \mathcal{J}_\lambda$ be the corresponding maximal ideal. Observe that the projection $\mathfrak{h}^*/W_\lambda \rightarrow \mathfrak{h}^*/W$ is unramified over $|\lambda|$. Hence, the $I_{|\lambda|}$ -adic completion of the algebra $Z(\mathfrak{g})$ is isomorphic *canonically* to the $\mathcal{J}_\lambda^{W_\lambda}$ -adic completion of the algebra $\mathbb{C}[\mathfrak{h}^*]^{W_\lambda}$. This yields a canonical algebra isomorphism between the $I_{|\lambda|}$ -adic completion of $U\mathfrak{g}$ and the $\mathcal{J}_\lambda^{W_\lambda}$ -adic completion of \tilde{U}^{W_λ} . Now, let $\text{Mod}_\lambda(\tilde{U}^{W_\lambda})$ be the category of finitely-generated \tilde{U}^{W_λ} -modules annihilated by a great enough power of the ideal $\mathcal{J}_\lambda^{W_\lambda}$. For any $M \in \text{Mod}_\lambda(\tilde{U}^{W_\lambda})$, the \tilde{U}^{W_λ} -action on M can be uniquely extended, by continuity, to an action of the $\mathcal{J}_\lambda^{W_\lambda}$ -adic completion; similarly, the $U\mathfrak{g}$ -action on any $M \in \text{Mod}_{|\lambda|}(U\mathfrak{g})$ can be extended to an action of the $I_{|\lambda|}$ -adic completion. The completions being isomorphic, we obtain

Lemma 1.2. *The $U\mathfrak{g}$ -action on any module $M \in \text{Mod}_{|\lambda|}(U\mathfrak{g})$ can be extended canonically to a \tilde{U}^{W_λ} -action so that the following restriction functor is an equivalence of categories:*

$$\text{Mod}_\lambda(\tilde{U}^{W_\lambda}) \longrightarrow \text{Mod}_{|\lambda|}(U\mathfrak{g}).$$

We will often view $U\mathfrak{g}$ -modules as \tilde{U}^{W_λ} -modules via the lemma.

Observe next that for the projection $\pi : \mathfrak{h}^* \rightarrow \mathfrak{h}^*/W_\lambda$ we have $\pi^{-1} \circ \pi(\lambda) = \lambda$. It follows that for any $M \in \text{Mod}_\lambda(\tilde{U}^{W_\lambda})$ one has: $\tilde{U} \otimes_{\tilde{U}^{W_\lambda}} M \in \text{Mod}_\lambda(\tilde{U})$. The

functor $\tilde{U} \otimes_{\tilde{U}^{W_\lambda}} (\cdot)$ is clearly the left adjoint of the restriction functor $\text{Mod}_\lambda(\tilde{U}) \longrightarrow \text{Mod}_\lambda(\tilde{U}^{W_\lambda})$. Thus, decomposing the functor $\text{Res}_\lambda : \text{Mod}_\lambda(\tilde{U}) \longrightarrow \text{Mod}_{|\lambda|}(\mathbf{U}\mathfrak{g})$ as the composition $\text{Mod}_\lambda(\tilde{U}) \rightarrow \text{Mod}_\lambda(\tilde{U}^{W_\lambda}) \rightarrow \text{Mod}_{|\lambda|}(\mathbf{U}\mathfrak{g})$ and taking adjoints, we get

Lemma 1.3. *The functor $M \mapsto \tilde{U} \otimes_{\tilde{U}^{W_\lambda}} M$ is the left adjoint of the functor Res_λ , where M is viewed as a \tilde{U}^{W_λ} -module via Lemma 1.2.*

We will call a weight $\lambda \in \mathfrak{h}^*$

- *integral* if $\langle \lambda, \check{\alpha} \rangle \in \mathbb{Z}$, for any coroot $\check{\alpha}$;
- *regular* if $W_\lambda = \{1\}$.

Let E be a finite-dimensional \mathfrak{g} -module and $M \in \text{Mod}_{|\lambda|}(\mathbf{U}\mathfrak{g})$, $\lambda \in \mathfrak{h}^*$. Then, the $\mathbf{U}\mathfrak{g}$ -module $E \otimes_{\mathbb{C}} M$ is annihilated by an ideal in $Z(\mathfrak{g})$ of finite codimension [Ko]. Hence there is a canonical finite direct sum decomposition:

$$E \otimes M = \bigoplus_{|\mu| \in \mathfrak{h}^*/W} \text{pr}_{|\mu|}(E \otimes M), \quad \text{where } \text{pr}_{|\mu|}(E \otimes M) \in \text{Mod}_{|\mu|}(\mathbf{U}\mathfrak{g}).$$

Fix integral ρ -dominant weights $\lambda, \mu \in \mathfrak{h}^*$, and let $E_{\lambda-\mu}$ be an irreducible finite-dimensional \mathfrak{g} -module with extreme weight $\lambda - \mu$. Following [Ja], define the translation functor

$$\theta_\mu^\lambda : \text{Mod}_{|\mu|}(\mathbf{U}\mathfrak{g}) \longrightarrow \text{Mod}_{|\lambda|}(\mathbf{U}\mathfrak{g}) \quad \text{by} \quad \theta_\mu^\lambda M = \text{pr}_{|\lambda|}(E_{\lambda-\mu} \otimes M).$$

The functor θ_λ^μ is both the left and the right adjoint of θ_μ^λ . The functors θ_μ^λ and θ_λ^μ are both exact.

By Lemma 1.2, the functor θ_μ^λ can be viewed as a functor $\text{Mod}_\mu(\tilde{U}^{W_\mu}) \rightarrow \text{Mod}_\lambda(\tilde{U}^{W_\lambda})$. The action of any $z \in \mathbb{C}[\mathfrak{h}^*]^{W_\mu}$ on $M \in \text{Mod}_\mu(\tilde{U}^{W_\mu})$ is a $\mathbf{U}\mathfrak{g}$ -module endomorphism, hence induces by functoriality an endomorphism $\theta_\mu^\lambda(z) : \theta_\mu^\lambda M \rightarrow \theta_\mu^\lambda M$. One can describe this endomorphism in terms of the algebra automorphism $T_{\lambda-\mu} : \mathbb{C}[\mathfrak{h}^*] \rightarrow \mathbb{C}[\mathfrak{h}^*]$ induced by the affine translation: $x \mapsto x + (\lambda - \mu)$, $x \in \mathfrak{h}^*$, as follows.

Assume in addition that $W_\lambda \subset W_\mu$. Then, the automorphisms $T_{\pm(\lambda-\mu)}$ preserve the subalgebra $\mathbb{C}[\mathfrak{h}^*]^{W_\lambda}$. We also have $\mathbb{C}[\mathfrak{h}^*]^{W_\mu} \subset \mathbb{C}[\mathfrak{h}^*]^{W_\lambda}$, hence $T_{\pm(\lambda-\mu)}\mathbb{C}[\mathfrak{h}^*]^{W_\mu} \subset \mathbb{C}[\mathfrak{h}^*]^{W_\lambda}$. In the next section we will prove the following result of Soergel [S1, Thm. 8] using the \mathcal{D} -module approach.

Proposition 1.4. *Let $z \in \mathbb{C}[\mathfrak{h}^*]^{W_\mu}$. Then*

- (i) *For any $M \in \text{Mod}_\mu(\tilde{U}^{W_\mu})$ and $m \in \theta_\mu^\lambda M$ we have $z \cdot m = [T_{\lambda-\mu}\theta_\mu^\lambda(z)] \cdot m$;*
- (ii) *For any $M \in \text{Mod}_\lambda(\tilde{U}^{W_\lambda})$ and $m \in \theta_\lambda^\mu M$ we have $z \cdot m = [\theta_\lambda^\mu(T_{\lambda-\mu}z)] \cdot m$.*

From now on, fix integral ρ -dominant weights $\lambda, \mu \in \mathfrak{h}^*$, such that $W_\lambda \subset W_\mu$, and set $\theta^+ := \theta_\mu^\lambda$, and $\theta^- := \theta_\lambda^\mu$. We are going to extend the functors θ^\pm to \tilde{U} -modules.

First, consider the composition of functors

$$\text{Mod}_\lambda(\tilde{U}) \xrightarrow{\text{Res}_\lambda} \text{Mod}_{|\lambda|}(\mathbf{U}\mathfrak{g}) \xrightarrow{\theta^-} \text{Mod}_{|\mu|}(\mathbf{U}\mathfrak{g}).$$

The action of an element $a \in \mathbb{C}[\mathfrak{h}^*] \subset \tilde{U}$ on $M \in \text{Mod}_\lambda(\tilde{U})$ induces, by functoriality, an endomorphism $\theta^-(a) : \theta^-(\text{Res}_\lambda M) \rightarrow \theta^-(\text{Res}_\lambda M)$. We define a $\mathbb{C}[\mathfrak{h}^*]$ -action on $\theta^-(\text{Res}_\lambda M)$ by the formula:

$$a * m := \theta^-(T_{\mu-\lambda}a) \cdot m, \quad m \in \theta^-(\text{Res}_\lambda M).$$

When restricted to the subalgebra $\mathbb{C}[\mathfrak{h}^*]^{W_\mu} \subset \mathbb{C}[\mathfrak{h}^*]$, this action coincides, by Proposition 1.4(ii), with the $\mathbb{C}[\mathfrak{h}^*]^{W_\mu}$ -action on $\theta^-(\text{Res}_\lambda M)$ arising from the \tilde{U}^{W_μ} -module structure. Therefore, combining together the $\mathbb{C}[\mathfrak{h}^*]$ - and the \tilde{U}^{W_μ} -actions we get a $\tilde{U}^{W_\mu} \otimes_{\mathbb{C}} \mathbb{C}[\mathfrak{h}^*]$ -action which factors through an action of the algebra $\tilde{U}^{W_\mu} \otimes_{\mathbb{C}[\mathfrak{h}^*]^{W_\mu}} \mathbb{C}[\mathfrak{h}^*]$. In view of isomorphism (1.1), that gives a \tilde{U} -action on $\theta^-(\text{Res}_\lambda M)$. This way we obtain an exact functor $\tilde{\theta}^- : \text{Mod}_\lambda(\tilde{U}) \rightarrow \text{Mod}_\mu(\tilde{U})$.

Next, consider the composition $\text{Mod}_\mu(\tilde{U}) \xrightarrow{\text{Res}_\mu} \text{Mod}_{|\mu|}(\mathbf{U}\mathfrak{g}) \xrightarrow{\theta^+} \text{Mod}_{|\lambda|}(\mathbf{U}\mathfrak{g})$. For any $N \in \text{Mod}_\mu(\tilde{U})$, define a $\mathbb{C}[\mathfrak{h}^*]$ -action on $\theta^+(\text{Res}_\mu N)$ by

$$a * m := [T_{\lambda-\mu}\theta^+(a)] \cdot m, \quad m \in \theta^+(\text{Res}_\mu N).$$

Again, when restricted to the subalgebra $\mathbb{C}[\mathfrak{h}^*]^{W_\mu}$, this action coincides, by Proposition 1.4(i), with the action arising from the natural \tilde{U}^{W_λ} -module structure on $\theta^+(\text{Res}_\mu N)$. This is not necessarily the case, however, for the full algebra $\mathbb{C}[\mathfrak{h}^*]^{W_\lambda} \supset \mathbb{C}[\mathfrak{h}^*]^{W_\mu}$; the two actions might be different! Thus, we put

$$\tilde{\theta}_r^+ N = \{n \in \theta^+(\text{Res}_\mu N) \mid a * n = a \cdot n, \quad \forall a \in \mathbb{C}[\mathfrak{h}^*]^{W_\lambda}\},$$

and dually,

$$\tilde{\theta}_\ell^+ N = \theta^+(\text{Res}_\mu N) / \{a * n - a \cdot n \mid a \in \mathbb{C}[\mathfrak{h}^*]^{W_\lambda}, \quad n \in \theta^+(\text{Res}_\mu N)\}.$$

The $\mathbf{U}\mathfrak{g}$ -action and the $\mathbb{C}[\mathfrak{h}^*]$ -action on $\theta^+(\text{Res}_\mu N)$ clearly induce similar actions on both $\tilde{\theta}_\ell^+ N$ and $\tilde{\theta}_r^+ N$. Moreover, by construction, these actions fit together, making $\tilde{\theta}_\ell^+ N$ and $\tilde{\theta}_r^+ N$ into \tilde{U} -modules. This way we obtain two functors

$$\tilde{\theta}_\ell^+, \tilde{\theta}_r^+ : \text{Mod}_\mu(\tilde{U}) \rightarrow \text{Mod}_\lambda(\tilde{U}).$$

These functors are not exact in general. Notice however that if both λ and μ are regular, then one has $\tilde{\theta}_r^+ = \tilde{\theta}_\ell^+ = \theta^+$.

Proposition 1.5. *The functors $\tilde{\theta}_\ell^+$ and $\tilde{\theta}_r^+$ are, respectively, the left and the right adjoint of the functor $\tilde{\theta}^-$.*

Proof. Let $\tilde{M} \in \text{Mod}_\lambda(\tilde{U})$ and $\tilde{N} \in \text{Mod}_\mu(\tilde{U})$. Set $M := \text{Res}_\lambda \tilde{M}$, $N := \text{Res}_\mu \tilde{N}$. Then we have

$$\begin{aligned} \text{Hom}_{\text{Mod}_\mu(\tilde{U})}(\tilde{\theta}^- \tilde{M}, \tilde{N}) &= \{f \in \text{Hom}_{\mathbf{U}\mathfrak{g}}(\theta^- M, N) \mid f(a * x) = a \cdot f(x), \quad \forall a \in \mathbb{C}[\mathfrak{h}^*]\} \\ &= \{g \in \text{Hom}_{\mathbf{U}\mathfrak{g}}(M, \theta^+ N) \mid g(a \cdot m) = a * g(m), \quad \forall a \in \mathbb{C}[\mathfrak{h}^*]\}. \end{aligned}$$

By $\mathbf{U}\mathfrak{g}$ -linearity of the morphism g in the last line above, and Lemma 1.2, we get $g(a \cdot m) = a \cdot g(m)$, $\forall a \in \mathbb{C}[\mathfrak{h}^*]^{W_\lambda}$. Hence, $a * g(m) = a \cdot g(m)$, $\forall a \in \mathbb{C}[\mathfrak{h}^*]^{W_\lambda}$, and $\text{Image}(g) \subset \tilde{\theta}_r^+ N$. Thus, $g \in \text{Hom}_{\text{Mod}_\lambda(\tilde{U})}(\tilde{M}, \tilde{\theta}_r^+ \tilde{N})$.

The second adjunction is proved in a similar way. \square

2. TRANSLATION FUNCTORS VIA \mathcal{D} -MODULES

For a complex variety X let \mathcal{O}_X and \mathcal{D}_X denote the sheaves of regular functions and regular differential operators on X respectively.

Let $G \supset B \supset U$ be the simply-connected semisimple Lie-group corresponding to \mathfrak{g} , a Borel subgroup of G , and the unipotent radical of B respectively. Let $T = B/U$ be the abstract maximal torus; see [CG, p. 303]. Set $\tilde{\mathcal{B}} = G/U$ and $\mathcal{B} = G/B$, the

flag manifold of G . There is a natural T -action on $\tilde{\mathcal{B}}$ on the right that makes the canonical projection $\pi : G/U \rightarrow G/B$ a principal G -equivariant T -bundle.

Let $\pi_{\bullet} \mathcal{D}_{\tilde{\mathcal{B}}}$ be the sheaf-theoretic direct image of $\mathcal{D}_{\tilde{\mathcal{B}}}$ to \mathcal{B} . The right T -action on $\tilde{\mathcal{B}}$ induces a T -action on the sheaf $\pi_{\bullet} \mathcal{D}_{\tilde{\mathcal{B}}}$ by algebra automorphisms and we let $\tilde{\mathcal{D}} \subset \pi_{\bullet} \mathcal{D}_{\tilde{\mathcal{B}}}$ denote the subsheaf of T -invariant sections of $\pi_{\bullet} \mathcal{D}_{\tilde{\mathcal{B}}}$. Thus, $\tilde{\mathcal{D}}$ is a sheaf of algebras on \mathcal{B} .

The infinitesimal left \mathfrak{g} -action and the infinitesimal right \mathfrak{h} -action on $\tilde{\mathcal{B}}$ commute, giving rise to a homomorphism of the Lie algebra $\mathfrak{g} \times \mathfrak{h}$ into global algebraic vector fields on $\tilde{\mathcal{B}}$. This Lie algebra homomorphism can be extended to an associative algebra homomorphism $U\mathfrak{g} \otimes_{\mathbb{C}} U\mathfrak{h} \rightarrow \Gamma(\tilde{\mathcal{B}}, \mathcal{D}_{\tilde{\mathcal{B}}})^T$ (= the algebra of right T -invariant global differential operators on $\tilde{\mathcal{B}}$). It turns out that, for any $z \in Z(\mathfrak{g})$, the differential operator on $\tilde{\mathcal{B}}$ corresponding under the homomorphism above to the element $z \otimes 1 \in U\mathfrak{g} \otimes U\mathfrak{h}$ is equal to the differential operator corresponding to the element $1 \otimes \chi(z) \in U\mathfrak{g} \otimes U\mathfrak{h}$, where $\chi : Z(\mathfrak{g}) \xrightarrow{\sim} U\mathfrak{h}^W$ is the Harish-Chandra isomorphism. It follows that the homomorphism above factors through $U\mathfrak{g} \otimes_{Z(\mathfrak{g})} U\mathfrak{h} \simeq \tilde{U}$ (we identify $U\mathfrak{h}$ with $\mathbb{C}[\mathfrak{h}^*]$). Furthermore, the associated graded map of “principal symbols” has been shown [BoBr] to be a bijection, so that one obtains an algebra isomorphism:

$$(2.1) \quad \tilde{U} \xrightarrow{\sim} \Gamma(\tilde{\mathcal{B}}, \mathcal{D}_{\tilde{\mathcal{B}}})^T = \Gamma(\mathcal{B}, \tilde{\mathcal{D}}).$$

Observe that since the torus T commutes with its own Lie algebra action, the image of $1 \otimes U\mathfrak{h} \simeq \mathbb{C}[\mathfrak{h}^*]$ is contained in the stalk of the sheaf $\tilde{\mathcal{D}}$ at any point of \mathcal{B} . Notice further that the embedding $\mathbb{C}[\mathfrak{h}^*] \hookrightarrow \tilde{\mathcal{D}}$ thus defined is central. Hence, for any $\lambda \in \mathfrak{h}^*$, we may define the category $\text{Mod}_{\lambda}(\tilde{\mathcal{D}})$ of coherent (sheaves on \mathcal{B} of) $\tilde{\mathcal{D}}$ -modules M such that $\mathcal{J}_{\lambda}^n \cdot M = 0$ for big enough $n = n(M) \gg 0$. Taking global sections of $\tilde{\mathcal{D}}$ -modules defines (via (2.1)) a functor $\Gamma_{\lambda} : \text{Mod}_{\lambda}(\tilde{\mathcal{D}}) \rightarrow \text{Mod}_{\lambda}(\tilde{U})$. Let $\Delta_{\lambda} : M \mapsto \tilde{\mathcal{D}} \otimes_{\Gamma(\mathcal{B}, \tilde{\mathcal{D}})} M$ be the *localization functor*, which is the left adjoint of Γ_{λ} . We recall (see [BB1] and also [BB3, Thm. 3.3.1]) the following important theorem:

Localization theorem 2.2. (i) *If λ is ρ -dominant, then the functor Γ_{λ} is exact and the canonical adjunction morphism $\Gamma_{\lambda} \cdot \Delta_{\lambda} \rightarrow \text{Id}_{\text{Mod}_{\lambda}(\tilde{U})}$ is an isomorphism;*

(ii) *If λ is regular, then Γ_{λ} gives an equivalence of the categories $\text{Mod}_{\lambda}(\tilde{\mathcal{D}})$ and $\text{Mod}_{\lambda}(\tilde{U})$, and the functor Δ_{λ} is the inverse of Γ_{λ} .*

Remark 2.3. To $M \in \text{Mod}_{\lambda}(\tilde{\mathcal{D}})$ assign a $\mathcal{D}_{\tilde{\mathcal{B}}}$ -module by the formula $\mathcal{D}_{\tilde{\mathcal{B}}} \otimes_{\tilde{\mathcal{D}}} M$. This $\mathcal{D}_{\tilde{\mathcal{B}}}$ -module is clearly smooth along the fibres of the projection $\pi : \tilde{\mathcal{B}} \rightarrow \mathcal{B}$. That gives, for regular λ , a fully faithful imbedding of the category $\text{Mod}_{\lambda}(\tilde{U})$ into the category of T -monodromic $\mathcal{D}_{\tilde{\mathcal{B}}}$ -coherent modules. For more details see [BB3, §§2.5, 3.3], and [Ka].

Recall that a subcategory \mathcal{A} of an abelian category \mathcal{C} is called a *Serre subcategory* if \mathcal{A} is a full abelian subcategory stable under taking extensions and subquotients in \mathcal{C} . Given a Serre subcategory $\mathcal{A} \subset \mathcal{C}$, one can define a quotient-category \mathcal{C}/\mathcal{A} . This is an abelian category equipped with an exact functor $quot : \mathcal{C} \rightarrow \mathcal{C}/\mathcal{A}$ such that $\mathcal{A} = \text{Ker}(quot)$, where $\text{Ker}(quot)$ stands for the full subcategory of \mathcal{C} formed by the objects A such that $quot(A) = 0$.

Conversely, let $F : \mathcal{C} \rightarrow \mathcal{C}'$ be an exact functor between abelian categories. Then, $\text{Ker } F$ is a Serre subcategory. Moreover, there exists a unique functor $\overline{F} : \mathcal{C}/\text{Ker } F \rightarrow \mathcal{C}'$, such that the functor F factors as the composition

$$\mathcal{C} \xrightarrow{\text{quot}} \mathcal{C}/\text{Ker } F \xrightarrow{\overline{F}} \mathcal{C}' .$$

Lemma 2.4. *Let $F : \mathcal{C} \rightarrow \mathcal{C}'$ be an exact functor between abelian categories which has a left (resp. right) adjoint functor $F^\dagger : \mathcal{C}' \rightarrow \mathcal{C}$. Then, $\overline{F} : \mathcal{C}/\text{Ker } F \rightarrow \mathcal{C}'$ is an equivalence of categories if and only if the canonical morphism $\text{Id}_{\mathcal{C}'} \rightarrow F \cdot F^\dagger$, (resp. $F^\dagger \cdot F \rightarrow \text{Id}_{\mathcal{C}}$) is an isomorphism.*

From this lemma applied to the functor $F = \Gamma_\lambda$, and Localization theorem 2.2, we deduce

Corollary 2.5. *Let λ be ρ -dominant. Then, the functor Γ_λ induces an equivalence*

$$\overline{\Gamma}_\lambda : \text{Mod}_\lambda(\widetilde{\mathcal{D}})/\text{Ker } \Gamma_\lambda \xrightarrow{\sim} \text{Mod}_\lambda(\widetilde{U}) . \quad \square$$

Remark. See [Ka] for a more detailed description of the category $\text{Ker } \Gamma_\lambda$ in the case of a non-regular λ .

Assume further that $\lambda \in \mathfrak{h}^*$ is integral. Then λ gives rise to a homomorphism $\dot{\lambda} : T \rightarrow \mathbb{C}^*$. Let $\mathcal{O}(\lambda)$ denote the sheaf on \mathcal{B} formed by all functions f on $\widetilde{\mathcal{B}}$ such that

$$(2.6) \quad f(x \cdot t) = \dot{\lambda}(t)f(x) \quad \forall x \in \widetilde{\mathcal{B}}, \quad t \in T .$$

Note that $\mathcal{O}(0) = \mathcal{O}_{\mathcal{B}}$, is the structure sheaf of \mathcal{B} . More generally, $\mathcal{O}(\lambda)$ is the sheaf of sections of a line bundle on \mathcal{B} . For a ρ -dominant λ , we have $\Gamma(\mathcal{B}, \mathcal{O}(\lambda)) = E_\lambda$, an irreducible finite-dimensional \mathfrak{g} -module with highest weight λ .

Let $\mathcal{D}_{\mathcal{B}}(\lambda)$ denote the sheaf of differential operators acting on $\mathcal{O}(\lambda)$ (i.e., a sheaf of twisted differential operators on \mathcal{B}). Observe next that the sheaf $\mathcal{O}(\lambda)$ is stable under the natural action on functions of T -invariant differential operators on $\widetilde{\mathcal{B}}$. This gives an algebra morphism $\widetilde{\mathcal{D}} \rightarrow \mathcal{D}_{\mathcal{B}}(\lambda)$. One can show (see e.g., [BoBr]) that this morphism induces the following isomorphisms:

$$(2.7) \quad \mathcal{D}_{\mathcal{B}}(\lambda) \simeq \widetilde{\mathcal{D}}/\widetilde{\mathcal{D}} \cdot \mathcal{J}_\lambda \quad \text{and} \quad \Gamma(\mathcal{B}, \mathcal{D}_{\mathcal{B}}(\lambda)) = \text{U}\mathfrak{g}/\text{U}\mathfrak{g} \cdot I_{|\lambda|} .$$

Given $\lambda, \mu \in \mathfrak{h}^*$ such that $\mu - \lambda$ is integral, define a geometric translation functor $\Theta_\mu^\lambda : \text{Mod}_\mu(\widetilde{\mathcal{D}}) \rightarrow \text{Mod}_\lambda(\widetilde{\mathcal{D}})$ by the formula $\Theta_\mu^\lambda : M \mapsto \mathcal{O}(\lambda - \mu) \otimes_{\mathcal{O}_{\mathcal{B}}} M$. We have $\mathcal{O}(\nu) \otimes_{\mathcal{O}_{\mathcal{B}}} \mathcal{O}(-\nu) = \mathcal{O}_{\mathcal{B}}$, hence $\Theta_\mu^\lambda \cdot \Theta_\lambda^\mu = \text{Id}_{\text{Mod}_\lambda(\widetilde{\mathcal{D}})}$. Thus, the functor Θ_μ^λ is always an equivalence of categories.

Assume now that λ and μ are integral ρ -dominant weights such that $W_\lambda \subset W_\mu$. Although the following result seems to be well known, we could not find its proof in the literature.

Proposition 2.8. *The following diagram commutes (up to canonical equivalence of functors)*

$$\begin{array}{ccc} \text{Mod}_\lambda(\widetilde{\mathcal{D}}) & \xrightarrow{\Gamma_\lambda} & \text{Mod}_\lambda(\widetilde{U}) \\ \downarrow \Theta_\lambda^\mu & & \downarrow \tilde{\theta}_\lambda^\mu \\ \text{Mod}_\mu(\widetilde{\mathcal{D}}) & \xrightarrow{\Gamma_\mu} & \text{Mod}_\mu(\widetilde{U}) \end{array}$$

Proof (essentially borrowed from [BB1]). Given a finite-dimensional \mathfrak{g} -module E , let $E_{\mathcal{B}} = E \otimes \mathcal{O}_{\mathcal{B}}$ be the trivial sheaf of E -valued regular functions on \mathcal{B} . For such a function f define a function $\varphi_f : G \rightarrow E$ by the formula $\varphi_f(g) = g^{-1} \cdot f(g)$. The assignment $f \mapsto \varphi_f$ identifies $E_{\mathcal{B}}$ with the sheaf $Ind_{\mathcal{B}}^G E$ (on \mathcal{B}) of germs of functions:

$$(2.9) \quad \{\varphi : G \rightarrow E \mid \varphi(g \cdot b) = b^{-1} \cdot \varphi(g), \quad \forall b \in B\}.$$

The infinitesimal action of \mathfrak{g} on G by left translation makes $Ind_{\mathcal{B}}^G E$ a \mathfrak{g} -module. \square

By Lie's theorem, one can find a B -stable filtration $E_0 \subset E_1 \subset \dots \subset E_n = E$ such that $\dim(E_i/E_{i-1}) = 1$, for all i . This gives a filtration on $Ind_{\mathcal{B}}^G E$ by the \mathfrak{g} -stable coherent subsheaves $Ind_{\mathcal{B}}^G E_i$ defined by replacing E by E_i in (2.9). We have

$$Ind_{\mathcal{B}}^G E_i / Ind_{\mathcal{B}}^G E_{i-1} = Ind_{\mathcal{B}}^G (E_i/E_{i-1}) = \mathcal{O}(\nu_i)$$

where ν_i is the character of B corresponding to the 1-dimensional B -module E_i/E_{i-1} .

Assume now that $E = E_{\mu-\lambda}$ is an irreducible \mathfrak{g} -module with extreme weight $\mu - \lambda$ and $M \in \text{Mod}_{\lambda}(\tilde{D})$. We endow the sheaf $(Ind_{\mathcal{B}}^G E) \otimes_{\mathcal{O}_{\mathcal{B}}} M$ with the tensor product \mathfrak{g} -module structure. The filtration $E_0 \subset E_1 \subset \dots \subset E$ as above gives a \mathfrak{g} -stable filtration on $(Ind_{\mathcal{B}}^G E) \otimes_{\mathcal{O}_{\mathcal{B}}} M$ by the subsheaves $(Ind_{\mathcal{B}}^G E_i) \otimes_{\mathcal{O}_{\mathcal{B}}} M$ with quotients of the form $\mathcal{O}(\nu) \otimes_{\mathcal{O}_{\mathcal{B}}} M$ where ν is a weight of $E_{\mu-\lambda}$. Clearly, for any $a \in \mathbb{C}[\mathfrak{h}^*]$, for the action of a on $\mathcal{O}(\nu) \otimes_{\mathcal{O}_{\mathcal{B}}} M$ we have the formula:

$$(2.10) \quad a \cdot (f \otimes m) = f \otimes (T_{\nu} a) \cdot m, \quad f \in \mathcal{O}(\nu), \quad m \in M,$$

where T_{ν} denotes the affine translation by ν on $\mathbb{C}[\mathfrak{h}^*]$ introduced before Proposition 1.4. In particular, $Z(\mathfrak{g})$ acts on $\mathcal{O}(\nu) \otimes M$ via the (generalized) infinitesimal character $|\lambda + \nu| \in \mathfrak{h}^*/W$.

We now use the following result [BeGe, Lemma 1.5(iii)]:

Let $\nu, \nu' \in \mathfrak{h}^$ be such that $\lambda + \nu$ is dominant, $\lambda + \nu'$ is W -conjugate to $\lambda + \nu$, and $\|\nu'\| \leq \|\nu\|$ in some euclidean W -invariant metric $\|\cdot\|$ on \mathfrak{h}^* . Then $\nu' \in W_{\lambda} \cdot \nu$.*

The result above implies that, for any weight ν' of $E_{\mu-\lambda}$ other than $\mu - \lambda$, the point $\lambda + \nu'$ is not W -conjugate to μ . Hence, the subquotient sheaf $\mathcal{O}(\mu - \lambda) \otimes_{\mathcal{O}_{\mathcal{B}}} M$ splits off from $Ind_{\mathcal{B}}^G E_{\mu-\lambda} \otimes_{\mathcal{O}_{\mathcal{B}}} M$ as a sheaf of $Z(\mathfrak{g})$ -modules, and moreover we have

$$(2.11) \quad \text{pr}_{|\mu|}(Ind_{\mathcal{B}}^G E_{\mu-\lambda} \otimes_{\mathcal{O}_{\mathcal{B}}} M) = \mathcal{O}(\mu - \lambda) \otimes_{\mathcal{O}_{\mathcal{B}}} M, \quad M \in \text{Mod}_{\lambda}(\tilde{U})$$

where $\text{pr}_{|\mu|}$ stands for the projection to the $|\mu|$ -isotypic component of a $Z(\mathfrak{g})$ -module.

Next, observe that, for any E one has

$$E \otimes \Gamma(M) = \Gamma(E_{\mathcal{B}} \otimes_{\mathcal{O}_{\mathcal{B}}} M) = \Gamma(Ind_{\mathcal{B}}^G E \otimes_{\mathcal{O}_{\mathcal{B}}} M).$$

Thus, from (2.11) one obtains

$$\begin{aligned} \tilde{\theta}_{\lambda}^{\mu} \cdot \Gamma_{\lambda}(M) &= \text{pr}_{|\mu|}(E_{\mu-\lambda} \otimes \Gamma(M)) \\ &= \text{pr}_{|\mu|} \Gamma(Ind_{\mathcal{B}}^G E_{\mu-\lambda} \otimes_{\mathcal{O}_{\mathcal{B}}} M) \\ &= \Gamma(\mathcal{O}(\mu - \lambda) \otimes_{\mathcal{O}_{\mathcal{B}}} M) = \Gamma_{\mu} \cdot \Theta_{\lambda}^{\mu} M. \end{aligned}$$

Finally, formula (2.10) shows that the $\mathbb{C}[\mathfrak{h}^*]$ -action on $\theta_{\lambda}^{\mu} \cdot \Gamma_{\lambda}(M)$ defined in Section 1 corresponds to the natural $\mathbb{C}[\mathfrak{h}^*]$ -action on $\Theta_{\lambda}^{\mu} M$.

3. PROPERTIES OF TRANSLATION FUNCTORS

Throughout this section we fix integral ρ -dominant weights $\lambda, \mu \in \mathfrak{h}^*$ such that $W_\lambda \subset W_\mu$, and use the notation $\theta^+ := \theta_\mu^\lambda$ and $\theta^- := \theta_\lambda^\mu$, and similarly, $\Theta^+ := \Theta_\mu^\lambda$ and $\Theta^- := \Theta_\lambda^\mu$.

Proof of Proposition 1.4. Let $M \in \text{Mod}_\lambda(\tilde{U})$. By Theorem 2.2(i), we can find a $\tilde{\mathcal{D}}$ -module \mathcal{M} such that $M = \Gamma_\lambda(\mathcal{M})$. Part (ii) of the Proposition now follows from Proposition 2.8 and formula (2.10) for $\nu = \mu - \lambda$. \square

To prove part (i) take $\tilde{M} \in \text{Mod}_\mu(\tilde{U})$, $\tilde{N} \in \text{Mod}_\lambda(\tilde{U})$ and set $M := \text{Res}_\mu \tilde{M}$, $N := \text{Res}_\lambda \tilde{N}$. An element $a \in \mathbb{C}[\mathfrak{h}]^{W_\mu}$ gives rise to an endomorphism $\theta^+(a) \in \text{End}(\theta^+M)$, hence, to an endomorphism $\theta^+(a)_M$ of $\text{Hom}_{\text{Ug}}(\theta^+M, N)$, via composition with $\theta^+(a)$. We have a commutative diagram:

$$\begin{array}{cccccc} \text{Hom}(\theta^+M, N) & = & \text{Hom}(M, \theta^-N) & = & \text{Hom}(M, \theta^-N) & = & \text{Hom}(M, \theta^-N) & = & \text{Hom}(\theta^+M, N) \\ \downarrow \theta^+(a)_M & & \downarrow a_M & & \downarrow a_{\theta^-N} & & \downarrow \theta^-(T_{\mu-\lambda}a) & & \downarrow (T_{\mu-\lambda}a)_{\theta^+M} \\ \text{Hom}(\theta^+M, N) & = & \text{Hom}(M, \theta^-N) & = & \text{Hom}(M, \theta^-N) & = & \text{Hom}(M, \theta^-N) & = & \text{Hom}(\theta^+M, N). \end{array}$$

Thus, we deduce that $\theta^+(a) = T_{\mu-\lambda}a$.

Here are the most important properties of the translation functors.

Proposition 3.1. (i) *The functor $\tilde{\theta}^- : \text{Mod}_\lambda(\tilde{U}) \rightarrow \text{Mod}_\mu(\tilde{U})$ induces an equivalence $\text{Mod}_\lambda(\tilde{U})/\text{Ker } \tilde{\theta}^- \xrightarrow{\sim} \text{Mod}_\mu(\tilde{U})$;*

(ii) *The adjunction morphisms induce isomorphisms of functors:*

$$\tilde{\theta}^- \cdot \tilde{\theta}_\ell^+ = \text{Id}_{\text{Mod}_\mu(\tilde{U})} = \tilde{\theta}^- \cdot \tilde{\theta}_r^+;$$

(iii) *We have a natural isomorphism of functors: $\tilde{\theta}_\ell^+ = \Gamma_\lambda \cdot \Theta^+ \cdot \Delta_\mu$.*

Proof. We begin with a general remark. Let \mathcal{C} be an abelian category, and let $\mathcal{C}_2 \subset \mathcal{C}_1 \subset \mathcal{C}$ be two Serre subcategories. Then the various quotient categories are related by the following canonical transitivity isomorphism

$$(3.1.1) \quad \mathcal{C}/\mathcal{C}_1 \simeq (\mathcal{C}/\mathcal{C}_2)/(\mathcal{C}_1/\mathcal{C}_2).$$

We set $\mathcal{C} := \text{Mod}_\lambda(\tilde{\mathcal{D}})$. Since the geometric translation functor $\Theta^- : \text{Mod}_\lambda(\tilde{\mathcal{D}}) \xrightarrow{\sim} \text{Mod}_\mu(\tilde{\mathcal{D}})$ is an equivalence of categories, we may (and will) identify the category $\text{Mod}_\mu(\tilde{\mathcal{D}})$ with \mathcal{C} . Further, by Corollary 2.5, we have equivalences

$$\text{Mod}_\lambda(\tilde{\mathcal{D}})/\text{Ker } \Gamma_\lambda \xrightarrow{\sim} \text{Mod}_\lambda(\tilde{U}) \quad , \quad \text{Mod}_\mu(\tilde{\mathcal{D}})/\text{Ker } \Gamma_\mu \xrightarrow{\sim} \text{Mod}_\mu(\tilde{U}).$$

Thus, setting $\mathcal{C}_1 := \text{Ker } \Gamma_\mu$ and $\mathcal{C}_2 := \text{Ker } \Gamma_\lambda$ we can rewrite the above as follows:

$$(3.1.2) \quad \mathcal{C}/\mathcal{C}_1 \simeq \text{Mod}_\mu(\tilde{U}), \quad \mathcal{C}/\mathcal{C}_2 \simeq \text{Mod}_\lambda(\tilde{U}).$$

Moreover, Proposition 2.8 insures that $\mathcal{C}_2 \subset \mathcal{C}_1$ and that the natural functor $\mathcal{C}/\mathcal{C}_1 \rightarrow \mathcal{C}/\mathcal{C}_2$ gets identified under the equivalences (3.1.2) with the functor $\tilde{\theta}^- : \text{Mod}_\lambda(\tilde{U}) \rightarrow \text{Mod}_\mu(\tilde{U})$. Part (i) of Proposition 3.1 now follows from the transitivity equivalence (3.1.1) applied to our categories. With part (i) being established, Lemma 2.4 yields part (ii) of Proposition 3.1.

To prove (iii) we take adjoints in the diagram of Proposition 2.8. This yields the following commutative diagram:

$$(3.2) \quad \begin{array}{ccc} \mathrm{Mod}_\lambda(\tilde{\mathcal{D}}) & \xleftarrow{\Delta_\lambda} & \mathrm{Mod}_\lambda(\tilde{U}) \\ \Theta^+ \uparrow & & \uparrow \tilde{\theta}_\ell^+ \\ \mathrm{Mod}_\mu(\tilde{\mathcal{D}}) & \xleftarrow{\Delta_\mu} & \mathrm{Mod}_\mu(\tilde{U}). \end{array}$$

The diagram implies readily $\tilde{\theta}_\ell^+ = \Gamma_\lambda \cdot \Theta^+ \cdot \Delta_\mu$, where we have used that every object of the category $\mathrm{Mod}_\mu(\tilde{\mathcal{D}})$ is isomorphic to an object of the form $\Delta_\mu(M)$, by Theorem 2.2(i). Part (iii) of Proposition 2.8 is proved.

One may reverse the logic slightly to derive parts (i) and (ii) of Proposition 3.1 in an alternative way, as follows. First of all one establishes part (iii) of the proposition the same way as above. With the formula of part (iii) at hand we calculate:

$$\tilde{\theta}^- \cdot \tilde{\theta}_\ell^+ = \tilde{\theta}^- \cdot \Gamma_\lambda \cdot \Theta^+ \cdot \Delta_\mu = \Gamma_\mu \cdot \Theta^- \cdot \Theta^+ \cdot \Delta_\mu = \Gamma_\mu \cdot \Delta_\mu = \mathrm{Id}_{\mathrm{Mod}_\mu(\tilde{U})},$$

where the first equality holds by Proposition 3.1(iii), the second is Proposition 2.8, and the last one is due to Theorem 2.2(i). This gives part (ii), and part (i) now follows from the “if” part of Lemma 2.4. \square

Corollary 3.3. *For $M \in \mathrm{Mod}_\lambda(\tilde{U})$, the following conditions are equivalent:*

(i) *For any $N \in \mathrm{Mod}_\lambda(\tilde{U})$, the following natural morphism is an isomorphism*

$$\mathrm{Hom}_{\mathrm{Mod}_\lambda(\tilde{U})}(M, N) \xrightarrow{\sim} \mathrm{Hom}_{\mathrm{Mod}_\mu(\tilde{U})}(\tilde{\theta}^- M, \tilde{\theta}^- N);$$

(ii) *There exists an $M' \in \mathrm{Mod}_\mu(\tilde{U})$ such that $M = \tilde{\theta}_\ell^+(M')$.*

Proof. Observe that the canonical adjunction morphism $\tilde{\theta}_\ell^+ \cdot \tilde{\theta}^- M \rightarrow M$ is an isomorphism if and only if the induced morphism

$$\mathrm{Hom}(M, N) \rightarrow \mathrm{Hom}(\tilde{\theta}_\ell^+ \cdot \tilde{\theta}^- M, N) = \mathrm{Hom}(\tilde{\theta}^- M, \tilde{\theta}^- N)$$

is an isomorphism, for every N . In the latter case we may put $M' = \tilde{\theta}^- M$. Conversely, if $M = \tilde{\theta}_\ell^+(M')$, then we have $\tilde{\theta}_\ell^+ \cdot \tilde{\theta}^- M = \tilde{\theta}_\ell^+ \cdot (\tilde{\theta}^- \cdot \tilde{\theta}_\ell^+ M') = \tilde{\theta}_\ell^+ M' = M$ (by Proposition 3.1(ii)), and the result follows. \square

Recall that to any $M \in \mathrm{Mod}_{|\mu|}(\mathrm{Ug})$ we can associate (cf. Lemma 1.3) the \tilde{U} -module $\tilde{M} := \tilde{U} \otimes_{\tilde{U}^{W_\mu}} M$. Note that the module \tilde{M} has a natural W_λ -action induced from the W_λ -action on \tilde{U} , which commutes with the Ug -action. By functoriality, this gives a W_λ -action on the Ug -module $\mathrm{Res}_\lambda \cdot \tilde{\theta}_\ell^+(\tilde{M})$. We have

Proposition 3.4. *There is a functorial isomorphism:*

$$\theta^+ M = \left(\mathrm{Res}_\lambda \cdot \tilde{\theta}_\ell^+(\tilde{U} \otimes_{\tilde{U}^{W_\mu}} M) \right)^{W_\lambda}, \quad M \in \mathrm{Mod}_{|\mu|}(U).$$

Proof. It is clear that there is a functor isomorphism $\theta^- \cdot \mathrm{Res}_\lambda = \mathrm{Res}_\mu \cdot \tilde{\theta}^- : \mathrm{Mod}_\lambda(\tilde{U}) \xrightarrow{\sim} \mathrm{Mod}_{|\mu|}(\mathrm{Ug})$. Taking adjoints on each side, and using Lemma 1.3 we obtain

$$\tilde{\theta}^+ \tilde{M} = \tilde{U} \otimes_{\tilde{U}^{W_\lambda}} (\theta^+ M) = \tilde{\theta}_\ell^+(\tilde{U} \otimes_{\tilde{U}^{W_\mu}} M) = \tilde{\theta}_\ell^+(\tilde{M}).$$

The isomorphisms above are compatible with the W_λ -actions. We take W_λ -invariants on each side of the isomorphism. Note that \tilde{U} is a free \tilde{U}^{W_λ} -module isomorphic to $\mathbb{C}[W_\lambda] \otimes_{\mathbb{C}} \tilde{U}^{W_\lambda}$ as a W_λ -module. Hence taking W_λ -invariants of the leftmost term

in the above isomorphisms yields $(\tilde{U} \otimes_{\tilde{U}^{W_\lambda}} (\theta^+ M))^{W_\lambda} = \theta^+ M$. Comparing with the W_λ -invariants of the rightmost term completes the proof. \square

Proposition 3.5. *There is a functorial isomorphism*

$$\theta^- \cdot \theta^+ M \simeq \text{Res}_\mu(\tilde{U}^{W_\lambda} \otimes_{\tilde{U}^{W_\mu}} M).$$

Proof. As in the proof of Proposition 3.4 we have $\widetilde{\theta^+ M} = \tilde{\theta}_\ell^+(\tilde{M})$. The functor θ^- clearly commutes with the functor $\text{Res}_\bullet(\tilde{U} \otimes_{\tilde{U}^{W_\mu}} \bullet)$ and with the W -actions. Hence, we obtain

$$\begin{aligned} \text{Res}_\mu(\tilde{U} \otimes_{\tilde{U}^{W_\lambda}} (\theta^- \cdot \theta^+ M)) &= \theta^- \cdot \text{Res}_\lambda \cdot \tilde{\theta}_\ell^+(\tilde{M}) \\ &= \text{Res}_\mu \cdot \tilde{\theta}^- \cdot \tilde{\theta}_\ell^+(\tilde{M}) = \text{Res}_\mu(\tilde{M}). \end{aligned}$$

Taking W_λ -invariants on each side completes the proof. \square

4. APPLICATIONS TO THE CATEGORY \mathcal{O}

Fix a Cartan and Borel subalgebras $\mathfrak{h} \subset \mathfrak{b} \subset \mathfrak{g}$. Thus we may identify this Cartan subalgebra with the abstract Cartan subalgebra $\mathfrak{b}/[\mathfrak{b}, \mathfrak{b}]$ via the composition $\mathfrak{h} \hookrightarrow \mathfrak{b} \twoheadrightarrow \mathfrak{b}/[\mathfrak{b}, \mathfrak{b}]$. Given $\lambda \in \mathfrak{h}^*$, we define the category \mathcal{O}_λ as the full subcategory of $\text{Mod}_{|\lambda|}(\text{U}\mathfrak{g})$ formed by the $\text{U}\mathfrak{g}$ -modules M such that

- $\text{U}\mathfrak{b}$ -action on M is locally finite, and
- $\text{U}\mathfrak{h}$ -action on M is diagonalizable.

Given a weight $\mu \in \mathfrak{h}^*$, view it as a 1-dimensional \mathfrak{b} -module, \mathbb{C}_μ , via the projection $\mathfrak{b} \twoheadrightarrow \mathfrak{b}/[\mathfrak{b}, \mathfrak{b}] = \mathfrak{h} \xrightarrow{\mu} \mathbb{C}$. Let $M_\mu := \text{U}\mathfrak{g} \otimes_{\text{U}\mathfrak{b}} \mathbb{C}_\mu$ be the Verma module with highest weight μ . We write L_μ for its simple quotient. Then, for any $w \in W$, we have $M_{w \cdot \lambda}, L_{w \cdot \lambda} \in \mathcal{O}_\lambda$. The category \mathcal{O}_λ is known ([BGG]) to have enough projectives, and we let $P_{w \cdot \lambda}$ denote the indecomposable projective cover of $L_{w \cdot \lambda}$ in \mathcal{O}_λ . We refer to [BGG] for more properties of the category \mathcal{O}_λ .

Clearly $-\rho$ is the fixed point of the dot-action. The Verma module $M_{-\rho}$ is simple and is the unique simple object of the category $\mathcal{O}_{-\rho}$. Moreover, $M_{-\rho}$ is also a projective in $\mathcal{O}_{-\rho}$, for $-\rho$ it is a ρ -dominant weight. Thus, $\mathcal{O}_{-\rho}$ is a semisimple category and any object of $\mathcal{O}_{-\rho}$ is isomorphic to a direct sum of finitely many copies of $M_{-\rho}$.

Now let λ and μ denote integral ρ -dominant weights such that $W_\lambda \subset W_\mu$. The adjoint functors $(\theta_\lambda^\mu, \theta_\mu^\lambda) : \mathcal{O}_\lambda \rightleftarrows \mathcal{O}_\mu$ are exact, hence, take projectives into projectives. Further, it is known and easy to prove (see e.g. [Ja]) that, for any $w \in W$, one has $\theta_\lambda^\mu M_{w \cdot \lambda} = M_{w \cdot \mu}$. Therefore, by the exactness, one gets:

$$\theta_\lambda^\mu L_{w \cdot \lambda} = \begin{cases} L_{w \cdot \mu} & \text{if } w \in W_\mu \cdot w_0 \\ 0 & \text{if } w \neq W_\mu \cdot w_0. \end{cases}$$

It follows, by adjunction, that the exact functor $\text{Hom}_{\mathcal{O}_\lambda}(\theta_\mu^\lambda P_\mu, \bullet)$ on \mathcal{O}_λ kills all simple modules but L_λ . Thus, we obtain

Proposition 4.1. $\theta_\mu^\lambda P_\mu = P_\lambda$.

From now on let λ be an integral ρ -dominant weight. Proposition 4.1 yields (for $\mu = -\rho$):

$$(4.2) \quad P_\lambda = \theta_{-\rho}^\lambda M_{-\rho}.$$

Let I^W be the ideal in $\mathbb{C}[\mathfrak{h}^*]$ generated by all W -invariant polynomials without constant term. Set $C = \mathbb{C}[\mathfrak{h}^*]/I^W$, the coinvariant algebra. The W -action on $\mathbb{C}[\mathfrak{h}^*]$ induces a W -action on C making it a regular representation of W . For each $\lambda \in \mathfrak{h}^*$ we have the subalgebra $C^{W_\lambda} \subset C$.

The results of the previous section enable us to give a short proof of the following important theorem that was implicitly conjectured in [BG] and was first proved in [S1, Thm. 3] (see also [Be]).

Endomorphism-theorem 4.3. *There is a canonical algebra isomorphism*

$$\text{End}_{\text{Ug}} P_\lambda \simeq C^{W_\lambda}.$$

Proof. Set $\theta^+ = \theta_{-\rho}^\lambda$ and $\theta^- = \theta_\lambda^{-\rho}$. From (4.2) and Proposition 3.5 we get

$$\begin{aligned} \text{Hom}_{\text{Ug}}(P_\lambda, P_\lambda) &= \text{Hom}_{\text{Ug}}(\theta^+ M_{-\rho}, \theta^+ M_{-\rho}) \\ &= \text{Hom}_{\text{Ug}}(M_{-\rho}, \theta^- \cdot \theta^+ M_{-\rho}) \\ &= \text{Hom}_{\text{Ug}}\left(M_{-\rho}, \text{Res}_{-\rho}(\tilde{U}^{W_\lambda} \otimes_{\text{Ug}} M_{-\rho})\right) \\ &= \tilde{U}^{W_\lambda} / \tilde{U}^{W_\lambda} \cdot \mathcal{J}_{-\rho} \simeq C^{W_\lambda}. \end{aligned}$$

To complete the proof we must show that the chain of isomorphisms above gives rise to an algebra map $\text{End}_{\text{Ug}} P_\lambda \rightarrow C^{W_\lambda}$. To this end, observe that the functor θ^- induces a ring homomorphism:

(4.3.1)

$$\text{End}_{\text{Ug}} P_\lambda \rightarrow \text{End}_{\tilde{U}^{W_\lambda}}(\theta^- P_\lambda) = \text{End}_{\tilde{U}^{W_\lambda}}(\theta^- \theta^+ M_{-\rho}) = \text{End}_{\tilde{U}^{W_\lambda}}(\tilde{U}^{W_\lambda} \otimes_{\text{Ug}} M_{-\rho}).$$

Observe further that the action of the central subalgebra $S\mathfrak{h}^{W_\lambda} \subset \tilde{U}^{W_\lambda}$ on $\tilde{U}^{W_\lambda} \otimes_{\text{Ug}} M_{-\rho}$ gives an algebra isomorphism $\tau : C^{W_\lambda} \xrightarrow{\sim} \text{End}_{\tilde{U}^{W_\lambda}}(\tilde{U}^{W_\lambda} \otimes_{\text{Ug}} M_{-\rho})$, since $M_{-\rho}$ is a simple Ug -module. The result now follows by composing (4.3.1) with τ^{-1} . \square

Let $B \subset G$ be the Borel subgroup corresponding to the Borel subalgebra \mathfrak{b} . Write $\mathcal{B}_w = B \cdot w \cdot B/B \subset \mathcal{B}$ for the Bruhat cell in the Flag manifold corresponding to $w \in W$. Let $j_w : \mathcal{B}_w \hookrightarrow \mathcal{B}$ denote the embedding, and $\mathcal{M}_w^* := (j_w)_* \mathcal{O}_{\mathcal{B}_w}$, the standard $\mathcal{D}_{\mathcal{B}}$ -module “of holomorphic distributions,” supported on $\overline{\mathcal{B}}_w$. Let $\mathbf{D}M_{w,\lambda}$ denote the dual (in the category \mathcal{O}) of the Verma module $M_{w,\lambda}$.

The following result is known; see e.g. [BB3].

Proposition 4.4. *Let λ be an integral ρ -dominant weight. Then, for any $w \in W^\lambda$ and $y \in W$, there are natural isomorphisms of $\mathcal{D}(\lambda)$ - and Ug -modules respectively:*

$$\Delta_\lambda(\mathbf{D}M_{w,\lambda}) \simeq \mathcal{O}(\lambda) \otimes_{\mathcal{O}_{\mathcal{B}}} \mathcal{M}_{ww_0}^* \quad \text{and} \quad \Gamma(\mathcal{B}, \mathcal{O}(\lambda) \otimes \mathcal{M}_{yw_0}^*) \simeq \mathbf{D}M_{y,\lambda}. \quad \square$$

Following [S1], define an exact functor $\mathbb{V} : \mathcal{O}_\lambda \rightarrow C^{W_\lambda}\text{-Mod}$ by the formula $\mathbb{V} : M \mapsto \text{Hom}_{\text{Ug}}(P_\lambda, M)$, where the Hom-space is viewed as an $\text{End}_{\text{Ug}} P_\lambda$ -module, hence a C^{W_λ} -module, via composition.

One may reinterpret the functor \mathbb{V} as follows. We have by (4.2), $\mathbb{V}(M) = \text{Hom}_{\text{Ug}}(M_{-\rho}, \theta_\lambda^{-\rho} M)$. The C^{W_λ} -module structure on $\mathbb{V}(M)$ arises from a natural $\mathbb{C}[\mathfrak{h}^*]^{W_\lambda}$ -action on $\theta_\lambda^{-\rho} M$. The latter action was actually defined in Section 1. Namely, view M as a \tilde{U}^{W_λ} -module, via Lemma 1.2. This gives a $\mathbb{C}[\mathfrak{h}]^{W_\lambda}$ -action on M which induces a $\mathbb{C}[\mathfrak{h}^*]^{W_\lambda}$ -action on $\theta_\lambda^{-\rho} M$, by functoriality. Furthermore, the

subalgebra $Z(\mathfrak{g}) \simeq \mathbb{C}[\mathfrak{h}]^W \subset \mathbb{C}[\mathfrak{h}^*]^{W_\lambda}$ acts trivially on $\mathrm{Hom}_{\mathrm{U}\mathfrak{g}}(M_{-\rho}, \theta_\lambda^{-\rho} M)$, for it obviously acts trivially on $M_{-\rho}$. Thus, the action on $\mathbb{V}(M)$ factors through C^{W_λ} .

Using the results of the previous sections we can simplify the proof of the key theorem ([S1, Struktursatz 2]) saying that the functor \mathbb{V} is faithful on projectives and injectives; we have

Theorem 4.5. *For any injective module $I \in \mathcal{O}_\lambda$ and any $M \in \mathcal{O}_\lambda$, the natural morphism*

$$\mathrm{Hom}_{\mathrm{U}\mathfrak{g}}(I, M) \longrightarrow \mathrm{Hom}_{C^{W_\lambda}}(\mathbb{V}(I), \mathbb{V}(M))$$

is an isomorphism.

Remark 4.6. The functor $\theta_\lambda^{-\rho}$, hence \mathbb{V} , clearly commutes with the standard duality $N \mapsto \mathbf{D}N$ on the category \mathcal{O} . Furthermore, I is an injective in \mathcal{O}_λ if and only if $\mathbf{D}I$ is a projective in \mathcal{O}_λ . Thus, dualizing Theorem 4.5 we get that, for any projective $P \in \mathcal{O}_\lambda$ and any $M \in \mathcal{O}_\lambda$, the natural morphism

$$\mathrm{Hom}_{\mathrm{U}\mathfrak{g}}(M, P) \longrightarrow \mathrm{Hom}_{C^{W_\lambda}}(\mathbb{V}(M), \mathbb{V}(P))$$

is an isomorphism.

Let $M \mapsto \widetilde{M} = \widetilde{U} \otimes_{\widetilde{C}^{W_\lambda}} M$ be the left adjoint of the functor $\mathrm{Res}_\lambda : \mathrm{Mod}_\lambda(\widetilde{U}) \rightarrow \mathrm{Mod}_{|\lambda|}(\mathrm{U}\mathfrak{g})$; cf. Lemma 1.3. The W_λ -action on \widetilde{U} induces a W_λ -action on \widetilde{M} and, for any $M, N \in \mathrm{Mod}_{|\lambda|}(\mathrm{U}\mathfrak{g})$, there is a canonical isomorphism

$$(4.7) \quad \mathrm{Hom}_{\mathrm{U}\mathfrak{g}}(M, N) \xrightarrow{\sim} \mathrm{Hom}_{\widetilde{C}^{W_\lambda}}(\widetilde{M}, \widetilde{N})^{W_\lambda}.$$

We will deduce Theorem 4.5 from the following result.

Lemma 4.8. *For any injective $I \in \mathcal{O}_\lambda$, the adjunction morphism $\widetilde{\theta}_\ell^+ \cdot \widetilde{\theta}^-(\widetilde{I}) \rightarrow \widetilde{I}$ (where $\theta^+ = \theta_{-\rho}^\lambda$) is an isomorphism.*

Proof of the Lemma. Let $\mathbf{D}M_\lambda$ be the dominant dual Verma module, viewed as a \widetilde{U} -module via the projection $\widetilde{U} \rightarrow \widetilde{U}/\widetilde{U} \cdot \mathcal{J}_\lambda \simeq \mathrm{U}\mathfrak{g}/\mathrm{U}\mathfrak{g} \cdot \mathcal{J}_{|\lambda|}$, and let $M_{-\rho}$ be viewed as a \widetilde{U} -module in a similar way. By Proposition 3.1(iii) we have, $\widetilde{\theta}_\ell^+ M_{-\rho} = \widetilde{\theta}_\ell^+(\mathbf{D}M_{-\rho}) = \Gamma_\lambda \cdot \Theta_{-\rho}^\lambda \cdot \Delta_{-\rho}(\mathbf{D}M_{-\rho})$. Hence, Proposition 4.4 yields

$$\widetilde{\theta}_\ell^+(M_{-\rho}) = \mathbf{D}M_\lambda.$$

It follows from Corollary 3.3 that the adjunction morphism $\widetilde{\theta}_\ell^+ \cdot \widetilde{\theta}^-(\mathbf{D}M_\lambda) \rightarrow \mathbf{D}M_\lambda$ is an isomorphism.

Now, $\mathbf{D}M_\lambda$ is an injective in \mathcal{O}_λ . Furthermore, it was shown in [BeGe, Thm. 3.3(b)] that any injective in \mathcal{O}_λ can be written in the form $I = \Phi(\mathbf{D}M_\lambda)$ for an appropriate projective functor Φ (see Section 6 below). Hence, $\widetilde{I} = \widetilde{\Phi}(\mathbf{D}M_\lambda)$ and the lemma follows from Proposition 6.6 (in Section 6), which is independent of the intervening material. \square

Proof of Theorem 4.5. Let I be an injective in \mathcal{O}_λ . Then, for any $M \in \mathcal{O}_\lambda$, from Lemma 4.8 and Corollary 3.3 we obtain an isomorphism

$$\mathrm{Hom}_{\widetilde{C}^{W_\lambda}}(\widetilde{I}, \widetilde{M}) \xrightarrow{\sim} \mathrm{Hom}_C(\widetilde{\theta}^-(\widetilde{I}), \widetilde{\theta}^-(\widetilde{M})).$$

Taking W_λ -invariants on each side and using (4.7) completes the proof. \square

5. CONVOLUTION OF HARISH-CHANDRA MODULES

Throughout this section λ and μ stand for integral ρ -dominant weights.

Let $U_{\mathfrak{g}}_{|\lambda|}$ and \tilde{U}_λ denote the completions of the algebras $U_{\mathfrak{g}}$ and \tilde{U} with respect to the $I_{|\lambda|}$ -adic and \mathcal{J}_λ -adic topology, respectively (recall that $I_{|\lambda|} \subset Z(\mathfrak{g})$ and $\mathcal{J}_\lambda \subset \mathbb{C}[\mathfrak{h}^*]$ are maximal ideals). Since $U_{\mathfrak{g}}$ is a Noetherian algebra, its $I_{|\lambda|}$ -adic completion, $U_{\mathfrak{g}}_{|\lambda|}$ is also Noetherian. Hence, the Artin-Rees lemma implies that any finitely-generated $U_{\mathfrak{g}}_{|\lambda|}$ -module is complete with respect to the $I_{|\lambda|}$ -adic topology. Therefore, finitely-generated $U_{\mathfrak{g}}_{|\lambda|}$ -modules form an abelian category. Similar considerations apply to the algebra \tilde{U}_λ .

By a *complete* $U_{\mathfrak{g}}$ -module, resp. \tilde{U} -module, we will mean an $U_{\mathfrak{g}}$ -module M which is complete in the $I_{|\lambda|}$ -adic, resp. \mathcal{J}_λ -adic, topology, i.e., such that

$$M = \lim \operatorname{proj} (M/I_{|\lambda|}^n \cdot M).$$

A complete $U_{\mathfrak{g}}$ -module is not necessarily annihilated by some power of the ideal $I_{|\lambda|}$, but it is isomorphic to a limit of the projective system of $U_{\mathfrak{g}}$ -bimodules: $M/I_{|\lambda|}^1 \cdot M \leftarrow M/I_{|\lambda|}^2 \cdot M \leftarrow \dots$. The algebra $U_{\mathfrak{g}}_{|\lambda|}$ is an example of a complete $U_{\mathfrak{g}}$ -module. Furthermore, any complete $U_{\mathfrak{g}}$ -module M has a natural structure of a $U_{\mathfrak{g}}_{|\lambda|}$ -module, and we will often make no distinction between these $U_{\mathfrak{g}}$ -modules and $U_{\mathfrak{g}}_{|\lambda|}$ -module structures on M .

Below, finitely generated left $U_{\mathfrak{g}}_{|\lambda|} \otimes_{\mathbb{C}} U_{\mathfrak{g}}_{|\lambda|}^{opp}$ -modules will be referred to as finitely generated $U_{\mathfrak{g}}_{|\lambda|}$ -bimodules, and similar terminology will be used for $\tilde{U}_\lambda \otimes_{\mathbb{C}} \tilde{U}_\lambda^{opp}$ -modules. Let $\operatorname{HC}_{|\lambda|}$, resp. $\widetilde{\operatorname{HC}}_\lambda$, be the category of finitely-generated *complete* $U_{\mathfrak{g}}_{|\lambda|}$ -bimodules (resp. \tilde{U}_λ -bimodules) M such that the adjoint \mathfrak{g} -action $\operatorname{ad} x : m \mapsto x \cdot m - m \cdot x$ on M is locally-finite. Any object of $\operatorname{HC}_{|\lambda|}$ is finitely-generated both as a left and as a right $U_{\mathfrak{g}}_{|\lambda|}$ -module. Therefore, the Artin-Rees lemma insures, as explained in the first paragraph of this section, that $\operatorname{HC}_{|\lambda|}$ is an abelian category. Similar remarks apply to the category $\widetilde{\operatorname{HC}}_\lambda$.

To localize complete $U_{\mathfrak{g}}_{|\lambda|}$ -bimodules we recall the imbedding $\mathbb{C}[\mathfrak{h}^*] \subset \tilde{\mathcal{D}}$, and form the sheaf $\tilde{\mathcal{D}}_\lambda$, the \mathcal{J}_λ -adic completion of the algebra $\tilde{\mathcal{D}}$. There is an algebra isomorphism $\Gamma(\mathcal{B}, \tilde{\mathcal{D}}_\lambda) = \tilde{U}_\lambda$. For any sheaf \mathcal{M} of coherent $\tilde{\mathcal{D}} \boxtimes \tilde{\mathcal{D}}^{opp}$ -modules on $\mathcal{B} \times \mathcal{B}$, the space $\Gamma(\mathcal{B} \times \mathcal{B}, \mathcal{M})$ has a natural $\tilde{U} \otimes \tilde{U}^{opp}$ -module structure. However, the assignment $\mathcal{M} \mapsto \Gamma(\mathcal{B} \times \mathcal{B}, \mathcal{M})$ is not an exact functor, for Γ is not exact on *right* $\tilde{\mathcal{D}}$ -modules. Thus, one has to localize \tilde{U} -bimodules in a different way, as we now explain.

We begin with a well known observation that if \mathcal{M} is a left \mathcal{D}_X -module on an algebraic variety X , and Ω_X is the line bundle of top-degree regular forms on X , then the sheaf $\Omega_X \otimes_{\mathcal{O}_X} \mathcal{M}$ has a natural right \mathcal{D}_X -module structure. Put another way, there is a canonical algebra isomorphism

$$(5.1) \quad \mathcal{D}_X^{opp} \xrightarrow{\sim} \Omega_X \otimes_{\mathcal{O}_X} \mathcal{D}_X \otimes_{\mathcal{O}_X} \Omega_X^{-1}.$$

Now, let $X = \mathcal{B}$ be the flag manifold. Then $\Omega_{\mathcal{B}} = \mathcal{O}(-2\rho)$; further, it is clear that, for any integral λ , there is a natural isomorphism

$$(5.2) \quad \tilde{\mathcal{D}}_\lambda \simeq \mathcal{O}(\lambda) \otimes_{\mathcal{O}_{\mathcal{B}}} \mathcal{D}_{\mathcal{B}} \otimes_{\mathcal{O}_{\mathcal{B}}} \mathcal{O}(-\lambda).$$

Formulas (5.1)–(5.2) yield an isomorphism $\tilde{\mathcal{D}}_\lambda^{opp} \xrightarrow{\sim} \tilde{\mathcal{D}}_{-\lambda-2\rho}$. Taking global sections we get canonical algebra isomorphisms

$$(5.3) \quad \tau : \tilde{U}_\lambda^{opp} \simeq \Gamma(\mathcal{B}, \tilde{\mathcal{D}}_\lambda^{opp}) \xrightarrow{\sim} \Gamma(\mathcal{B}, \tilde{\mathcal{D}}_{-\lambda-2\rho}) \simeq \tilde{U}_{-\lambda-2\rho}.$$

It was shown in [BB3, 3.2.1] that isomorphisms (5.3) are obtained by specializing at λ a “universal” isomorphism $\tilde{U}^{opp} \xrightarrow{\sim} \tilde{U}$. This way one deduces the following result.

Lemma 5.4. (i) *There is a principal anti-involution $\tau : \tilde{U} \rightarrow \tilde{U}$ that, for any λ , induces the isomorphism $\tilde{U}_\lambda^{opp} \xrightarrow{\sim} \tilde{U}_{-\lambda-2\rho}$ of (5.3).*

(ii) *The anti-involution τ induces anti-involutions $\tau_{\mathfrak{g}}$ and $\tau_{\mathfrak{h}}$ on the subalgebras $\mathbf{U}_{\mathfrak{g}} \subset \tilde{U}$ and $\mathbb{C}[\mathfrak{h}^*] \subset \tilde{U}$, respectively. The anti-involutions $\tau_{\mathfrak{g}}$ and $\tau_{\mathfrak{h}}$ are given on generators $x \in \mathfrak{g} \subset \mathbf{U}_{\mathfrak{g}}$ and $h \in \mathfrak{h} \subset \mathbb{C}[\mathfrak{h}^*]$, by the formulas:*

$$x \mapsto -x, \quad h \mapsto -h - 2 \cdot \rho(h).$$

(iii) *The Harish-Chandra isomorphism $Z(\mathfrak{g}) \xrightarrow{\sim} \mathbb{C}[\mathfrak{h}^*]^W$ intertwines the restriction of $\tau_{\mathfrak{g}}$ to $Z(\mathfrak{g})$ with the restriction of $\tau_{\mathfrak{h}}$ to $\mathbb{C}[\mathfrak{h}^*]^W$. \square*

Recall next that the Weyl group W acts on \tilde{U} and the action of $w \in W$ induces an isomorphism of completed algebras $w : \tilde{U}_{w \cdot \mu} \xrightarrow{\sim} \tilde{U}_\mu$. Let w_0 be the longest element of W . Observe that, for any λ , we have $w_0(\lambda) - 2\rho = w_0(\lambda + \rho) - \rho = w_0 \cdot \lambda$. Therefore, composing the automorphism of \tilde{U} induced by w_0 with isomorphism (5.3) we obtain the following isomorphisms

$$(5.5) \quad \Gamma(\mathcal{B}, \tilde{\mathcal{D}}_{-w_0(\lambda)}) \xrightarrow{\sim} \tilde{U}_{-w_0(\lambda)} \xrightarrow{\tau} \tilde{U}_{w_0(\lambda)-2\rho}^{opp} = \tilde{U}_{w_0 \cdot \lambda}^{opp} \xrightarrow{w_0} \tilde{U}_\lambda^{opp}.$$

Thus, for any $\tilde{\mathcal{D}}_{-w_0(\lambda)}$ -module \mathcal{N} , the space $\Gamma(\mathcal{B}, \mathcal{N})$ acquires, via (5.5) a natural right \tilde{U}_λ -module structure. Similarly, the assignment $\mathcal{M} \mapsto \Gamma(\mathcal{B} \times \mathcal{B}, \mathcal{M})$ gives a functor from the category of left $\tilde{\mathcal{D}}_\lambda \boxtimes \tilde{\mathcal{D}}_{-w_0(\lambda)}$ -modules to the category of \tilde{U}_λ -bimodules. Since both λ and $-w_0(\lambda)$ are ρ -dominant weights, this functor is exact, by Theorem 2.2. It will be denoted by Γ^\diamond in the future.

Next, recall that a \mathcal{D} -module \mathcal{M} on $\mathcal{B} \times \mathcal{B}$ is said to be G -equivariant if \mathcal{M} is a G -equivariant sheaf of $\mathcal{O}_{\mathcal{B} \times \mathcal{B}}$ -modules and the differential of the G -action coincides with the \mathfrak{g} -action on \mathcal{M} arising from the \mathcal{D} -module structure via the imbedding $\mathfrak{g} \hookrightarrow \tilde{\mathcal{D}}_\lambda \boxtimes \tilde{\mathcal{D}}_{-w_0(\lambda)}$. Let \mathcal{HC}_λ be the category of sheaves of complete coherent $\tilde{\mathcal{D}}_\lambda \boxtimes \tilde{\mathcal{D}}_{-w_0(\lambda)}$ -modules on $\mathcal{B} \times \mathcal{B}$ that are G -equivariant with respect to the diagonal G -action on $\mathcal{B} \times \mathcal{B}$. For any $\mathcal{M} \in \mathcal{HC}_\lambda$, the adjoint \mathfrak{g} -action on the \tilde{U}_λ -bimodule $\Gamma(\mathcal{B} \times \mathcal{B}, \mathcal{M})$ is locally-finite because of the G -equivariance of \mathcal{M} . Thus, $\Gamma^\diamond(\mathcal{M})$ becomes an object of $\widetilde{\mathcal{HC}}_\lambda$. Moreover, the functor $\Gamma^\diamond : \mathcal{HC}_\lambda \rightarrow \widetilde{\mathcal{HC}}_\lambda$ is an equivalence of categories provided λ is regular (Theorem 2.2).

Given an algebra A , resp. algebras A and B , we write $A\text{-Mod}$ for the category of left A -modules, resp. $\text{Mod-}A$ or $A\text{-Mod-}B$ for the category of right A -modules, or left $A \boxtimes B^{opp}$ -modules (= $A - B$ -bimodules). We form the bounded derived categories $D^b(\tilde{U}_\lambda\text{-Mod})$, $D^b(\tilde{U}_\lambda\text{-Mod-}\tilde{U}_\mu)$, and $D^b(\tilde{\mathcal{D}}_\lambda \boxtimes \tilde{\mathcal{D}}_\mu\text{-Mod})$. Let $\mathbb{D}\text{Mod}_\lambda(\tilde{U})$, $\mathbb{D}\widetilde{\mathcal{HC}}_\lambda$, and $\mathbb{D}\mathcal{HC}_\lambda$ be the full triangulated subcategories of the categories $D^b(\tilde{U}\text{-Mod})$, (resp. $D^b(\tilde{U}_\lambda\text{-Mod-}\tilde{U}_\lambda)$, and $D^b(\tilde{\mathcal{D}}_\lambda \boxtimes \tilde{\mathcal{D}}_{-w_0(\lambda)}\text{-Mod})$) formed by the objects whose cohomology belong to the categories $\text{Mod}_\lambda(\tilde{U})$, (resp. $\widetilde{\mathcal{HC}}_\lambda$ and \mathcal{HC}_λ). The functor Γ^\diamond has a natural extension to a derived functor $R\Gamma^\diamond : \mathbb{D}\mathcal{HC}_\lambda \rightarrow \mathbb{D}\widetilde{\mathcal{HC}}_\lambda$.

We now introduce certain convolution functors on our categories. First, define a functor $D^b(\widetilde{U}_\lambda\text{-Mod-}\widetilde{U}_\mu) \times D^b(\widetilde{U}_\mu\text{-Mod-}\widetilde{U}_\nu) \rightarrow D^b(\widetilde{U}_\lambda\text{-Mod-}\widetilde{U}_\nu)$ by the formula $M \star N = M \overset{L}{\otimes}_{\widetilde{U}_\mu} N$. It is easy to verify that the functor so defined restricts to a functor

$$\star : \mathbb{D}\widetilde{\mathcal{H}}\mathcal{C}_\lambda \times \mathbb{D}\widetilde{\mathcal{H}}\mathcal{C}_\lambda \rightarrow \mathbb{D}\widetilde{\mathcal{H}}\mathcal{C}_\lambda .$$

Observe next that one may regard any object $A \in \text{Mod}_\lambda(\widetilde{U})$ as an \widetilde{U}_λ -module on which \widetilde{U}_λ acts through a certain quotient $\widetilde{U}_\lambda/\mathcal{J}_\lambda^n \cdot \widetilde{U}_\lambda$. Furthermore, given $M \in \mathbb{D}\widetilde{\mathcal{H}}\mathcal{C}_\lambda$, one verifies that if all the cohomology of an object $A \in \mathbb{D}\text{Mod}_\lambda(\widetilde{U})$ are killed by some power of the ideal \mathcal{J}_λ , then a similar vanishing holds for the cohomology of $M \star A$. This allows us to define a functor $\mathbb{D}\widetilde{\mathcal{H}}\mathcal{C}_\lambda \times \mathbb{D}\text{Mod}_\lambda(\widetilde{U}) \rightarrow \mathbb{D}\text{Mod}_\lambda(\widetilde{U})$ by the formula $M, A \mapsto M \star A = M \overset{L}{\otimes}_{\widetilde{U}_\lambda} A$. For any $M, N, L \in \mathbb{D}\widetilde{\mathcal{H}}\mathcal{C}_\lambda$, and $A \in \mathbb{D}\text{Mod}_\lambda(\widetilde{U})$, there are natural functorial isomorphisms (associativity):

$$(M \star N) \star L \simeq M \star (N \star L) \quad \text{and} \quad M \star (N \star A) \simeq (M \star N) \star A .$$

To define analogous functors on \mathcal{D} -modules we introduce the projections $p_{ij} : \mathcal{B} \times \mathcal{B} \times \mathcal{B} \rightarrow \mathcal{B} \times \mathcal{B}$, and $p_i : \mathcal{B} \times \mathcal{B} \times \mathcal{B} \rightarrow \mathcal{B}$, $i, j \in \{1, 2, 3\}$, along the factors not named. Let p_{ij}^* , p_i^* stand for the corresponding inverse image functors in the category of \mathcal{O} -sheaves. Given a triple of weights (λ, μ, ν) , and two objects $\mathcal{M} \in D^b(\widetilde{\mathcal{D}}_\lambda \boxtimes \widetilde{\mathcal{D}}_\mu\text{-Mod})$, and $\mathcal{N} \in D^b(\widetilde{\mathcal{D}}_{-\mu-2\rho} \boxtimes \widetilde{\mathcal{D}}_\nu\text{-Mod})$, define an object $\mathcal{M} \star \mathcal{N} \in D^b(\widetilde{\mathcal{D}}_\lambda \boxtimes \widetilde{\mathcal{D}}_\nu\text{-Mod})$ by the formula:

$$(5.6) \quad \mathcal{M} \star \mathcal{N} := (Rp_{13})_\bullet (p_{12}^* \mathcal{M} \overset{L}{\otimes}_{p_2^* \widetilde{\mathcal{D}}_\mu} p_{23}^* \mathcal{N}) ,$$

where $(Rp_{13})_\bullet$ stands for the sheaf-theoretic derived direct image. It is easy to verify that (5.6) restricts to a functor $\star : \mathbb{D}\mathcal{H}\mathcal{C}_\lambda \times \mathbb{D}\mathcal{H}\mathcal{C}_\lambda \rightarrow \mathbb{D}\mathcal{H}\mathcal{C}_\lambda$.

Using the two projections $\text{pr}_i : \mathcal{B} \times \mathcal{B} \rightarrow \mathcal{B}$, one also defines a convolution-functor $\star : D^b(\widetilde{\mathcal{D}}_\lambda \boxtimes \widetilde{\mathcal{D}}_\mu\text{-Mod}) \times D^b(\widetilde{\mathcal{D}}_{-\mu-2\rho}\text{-Mod}) \rightarrow D^b(\widetilde{\mathcal{D}}_\lambda\text{-Mod})$ by the formula

$$\mathcal{M} \star \mathcal{A} := (R\text{pr}_1)_\bullet (\mathcal{M} \overset{L}{\otimes}_{\text{pr}_2^* \widetilde{\mathcal{D}}_\mu} \text{pr}_2^* \mathcal{A}) .$$

This convolution restricts to a functor $\star : \mathbb{D}\mathcal{H}\mathcal{C}_\lambda \times \mathbb{D}\text{Mod}_\lambda(\widetilde{\mathcal{D}}_\lambda) \rightarrow \mathbb{D}\text{Mod}_\lambda(\widetilde{\mathcal{D}}_\lambda)$.

The following observation will be useful for us in the next section.

Remark 5.7. Any object $M \in \mathbb{D}\widetilde{\mathcal{H}}\mathcal{C}_\lambda$ is completely determined by the corresponding convolution functor

$$M \star : \mathbb{D}\text{Mod}_\lambda(\widetilde{U}) \rightarrow \mathbb{D}\text{Mod}_\lambda(\widetilde{U}), \quad A \mapsto M \star A .$$

To see this, we first have to enlarge the category $\text{Mod}_\lambda(\widetilde{U})$ by adjoining all \widetilde{U}_λ -modules which are complete in the \mathcal{J}_λ -adic topology. Write $\widehat{\text{Mod}}_\lambda(\widetilde{U})$ for the latter category, and $\widehat{\mathbb{D}\text{Mod}}_\lambda(\widetilde{U})$ for the corresponding derived category enlargement of $\mathbb{D}\text{Mod}_\lambda(\widetilde{U})$. The functor $M \star (\cdot)$ is continuous in the \mathcal{J}_λ -adic topology, hence extends uniquely to a well defined functor $M \star : \widehat{\mathbb{D}\text{Mod}}_\lambda(\widetilde{U}) \rightarrow \widehat{\mathbb{D}\text{Mod}}_\lambda(\widetilde{U})$.

Observe that $\widetilde{U}_\lambda \in \widehat{\text{Mod}}_\lambda(\widetilde{U})$, as a left module, and clearly we have

$$(5.8) \quad M = M \star \widetilde{U}_\lambda = \varprojlim_n \left(M \star (\widetilde{U}/\mathcal{J}_\lambda^n \cdot \widetilde{U}) \right), \quad \forall M \in \mathbb{D}\widetilde{\mathcal{H}}\mathcal{C}_\lambda .$$

Further, notice that multiplication by $u \in \tilde{U}$ on the right gives an endomorphism of the left module $\tilde{U}/\mathcal{J}_\lambda^n \cdot \tilde{U}$, hence induces by functoriality a right \tilde{U}_λ -action on $M \star (\tilde{U}/\mathcal{J}_\lambda^n \cdot \tilde{U})$. This way, formula (5.8) recovers M from the functor $M \star (\cdot)$, as a \tilde{U}_λ -bimodule. \square

Let $\mathcal{D}(T)$ be the algebra of globally-defined regular differential operators on the torus T . View $S\mathfrak{h} \simeq \mathbb{C}[\mathfrak{h}^*]$ as the subalgebra of $\mathcal{D}(T)$ consisting of T -invariant differential operators. For each integer $n = 1, 2, \dots$, define the $\mathcal{D}(T)$ -module $\mathcal{E}^n = \mathcal{D}(T)/\mathcal{D}(T) \cdot \mathcal{J}_0^n$ where \mathcal{J}_0 denotes the augmentation ideal in $\mathbb{C}[\mathfrak{h}^*]$. Then, \mathcal{E}^n gives rise to a sheaf of regular holonomic \mathcal{D} -modules on T with unipotent monodromy, for T is an affine variety. These \mathcal{D} -modules form a natural projective system: $\mathcal{E}^1 \leftarrow \mathcal{E}^2 \leftarrow \dots$.

Now, let \mathbb{O} be the unique open G -orbit in $\mathcal{B} \times \mathcal{B}$, $\tilde{\mathbb{O}} := \pi^{-1}(\mathbb{O})$ the inverse image of \mathbb{O} in $\tilde{\mathcal{B}} \times \tilde{\mathcal{B}}$, and $j : \tilde{\mathbb{O}} \hookrightarrow \tilde{\mathcal{B}} \times \tilde{\mathcal{B}}$ the imbedding. The group G acts *freely* on $\tilde{\mathbb{O}}$ on the left and the space of left cosets, $G \backslash \tilde{\mathbb{O}}$, is canonically isomorphic to T , due to the Bruhat decomposition. Thus, we have a diagram:

$$(5.9.1) \quad \begin{array}{ccc} & \tilde{\mathbb{O}} & \\ & \mu \swarrow & \searrow \pi \\ T = G \backslash \tilde{\mathbb{O}} & & \mathbb{O} \end{array}$$

For each $n = 1, 2, \dots$, define a $\mathcal{D}_{\tilde{\mathcal{B}} \times \tilde{\mathcal{B}}}$ -module of “multivalued functions” on $\tilde{\mathbb{O}} \subset \tilde{\mathcal{B}} \times \tilde{\mathcal{B}}$ by $\mathcal{O}_*^n = j_*(\mu^* \mathcal{E}^n)$. The sheaf $\pi_* \tilde{\mathcal{O}}_*^n$ has a natural $\tilde{\mathcal{D}} \boxtimes \tilde{\mathcal{D}}$ -module structure. For any weight $\mu \in \mathfrak{h}^*$, we set

$$(5.9.2) \quad \mathcal{R}_\mu := \varprojlim_n \left((\tilde{\mathcal{D}}_\mu \boxtimes \tilde{\mathcal{D}}_{-w_0(\mu)}) \bigotimes_{\tilde{\mathcal{D}} \boxtimes \tilde{\mathcal{D}}} \pi_* \mathcal{O}_*^n \right).$$

We have $\mathcal{R}_\mu \in \mathcal{HC}_\mu$. The module \mathcal{R}_μ has the following local description. Locally on \mathbb{O} , one can trivialize the projection $\pi : \tilde{\mathcal{B}} \times \tilde{\mathcal{B}} \rightarrow \mathcal{B} \times \mathcal{B}$. Using such a trivialization, one can write

$$\tilde{\mathcal{B}} \times \tilde{\mathcal{B}} \cong \mathcal{B} \times \mathcal{B} \times T \times T \quad \text{and} \quad \tilde{\mathcal{D}}_\mu \boxtimes \tilde{\mathcal{D}}_{-w_0(\mu)} \cong \tilde{\mathcal{D}}_{\mathcal{B} \times \mathcal{B}} \otimes \mathbb{C}[\mathfrak{h}^*]_\mu \otimes \mathbb{C}[\mathfrak{h}^*]_{-w_0(\mu)},$$

where $\mathbb{C}[\mathfrak{h}^*]_\mu$ denotes the \mathcal{J}_μ -adic completion of $\mathbb{C}[\mathfrak{h}^*]$. Accordingly, locally on \mathbb{O} , one has $\mathcal{R}_\mu = \mathcal{O}_\circ \otimes \mathbb{C}[\mathfrak{h}^*]_\mu$. An element $1 \otimes a_1 \otimes a_2 \in \tilde{\mathcal{D}}_{\mathcal{B} \times \mathcal{B}} \otimes \mathbb{C}[\mathfrak{h}^*] \otimes \mathbb{C}[\mathfrak{h}^*]$, viewed as an element of $\tilde{\mathcal{D}}_\mu \boxtimes \tilde{\mathcal{D}}_{-w_0(\mu)}$ via the local factorization above, acts on $\mathcal{R}_\mu = \mathcal{O}_\circ \otimes \mathbb{C}[\mathfrak{h}^*]_\mu$ as multiplication by $1 \otimes a_1 \cdot w_0(\tau(a_2))$; see (5.3) for the meaning of $\tau(a_2)$.

We now define convolution functors $\mathbb{DHC}_\lambda \times \mathbb{DHC}_\lambda \rightarrow \mathbb{DHC}_\lambda$ and $\mathbb{DHC}_\lambda \times \mathbb{DMod}_\lambda(\tilde{D}) \rightarrow \mathbb{DMod}_\lambda(\tilde{D})$ by the formula:

$$(5.10) \quad \mathcal{M} \star \mathcal{N} = \mathcal{M} * \mathcal{R}_{w_0(\lambda)-2\rho} * \mathcal{N},$$

where $\mathcal{M} \in \mathbb{DHC}_\lambda$ and \mathcal{N} is an object of either \mathbb{DHC}_λ or $\mathbb{DMod}_\lambda(\tilde{D})$. Notice that unlike the \star -convolution in (5.10), the $*$ -convolution, $\mathcal{M} * \mathcal{N}$, without the middle factor $\mathcal{R}_{w_0(\lambda)-2\rho}$ is undefined, for the corresponding parameters λ, μ, ν in (5.6) do not match in the right way.

It turns out that the convolution (5.10) on \mathcal{D} -modules corresponds to the algebraic \star -convolution on \tilde{U}_λ -bimodules, i.e., we have the following:

Proposition 5.11. *For any $\mathcal{M} \in \mathbb{D}\mathcal{H}\mathcal{C}_\lambda$ and $\mathcal{N} \in \mathbb{D}\mathcal{H}\mathcal{C}_\lambda$, resp. $\mathcal{N} \in \mathbb{D}\text{Mod}_\lambda(\tilde{U})$, there is a natural isomorphism*

$$R\Gamma^\diamond(\mathcal{M} \star \mathcal{N}) \simeq R\Gamma^\diamond(\mathcal{M}) \star R\Gamma^\diamond(\mathcal{N}), \quad \text{resp. } R\Gamma(\mathcal{M} \star \mathcal{N}) \simeq R\Gamma(\mathcal{M}) \star R\Gamma(\mathcal{N}).$$

Proof. We have

$$\begin{aligned} R\Gamma(\mathcal{M} \star \mathcal{N}) &= R\Gamma(\mathcal{M} \star \mathcal{R}_{w_0(\lambda)-2\rho} \star \mathcal{N}) \\ (5.12) \qquad &= R\Gamma(\mathcal{M} \star \mathcal{R}_{w_0(\lambda)-2\rho}) \bigotimes_{\tilde{U}_\lambda}^L R\Gamma(\mathcal{N}), \end{aligned}$$

where in the second isomorphism we have that $\Gamma(\mathcal{B}, \tilde{\mathcal{D}}_\lambda) = \tilde{U}_\lambda$ and that the isomorphism clearly holds if $\mathcal{M} \star \mathcal{R}_{w_0(\lambda)-2\rho}$ and \mathcal{N} are replaced by free $\tilde{\mathcal{D}}$ -modules. \square

To compute the factor $R\Gamma(\mathcal{M} \star \mathcal{R}_{w_0(\lambda)-2\rho} \star \mathcal{N})$ in (5.12) we apply an extension of the localization theorem which holds for the derived category of $\tilde{\mathcal{D}}_\mu$ -modules even if μ is *not* dominant; see [BB2, §12]. Recall first that we have constructed an algebra isomorphism $\tilde{U}_\lambda^{\text{opp}} \simeq \tilde{U}_{-w_0(\lambda)}$; see (5.3). It gives rise to an exact functor $\mathbf{opp} : \tilde{U}_{-w_0(\lambda)}\text{-Mod} \rightarrow \text{Mod-}\tilde{U}_\lambda$ that can be extended to derived categories. On the other hand, the functor $\mathcal{M} \mapsto \mathcal{M} \star \mathcal{R}_{w_0(\lambda)-2\rho}$ takes $D^b(\tilde{\mathcal{D}}_{-w_0(\lambda)}\text{-Mod})$ to $D^b(\tilde{\mathcal{D}}_{-\lambda-2\rho}\text{-Mod})$. Using the canonical isomorphism $\tilde{\mathcal{D}}_{-\lambda-2\rho} \simeq \tilde{\mathcal{D}}_\lambda^{\text{opp}}$ we can view the latter functor as a functor $\star\mathcal{R} : D^b(\tilde{\mathcal{D}}_{-w_0(\lambda)}\text{-Mod}) \rightarrow D^b(\text{Mod-}\tilde{\mathcal{D}}_\lambda)$. Thus, Lemma 5.4 combined with [BB2, theorem in §12] yield

For any ρ -dominant λ the following diagram commutes:

$$\begin{array}{ccc} D^b(\tilde{\mathcal{D}}_{-w_0(\lambda)}\text{-Mod}) & \xrightarrow{\star\mathcal{R}} & D^b(\text{Mod-}\tilde{\mathcal{D}}_\lambda) \\ \downarrow R\Gamma & & \downarrow R\Gamma \\ D^b(\tilde{U}_{-w_0(\lambda)}\text{-Mod}) & \xrightarrow{\mathbf{opp}} & D^b(\text{Mod-}\tilde{U}_\lambda). \end{array}$$

Applying to (5.12) a bimodule version of this theorem we obtain

$$R\Gamma(\mathcal{B} \times \mathcal{B}, \mathcal{M} \star \mathcal{R}_{w_0(\lambda)-2\rho}) = \mathbf{opp}(R\Gamma(\mathcal{B} \times \mathcal{B}, \mathcal{M})).$$

Hence, $R\Gamma(\mathcal{M} \star \mathcal{N}) \simeq R\Gamma(\mathcal{M}) \bigotimes_{\tilde{U}_\lambda}^L R\Gamma(\mathcal{N})$, and the proposition follows.

It should be mentioned, perhaps, that the \star -convolution given by formula (5.6) is a $\tilde{\mathcal{D}}$ -analogue of a more familiar convolution of holonomic $\mathcal{D}_{\mathcal{B} \times \mathcal{B}}$ -modules. This latter convolution is defined by

$$(5.13) \quad \mathcal{M} \star \mathcal{N} := (p_{13})_* (p_{12}^* \mathcal{M} \bigotimes_{p_{23}^*}^L \mathcal{N}) \left(= \int_{p_{13}} p_{12}^* \mathcal{M} \bigotimes_{\mathcal{O}_{\mathcal{B} \times \mathcal{B} \times \mathcal{B}}}^L p_{23}^* \mathcal{N} \right),$$

where we are using the same notation for the projections $p_{ij} : \mathcal{B} \times \mathcal{B} \times \mathcal{B} \rightarrow \mathcal{B} \times \mathcal{B}$, as in (5.6), and $\int_{p_{13}}$ stands for a direct image of \mathcal{D} -modules. Notice that (5.13) goes under the Riemann-Hilbert correspondence to the convolution on $D^b(\mathcal{B} \times \mathcal{B})$ defined by the expression in the middle of (5.13).

To proceed further we have to recall some generalities. Recall that there are Verdier duality functors \mathbf{D} both on the category of holonomic \mathcal{D} -modules and on the category of perverse sheaves, and that these two functors go to each other under the Riemann-Hilbert correspondence. It is well known that convolution (5.13) commutes with Verdier duality. This is no longer the case for the convolution (5.6) of $\tilde{\mathcal{D}}$ -modules, due to the fact that $\tilde{\mathcal{B}}$ is not compact. In more detail, let us restrict to the special case $\lambda = 0$, the only case we need. Verdier duality takes an object of

$\mathbb{D}\mathcal{H}\mathcal{C}_0$ into (in general) a direct limit of objects of $\mathbb{D}\mathcal{H}\mathcal{C}_0$, an *ind*-object of $\mathbb{D}\mathcal{H}\mathcal{C}_0$. One can easily find the commutation relation between convolution (5.6) and Verdier duality on $\mathbb{D}\mathcal{H}\mathcal{C}_0$ using the fact that all objects of this category are smooth along the fibers of the projection $\tilde{\mathcal{B}} \times \tilde{\mathcal{B}} \rightarrow \mathcal{B} \times \mathcal{B}$. We have the following result whose proof is left to the reader.

Lemma 5.14. *For any $\mathcal{M}, \mathcal{N} \in \mathbb{D}\mathcal{H}\mathcal{C}_0$ there is a functorial isomorphism:*

$$(\mathbf{D}\mathcal{M}) * (\mathbf{D}\mathcal{N}) \simeq \mathbf{D}(\mathcal{M} * \mathcal{N})[\dim_{\mathbb{C}} T],$$

where $[\dim_{\mathbb{C}} T]$ stands for the shift in the derived category by the dimension of the fiber of the projection $\tilde{\mathcal{B}} \rightarrow \mathcal{B}$. \square

Next, let $\mathcal{M} \mapsto \mathcal{M}^t$ be the functor on $\mathbb{D}\mathcal{H}\mathcal{C}_0$ induced by the flip of the two factors of the manifold $\mathcal{B} \times \mathcal{B}$. This functor clearly commutes with Verdier duality, and we let \mathbf{D}^t denote the composition functor $\mathcal{M} \mapsto \mathbf{D}(\mathcal{M}^t) = (\mathbf{D}\mathcal{M})^t$. With this understood, one has the following standard result.

Lemma 5.15. *For any $\mathcal{M}, \mathcal{N}, \mathcal{L} \in \mathbb{D}\mathcal{H}\mathcal{C}_0$ there is a functorial isomorphism:*

$$\mathrm{Hom}(\mathcal{M} * \mathcal{N}, \mathcal{L}) \simeq \mathrm{Hom}\left(\mathcal{M}, \mathcal{L} * (\mathbf{D}^t \mathcal{N})\right).$$

In the remainder of this section we are going to formulate a conjecture relating the various convolution functors introduced above to Koszul duality. More precisely, we will be dealing with an extension of the Koszul duality considered in [BGS] to the Harish-Chandra setup; see [S3].

Write $D_G^b(\mathcal{B} \times \mathcal{B})$ for the G -equivariant derived category on $\mathcal{B} \times \mathcal{B}$, as defined in [BeLu]. Let $\mathbb{D}_G^{\mathrm{mix}}(\mathcal{B} \times \mathcal{B})$ be its *mixed* version (see [BGS, §4.3] where it is referred to as a *graded* version). On the other hand, let $\mathcal{H}\mathcal{C}_0^{\mathrm{mix}}$ denote the *mixed* version of the category $\mathcal{H}\mathcal{C}_0$. The following result, conjectured in [BG], is an extension of the main theorem of [BGS] to an equivariant setup.

Theorem 5.16. *There is a contravariant equivalence of triangulated categories*

$$\mathbf{K} : \mathbb{D}_G^{\mathrm{mix}}(\mathcal{B} \times \mathcal{B}) \xrightarrow{\sim} \mathbb{D}\mathcal{H}\mathcal{C}_0^{\mathrm{mix}},$$

that takes pure perverse sheaves in $\mathbb{D}_G^{\mathrm{mix}}(\mathcal{B} \times \mathcal{B})$ to indecomposable projectives in $\mathcal{H}\mathcal{C}_0^{\mathrm{mix}}$.

Unfortunately, so far no complete proof of this theorem has been written down (although, see [S3]). The remarks below should hopefully help the reader to better understand the result.

Remark 5.17. (a) Let $\mathcal{P}er_G(\mathcal{B} \times \mathcal{B})$ denote the abelian category of G -equivariant perverse sheaves on $\mathcal{B} \times \mathcal{B}$. This is an abelian subcategory of $D_G^b(\mathcal{B} \times \mathcal{B})$, yet the natural functor $D^b(\mathcal{P}er_G(\mathcal{B} \times \mathcal{B})) \rightarrow D_G^b(\mathcal{B} \times \mathcal{B})$ is not an equivalence.

(b) The category $\mathcal{P}er_G(\mathcal{B} \times \mathcal{B})$ is known to be equivalent to the (proper) subcategory of the category \mathcal{O}_0 (= category \mathcal{O}_λ with $\lambda = 0$) formed by the finitely generated $U\mathfrak{g}$ -modules M such that

- The augmentation ideal $Z_+(\mathfrak{g}) = Z(\mathfrak{g}) \cap U\mathfrak{g}_+$ annihilates M ;
- $U\mathfrak{b}$ -action on M is locally finite;
- $U\mathfrak{h}$ -action on M is diagonalizable.

(c) The category \mathcal{O}_0 , in its turn, is known to be equivalent to the (proper) subcategory of the category \mathcal{HC}_0 formed by the $\tilde{\mathcal{D}}$ -modules with the trivial monodromy along the fiber of the projection $\tilde{\mathcal{B}} \rightarrow \mathcal{B}$, the first factor of the map $\tilde{\mathcal{B}} \times \tilde{\mathcal{B}} \rightarrow \mathcal{B} \times \mathcal{B}$. \square

(d) In [BGS] we used instead of the contravariant duality \mathbf{K} , a covariant duality that sends pure perverse sheaves in $\mathbb{D}_G^{\text{mix}}(\mathcal{B} \times \mathcal{B})$ to indecomposable injectives in $\mathcal{HC}_0^{\text{mix}}$. The duality \mathbf{K} used here is obtained from that of [BGS] via composition with Verdier duality \mathbf{D} . \square

Our definitions of the $*$ -convolution, resp. \star -convolution, extend verbatim to the category $\mathbb{D}_G^{\text{mix}}(\mathcal{B} \times \mathcal{B})$, resp. $\mathbb{D}\mathcal{HC}_0^{\text{mix}}$, and we have

Conjecture 5.18. *The Koszul duality functor \mathbf{K} intertwines the $*$ -convolution (5.13) on $\mathbb{D}_G^{\text{mix}}(\mathcal{B} \times \mathcal{B})$ with the \star -convolution on $\mathbb{D}\mathcal{HC}_0^{\text{mix}}$, that is,*

$$\mathbf{K}(\mathcal{M} * \mathcal{N}) = \mathbf{K}(\mathcal{M}) \star \mathbf{K}(\mathcal{N}), \quad \forall \mathcal{M}, \mathcal{N} \in \mathbb{D}_G^{\text{mix}}(\mathcal{B} \times \mathcal{B}).$$

Note that Decomposition theorem ([BBD]) implies that $*$ -convolution of pure perverse sheaves is pure. On the other hand, we will see in the next section that \star -convolution of two projectives in \mathcal{HC}_0 is again a projective. This gives a supporting evidence for our conjecture.

We need some more notation. Let $i : \Delta \hookrightarrow \mathcal{B} \times \mathcal{B}$ denote the diagonal, and let $\tilde{\Delta}$ be its inverse image under the projection $\tilde{\mathcal{B}} \times \tilde{\mathcal{B}} \rightarrow \mathcal{B} \times \mathcal{B}$. For any point $(x_1, x_2) \in \tilde{\Delta} \subset \tilde{\mathcal{B}} \times \tilde{\mathcal{B}}$ there is a unique element $t = t(x_1, x_2) \in T$ such that the point $(x_1 \cdot t, x_2)$ belongs to the diagonal of $\tilde{\mathcal{B}} \times \tilde{\mathcal{B}}$. The assignment $(x_1, x_2) \mapsto t(x_1, x_2)$ gives a map $\nu : \tilde{\Delta} \rightarrow T$, which is analogous to the map μ in (5.9.1). Let $\tilde{i} : \tilde{\Delta} \hookrightarrow \tilde{\mathcal{B}} \times \tilde{\mathcal{B}}$ denote the imbedding. We define the following objects of the categories $\mathbb{D}_G^{\text{mix}}(\mathcal{B} \times \mathcal{B})$ and $\mathcal{HC}_0^{\text{mix}}$, respectively:

$$(5.19) \quad \bar{\mathcal{L}}_\Delta := i_* \mathcal{O}_\Delta, \quad \mathcal{L}_\Delta := \lim_{\leftarrow n} \left((\tilde{\mathcal{D}}_0 \boxtimes \tilde{\mathcal{D}}_0) \bigotimes_{\tilde{\mathcal{D}} \boxtimes \tilde{\mathcal{D}}} \pi_\bullet(\tilde{i}_* \nu^* \mathcal{E}^n) \right).$$

These objects are the units with respect to $*$ -convolutions, that is, for any $\bar{\mathcal{M}} \in \mathbb{D}_G^{\text{mix}}(\mathcal{B} \times \mathcal{B})$ and $\mathcal{M} \in \mathbb{D}\mathcal{HC}_0^{\text{mix}}$, we have

$$(5.20) \quad \bar{\mathcal{M}} * \bar{\mathcal{L}}_\Delta = \bar{\mathcal{M}} = \bar{\mathcal{L}}_\Delta * \bar{\mathcal{M}}, \quad \mathcal{M} * \mathcal{L}_\Delta = \mathcal{M} = \mathcal{L}_\Delta * \mathcal{M}.$$

Further, there is the unique open G -orbit $j : \mathbb{O} \hookrightarrow \mathcal{B} \times \mathcal{B}$. We define the following objects of $\mathbb{D}_G^{\text{mix}}(\mathcal{B} \times \mathcal{B})$:

$$\bar{\mathcal{R}}_! := j_! \mathcal{O}_\mathbb{O}, \quad \bar{\mathcal{R}}_* := j_* \mathcal{O}_\mathbb{O}.$$

These objects are $*$ -inverse to each other, i.e., it is easy to check that

$$(5.21) \quad \bar{\mathcal{R}}_! * \bar{\mathcal{R}}_* = \bar{\mathcal{R}}_* * \bar{\mathcal{R}}_! = \bar{\mathcal{L}}_\Delta.$$

Next, recall diagram (5.9.1), write $\tilde{j}_!$ for the direct image with compact support functor corresponding to the open imbedding $\tilde{j} : \tilde{\mathbb{O}} \hookrightarrow \tilde{\mathcal{B}} \times \tilde{\mathcal{B}}$. We define the following object of the category $\mathcal{HC}_\lambda^{\text{mix}}$

$$(5.22) \quad \mathcal{R}_! := \lim_{\leftarrow n} \left((\tilde{\mathcal{D}}_\lambda \boxtimes \tilde{\mathcal{D}}_{-w_0(\lambda)}) \bigotimes_{\tilde{\mathcal{D}} \boxtimes \tilde{\mathcal{D}}} \pi_\bullet \tilde{j}_!(\mu^* \mathcal{E}^n) \right).$$

Then, one can show that the sheaf $\mathcal{R}_! * \mathcal{R}_{w_0(\lambda)-2\rho}$ is supported on the diagonal in $\mathcal{B} \times \mathcal{B}$ and, moreover, the corresponding sheaf on the diagonal is the indecomposable projective local system with the monodromy representation (viewed as $\mathbb{C}[\mathfrak{h}]$ -module) isomorphic to $\mathbb{C}[\mathfrak{h}]_{-\lambda-2\rho}$. It follows that the object $\mathcal{R}_!$ is the $*$ -inverse of $\mathcal{R}_{w_0(\lambda)-2\rho}$, i.e., one has

$$(5.23) \quad \mathcal{R}_! * \mathcal{R}_{w_0(\lambda)-2\rho} = \mathcal{R}_{w_0(\lambda)-2\rho} * \mathcal{R}_! = \mathcal{L}_\Delta.$$

If $\lambda = 0$, as we will assume below, we will write \mathcal{R}_* instead of $\mathcal{R}_{w_0(\lambda)-2\rho}$, and view it as an object of $\mathbb{D}\mathcal{H}\mathcal{C}_0^{\text{mix}}$, using a twist by $\mathcal{O}(2\rho)$.

Remark. An important motivation for Theorem 5.16 comes from the equation $\mathbf{K}(\overline{\mathcal{L}}_\Delta) = \mathcal{R}_!$ (see (5.25) below) in view of the following. The object $\overline{\mathcal{L}}_\Delta$ is a simple G -equivariant perverse sheaf supported on the diagonal of $\mathcal{B} \times \mathcal{B}$, and one has:

$$\text{Ext}_{D_G^b(\mathcal{B} \times \mathcal{B})}^\bullet(\overline{\mathcal{L}}_\Delta, \overline{\mathcal{L}}_\Delta) \simeq H_G^\bullet(\mathcal{B}) \simeq S\mathfrak{h}^*.$$

This corresponds, on the other side of Koszul duality, to the fact that $\mathcal{R}_!$ is a projective object of the category $\mathcal{H}\mathcal{C}_0^{\text{mix}}$, and moreover one has an isomorphism $\text{End}_{\mathcal{H}\mathcal{C}_0^{\text{mix}}}(\mathcal{R}_!) \simeq S\mathfrak{h}$. Similarly, let $\mathbb{C}_{\mathcal{B} \times \mathcal{B}}$ be the constant sheaf, viewed as a simple object of $D_G^b(\mathcal{B} \times \mathcal{B})$, and let \mathcal{P}_Δ be the indecomposable projective cover of \mathcal{L}_Δ in $\mathcal{H}\mathcal{C}_0^{\text{mix}}$. Then one has graded algebra isomorphisms:

$$\begin{aligned} \text{Ext}_{D_G^b(\mathcal{B} \times \mathcal{B})}^\bullet(\mathbb{C}_{\mathcal{B} \times \mathcal{B}}, \mathbb{C}_{\mathcal{B} \times \mathcal{B}}) &\simeq H_G^\bullet(\mathcal{B} \times \mathcal{B}) \\ &\simeq S\mathfrak{h}^* \otimes_{S(\mathfrak{h}^*)^w} S\mathfrak{h}^* \simeq \text{End}_{\mathcal{H}\mathcal{C}_0^{\text{mix}}}(\mathcal{P}_\Delta, \mathcal{P}_\Delta). \end{aligned}$$

Assuming Conjecture 5.18 holds true, we establish the following commutation relation between the Koszul duality functor \mathbf{K} and the Verdier duality functor \mathbf{D} .

Theorem 5.24. *If Conjecture 5.18 holds, then, for any $\mathcal{M} \in \mathbb{D}_G^{\text{mix}}(\mathcal{B} \times \mathcal{B})$, one has a functorial isomorphism*

$$\mathbf{K}(\mathbf{D}^\dagger \mathcal{M}) \simeq \mathbf{D}^\dagger \left(\mathcal{R}_* * \mathbf{K}(\mathcal{M} * \overline{\mathcal{R}}_!) \right) \quad \text{in } \mathbb{D}\mathcal{H}\mathcal{C}_0^{\text{mix}}.$$

Proof. We begin with the following two easy isomorphisms, which we leave for the reader to prove:

$$(5.25) \quad \mathbf{K}(\overline{\mathcal{L}}_\Delta) = \mathcal{R}_!, \quad \mathbf{K}(\overline{\mathcal{R}}_!) = \mathcal{L}_\Delta.$$

Now, for any $\mathcal{M}, \mathcal{N} \in \mathbb{D}_G^{\text{mix}}(\mathcal{B} \times \mathcal{B})$ we have

$$\text{Hom}(\mathcal{N}, \mathcal{M}) = \quad \text{by (5.20)}$$

$$\text{Hom}(\mathcal{N}, \overline{\mathcal{L}}_\Delta * \mathcal{M}) = \quad \text{by Lemma 5.15}$$

$$\text{Hom}(\mathcal{N} * (\mathbf{D}^t \mathcal{M}), \overline{\mathcal{L}}_\Delta) = \quad \text{by Theorem 5.16}$$

$$\text{Hom}\left((\mathbf{K}\overline{\mathcal{L}}_\Delta), \mathbf{K}(\mathcal{N} * (\mathbf{D}^t \mathcal{M}))\right) = \quad \text{by Conjecture 5.18}$$

$$\text{Hom}\left(\mathbf{K}\overline{\mathcal{L}}_\Delta, \mathbf{K}\mathcal{N} * \mathcal{R}_* * \mathbf{K}(\mathbf{D}^t \mathcal{M})\right) = \quad \text{by Lemma 5.15}$$

$$\text{Hom}\left((\mathbf{K}\overline{\mathcal{L}}_\Delta) * (\mathbf{D}^t \cdot \mathbf{K} \cdot \mathbf{D}^t \mathcal{M}), (\mathbf{K}\mathcal{N}) * \mathcal{R}_*\right) = \quad \text{by (5.20)}$$

$$\text{Hom}\left((\mathbf{K}\overline{\mathcal{L}}_\Delta) * (\mathbf{D}^t \cdot \mathbf{K} \cdot \mathbf{D}^t \mathcal{M}), (\mathbf{K}\mathcal{N}) * \mathcal{R}_* * \mathcal{L}_\Delta\right) = \quad \text{by Conjecture 5.18}$$

$$\text{Hom}\left((\mathbf{K}^{-1}\mathbf{K}\mathcal{N}) * (\mathbf{K}^{-1}\mathcal{L}_\Delta), \mathbf{K}^{-1}(\mathbf{K}\overline{\mathcal{L}}_\Delta * (\mathbf{D}^t \cdot \mathbf{K} \cdot \mathbf{D}^t \mathcal{M}))\right) =$$

$$\text{Hom}\left(\mathcal{N} * (\mathbf{K}^{-1}\mathcal{L}_\Delta), \mathbf{K}^{-1}(\mathbf{K}(\overline{\mathcal{L}}_\Delta) * (\mathbf{D}^t \cdot \mathbf{K} \cdot \mathbf{D}^t \mathcal{M}))\right) = \quad \text{by (5.25)}$$

$$\text{Hom}\left(\mathcal{N} * \overline{\mathcal{R}}_!, \mathbf{K}^{-1}(\mathbf{K}(\overline{\mathcal{L}}_\Delta) * (\mathbf{D}^t \cdot \mathbf{K} \cdot \mathbf{D}^t \mathcal{M}))\right) = \quad \text{convolve with } \overline{\mathcal{R}}_*, \text{ use (5.21)}$$

$$\text{Hom}\left(\mathcal{N}, \mathbf{K}^{-1}(\mathbf{K}\overline{\mathcal{L}}_\Delta * (\mathbf{D}^t \cdot \mathbf{K} \cdot \mathbf{D}^t \mathcal{M})) * \overline{\mathcal{R}}_*\right).$$

Since these isomorphisms hold for any \mathcal{N} , we deduce that

$$\mathcal{M} = \mathbf{K}^{-1}\left(\mathbf{K}\overline{\mathcal{L}}_\Delta * (\mathbf{D}^t \cdot \mathbf{K} \cdot \mathbf{D}^t \mathcal{M})\right) * \overline{\mathcal{R}}_*.$$

Hence, convolving each side with $\overline{\mathcal{R}}_!$, using (5.21) and applying \mathbf{K} we get

$$\mathbf{K}(\mathcal{M} * \overline{\mathcal{R}}_!) = (\mathbf{K}\overline{\mathcal{L}}_\Delta) * (\mathbf{D}^t \cdot \mathbf{K} \cdot \mathbf{D}^t \mathcal{M}).$$

Thus, using (5.23), (5.25) we calculate $(\mathbf{K}\overline{\mathcal{L}}_\Delta) * \mathcal{R}_* = \mathcal{R}_! * \mathcal{R}_* = \mathcal{L}_\Delta$. Hence, convolving each side with \mathcal{R}_* , we obtain

$$\mathbf{D}^t(\mathcal{R}_* * \mathbf{K}(\mathcal{M} * \overline{\mathcal{R}}_!)) = \mathbf{K}(\mathbf{D}^t \mathcal{M}).$$

The theorem is proved \square

6. PROJECTIVE FUNCTORS

Fix a ρ -dominant integral weight λ and write $\text{pr}_{|\lambda|}$ for the projection of a locally finite $Z(\mathfrak{g})$ -module to its $|\lambda|$ -isotypic component (see after Lemma 1.3).

Following [BeGe, §3.1], we say that a functor $\Phi : \text{Mod}_{|\lambda|}(\text{U}\mathfrak{g}) \rightarrow \text{Mod}_{|\lambda|}(\text{U}\mathfrak{g})$ is a *projective functor* if Φ is a direct summand of the functor $M \mapsto \text{pr}_{|\lambda|}(E \otimes_{\mathbb{C}} M)$, for a certain finite-dimensional \mathfrak{g} -module E . Any such functor can be written in terms of \star -convolution as follows. Given E as above, make $E \otimes_{\mathbb{C}} \text{U}\mathfrak{g}$ into a $\text{U}\mathfrak{g}$ -bimodule by the formulas

$$x \cdot (e \otimes u) = x \cdot e \otimes u + e \otimes x \cdot u$$

and

$$(e \otimes u) \cdot x = e \otimes u \cdot x, \quad x \in \mathfrak{g}, \quad e \in E, \quad u \in \text{U}\mathfrak{g}.$$

Similar formulas make $E \otimes_{\mathbb{C}} \mathbf{Ug}_{|\lambda|}$ into a $\mathbf{Ug}\text{-}\mathbf{Ug}_{|\lambda|}$ -bimodule, and for any $M \in \text{Mod}_{\lambda}(\mathbf{Ug})$, we clearly have

$$\begin{aligned} \Phi(M) &= \text{pr}_{|\lambda|}(E \otimes_{\mathbb{C}} M) \\ &= \text{pr}_{|\lambda|}\left((E \otimes_{\mathbb{C}} \mathbf{Ug}) \otimes_{\mathbf{Ug}} M\right) = \left(\text{pr}_{|\lambda|}(E \otimes_{\mathbb{C}} \mathbf{Ug}_{|\lambda|})\right) \otimes_{\mathbf{Ug}_{|\lambda|}} M. \end{aligned}$$

We see that our functor Φ is a direct summand of the \star -convolution with the bimodule $\text{pr}_{|\lambda|}(E \otimes_{\mathbb{C}} \mathbf{Ug}_{|\lambda|}) \in \text{HC}_{|\lambda|}$. Notice that no (higher) *derived* tensor product is required above, because the bimodule $\text{pr}_{|\lambda|}(E \otimes_{\mathbb{C}} \mathbf{Ug}_{|\lambda|})$ is a projective right $\mathbf{Ug}_{|\lambda|}$ -module, as a direct summand of the free right $\mathbf{Ug}_{|\lambda|}$ -module $E \otimes_{\mathbb{C}} \mathbf{Ug}_{|\lambda|}$.

It is known that the category $\text{HC}_{|\lambda|}$ has enough projectives; cf. [BGG]. Furthermore, it was shown in [BeGe, §§2.2, 4.1] that any $\mathbf{Ug}_{|\lambda|}$ -bimodule of the form $\text{pr}_{|\lambda|}(E \otimes_{\mathbb{C}} \mathbf{Ug}_{|\lambda|})$ is a projective in the category $\text{HC}_{|\lambda|}$, and conversely, any projective in $\text{HC}_{|\lambda|}$ is a direct summand of $\text{pr}_{|\lambda|}(E \otimes_{\mathbb{C}} \mathbf{Ug}_{|\lambda|})$, for an appropriate finite dimensional E . Convolution with such a direct summand is by definition a projective functor. Moreover, we have the following:

Proposition 6.1. (i) *For any projective functor Φ there exists a uniquely determined projective object $P_{\Phi} \in \text{HC}_{|\lambda|}$ such that*

$$\Phi(M) = P_{\Phi} \star M \quad \text{where} \quad P_{\Phi} = \varprojlim_n \Phi(\mathbf{Ug}/I_{|\lambda|}^n \cdot \mathbf{Ug});$$

The assignment $\Phi \mapsto P_{\Phi}$ sets up a (1-1)-correspondence between projective functors and projectives in $\text{HC}_{|\lambda|}$.

(ii) *Any projective functor is exact.*

(iii) *Composition of projective functors is a projective functor.*

Proof. Part (i) follows from an analogue of Remark 5.7 (for \mathbf{Ug} instead of \tilde{U}). Part (ii) follows from the exactness of the functor $\text{pr}_{|\lambda|}(E \otimes_{\mathbb{C}} \bullet)$. Part (iii) is clear from definition. \square

For any projective functor Φ , there is a projective functor Φ^{\dagger} which is both the left and the right adjoint of Φ ; see [BeGe, Lemma 3.2(v)].

Next we extend projective functors to the category $\text{Mod}_{\lambda}(\tilde{U})$ as follows. To any $\mathbf{Ug}_{|\lambda|}$ -bimodule $M \in \text{HC}_{|\lambda|}$ associate the \tilde{U}_{λ} -bimodule

$$\tilde{M} := \tilde{U}_{\lambda} \otimes_{\tilde{U}_{\lambda}^{w_{\lambda}}} M \otimes_{\tilde{U}_{\lambda}^{w_{\lambda}}} \tilde{U}_{\lambda} \in \widetilde{\text{HC}}_{\lambda};$$

see Lemma 1.3. If M is a projective in $\text{HC}_{|\lambda|}$, then \tilde{M} is a projective in $\widetilde{\text{HC}}_{\lambda}$ (by adjunction of Res_{λ} and $\tilde{U}_{\lambda} \otimes (\cdot)$). Given a projective functor Φ on $\text{Mod}_{|\lambda|}(\mathbf{Ug})$, define a functor $\tilde{\Phi}$ on $\text{Mod}_{\lambda}(\tilde{U})$ by

$$\tilde{\Phi}(M) := \overbrace{P_{\Phi} \otimes_{\mathbf{Ug}_{|\lambda|}} \text{Res}_{\lambda} M}^{\sim} = \tilde{P}_{\Phi} \otimes_{\tilde{U}_{\lambda}} M.$$

Clearly, $\tilde{\Phi} : \text{Mod}_{\lambda}(\tilde{U}) \rightarrow \text{Mod}_{\lambda}(\tilde{U})$ is an exact functor and, moreover, $\tilde{\Phi}(M) = \tilde{P}_{\Phi} \star M$.

The following result describes the relation between projective functors and \mathcal{D} -modules.

Proposition 6.2. (i) For any projective $\mathcal{P} \in \mathcal{HC}_\lambda$ the assignment $\mathcal{M} \mapsto \mathcal{P} \star \mathcal{M}$ gives an exact functor $\text{Mod}_\lambda(\widetilde{\mathcal{D}}) \rightarrow \text{Mod}_\lambda(\widetilde{\mathcal{D}})$ and there is a projective functor $\widetilde{\Phi} = \widetilde{\Phi}_{\mathcal{P}}$ on $\text{Mod}_\lambda(\widetilde{U})$ such that the following diagram commutes:

$$\begin{array}{ccc} \text{Mod}_\lambda(\widetilde{\mathcal{D}}) & \xrightarrow{\Gamma} & \text{Mod}_\lambda(\widetilde{U}) \\ \mathcal{P} \star (\cdot) \downarrow & & \downarrow \widetilde{\Phi} \\ \text{Mod}_\lambda(\widetilde{\mathcal{D}}) & \xrightarrow{\Gamma} & \text{Mod}_\lambda(\widetilde{U}) \end{array}$$

(ii) For any projective functor $\widetilde{\Phi}$ on $\text{Mod}_\lambda(\widetilde{U})$ there exists a projective $\mathcal{P} \in \mathcal{HC}_\lambda$ such that $\widetilde{\Phi} = \widetilde{\Phi}_{\mathcal{P}}$, i.e., such that the above diagram commutes.

Proof. By Proposition 5.11 we have $\Gamma(\mathcal{P} \star \mathcal{M}) = \Gamma^\diamond(\mathcal{P}) \star \Gamma(M)$. If λ is regular, then Γ^\diamond gives an equivalence of the categories \mathcal{HC}_λ and $\widetilde{\text{HC}}_\lambda$. This proves the proposition if λ is regular.

To complete the proof of part (i) in the general case choose a regular ρ -dominant weight μ . Using the geometric translation functor Θ_λ^μ we obtain the following equivalences of categories:

$$\mathcal{HC}_\lambda \simeq \mathcal{HC}_\mu \simeq \widetilde{\text{HC}}_\mu \simeq \text{HC}_{|\mu|}, \quad \text{Mod}_\lambda(\widetilde{\mathcal{D}}) \simeq \text{Mod}_\mu(\widetilde{\mathcal{D}}) \simeq \text{Mod}_\mu(\widetilde{U}) \simeq \text{Mod}_{|\mu|}(\text{Ug}).$$

The proposition being already known for regular μ 's, it follows that the functor $\mathcal{P} \star (\cdot)$ is an exact functor on $\text{Mod}_\lambda(\widetilde{\mathcal{D}})$ and that there exists a projective $\mathcal{P}^\dagger \in \mathcal{HC}_\lambda$ such that the functor $\mathcal{P}^\dagger \star (\cdot)$ is both the left and the right adjoint of $\mathcal{P} \star (\cdot)$. Now, the category $\text{Mod}_\lambda(\widetilde{U})$ may be viewed as a quotient of the category $\text{Mod}_\lambda(\widetilde{\mathcal{D}})$. By [S1, Lemma 6, p. 432], the pair of adjoint functors $(\mathcal{P} \star (\cdot), \mathcal{P}^\dagger \star (\cdot))$ descends to the pair $(\Gamma^\diamond(\mathcal{P}) \star (\cdot), \Gamma^\diamond(\mathcal{P}^\dagger) \star (\cdot))$ of adjoint (exact) functors on $\text{Mod}_\lambda(\widetilde{U})$. We have in particular

$$(6.3) \quad \begin{aligned} \text{Hom}_{\text{Mod}_\lambda(\widetilde{U})}(\widetilde{U}_\lambda, \Gamma^\diamond(\mathcal{P}^\dagger) \star M) &= \text{Hom}_{\text{Mod}_\lambda(\widetilde{U})}(\Gamma^\diamond(\mathcal{P}) \star \widetilde{U}_\lambda, M) \\ &= \text{Hom}_{\text{Mod}_\lambda(\widetilde{U})}(\Gamma^\diamond(\mathcal{P}), M). \end{aligned}$$

One checks from the construction that isomorphisms (6.3) still hold if M is taken to be a bimodule from $\widetilde{\text{HC}}_\lambda$ and Hom's are taken in $\widetilde{\text{HC}}_\lambda$. But then the functor $\text{Hom}_{\text{Mod}_\lambda(\widetilde{U})}(\widetilde{U}_\lambda, \Gamma^\diamond(\mathcal{P}^\dagger) \star (\bullet))$ on the left-hand side of (6.3) is an exact functor. Hence, the functor $\text{Hom}_{\text{Mod}_\lambda(\widetilde{U})}(\Gamma^\diamond(\mathcal{P}), \bullet)$ on the right-hand side is an exact functor again. Thus, $\Gamma^\diamond(\mathcal{P})$ is a projective in $\widetilde{\text{HC}}_\lambda$ and part (i) follows.

To prove part (ii) it suffice to show, by part (i), that any projective $P \in \widetilde{\text{HC}}_\lambda$ can be written in the form $P = \Gamma^\diamond(\mathcal{P})$, where \mathcal{P} is a projective in \mathcal{HC}_λ . We may put $\mathcal{P} := \Delta_\lambda P$ where Δ_λ is the localization functor. Then \mathcal{P} is projective, by adjunction, and the result follows from Theorem 2.2(i). \square

As a simple application of our analysis we get

Corollary 6.4. *There is a natural isomorphism $\Gamma^\diamond(\mathcal{R}_!) \simeq \widetilde{U}_\lambda$.*

Proof. Observe that formula (5.23) yields

$$\mathcal{R}_! \star \mathcal{R}_{w_0(\lambda)-2\rho} \star \mathcal{M} = \mathcal{M}, \quad \text{for any } \mathcal{M} \in \text{Mod}_\lambda(\widetilde{\mathcal{D}}).$$

This equation shows that \star -convolution with $\mathcal{R}_!$ gives the identity functor on $\widetilde{\text{HC}}_\lambda$. On the other hand, view the bimodule \widetilde{U}_λ as an object of $\widetilde{\text{HC}}_\lambda$. We know that the

functor $\widetilde{U}_\lambda \star (\cdot)$ is the identity functor on \widetilde{HC}_λ , cf. (5.8). Thus, Proposition 5.11 yields the result. \square

Now let λ, μ be integral ρ -dominant weights such that $W_\lambda \subset W_\mu$ and let $\widetilde{\theta}^- : \text{Mod}_\lambda(\widetilde{U}) \rightarrow \text{Mod}_\mu(\widetilde{U})$ be the translation functor. Repeating the definitions, one gets a similar functor $\widetilde{\theta}^- : \widetilde{HC}_\lambda \rightarrow \widetilde{HC}_\mu$ on bimodules.

Corollary 6.5. *Let P be a projective in \widetilde{HC}_λ . Then $\widetilde{\theta}^- P$ is a projective in \widetilde{HC}_μ and the following diagram of functors commutes:*

$$\begin{array}{ccc} \text{Mod}_\lambda(\widetilde{U}) & \xrightarrow{P\star(\cdot)} & \text{Mod}_\lambda(\widetilde{U}) \\ \widetilde{\theta}^- \downarrow & & \downarrow \widetilde{\theta}^- \\ \text{Mod}_\mu(\widetilde{U}) & \xrightarrow{(\widetilde{\theta}^- P)\star(\cdot)} & \text{Mod}_\mu(\widetilde{U}). \end{array}$$

Proof. Let $\Theta^- : \mathcal{HC}_\lambda \rightarrow \mathcal{HC}_\mu$ be the geometric translation functor. By the proof of part (ii) of Proposition 6.2, there is a projective $\mathcal{P} \in \mathcal{HC}_\lambda$ such that $P = \Gamma^\diamond(\mathcal{P})$. Then, by Proposition (2.8), we have $\theta^- P = \Gamma^\diamond(\Theta^- \mathcal{P})$. The proof of part (i) of Proposition 6.2 now shows that $\Gamma^\diamond(\Theta^- \mathcal{P})$ is a projective in \widetilde{HC}_μ . Furthermore, one checks easily that the following \mathcal{D} -module counterpart of the above diagram commutes:

$$\begin{array}{ccc} \text{Mod}_\lambda(\widetilde{\mathcal{D}}) & \xrightarrow{P\star(\cdot)} & \text{Mod}_\lambda(\widetilde{\mathcal{D}}) \\ \widetilde{\theta}^- \downarrow & & \downarrow \widetilde{\theta}^- \\ \text{Mod}_\mu(\widetilde{\mathcal{D}}) & \xrightarrow{(\widetilde{\theta}^- P)\star(\cdot)} & \text{Mod}_\mu(\widetilde{\mathcal{D}}). \end{array}$$

The result now follows from Proposition 6.2. \square

Recall the functor $\widetilde{\theta}_\ell^+ : \text{Mod}_\mu(\widetilde{U}) \rightarrow \text{Mod}_\lambda(\widetilde{U})$, the left adjoint of $\widetilde{\theta}^-$. We have

Proposition 6.6. *Assume that for some $M \in \text{Mod}_\lambda(\widetilde{U})$ the adjunction morphism $\widetilde{\theta}_\ell^+ \cdot \widetilde{\theta}^- M \rightarrow M$ is an isomorphism. Then, for any projective functor $\widetilde{\Phi}$ on $\text{Mod}_\lambda(\widetilde{U})$, the adjunction morphism $\widetilde{\theta}_\ell^+ \cdot \widetilde{\theta}^- (\widetilde{\Phi} M) \rightarrow \widetilde{\Phi} M$ is an isomorphism again.*

Proof. Let $\widetilde{\Phi}^\dagger$ be the adjoint of a projective functor $\widetilde{\Phi}$. The pair $(\widetilde{\Phi}, \widetilde{\Phi}^\dagger)$ descends, by Corollary 6.5 and [S1, Lemma 6, p. 432] to an adjoint pair of exact functors $(\widetilde{\Psi}, \widetilde{\Psi}^\dagger)$ on $\text{Mod}_\mu(\widetilde{U})$. For any $M, N \in \text{Mod}_\lambda(\widetilde{U})$ we have, by adjunction,

$$\begin{aligned} \text{Hom}(\widetilde{\theta}_\ell^+ \cdot \widetilde{\theta}^- \widetilde{\Phi} M, N) &= \text{Hom}(\widetilde{\theta}^- \widetilde{\Phi} M, \widetilde{\theta}^- N) \\ &= \text{Hom}(\widetilde{\Psi} \widetilde{\theta}^- M, \widetilde{\theta}^- N) \\ &= \text{Hom}(\widetilde{\theta}^- M, \widetilde{\Psi}^\dagger \widetilde{\theta}^- N) \\ &= \text{Hom}(\widetilde{\theta}^- M, \widetilde{\theta}^- \widetilde{\Phi}^\dagger N) \\ &= \text{Hom}(\widetilde{\theta}_\ell^+ \cdot \widetilde{\theta}^- M, \widetilde{\Phi}^\dagger N). \end{aligned}$$

If $\widetilde{\theta}_\ell^+ \cdot \widetilde{\theta}^- M = M$, then the last Hom can be rewritten as

$$\text{Hom}(M, \widetilde{\Phi}^\dagger N) = \text{Hom}(\widetilde{\Phi} M, N),$$

and the proposition follows.

Next let $\lambda = 0$. Since 0 is a ρ -dominant regular weight, we may identify the categories \mathcal{HC}_0 and \widetilde{HC}_0 . There is a standard duality functor on the category \widetilde{HC}_0 . Any projective functor is known to commute with that duality, so does tensoring

with a finite dimensional representation; see [BeGe]. Moreover, it is known that the standard duality on $\widetilde{\mathcal{HC}}_0$ goes into the Verdier duality on \mathcal{HC}_0 (this is false for the Harish-Chandra category over the general *real* reductive group: the two dualities may, in general, act differently already on the simple objects). The two dualities coincide, however, in the special case of a *complex* reductive group, the case we are interested in. Then the claim boils down to a similar result for the category \mathcal{O} . In the category \mathcal{O} case the result is known, although we could not find any written account of it.

Here is a sketch of proof. First, it is immediate to verify that the two dualities agree on every simple object of \mathcal{O} . Second, each duality is an exact functor, hence, agreement on simple objects implies that any object $M \in \mathcal{O}$ is sent by both dualities to isomorphic objects, call it M^\dagger . The remaining (most delicate) part of the proof is to verify that, for any $M, N \in \mathcal{O}$, the two dualities induce the same maps: $\text{Hom}_{\mathcal{O}}(M, N) \rightarrow \text{Hom}_{\mathcal{O}}(N^\dagger, M^\dagger)$. This is equivalent, by a standard homological algebra, to the claim that, for any *simple* objects $M, N \in \mathcal{O}$ and any $i \geq 0$, the two dualities induce the same maps: $\text{Ext}_{\mathcal{O}}^i(M, N) \rightarrow \text{Ext}_{\mathcal{O}}^i(N^\dagger, M^\dagger)$. The latter claim is obvious for $i = 0$, and can be verified by hand for $i = 1$, using that for $i = 1$ one can take M to be a Verma module instead of a simple module. The general case $i \geq 1$ now follows from the main theorem of [BGS] which implies that the *Ext*-algebra of all the simple objects is Koszul, in particular, is generated by *Ext*⁰'s and *Ext*¹'s. \square

We now return to our convolutions and note that for $\lambda = 0$ we may assume, twisting by the canonical bundle if necessary, that both \star -convolution, and the \star -convolution take the category $\mathbb{D}\mathcal{HC}_0$ into itself and, moreover, that the object \mathcal{R} involved in formula (5.10) is also an object of \mathcal{HC}_0 . Recall further that the Verdier duality functor, \mathbf{D} , is a contravariant exact functor on \mathcal{HC}_0 . Hence, it takes projective objects into injective objects.

Proposition 6.7. *For any projective $\mathcal{P} \in \mathcal{HC}_0$, in the notation of (5.9.1)–(5.9.2) we have*

$$\mathbf{D}\mathcal{P} = \mathcal{P} \star \mathcal{R}^\vee \quad \text{where} \quad \mathcal{R}^\vee := \lim_{\substack{\longrightarrow \\ n}} \left((\widetilde{\mathcal{D}}_0 \boxtimes \widetilde{\mathcal{D}}_0) \bigotimes_{\substack{\otimes \\ \mathbb{D}\boxtimes\mathbb{D}}} \pi_{\bullet, j_* \mu^*}(\mathbf{D}\mathcal{E}^n) \right).$$

Proof. Let $\Phi_{\mathcal{P}} : M \mapsto \mathcal{P} \star M$ be the projective functor corresponding to \mathcal{P} . It follows from the discussion of two dualities above the proposition that the functor $\Phi_{\mathcal{P}}$ on \mathcal{HC}_0 commutes with Verdier duality. Therefore, setting $r = \dim T$ and using Lemma 5.14, for any $\mathcal{M} \in \mathcal{HC}_0$, we find

$$\begin{aligned} \mathcal{P} \star \mathcal{R} \star (\mathbf{D}\mathcal{M}) &= \mathcal{P} \star (\mathbf{D}\mathcal{M}) \\ &= \Phi_{\mathcal{P}}(\mathbf{D}\mathcal{M}) \quad \text{since } \Phi \text{ commutes with Verdier duality} \\ &= \mathbf{D} \cdot \Phi_{\mathcal{P}}(\mathcal{M}) = \mathbf{D}(\mathcal{P} \star \mathcal{M}) \\ &= \mathbf{D}(\mathcal{P} \star \mathcal{R} \star \mathcal{M}) \quad \text{by Lemma 5.14} \\ &= \mathbf{D}(\mathcal{P} \star \mathcal{R}) \star \mathbf{D}(\mathcal{M})[r]. \end{aligned}$$

The above isomorphisms imply, by Remark 5.7, that

$$(6.8) \quad \mathcal{P} \star \mathcal{R} = \mathbf{D}(\mathcal{P} \star \mathcal{R})[r] = (\mathbf{D}\mathcal{P}) \star (\mathbf{D}\mathcal{R}),$$

where in the last equality we have used Lemma 5.14 once more. Now, it is easy to show that the object \mathcal{R}^\vee is \star -inverse to $\mathbf{D}\mathcal{R}$. Therefore, convolving both sides of

(6.8) with \mathcal{R}^\vee yields

$$\mathbf{DP} = \mathcal{P} * \mathcal{R} * \mathcal{R}^\vee = \mathcal{P} \star \mathcal{R}^\vee,$$

and the claim follows. \square

Tilting \mathcal{D} -modules. We continue to assume that $\lambda = 0$, and use the notation of Section 4. Let \mathfrak{b} be the fixed Borel subalgebra, and $B \subset G$ the corresponding Borel group. Since 0 is regular we may (and will) identify $\mathbf{U}\mathfrak{g}$ -modules with $\tilde{\mathcal{D}}$ -modules via the equivalences: $\text{Mod}_{|0|}(\mathbf{U}\mathfrak{g}) \simeq \text{Mod}_0(\tilde{U}) \simeq \text{Mod}_0(\tilde{\mathcal{D}})$. Further, there is a natural equivalence (see e.g. [BeGe, §5]) between the category of $\mathbf{U}\mathfrak{b}$ -locally finite $\mathbf{U}\mathfrak{g}_{|0|}$ -modules, resp. B -monodromic $\tilde{\mathcal{D}}$ -modules on \mathcal{B} (see [BB3]) and the category \mathcal{HC}_0 . We will write $\tilde{\Delta}(M)$ for the object of \mathcal{HC}_0 corresponding to $M \in \text{Mod}_{|0|}(\mathbf{U}\mathfrak{g})$. As we mentioned in Remark 5.17(c), the category \mathcal{O}_0 goes under the above equivalence to the subcategory of \mathcal{HC}_0 formed by the $\tilde{\mathcal{D}}$ -modules with the trivial monodromy along the fiber of the projection $\tilde{\mathcal{B}} \rightarrow \mathcal{B}$, the first factor of the map $\tilde{\mathcal{B}} \times \tilde{\mathcal{B}} \rightarrow \mathcal{B} \times \mathcal{B}$. In particular, for each $w \in W$, there are objects $\tilde{\Delta}(M_{w,0}), \tilde{\Delta}(P_{w,0}) \in \mathcal{HC}_0$ corresponding to the Verma module and its indecomposable projective cover, respectively.

Given $\mu \in \mathfrak{h}^*$, let Sh_μ denote the completion of the Symmetric algebra at the corresponding point. We introduce the “universal” Verma module $\mathbb{M} := \mathbf{U}\mathfrak{g}/\mathbf{U}\mathfrak{g} \cdot \mathfrak{n}$, where $\mathfrak{n} := [\mathfrak{b}, \mathfrak{b}]$ is the nilradical of \mathfrak{b} . This module has a natural *right* Sh -action, and we set $\mathbb{M}_\mu := \mathbb{M} \otimes_{\text{sh}} \text{Sh}_\mu$, the completion of \mathbb{M} at the point μ . In particular, for each $w \in W$, there is an $\mathbf{U}\mathfrak{g}$ -module $\mathbb{M}_{w,0}$, which is not an object of the category \mathcal{O}_0 . We let $\mathcal{M}_w := \tilde{\Delta}(\mathbb{M}_{w,0})$ denote the corresponding object in \mathcal{HC}_0 , and let \mathcal{P}_w be its indecomposable projective cover in \mathcal{HC}_0 . We note that, if $w = w_0$, then $\mathcal{M}_{w_0} = \mathcal{R}_!$ is the Harish-Chandra module defined in (5.22). In general, \mathcal{M}_w has a pro-unipotent monodromy (i.e., free $\text{Sh}_{w,0}$ -action) along the fiber of the projection $\tilde{\mathcal{B}} \rightarrow \mathcal{B}$ (the first factor of the map $\tilde{\mathcal{B}} \times \tilde{\mathcal{B}} \rightarrow \mathcal{B} \times \mathcal{B}$), and $\tilde{\Delta}(\mathbb{M}_{w,0})$ is obtained from \mathcal{M}_w by “killing” (= taking co-invariants of) the monodromy along the first factor (very much the same way as the Verma module M_μ is obtained from \mathbb{M}_μ). Similarly, the projective \mathcal{P}_w has a flag formed by the various \mathcal{M}_y ’s, hence has free pro-unipotent monodromy; again, $\tilde{\Delta}(P_{w,0})$ is obtained from \mathcal{P}_w by “killing” that monodromy. In particular, one has an equality of multiplicities:

$$(6.9) \quad [\mathcal{P}_w : \mathcal{M}_y] = [\tilde{\Delta}(P_{w,0}) : \tilde{\Delta}(M_{y,0})].$$

We will also need Harish-Chandra modules \mathcal{M}_w^\vee which are in a sense dual analogues of \mathcal{M}_w . We first define the corresponding $\mathbf{U}\mathfrak{g}$ -modules as follows. For $w \in W$, and each $n = 1, 2, \dots$, we have a well defined object $\mathbb{M}_\mu/\mathcal{J}_\mu^n \mathbb{M}_\mu \in \mathcal{O}_0$, where $\mathcal{J}_\mu \subset \text{Sh}_\mu$ is the maximal ideal. Let $\mathbf{D}(\mathbb{M}_\mu/\mathcal{J}_\mu^n \mathbb{M}_\mu)$ be its dual in \mathcal{O}_0 . There are natural projections (not imbeddings!): $\mathbf{D}(\mathbb{M}_\mu/\mathcal{J}_\mu^{n+1} \mathbb{M}_\mu) \rightarrow \mathbf{D}(\mathbb{M}_\mu/\mathcal{J}_\mu^n \mathbb{M}_\mu)$, $n = 1, 2, \dots$, and we set $\mathbb{M}_\mu^\vee := \varprojlim_n \mathbf{D}(\mathbb{M}_\mu/\mathcal{J}_\mu^n \mathbb{M}_\mu)$. Let $\mathcal{M}_w^\vee := \tilde{\Delta}(\mathbb{M}_{w,0}^\vee)$ be the corresponding Harish-Chandra modules. We note that for $w = w_0$ we have $\mathcal{M}_{w_0}^\vee = \mathcal{R}_{-2\rho}$ is the object introduced in (5.9.2); see also (5.23).

It was shown in [BeGe] that, for each $w \in W$, there is a unique indecomposable projective functor Φ_w (see [BeGe, Thm. 3.3]) such that $\Phi_w(M_0) = P_{w,0}$. Hence, for each $w \in W$, there is, by Propositions 6.1 and 6.2, a unique projective $\mathcal{P}_w \in \mathcal{HC}_0$,

such that the functor Φ_w corresponds on $\tilde{\mathcal{D}}$ -modules to the convolution functor $\mathcal{M} \mapsto \mathcal{P}_w * \mathcal{R} * \mathcal{M}$, where $\mathcal{R} := \mathcal{R}_{-2\rho}$ is as above. It is not difficult to see that \mathcal{P}_w is the indecomposable projective cover of \mathcal{M}_w (considered 2 paragraphs above), equivalently, the projective cover of the simple $\tilde{\mathcal{D}}$ -module supported on the G -diagonal orbit in $\mathcal{B} \times \mathcal{B}$, corresponding to the element w .

Recall that an object of the category \mathcal{O} is called a *tilting* module, if it is self-dual and has a Verma-flag. Observe that projective functors take self-dual objects into self-dual ones, and modules with Verma-flag into modules with Verma-flag; cf. [BeGe]. Since $M_{-2\rho}$ is a simple, hence, self-dual Verma module, it follows that, for each $w \in W$, the module $Q_{w,0} := \Phi_w(M_{-2\rho})$ is tilting. In fact, the $\{Q_{w,0}, w \in W\}$, are exactly all the indecomposable tilting modules in the category \mathcal{O}_0 .

Definition. A finite direct sum of objects of $\mathbb{D}\mathcal{HC}_0$ of the form $\mathcal{Q}_w := \mathcal{P}_w * \mathcal{R}$, $w \in W$ (cf. (5.10)) will be called a *tilting* Harish-Chandra module.

Note that this definition gives a natural mixed structure on \mathcal{Q}_w ; cf. [BGS, §4].

Theorem 6.10. (i) For any $w \in W$, the complex $\mathcal{Q}_w := \mathcal{P}_w * \mathcal{R}$ is actually a $\tilde{\mathcal{D}}$ -module (not just an object of the derived category), $\mathcal{Q}_w \in \mathcal{HC}_0$.

(ii) For any $y, w \in W$ we have $\mathcal{Q}_y * \mathcal{Q}_w \in \mathcal{HC}_0$ is again a tilting module.

(iii) For any $w, y \in W$ the following multiplicity formula holds:

$$[Q_w : \mathcal{M}_y^\vee] = [P_{w,0} : M_{yw_0,0}].$$

Moreover,

$$\text{Hom}_{\mathcal{HC}_0}(\mathcal{Q}_w, \mathcal{Q}_y) \simeq \text{Hom}_{\mathcal{HC}_0}(\mathcal{P}_w, \mathcal{P}_y),$$

and

$$\text{Hom}_{\mathcal{O}}(Q_{w,0}, Q_{y,0}) \simeq \text{Hom}_{\mathcal{O}}(P_{w,0}, P_{y,0}).$$

Before going into the proof we note that Proposition 6.2 yields $\mathcal{Q}_w * \tilde{\Delta}(M_{-2\rho}) = \mathcal{P}_w * \mathcal{R} * \tilde{\Delta}(M_{-2\rho}) = \mathcal{P}_w * \tilde{\Delta}(M_{-2\rho}) = \tilde{\Delta}(\Phi_w(M_{-2\rho})) = \tilde{\Delta}(Q_{w,0})$. Furthermore, one shows as above that the Harish-Chandra module $\tilde{\Delta}(Q_{w,0})$ is obtained from the tilting Harish-Chandra module \mathcal{Q}_w by taking coinvariants of the monodromy along the fiber of the projection $\tilde{\mathcal{B}} \rightarrow \mathcal{B}$ (the first factor of the map $\tilde{\mathcal{B}} \times \tilde{\mathcal{B}} \rightarrow \mathcal{B} \times \mathcal{B}$).

Sketch of the proof of Theorem 6.10. To prove (i) we exploit an interpretation of the functor $\mathcal{M} \mapsto \mathcal{P}_y * \mathcal{R} * \mathcal{M}$, as of a projective functor. Such functors take $\tilde{\mathcal{D}}$ -modules into $\tilde{\mathcal{D}}$ -modules, because of the global interpretation via bi-modules over the extended enveloping algebra \tilde{U} . Hence, using the notation of (5.19) we get $\mathcal{Q}_y = \mathcal{Q}_y * \mathcal{L}_\Delta = \mathcal{P}_w * \mathcal{R} * \mathcal{L}_\Delta = \mathcal{P}_y * \mathcal{L}_\Delta \in \mathcal{HC}_0$, and part (i) is proved. Further, for any $M \in \text{Mod}_{|0|}(\mathbf{U}\mathfrak{g})$, one has $(\mathcal{Q}_y * \mathcal{Q}_w) * M = \mathcal{P}_y * \mathcal{R} * \mathcal{P}_w * \mathcal{R} * M = (\mathcal{P}_y * \mathcal{P}_w) * M$. Part (ii) now follows from Proposition 6.1(iii) and Remark 5.7.

Observe next that, for any $y \in W$, we have $\ell(y^{-1}) + \ell(yw_0) = \ell(w_0)$. It follows that $\mathcal{M}_{y^{-1}}^\vee * \mathcal{M}_{yw_0}^\vee = \mathcal{M}_{w_0}^\vee = \mathcal{R}$. Also, one knows that $\mathcal{M}_y * \mathcal{M}_{y^{-1}}^\vee = \mathcal{L}_\Delta$; cf. (5.23). Therefore, we deduce:

$$\mathcal{M}_y * \mathcal{R} = \mathcal{M}_y * (\mathcal{M}_{y^{-1}}^\vee * \mathcal{M}_{yw_0}^\vee) = (\mathcal{M}_y * \mathcal{M}_{y^{-1}}^\vee) * \mathcal{M}_{yw_0}^\vee = \mathcal{L}_\Delta * \mathcal{M}_{yw_0}^\vee = \mathcal{M}_{yw_0}^\vee.$$

It follows that the functor of $*$ -convolution with \mathcal{R} takes modules with \mathcal{M}_y -flag to modules (not just complexes in the derived category) with $\mathcal{M}_{yw_0}^\vee$ -flag and, moreover, the multiplicities in the two flags correspond. Since \mathcal{P}_w has a \mathcal{M}_y -flag, we

deduce that $\mathcal{Q}_w = \mathcal{P}_w * \mathcal{R}$ is an actual module. Furthermore, \mathcal{Q}_w has a $\mathcal{M}_{y_{w_0}}^\vee$ -flag, and we have $[\mathcal{Q}_w : \mathcal{M}_{y_{w_0}}^\vee] = [\mathcal{P}_w : \mathcal{M}_y]$. This, together with (6.9) and the remark preceding the proof of the theorem, implies the multiplicity formula of part (iii).

Finally, we observe that the functor of $*$ -convolution with \mathcal{R} is an equivalence of derived categories, for it has $*$ -convolution with $\mathcal{R}!$ as its inverse. Therefore, we have $\mathrm{Hom}_{\mathcal{H}\mathcal{C}_0}(\mathcal{Q}_w, \mathcal{Q}_y) \simeq \mathrm{Hom}_{\mathcal{H}\mathcal{C}_0}(\mathcal{P}_w, \mathcal{P}_y)$. The last equation of part (iii) is proved similarly (cf. discussion preceding the proof of the theorem). \square

Remarks. (i) The multiplicity formula of Theorem 6.10(iii) may be viewed as a character formula for tilting modules. An analogous formula for tilting modules over an affine Lie algebra is the main result of [S2]. We observe further that, in the affine setup there are two affine flag manifolds, \mathcal{B}_+ and \mathcal{B}_- , with strata of finite co-dimension and strata of finite dimension, respectively. Category \mathcal{O} at a positive level has enough projectives and corresponds $\tilde{\mathcal{D}}$ -modules on \mathcal{B}_+ , while category \mathcal{O} at a negative level corresponds $\tilde{\mathcal{D}}$ -modules on \mathcal{B}_- . The latter category has *no* projectives, but has well defined tilting modules instead. In the affine setup, the “kernel” \mathcal{R} lives naturally on $\mathcal{B}_+ \times \mathcal{B}_-$. Our formula $\mathcal{Q}_w := \mathcal{P}_w * \mathcal{R}$ says that the tiltings on \mathcal{B}_- are obtained from the projectives on \mathcal{B}_+ by convolving the latter with \mathcal{R} , the most natural way to “transport” projectives from \mathcal{B}_+ to \mathcal{B}_- .

(ii) It would be very interesting to prove an affine analogue of Corollary 6.4 saying that $\Gamma(\mathcal{B}_+ \times \mathcal{B}_-, \mathcal{R})$ is the *semi-infinitely induced* module that played a crucial role in [S2].

(iii) The same argument as the one used in the proof of the Hom-equality of Theorem 6.10(iii) also yields

$$\mathrm{Ext}_{\mathcal{H}\mathcal{C}_0}^i(\mathcal{Q}_y, \mathcal{Q}_w) = \mathrm{Ext}_{\mathcal{H}\mathcal{C}_0}^i(\mathcal{P}_y, \mathcal{P}_w) = 0, \quad \forall i > 0.$$

REFERENCES

- [BBD] A. Beilinson, J. Bernstein, and P. Deligne, *Faisceaux pervers*, Astérisque **100** (1982). MR **86g**:32015
- [BB1] A. Beilinson, J. Bernstein, *Localisation de \mathfrak{g} -modules*, C.R. Acad. Sci. Paris, Sér. 1 **292** (1981), 15–18. MR **82k**:14015
- [BB2] A. Beilinson, J. Bernstein, *A generalization of Casselman’s submodule theorem*. Birkhäuser, Progress in Math. **40**, Boston, 1983, pp. 35–52. MR **85e**:22024
- [BB3] A. Beilinson, J. Bernstein, *Proof of Jantzen conjectures*, Advances in Soviet Math. **16**, Gelfand Seminar, 1983, pp. 1–50. MR **95a**:22022
- [BG] A. Beilinson and V. Ginzburg, *Mixed categories, Ext-duality and representations (results and conjectures)*, Preprint, Moscow (1986).
- [BGS] A. Beilinson, V. Ginzburg, and V. Soergel, *Koszul duality patterns in representation theory*, J. Amer. Math. Soc. **9** (1996), 473–527. MR **96k**:17010
- [Be] J. Bernstein, *Trace in categories*, in: Proceedings of the Dixmier Colloquium, Birkhäuser, Progress in Math. **92**, Boston, 1990, pp. 417–423. MR **92d**:17010
- [BGG] J. Bernstein, I. Gelfand, and S. Gelfand, *A certain category of \mathfrak{g} -modules*, Funct. Anal. and its Appl. **10** (1976), 1–8. MR **53**:10880
- [BeGe] J. Bernstein, S. Gelfand, *Tensor products of finite and infinite dimensional representations of semisimple Lie algebras*, Comp. Math. **41** (1980), 245–285. MR **82c**:17003
- [BeLu] J. Bernstein, V. Lunts, *Equivariant sheaves and functors*, Springer Lecture Notes in Math. **1578** (1994). MR **95k**:55012
- [BoBr] W. Borho, J.-L. Brylinski, *Differential operators on flag manifolds II*, MPI Preprint (1989).
- [CG] N. Chriss, V. Ginzburg, *Representation theory and complex geometry*, Birkhäuser, Boston, 1997. MR **98i**:22021

- [FM] M. Finkelberg, I. Mirkovic, *Semi-infinite flags I. Case of global curve \mathbb{P}^1* , Preprint 1997, alg-geom/9707010.
- [FrMa] I. Frenkel, F. Malikov, *Kazhdan-Lusztig tensoring and Harish-Chandra categories*, Preprint 1997, q-alg/9703010.
- [Gi1] V. Ginzburg, *\mathfrak{g} -modules, Springer representations, and bivariant Chern classes*, Adv. Math. **61** (1986), 1–48.
- [Gi2] V. Ginzburg, *Perverse sheaves and \mathbb{C}^* -actions*, J. Amer. Math. Soc. **4** (1991), 483–490. MR **92d**:14013
- [Ja] J. C. Jantzen, *Moduln mit Einem Höchsten Gewicht*, Lecture Notes in Math. **750** (1979). MR **81m**:17011
- [Ka] M. Kashiwara, *Representation theory and D -modules*, in: Orbits Unipotentes et Représentations, III, Astérisque **173–174** (1989), pp. 55–109. MR **90k**:17029
- [KT1] M. Kashiwara, T. Tanisaki, *The Kazhdan-Lusztig conjecture for symmetrizable Kac-Moody algebras*, in: The Grothendieck Festschrift, II, Birkhäuser, Progress in Mathem. **87**, Boston, 1990, pp. 407–433. MR **93a**:17026
- [KT2] M. Kashiwara, T. Tanisaki, *The Kazhdan-Lusztig conjecture for affine Lie algebras with negative level*, Duke Math. J. **77** (1995), 21–62. MR **96j**:17016
- [Ko] B. Kostant, *On the tensor product of a finite and an infinite dimensional representation*, Journ. Funct. Anal. **20** (1975), 257–285. MR **54**:2888
- [LW] J. Lepowsky, N. Wallach, *Finite and infinite-dimensional representations of linear semi-simple groups*, Trans. Amer. Math. Soc. **184** (1973), 223–246. MR **48**:6320
- [S1] W. Soergel, *Kategorie \mathcal{O} , perverse Garben und Moduln über den Koinvarianten zur Weyl-gruppe*, J. Amer. Math. Soc. **3** (1990), 421–445. MR **91e**:17007
- [S2] W. Soergel, *Charakterformeln Für Kipp-Moduln Über Kac-Moody-Algebren*, Representation Theory (Electronic Journal of the AMS) **1** (1997), 115–132. MR **98f**:17016
- [S3] W. Soergel, *The combinatorics of Harish-Chandra bimodules*, J. Reine Angew. Math. **429** (1992), 49–74 (see also: *Langlands’ philosophy and Koszul duality*, Preprint, 1992). MR **94b**:17011
- [W] N. Wallach, *Cyclic vectors and irreducibility for principal series representations*, Trans. Amer. Math. Soc. **158** (1971), 107–113. MR **43**:7558
- [Z] G. Zuckerman, *Tensor products of finite and infinite-dimensional representations of semisimple Lie groups*, Annals Math. **106** (1977), 295–308. MR **56**:15841

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