

TRANSFER FACTORS FOR LIE ALGEBRAS

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ABSTRACT. Let G be a quasi-split connected reductive group over a local field of characteristic 0, and fix a regular nilpotent element in the Lie algebra \mathfrak{g} of G . A theorem of Kostant then provides a canonical conjugacy class within each stable conjugacy class of regular semisimple elements in \mathfrak{g} . Normalized transfer factors take the value 1 on these canonical conjugacy classes.

1. INTRODUCTION

Let k_0 be a local field of characteristic 0, let k be an algebraic closure of k_0 , and let Γ denote the Galois group of k over k_0 . Let G be a connected reductive group over k_0 , with Lie algebra \mathfrak{g} . We write $\mathfrak{g}(k_0)$ for the set of k_0 -rational points on \mathfrak{g} . Let (H, \mathcal{H}, s, ξ) be endoscopic data [LS87] for G . In [LS87] Langlands and Shelstad define transfer factors for G relative to (H, \mathcal{H}, s, ξ) , and they conjecture that their transfer factors can be used to define a notion of endoscopic induction (analogous to parabolic induction), associating to any stably invariant distribution on $H(k_0)$ an invariant distribution on $G(k_0)$.

Waldspurger [Wal97] has shown that the conjecture of Langlands and Shelstad would follow from another conjecture, known as the fundamental lemma. Waldspurger's method is to study endoscopy on the Lie algebra \mathfrak{g} , and he starts by defining transfer factors for Lie algebras, analogous to those of Langlands and Shelstad.

The main result of this paper gives a new way to express these transfer factors for Lie algebras in the case that the group G is quasi-split over k_0 , which we now assume. In order to state this result precisely we must specify how we are normalizing our transfer factors. Therefore we need to fix a k_0 -splitting $\mathbf{spl} = (B_0, T, \{X_\alpha\})$ for G . Thus B_0 is a Borel subgroup of G over k_0 , T is a maximal k_0 -torus in B_0 , and $\{X_\alpha\}$ is a collection of simple root vectors $X_\alpha \in \mathfrak{g}_\alpha$, one for each simple root α of T in the Lie algebra of B_0 , having the property that $X_{\sigma\alpha} = \sigma(X_\alpha)$ for all $\sigma \in \Gamma$. (As usual, for any root β of T in \mathfrak{g} we write \mathfrak{g}_β for the corresponding root subspace of \mathfrak{g} .)

Waldspurger's factors are analogous to the transfer factors $\Delta(\gamma_H, \gamma_G; \bar{\gamma}_H, \bar{\gamma}_G)$ on p. 248 of [LS87] with the factor Δ_{IV} removed. On the quasi-split group G Langlands and Shelstad also define transfer factors $\Delta_0(\gamma_H, \gamma_G)$ (again on p. 248 of [LS87]). These depend on the chosen k_0 -splitting \mathbf{spl} . The transfer factors $\Delta'_0(X_H, X_G)$ in this paper are complex roots of unity, analogous to $\Delta_0(\gamma_H, \gamma_G)$ with the factor Δ_{IV} removed, and they too depend on the choice of k_0 -splitting.

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We write \mathfrak{b}_0 for the Lie algebra of the Borel subgroup B_0 . For each simple root α we define a root vector $X_{-\alpha} \in \mathfrak{g}_{-\alpha}$ by requiring that $[X_\alpha, X_{-\alpha}]$ be the coroot for α , viewed as an element in the Lie algebra of T . We put $X_- := \sum_\alpha X_{-\alpha}$, where α runs over the set of simple roots of T in B_0 . Of course X_- lies in $\mathfrak{g}(k_0)$ and depends on the choice of k_0 -splitting.

The main result of this paper, Theorem 5.1, is that $\Delta'_0(X_H, X_G) = 1$ whenever X_G lies in $\mathfrak{b}_0(k_0) + X_-$. By results of Kostant [Kos63] every stable conjugacy class of regular semisimple elements in $\mathfrak{g}(k_0)$ meets the set $\mathfrak{b}_0(k_0) + X_-$. Since the values of $\Delta'_0(X_H, X_G)$ and $\Delta'_0(X_H, X'_G)$ are related by a simple Galois-cohomological factor whenever X_G and X'_G are stably conjugate, this main result also yields a simple formula (see Corollary 5.2) for $\Delta'_0(X_H, X_G)$ for arbitrary X_G .

The methods used in this paper are variants of ones used in [Lan83], [LS87], [She89]. In particular we rely on a key result of Langlands [Lan83], namely Proposition 5.2. However, since we need this result in a slightly different form, we have included its proof, which is essentially just a rearrangement of the one given by Langlands. The new ingredient in this paper is the connection with Kostant's section. Kostant's section [Kos63] is reviewed in 2.4, following Drinfeld's exposition in a lecture at the IAS in February, 1997.

Throughout this paper we follow the convention that Lie algebras of groups G , B , T are denoted by the corresponding gothic letters \mathfrak{g} , \mathfrak{b} , \mathfrak{t} , and so on. We also use $N_G(T)$ to denote the normalizer in G of T . In the first several sections we work over an algebraically closed field k of characteristic 0; later we work over a local field k_0 with algebraic closure k .

2. REVIEW OF KOSTANT'S SECTIONS OF $\mathfrak{g} \rightarrow \mathfrak{t}/W$

2.1. Basic definitions. Let G be a connected reductive group over an algebraically closed field k of characteristic 0. We fix a maximal torus T in G and a Borel subgroup B_0 of G containing T ; thus $B_0 = TN_0$ where N_0 denotes the unipotent radical of B_0 . We write B_∞ for the unique Borel subgroup of G containing T that is opposed to B_0 ; thus $B_\infty = TN_\infty$ where N_∞ denotes the unipotent radical of B_∞ . For any root α of T in G we write \mathfrak{g}_α for the root space of T in \mathfrak{g} corresponding to α . Then $\mathfrak{n}_0 = \bigoplus_{\alpha>0} \mathfrak{g}_\alpha$ and $\mathfrak{n}_\infty = \bigoplus_{\alpha<0} \mathfrak{g}_\alpha$. For each simple positive root α we fix a non-zero root vector $X_\alpha \in \mathfrak{g}_\alpha$. The triple $\mathbf{spl} := (B_0, T, \{X_\alpha\})$ is called a *splitting* of G .

Let α be a simple positive root. Let H_α be the coroot for α regarded as an element in \mathfrak{t} , and fix $X_{-\alpha} \in \mathfrak{g}_{-\alpha}$ by the requirement that $[X_\alpha, X_{-\alpha}] = H_\alpha$. There is a unique homomorphism $\phi_\alpha : SL(2) \rightarrow G$ whose differential sends the elements

$$h_0 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, e_+ = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, e_- = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \in \mathfrak{sl}(2)$$

to $H_\alpha, X_\alpha, X_{-\alpha}$ respectively.

Let $X_+ := \sum_\alpha X_\alpha$, where the index α runs over all simple positive roots. Let $\mu \in X_*(T)$ be the sum of the positive coroots for T ; recall that $\langle \alpha, \mu \rangle = 2$ for every simple positive root α . We now decompose \mathfrak{g} as $\mathfrak{g} = \bigoplus_m \mathfrak{g}(m)$, where $\mathfrak{g}(m)$ is the $(-2m)$ -th weight subspace

$$\{Z \in \mathfrak{g} \mid \text{Ad}(\mu(a))Z = a^{-2m}Z \text{ for all } a \in \mathbf{G}_m\}$$

for the adjoint action $\text{Ad} \circ \mu$ of \mathbf{G}_m on \mathfrak{g} . For any $d \in \mathbf{Z}$ we put $F^d \mathfrak{g} = \bigoplus_{m \geq d} \mathfrak{g}(m)$. Thus $F^0 \mathfrak{g} = \mathfrak{b}_\infty$ and $F^1 \mathfrak{g} = \mathfrak{n}_\infty$. Note also that $X_+ \in \mathfrak{g}(-1)$ and hence that $\text{ad}(X_+)(\mathfrak{n}_\infty) \subset \mathfrak{b}_\infty$.

2.2. Chevalley’s Theorem. We denote by $k[\mathfrak{g}]$ (respectively, $k[\mathfrak{t}]$) the k -algebra of polynomial functions on \mathfrak{g} (respectively, \mathfrak{t}). The adjoint action of G on \mathfrak{g} induces an action of G on $k[\mathfrak{g}]$, and the action of the Weyl group W (of T in G) on \mathfrak{t} induces an action of W on $k[\mathfrak{t}]$. A theorem of Chevalley (see [CG97] for example) states that the restriction mapping $k[\mathfrak{g}] \rightarrow k[\mathfrak{t}]$ induces an isomorphism $k[\mathfrak{g}]^G \rightarrow k[\mathfrak{t}]^W$ from the ring of G -invariants in $k[\mathfrak{g}]$ to the ring of W -invariants in $k[\mathfrak{t}]$. We denote the affine k -variety corresponding to $k[\mathfrak{t}]^W$ by \mathfrak{t}/W . The morphism $u : \mathfrak{g} \rightarrow \mathfrak{t}/W$ dual to $k[\mathfrak{t}]^W = k[\mathfrak{g}]^G \hookrightarrow k[\mathfrak{g}]$ sends Z to the W -orbit in \mathfrak{t} consisting of elements that are G -conjugate to the semisimple part Z_s of the Jordan decomposition $Z = Z_s + Z_n$ of Z . (Thus Z_s is semisimple, Z_n is nilpotent, and $[Z_s, Z_n] = 0$.)

2.3. Regular elements of \mathfrak{g} . An element $Z \in \mathfrak{g}$ is said to be *regular* if the dimension of its centralizer in \mathfrak{g} is equal to the dimension of \mathfrak{t} . Recall [Kos63] that Z is regular if and only if the nilpotent part Z_n of the Jordan decomposition of Z is a regular element in the centralizer in \mathfrak{g} of the semisimple part Z_s of Z . The map $Z \mapsto Z_s$ induces (see [Kos63]) a bijection from the set of regular $\text{Ad}(G)$ -orbits in \mathfrak{g} to the set of semisimple $\text{Ad}(G)$ -orbits in \mathfrak{g} , and via the map u both sets of orbits can be identified with \mathfrak{t}/W .

2.4. Kostant’s section. Kostant proved that every element in the affine subspace $\mathfrak{b}_\infty + X_+$ of \mathfrak{g} is regular (see Lemma 10 in [Kos63]). For any $H \in \mathfrak{t} \subset \mathfrak{b}_\infty$, the semisimple part of $H + X_+$ is conjugate to H (see Lemma 11 in [Kos63]), and hence $\mathfrak{t} + X_+$ meets every regular $\text{Ad}(G)$ -orbit in \mathfrak{g} .

Now let \mathfrak{a} be any linear subspace of \mathfrak{b}_∞ that is complementary to $\text{ad}(X_+)(\mathfrak{n}_\infty)$ and stable under the action $\text{Ad} \circ \mu$ of \mathbf{G}_m (with μ as in 2.1). Then Kostant proved (see Remark 19’ in [Kos63]) that $\mathfrak{a} + X_+$ meets every regular $\text{Ad}(G)$ -orbit exactly once, and that the composition of the closed embedding $\mathfrak{a} + X_+ \hookrightarrow \mathfrak{g}$ and the morphism $u : \mathfrak{g} \rightarrow \mathfrak{t}/W$ is an isomorphism $\mathfrak{a} + X_+ \rightarrow \mathfrak{t}/W$ of algebraic varieties. Moreover Kostant proved (see Proposition 19 in [Kos63]) that for any $Z \in \mathfrak{t} + X_+$ there exists a unique element $n(Z) \in N_\infty$ such that $\text{Ad}(n(Z))(Z) \in \mathfrak{a} + X_+$, and that the map $Z \rightarrow n(Z)$ is a morphism of algebraic varieties from $\mathfrak{t} + X_+$ to N_∞ . But in fact the method of proof of Kostant’s Proposition 19 shows more, namely that for any $Z \in \mathfrak{b}_\infty + X_+$ there exists a unique element $n(Z) \in N_\infty$ such that $\text{Ad}(n(Z))(Z) \in \mathfrak{a} + X_+$, and that the map $Z \rightarrow n(Z)$ is a morphism of algebraic varieties from $\mathfrak{b}_\infty + X_+$ to N_∞ . It follows that the map $(n, Y) \mapsto \text{Ad}(n)(Y)$ is an isomorphism of varieties from $N_\infty \times (\mathfrak{a} + X_+)$ to $\mathfrak{b}_\infty + X_+$.

The results discussed above can be summarized as follows. Every element of $\mathfrak{b}_\infty + X_+$ is regular. The group N_∞ acts on $\mathfrak{b}_\infty + X_+$ (by the adjoint action), and the morphism $\mathfrak{b}_\infty + X_+ \rightarrow \mathfrak{t}/W$ (obtained by restricting u to $\mathfrak{b}_\infty + X_+$) is a principal N_∞ -bundle. Moreover, this principal bundle admits sections and is therefore trivial.

3. REVIEW OF THE VARIETY S OF STARS

3.1. Notation. In this section G is again a connected reductive group over an algebraically closed field k of characteristic 0. We fix a maximal torus T in G . We write $X_*(T)$ for the lattice of cocharacters of T , and we write $X_*(T)_{\mathbf{R}}$ for the real vector space obtained from $X_*(T)$ by extending scalars from \mathbf{Z} to \mathbf{R} . We write \mathcal{C} for

the set of Weyl chambers in $X_*(T)_{\mathbf{R}}$. For any Weyl chamber $C \in \mathcal{C}$ we write B_C for the corresponding Borel subgroup of G containing T . Thus any root of T appearing in the unipotent radical of B_C takes positive values on the chamber C . For any pair C, D of adjacent Weyl chambers, we let $P_{C,D}$ be the unique parabolic subgroup of G that contains both B_C and B_D and whose Levi component has semisimple rank 1. Thus the Lie algebra of $P_{C,D}$ is the direct sum of the Lie algebra of B_C and the root space for the unique root of T that is negative for C and positive for D .

We denote by W the Weyl group of T in G . The group W acts on the left of $X_*(T)$ and \mathcal{C} .

3.2. Definition of the variety of stars. We now review the definition of the variety S of stars, introduced by Langlands in [Lan83]. Let \mathcal{B} be the flag variety of G . Thus elements of \mathcal{B} are Borel subgroups in G , and G acts on the left of \mathcal{B} by conjugation. Consider a map $\mathcal{C} \rightarrow \mathcal{B}$, which we think of as a collection $(B(C))_{C \in \mathcal{C}}$ of Borel subgroups $B(C)$ of G indexed by the set of chambers $C \in \mathcal{C}$. We say that $(B(C))_{C \in \mathcal{C}}$ is a *star* if for every pair C, D of adjacent chambers there exists $g \in G$ such that $gB(C)g^{-1} = B_C$ and $gB(D)g^{-1} \subset P_{C,D}$ (equivalently, for every such pair C, D either $B(C) = B(D)$ or there exists $g \in G$ such that conjugation by g carries the pair $(B(C), B(D))$ into (B_C, B_D)). The set of stars is a Zariski closed subset of the Cartesian product $\mathcal{B} \times \cdots \times \mathcal{B}$, where the factors in the product are indexed by \mathcal{C} . Thus the set S of stars is a projective algebraic variety, and the diagonal left action of G on $\mathcal{B} \times \cdots \times \mathcal{B}$ preserves the subset of stars, so that G acts on the left of S . There is also an obvious right action of W on S : an element $w \in W$ sends a star $(B(C))_{C \in \mathcal{C}}$ to the star $C \mapsto B(w(C))$. The actions of G and W on S commute.

A star is said to be *regular* if $B(C) \neq B(D)$ for every pair C, D of adjacent chambers. The set S^0 of regular stars is a Zariski open subset of S , and it is preserved by the actions of G and W . There is an obvious base-point $s_0 \in S^0$, namely the regular star $C \mapsto B_C$. The action of G on S^0 is transitive, and the stabilizer in G of the base-point is T , so that $S^0 \simeq G/T$ as (G, W) -varieties.

3.3. Some rational functions $z(C, \beta)$ on S . At this point we need to fix a splitting $(B_0, T, \{X_\alpha\})$ (see 2.1) whose torus component is the torus T we fixed at the beginning of this section. As in 2.1 we write $B_0 = TN_0$ and $B_\infty = TN_\infty$, where B_∞ is the unique Borel subgroup containing T and opposite to B_0 . The group B_∞ has a unique open orbit on \mathcal{B} , namely the big cell consisting of all Borel subgroups $B \in \mathcal{B}$ opposed to B_∞ . The Borel subgroup B_0 lies in the big cell, and the group N_∞ acts simply transitively on the big cell, so that every Borel subgroup B opposed to B_∞ can be written as $B = nB_0n^{-1}$ for a unique element $n \in N_\infty$.

Following Langlands [Lan83], we let $S(B_\infty)$ denote the Zariski open subset of S consisting of all stars $C \mapsto B(C)$ such that $B(C)$ is opposed to B_∞ for all $C \in \mathcal{C}$. For each pair (C, β) consisting of a chamber $C \in \mathcal{C}$ and a root β of T that is simple relative to C (that is, β is positive on C and its kernel is a wall of C), Langlands [Lan83] defines a regular function $z(C, \beta)$ on $S(B_\infty)$. This regular function $z(C, \beta)$ depends on the choice of splitting $(B_0, T, \{X_\alpha\})$. Let w be the unique element in W such that $C = w(C_0)$, where C_0 is the unique chamber in \mathcal{C} such that $B_{C_0} = B_0$, and put $\alpha := w^{-1}(\beta)$; note that α is simple relative to B_0 . The value of the regular function $z(C, \beta)$ at a star $(B(C))$ in $S(B_\infty)$ is defined to be the unique element z in the field k such that

$$(3.1) \quad nB(D)n^{-1} = \exp(-zX_{-\alpha})B_0 \exp(zX_{-\alpha}),$$

where $D \in \mathcal{C}$ is the unique chamber adjacent to C across the wall defined by β , and where $n \in N_\infty$ is uniquely determined by the condition $nB(C)n^{-1} = B_0$.

It is obvious that the right W -action on S preserves the subset $S(B_\infty)$. Moreover, the W -action permutes the regular functions $z(C, \beta)$. More precisely, for $w \in W$ we have

$$(3.2) \quad z(C, \beta) \circ r_w = z(w(C), w(\beta)),$$

where $r_w : S(B_\infty) \rightarrow S(B_\infty)$ denotes the right action of w .

The left action of $B_\infty = TN_\infty$ on S preserves the subset $S(B_\infty)$. The regular functions $z(C, \beta)$ on $S(B_\infty)$ are invariant under N_∞ and transform as follows under T . Let $t \in T$ and let $l_t : S(B_\infty) \rightarrow S(B_\infty)$ denote the left action of t . We use (C, β) to define a B_0 -simple root α as before, so that there exists $w \in W$ carrying (C_0, α) into (C, β) . Then the transformation law is

$$(3.3) \quad z(C, \beta) \circ l_t = \alpha(t)^{-1}z(C, \beta).$$

We should note that Langlands only defines $z(C, \beta)$ on a certain cross-section for the N_∞ -action on $S(B_\infty)$, but since there is no essential difference between a regular function on this cross-section and an N_∞ -invariant regular function on $S(B_\infty)$, we have retained the notation used by Langlands for our extended functions.

3.4. Coordinates on $N_\infty \backslash S(B_\infty)$. Let $C \mapsto B(C)$ be a star in $S(B_\infty)$. Thus there exists a unique element $n \in N_\infty$ such that $B(C_0) = nB_0n^{-1}$. Moreover for each element $w \in W$ there is a unique element $n_w \in N_\infty$ such that $B(wC_0) = nn_wB_0(nn_w)^{-1}$. The elements n_w can be easily expressed [Lan83] in terms of the values of the regular functions $z(C, \beta)$ at the given star. Indeed, let $w = s_{\alpha_1}s_{\alpha_2}\dots s_{\alpha_p}$ be a reduced decomposition for w , where, as usual, we write s_α for the simple reflection associated to the simple root α . For $i = 1, \dots, p$ put $C_i := s_{\alpha_1}\dots s_{\alpha_i}C_0$ and $\beta_i = s_{\alpha_1}\dots s_{\alpha_{i-1}}\alpha_i$. Thus for $i = 1, \dots, p$ the chambers C_{i-1}, C_i are adjacent and separated by β_i . For $i = 1, \dots, p$ let z_i denote the value of the function $z(C_{i-1}, \beta_i)$ on the given star. Then it follows immediately from the definition of the regular functions $z(C_{i-1}, \beta_i)$ that

$$(3.4) \quad n_w = \exp(-z_1X_{-\alpha_1}) \exp(-z_2X_{-\alpha_2}) \dots \exp(-z_pX_{-\alpha_p}).$$

3.5. Refined Bruhat decomposition. We will soon need the following two well-known statements (part of Bruhat theory). The first statement is that the inclusion of $N_G(T)$ in G induces a bijection from $N_G(T)$ to $N_0 \backslash G/N_0$. The second statement is that $N_0x_1N_0 \cdot N_0x_2N_0 = N_0x_1x_2N_0$ whenever $x_1, x_2 \in N_G(T)$ satisfy the condition that the length of the image of x_1x_2 in W is the sum of the lengths of the images of x_1, x_2 .

3.6. Analysis of the W -action on $S(B_\infty)$. Now let $C \mapsto B(C)$ be a regular star in $S(B_\infty)$. Denote this star by s and as usual denote the standard star $C \mapsto B_C$ by s_0 . We claim that there exists a unique element $g \in N_\infty N_0$ such that $s = gs_0$. Indeed, let g_1 be any element in G such that $s = g_1s_0$. Since $B(C_0)$ lies in the big cell, there exists $n \in N_\infty$ such that $g_1 \in nB_0$. Modifying the element g_1 on the right by a suitable element in T , we see that there exists $u \in N_0$ such that $s = nus_0$. This proves the existence of g ; its uniqueness is obvious. The uniqueness of the factorization $g = nu$ is also obvious. Note that the element n agrees with the element denoted by n in 3.4.

Recall that there is right action of W on $S(B_\infty)$. Let $w \in W$. Then $s' := sw$ is another regular star in $S(B_\infty)$, so there exists unique $g' = n'u' \in N_\infty N_0$ such that $s' = g's_0$. Let $x \in G$ be defined by the equation $g' = gx$. It is obvious that $x \in N_G(T)$ and that $x \mapsto w$ under the canonical surjection $N_G(T) \rightarrow W$, and it is also obvious that x depends only on w and the N_∞ -orbit of s . Thus one would expect to be able to express x in terms of w and the values of the regular functions $z(C, \beta)$ on s . Such a formula is implicit in [Lan83] (see the proof of Proposition 5.2 of that paper), and is stated explicitly in the next lemma.

As in 3.4 we choose a reduced decomposition $w = s_{\alpha_1} \dots s_{\alpha_p}$ and use it and the star s to define scalars z_1, \dots, z_p . For any simple root α we put

$$(3.5) \quad \dot{s}_\alpha := \phi_\alpha \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix},$$

where $\phi_\alpha : SL(2) \rightarrow G$ is the homomorphism defined in 2.1; of course \dot{s}_α lies in $N_G(T)$ and maps to the simple reflection s_α in W .

Lemma 3.1. *The element x defined above is given by*

$$x = (\alpha_1^\vee(z_1)^{-1} \dot{s}_{\alpha_1})(\alpha_2^\vee(z_2)^{-1} \dot{s}_{\alpha_2}) \dots (\alpha_p^\vee(z_p)^{-1} \dot{s}_{\alpha_p}).$$

Proof. The star s' is equal to $C \mapsto B'(C) := B(w(C))$. On the one hand,

$$B'(C_0) = n' B_0(n')^{-1}.$$

On the other hand,

$$\begin{aligned} B'(C_0) &= B(w(C_0)) \\ &= nn_w B_0(nn_w)^{-1}, \end{aligned}$$

with $n_w \in N_\infty$ as in 3.4 (relative to the star s). Therefore $n^{-1}n' = n_w$, and hence

$$(3.6) \quad x = g^{-1}g' = u^{-1}n^{-1}n'u' \in N_0 n_w N_0.$$

A simple calculation in $SL(2)$ shows that for any simple root α

$$N_0 \exp(-zX_{-\alpha})N_0 = N_0 \alpha^\vee(z)^{-1} \dot{s}_\alpha N_0.$$

Combining this with equation (3.4) and the second statement in section 3.5, we find that

$$(3.7) \quad N_0 n_w N_0 = N_0 y N_0,$$

where

$$y = (\alpha_1^\vee(z_1)^{-1} \dot{s}_{\alpha_1})(\alpha_2^\vee(z_2)^{-1} \dot{s}_{\alpha_2}) \dots (\alpha_p^\vee(z_p)^{-1} \dot{s}_{\alpha_p}).$$

Comparing (3.6) and (3.7), we see that $N_0 x N_0 = N_0 y N_0$, and it then follows from the first statement in section 3.5 that $x = y$, as we needed to show. \square

3.7. Some 1-cocycles in $N_G(T)$. Now suppose that G, T, B_0, B_∞ are defined over a subfield k_0 of k such that k is algebraic over k_0 . Thus k is an algebraic closure of k_0 , and we put $\Gamma := \text{Gal}(k/k_0)$. We write $\mathfrak{g}(k_0)$ for the subset of \mathfrak{g} consisting of k_0 -rational points. Let $Y \in \mathfrak{g}(k_0)$ and suppose that Y is regular semisimple. We choose an element $H \in \mathfrak{t}$ in the G -conjugacy class of Y . Of course H need not be defined over k_0 .

There exists $g \in G$, unique up to right multiplication by T , such that $Y = \text{Ad}(g)(H)$. For $\sigma \in \Gamma$ put $x_\sigma := g^{-1}\sigma(g)$. Then $\sigma \mapsto x_\sigma$ is a 1-cocycle of Γ in $N_G(T)$, and replacing g by gt (for $t \in T$) replaces x_σ by $t^{-1}x_\sigma\sigma(t)$.

The star $s := gs_0$ is well-defined. We now assume that s lies in $S(B_\infty)$. As in 3.6 we normalize g within its coset gT by insisting that g lie in $N_\infty N_0$. The next lemma gives a formula for the 1-cocycle x_σ (obtained from this particular g) in terms of the values of the functions $z(C, \beta)$ on the star s . This lemma is a variant of Proposition 5.2 in [Lan83], and our proof (including that of Lemma 3.1) is a rearrangement of Langlands's.

Lemma 3.2. *Let $\sigma \in \Gamma$ and let w_σ be the image of x_σ in W . Choose a reduced decomposition $w_\sigma = s_{\alpha_1} \dots s_{\alpha_p}$ of w_σ , and as in section 3.4 use it and the star s to define scalars z_1, \dots, z_p . Then x_σ is given by*

$$x_\sigma = (\alpha_1^\vee(z_1)^{-1} \dot{s}_{\alpha_1})(\alpha_2^\vee(z_2)^{-1} \dot{s}_{\alpha_2}) \dots (\alpha_p^\vee(z_p)^{-1} \dot{s}_{\alpha_p}).$$

Proof. Since N_∞, N_0 are defined over k_0 , the element $\sigma(g)$ also lies in $N_\infty N_0$. The lemma now follows from Lemma 3.1, applied to the elements g and $g' = \sigma(g)$. \square

4. 1-COCYCLES COMING FROM KOSTANT'S SECTION

4.1. Goal. Our next goal is to calculate the 1-cocycles of section 3.7 for elements Y lying in the image of Kostant's section $v : \mathfrak{t}/W \rightarrow \mathfrak{g}$. By Lemma 3.2 what we must do is calculate certain values of the functions $z(C, \beta)$.

4.2. Values of $z(C, \beta)$ on stars coming from Kostant's section. Let $H \in \mathfrak{t}'$, where \mathfrak{t}' denotes the set of regular elements in \mathfrak{t} . We write $p : \mathfrak{t} \rightarrow \mathfrak{t}/W$ for the map dual to the inclusion $k[\mathfrak{t}]^W \hookrightarrow k[\mathfrak{t}]$. Let Y be any element in $\mathfrak{b}_\infty + X_+$ such that $u(Y) = p(H)$ (where $u : \mathfrak{g} \rightarrow \mathfrak{t}/W$ is as in 2.2). Recall from section 2.4 that the $\text{Ad}(N_\infty)$ -orbit of Y is uniquely determined by H .

Now choose $g \in G$ such that $Y = \text{Ad}(g)(H)$ and define a regular star $s := gs_0$ (as in 3.7). The star s depends only on Y , and the N_∞ -orbit of s depends only on H .

Lemma 4.1. *The star s lies in $S(B_\infty)$. The value of the regular function $z(C, \beta)$ on s is $\beta(H)$.*

Proof. The construction $H \mapsto s$ is a well-defined map $\delta : \mathfrak{t}' \rightarrow N_\infty \backslash S$. The right action of W on S induces a right action of W on $N_\infty \backslash S$, and we define a right action of W on \mathfrak{t}' by converting the usual left action into a right action:

$$Hw := w^{-1}(H).$$

It is then immediate that our map $\delta : \mathfrak{t}' \rightarrow N_\infty \backslash S$ is W -equivariant.

The truth of the two statements of the lemma depends only on the N_∞ -orbit of s . Therefore we are free to take $Y = H + X_+$. We write the star s obtained from Y and H as $C \mapsto B(C)$. It is well-known (and easy to prove) that since H is regular, the map $u \mapsto \text{Ad}(u)(H)$ is an isomorphism from N_0 to $H + \mathfrak{n}_0$. Therefore there exists $u \in N_0$ such that $Y = \text{Ad}(u)(H)$. Thus $s = us_0$, and it follows that $B(C_0) = B_0$. Since the map $\delta : \mathfrak{t}' \rightarrow N_\infty \backslash S$ is W -equivariant, we conclude that $B(C)$ lies in the N_∞ -orbit of B_0 for all chambers C , which proves the first statement of the lemma.

Again using the W -equivariance of δ , and using (3.2) as well, we see that in order to prove the second statement of the lemma, it is enough to prove that for every simple root α the function $z(C_0, \alpha)$ takes the value $\alpha(H)$ on s . So fix a simple root α and let $P_\alpha = M_\alpha N_\alpha$ be the associated parabolic subgroup containing B_0 ;

thus \mathfrak{m}_α is spanned by \mathfrak{t} , \mathfrak{g}_α and $\mathfrak{g}_{-\alpha}$, and N_α is the unipotent radical of P_α . The homomorphism $\phi_\alpha : SL(2) \rightarrow G$ factors through M_α .

Note that for any scalar z

$$\text{Ad}(\exp(zX_\alpha))(H + X_+) \in H + X_\alpha + [zX_\alpha, H] + \mathfrak{n}_\alpha.$$

Therefore $\text{Ad}(\exp(\alpha(H)^{-1}X_\alpha))(H + X_+)$ lies in $H + \mathfrak{n}_\alpha$. But (again since H is regular) the map

$$n_\alpha \mapsto \text{Ad}(n_\alpha)(H)$$

is an isomorphism from N_α to $H + \mathfrak{n}_\alpha$. Therefore there exists $n_\alpha \in N_\alpha$ such that

$$H + X_+ = \text{Ad}(\exp(-\alpha(H)^{-1}X_\alpha) \cdot n_\alpha)(H).$$

It follows from this equality that

$$B(s_\alpha(C_0)) = \exp(-\alpha(H)^{-1}X_\alpha) \cdot \dot{s}_\alpha \cdot B_0 \cdot (\exp(-\alpha(H)^{-1}X_\alpha) \cdot \dot{s}_\alpha)^{-1}.$$

A simple calculation in $SL(2)$ then shows that

$$B(s_\alpha(C_0)) = \exp(-\alpha(H)X_{-\alpha}) \cdot B_0 \cdot \exp(\alpha(H)X_{-\alpha}),$$

and this (together with the fact that $B(C_0) = C_0$) shows that the value of $z(C_0, \alpha)$ on s is $\alpha(H)$, as desired. \square

4.3. 1-cocycles in $N_G(T)$ coming from Kostant's section. We return to the situation of 3.7. We assume further that the family $\{X_\alpha\}$ is defined over k_0 , in the sense that $\sigma(X_\alpha) = X_{\sigma(\alpha)}$ for all $\sigma \in \Gamma$. Thus $(B_0, T, \{X_\alpha\})$ is a k_0 -splitting in the terminology of [LS87]. Moreover the element $X_+ = \sum_\alpha X_\alpha$ lies in $\mathfrak{g}(k_0)$.

To define Kostant's section we need to choose a complementary subspace \mathfrak{a} as in 2.4. Of course we may assume that \mathfrak{a} is defined over k_0 . Then $\mathfrak{a} + X_+$ is defined over k_0 .

Now let $H \in \mathfrak{t}'$ and assume that the image $p(H)$ of H in \mathfrak{t}/W is k_0 -rational. Let Y be any k_0 -rational element of $\mathfrak{b}_\infty + X_+$ such that $u(Y) = p(H)$. It is clear from the discussion above that such a k_0 -rational element exists, and it follows from 2.4 that the $\text{Ad}(N_\infty(k_0))$ -orbit of Y is uniquely determined by H . The first statement of Lemma 4.1 implies that the star s defined in 3.7 (for H, Y as above) lies in $S(B_\infty)$. Therefore there exists a unique element $g \in N_\infty N_0$ such that $Y = \text{Ad}(g)(H)$. As in 3.7 we define a 1-cocycle x_σ of Γ in $N_G(T)$ by $x_\sigma = g^{-1}\sigma(g)$. Note that this 1-cocycle is independent of the choice of Y ; it depends only on H and our chosen k_0 -splitting $\mathfrak{spl} = (B_0, T, \{X_\alpha\})$.

Lemma 4.2. *Let $\sigma \in \Gamma$ and let w_σ be the image of x_σ in W . Choose a reduced decomposition $w_\sigma = s_{\alpha_1} s_{\alpha_2} \dots s_{\alpha_p}$ of w_σ and for $i = 1, \dots, p$ put $\beta_i = s_{\alpha_1} s_{\alpha_2} \dots s_{\alpha_{i-1}}(\alpha_i)$. Then*

$$x_\sigma = \left(\prod_{i=1}^p \beta_i^\vee(\beta_i(H)) \right)^{-1} \cdot \dot{w}_\sigma,$$

where $\dot{w}_\sigma = \dot{s}_{\alpha_1} \dots \dot{s}_{\alpha_p}$.

Proof. This follows immediately from Lemmas 3.2 and 4.1. \square

However the 1-cocycle we really need is a variant of x_σ . We replace $\mathfrak{b}_\infty + X_+$ by $\mathfrak{b}_0 + X_-$, where $X_- := \sum_\alpha X_{-\alpha}$, the sum ranging over simple roots α . In this context Kostant's theory provides an isomorphism (over k_0) from the quotient of $\mathfrak{b}_0 + X_-$ by the adjoint action of N_0 to the space \mathfrak{t}/W .

Let $H \in \mathfrak{t}'$ be as above and let Z be any k_0 -rational element of $\mathfrak{b}_0 + X_-$ such that $u(Z) = p(H)$. Let h be any element of G such that $Z = \text{Ad}(h)(H)$. Define a 1-cocycle y_σ of Γ in $N_G(T)$ by $y_\sigma = h^{-1}\sigma(h)$. Replacing h by ht , for $t \in T$, has the effect of replacing y_σ by $t^{-1}y_\sigma\sigma(t)$. Thus the 1-cocycle y_σ is well-defined up to such equivalences.

Lemma 4.3. *For a suitable choice of h , the 1-cocycle y_σ is given by the following formula. Let $\sigma \in \Gamma$ and let w_σ be the image of y_σ in W . Choose a reduced decomposition $w_\sigma = s_{\alpha_1}s_{\alpha_2}\dots s_{\alpha_p}$ of w_σ and for $i = 1, \dots, p$ put $\beta_i = s_{\alpha_1}s_{\alpha_2}\dots s_{\alpha_{i-1}}(\alpha_i)$. Then y_σ is given by*

$$y_\sigma = \left(\prod_{i=1}^p \beta_i^\vee(\beta_i(H)) \right) \cdot \dot{w}_\sigma,$$

where $\dot{w}_\sigma = \dot{s}_{\alpha_1} \dots \dot{s}_{\alpha_p}$.

Proof. Let θ be the automorphism of G that induces -1 on \mathfrak{t} and exchanges X_α and $X_{-\alpha}$ for all simple roots α . It is clear that θ is defined over k_0 and that its square is the identity. Since θ exchanges $\mathfrak{b}_\infty + X_+$ and $\mathfrak{b}_0 + X_-$, it is clear that we get a suitable 1-cocycle y_σ by applying θ to the 1-cocycle x_σ associated to $\theta(H) = -H$. Note that $\theta(\dot{s}_\alpha) = \alpha^\vee(-1)\dot{s}_\alpha$ for any simple root α (a simple calculation in $SL(2)$). The desired result then follows from the previous lemma. \square

5. TRANSFER FACTORS $\Delta'_0(X_H, X_G)$ ON \mathfrak{g}

5.1. Notation. As in 3.7 we assume that G is quasi-split over k_0 , and we fix a k_0 -splitting $\mathbf{spl} = (B_0, T, \{X_\alpha\})$ for G . We assume further that k_0 is a local field (of characteristic 0). Let (H, \mathcal{H}, s, ξ) be endoscopic data for G (see [LS87]).

5.2. General discussion of transfer factors. Following Waldspurger [Wal97] we consider transfer factors on \mathfrak{g} analogous to the transfer factors on G defined by Langlands and Shelstad [LS87]. Waldspurger's factors are analogous to the transfer factors $\Delta(\gamma_H, \gamma_G; \bar{\gamma}_H, \bar{\gamma}_G)$ on p. 248 of [LS87] with the factor Δ_{IV} removed.

On the quasi-split group G Langlands and Shelstad also define transfer factors $\Delta_0(\gamma_H, \gamma_G)$ (again on p. 248 of [LS87]). These depend on the chosen k_0 -splitting \mathbf{spl} . The transfer factors $\Delta'_0(X_H, X_G)$ in this paper are analogous to $\Delta_0(\gamma_H, \gamma_G)$ with the factor Δ_{IV} removed.

5.3. a -data. Let T_H be a maximal torus in H . There is a canonical G -conjugacy class of embeddings $T_H \rightarrow G$, and this G -conjugacy class contains members that are defined over k_0 . Fix such a k_0 -embedding $T_H \rightarrow G$, and let T_G denote the image of T_H in G , a maximal torus of G , defined over k_0 . We identify T_H with T_G , so that the set R_H of roots of T_H in H becomes a subset of the set R_G of roots of T_G in G .

Recall from [LS87], 2.2 that a -data for T_G consists of elements $a_\alpha \in k^\times$, one for each $\alpha \in R_G$, satisfying

1. $a_{\sigma\alpha} = \sigma(a_\alpha)$ for all $\alpha \in R_G$ and all $\sigma \in \Gamma$,
2. $a_{-\alpha} = -a_\alpha$ for all $\alpha \in R_G$.

We now choose a -data for T_G .

Let G_{sc} denote the simply connected cover of the derived group of G , and let T_G^{sc} denote the inverse image of T_G under the canonical homomorphism from G_{sc} to G . Recall that Langlands and Shelstad (see 2.3 in [LS87]) define an invariant

$\lambda(T_G^{\text{sc}}) \in H^1(k_0, T_G^{\text{sc}})$, which depends on **spl** and the a -data $\{a_\alpha\}$, as well as T_G . We denote by $\lambda(T_G) \in H^1(k_0, T_G)$ the image of $\lambda(T_G^{\text{sc}})$ under the map induced by the canonical homomorphism $T_G^{\text{sc}} \rightarrow T_G$.

5.4. χ -data. For any root $\alpha \in R_G$ we let k_α (respectively, $k_{\pm\alpha}$) denote the field of definition of α (respectively, the set $\{\alpha, -\alpha\}$). Thus $k_0 \subset k_{\pm\alpha} \subset k_\alpha \subset k$ and $[k_\alpha : k_{\pm\alpha}]$ is 1 or 2. As in [LS87], if $[k_\alpha : k_{\pm\alpha}] = 2$, we say that α (and its Γ -orbit in R_G) is *symmetric*, and we let χ_α denote the quadratic character on $k_{\pm\alpha}^\times$ associated to the quadratic extension $k_\alpha/k_{\pm\alpha}$ by local classfield theory. Transfer factors on \mathfrak{g} are simpler than those on G , in that it is not necessary to extend χ_α to a quasi-character on k_α^\times . Consequently, when defining transfer factors on \mathfrak{g} , it is irrelevant whether or not \mathcal{H} is isomorphic to ${}^L H$.

5.5. Definition of $\Delta'_0(X_H, X_G)$. Let $X_H \in \mathfrak{t}_H$ and assume that its image X_G in \mathfrak{t}_G is regular. We are going to define our transfer factor $\Delta'_0(X_H, X_G)$ by

$$\Delta'_0(X_H, X_G) = \Delta_I(X_H, X_G)\Delta_{II}(X_H, X_G),$$

where the factors $\Delta_I(X_H, X_G)$, $\Delta_{II}(X_H, X_G)$, to be defined below, are analogous to the factors $\Delta_I(\gamma_H, \gamma_G)$, $\Delta_{II}(\gamma_H, \gamma_G)$ of [LS87].

5.6. Definition of $\Delta_I(X_H, X_G)$. Let \hat{T}_G denote the complex torus dual to T_G (Langlands duality). The element s appearing in our endoscopic data is a Γ -fixed element in the center of the Langlands dual group \hat{H} of H , and thus can be regarded as a Γ -fixed element \mathfrak{s}_{T_G} of $\hat{T}_H = \hat{T}_G$. There is a Tate-Nakayama pairing (see [Kot86] for example)

$$\langle \cdot, \cdot \rangle : H^1(k_0, T_G) \times \hat{T}_G^\Gamma \rightarrow \mathbf{C}^\times,$$

where \hat{T}_G^Γ denotes the group of fixed points of Γ in \hat{T}_G . We define Δ_I by

$$\Delta_I(X_H, X_G) := \langle \lambda(T_G), \mathfrak{s}_{T_G} \rangle.$$

5.7. Definition of $\Delta_{II}(X_H, X_G)$. We define Δ_{II} by

$$\Delta_{II}(X_H, X_G) := \prod_\alpha \chi_\alpha \left(\frac{\alpha(X_G)}{a_\alpha} \right),$$

where the product is taken over a set of representatives for the symmetric orbits of Γ in the set $R_G \setminus R_H$.

5.8. Discussion of $\Delta'_0(X_H, X_G)$. It is immediate from (the proof of) Lemma 3.2.C of [LS87] that $\Delta'_0(X_H, X_G)$ is independent of the choice of a -data. Thus $\Delta'_0(X_H, X_G)$ depends only on the choice of k_0 -splitting **spl**.

Put $\Delta'_0(\gamma_H, \gamma_G) := \Delta_0(\gamma_H, \gamma_G) \cdot \Delta_{IV}(\gamma_H, \gamma_G)^{-1}$, with Δ_0 and Δ_{IV} as in [LS87]. It is easy to see that for X_H sufficiently close to 0

$$\Delta'_0(X_H, X_G) = \Delta'_0(\exp(X_H), \exp(X_G)).$$

Moreover it is obvious that

$$\Delta'_0(a^2 X_H, a^2 X_G) = \Delta'_0(X_H, X_G)$$

for all $a \in k_0^\times$. These two properties characterize the transfer factors Δ'_0 on \mathfrak{g} .

Now suppose that $X'_G \in \mathfrak{g}(k_0)$ is stably conjugate to X_G , so that there exists $h \in G$ such that $\text{Ad}(h)(X'_G) = X_G$. Then $\sigma \mapsto h\sigma(h)^{-1}$ is a 1-cocycle of Γ in T_G whose class we denote by $\text{inv}(X_G, X'_G)$. Then

$$(5.1) \quad \Delta'_0(X_H, X'_G) = \Delta'_0(X_H, X_G) \cdot \langle \text{inv}(X_G, X'_G), s_{T_G} \rangle^{-1}.$$

This follows from Lemmas 3.2.B and 3.4.A of [LS87], or rather from their (easy) Lie algebra analogs.

5.9. Main result. As above we use our fixed splitting **spl** to define transfer factors $\Delta'_0(X_H, X_G)$ on \mathfrak{g} and to define an element $X_- \in \mathfrak{g}(k_0)$.

Theorem 5.1. *The transfer factor $\Delta'_0(X_H, X_G)$ is equal to 1 whenever X_G lies in the set of k_0 -rational elements in $\mathfrak{b}_0 + X_-$.*

Proof. Note that $a_\alpha := \alpha(X_G)$ is a valid choice of a -data for T_G . With this choice of a -data it is obvious that $\Delta_{\text{II}}(X_H, X_G) = 1$, and what we must show is that $\Delta_{\text{I}}(X_H, X_G) = 1$. Since this must be true for all endoscopic data, we must show that $\lambda(T_G^{\text{sc}}) = 1$ for this particular choice of a -data. It is harmless to assume that $G = G_{\text{sc}}$, and we do so in order to simplify notation. Choose $h \in G$ and $H \in \mathfrak{t}$ such that $X_G = \text{Ad}(h)(H)$. (Since the endoscopic data are now irrelevant, it should cause no confusion to use our usual convention of denoting elements of \mathfrak{t} by H .) Thus $hTh^{-1} = T_G$ and we use the inner automorphism $x \mapsto h x h^{-1}$ to identify T with T_G over k . As in 4.3 $y_\sigma := h^{-1}\sigma(h)$ is a 1-cocycle of Γ in $N_G(T)$, and we denote by w_σ the image of y_σ in the Weyl group W . On p. 231 of [LS87] Langlands and Shelstad define a 1-cocycle

$$m_\sigma := \left(\prod_{\beta} \beta^\vee(a_\beta) \right) \cdot \dot{w}_\sigma$$

of Γ in $N_G(T)$, where \dot{w}_σ is defined as in Lemma 4.3, and where the product is taken over all positive roots β of T in G such that $w_\sigma^{-1}(\beta)$ is negative. We are using our identification of T and T_G over k to view β as a root of T_G , so that a_β is defined. For our particular choice of a -data, we have $a_\beta = \beta(H)$, and therefore m_σ is exactly the same as the 1-cocycle appearing in Lemma 4.3. It follows from Lemma 4.3 that by choosing h correctly within its coset hT , we may assume that

$$(5.2) \quad h^{-1}\sigma(h) = m_\sigma.$$

Langlands and Shelstad now define a 1-cocycle of Γ in T_G by $\sigma \mapsto h m_\sigma \sigma(h)^{-1}$, and they define $\lambda(T_G^{\text{sc}})$ to be the class of this 1-cocycle. Since (5.2) shows that this 1-cocycle is trivial, we see that $\lambda(T_G^{\text{sc}})$ is trivial, as desired. \square

Corollary 5.2. *The transfer factor $\Delta'_0(X_H, X_G)$ is equal to $\langle \text{inv}(X_G, X'_G), s_{T_G} \rangle$, where X'_G is any k_0 -rational element in $\mathfrak{b}_0 + X_-$ that is stably conjugate to X_G .*

Proof. This follows from the theorem together with (5.1). \square

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