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KOSZUL DUALITY FOR PARABOLIC AND SINGULAR CATEGORY $\mathcal{O}$

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ABSTRACT. This paper deals with a generalization of the "Koszul duality theorem" for the Bernstein-Gelfand-Gelfand category $\mathcal{O}$ over a complex semisimple Lie-algebra, established by Beilinson, Ginzburg and Soergel in *Koszul duality patterns in representation theory*, J. Amer. Math. Soc. 9 (1996), 473–527. In that paper it was proved that any "block" in $\mathcal{O}$, determined by an integral, but possibly singular weight, is Koszul (i.e. equivalent to the category of finitely generated modules over some Koszul ring) and, moreover, that the "Koszul dual" of such a block is isomorphic to a "parabolic subcategory" of the trivial block in $\mathcal{O}$.

We extend these results to prove that a parabolic subcategory of an integral and (possibly) singular block in $\mathcal{O}$ is Koszul and we also calculate the Koszul dual of such a category.

1. INTRODUCTION

(Cf. the subsection about category $\mathcal{O}$ in section 2.) Let $\mathfrak{g} \supset \mathfrak{b} \supset \mathfrak{h}$ be a complex semi-simple Lie-algebra, a Borel subalgebra and a Cartan subalgebra. Let $\phi, \psi \in \mathfrak{h}^*$ be integral and dominant. Let $\mathcal{O}_\phi$ be the block in the category $\mathcal{O}$ defined in [BGG1] corresponding to the weight ϕ. Let $\mathfrak{q}(\psi) \supset \mathfrak{b}$ be the parabolic subalgebra of $\mathfrak{g}$ corresponding to ψ and denote by $\mathcal{O}_\phi^\psi$ the subcategory of $\mathcal{O}_\phi$ whose objects are locally finite over $\mathfrak{q}(\psi)$. Denote by R_ϕ^ψ the endomorphism ring of a (minimal) projective generator of $\mathcal{O}_\phi^\psi$. The conclusion of this paper is the parabolic and singular-singular and parabolic Koszul duality:

Theorem 1.1. *The ring R_ϕ^ψ is Koszul. Its Koszul dual $(R_\phi^\psi)^!$ is isomorphic to $R_{-w_0\psi}^\phi$.*

This theorem generalizes the main result of [BGS], Theorem 1.1.3, where $R_0^\phi \cong (R_\phi^0)^!$ is proved. My proof depends heavily on their results but may have the advantage that I am (once an isomorphism $(R_0^0)^! \cong R_0^0$ is chosen) able to construct the isomorphism and not just prove its existence as in [BGS]. Actually, this isomorphism is induced by "translation onto the wall".

The methods used in this paper are elementary except for the proof of Proposition 3.5 where some theory of perverse sheaves is needed.

At the end of section 2 we consider as an example — in order to give the non-expert reader a feeling for these things — the $\mathfrak{sl}(2, \mathbf{C})$ case.

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Motivation. Let me start by mentioning some applications of the Koszul duality of [BGS]. First of all, the fact that $\mathcal{O}_0$ is Koszul and selfdual implies ([BGS]) that the Kazhdan-Lusztig conjecture, which is one of the deepest results in representation theory, holds for this category. However, there is no way to prove the Koszulity of $\mathcal{O}_0$ without using the Kazhdan-Lusztig conjectures. One could think of the Koszulity property as a strengthening of the Kazhdan-Lusztig conjectures.

The Koszul duality theorem for $\mathcal{O}_\phi$ enables us to solve some questions working with the Koszul dual category $\mathcal{O}_0^{\mathfrak{q}(\phi)}$. A famous theorem of Soergel ([Soe2]) states that the ext groups between a Verma module and a simple module vanish in every second degree. From the standpoint of geometry, which applies to $\mathcal{O}_0^{\mathfrak{q}(\phi)}$ (but not to $\mathcal{O}_\phi$) — via $\mathcal{D}$-modules on the flag variety of G/Q, where G is the Lie-group corresponding to $\mathfrak{g}$ and Q is the subgroup with $Lie Q = \mathfrak{q}$ — it is not so hard to show that this theorem actually holds for $\mathcal{O}_0^{\mathfrak{q}(\phi)}$. One then deduces Soergel's theorem from the Koszul duality theorem.

A possible important application of the Koszul duality theorem in this paper is the following: There exists an equivalence between the category of finite-dimensional representations of a quantum group at a root of unity and a parabolic category $\mathcal{O}$ of the corresponding affine Kac-Moody algebra; cf. [KL1], [KL2]. So in order to establish a singular-parabolic duality for the category of finite-dimensional representations of the quantum group we need a parabolic and singular-singular and parabolic duality for $\mathcal{O}$ of the Kac-Moody algebra. Theorem 1.1 indicates that such a duality should exist.

2. Review of category $\mathcal{O}$ and Koszul rings

Koszul rings. We refer to [BGS] for a detailed review of the theory of Koszul rings. Here we just recall

Definition 2.1. Let $R = \bigoplus_{i \geq 0} R_i$ be a positively graded ring with R_0 semi-simple and put $R_+ = \bigoplus_{i>0} R_i$. R is called Koszul if the right module $R_0 \cong R/R_+$ admits a graded projective resolution $P^\bullet \twoheadrightarrow R_0$ such that P^i is generated by its component P^i_i in degree i.

For a positively graded ring R put $R^! := \mathrm{Ext}_R(R_0, R_0)$. This has the structure of a graded ring, with $(R^!)_i = \mathrm{Ext}^i_R(R_0, R_0)$ and multiplication given by the cup-product. If R is Koszul, we call $R^!$ the Koszul dual of R. Denote by $\mathrm{ext}^i_R(\ , \ )$ extensions in the category of graded (left) R-modules and for a graded R-module M and an integer j define the graded R-module $M\langle j\rangle$ by $M\langle j\rangle_i = M_{i-j}$. Let us say that a graded R-module M is pure of weight i if $M = M_{-i}$.

Proposition 2.2. *Let R be a positively graded ring with R_0 semi-simple. Then R is Koszul iff $\mathrm{ext}^i_R(R_0, R_0\langle j\rangle) = 0$ whenever $i \neq j$. If R is Koszul, then R is generated in degree 1 over its degree 0 part and has relations only in degree 2. Then also $R^!$ is Koszul and, if in addition R_1 is right finitely generated over R_0, $(R^!)^! \cong R$ canonically.*

Remark 2.3. (i) Each ring R considered in this paper is finite-dimensional over $\mathbf{C}$ so the condition R_1 is right finitely generated over R_0 is automatically fulfilled.

(ii) The standard definition of $R^!$ is $\mathrm{Ext}_R(R_0, R_0)^{opp}$. However, the category $\mathcal{O}$ defined below admits a duality (e.g. [BGG1]) which restricts to a duality on each subcategory of $\mathcal{O}$ we shall consider. This implies that the endomorphism ring of a projective generator in such a category is isomorphic to its own opposite ring and each ring R in this paper is either an endomorphism ring of this type or the ext-algebra of such a ring, hence $R^{opp} \cong R$. Thus, our definition of $R^!$ coincides with the standard definition.

Anyway, Proposition 2.2 holds with our definition of $R^!$ as well as with the standard definition.

A convention. *Each ring R under consideration in this paper will be indexed by a pair of weights in $\mathfrak{h}^*$, e.g. $R = R_0^\phi$. In order to not confuse the index $0 \in \mathbb{N}$ referring to grading degree with index $0 \in \mathfrak{h}^*$ referring to weight we shall in the following always write $(R)_i$ when referring to grading degree $i \in \mathbb{N}$, e.g. $(R_0^\phi)_0$.*

Category $\mathcal{O}$. Let $\mathfrak{g} \supset \mathfrak{b} \supset \mathfrak{h}$ be a semi-simple Lie-algebra, a Borel subalgebra and a Cartan subalgebra and let $U(\mathfrak{g})$ be the enveloping algebra of $\mathfrak{g}$. Let R be the root-system, R_+ the positive roots, $B \subset R_+$ the basis, $R_- := -R_+$ and let (W, S) be the corresponding Coxeter-system and let $w_0 \in W$ be the longest element. Let ρ denote the half-sum of the positive roots; for $\phi \in \mathfrak{h}^*, w \in W$ we put $w \cdot \phi := w(\phi + \rho) - \rho$. Denote by H_α the coroot corresponding to $\alpha \in R$.

In this paper each weight $\phi \in \mathfrak{h}^*$ is assumed to be integral, i.e. $\phi(H_\alpha) \in \mathbb{Z}, \forall \alpha \in B$. ϕ is called dominant if $\phi(H_\alpha) \geq -1, \forall \alpha \in B$ and anti-dominant if $\phi(H_\alpha) \leq -1, \forall \alpha \in B$. Denote by $L(\phi)$ the simple module with highest weight ϕ. (*This terminology differs from that of* [BGG1]*; they denote by L_ϕ the simple module of highest weight $\phi - \rho$.*)

Let $\mathcal{O}$ be the category of all finitely generated $U(\mathfrak{g})$-modules which are locally finite over $\mathfrak{b}$ and semi-simple over $\mathfrak{h}$. Each object in $\mathcal{O}$ has finite length and $\mathcal{O}$ has enough projectives, e.g. [BGG1]; denote by $P(\phi)$ a projective cover of $L(\phi)$ in $\mathcal{O}$. Let $\mathcal{O}_\phi$ be the subcategory, so called block, of $\mathcal{O}$ whose objects have composition factors isomorphic to $L(w \cdot \phi)$, $w \in W$. The simple objects in $\mathcal{O}_\phi$ are parametrized by the cosets W/W_ϕ, where $W_\phi := \{w \in W; w \cdot \phi = \phi\}$. Denote by W^ϕ the set of longest representatives of the elements in W/W_ϕ. Put $S_\phi = S \cap W_\phi$. We refer to (W, S, S_ϕ) as the Coxeter-system of $(\mathfrak{g}, \mathfrak{b}, \phi)$. A block in category $\mathcal{O}$ depends only on this Coxeter-system:

Proposition 2.4 ([Soe1], Theorem 11)**.** [1] *Assume we are given a Coxeter-system (W, S, S_ϕ) where ϕ is any dominant weight. Let $\mathfrak{g}' \supset \mathfrak{b}' \supset \mathfrak{h}'$ be another semi-simple Lie-algebra, a Borel subalgebra and a Cartan subalgebra and let $\phi' \in \mathfrak{h}'^*$ be any dominant weight. Then any isomorphism $(W, S, S_\phi) \tilde{\to} (W', S', S'_{\phi'})$, $w \to w'$ induces an equivalence of categories $\mathcal{O}_\phi(\mathfrak{g}, \mathfrak{b}) \tilde{\to} \mathcal{O}_{\phi'}(\mathfrak{g}', \mathfrak{b}')$ with $L(w \cdot \phi) \to L(w' \cdot \phi')$.*

Let $\phi, \psi \in \mathfrak{h}^*$ be dominant weights. ψ defines a set of singular simple roots $B_\psi := \{\alpha \in B; \psi(H_\alpha) = -1\}$, and hence the parabolic subalgebra $\mathfrak{q}(\psi) \supseteq \mathfrak{b}$, generated by $\mathfrak{b}$ and the weight-spaces $\mathfrak{g}^{-\alpha}$ for $\alpha \in B_\psi$. Let $\mathcal{O}_\phi^\psi$ be the subcategory of $\mathcal{O}_\phi$ consisting of locally $\mathfrak{q}(\psi)$-finite objects.

Translation functors. Recall the two translation functors $T_0^\phi \colon \mathcal{O}_0 \to \mathcal{O}_\phi$ and $T_\phi^0 \colon \mathcal{O}_\phi \to \mathcal{O}_0$, translation onto and out of the wall, respectively, e.g. [Jan]. By definition $T_0^\phi = pr_\phi(E \otimes_{\mathbf{C}} \cdot)$, where E is an irreducible finite-dimensional $\mathfrak{g}$-module

[1] This proposition actually holds for non-integral weights, with W replaced by the integral Weyl group of ϕ, $\mathcal{O}_\phi$ replaced by... etc; so all results in this paper generalize to the non-integral case.

with extremal weight ϕ and pr_ϕ is projection from $\mathcal{O}$ onto the block $\mathcal{O}_\phi$; $T_\phi^0 = pr_0(F \otimes_{\mathbf{C}} \cdot\,)$, where F is an irreducible finite-dimensional $\mathfrak{g}$-module with extremal weight $-\phi$, pr_0 is projection onto $\mathcal{O}_0$. They satisfy the following properties:

Lemma 2.5. 1) *T_0^ϕ and T_ϕ^0 are exact functors adjoint to each other. There is an isomorphism of functors $id_{\mathcal{O}_\phi} \oplus \ldots \oplus id_{\mathcal{O}_\phi} \cong T_0^\phi \circ T_\phi^0$ ($\mathrm{card}(W_\phi)$ number of copies).*

2) *For each simple in $\mathcal{O}_\phi$ there is exactly one simple in $\mathcal{O}_0$ mapped to it by T_0^ϕ. More precisely, $T_0^\phi L(w \cdot 0) \cong L(w \cdot \phi)$, if $w \in W^\phi$, else $T_0^\phi L(w \cdot 0) = 0$.*

3) *T_ϕ^0 and T_0^ϕ map projectives to projectives. Moreover, $T_\phi^0 P(w \cdot \phi) \cong P(w \cdot 0)$.*

4) *T_0^ϕ and T_ϕ^0 induce functors $T_0^\phi\colon \mathcal{O}_0^\psi \to \mathcal{O}_\phi^\psi$ and $T_\phi^0\colon \mathcal{O}_\phi^\psi \to \mathcal{O}_0^\psi$. Moreover, if $T_0^\phi L(w \cdot 0) \neq 0$ (hence $\cong L(w \cdot \phi)$) and belongs to $\mathcal{O}_\phi^\psi$, then $L(w \cdot 0) \in \mathcal{O}_0^\psi$.*

Proof. 1) The exactness of T_0^ϕ and T_ϕ^0 follows from the fact that E and F are flat modules over $\mathbf{C}$ and that projecting to a block in $\mathcal{O}$ is an exact functor. Next, note that $F \cong \mathrm{Hom}_{\mathbf{C}}(E, \mathbf{C})$ with the $\mathfrak{g}$-module structure given by $(x\phi)(e) := -\phi(xe)$ for $x \in \mathfrak{g}, \phi \in \mathrm{Hom}_{\mathbf{C}}(E, \mathbf{C}), e \in E$. Then, for $M \in \mathcal{O}_0, N \in \mathcal{O}_\phi$, we have

$$\begin{aligned}\mathrm{Hom}_{\mathcal{O}}(T_0^\phi M, N) &= \mathrm{Hom}_{\mathcal{O}}(pr_\phi(E \otimes_{\mathbf{C}} M), N) = \mathrm{Hom}_{\mathcal{O}}(E \otimes_{\mathbf{C}} M, N)\\ &= \mathrm{Hom}_{\mathcal{O}}(M, F \otimes_{\mathbf{C}} N) = \mathrm{Hom}_{\mathcal{O}}(M, pr_0(F \otimes_{\mathbf{C}} N)) = \mathrm{Hom}_{\mathcal{O}}(M, T_\phi^0 N).\end{aligned}$$

This means that T_0^ϕ is left adjoint to T_ϕ^0. A similar argument shows that T_0^ϕ is also right adjoint to T_ϕ^0.

The last statement follows from [Jan], 4.13 (2), and [BG], Theorem 3.3

2) is proved in [Jan], 4.12 (3).

3) Let $P \in \mathcal{O}_\phi$ be projective. We have $\mathrm{Hom}_{\mathfrak{g}}(T_\phi^0 P, \ ) = \mathrm{Hom}_{\mathfrak{g}}(P, T_0^\phi(\ ))$. Since P is projective and T_0^ϕ is exact, we conclude that this functor is exact; hence $T_\phi^0 P$ is projective. Analogously, T_0^ϕ maps projectives to projectives. Next, from 2) we get

$$\mathrm{Hom}_{\mathfrak{g}}(T_\phi^0 P(w \cdot \phi), L(x \cdot \phi)) = \mathrm{Hom}_{\mathfrak{g}}(P(w \cdot \phi), T_0^\phi L(x \cdot \phi)) \cong \delta_{w,x}\mathbf{C}$$

for each $x \in W$. Since we know that $T_\phi^0 P(w \cdot \phi)$ is projective, this formula implies that $T_\phi^0 P(w \cdot \phi) \cong P(w \cdot 0)$.

For 4) we note that tensor product with a finite-dimensional module clearly preserves locally finiteness over $q(\psi)$; hence we get the asserted induced functors.

Assume $0 \neq T_0^\phi L(w \cdot 0)$. Then

$$\mathrm{Hom}_{\mathcal{O}}(L(w \cdot 0), T_\phi^0 T_0^\phi L(w \cdot 0)) = \mathrm{Hom}_{\mathcal{O}}(T_0^\phi L(w \cdot 0), T_0^\phi L(w \cdot 0)) \neq 0,$$

which implies that $L(w \cdot 0)$ is isomorphic to a submodule of $T_\phi^0 T_0^\phi L(w \cdot 0)$. Now, if $T_0^\phi L(w \cdot 0)$ is locally finite over $\mathfrak{q}(\psi)$, then $T_\phi^0 T_0^\phi L(w \cdot 0)$, and hence also $L(w \cdot 0)$, have this property. □

Parabolic inclusion and truncation. Let $i := i_\phi^\psi\colon \mathcal{O}_\phi^\psi \to \mathcal{O}_\phi$ be the inclusion. We shall refer to this exact functor i as the parabolic inclusion functor. We define the parabolic truncation functor $\tau := \tau_\phi^\psi\colon \mathcal{O}_\phi \to \mathcal{O}_\phi^\psi$ by $\tau(M) :=$ "the maximal quotient of M which is locally finite over $\mathfrak{q}(\psi)$". Then τ is right exact and left adjoint to i and it follows that τ maps projectives to projectives. Since a projective object is indecomposable if and only if it has a unique simple quotient, we see that τ maps indecomposable projectives to indecomposable projectives. (τ doesn't map arbitrary indecomposables to indecomposables; a counterexample can be given for $\mathfrak{sp}(4, \mathbf{C})$.) It follows readily that τ maps a minimal projective generator to a minimal projective generator. It is also easy to verify that if $P, Q \in \mathcal{O}_\phi$ are projective, then $\tau\colon \operatorname{Hom}_{\mathcal{O}_\phi}(P, Q) \to \operatorname{Hom}_{\mathcal{O}_\phi^\psi}(\tau P, \tau Q)$ is surjective.

Lemma 2.6. **a**) *i and τ commutes with translation functors.*
b) *The functor $D^b(\mathcal{O}_\phi^\psi) \to D^b(\mathcal{O}_\phi)$ induced by i is faithful.*

Proof. **a**) Clearly i commutes with translations functor: we have for each $M \in \mathcal{O}_\phi$ and $N \in \mathcal{O}_\phi^\psi$ that

$$\begin{aligned}\operatorname{Hom}_{\mathcal{O}_\phi^\psi}(\tau T_\phi^0 M, N) &= \operatorname{Hom}_{\mathcal{O}_\phi}(T_\phi^0 M, iN)\\ &= \operatorname{Hom}_{\mathcal{O}_\phi}(M, T_0^\phi iN) = \operatorname{Hom}_{\mathcal{O}_\phi}(M, iT_0^\phi N)\\ &= \operatorname{Hom}_{\mathcal{O}_\phi^\psi}(\tau M, T_0^\phi N) = \operatorname{Hom}_{\mathcal{O}_\phi^\psi}(T_\phi^0 \tau M, N)\end{aligned}$$

so that $\tau T_\phi^0 = T_\phi^0 \tau$. Analogously $\tau T_0^\phi = T_0^\phi \tau$.

b) The case when $\phi = 0$ is proved by geometry in [BGS], Theorem 3.5.3. We have a commutative diagram of bounded derived categories:

$$\begin{array}{ccc} D^b(\mathcal{O}_0^\psi) & \xrightarrow{i_0^\psi} & D^b(\mathcal{O}_0) \\ \uparrow{\scriptstyle T_\phi^0} & & \uparrow{\scriptstyle T_\phi^0} \\ D^b(\mathcal{O}_\phi^\psi) & \xrightarrow{i_\phi^\psi} & D^b(\mathcal{O}_\phi) \end{array}$$

Since $T_0^\phi \circ T_\phi^0$ is isomorphic to $\operatorname{card}(W_\phi)$ copies of the identity functor we see that T_ϕ^0 is faithful on derived categories and we conclude that also i_ϕ^ψ is faithful. $\square$

Endomorphism rings, ext-algebras and canonicity. The fact that projective covers are unique only up to non-unique isomorphism makes our constructions non-canonical and causes delicate problems with commutative diagrams. We can avoid these problems to some extent by choosing projective covers and maps between them commuting with various functors. Fix dominant weights ϕ and ψ. Fix projective covers $P(w\cdot\phi)$ of $L(w\cdot\phi)$ in $\mathcal{O}_\phi$ and put $P_\phi := \sum_{w \in W/W_\phi} P(w\cdot\phi)$. Analogously, fix $P(w \cdot \psi)$ and P_ψ and fix $P(w \cdot 0)$ and P_0. According to Lemma 2.5 3) we fix for $w \in W^\phi$ isomorphisms $T_\phi^0 P(w\cdot\phi) \cong P(w\cdot 0)$. We get the (now) canonical isomorphism $P_0 \cong T_\phi^0 P_\phi \oplus (T_\phi^0 P_\phi)^\perp$ where $(T_\phi^0 P_\phi)^\perp = \bigoplus_{w \in W \setminus W^\phi} P(w \cdot 0)$.

It follows from the results in the previous paragraph that $P^\psi(w\cdot 0) := \tau^\psi P(w\cdot 0)$ is a projective cover of $L(w\cdot 0)$ in $\mathcal{O}_0^\psi$, if $L(w\cdot 0) \in \mathcal{O}_0^\psi$ and $\tau^\psi P(w\cdot 0) = 0$ else. Put $P_0^\psi = \tau^\psi P_0$ which is a (minimal) projective generator of $\mathcal{O}_0^\psi$. Define analogously $P_\phi^\psi(w \cdot \phi) := \tau_\phi^\psi P(w \cdot \phi)$ and $P_\phi^\psi := \tau_\phi^\psi P_\phi$. We get the canonical isomorphism $P_0^\psi \cong T_\phi^0 P_\phi^\psi \oplus (T_\phi^0 P_\phi^\psi)^\perp$ where $(T_\phi^0 P_\phi^\psi)^\perp = \tau_\phi^\psi (T_\phi^0 P_\phi)^\perp$. Put $L_0 = \bigoplus_{w \in W} L(w \cdot 0)$ etc.

Definition 2.7. Let $R_\phi^\psi := \operatorname{End}_{\mathfrak{g}}(P_\phi^\psi)$ and write for simplicity $R_\phi := R_\phi^0$. Next, consider the ring $\operatorname{Ext}_{\mathcal{O}_\psi^\phi}(L_\psi^\phi, L_\psi^\phi)$. This is a canonically graded ring with degree i

part $\mathrm{Ext}^i_{\mathcal{O}^\phi_\psi}(L^\phi_\psi, L^\phi_\psi)$ and multiplication given by the cup-product. In [BGS], Theorem 1.1.3, it is proved that $\mathrm{Ext}_{\mathcal{O}^\phi_0}(L^\phi_0, L^\phi_0)$ is Koszul and they also prove the existence of an isomorphism between $\mathrm{Ext}_{\mathcal{O}^\phi_0}(L^\phi_0, L^\phi_0)$ and R_ϕ. Under this isomorphism is $\mathrm{Hom}_{\mathcal{O}^\phi_0}(L^\phi_0, L^\phi_0)$ mapped onto the subring of R_ϕ spanned by projections onto indecomposable projectives. We fix such an isomorphism. This defines a grading on R_ϕ making it a Koszul ring. The parabolic truncation functor gives in Proposition 3.2 a graded surjection $R_\phi \twoheadrightarrow R^\psi_\phi$ defining a grading $(R^\psi_\phi)_i$ on R^ψ_ϕ. With this grading we define a (canonically graded) algebra.

Definition 2.8. $(R^\psi_\phi)^! := \mathrm{Ext}_{R^\psi_\phi}((R^\psi_\phi)_0, (R^\psi_\phi)_0)$.

Since each object in $\mathcal{O}$ has finite length, it is known and easy to verify that the functor $\mathrm{Hom}_{\mathfrak{g}}(P^\psi_\phi, \cdot)$ induces an equivalence of categories between $\mathcal{O}^\psi_\phi$ and the category of finitely generated right R^ψ_ϕ-modules. Now clearly L^ψ_ϕ is mapped to the right R^ψ_ϕ-module $(R^\psi_\phi)_0 \cong R^\psi_\phi/(R^\psi_\phi)_+$ under this equivalence; this gives the ring isomorphism

$$(R^\psi_\phi)^! = \mathrm{Ext}_{R^\psi_\phi}((R^\psi_\phi)_0,\, (R^\psi_\phi)_0) \cong \mathrm{Ext}_{\mathcal{O}^\psi_\phi}(L^\psi_\phi, L^\psi_\phi). \tag{2.1}$$

We have the surjection

$$\tau^\psi_\phi \colon R_\phi \twoheadrightarrow R^\psi_\phi. \tag{2.2}$$

The composition $\mathrm{End}_{\mathcal{O}^\psi_\phi}(P^\psi_\phi) \to \mathrm{End}_{\mathcal{O}^\psi_0}(T^0_\phi P^\psi_\phi) \to \mathrm{End}_{\mathcal{O}^\psi}(T^0_\phi P^\psi_\phi \oplus T^0_\phi P^\psi_\phi)^\perp) = \mathrm{End}_{\mathcal{O}^\psi_0}(P^\psi_0)$ gives the injection

$$T^0_\phi \colon R^\psi_\phi \hookrightarrow R^\psi_0. \tag{2.3}$$

Similary, we get a map (which by Lemma 2.6 is injective)

$$i^\psi_\phi \colon (R^\psi_\phi)^! \hookrightarrow (R_\phi)^!. \tag{2.4}$$

By Lemma 2.5 $T^\phi_0 L^\psi_0 \cong L^\psi_\phi$. Fix such an isomorphism and fix an analogous isomorphism when ψ is replaced by 0 such that these two isomorphisms commute with parabolic inclusion. We get from (2.1) the ring homomorphism

$$T^\phi_0 \colon (R^\psi_0)^! \to (R^\psi_\phi)^! \tag{2.5}$$

which will be shown to be a surjection in Lemma 3.4 below.

The easiest example. Let $\mathfrak{g} = \mathfrak{sl}(2, C)$, X_-, X_+ and H the standard Chevalley generators. $\mathcal{O}_0$ has up to isomorphism two simple objects: $L(0) = \mathbf{C}$ and $L(-2) = U(\mathfrak{g})/(U(\mathfrak{g})(H+2) + U(\mathfrak{g})X_+)$. Then $P(0) := U(\mathfrak{g})/(U(\mathfrak{g})H + U(\mathfrak{g})X_+)$ and $P(-2) := U(\mathfrak{g})/(U(\mathfrak{g})(H+2) + U(\mathfrak{g})X_+^2)$ is a projective cover of $L(0)$ and $L(-2\rho)$ in $\mathcal{O}_0$, respectively. Put $P := P(0) \oplus P(-2)$. This is a projective generator of $\mathcal{O}_0$. Put $R = \mathrm{End}_{\mathfrak{g}}(P)$. Then R is a five-dimensional algebra over $\mathbf{C}$. Denote by $\bar{1}$ the image of $1 \in U(\mathfrak{g})$ in $P(0)$ and in $P(-2)$, respectively. A vector space basis of R is then given by: $\pi_i := id_{P(i)}$ for $i = 0, -2$; $\alpha \colon P(0) \to P(-2) : \bar{1}) \to X_+\bar{1}$; $\beta \colon P(-2) \to P(0) : \bar{1} \to X_-\bar{1}$ and $\gamma \colon P(0) \to P(0) : \bar{1} \to X_-X_+\bar{1}$.

This makes $R = R[0] \oplus R[1] \oplus R[2]$ a graded algebra with $R[0] := \{\pi_0, \pi_{-2}\}$, $R[1] = \{\alpha, \beta\}$ and $R[2] := \{\gamma\}$. This defines also an R-module structure on $R[0]$

by letting $R[1] \oplus R[2]$ act trivially. The Koszul dual of the graded ring R is by definition $R^! := \mathrm{Ext}_R(R[0], R[0])$.

Put $L := L(0) \oplus L(-2)$ and $E := \mathrm{Ext}_{\mathcal{O}}(L, L)$. Let $E[i] := \mathrm{Ext}^i_{\mathcal{O}}(L, L)$, then $E = E[0] \oplus E[1] \oplus E[2]$ is a graded algebra with multiplication given by the cup product.

Now, $E[0]$ has basis j_0, j_{-2} corresponding to the identity on $L(0), L(-2)$. $E[1]$ has basis corresponding to the extension $a\colon L(-2) \hookrightarrow M(0) \twoheadrightarrow L(0)$, where $M(0) = U(\mathfrak{g})/(U(\mathfrak{g})H + U(\mathfrak{g})X_+)$ is the Verma module of highest weight zero, and $b\colon L(0) \hookrightarrow M(0)^* \twoheadrightarrow L(-2)$, where $M(0)^*$ is the dual of $M(0)$ ([BGG1]) $E[2]$ has basis $c := ba \in \mathrm{Ext}^2_{\mathcal{O}}(L(0), L(0))$.

As we have seen before $E \cong R^!$. The Koszul duality theorem in our case then states that $R \cong E$. Under this isomorphism, $\pi_0 \to j_0, \pi_{-2} \to j_{-2}, \alpha \to a, \beta \to b$ and $\gamma \to c$.

3. Parabolic and singular-singular and parabolic Koszul duality

We have our fixed dominant weights ϕ and ψ. As mentioned before the set of isomorphism classes of simple objects in $\mathcal{O}_\phi$, which we denote by $Irr\mathcal{O}_\phi$, is naturally parametrized by $W^\phi \ni w \leftrightarrow L(w \cdot \phi) \in Irr\mathcal{O}_\phi$. Note that $w \cdot 0(H_\alpha) \geq 0,\ \forall \alpha \in B_\phi \iff w \in (W^\phi)^{-1} w_0$; this means precisely that $L(w \cdot 0)$ is locally finite over $\mathfrak{q}(\phi)$ if and only if $w \in (W^\phi)^{-1} w_0$. This gives the bijection $W^\phi \ni w \leftrightarrow L(w^{-1} w_0 \cdot 0) \in Irr\mathcal{O}_0^\phi$.

The projection $P_\phi \twoheadrightarrow P(w \cdot \phi)$ defines an idempotent $e_w^\phi \in (R_\phi)_0,\ w \in W^\phi$. Similary, the projection $L_0^\phi \twoheadrightarrow L(w \cdot 0)$ defines an idempotent $f_w^\phi \in ((R_0^\phi)^!)_0 = \mathrm{Hom}_{\mathcal{O}_\phi}(L_0^\phi, L_0^\phi)$, for $w \in (W^\phi)^{-1} w_0$.

The following theorem was proved in [BGS], Theorem 1.1.3. (Their proof is partly reconstructed in our proof of Lemma 3.5.)

Proposition 3.1. *The rings R_ϕ and R_0^ϕ are Koszul and there is an isomorphism $R_\phi \tilde{\to} (R_0^\phi)^!$ such that $e_w^\phi \to f_{w^{-1}w_0}^\phi$, $w \in W^\phi$.*

Clearly, the $f_{w^{-1}w_0}^\phi$, $w \in W^\phi$ forms a basis of $((R_0^\phi)^!)_0$ over $\mathbf{C}$. Hence the e_w^ϕ, $w \in W^\phi$ forms a basis of $(R_\phi)_0$, by the choice of grading on R_ϕ coming from the grading of $(R_0^\phi)^!$ and the above isomorphism.

Proposition 3.2. *The ring R_ϕ^ψ is Koszul.*

Proof. Put $\Omega := \{w \in W;\ \exists \alpha \in B_\psi :\ w \cdot \phi(H_\alpha) < 0\}$ and let $I \subset R_\phi$ be the ideal generated by the e_w^ϕ for $w \in \Omega$. We have (where mod-R_ϕ^ψ means finitely generated right R_ϕ^ψ-modules)

$$\begin{aligned} \text{mod-}R_\phi^\psi \cong \mathcal{O}_\phi^\psi &= \{M \in \mathcal{O}_\phi : [M : L(w \cdot \phi)] = 0,\ \forall w \in \Omega\} \\ &= \{M \in \mathcal{O}_\phi : \mathrm{Hom}_{\mathfrak{g}}(P(w \cdot \phi), M) = 0, \forall w \in \Omega\} \cong \text{mod-}R_\phi/I. \end{aligned}$$

Since R_ϕ^ψ and R_ϕ/I are basic algebras, we now conclude that they are isomorphic.

By Proposition 3.1 R_ϕ is Koszul and we conclude that R_ϕ^ψ is a graded quotient of R_ϕ with respect to the Koszul grading of R_ϕ. To show that R_ϕ^ψ is Koszul we just have to prove (by Proposition 2.2) that $\mathrm{ext}^i_{R_\phi^\psi}((R_\phi^\psi)_0, (R_\phi^\psi)_0\langle j \rangle) = 0$ when $i \neq j$. We have the morphism $D^b(\text{mod-}R_\phi^\psi) \to D^b(\text{mod-}R_\phi)$ corresponding to

$i^{\psi}_{\phi}\colon D^b(\mathcal{O}^{\psi}_{\phi}) \to D^b(\mathcal{O}_{\phi})$, which is faithful by Lemma 2.6. Since this morphism is also induced by a *graded* ring homomorphism $R_{\phi} \to R^{\psi}_{\phi}$, we conclude that it gives injections $\mathrm{ext}^i_{R^{\psi}_{\phi}}((R^{\psi}_{\phi})_0, (R^{\psi}_{\phi})_0\langle j\rangle) \hookrightarrow \mathrm{ext}^i_{R_{\phi}}((R_{\phi})_0, (R_{\phi})_0\langle j\rangle)$. The Koszulity of R_{ϕ} ensures that the latter expression vanishes for $i \neq j$. □

The modular representation theoretic analogy of the following key lemma was proved in [AJS].

Lemma 3.3. *Let L and L' be two simples in $\mathcal{O}_0$. Then T^{ϕ}_0 induces a surjection $\mathrm{Ext}^1_{\mathcal{O}_0}(L, L') \to \mathrm{Ext}^1_{\mathcal{O}_{\phi}}(T^{\phi}_0 L, T^{\phi}_0 L')$.*

Proof. (**a**) Put $\nu = T^0_{\phi} T^{\phi}_0$. Clearly the assertion of the lemma is equivalent to showing that the homomorphism $\mathrm{Ext}^1_{\mathcal{O}_0}(L, L') \to \mathrm{Ext}^1_{\mathcal{O}_{\phi}}(L, \nu L')$ given by the canonical adjunction $id \to \nu$ is surjective. We assume without restriction that $T^{\phi}_0 L' \neq 0$; then the morphism $L' \to \nu L'$ gives an embedding which identifies L' with $soc(\nu L')$ as follows from the formula (for any simple $L'' \in \mathcal{O}_0$)

$$\mathrm{Hom}_{\mathfrak{g}}(L'', \nu L') = \mathrm{Hom}_{\mathfrak{g}}(T^{\phi}_0 L'', T^{\phi}_0 L') = \delta_{L'',L'} \cdot \mathbf{C}.$$

We want to define a grading on ν and for this purpose we must pass to derived categories and modules over the coinvariant algebra. The discussion below should rather have been presented as a part of a comprehensive exposition of the theory.

(**b**) Put $S = S(\mathfrak{h}^*)$; we define a grading on S by putting $\mathfrak{h}^*$ in degree 1. Its graded quotient $C := S/S_+ \cdot S^W$ is called the coinvariant algebra. Let $C^{\phi} := C^{W_{\phi}} \subset C$ be the graded subalgebra of W_{ϕ} invariants. In [Soe1], Endomorphismensatz 7, an isomorphism $C^{\phi} \tilde{\to} \mathrm{End}_{\mathcal{O}_{\phi}}(P(w_0 \cdot \phi))$ of rings is constructed.

Let $Proj(\mathcal{O}_{\phi})$ be the category of projective objects in $\mathcal{O}_{\phi}$. Define the functor $\mathbb{V}_{\phi}\colon Q \to \mathrm{Hom}_{\mathcal{O}_{\phi}}(P(w_0 \cdot \phi), Q)$ from $Proj(\mathcal{O}_{\phi})$ to mod-C^{ϕ} := the category of right modules over C^{ϕ} (by means of the above ring isomorphism). It is proved in [Soe1], Struktursatz 9, that $\mathbb{V}_{\phi}$ is fully faithful so that $\mathbb{V}_{\phi}$ induces an equivalence of categories between $Proj(\mathcal{O}_{\phi})$ and its image category $Im(\mathbb{V}_{\phi})$.

(**c**) Given a category $\mathcal{A}$ of some modules over a graded ring R, we define its graded version $\widetilde{\mathcal{A}}$ to be the category of those graded modules over R whose underlying R-modules belong to $\mathcal{A}$. There is the grading-forgetting functor $for\colon \widetilde{\mathcal{A}} \to \mathcal{A}$. Assume in addition we are given the data $(\widetilde{\mathcal{B}}, \mathcal{B}, for)$. We say that a functor $\widetilde{F}\colon \widetilde{\mathcal{A}} \to \widetilde{\mathcal{B}}$ is a grading of the functor $F\colon \mathcal{A} \to \mathcal{B}$ if $for \circ \widetilde{F} = F \circ for$.

Consider the grading of R_{ϕ} given by Proposition 3.1 and identify $\mathcal{O}_{\phi}$ with the category of finitely generated right modules over R_{ϕ}; we get graded categories $\widetilde{\mathcal{O}_{\phi}}$ and $\widetilde{Proj(\mathcal{O}_{\phi})} \subset \widetilde{\mathcal{O}_{\phi}}$ and also $\widetilde{\text{mod}}\text{-}C^{\phi}$ and $\widetilde{Im(\mathbb{V}_{\phi})} \subset \widetilde{\text{mod}}\text{-}C^{\phi}$.

The eqivalence $\mathbb{V}_{\phi}\colon Proj(\mathcal{O}_{\phi}) \tilde{\to} Im(\mathbb{V}_{\phi})$ induces (by definition!) an equivalence $\widetilde{Proj(\mathcal{O}_{\phi})} \tilde{\to} \widetilde{Im(\mathbb{V}_{\phi})}$, which in turn induces an equivalence between homotopy categories $K^b(\widetilde{Proj(\mathcal{O}_{\phi})}) \tilde{\to} K^b(\widetilde{Im(\mathbb{V}_{\phi})})$. Remember that $K^b(\widetilde{Proj(\mathcal{O}_{\phi})})$ is canonically isomorphic to the bounded derived category $D^b(\widetilde{\mathcal{O}_{\phi}})$; we get the equivalence

$$\mathbb{V}_{\phi}\colon D^b(\widetilde{\mathcal{O}_{\phi}}) \tilde{\to} K^b(\widetilde{Im(\mathbb{V}_{\phi})}). \tag{3.1}$$

(**d**) There are the standard functors

$$res_C^{C^\phi} : \text{mod-}C \ni M \to M|C^\phi \in \text{mod-}C^\phi,$$
$$ind_{C^\phi}^{C} : \text{mod-}C^\phi \ni M \to C \otimes_{C^\phi} M \in \text{mod-}C.$$

The fact that C^ϕ is a graded subalgebra enables us to define $\widetilde{res_C^{C^\phi}}$ and $\widetilde{ind_{C^\phi}^{C}}$ in such a way that they become adjoint to each other and $\widetilde{res_C^{C^\phi}}$ preserves pure objects of any weight.

It is proved in [Soe1], Theorem 10, that $\mathbb{V}_\phi \circ T_0^\phi = res_C^{C^\phi} \circ \mathbb{V}_0 \colon Proj(\mathcal{O}_0) \to Im(\mathbb{V}_\phi)$ and that $\mathbb{V}_0 \circ T_\phi^0 = ind_{C^\phi}^{C} \circ \mathbb{V}_\phi : \mathcal{O}_\phi \to Im(\mathbb{V}_0)$.

Thus formula (3.1) gives us (abusing some notation) functors $\widetilde{T}_\phi^0 \colon D^b(\widetilde{\mathcal{O}}_\phi) \to D^b(\widetilde{\mathcal{O}}_0)$ and $\widetilde{T}_0^\phi \colon D^b(\widetilde{\mathcal{O}}_0) \to D^b(\widetilde{\mathcal{O}}_\phi)$.

However, since T_0^ϕ is clearly t exact (with respect to the standard t-structures on $D^b(\mathcal{O}_0)$ and $D^b(\mathcal{O}_\phi)$), it follows that also $\widetilde{T}_0^\phi$ is t exact. This gives us the graded translation functor $\widetilde{T}_\phi^0 \colon \widetilde{\mathcal{O}}_\phi \to \widetilde{\mathcal{O}}_0$ and analogously we get $\widetilde{T}_0^\phi \colon \widetilde{\mathcal{O}}_0 \to \widetilde{\mathcal{O}}_\phi$. These latter two functors are adjoint and $\widetilde{T}_0^\phi$ preserves pure objects of any weight. We also get the graded functor $\widetilde{\nu} := \widetilde{T}_\phi^0 \circ \widetilde{T}_0^\phi$ on $\widetilde{\mathcal{O}}_0$.

(**e**) A simple module in $\mathcal{O}_0$ can be given the (unique up to a shift) grading of being pure. Denote by $\widetilde{L}$ and $\widetilde{L}'$ the simple objects in $\widetilde{\mathcal{O}}_0$ which are pure of weight 0 satisfying $for(\widetilde{L}) = L$ and $for(\widetilde{L}') = L'$. Since $\widetilde{T}_\phi^0$ and $\widetilde{T}_0^\phi$ are adjoint and $\widetilde{T}_0^\phi$ preserves pure objects of weight 0, we see that the image of $\widetilde{L}' \to \widetilde{\nu}\widetilde{L}'$ lives in degree 0. Note also that it follows from (**a**) that the morphism $id \to \widetilde{\nu}$ identifies $\widetilde{L}'$ with $soc(\widetilde{\nu}\widetilde{L}')$.

Let i be maximal such that $(\widetilde{\nu}\widetilde{L}')_i \neq 0$. Then $(\widetilde{\nu}\widetilde{L}')_i$ is semi-simple (since it is annihilated by the radical of R_0 when we identify the category $\mathcal{O}_0$ with some modules over R_0), hence $(\widetilde{\nu}\widetilde{L}')_i \subseteq soc(\widetilde{\nu}\widetilde{L}')$, hence $i = 0$.

Since $soc(\widetilde{\nu}\widetilde{L}') = \widetilde{L}'$ is simple and $(\widetilde{\nu}\widetilde{L}')_0 \subseteq soc(\widetilde{\nu}\widetilde{L}')$ is non-zero we now get $(\widetilde{\nu}\widetilde{L}')_0 = \widetilde{L}'$. We conclude that $\widetilde{\nu}\widetilde{L}'/\widetilde{L}'$ lives only in degrees ≤ -1.

(**f**) Consider the short exact sequence

$$0 \to L' \to \nu L' \to \nu L'/L' \to 0.$$

It induces the exact sequence

$$\mathrm{Ext}^1_{\mathcal{O}_0}(L, L') \to \mathrm{Ext}^1_{\mathcal{O}_0}(L, \nu L') \to \mathrm{Ext}^1_{\mathcal{O}_0}(L, \nu L'/L'). \tag{3.2}$$

Now let $\mathrm{Ext}^1_{\widetilde{\mathcal{O}_0}}(\ ,\ )$ denote the first left derived functor of the (grading-preserving) hom-functor in the graded category $\widetilde{\mathcal{O}_0}$; we get

$$\mathrm{Ext}^1_{\mathcal{O}_0}(for(M), for(N)) = \prod_{i \in \mathbb{Z}} \mathrm{Ext}^1_{\widetilde{\mathcal{O}_0}}(M, N\langle i \rangle)$$

for each $M, N \in \widetilde{\mathcal{O}_0}$. Thus (3.2) is given by exact sequences

$$\mathrm{Ext}^1_{\widetilde{\mathcal{O}_0}}(\widetilde{L}, \widetilde{L}'\langle i \rangle) \to \mathrm{Ext}^1_{\widetilde{\mathcal{O}_0}}(\widetilde{L}, \widetilde{\nu}\widetilde{L}'\langle i \rangle) \to \mathrm{Ext}^1_{\widetilde{\mathcal{O}_0}}(\widetilde{L}, \widetilde{\nu}\widetilde{L}'/\widetilde{L}'\langle i \rangle)$$

and we have to prove that the first map is surjective for every i.

We have $\mathrm{Ext}^1_{\widetilde{\mathcal{O}_0}}(\widetilde{L}, \widetilde{\nu}\widetilde{L}'\langle i \rangle) = \mathrm{Ext}^1_{\widetilde{\mathcal{O}_\phi}}(\widetilde{T}_0^\phi L, \widetilde{T}_0^\phi \widetilde{L}'\langle i \rangle)$ and $\widetilde{T}_0^\phi \widetilde{L}'$ is pure of weight 0. Koszulity of $\mathcal{O}_\phi$ implies that $\widetilde{T}_0^\phi \widetilde{L}$ admits a graded projective resolution whose jth component is generated in degree j. It follows easily that $\mathrm{Ext}^1_{\widetilde{\mathcal{O}_0}}(\widetilde{L}, \widetilde{\nu}\widetilde{L}'\langle i \rangle) = 0$

if $i \neq 1$. Hence it suffices to show that $\operatorname{Ext}^1_{\widetilde{\mathcal{O}_0}}(\widetilde{L}, \widetilde{L}'\langle 1\rangle) \to \operatorname{Ext}^1_{\widetilde{\mathcal{O}_0}}(\widetilde{L}, \widetilde{\nu}\widetilde{L}'\langle 1\rangle)$ is surjective.

This is true, because by Koszulity again $\widetilde{L}$ admits a graded projective resolution whose jth component is generated in degree j and since $\widetilde{\nu}\widetilde{L}'/\widetilde{L}'$ lives in degrees < 0, we get $\operatorname{Ext}^1_{\widetilde{\mathcal{O}_0}}(\widetilde{L}, \widetilde{\nu}\widetilde{L}'/\widetilde{L}'\langle i\rangle) = 0$ unless $i > 1$. In particular this group vanishes for $i = 1$. ☐

Lemma 3.4. *T_0^ϕ induces a surjective algebra homomorphism $(R_0^\psi)^! \twoheadrightarrow (R_\phi^\psi)^!$.*

Proof. Recall from (2.1) that this means that the homomorphism $\operatorname{Ext}_{\mathcal{O}_0^\psi}(L_0^\psi, L_0^\psi) \to \operatorname{Ext}_{\mathcal{O}_\phi^\psi}(L_\phi^\psi, L_\phi^\psi)$ induced from T_0^ϕ (and our fixed isomorphism $T_0^\phi L_0^\psi \cong L_\phi^\psi$) is surjective. Surjectivity in degree 0 is obvious.

The category $\mathcal{O}_0^\psi$ (resp., $\mathcal{O}_\phi^\psi$) is closed under extensions in $\mathcal{O}_0$ (resp., in $\mathcal{O}_\phi$) so if we interpret Lemma 3.3 as surjections between Yoneda extensions, the surjectivity in degree 1 is clear.

Since the second algebra in the statement of the lemma is Koszul, it is generated in degree 1 over its degree 0 part and we are done. ☐

Proposition 3.5. *One can choose isomorphisms $R_0 \tilde{\to} (R_0)^!$ and $R_\phi \tilde{\to} (R_0^\phi)^!$ satisfying the properties of Proposition 3.1 which make the diagram below commute:*

$$\begin{array}{ccc} R_0 & \xrightarrow{\sim} & (R_0)^! \\ \uparrow{\scriptstyle T_\phi^0} & & \uparrow{\scriptstyle i} \\ R_\phi & \xrightarrow{\sim} & (R_0^\phi)^! \end{array}$$

Proof. (**a**) We keep the notations of the proof of Lemma 3.3. We get from the discussion in (**d**) in the proof of Lemma 3.3 the commutative diagram

$$\begin{array}{ccc} R_0 & \xrightarrow{\mathbb{V}} & \operatorname{End}_C(\mathbb{V}P_0) \\ \uparrow{\scriptstyle T_\phi^0} & & \uparrow{\scriptstyle ind_{C^\phi}^C} \\ R_\phi & \xrightarrow{\mathbb{V}_\phi} & \operatorname{End}_{C^\phi}(\mathbb{V}_\phi P_\phi) \end{array}$$

(**b**) Let X be a smooth complex variety equipped with an algebraic B-action, such that the B-orbits form a stratification of X, consisting of finitely many linear affine spaces. We denote by $\mathcal{D}(X)$ the bounded derived category of $\mathbf{C}$-constructible sheaves on X, whose cohomologies are constant on B-orbits. Let $\mathcal{P}(X) \subset \mathcal{D}(X)$ be the subcategory of perverse sheaves. It follows from the Riemann-Hilbert correspondence that $\mathcal{D}(X)$ is canonically isomorphic to $D^b(\mathcal{P}(X))$, (e.g. [BGS], Corollary 3.3.2). For $\mathcal{F} \in \mathcal{D}(X)$, the hypercohomology complex $\mathbb{H}(\mathcal{F}) := R\Gamma(X; \mathcal{F})$ is an object in $D^b(\mathbf{C})$, the derived category of vector spaces. Let $\mathbf{C}_X$ denote the constant sheaf on X. The canonical isomorphism $\mathcal{F} \cong \mathbf{C}_X \otimes \mathcal{F}$ gives us a ring-homomorphism $\operatorname{End}^\bullet_{\mathcal{D}(X)}(\mathbf{C}_X) \to \operatorname{End}^\bullet_{D^b(\mathbf{C})}(\mathbb{H}\mathcal{F})$. This defines an action of the cohomology ring $H^\bullet(X) = \operatorname{End}^\bullet_{\mathcal{D}(X)}(\mathbf{C}_X)$ on $\mathbb{H}\mathcal{F}$; hence, hypercohomology gives a functor $\mathbb{H}\colon \mathcal{D}(X) \to H^\bullet(X)$-Mod.

A (minimal) simple generator of the category $\mathcal{P}(X)$ is given by $\bigoplus \mathrm{IC}^\bullet(\overline{Y})$ where Y runs over the set of B-orbits on X and $\mathrm{IC}^\bullet(\overline{Y})$ denotes the intersection cohomology sheaf on $\overline{Y}$ (recall that Y is simply connected).

(**c**) Let G be the group of inner automorphisms of $\mathfrak{g}$, $B \subset Q$ the subgroups such that $LieB = \mathfrak{b}$ and $LieQ = \mathfrak{q}(\phi)$. We put $\mathcal{D} := \mathcal{D}(G/B), \mathcal{D}^Q := \mathcal{D}(G/Q), \mathcal{P} := \mathcal{P}(G/B)$ and $\mathcal{P}^Q := \mathcal{P}(G/Q)$. We fix isomorphisms $C \cong H^\bullet(G/B)$ and $C^\phi \cong H^\bullet(G/Q)$ such that the inclusion $C^\phi \hookrightarrow C$ corresponds to the pullback morphism $\pi^\star\colon \mathrm{End}^\bullet_{\mathcal{D}}(\mathbf{C}_{G/Q}) \to \mathrm{End}^\bullet_{\mathcal{D}}(\mathbf{C}_{G/B})$, where $\pi\colon G/B \to G/Q$ is the projection, (cf. [BGG2]).

Let $\mathcal{L}_0(w\cdot 0) := \mathrm{IC}^\bullet(\overline{BwB/B}) \in \mathcal{P}$ and $\mathcal{L}_0 := \bigoplus_{w\in W} \mathcal{L}_0(w\cdot 0)$; analogously, put $\mathcal{L}^\phi_0(w\cdot 0) := \mathrm{IC}^\bullet(\overline{BwQ/Q}) \in \mathcal{P}^Q$ and $\mathcal{L}^\phi_0 := \bigoplus_{w\in W^\phi} \mathcal{L}^\phi_0(w\cdot 0)$.

Recall $(R_0)^! \cong \mathrm{Ext}_{\mathcal{O}_0}(L_0, L_0)$ and $(R^\phi_0)^! \cong \mathrm{Ext}_{\mathcal{O}^\phi_0}(L^\phi_0, L^\phi_0)$ canonically. We have $\mathrm{Ext}_{\mathcal{O}^\phi_0}(L^\phi_0, L^\phi_0) \cong \mathrm{Ext}_{\mathcal{P}^Q}(\mathcal{L}^\phi_0, \mathcal{L}^\phi_0)$, by [BGS], Theorem 3.5.1. Also,

$$\mathrm{End}^\bullet_{\mathcal{D}^Q}(\mathcal{L}^\phi_0, \mathcal{L}^\phi_0) \cong \mathrm{End}^\bullet_{\mathcal{D}^b(\mathcal{P}^Q)}(\mathcal{L}^\phi_0, \mathcal{L}^\phi_0) \cong \mathrm{Ext}_{\mathcal{P}^Q}(\mathcal{L}^\phi_0, \mathcal{L}^\phi_0).$$

Thus, $(R^\phi_0)^! \cong \mathrm{End}^\bullet_{\mathcal{D}^Q}(\mathcal{L}^\phi_0, \mathcal{L}^\phi_0)$. In particular, $(R_0)^! \cong \mathrm{End}^\bullet_{\mathcal{D}}(\mathcal{L}_0, \mathcal{L}_0)$. By [Soe1], Erweiterungssatz 17, the hypercohomology induces ring isomorphisms

$$\mathrm{End}^\bullet_{\mathcal{D}}(\mathcal{L}_0, \mathcal{L}_0) \to \mathrm{End}_C(\mathbb{H}\mathcal{L}_0, \mathbb{H}\mathcal{L}_0) \;\&\; \mathrm{End}^\bullet_{\mathcal{D}^Q}(\mathcal{L}^\phi_0, \mathcal{L}^\phi_0) \to \mathrm{End}_{C^\phi}(\mathbb{H}\mathcal{L}_\phi, \mathbb{H}\mathcal{L}^\phi_0).$$

Consider the diagram (where $d = \dim_{\mathbf{C}} G/B - \dim_{\mathbf{C}} G/Q$):

$$\begin{array}{ccccc}
\mathrm{End}_C(\mathbb{H}\mathcal{L}_0, \mathbb{H}\mathcal{L}_0) & \xleftarrow[\sim]{} & \mathrm{End}^\bullet_{\mathcal{D}}(\mathcal{L}_0, \mathcal{L}_0) & \xrightarrow{\sim} & (R_0)^! \\
\uparrow{\scriptstyle ind^C_{C^\phi}} & & \uparrow{\scriptstyle \pi^\star[d]} & & \uparrow{\scriptstyle i} \\
\mathrm{End}_{\mathbf{C}^\phi}(\mathbb{H}\mathcal{L}^\phi_0, \mathbb{H}\mathcal{L}_\phi) & \xleftarrow[\sim]{} & \mathrm{End}^\bullet_{\mathcal{D}^Q}(\mathcal{L}^\phi_0, \mathcal{L}^\phi_0) & \xrightarrow{\sim} & (R^\phi_0)^!
\end{array}$$

The second square commutes by [BGS], Theorem 3.5.3. The first square commutes by definition if we replace $ind^C_{C^\phi}$ by $\mathbb{H}\pi^\star[d]$. However, in [Soe1], Theorem 14, an isomorphism of functors $\mathbb{H}\pi^\star[d] \cong ind^C_{C^\phi} \circ \mathbb{H}\colon \mathcal{D}^Q \to C$-Mod is constructed, so the diagram commutes as asserted.

(**d**) Isomorphisms $\mathbb{V}_0 P_0(w\cdot 0) \cong \mathbb{H}\mathcal{L}_0(w\cdot 0)$, $w \in W$, of C-modules are constructed in [Soe1], Proposition 10 and Lemma 9.

That proof can be modified to the parabolic case (an *ad hoc* proof is also given in [BGS], Lemma 3.7.2) which gives the isomorphisms $\mathbb{V}_\phi P_\phi(w\cdot\phi) \cong \mathbb{H}\mathcal{L}^\phi_0(w\cdot 0)$, $w \in W^\phi$, of C^ϕ-modules; we fix these latter isomorphisms.

Now, according to (**d**) in the proof of Proposition 3.5 and [Soe1], Theorem 14, there are isomorphisms $ind^C_{C^\phi}\mathbb{V}_\phi P_\phi(w\cdot\phi) \cong \mathbb{V}_0 P_0(w\cdot 0)$ and $ind^C_{C^\phi}\mathbb{H}\mathcal{L}^\phi_0(w\cdot 0) \cong \mathbb{H}\mathcal{L}_0(w\cdot 0)$, $w \in W^\phi$. Fix such isomorphisms.

Then we choose those isomorphisms $\mathbb{V}_0 P_0(w\cdot 0) \cong \mathbb{H}\mathcal{L}_0(w\cdot 0)$ for $w \in W^\phi$ which are compatible with all our fixed isomorphisms above; finally fix any isomorphisms $\mathbb{V}_0 P_0(w\cdot 0) \cong \mathbb{H}\mathcal{L}_0(w\cdot 0)$ for $w \in W \setminus W^\phi$. By construction this gives us a commutative diagram:

$$\begin{array}{ccc}
\mathrm{End}_C(\mathbb{V}_0 P_0, \mathbb{V}_0 P_0) & \xrightarrow{\sim} & \mathrm{End}_C(\mathbb{H}\mathcal{L}_0, \mathbb{H}\mathcal{L}_0) \\
\uparrow{\scriptstyle ind^C_{C^\phi}} & & \uparrow{\scriptstyle ind^C_{C^\phi}} \\
\mathrm{End}_{C^\phi}(\mathbb{V}_\phi P_\phi, \mathbb{V}_\phi P_\phi) & \xrightarrow{\sim} & \mathrm{End}_{C^\phi}(\mathbb{H}\mathcal{L}_\phi, \mathbb{H}\mathcal{L}_\phi)
\end{array}$$

Arranging the three commutative diagrams that occur in this proof into one big commutative diagram the theorem is proved. ∎

Recall the idempotents $e^\psi_w \in (R_\psi)_0$ and $f^\psi_{w^{-1}w_0} \in ((R^\psi_0)^!)_0$, $w \in W^\psi$, from the beginning of this section which correspond under the isomorphism $R_\psi \tilde{\to} (R^\psi_0)^!$ given by Proposition 3.1.

Put $\bar\psi := -w_0\psi$. (*Note that there is no "$\cdot$" in this product.*) Then $\bar\psi$ is a dominant weight such that $w_0 S_{\bar\psi} w_0 = S_\psi$ so that we have an isomorphism of Coxeter-systems $(W, S, S_{\bar\psi}) \tilde{\to} (W, S, S_\psi)$, $w_0 w w_0 \to w$. It follows from Proposition 2.4 that this gives (after choosing a projective generator $P_{\bar\psi}$ and putting $R_{\bar\psi} := \mathrm{End}_{\mathfrak{g}}(P_{\bar\psi})$) a ring isomorphism $R_{\bar\psi} \tilde{\to} R_\psi$ such that $e^{\bar\psi}_{w_0 w w_0}$ maps to e^ψ_w, $w \in W^\psi$. Composing the surjection in Lemma 3.4 with the isomorphism $R_\psi \tilde{\to} (R^\psi_0)^!$ and the isomorphism $R_{\bar\psi} \tilde{\to} R_\psi$ we get a surjection

$$\pi\colon R_{\bar\psi} \twoheadrightarrow (R^\psi_\phi)^!, \text{ with } \pi(e^{\bar\psi}_{w_0 w w_0}) = T^\phi_0(f^\psi_{w^{-1}w_0}), \text{ for } w \in W^\psi. \tag{3.3}$$

Put $\Omega^\phi_{\bar\psi} := \{w \in w_0 W^\psi w_0;\ \exists \alpha \in B_\phi : w \cdot \bar\psi(H_\alpha) < 0\}$ and let $I^\phi_{\bar\psi}$ be the ideal in $R_{\bar\psi}$ generated by the $e^{\bar\psi}_w$ for $w \in \Omega^\phi_{\bar\psi}$. Then $R_{\bar\psi}/I^\phi_{\bar\psi}$ and $R^\phi_{\bar\psi}$ have — in analogy with what we saw in the proof of Proposition 3.2 — isomorphic categories of finitely generated right modules ($\cong \mathcal{O}^\phi_{\bar\psi}$) and these algebras are basic so they must be isomorphic. We have the surjection $\tau^\phi_{\bar\psi}\colon R_{\bar\psi} \twoheadrightarrow R^\phi_{\bar\psi}$ (after putting $R^\phi_{\bar\psi} := \mathrm{End}_{\mathfrak{g}}(\tau^\phi_{\bar\psi} P_{\bar\psi})$) which clearly vanishes on $I^\phi_{\bar\psi}$ so we get the induced surjection $\tau^\phi_{\bar\psi}\colon R_{\bar\psi}/I^\phi_{\bar\psi} \twoheadrightarrow R^\phi_{\bar\psi}$ which must be an isomorphism.

Lemma 3.6. *π vanishes on $I^\phi_{\bar\psi}$; thus π factors as*

$$R_{\bar\psi} \overset{\tau^\phi_{\bar\psi}}{\twoheadrightarrow} R^\phi_{\bar\psi} \twoheadrightarrow (R^\psi_\phi)^!.$$

Proof. (**a**) We know that $\pi(e^{\bar\psi}_{w_0 w w_0}) = T^\phi_0(f^\psi_{w^{-1}w_0})$, for $w \in W^\psi$. Put $\Gamma := \{w \in W^\psi;\ w^{-1}w_0(\alpha) \in R_-,\ \forall\, \alpha \in B_\phi\}$. Thus $\pi(e_{w_0 w w_0}) \neq 0 \iff w \in \Gamma$, by Lemma 2.5 3) and the fact that $w \in W^\phi \iff w(\alpha) \in R_-, \forall \alpha \in B_\phi$. Put $\Lambda := \{w \in W^\psi;\ w_0 w w_0 \cdot \bar\psi(H_\alpha) \geq 0\}$. If we can show that $\Gamma \subseteq \Lambda$, the lemma is proved. We prove $\Gamma = \Lambda$.

(**b**) $\Lambda \subseteq \Gamma$. Let $w \in \Lambda$. Then for each $\alpha \in B_\phi$ we have (where $(\ ,\ )$ denotes the Killing-form):

$$\begin{aligned}
0 &\leq w_0 w w_0 \cdot \bar\psi(H_\alpha) = -w_0 w \cdot \psi(H_\alpha) \\
&\iff 0 \leq (-w_0 w \cdot \psi\,, \alpha) = (-w_0 w(\psi + \rho) - \rho\,, \alpha) \\
&= (-w_0 w(\psi + \rho)\,, \alpha) - 1 = -(\psi + \rho,\ w^{-1}w_0(\alpha)) - 1 \\
&\iff (\psi + \rho,\ w^{-1}w_0(\alpha)) \leq -1 \implies w^{-1}w_0(\alpha) \in R_-,
\end{aligned}$$

since ψ is dominant. Hence $w \in \Gamma$.

(**c**) We conclude from Lemma 2.5 that we have the bijection $\Gamma \ni w \leftrightarrow L(w \cdot \phi) \in Irr\mathcal{O}^\psi_\phi$ and we clearly have the bijection $\Lambda \ni w \leftrightarrow L(w_0 w w_0 \cdot \bar\psi) \in Irr\mathcal{O}^\phi_{\bar\psi}$. Thus we have by (**b**) the inequality $\mathrm{card}(Irr\mathcal{O}^\phi_{\bar\psi}) \leq \mathrm{card}(Irr\mathcal{O}^\psi_\phi)$ and replacing ψ by ϕ and ϕ by $\bar\psi$ we get a reserve inequality which now gives $\Gamma = \Lambda$. ☐

Theorem 3.7. *$R^\phi_{-w_0\psi}$ is isomorphic to $(R^\psi_\phi)^!$.*

Remark 3.8. The obvious idempotents in $(R^\phi_{-w_0\psi})_0$ resp., in $((R^\psi_\phi)^!)_0$ correspond under this isomorphism.

Remark 3.9. One could probably combinatorially show that both rings in the theorem have the same dimension by constructing an inversion formula for the generalized Kazhdan-Lusztig polynomials associated to the highest weight category $\mathcal{O}_\phi^\psi$ (cf. [CPS]) which according to Lemma 3.6 would prove the theorem.

Proof. Consider the diagram (with $\bar{\psi} = -w_0\psi$)

$$\begin{array}{ccccc} R_0 & \xrightarrow{\sim} & (R_0)^! & \xrightarrow{T_0^\phi} & (R_\phi)^! \\ \uparrow{\scriptstyle T_{\bar\psi}^0} & & \uparrow{\scriptstyle i_0^\psi} & & \uparrow{\scriptstyle i_\phi^\psi} \\ R_{\bar\psi} & \xrightarrow{\sim} & (R_0^\psi)^! & \xrightarrow{T_0^\phi} & (R_\phi^\psi)^! \end{array} \tag{3.4}$$

The horizontal isomorphisms to the left can be chosen making the left square commutative. Indeed, if in the left lower corner we replace $R_{\bar\psi}$ by R_ψ and its upgoing arrow $T_{\bar\psi}^0$ by T_ψ^0, this is the statement of Proposition 3.5. The reader may then verify that $T_{\bar\psi}^0 : R_{\bar\psi} \hookrightarrow R_0$ equals the composition $R_{\bar\psi} \tilde{\to} R_\psi \overset{T_\phi^0}{\hookrightarrow} R_0$ where the first map is induced from the isomorphism $(W, S, S_{\bar\psi}) \ni w \to w_0ww_0 \in (W, S, S_\psi)$ of Coxeter-systems.[2] The right square clearly commutes and its horizontal maps are surjections by Lemma 3.4.

Applying Lemma 3.6 to the composition of the maps in the bottom row of (3.4) (resp., in the top row with $\psi = 0$) we conclude — since parabolic truncation functors commute with translation functors — that the outer rectangle of (3.4) factors through the commutative diagram

$$\begin{array}{ccc} R_0^\phi & \twoheadrightarrow & (R_\phi)^! \\ \uparrow{\scriptstyle T_{\bar\psi}^0} & & \uparrow{\scriptstyle i_\phi^\psi} \\ R_{\bar\psi}^\phi & \twoheadrightarrow & (R_\phi^\psi)^! \end{array}$$

The upper horizontal map must be an isomorphism since we know that R_0^ϕ and $(R_\phi)^!$ (being isomorphic) have the same dimensions. Since the vertical maps are injective, we conclude that the lower horizontal map is injective; hence an isomorphism.

□

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[2]Note that $T_{\bar\psi}^0 : R_{\bar\psi} \hookrightarrow R_0$ is only well-defined after choosing an isomorphism between $T_{\bar\psi}^0 P_{\bar\psi}$ and a summand of P_0 it is isomorphic to. What we really claim is that this isomorphism can be chosen such that $T_{\bar\psi}^0 : R_{\bar\psi} \hookrightarrow R_0$ equals the required composition.

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