

THE HOWE DUALITY AND THE PROJECTIVE REPRESENTATIONS OF SYMMETRIC GROUPS

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ABSTRACT. The symmetric group \mathfrak{S}_k possesses a nontrivial central extension, whose irreducible representations, different from the irreducible representations of \mathfrak{S}_k itself, coincide with the irreducible representations of the algebra \mathfrak{A}_k generated by indeterminates $\tau_{i,j}$ for $i \neq j$, $1 \leq i, j \leq n$ subject to the relations

$$\begin{aligned} \tau_{i,j} &= -\tau_{j,i}, \quad \tau_{i,j}^2 = 1, \quad \tau_{i,j}\tau_{m,l} = -\tau_{m,l}\tau_{i,j} \text{ if } \{i,j\} \cap \{m,l\} = \emptyset; \\ \tau_{i,j}\tau_{j,m}\tau_{i,j} &= \tau_{j,m}\tau_{i,j}\tau_{j,m} = -\tau_{i,m} \text{ for any } i, j, l, m. \end{aligned}$$

Recently M. Nazarov realized irreducible representations of \mathfrak{A}_k and Young symmetrizers by means of the Howe duality between the Lie superalgebra $\mathfrak{q}(n)$ and the Hecke algebra $H_k = \mathfrak{S}_k \circ Cl_k$, the semidirect product of \mathfrak{S}_k with the Clifford algebra Cl_k on k indeterminates.

Here I construct one more analog of Young symmetrizers in H_k as well as the analogs of Specht modules for \mathfrak{A}_k and H_k .

1. SUMMARY

Lately, we have witnessed an increase of interest in the study of representations of symmetric groups. In particular, in their projective representations.

Recall that the symmetric group \mathfrak{S}_k has a nontrivial central extension whose irreducible representations do not reduce to those of \mathfrak{S}_k but coincide (may be identified) with the irreducible representations of the algebra \mathfrak{A}_k determined by generators $\tau_{i,j}$ for $i \neq j$, $1 \leq i, j \leq n$ subject to the relations

$$(1.1) \quad \begin{aligned} \tau_{i,j} &= -\tau_{j,i}, \quad \tau_{i,j}^2 = 1, \quad \tau_{i,j}\tau_{m,l} = -\tau_{m,l}\tau_{i,j} \text{ if } \{i,j\} \cap \{m,l\} = \emptyset; \\ \tau_{i,j}\tau_{j,m}\tau_{i,j} &= \tau_{j,m}\tau_{i,j}\tau_{j,m} = -\tau_{i,m} \text{ for any } i, j, l, m. \end{aligned}$$

In [N1] Nazarov realized irreducible representations of \mathfrak{A}_k by means of an orthogonal basis constructed in each of the spaces of the representations and indicating the action of the generators $\tau_{i,i+1}$ on them (an analog of the Young orthogonal form). In [N2], with the help of an “odd” analog of the degenerate affine Hecke algebra Nazarov constructed elements of the algebra $H_k = \mathfrak{S}_k \circ Cl_k$, the semidirect product of \mathfrak{S}_k with the Clifford algebra Cl_k on k indeterminates. Certain elements of H_k serve as analogs of Young symmetrizers.

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Here I construct one more analog of Young symmetrizers in H_k as well as analogs of Specht modules (cf. [Ja]) for the algebras \mathfrak{A}_k and H_k . This construction is based on another form of expression of Young symmetrizers based on the notion of seminormal representation (an analog of Gelfand-Tsetlin basis) for the symmetric group (see [OV], [R], [Mu]) one can express Young symmetrizers in the form distinct from both the classical one (that of Weyl [W]) and that suggested in [Ju2]. Observe that the structure of Nazarov's symmetrizers is rather complicated; Jones simplified them [Jo1], [Jo2] but not completely. The symmetrizers constructed in this paper are simpler in structure. Our method is also applicable to the q -analogs of the Young symmetrizers of the usual and projective representations of \mathfrak{S}_k ; cf. [JN].

This paper is an expanded and rewritten (thanks to a shrewd referee) version of my short note math.RT/9810148. The reader might also be interested in the results of [Ya1] and [Ya2], slightly related with this paper; namely, [Ya1] describes \mathfrak{A}_k as a Howe dual of $\mathfrak{q}(n)$ and in [Ya2] a Howe dual of $\mathfrak{q}(m) \oplus \mathfrak{q}(n)$ is established to be the hyperoctahedral group.

1.1. For the general notion of a seminormal representation see [R]. In this section I will formulate it for \mathfrak{S}_k in convenient terms (cf. [OV]).

Let \mathfrak{S}_k be the symmetric group on k symbols, $s_{ij} \in \mathfrak{S}_k$ the transposition. The Jucys–Murphy elements are defined as follows (cf. [Ju1], [Mu]):

$$x_1 = 0, \quad x_2 = s_{12}, \quad x_3 = s_{13} + s_{23}, \quad \dots \quad x_k = s_{1k} + s_{2k} + \dots + s_{k-1,k}.$$

As in [Ma], we will identify the partition λ of a positive integer k with a Young diagram of k cells. A *Young tableau of the form* λ is the Young diagram corresponding to λ filled in with the integers 1 to k . The tableau is called a *standard* one if the numbers that fill it do not decrease along the rows left to right and along the columns downwards.

For any partition λ denote by $\mathcal{T}(\lambda)$ the set of standard tableaux of shape λ . For $\Lambda \in \mathcal{T}(\lambda)$ and $1 \leq p \leq k$, we set $c_\Lambda(p) = j - i$, where (i, j) is the slot in Λ occupied by p .

1.2. Theorem ([Mu], [OV], [R]). *Let S^λ be the irreducible representation of \mathfrak{S}_k corresponding to the partition λ . Then in S^λ there exists a common eigenbasis for the Jucys–Murphy elements x_1, \dots, x_k ; the basis can be indexed with the elements of $\mathcal{T}(\lambda)$ as follows:*

$$x_p v_\Lambda = c_\Lambda(p) v_\Lambda \quad \text{for any } \Lambda \in \mathcal{T}(\lambda) \text{ and } 1 \leq p \leq k.$$

Let us partially order the partitions with respect to *dominance*, i.e., we set

$$\mu \geq \lambda \iff \sum_{i=1}^l \mu_i \geq \sum_{i=1}^l \lambda_i \text{ for } l = 1, 2, \dots$$

Let Λ_c be a Young tableau of shape λ consecutively filled in along columns (that is where the subscript comes from) from left to right. Set

$$(1.1.1) \quad \kappa_{\Lambda_c} = \prod_{i=1}^k (j - x_i),$$

where j is the number of the column occupied by i . Since the elements x_1, \dots, x_k pairwise commute, the order of factors in (1.1.1) is irrelevant.

Example 1. For

$$\Lambda_c = \begin{matrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & \end{matrix}$$

we have

$$\kappa_{\Lambda_c} = 1 \cdot (1 - x_2)(1 - x_3)(2 - x_4)(2 - x_5)(2 - x_6)(3 - x_7)(3 - x_8).$$

1.3. Theorem. *Let S^μ be the irreducible representation of \mathfrak{S}_k corresponding to the partition μ of k . Then*

- i) $\kappa_{\Lambda_c}(S^\mu) = 0$ if $\mu > \Lambda_c$;
- ii) $\kappa_{\Lambda_c}(v_\Lambda) = 0$ if $\Lambda_c \neq \Lambda$ for $\Lambda \in \mathcal{T}(\lambda)$ and the basis vectors v_{Λ_c} as in Theorem 1.2;
- iii) $\kappa_{\Lambda_c}(v_{\Lambda_c}) = 1^{\lambda_1} 2^{\lambda_2} \dots n^{\lambda_n} v_{\Lambda_c}$, where $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n$ are nonzero integers of the partition λ .

Proof. Let $\mu \geq \lambda$ and Λ be a standard tableau of shape μ ; let $v_\Lambda \in S^\mu$ be the corresponding vector for the seminormal basis (i.e., as in section 1.2). If $\Lambda \neq \Lambda_c$, then there exists a j such that the j -th columns of the tableaux Λ and Λ_c are distinct whereas all the columns with lesser numbers are identical. Since the orders of the tableaux coincide and $\mu \geq \lambda$, it follows that $\mu'_j \leq \lambda'_j$ (these are the lengths of the j -th columns of the tableaux Λ and Λ_c , respectively).

Let p be the least of the numbers from the j -th column of Λ (such a number exists because $\Lambda \neq \Lambda_c$ and $\mu'_j \leq \lambda'_j$). Since Λ is standard, the slot that p occupies is $(1, j + 1)$. Hence,

$$x_p v_\Lambda = c_\Lambda(p) v_\Lambda = (j + 1 - 1) v_s = j v_\Lambda, \quad \text{i.e., } (j - x_p) v_\Lambda = 0.$$

Since $j - x_p$ enters κ_{Λ_c} as a factor, $\kappa_{\Lambda_c} v_\Lambda = 0$. This proves i) and ii).

To prove iii), observe that

$$(j - x_p) v_{\Lambda_c} = (j - (j - i)) v_{\Lambda_c} = i v_{\Lambda_c},$$

where i is the number of the row occupied by x_p . This proves iii). □

1.4. Lemma. *Let $p < k$ and let \mathfrak{S}_p be considered as the subgroup of \mathfrak{S}_k that preserves all but the first p symbols. Let U be a \mathfrak{S}_k -module, ε the nontrivial one-dimensional representation and $a = \sum_{\sigma \in \mathfrak{S}_p} \varepsilon(\sigma) \sigma$. Let $v \in U$ be a nonzero vector fixed by a transposition s_{lk} for $l \leq p$. Then*

$$a \left(\sum_{i \in \mathfrak{S}_p} s_{ik} \right) v = av.$$

Proof. Let $P = \{1, \dots, p\}$. Then

$$\begin{aligned} a \left(\sum_{i \in P} s_{ik} \right) v &= a \left(\sum_{i \in P} s_{ik} \right) s_{lk} v \\ &= a \left(1 + \sum_{i \in P \setminus \{l\}} s_{ik} s_{kl} \right) v = a \left(1 + \sum_{i \in P \setminus \{l\}} s_{li} s_{ik} \right) v \\ &= a \left(1 - \sum_{i \in P \setminus \{l\}} s_{ik} \right) v = a \left(2 - \sum_{i \in P} s_{ik} \right) v; \end{aligned}$$

hence, $a \left(\sum_{i \in P} s_{ik} \right) v = av$. □

1.5. Corollary. *Let R_{Λ_c} and C_{Λ_c} be the row and column stabilizers of the tableau Λ_c ; let U be a \mathfrak{S}_k -module and $v \in U$ a nonzero R_{Λ_c} -invariant vector. Set $a_{\Lambda_c} = \sum_{\sigma \in C_{\Lambda_c}} \varepsilon(\sigma)\sigma$. Then $\kappa_{\Lambda_c}v = a_{\Lambda_c}v$.*

Proof. Induction on the number of columns in Λ_c . If Λ_c consist of one column, then by induction on k we easily verify the identity

$$(1 - s_{12})(1 - s_{13} - s_{23}) \dots (1 - s_{1k} - \dots - s_{k-1,k}) = \sum_{\sigma \in \mathfrak{S}_k} \varepsilon(\sigma)\sigma.$$

So for one column the statement is true.

Let tableau Λ_c consist of n columns $\Lambda_1, \dots, \Lambda_n$; let Λ_c^* be obtained from Λ_c by deleting the last column. Then

$$\begin{aligned} \kappa_{\Lambda_c}v &= \kappa_{\Lambda_c^*} \prod_{i \in \Lambda_n} (n - x_i)v = \prod_{i \in \Lambda_n} (n - x_i) \cdot \kappa_{\Lambda_c^*}v \\ &= (\text{by induction}) = \prod_{i \in \Lambda_n} (n - x_i) \cdot a_{\Lambda_c^*}v = a_{\Lambda_c^*} \prod_{i \in \Lambda_n} (n - x_i)v. \end{aligned}$$

Let $i \in \Lambda_n$; then

$$a_{\Lambda_c^*}(n - x_i)v = a_{\Lambda_c^*}(n - \sum_{\alpha \in \Lambda_1} s_{\alpha i} - \sum_{\alpha \in \Lambda_2} s_{\alpha i} - \dots - \sum_{\alpha \in \Lambda_n, \alpha < i} s_{\alpha i})v.$$

By Lemma 1.4 $a_{\Lambda_c^*}(\sum_{\alpha \in \Lambda_j} s_{\alpha i})v = a_{\Lambda_c^*}v$ for $j = 1, \dots, n - 1$, so $a_{\Lambda_c^*}(n - x_i)v = a_{\Lambda_c^*}(1 - \sum_{\alpha \in \Lambda_n, \alpha < i} s_{\alpha i})v$. Hence, if $\Lambda_n = \{i, i + 1, \dots, i + l\}$, then

$$\begin{aligned} \kappa_{\Lambda_c}v &= a_{\Lambda_c^*}(1 - s_{i,i+1})(1 - s_{i,i+2} - s_{i+1,i+2}) \dots (1 - s_{i,i+l} - \dots - s_{i+l-1,i+l})v \\ &= a_{\Lambda_c^*}(\sum_{\sigma \in \Lambda_n} \varepsilon(\sigma)\sigma)v = a_{\Lambda_c}v. \end{aligned}$$

□

Thus, if $v = \sum_{\sigma \in R_{\Lambda_c}} \sigma = b_{\Lambda_c}$, then

$$e_{\Lambda_c} = a_{\Lambda_c}b_{\Lambda_c} = \kappa_{\Lambda_c}b_{\Lambda_c},$$

i.e., we have expressed the Young symmetrizer e_{Λ_c} as $\kappa_{\Lambda_c}b_{\Lambda_c}$. In §3 a similar construction will be used to construct projective analogs of Young symmetrizers.

2. AUXILIARY DATA

Let \mathfrak{S}_k be the symmetric group, Cl_k the Clifford algebra generated by k indeterminates p_1, \dots, p_k subject to the relations

$$p_i^2 = -1, p_i p_j + p_j p_i = 0 \text{ for } i \neq j.$$

The symmetric group acts on Cl_n permuting the generators, so we can form a semidirect product $H_k = \mathfrak{S}_k \circ Cl_k$, i.e., the linear span of the elements $f\sigma$, where $f \in Cl_k$ and $\sigma \in \mathfrak{S}_k$ with the multiplication satisfying

$$\sigma f = f^\sigma \cdot \sigma \text{ (here } f^\sigma \text{ is the result of the action of } \sigma \in \mathfrak{S}_k \text{ on } f \in Cl_k).$$

Set

$$\tau_{i,j} = \frac{1}{\sqrt{2}}(p_i - p_j)s_{i,j}.$$

As is not difficult to verify, the relations (1.1) hold; hence, the algebra generated by the $\tau_{i,j}$ is isomorphic to \mathfrak{A}_n . Besides, \mathfrak{A}_n supercommutes with Cl_n ; hence, $H_n = \mathfrak{A}_n \otimes Cl_n$, as *superalgebras* if we define parity in \mathfrak{A}_n by setting $p(\tau_{i,j}) = \bar{1}$.

Let \mathbb{N} be the set of positive integers, $\bar{\mathbb{N}}$, another, “odd”, copy of \mathbb{N} ; its elements are barred. Let V be a superspace of superdimension (n, n) with the fixed basis $\{e_i\}_{i=1}^n \amalg \{e_{\bar{i}}\}_{\bar{i}=\bar{1}}^{\bar{n}}$. Define the odd operator $Q : V \rightarrow V$ by setting

$$Q(e_i) = e_{\bar{i}}; \quad Q(e_{\bar{i}}) = -e_i.$$

The (super)centralizer of Q in $\text{Mat}(V)$ is denoted by $Q(V)$; cf. [BL]. We denote the Lie superalgebras associated with the associative superalgebras $\text{Mat}(V)$ and $Q(V)$ by $\mathfrak{gl}(V)$ and $\mathfrak{q}(V)$, respectively. We set $\mathfrak{q}(n) = \mathfrak{q}(V)$ for $n = \dim V$.

Select a basis in $\mathfrak{q}(n)$: let the e_i^* be the left dual basis to the $\{e_i : i = 1, \dots, n; \bar{1}, \dots, \bar{n}\}$; set

$$E_{i,j} = e_i \otimes e_j^* + e_{\bar{i}} \otimes e_{\bar{j}}^*; \quad F_{i,j} = e_i \otimes e_{\bar{j}}^* + e_{\bar{i}} \otimes e_j^*.$$

Then $\mathfrak{h} = \text{Span}(E_{i,i} \text{ and } F_{i,i} : i = 1, \dots, n)$ is a Cartan subalgebra in $\mathfrak{q}(n)$ and $\mathfrak{b} = \text{Span}(E_{i,j} \text{ and } F_{i,j} : i \leq j)$ is a Borel subalgebra; cf. [Pe]. Set further $\mathfrak{n}_- = \text{Span}(E_{i,j} \text{ and } F_{i,j} : i > j)$ and $\mathfrak{n}_+ = \text{Span}(E_{i,j} \text{ and } F_{i,j} : i < j)$. Then, clearly, $\mathfrak{q}(n) = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$; any finite dimensional irreducible $\mathfrak{q}(n)$ -module $U = U^\lambda$ is uniquely, up to an isomorphism, determined by its highest weight and the irreducible \mathfrak{h} -module U^+ of highest weight vectors annihilated by \mathfrak{n}_+ , and for any $u \in U^+$ we have $hu = \lambda(h)u$ for any $h \in \mathfrak{h}_0$; cf. [Pe]. Observe that since \mathfrak{h} is, unlike the case of Lie algebras, not commutative, U^+ is seldom one-dimensional; cf. [S3]. This module U will be denoted by U^λ .

We will stick to the following terminology; cf [BL], [L]. The representation of a superalgebra A in the superspace V is *irreducible* if it does not contain homogeneous (with respect to parity) subrepresentations distinct from 0 and V itself.

Thus, there are two types of irreducible representations: those that do not contain any nontrivial subrepresentations (called of *general type* or of type G) and those that contain *inhomogeneous* invariant subspaces (call them of *type Q*). If V is of finite dimension, then in the first case its centralizer, as of A -module, is isomorphic to $\mathfrak{gl}(1)$, whereas in the second case to $\mathfrak{q}(1)$.

Let V_1 and V_2 be finite dimensional irreducible modules over A_1 and A_2 , respectively. Then $V_1 \otimes V_2$ is an irreducible $A_1 \otimes A_2$ -module except for the case where both V_1 and V_2 are of type Q . In the latter case, the centralizer of the $A_1 \otimes A_2$ -module $V_1 \otimes V_2$ is isomorphic to Cl_2 , the Clifford superalgebra with 2 generators. (For further interesting details see [Sch].)

If $e \in Cl_2$ is a minimal idempotent, then $e(V_1 \otimes V_2)$ is an irreducible $A_1 \otimes A_2$ -module of type G that we will denote by $2^{-1}(V_1 \otimes V_2)$.

Let $\mathbb{M} = \mathbb{N} \amalg \bar{\mathbb{N}}$ be ordered as follows:

$$\bar{1} < 1 < \bar{2} < 2 < \bar{3} < 3 \dots$$

We will call the elements from \mathbb{N} “even” and those from $\bar{\mathbb{N}}$ “odd” ones; so we can encounter an “even” odd element, etc.

The partition λ of k is called a *strict* one if its parts are pairwise distinct. To a strict partition λ we assign a *shifted diagram* (see [Ma]) obtained from the ordinary diagram by shifting its i -th row by $i - 1$ cell to the right for $i = 1, 2, \dots$. A shifted tableau of shape λ is a filling of a shifted diagram of shape λ with numbers from \mathbb{M} . The shifted tableau is called a *standard* one if:

- i) the numbers that fill it do not decrease from left to right along rows and downwards along columns;
- ii) the symbols from \mathbb{N} strictly increase along columns (downwards);
- iii) the symbols from $\overline{\mathbb{N}}$ strictly increase along rows (from left to right).

For the strict partition λ denote by $ST(\lambda)$ the set of strict standard tableaux of shape λ filled with the numbers $1, \dots, k$, where $k = \lambda_1 + \lambda_2 + \dots$. If $\Lambda \in ST(\lambda)$ and $1 \leq p \leq k$, then set $c_\Lambda(p) = \sqrt{(j-i)(j-i+1)}$, where (i, j) is the slot in Λ occupied by p .

Denote by $l(\lambda)$ the number of nonzero parts of the partition λ and set

$$\delta(\lambda) = \begin{cases} 0 & \text{if the number of nonzero parts of } \lambda \text{ is even,} \\ 1 & \text{otherwise.} \end{cases}$$

As is shown in [S1], the algebras $\mathfrak{q}(n)$ and H_k constitute a Howe-dual pair in the superspace $W_k = V^{\otimes k}$. The $\mathfrak{q}(n)$ -action is uniquely determined by the identity action in V extended as a superdifferentiation onto the tensor algebra $T(V) = \sum_{k \geq 0} W_k$. The symmetric group \mathfrak{S}_k acts on W_k by the formula

$$\begin{aligned} & s_{i,i+1}(v_1 \otimes \dots \otimes v_i \otimes v_{i+1} \otimes \dots \otimes v_k \\ &= (-1)^{p(v_i)p(v_{i+1})} v_1 \otimes \dots \otimes v_{i+1} \otimes v_i \otimes \dots \otimes v_k, \end{aligned}$$

where $p(v_i)$ and $p(v_{i+1})$ are parities of respective vectors. Recall that $\mathfrak{q}(n)$ is defined as the centralizer of an odd operator Q whose square is $-\text{id}$. The element $p_i \in H_k$ acts in W_k as the operator

$$1 \otimes \dots \otimes 1 \otimes Q \otimes 1 \otimes \dots \otimes 1 \quad (Q \text{ occupies the } i\text{-th slot}).$$

2.1. Lemma. *Let $w \in W_{k-1}$, set $\pi_k = \sum_{\alpha < k} \tau_{\alpha k}$, and also*

$$E_{im}^{(2)} = \sum_{j=1}^n (E_{ij}E_{jm} - F_{ij}F_{jm}), \quad F_{im}^{(2)} = \sum_{j=1}^n (F_{ij}E_{jm} - E_{ij}F_{jm}).$$

Then

- i) $\pi_k(w \otimes e_i) = \frac{1}{\sqrt{2}} \sum_{j=1}^n (F_{ij} - p_k E_{ij})(w \otimes e_j)$;
- ii) $\pi_k^2(w \otimes e_i) = \frac{1}{2} \sum_{m=1}^n \left((E_{im}^{(2)} w) \otimes e_m + (F_{im}^{(2)} w) \otimes e_{\overline{m}} \right) + \sum_{m=1}^n (E_{mm} w) \otimes e_i$.

Proof. i) It suffices to assume that $w = e_{i_1} \otimes \dots \otimes e_{i_{k-1}}$, where all the indices i_1, \dots, i_{k-1} are ‘‘even’’, i.e., belong to \mathbb{N} , not $\overline{\mathbb{N}}$. Then

$$\begin{aligned} \pi_k(w \otimes e_i) &= \sum_{\alpha < k} \tau_{\alpha k}(w \otimes e_i) = \frac{1}{\sqrt{2}} \sum_{\alpha < k} (p_\alpha - p_k) s_{\alpha k}(w \otimes e_j) \\ &= \frac{1}{\sqrt{2}} \sum_{\alpha < k} (p_\alpha - p_k) w_\alpha^i \otimes e_{i_\alpha}, \end{aligned}$$

where w_α^i is obtained from $w = e_{i_1} \otimes \dots \otimes e_{i_{k-1}}$ by replacing e_{i_α} with e_i . Hence,

$$\pi_k(w \otimes e_i) = \frac{1}{\sqrt{2}} \sum_{j=1}^n \sum_{i_\alpha=j} (p_\alpha - p_k) w_\alpha^i \otimes e_j.$$

But, on the other hand, $\sum_{i_\alpha=j} w_\alpha^i = E_{ij}w$ and $\sum_{i_\alpha=j} p_\alpha w_\alpha^i = F_{ij}w$ implying that

$$\begin{aligned} \pi_k(w \otimes e_i) &= \frac{1}{\sqrt{2}} \sum_{j=1}^n (F_{ij}w \otimes e_j - E_{ij}w \otimes e_{\bar{j}}) \\ &= \frac{1}{\sqrt{2}} \sum_{j=1}^n (F_{ij}(w \otimes e_j) - w \otimes e_{\bar{i}} - p_k E_{ij}(w \otimes e_{\bar{i}}) + w \otimes e_{\bar{i}}) \\ &= \frac{1}{\sqrt{2}} \sum_{j=1}^n (F_{ij} - p_k E_{ij})(w \otimes e_j) \end{aligned}$$

and heading i) is proven.

ii)

$$\begin{aligned} \pi_k^2(w \otimes e_i) &= \pi_k \left(\frac{1}{\sqrt{2}} \sum_{j=1}^n (F_{ij} - p_k E_{ij})(w \otimes e_j) \right) \\ &= -\frac{1}{\sqrt{2}} \sum_{j=1}^n (F_{ij} - p_k E_{ij}) \pi_k(w \otimes e_j) \\ &= -\frac{1}{2} \sum_{j=1}^n (F_{ij} - p_k E_{ij}) \sum_{m=1}^n (F_{jm} - p_k E_{jm})(w \otimes e_m) \\ &= -\frac{1}{2} \sum_{j,m=1}^n (F_{ij} - p_k E_{ij})(F_{jm} - p_k E_{jm})(w \otimes e_m) \\ &= -\frac{1}{2} \sum_{j,m=1}^n ((F_{ij}F_{jm} - E_{ij}E_{jm}) - p_k(E_{ij}F_{jm} - F_{ij}E_{jm}))(w \otimes e_m) \\ &= \frac{1}{2} \sum_{j,m=1}^n ((E_{ij}E_{jm} - F_{ij}F_{jm}) + p_k(E_{ij}F_{jm} - F_{ij}E_{jm}))(w \otimes e_m). \end{aligned}$$

Let us calculate the sum by parts. We have

$$\begin{aligned} &(E_{ij}E_{jm} - F_{ij}F_{jm})(w \otimes e_m) \\ &= E_{ij}(E_{jm}w \otimes e_m + w \otimes e_j) - F_{ij}(F_{jm}w \otimes e_m + w \otimes e_{\bar{j}}) \\ &= E_{ij}E_{jm}w \otimes e_m + E_{jm}w \otimes E_{ij}e_m + E_{ij}w \otimes e_j + w \otimes e_i \\ &\quad - F_{ij}F_{jm}w \otimes e_m + F_{jm}w \otimes F_{ij}e_m - F_{ij}w \otimes e_{\bar{j}} - w \otimes e_i \\ &= E_{ij}E_{jm}w \otimes e_m - F_{ij}F_{jm}w \otimes e_m + E_{mm}w \otimes e_i \\ &\quad + F_{mm}w \otimes e_{\bar{i}} + E_{ij}w \otimes e_j - F_{ij}w \otimes e_{\bar{j}}. \end{aligned}$$

and

$$\begin{aligned} &p_k(E_{ij}F_{jm} - F_{ij}E_{jm})(w \otimes e_m) \\ &= p_k E_{ij}(F_{jm}w \otimes e_m + w \otimes e_{\bar{j}}) - p_k F_{ij}(E_{jm}w \otimes e_m + w \otimes e_j) \\ &= -E_{ij}E_{jm}w \otimes e_{\bar{m}} - F_{mm}w \otimes e_{\bar{i}} - E_{ij}w \otimes e_j - w \otimes e_i \\ &\quad + F_{ij}E_{jm}w \otimes e_{\bar{m}} + E_{mm}w \otimes e_i + F_{ij}w \otimes e_{\bar{j}} + w \otimes e_i \\ &= (F_{ij}E_{jm} - E_{ij}F_{jm})w \otimes e_{\bar{m}} + E_{mm}w \otimes e_i \\ &\quad + F_{ij}w \otimes e_{\bar{j}} - E_{ij}w \otimes e_j - F_{mm}w \otimes e_{\bar{i}}. \end{aligned}$$

Therefore,

$$\begin{aligned} & \frac{1}{2} \sum_{j,m=1}^n ((E_{ij}E_{jm} - F_{ij}F_{jm}) + p_k(E_{ij}F_{jm} - F_{ij}E_{jm}))(w \otimes e_m) \\ &= \frac{1}{2} \sum_{j,m=1}^n ((E_{ij}E_{jm}w \otimes e_m - F_{ij}F_{jm}w \otimes e_m) \\ & \quad + (F_{ij}E_{jm} - E_{ij}F_{jm})w \otimes e_{\bar{m}} + 2E_{mm}w \otimes e_i) \\ &= \frac{1}{2} \sum_{m=1}^n (E_{im}^{(2)}w \otimes e_m + F_{im}^{(2)}w \otimes e_{\bar{m}}) + \sum_{m=1}^n E_{mm}w \otimes e_i. \end{aligned}$$

□

2.2. For the convenience of the reader we formulate the main result of [S1] in the following suitable form. Observe that if λ is a strict partition of k and $\lambda_{n+1} = 0$, then λ can be interpreted as a linear functional on $\mathfrak{h}_{\bar{0}}$: set

$$(2.2.1) \quad \lambda(E_{ii}) = \lambda_i.$$

Theorem. *The superalgebras H_k and $\mathfrak{q}(n)$ are mutual centralizers in the superspace W_k and the following decomposition takes place:*

$$(2.2.2) \quad W_k = \bigoplus_{\lambda} 2^{-\delta(\lambda)} T^{\lambda} \otimes V^{\lambda},$$

where λ runs over strict partitions of k such that $\lambda_{n+1} = 0$; here T^{λ} is an irreducible (in the super sense) H_k -module corresponding to λ , and V^{λ} an irreducible $\mathfrak{q}(n)$ -module.

If $\delta(\lambda) = 0$, then the corresponding representation T^{λ} is of type G , it is of type Q if $\delta(\lambda) = 1$.

The dimension of the space (superstructure ignored) of the highest weight vectors $(V^{\lambda})^+$ in V^{λ} is equal to $2^{(l(\lambda)+\delta(\lambda))/2}$ copies of T^{λ} .

2.3. Corollary. *The H_k -module of highest weight vectors in W_k is a free Cl_k -module.*

Proof. By Theorem $W_k^+ = \bigoplus_{\lambda} 2^{-\delta(\lambda)} T^{\lambda} \otimes (V^{\lambda})^+$, so the H_k -module of highest weight vectors in W_k is equal to $2^{-\delta(\lambda)} T^{\lambda} \otimes (V^{\lambda})^+$. By considering the dimension of $(V^{\lambda})^+$ and taking into account the type (G or Q) of its irreducibility we get the statement desired. □

2.4. Let us denote the H_k -module of highest weight vectors of weight λ in W_k by R^{λ} .

Lemma. *Let $\mathfrak{g} = \mathfrak{q}(n)$; let \mathfrak{b} and \mathfrak{h} be defined as above and V a \mathfrak{g} -module on which $\mathfrak{h}_{\bar{0}}$ acts diagonally. Let $(V^{\lambda})^+$ be the set of \mathfrak{b} -highest vectors of weight λ .*

If $u \in U(\mathfrak{g})$ and $u((V^{\lambda})^+) \subset (V^{\lambda})^+$, then there exists $w \in U(\mathfrak{h})$ such that $u|_{(V^{\lambda})^+} = w|_{(V^{\lambda})^+}$.

Proof. Let $u = \sum u_{\alpha}$ be the weight decomposition of $u \in U(\mathfrak{g})$ with respect to $\mathfrak{h}_{\bar{0}}$. Observe that $(V^{\lambda})^+$ is the λ weight space of V . Therefore, if $v \in V_{\lambda}$, then $u(v) = \sum_{\alpha} u_{\alpha}v$ and, if $u_{\alpha}v \neq 0$, then the weight of $u_{\alpha}v$ is equal to $\lambda + \alpha$. Thus, thanks to the condition $u(v) \in (V^{\lambda})^+$, we may assume that $u = u_0 \in U(\mathfrak{g})^{\mathfrak{h}_{\bar{0}}}$, where $U(\mathfrak{g})^{\mathfrak{h}_{\bar{0}}}$ is the centralizer of $\mathfrak{h}_{\bar{0}}$.

Further, thanks to [S2], we know that $U(\mathfrak{g})^{\mathfrak{h}_0} \cong U(\mathfrak{h}) \oplus L$, where $L = U(\mathfrak{g})^{\mathfrak{h}_0} \cap U(\mathfrak{g})\mathfrak{b}^+$ and where \mathfrak{b}^+ is the linear span of the positive roots in \mathfrak{b} , is a two-sided ideal in $U(\mathfrak{g})^{\mathfrak{h}_0}$.

Hence, $u = w + u_1$, where $w \in U(\mathfrak{h})$ and $u_1 \in L$; this implies that $uv = vw$ for $v \in (V^\lambda)^+$. □

Let T be a finite dimensional H_k -module. The $\mathfrak{q}(n)$ -module U corresponding to it via Howe duality is determined by the formula

$$U = \text{Hom}_{H_k}(T, W_k) \text{ for } W_k = V^{\otimes k}.$$

It is not difficult to deduce from Theorem 2.2 that likewise T is uniquely recoverable from U , namely,

$$T = \text{Hom}_{\mathfrak{q}(n)}(U, W_k).$$

Further on, since U is a $\mathfrak{q}(n)$ -submodule of W_k , it follows that all its weights with respect to \mathfrak{h}_0 are of the form $m_1\varepsilon_1 + \dots + m_n\varepsilon_n$, where the ε_i are the weights of the standard $\mathfrak{q}(n)$ -module V and the m_i are nonnegative integers. This implies that the formal character of the $\mathfrak{q}(n)$ -module U equal to $\sum m_\chi e^\chi$ (here χ runs over the set of weights of U with respect to \mathfrak{h}_0) is a polynomial in $x_1 = e^{\varepsilon_1}, \dots, x_n = e^{\varepsilon_n}$. This polynomial is called the *characteristic* of T .

Example 2. Consider the Clifford superalgebra Cl_k as a module over H_k (\mathfrak{S}_k acts by permuting the generators of Cl_k while Cl_k acts on itself via the left regular representation). The corresponding characteristic q_k is the coefficient of t^k in the decomposition

$$\frac{\prod_{i=1}^n (1 + x_i t)}{\prod_{i=1}^n (1 - x_i t)} = \sum_{k \geq 0} q_k(x) t^k.$$

In [S1] I showed that the x -character of T^λ for the strict partition λ is equal to $2^{(l(\lambda) - \delta(\lambda))/2} Q_\lambda(x)$, where $Q_\lambda(x)$ is the projective Schur function (for its definition see [Ma]). Therefore, the characteristic of R^λ is equal precisely to $Q_\lambda(x)$.

2.5. Set $P_\lambda(x) = 2^{-l(\lambda)} Q_\lambda(x)$. Denote by $SST(\lambda)$ the set of standard tableaux of arbitrary shape with elements from \mathbb{M} and such that

- a) scanning the tableau from left to right and downwards the last of the symbols encountered should be an “even” one;
- b) for every $i = 1, 2, \dots$ the total number of occurrences of any of the symbols i or \bar{i} is equal to λ_i .

Lemma. *Let λ be a strict partition of k and $\lambda_1 > \dots > \lambda_n > 0$. Then*

$$P_{\lambda_1}(x) \dots P_{\lambda_n}(x) = \sum_{\mu} m_{\lambda\mu} P_{\mu}(x),$$

where the sum runs over strict partitions μ such that $\mu \geq \lambda$ (with respect to dominance partial order) and $m_{\lambda\mu}$ is equal to the number of tableaux from $SST(\lambda)$ of shape μ .

Proof of this statement is an easy corollary of the multiplication table for projective Schur functions; cf. [P], §4.

2.6. **Even and odd Jucys-Murphy elements** (cf. [N2]). Set

$$x_1 = 0, x_2 = s_{12}(1 + p_1p_2), \dots, x_k = \sum_{i < k} s_{ik}(1 + p_ip_k).$$

The *odd analogs* of the Jucys-Murphy elements π_i are defined as

$$\pi_1 = 0, \pi_2 = \tau_{12}, \dots, \pi_k = \sum_{i < k} \tau_{ik}.$$

Observe that $\pi_i = -\frac{1}{\sqrt{2}}p_ix_i$ and the distinction of π_i from x_i is that the former supercommute with the elements of the Clifford superalgebra Cl_k .

The following statement proven by M. Nazarov in [N2] (Th. 7.2) describes the action of x_i in R^λ .

Theorem. *The Cl_k -module R^λ is free with a basis $\{v_\Lambda : \Lambda \in ST(\lambda)\}$ and*

$$x_p v_\Lambda = c_\Lambda(p)v_\Lambda \text{ for } 1 \leq p \leq k.$$

3. SPECHT MODULES OVER H_k

Let λ be a strict partition of k and $\lambda_{n+1} = 0$. Define the functional λ on the Cartan subalgebra \mathfrak{h} by setting

$$\lambda(E_{ii}) = \lambda_i; \quad \lambda(F_{ii}) = 0.$$

In $W_k = V^{\otimes k}$, where V is the standard $\mathfrak{q}(n)$ -module of dimension (n, n) , consider the H_k -submodule M^λ consisting of the vectors of weight λ , i.e., $w \in W_k$ such that $E_{ii}w = \lambda_i w$. Let Λ_r be a shifted tableau of shape λ filled in consecutively along rows from left to right with the numbers 1 to k .

Let Λ be an arbitrary shifted tableau of shape λ and let R_Λ be the row stabilizer of Λ . Let I be the sequence obtained by reading the tableau Λ from left to right and downwards. For $i \in I$ define

$$\pi_i = \sum_{\alpha \in I, \alpha \text{ precedes } i} \tau_{\alpha,i}.$$

Set

$$\kappa_\Lambda = \prod_{i \in I} \left(\frac{1}{2}j(j+1) - \pi_i^2 \right),$$

where j is the number of the column occupied by i .

Example 3. For

$$\Lambda = \begin{array}{cccc} 1 & 2 & 4 & 7 \\ & 3 & 5 & 8 \\ & & & 6 \end{array}$$

we have

$$\kappa_\Lambda = 1 \cdot (3 - \pi_2^2)(3 - \pi_3^2)(6 - \pi_4^2)(6 - \pi_5^2)(6 - \pi_6^2)(10 - \pi_7^2)(10 - \pi_8^2).$$

For every shifted tableau Λ of shape λ denote by w_Λ the vector from W_k equal to $v_1 \otimes \dots \otimes v_k$, where $v_p = e_i$ if p occupies the i -th row. Clearly, $w_\Lambda \in M^\lambda$.

3.1. Theorem. $\kappa_\Lambda(M^\lambda) = Cl_k \kappa_\Lambda(w_\Lambda)$ and $\kappa_\Lambda(w_\Lambda) \in R^\lambda$ is nonzero.

Proof. Since M^λ is the module induced from the trivial representation of the row stabilizer of Λ_r , it follows that

$$\text{Hom}_{H_k}(T, W_k) = S^{\lambda_1}(V) \otimes \dots \otimes S^{\lambda_n}(V),$$

where S^p is the operator of the p -th symmetric power and V is the standard $\mathfrak{q}(n)$ -module. Hence, the characteristic of M^λ is equal to $q_{\lambda_1} \dots q_{\lambda_n}$ (see (2.4)). Lemma 2.5 implies that

$$q_{\lambda_1} \dots q_{\lambda_n} = \sum_{\mu \geq \lambda} 2^{l(\lambda)-l(\mu)} m_{\lambda\mu} Q_\mu.$$

Hence, M^λ is equal to the direct sum of modules isomorphic to either T^μ or $\Pi(T^\mu)$. Further, let $\varphi_i : M^\lambda \rightarrow M^\lambda$ be a homomorphism such that $\varphi_i(v_{\Lambda_r}) = p_{\Lambda_r}^i$ for $i = 1, \dots, l(\lambda)$, where $p_{\Lambda_r}^i = \sum_{j \text{ lies in the } i\text{-th row}} p_j$.

It is not difficult to verify that $\varphi_1, \dots, \varphi_{l(\lambda)}$ generate the Clifford algebra $Cl_{l(\lambda)}$. Therefore, M^λ is equal to the direct sum of modules isomorphic to R^μ (or $\Pi(R^\mu)$) for $\mu \geq \lambda$ and the multiplicity of R^λ in M^λ is equal to 1.

It suffices to consider the case when $\Lambda = \Lambda_c$ is a shifted tableau of shape λ filled from left to right and downwards, as usual. In each module R^μ , select a basis $\{v_\Lambda\}$, where Λ runs over standard shifted tableaux, as in Theorem 2.6. Let us demonstrate that $\kappa_{\Lambda_c}(R^\mu) = 0$ if $\mu > \lambda$ and $\kappa_{\Lambda_c}(R^\lambda) = Cl_k v_{\Lambda_c}$.

Consider a vector $v_\Lambda \in R^\mu$ for $\mu \geq \lambda$ and $\Lambda \neq \Lambda_c$. There exists a number j such that the j -th columns of the tableaux Λ and Λ_c are distinct and all the respective columns with lesser numbers are identical. Since the tableaux are of the same order and $\mu \geq \lambda$, it follows that the length of the j -th column of the *shifted* tableau Λ is not less than the length of the j -th column of the shifted tableaux Λ . Let p be the least of the numbers occupying the j -th column of Λ_c which is not contained in the j -th column of Λ . Since Λ is standard, we see that p occupies the $(j + 1)$ -st column and the first row of Λ .

By Theorem 2.6

$$\pi_p^2 v_\Lambda = \frac{1}{2} x_p^2 v_\Lambda = \frac{1}{2} (j + 1 - 1)(j + 1 - 1 + 1) v_\Lambda = \frac{1}{2} j(j + 1) v_\Lambda.$$

Hence,

$$\left(\frac{1}{2} j(j + 1) - \pi_p^2\right) v_\Lambda = 0.$$

Since $\frac{1}{2} j(j + 1) - \pi_p^2$ enters κ_{Λ_c} as a factor, $\kappa_{\Lambda_c}(v_\Lambda) = 0$. Moreover, since the π_p for $p = 1, \dots, k$ supercommute with Cl_k , it follows that $\kappa_{\Lambda_c}(R^\mu) = 0$ if $\mu > \lambda$ and $\kappa_{\Lambda_c}(v_\Lambda) = 0$ if $v_\Lambda \in R^\lambda$ and $\Lambda \neq \Lambda_c$. Moreover,

$$\left(\frac{1}{2} j(j + 1) - \pi_p^2\right) v_{\Lambda_c} = \left(\frac{1}{2} j(j + 1) - \frac{1}{2} (j - i)(j - i + 1)\right) v_{\Lambda_c} = \frac{1}{2} i(2j + i - 1) v_{\Lambda_c},$$

where (i, j) is the position of p . Hence, $\kappa_{\Lambda_c}(v_{\Lambda_c}) = \alpha v_{\Lambda_c}$ for a nonzero α .

Thus, it is shown that κ_{Λ_c} projects M^λ onto the subspace $Cl_k v_{\Lambda_c}$. To prove the theorem, it suffices to establish that $\kappa_{\Lambda_c}(w_{\Lambda_c}) \neq 0$. We induct on k . For $k = 1$ the statement is obvious. Let $k > 1$, let Λ_c^* be obtained from Λ_c by deleting the last cell in the last column and let (i, j) be the position of k . Then

$$\kappa_{\Lambda_c}(w_{\Lambda_c}) = \left(\frac{1}{2} j(j + 1) - \pi_k^2\right) \kappa_{\Lambda_c^*}(w_{\Lambda_c^*} \otimes e_i) = \left(\frac{1}{2} j(j + 1) - \pi_k^2\right) (\kappa_{\Lambda_c^*}(w_{\Lambda_c^*}) \otimes e_i).$$

By induction, $\kappa_{\Lambda_c^*}(w_{\Lambda_c^*}) \neq 0$ while by Lemma 2.1 and in notations of Lemma 2.1 we have

$$\left(\frac{1}{2}j(j+1) - \pi_k^2\right)(\kappa_{\Lambda_c^*}(w_{\Lambda_c^*}) \otimes e_i) = \sum_{m=1}^n (w_m \otimes e_m + \bar{w}_m \otimes e_{\bar{m}}),$$

where

$$w_m = \left(\frac{1}{2}j(j+1) - \frac{1}{2}E_{im}^{(2)} - \delta_{il} \sum_{l=1}^n E_{ll}\right) \kappa_{\Lambda_c^*}(w_{\Lambda_c^*})$$

and $\bar{w}_m = -\frac{1}{2}F_{im}^{(2)} \kappa_{\Lambda_c^*}(w_{\Lambda_c^*})$.

It remains to show that $w_i \neq 0$. Observe that since $\kappa_{\Lambda_c^*}(w_{\Lambda_c^*})$ is a highest weight vector with respect to $\mathfrak{q}(n)$, i.e.,

$$E_{ij} \kappa_{\Lambda_c^*}(w_{\Lambda_c^*}) = F_{ij} \kappa_{\Lambda_c^*}(w_{\Lambda_c^*}) = 0 \text{ for } i < j,$$

we have

$$\begin{aligned} w_i &= \left(\frac{1}{2}j(j+1) - \frac{1}{2} \sum_{r=1}^n (E_{ir} E_{ri} - F_{ir} F_{ri}) - \sum_{r=1}^n E_{rr}\right) \kappa_{\Lambda_c^*}(w_{\Lambda_c^*}) \\ &= \left(\frac{1}{2}j(j+1) - \frac{1}{2} \sum_{r>i}^n (E_{ir} E_{ri} - F_{ir} F_{ri}) - \frac{1}{2}(E_{ii}^2 - F_{ii}^2) - \sum_{r=1}^n E_{rr}\right) \kappa_{\Lambda_c^*}(w_{\Lambda_c^*}) \\ &= \left(\frac{1}{2}j(j+1) - \frac{1}{2} \sum_{r>i}^n ([E_{ir}, E_{ri}] - [F_{ir}, F_{ri}]) \right. \\ &\quad \left. - \frac{1}{2}(E_{ii}^2 - E_{ii}) - \sum_{r=1}^n E_{rr}\right) \kappa_{\Lambda_c^*}(w_{\Lambda_c^*}) \\ &= \left(\frac{1}{2}j(j+1) - \frac{1}{2} \sum_{r>i}^n (E_{ii} - E_{rr} - E_{ii} - E_{rr}) \right. \\ &\quad \left. - \frac{1}{2}(E_{ii}^2 - E_{ii}) - \sum_{r=1}^n E_{rr}\right) \kappa_{\Lambda_c^*}(w_{\Lambda_c^*}) \\ &= \left(\frac{1}{2}j(j+1) - \frac{1}{2} \sum_{r>i}^n E_{rr} - \frac{1}{2}(E_{ii}^2 - E_{ii}) - \sum_{r=1}^n E_{rr}\right) \kappa_{\Lambda_c^*}(w_{\Lambda_c^*}) \\ &= \left(\frac{1}{2}j(j+1) - \frac{1}{2} \sum_{r=1}^n i E_{rr} - \frac{1}{2}(E_{ii}^2 - E_{ii})\right) \kappa_{\Lambda_c^*}(w_{\Lambda_c^*}) \\ &= \left(\frac{1}{2}\lambda_1(\lambda_1 + 1) - (\lambda_1 + \dots + \lambda_i - 1) - \frac{1}{2}(\lambda_i - 1)(\lambda_i - 2)\right) \kappa_{\Lambda_c^*}(w_{\Lambda_c^*}) \\ &= \frac{1}{2}(\lambda_1(\lambda_1 + 1) - 2(\lambda_1 + \dots + \lambda_i) + 2 - (\lambda_i - 1)(\lambda_i - 2)) \kappa_{\Lambda_c^*}(w_{\Lambda_c^*}). \end{aligned}$$

If i is the number of the row occupied by k , then

$$\lambda_1 = \lambda_i + i - 1, \lambda_2 = \lambda_i + i - 2, \dots, \lambda_{i-1} = \lambda_i + 1,$$

and, therefore,

$$w_i = \frac{1}{2} ((\lambda_i + i - 1)(\lambda_i + i) - 2(i\lambda_i + 1 + 2 + \dots + i - 1) + 2 - (\lambda_i - 1)(\lambda_i - 2)) \kappa_{\Lambda_c^*}(w_{\Lambda_c^*}) = \lambda_i \kappa_{\Lambda_c^*}(w_{\Lambda_c^*}) \neq 0.$$

□

3.2. *Remark.* Observe that

$$\begin{aligned} \bar{w}_i &= -\frac{1}{2} F_{ii}^{(2)} \kappa_{\Lambda_c^*}(w_{\Lambda_c^*}) = -\frac{1}{2} \sum_{r=1}^n (F_{ir} E_{ri} - E_{ir} F_{ri}) \kappa_{\Lambda_c^*}(w_{\Lambda_c^*}) \\ &= \frac{1}{2} \sum_{i=1}^n (E_{ir} F_{ri} - F_{ir} E_{ri}) \kappa_{\Lambda_c^*}(w_{\Lambda_c^*}) = \frac{1}{2} \sum_{r>i} (E_{ir} F_{ri} - F_{ir} E_{ri}) \kappa_{\Lambda_c^*}(w_{\Lambda_c^*}) \\ &= \frac{1}{2} \sum_{r>i} ([E_{ir}, F_{ri}] - [F_{ir}, E_{ri}]) \kappa_{\Lambda_c^*}(w_{\Lambda_c^*}) = 0. \end{aligned}$$

Hence,

$$\kappa_{\Lambda_c}(w_{\Lambda_c}) = \lambda_i \kappa_{\Lambda_c^*}(w_{\Lambda_c^*}) \otimes e_i + \sum_{r \neq i} (w_r + \bar{w}_r p_k) \otimes e_r$$

and by the inductive hypothesis applied to $\kappa_{\Lambda_c^*}(w_{\Lambda_c^*})$ we deduce that

$$\kappa_{\Lambda_c}(w_{\Lambda_c}) = \lambda_1! \dots \lambda_n! w_{\Lambda_c} + \sum f_{\Lambda} w_{\Lambda}, \text{ where } f_{\Lambda} \in Cl_k \text{ and } w_{\Lambda} \neq w_{\Lambda_c}$$

and where the summation runs over the tableaux Λ of order k which contain λ_1 symbols 1 or $\bar{1}$, λ_2 symbols 2 or $\bar{2}$, etc., λ_n symbols n or \bar{n} .

3.3. **Specht modules.** Recall that for a strict partition λ of k the *Specht module* is the submodule in M^λ generated by the vectors $\kappa_{\Lambda} w_{\Lambda}$ for all shifted tableaux s of shape λ .

3.3.1. Theorem. *Specht module is equal to R^λ . It is isotypical and its H_k -centralizer is isomorphic to the Clifford algebra with $l(\lambda)$ generators.*

Proof. Recall that R^λ is the set of highest weight vectors in $W_k = V^{\otimes k}$. By the Howe duality between $U(\mathfrak{q}(n))$ and H_k the algebra of H_k -homomorphisms is generated by $U(\mathfrak{q}(n))$. But thanks to Lemma 2.4 we may assume that the algebra of H_k -homomorphisms is generated by $U(\mathfrak{h})$ for the Cartan subalgebra \mathfrak{h} of $\mathfrak{q}(n)$. If $\lambda_i = 0$, then, in our notations for the basis of \mathfrak{h} , we have

$$F_{ii}^2 = E_{ii} = \lambda_i = 0 \text{ in } R^\lambda.$$

But $\ker F_{ii}$ is an \mathfrak{h} -submodule in the set $(V^\lambda)^+$ of highest weight vectors. But $(V^\lambda)^+$ is irreducible as an \mathfrak{h} -module by Theorem 2.2. Hence, $F_{ii}|_{(V^\lambda)^+} = 0$ which implies $F_{ii}|_{R^\lambda} = 0$. Thus, the algebra of H_k -homomorphisms is generated by the F_{ii} for i such that $\lambda_i \neq 0$.

By Theorem 3.1 the Specht module is contained in R^λ . Moreover, if Λ is a shifted tableau of shape λ , then

$$F_{ii} \kappa_{\Lambda} w_{\Lambda} = \kappa_{\Lambda} F_{ii} w_{\Lambda} = \left(\sum_{\alpha \text{ lies in the } i\text{-th row of } \Lambda} p_{\alpha} \right) \kappa_{\Lambda} w_{\Lambda}.$$

Therefore, any homomorphism of M^λ sends R^λ into itself. Hence, the Specht module coincides with R^λ . □

3.3.2. Corollary. *Let Λ be a shifted tableau. Set $p_\Lambda^i = \sum_{\alpha \text{ lies in the } i\text{-th row of } \Lambda} p_\alpha$. Then for any H_k -module endomorphism φ of M^λ we have*

$$\varphi(\kappa_\Lambda w_\Lambda) = f \cdot \kappa_\Lambda w_\Lambda,$$

where f belongs to the subalgebra of Cl_k generated by $p_\Lambda^1, \dots, p_\Lambda^{l(\lambda)}$.

Proof. Recall that M^λ is the subset of vectors of weight λ in W_k . Therefore, any endomorphism of M^λ may be identified with an element of $U(\mathfrak{q}(n))$; the restriction of this endomorphism onto R^λ may be identified with an element of $U(\mathfrak{h})$.

But $U(\mathfrak{h})$ is generated by the F_{ii} and to prove the corollary, it suffices to verify it for these elements. We have

$$F_{ii}(\kappa_\Lambda w_\Lambda) = \kappa_\Lambda(F_{ii}w_\Lambda) = \kappa_\Lambda(p_\Lambda^i w_\Lambda) = p_\Lambda^i \kappa_\Lambda(w_\Lambda).$$

□

3.3.3. Corollary. *Let b_Λ be the sum of the elements of the row stabilizer of a shifted tableau Λ and $\varphi : M^\lambda \rightarrow M^\lambda$ be an H_k -module endomorphism given by the formula $\varphi(w_\Lambda) = b_\Lambda \kappa_\Lambda(w_\Lambda)$. Then $\varphi|_{R^\lambda} = c \in \mathbb{C}$.*

Proof. Let us show that φ commutes with the endomorphisms F_{ii} . Indeed,

$$\begin{aligned} \varphi \cdot F_{ii}(w_\Lambda) &= \varphi(p_\Lambda^i w_\Lambda) = p_\Lambda^i \varphi(w_\Lambda) = p_\Lambda^i b_\Lambda \kappa_\Lambda(w_\Lambda); \\ F_{ii} \cdot \varphi(w_\Lambda) &= F_{ii} b_\Lambda \kappa_\Lambda(w_\Lambda) = b_\Lambda \kappa_w(F_{ii}w_\Lambda) \\ &= b_\Lambda \kappa_\Lambda(p_\Lambda^i w_\Lambda) = b_\Lambda p_\Lambda^i \kappa_\Lambda(w_\Lambda) = p_\Lambda^i b_\Lambda \kappa_\Lambda(w_\Lambda). \end{aligned}$$

The latter identity holds thanks to the fact that p_Λ^i commutes with b_Λ .

Thus, $\varphi \cdot F_{ii}(w_\Lambda) = F_{ii} \cdot \varphi(w_\Lambda)$ and, since the elements w_Λ generate the H_k -module M^λ , we have

$$\varphi \cdot F_{ii} = F_{ii} \cdot \varphi.$$

Further, $\varphi(R^\lambda) \subset R^\lambda$ for any endomorphism φ of M^λ , so $\varphi|_{R^\lambda}$ is an element from the centralizer of R^λ . But by Theorem 3.3.1 the centralizer is the Clifford algebra $Cl_{l(\lambda)}$. But φ is an even central element of $Cl_{l(\lambda)}$, hence, φ is a constant. □

3.3.4. Corollary. *Set $e_\Lambda = \kappa_\Lambda b_\Lambda$. Then*

$$e_\Lambda^2 = c \cdot e_\Lambda \text{ for } c \in \mathbb{C}, c \neq 0;$$

the algebra $e_\Lambda H_k e_\Lambda$ is isomorphic to $Cl_{l(\lambda)}$ and is generated by the p_Λ^i for $1 \leq i \leq l(\lambda)$.

Proof. Thanks to Corollary 3.3.3

$$\varphi(\kappa_\Lambda(w_\Lambda)) = \kappa_\Lambda b_\Lambda \kappa_\Lambda(w_\Lambda) = c \kappa_\Lambda(w_\Lambda)$$

or, equivalently,

$$e_\Lambda^2 = c \cdot e_\Lambda.$$

If $c = 0$, then $\varphi(\kappa_\Lambda(w_\Lambda)) = 0$. But the H_k -submodule generated by $\kappa_\Lambda(w_\Lambda)$ coincides with the Specht module, which, thanks to Theorem 3.3.1 is equal to R^λ . So in this case $\varphi(R^\lambda) = 0$. On the other hand, Theorem 3.1 implies that $\kappa_\Lambda(w_\Lambda) = \alpha w_\Lambda + \sum \beta_T w_T$, where $\alpha \in \mathbb{C}$ and $\beta_T \in Cl_k$ and where $w_\Lambda \neq w_T$.

Therefore,

$$b_\Lambda \kappa_\Lambda(w_\Lambda) = \alpha \lambda_1! \dots \lambda_n! w_\Lambda + \sum \gamma_T w_T,$$

where $\gamma_T \in Cl_k$, the sum runs over the tableaux T of order k which contain λ_1 symbols 1 or $\bar{1}$, λ_2 symbols 2 or $\bar{2}$, etc., λ_n symbols n or \bar{n} , and where $w_\Lambda \neq w_T$ and $\varphi(w_\Lambda) = b_\Lambda \kappa_\Lambda(w_\Lambda) \neq 0$. This contradicts the requirement $\varphi(R^\lambda) = 0$. So $c \neq 0$.

The algebra $e_\Lambda H_k e_\Lambda$ is anti-isomorphic to the algebra H_k of endomorphisms of R^λ and, thanks to Theorem 3.3.1, it is the Clifford superalgebra $Cl_{l(\lambda)}$. \square

3.3.5. Let $\lambda = (k)$ be the partition consisting of one part, $\Lambda = 12 \dots k$ the corresponding one-row tableau. Then $b_\Lambda = \sum_{\sigma \in \mathfrak{S}_k} \sigma$ and

$$\kappa_\Lambda = 1 \cdot (3 - \pi_2^2)(6 - \pi_3^2)(10 - \pi_4^2) \dots \left(\frac{1}{2}k(k+1) - \pi_k^2\right).$$

Further, for any i, j and l we have $(\tau_{ij} + \tau_{jl} + \tau_{li})b_\Lambda = 0$, so

$$\begin{aligned} \pi_p^2 b_\Lambda &= \left(\sum_{i < p} \tau_{ip}\right)^2 b_\Lambda = \left(p - 1 - \sum_{i < j < p} (\tau_{ip}\tau_{jp} + \tau_{jp}\tau_{ip})\right) b_\Lambda \\ &= \left(p - 1 - \sum (\tau_{ij}(\tau_{ij} + \tau_{jp} + \tau_{pi}) - 1)\right) b_\Lambda = \frac{1}{2}p(p-1)b_\Lambda. \end{aligned}$$

Therefore, $\kappa_\Lambda b_\Lambda = k! b_\Lambda = e_\Lambda$ and, up to a factor, e_Λ coincides with the respective Young symmetrizer.

Observe that the symmetrizer Nazarov constructed for the one-row tableau is equal to

$$\frac{1}{k!} \prod_{j=2}^k \left(\prod_{i=1}^{j-1} \left(1 + \frac{s_{ij}}{u_j - u_i} - \frac{p_i p_j s_{ij}}{u_j + u_i} \right) \right),$$

where $u_p = \sqrt{p(p-1)}$ and s_{ij} is the transposition; so it does not coincide with b_Λ .

3.3.6. In the case $\lambda = (p+1, p)$ and

$$\Lambda = \begin{matrix} 1 & 2 & 4 & 6 & \dots & p \\ & 3 & 5 & 7 & \dots & 2p+1 \end{matrix}$$

we can simplify the expression for $\kappa_\Lambda b_\Lambda$. Namely, set

$$\begin{aligned} a_1 &= 1 + \tau_{12}\tau_{23} + \tau_{23}\tau_{12}, \\ a_2 &= 2 + (\tau_{14} + \tau_{24} + \tau_{34})\tau_{45} + \tau_{45}(\tau_{14} + \tau_{24} + \tau_{34}), \\ a_3 &= 3 + \pi_6\tau_{67} + \tau_{67}\pi_6, \\ &\dots\dots\dots \\ a_p &= p + \pi_{2p}\tau_{2p,2p+1} + \tau_{2p,2p+1}\pi_{2p}. \end{aligned}$$

Then it is possible to verify that

$$\kappa_\Lambda b_\Lambda = a_1 a_2 \dots a_p b_\Lambda.$$

The author does not know how to simplify the expression for $\kappa_\Lambda b_\Lambda$ in the general case similarly to the case of the symmetric group; cf. section 1.5.

4. THE SPECHT MODULES OVER \mathfrak{A}_k

In this section, we construct for \mathfrak{A}_k certain analogs of the modules M^λ and the Specht modules R^λ . First of all, we need the following statement.

4.1. Lemma. *Let $\pi_p = \sum_{\alpha < p} \tau_{\alpha p}$ for $p = 2, \dots, k$ be odd analogs of Jucys–Murphy’s elements. Then for $k \geq 2$*

$$e_k = \prod_{p \geq 2} \frac{2}{p(p-1)} \pi_p^2$$

is an idempotent and $e_k \mathfrak{A}_k e_k$ is isomorphic to the Clifford algebra Cl_{k-1} .

Proof. Set $e_1 = e_2 = 1$. It is easy to verify by induction that for $k \geq 2$ we have

$$e_k = \frac{1}{k!} \sum_{p \geq 1} (-1)^p 2^{k-p-1} \Sigma_{2p+1},$$

where Σ_{2p+1} is the sum of all elements from \mathfrak{A}_k of the form $\tau_{i_1 i_2} \tau_{i_2 i_3} \dots \tau_{i_{2\alpha} i_{2\alpha+1}}$. This implies that e_k is a central element that does not vary under automorphisms of \mathfrak{A}_k that send τ_{ij} into $\tau_{\sigma(i)\sigma(j)}$ for any $\sigma \in \mathfrak{S}_k$. It is not difficult to verify that

$$(\tau_{12} + \tau_{23} + \tau_{31})\pi_2^2 = 0.$$

Hence, $(\tau_{12} + \tau_{23} + \tau_{31})e_k = 0$. Having applied the above described automorphisms to this relation we deduce that

$$(\tau_{ij} + \tau_{il} + \tau_{li})e_k = 0.$$

Further, arguments as in section 3.3.5 yield

$$\pi_p^2 e_k = \frac{1}{2} p(p-1) e_k,$$

hence,

$$e_k^2 = \left(\prod_{p \geq 2} \frac{2}{p(p-1)} \pi_p^2 \right) e_k = e_k.$$

Furthermore, since e_k is a central element, then $e_k \mathfrak{A}_k e_k = \mathfrak{A}_k e_k$. Let I be the two-sided ideal in \mathfrak{A}_k generated by the elements $\tau_{ij} + \tau_{jl} + \tau_{li}$. Then $e_k I = 0$. Set $\bar{\mathfrak{A}} = \mathfrak{A}_k / I$. In $\bar{\mathfrak{A}}$, then, the following relations hold:

$$\tau_{12} = \pi_2, \tau_{23} = \frac{1}{2}(\pi_3 - \pi_2), \tau_{34} = \frac{1}{3}(\pi_4 - \pi_3), \dots, \tau_{k-1,k} = \frac{1}{k-1}(\pi_k - \pi_{k-1}).$$

Hence, $\bar{\mathfrak{A}}$ is generated by the π_p for $2 \leq p \leq k$. We showed above that condition $\tau_{ij} + \tau_{il} + \tau_{li} = 0$ in $\bar{\mathfrak{A}}$ implies that $\pi_i^2 = \frac{1}{2}(i-1)(i-2)$ in $\bar{\mathfrak{A}}$. Moreover, $\pi_i \pi_j + \pi_j \pi_i = 0$ for $i \neq j$. So $\bar{\mathfrak{A}}$ is the Clifford algebra generated by the images of the π_i for $2 \leq i \leq k$.

Further on, $(1 - e_k)I = I$; so $\mathfrak{A}_k(1 - e_k) \supset I$, and since $\bar{\mathfrak{A}}$ is simple as a superalgebra, we deduce that $\mathfrak{A}_k(1 - e_k) = I$. Therefore, $\mathfrak{A}_k e_k \cong \mathfrak{A}_k / \mathfrak{A}_k(1 - e_k) \cong \mathfrak{A}_k / I \cong \bar{\mathfrak{A}}$. \square

Let λ be a strict partition of k and Λ a shifted λ -tableau. Let $\Lambda^{(i)} = \{\alpha_1, \dots, \alpha_{\lambda_i}\}$ be the i -th row. Recall that

$$p_\Lambda^i = \sum_{\alpha \in \Lambda^{(i)}} p_\alpha.$$

Let further $\pi_p^{(i)} = \sum_{j < p} \tau_{\alpha_j \alpha_p}$ and

$$e_\Lambda^i = \prod_{p=2}^{\lambda_i} \frac{2}{p(p-1)} \left(\pi_p^{(i)}\right)^2, \quad e_\Lambda = \prod_{i=1}^{l(\lambda)} e_\Lambda^i.$$

Let V be an (n, n) -dimensional superspace with $n \geq l(\lambda)$ and $W_k = V^{\otimes k}$, where $k = \sum \lambda_i$. Let M^Λ be the subspace of W_k spanned by the vectors $w \in W_k$ such that

$$E_{ii}w = \lambda_i w; \quad F_{ii}w = p_\Lambda^i w \text{ for } 1 \leq i \leq l(\lambda).$$

Let $R^\Lambda = \{w \in M^\Lambda : E_{ij}w = F_{ij}w = 0\}$ be the set of highest weight vectors from M^Λ . Recall that the irreducible representations of superalgebra \mathfrak{A}_k are labelled by strict partitions λ of k ; moreover, the representation is of type G if k and $l(\lambda)$ are of the same parity, and of type Q , otherwise; see [N2].

4.2. Specht modules. Let λ be a strict partition of k and Λ a shifted tableau of shape λ . The Specht module for tableau Λ is the submodule in M^Λ generated by the vectors $\kappa_T w_\Lambda$, where the tableau T is of shape λ and its rows have the same numbers as the respective rows of Λ .

Theorem. i) As \mathfrak{A}_k -module, M^Λ is isomorphic to $\mathfrak{A}_k e_\Lambda$.

ii) The Specht submodule of M^Λ is equal to R^Λ and its centralizer is isomorphic to the Clifford superalgebra $Cl_{k-l(\lambda)}$.

Proof. i) Let \mathfrak{A}_i be the subalgebra of \mathfrak{A}_k generated by the $\tau_{\alpha\beta}$, where α and β lie in the i -th row of Λ . Define the homomorphism $\varphi_i : \mathfrak{A}_i \rightarrow Cl_{\lambda_i}$ into the Clifford superalgebra on λ_i generators $p_{\alpha_1}, \dots, p_{\alpha_{\lambda_i}}$, where α_i runs numbers in the i -th row of Λ by setting

$$\varphi_i(\tau_{\alpha\beta}) = \frac{1}{\sqrt{2}}(p_\alpha - p_\beta).$$

From Lemma 4.1 we deduce that this homomorphism induces an isomorphism of $\mathfrak{A}_i e_\Lambda^i$ with the subalgebra of Cl_{λ_i} generated by $p_\alpha - p_\beta$ for $\alpha, \beta \in \Lambda^{(i)}$ (recall that $\Lambda^{(i)}$ is the i -th row of Λ).

Therefore, we have an homomorphism

$$\varphi_1 \otimes \dots \otimes \varphi_{l(\lambda)} : \mathfrak{A}_1 \otimes \dots \otimes \mathfrak{A}_{l(\lambda)} \rightarrow Cl_{\lambda_1} \otimes \dots \otimes Cl_{\lambda_{l(\lambda)}}$$

which induces an isomorphism of $\mathfrak{A}_1 \otimes \dots \otimes \mathfrak{A}_{l(\lambda)} e_\Lambda$ with a subalgebra Cl_Λ in $Cl_{\lambda_1} \otimes \dots \otimes Cl_{\lambda_{l(\lambda)}} \cong Cl_k$. Moreover, we can see that this subalgebra is generated by the $p_\alpha - p_\beta$, where α and β lie in the same row of Λ . The dimension of this subalgebra is equal to $2^{\lambda_1-1} 2^{\lambda_2-1} \dots 2^{\lambda_{l(\lambda)}-1} = 2^{k-l(\lambda)}$. This implies that the ideal $\mathfrak{A}_k e_\Lambda$ is isomorphic, as an \mathfrak{A}_k -module, to the module induced from the module Cl_Λ over $\mathfrak{A}_1 \otimes \dots \otimes \mathfrak{A}_{l(\lambda)}$, is of dimension

$$\frac{k!}{\lambda_1! \lambda_2! \dots \lambda_{l(\lambda)}!} 2^{k-l(\lambda)}.$$

It is subject to direct verification that $\dim M^\Lambda$ is the same and the homomorphism $\varphi : \mathfrak{A}_k e_\Lambda \rightarrow M^\Lambda$ such that $\varphi(e_\Lambda) = w_\Lambda$ is surjective. Hence, φ is an isomorphism.

ii) By Theorem 3.1 $\kappa_T w_\Lambda = \kappa_T w_T \in R^\Lambda$, so the Specht module is contained in R^Λ .

Now, let us show that the centralizer of R^t is isomorphic to the Clifford algebra $Cl_{k-l(\lambda)}$.

Obviously, \mathfrak{A}_k and $Cl_k \otimes U(\mathfrak{q}(n))$ form a Howe-dual pair in W_k . By Theorem 3.3.1 the centralizer of the H_k -module R^λ is the Clifford algebra generated by the F_{ii} for $1 \leq i \leq l(\lambda)$. Therefore, the centralizer of the \mathfrak{A}_k -module R^λ is the Clifford algebra $Cl_{k+l(\lambda)}$ generated by the F_{ii} for $1 \leq i \leq l(\lambda)$ and p_1, \dots, p_k .

The condition $F_{ii}w = p_\Lambda^i w$ is equivalent to the condition $ew = w$ for

$$e = \prod_{1 \leq i \leq l(\lambda)} \frac{1}{2} \left(1 - \frac{1}{\lambda_i} p_\Lambda^i F_{ii} \right) \text{ and } e^2 = e.$$

Therefore, the centralizer of R^Λ is isomorphic to $eCl_{k+l(\lambda)}e$.

Let Cl_Λ be the subalgebra of Cl_k generated by the $p_i - p_j$ for i, j that belong to the same row of Λ . Then it is not difficult to see that $eCl_{k+l(\lambda)}e \cong Cl_\Lambda$ and this proves that the centralizer of R^Λ is isomorphic to $Cl_{k-l(\lambda)}$.

Let us prove now that the Specht module is actually equal to R^Λ . Indeed, Cl_Λ naturally acts on R^Λ by left multiplications and this action commutes with that of \mathfrak{A}_k . Hence, by the above, any endomorphism of the \mathfrak{A}_k -module R^Λ is of the form $w \mapsto fw$, where $w \in R^\Lambda$ and $f \in Cl_\Lambda$, and the algebra of such endomorphisms is generated by multiplications by $p_\alpha - p_\beta$, where α and β lie in the same row of Λ .

To prove that the Specht module is equal to R^Λ , it suffices to verify that every endomorphism of R^Λ fixes the Specht module. By the above, it suffices to consider the homomorphism φ given by multiplication by $p_\alpha - p_\beta$, where α and β lie in the same row of Λ . We have

$$\varphi(p_\alpha - p_\beta)\kappa_T w_\Lambda = (p_\alpha - p_\beta)\kappa_T w_\Lambda = \sqrt{2}\tau_{\alpha\beta}s_{\alpha\beta}\kappa_T w_\Lambda = \sqrt{2}\tau_{\alpha\beta}\kappa_{s_{\alpha\beta}T} w_\Lambda.$$

But the rows of $s_{\alpha\beta}T$ consist of the same elements that constitute the respective rows of Λ . \square

REFERENCES

- [BL] Bernstein J. and Leites D., The superalgebra $Q(n)$, the odd trace and the odd determinant. C.R. Acad. Bulgare Sci. v. 35, no.3, 1982, 285–286. MR **84c**:17003
- [Ja] James G., *The representation theory of the symmetric groups*, Lect. Notes Math. 682, 1978. MR **80g**:20019
- [Jo1] Jones A., The structure of the Young's symmetrizers for spin representations of the symmetric group. I., J. Algebra, 205, 1998, 626–660. MR **99j**:20017
- [Jo2] Jones A., The structure of the Young's symmetrizers for spin representations of the symmetric group. II., J. Algebra, 213, 1999, 381–404. CMP 99:10
- [JN] Jones A. and Nazarov M., Affine Sergeev algebra and q -analogs of the Young's symmetrizers for projective representations of the symmetric group, Proc. London Math. Soc., 78, 1999, 481–512. CMP 99:09
- [Ju1] Jucys A., Symmetric polynomials and the center of the symmetric group ring, Report Math. Phys. 5, 1974, 107–112. MR **54**:7597
- [Ju2] Jucys A., Factorization of Young's projection operators for symmetric groups, Litovsk. Fiz. Sb. 5, 1971, 1–10. MR **44**:7851
- [L] Leites D., Lie superalgebras. In: Modern Problems of Mathematics. Recent developments, v. 25, VINITI, Moscow, 1984, 3–49 (in Russian; the English translation: JOSMAR, v. 30 (6), 1985, 2481–2512). MR **86f**:17019
- [Ma] Macdonald I., *Symmetric functions and Hall polynomials*, Oxford Univ. Press, 1995. MR **96h**:05207
- [Mu] Murphy G., A new construction of the Young seminormal representations of the symmetric group, J. Algebra 69, 1981, 287–291. MR **82h**:20014

- [N1] Nazarov M., An orthogonal basis in the irreducible projective representations of the symmetric group, *Funct. Anal. Appl.*, 22, no. 2, 1988, 66–68. MR **89f**:20019
- [N2] Nazarov M., Young’s symmetrizers for projective representations of the symmetric group, *Adv. Math.*, 127, no. 2, 1997, 190–257. MR **98m**:20019
- [OV] Okunkov A. and Vershik A., A new approach to representation theory of symmetric groups. *Selecta Math.*, new series, 2, no. 4, 1996, 581–605. MR **99g**:20024
- [Pe] Penkov I., Characters of typical irreducible representations of finite dimensional $q(n)$ -modules (Russian), *Funktional. Anal. i Prilozhen.* 20, no. 1, 1986, 37–45. MR **87j**:17033
- [P] Pragacz P., Algebro-geometric applications of Schur S - and Q - polynomials, *Lect. Notes Math.* 1478, 1991, 130–191. MR **93h**:05170
- [R] Ram A., Seminormal representations of Weyl groups and Iwahori–Hecke algebras, *Proc. London Math. Soc.* (3), 75, 1997, 99–133. MR **98d**:20007
- [Sch] Schepochkina I., Maximal subalgebras of matrix Lie superalgebras, hep-th/9702122.
- [S1] Sergeev A., The tensor algebra of the standard representation as a module over the Lie superalgebras $GL(n, m)$ and $Q(n)$, *Math. Sbornik*, 123 (165), no.3, 1984, 422–430. MR **85h**:17010
- [S2] Sergeev A., The centre of the enveloping algebra for the Lie superalgebra $Q(n, \mathbf{C})$, *Lett. Math. Phys.*, 7, 1983, 177–179. MR **85i**:17004
- [S3] Sergeev A., Irreducible representations of solvable Lie superalgebras, math.RT/9810109.
- [W] Weyl H., *Classical groups, their invariants and representations*, Princeton Univ. Press, Princeton, 1939. MR **1**:42c
- [Ya1] Yamaguchi M., A duality of the twisted group algebra of the symmetric group and a Lie superalgebra, math.RT/9811090.
- [Ya2] Yamaguchi M., A duality of the twisted group algebra of the hyperoctahedral group and the queer Lie superalgebra, math.RT/9903159.

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