

THE ADJOINT REPRESENTATION OF A REDUCTIVE GROUP AND HYPERPLANE ARRANGEMENTS

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ABSTRACT. Let G be a connected reductive algebraic group with Lie algebra \mathfrak{g} defined over an algebraically closed field, k , with $\text{char } k = 0$. Fix a parabolic subgroup of G with Levi decomposition $P = LU$ where U is the unipotent radical of P . Let $\mathfrak{u} = \text{Lie}(U)$ and let \mathfrak{z} denote the center of $\text{Lie}(L)$. Let T be a maximal torus in L with Lie algebra \mathfrak{t} . Then the root system of $(\mathfrak{g}, \mathfrak{t})$ is a subset of \mathfrak{t}^* and by restriction to \mathfrak{z} , the roots of \mathfrak{t} in \mathfrak{u} determine an arrangement of hyperplanes in \mathfrak{z} we denote by \mathcal{A}^3 . In this paper we construct an isomorphism of graded $k[\mathfrak{z}]$ -modules $\text{Hom}_G(\mathfrak{g}^*, k[G \times^P (\mathfrak{z} + \mathfrak{u})]) \cong D(\mathcal{A}^3)$, where $D(\mathcal{A}^3)$ is the $k[\mathfrak{z}]$ -module of derivations of \mathcal{A}^3 . We also show that $\text{Hom}_G(\mathfrak{g}^*, k[G \times^P (\mathfrak{z} + \mathfrak{u})])$ and $k[\mathfrak{z}] \otimes \text{Hom}_G(\mathfrak{g}^*, k[G \times^P \mathfrak{u}])$ are isomorphic graded $k[\mathfrak{z}]$ -modules, so $D(\mathcal{A}^3)$ and $k[\mathfrak{z}] \otimes \text{Hom}_G(\mathfrak{g}^*, k[G \times^P \mathfrak{u}])$ are isomorphic, graded $k[\mathfrak{z}]$ -modules. It follows immediately that \mathcal{A}^3 is a free hyperplane arrangement. This result has been proved using case-by-case arguments by Orlik and Terao. By keeping track of the gradings involved, and recalling that \mathfrak{g} affords a self-dual representation of G , we recover a result of Sommers, Trapa, and Broer which states that the degrees in which the adjoint representation of G occurs as a constituent of the graded, rational G -module $k[G \times^P \mathfrak{u}]$ are the exponents of \mathcal{A}^3 . This result has also been proved, again using case-by-case arguments, by Sommers and Trapa and independently by Broer.

1. INTRODUCTION

Suppose k is an algebraically closed field with characteristic zero and G is a connected, reductive, algebraic group defined over k with Lie algebra \mathfrak{g} . Let \mathcal{N} denote the cone of nilpotent elements in \mathfrak{g} . Then G acts on \mathfrak{g} by the adjoint representation and \mathcal{N} is a closed, G -invariant subvariety, so G acts on $k[\mathcal{N}]$, the ring of regular functions on \mathcal{N} . Since \mathcal{N} is a cone, $k[\mathcal{N}]$ inherits a grading from $k[\mathfrak{g}]$ and each homogeneous component, $k[\mathcal{N}]_j$, is a G -stable subspace. Kostant [5] proved that \mathfrak{g} occurs as a constituent of $k[\mathcal{N}]_j$ if and only if j is an exponent of the Weyl group of G . Precisely, he proved the equality of polynomials in q :

$$\sum_{j \geq 0} \dim \text{Hom}_G(\mathfrak{g}, k[\mathcal{N}]_j) q^j = q^{n_1} + \cdots + q^{n_l}$$

where $\{n_1, \dots, n_l\}$ is the multiset of exponents of W (see [7, §6.2] for a definition of the exponents of W).

Sommers and Trapa [9] and also Broer [4] have generalized this result as follows. Fix a maximal torus, T , and a parabolic subgroup, P , of G with $T \subseteq P$. Let U denote the unipotent radical of P and let L be a Levi subgroup of P containing

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T . We follow the convention that lower case Fraktur letters will denote the Lie algebra of the group denoted by the same upper case Roman letter, so for example $\mathfrak{t} = \text{Lie}(T)$, and $\mathfrak{u} = \text{Lie}(U)$. Let \mathfrak{z} denote the center of \mathfrak{l} , so $\mathfrak{z} \subseteq \mathfrak{t}$. Then \mathfrak{z} acts on \mathfrak{u} by restricting the adjoint representation of \mathfrak{g} and the weights of \mathfrak{z} on \mathfrak{u} are the restrictions to \mathfrak{z} of the weights of \mathfrak{t} on \mathfrak{u} . The kernels of these weights determine a hyperplane arrangement in \mathfrak{z} we denote by $(\mathfrak{z}, \mathcal{A}^{\mathfrak{z}})$. Orlik and Terao [8] have proven, using case-by-case arguments, that $\mathcal{A}^{\mathfrak{z}}$ is a free hyperplane arrangement and Orlik and Solomon [6] have computed the exponents of $\mathcal{A}^{\mathfrak{z}}$ in all cases.

Now P acts on \mathfrak{u} and so we can form the associated fibre bundle, $G \times^P \mathfrak{u}$, over G/P with fibre \mathfrak{u} . Then G acts on $G \times^P \mathfrak{u}$ by left multiplication and so G acts on $k[G \times^P \mathfrak{u}]$, the ring of global regular functions on $G \times^P \mathfrak{u}$. Moreover, $k[G \times^P \mathfrak{u}]$ inherits a grading from the scalar action of k on \mathfrak{u} , $k[G \times^P \mathfrak{u}] \cong \bigoplus_{j \geq 0} k[G \times^P \mathfrak{u}]_j$, and each homogeneous component is a G -stable subspace. Sommers and Trapa [9], and independently Broer [4], have proven that \mathfrak{g} occurs as a constituent of $k[G \times^P \mathfrak{u}]_j$ if and only if j is an exponent of $\mathcal{A}^{\mathfrak{z}}$. Precisely, they prove the equality of polynomials in q :

$$(1.1) \quad \sum_{j \geq 0} \dim \text{Hom}_G(\mathfrak{g}, k[G \times^P \mathfrak{u}]_j) q^j = q^{n_1^{\mathfrak{z}}} + \cdots + q^{n_r^{\mathfrak{z}}}$$

where $\{n_1^{\mathfrak{z}}, \dots, n_r^{\mathfrak{z}}\}$ is the multiset of exponents of $\mathcal{A}^{\mathfrak{z}}$ as defined in [7, Definition 4.25].

In the special case when P is a Borel subgroup of G , then $\mathfrak{z} = \mathfrak{t}$, the exponents of $\mathcal{A}^{\mathfrak{t}}$ are the exponents of the Weyl group of G , and $k[\mathcal{N}]_j$ and $k[G \times^P \mathfrak{u}]_j$ afford equivalent representations of G for $j \geq 0$, so we recover Kostant's result.

The proofs given by Sommers and Trapa and by Broer of the equality (1.1) are both case-by-case arguments, with separate arguments for each type of root system.

In this paper, our main result is that there is an isomorphism of graded $k[\mathfrak{z}]$ -modules $\Gamma: \text{Hom}_G(\mathfrak{g}^*, k[G \times^P(\mathfrak{z} + \mathfrak{u})]) \rightarrow D(\mathcal{A}^{\mathfrak{z}})$, where $D(\mathcal{A}^{\mathfrak{z}})$ is the $k[\mathfrak{z}]$ -module of derivations of $\mathcal{A}^{\mathfrak{z}}$. We also show that there is an isomorphism of graded $k[\mathfrak{z}]$ -modules between $\text{Hom}_G(\mathfrak{g}^*, k[G \times^P(\mathfrak{z} + \mathfrak{u})])$ and $k[\mathfrak{z}] \otimes \text{Hom}_G(\mathfrak{g}^*, k[G \times^P \mathfrak{u}])$. Thus $D(\mathcal{A}^{\mathfrak{z}})$ and $k[\mathfrak{z}] \otimes \text{Hom}_G(\mathfrak{g}^*, k[G \times^P \mathfrak{u}])$ are isomorphic, graded $k[\mathfrak{z}]$ -modules, so by keeping track of the gradings involved, and recalling that \mathfrak{g} affords a self-dual representation of G , we see that the isomorphism Γ explains the computation, (1.1), of Sommers, Trapa, and Broer.

Our proof that $\text{Hom}_G(\mathfrak{g}^*, k[G \times^P(\mathfrak{z} + \mathfrak{u})])$ and $D(\mathcal{A}^{\mathfrak{z}})$ are isomorphic, graded $k[\mathfrak{z}]$ -modules does not involve any case-by-case computations. We define Γ explicitly and then show, using that the set of semisimple elements is dense in \mathfrak{g} and some results on the P -orbit structure of $\mathfrak{z} + \mathfrak{u}$, that it's a bijection.

Since $k[\mathfrak{z}] \otimes \text{Hom}_G(\mathfrak{g}^*, k[G \times^P \mathfrak{u}])$ and $D(\mathcal{A}^{\mathfrak{z}})$ are isomorphic $k[\mathfrak{z}]$ -modules, it follows immediately that the arrangement $\mathcal{A}^{\mathfrak{z}}$ is free. We thus obtain a proof, free of case-by-case considerations, of the result of Orlik and Terao [8] that the restrictions of the reflection arrangements arising from Weyl groups are free.

After learning of our ideas about the definition of Γ , Broer [3] has given another proof that it's surjective, the difficult part of showing that it's an isomorphism. His proof is elegant, using Saito's criterion, but relies on a previous result of his concerning sums of exponents. Our proof that Γ is surjective is completely different than Broer's and uses only some basic algebraic geometry and properties of root systems and algebraic groups.

Moreover, our technique can be extended to give a new proof of a theorem of Broer's [2, Theorem 1] that characterizes when the restriction mapping from $\text{Mor}_G(\mathfrak{g}, V)$ to $\text{Mor}_W(\mathfrak{t}, V^T)$ is an isomorphism for a rational G -module V . In a subsequent paper we hope to give a parabolic analog of Broer's theorem, where \mathfrak{g} is replaced by $\text{Ad}G(\mathfrak{z} + \mathfrak{u})$ (the closure of a Dixmier sheet in \mathfrak{g}), \mathfrak{t} is replaced by \mathfrak{z} , and V^T is replaced by V^L .

The rest of this paper is organized as follows. In §2 we collect several results about hyperplane arrangements in the form we need later; in §3 we discuss associated fibre bundles over G/P ; in §4 we collect some results about the adjoint action of U on $\mathfrak{z} + \mathfrak{u}$; in §5 we construct Γ and prove it's an isomorphism; and in §6 we explain how the fact that Γ is an isomorphism implies the results discussed above.

Finally, we use the following notation throughout this paper. If X and Y are varieties defined over k , then $k[X]$ denotes the k -algebra of global regular functions on X and $\text{Mor}(X, Y)$ denotes the set of all morphisms from X to Y . If a group, G , acts on X and Y , then $\text{Mor}_G(X, Y)$ denotes the set of all G -equivariant morphisms from X to Y . If V and W are k -vector spaces, then $V \otimes W$ and $\text{Hom}(V, W)$ denote $V \otimes_k W$ and $\text{Hom}_k(V, W)$ respectively. If V and W afford representations of G , then $\text{Hom}_G(V, W)$ denotes the set of all G -equivariant linear transformations from V to W .

2. HYPERPLANE ARRANGEMENTS

In this paper a "hyperplane arrangement" is a pair, (V, \mathcal{A}) , where V is a finite dimensional k -vector space and \mathcal{A} is a finite multiset of hyperplanes in V . This slight generalization of the usual meaning has no effect on the invariants of arrangements we consider.

Suppose (V, \mathcal{A}) is an arrangement with $\dim V = n$. Let $\{\mu_j \mid 1 \leq j \leq m\}$ be a multiset of linear functionals so that $\mathcal{A} = \{\ker \mu_1, \dots, \ker \mu_m\}$. Fix a basis, $\{e_1, \dots, e_n\}$, of V and let $\{x_1, \dots, x_n\}$ be the dual basis of V^* . Then for $1 \leq i \leq m$ we may write $\mu_j = \sum_{i=1}^n \xi_{i,j} x_i$ where the $\xi_{i,j}$'s are in k . We may identify $k[V]$ with the polynomial ring $k[x_1, \dots, x_n]$ and if ∂_i denotes partial differentiation with respect to x_i , then $\text{Der } k[V]$, the $k[V]$ -module of k -linear derivations of $k[V]$, has basis $\{\partial_1, \dots, \partial_n\}$.

With pointwise operations, $\text{Mor}(V, V)$ is naturally a $k[V]$ -module, isomorphic to $k[V] \otimes V$. Suppose $\theta = \sum_{i=1}^n f_i \partial_i$ is in $\text{Der } k[V]$. Define $\phi: V \rightarrow V$ by $\phi(v) = \sum_{i=1}^n f_i(v) e_i$. Then it is easily seen that ϕ is in $\text{Mor}(V, V)$ and that the mapping $\theta \mapsto \phi$ is an isomorphism of $k[V]$ -modules.

Recall that the module of \mathcal{A} -derivations, $D(\mathcal{A})$, is the $k[V]$ -module of all derivations, θ , of $k[V]$ with the property that $\theta(q)$ is in $k[V]q$ where $q = \prod_{j=1}^m \mu_j$ is the defining polynomial of \mathcal{A} . It's known that $D(\mathcal{A})$ may be characterized as the set of all θ in $\text{Der } k[V]$ with the property that $\theta(\mu_i)$ is in $k[V]\mu_i$ for $1 \leq i \leq m$ [7, Proposition 4.8]. If θ and ϕ are as in the preceding paragraph, then θ is in $D(\mathcal{A})$ if and only if $\sum_{i=1}^n \xi_{i,j} f_i$ is in $k[V]\mu_j$ for $1 \leq j \leq m$. On the other hand, if $\phi^\#$ is the comorphism of ϕ , then $\phi^\#(\mu_j) = \sum_{i=1}^n \xi_{i,j} f_i$. Therefore, the image of $D(\mathcal{A})$ under the isomorphism in the preceding paragraph is the set of all morphisms, $\phi: V \rightarrow V$, with the property that $\phi^\#(\mu_j)$ is in $k[V]\mu_j$ for $1 \leq j \leq m$.

In the rest of this paper, we'll use only morphisms, and not derivations, so from now on, we'll abuse notation and identify $D(\mathcal{A})$ with its image in $\text{Mor}(V, V)$, so $D(\mathcal{A}) = \{\phi \in \text{Mor}(V, V) \mid \phi^\#(\mu_j) \in k[V]\mu_j, 1 \leq j \leq m\}$.

Next, recall that the lattice of \mathcal{A} , $L(\mathcal{A})$, is the set of subspaces of V that arise as intersections of hyperplanes in \mathcal{A} . Suppose X is in the lattice of \mathcal{A} . Then if $\ker \mu_j$ is in \mathcal{A} , either $X \subseteq \ker \mu_j$ or $X \cap \ker \mu_j$ is a hyperplane in X . The “restricted” arrangement, (X, \mathcal{A}^X) , is defined to be the arrangement in X consisting of all hyperplanes, $X \cap \ker \mu_j$, where $X \not\subseteq \ker \mu_j$. Let

$$X_{\text{reg}} = \{v \in X \mid v \in \ker \mu_j \text{ if and only if } X \subseteq \ker \mu_j, 1 \leq j \leq m\}.$$

Then V is the disjoint union of the X_{reg} ’s as X runs through $L(\mathcal{A})$. Thus, given v in V we can define X_v to be the unique subspace in $L(\mathcal{A})$ with v in $(X_v)_{\text{reg}}$.

Proposition 2.1. *Suppose $\phi: V \rightarrow V$ is a morphism. Then the following are equivalent:*

1. ϕ is in $D(\mathcal{A})$,
2. $\phi(v) \in X_v$ for all v in V ,
3. $\phi(X) \subseteq X$ for all X in $L(\mathcal{A})$.

Proof. Suppose first that ϕ is in $D(\mathcal{A})$. Fix v in V . If $X_v = X$, then the result is obvious. Suppose $X_v \neq X$. Then relabeling if necessary we may assume that v is in $\ker \mu_j$ if and only if $1 \leq j \leq r$, for some r with $1 \leq r \leq m$. Then $X \subseteq \ker \mu_j$ if and only if $1 \leq j \leq r$ and $X_v = \bigcap_{j=1}^r \ker \mu_j$. By assumption, for $1 \leq j \leq r$, $\mu_j(v) = 0$ implies that $\mu_j \phi(v) = 0$, so $\phi(v)$ is in $\bigcap_{j=1}^r \ker \mu_j = X_v$.

The second statement clearly implies the third since if v is in X , then X_v is a subset of X .

Finally, suppose $\phi(X) \subseteq X$ for all X in $L(\mathcal{A})$. Then $\phi(\ker \mu_j) \subseteq \ker \mu_j$ for $1 \leq j \leq m$. It follows that $\ker \mu_j$ is contained in the zero set of $\phi^\#(\mu_j)$ and so by the Nullstellensatz, some power of $\phi^\#(\mu_j)$ is a multiple of μ_j . Since μ_j is irreducible, it must be the case that $\phi^\#(\mu_j)$ is a multiple of μ_j . Therefore ϕ is in $D(\mathcal{A})$. \square

3. FIBRE BUNDLES

In this section, G is an affine algebraic group and P is a closed subgroup of G .

Suppose W is a finite dimensional k -vector space that affords a rational representation of P . Then P acts on $G \times W$ by $p(g, w) = (gp^{-1}, pw)$ for p in P , g in G , and w in W . We can form the associated fibre bundle, $G \times^P W$, over G/P , which as a set is the set of P -orbits on $G \times W$. For g in G and w in W , $g * w$ will denote the image of (g, w) in $G \times^P W$.

For a non-negative integer, j , let $k[W]_j$ denote the vector space of all homogeneous, degree j polynomial functions on W . Since P acts linearly on W , $k[W]_j$ is a P -invariant subspace of $k[W]$. The grading, $k[W] \cong \bigoplus_{j \geq 0} k[W]_j$, induces a grading, $(k[G] \otimes k[W])^P \cong \bigoplus_{j \geq 0} (k[G] \otimes k[W]_j)^P$, of $(k[G] \otimes k[W])^P$. The natural map from $(k[G] \otimes k[W])^P$ to $k[G \times^P W]$ is an isomorphism of k -algebras and it’s straightforward to check that the image of $(k[G] \otimes k[W]_j)^P$ in $k[G \times^P W]$ is

$$\{f \in k[G \times^P W] \mid f(g * \xi w) = \xi^j f(g * w) \text{ for all } g \in G, w \in W, \xi \in k\}.$$

Denoting this last set by $k[G \times^P W]_j$ we see that $k[G \times^P W]$ is graded by $k[G \times^P W] = \bigoplus_{j \geq 0} k[G \times^P W]_j$.

Left multiplication defines a regular action of G on $G \times^P W$ and so the coordinate ring, $k[G \times^P W]$, affords a representation of G . Clearly each $k[G \times^P W]_j$ is a G -invariant subspace. Moreover, if $f = \sum_{j \geq 0} f_j$ is in $k[G \times^P W]^G$ with f_j

in $k[G \times^P W]_j$, then for fixed g and w , and any ξ in k , $f(g * \xi w)$ is a polynomial in ξ with coefficients $f_j(g * w)$. Since $f(g * \xi w) = f(1 * \xi w)$, it follows that the coefficients don't depend on g and so each f_j is in $k[G \times^P W]^G$. Therefore, $k[G \times^P W]^G \cong \bigoplus_{j \geq 0} k[G \times^P W]_j^G$ is a graded k -algebra.

Now suppose that V is a finite dimensional k -vector space that affords a rational representation of G . It's easy to see that with pointwise operations $\text{Mor}_G(G \times^P W, V)$ is a $k[G \times^P W]^G$ -module. Also, $\text{Mor}_G(G \times^P W, V)$ is isomorphic to the vector space of all G -equivariant linear transformations from V^* to $k[G \times^P W]$ since V is an affine space. Thus, $\text{Mor}_G(G \times^P W, V) \cong \text{Hom}_G(V^*, k[G \times^P W])$. It's straightforward to check that

$$\text{Hom}_G(V^*, k[G \times^P W]) \cong \bigoplus_{j \geq 0} \text{Hom}_G(V^*, k[G \times^P W]_j)$$

and that the image of $\text{Hom}_G(V^*, k[G \times^P W]_j)$ in $\text{Mor}_G(G \times^P W, V)$ is

$$\{ \phi \in \text{Mor}_G(G \times^P W, V) \mid \phi(g * \xi w) = \xi^j \phi(g * w) \text{ for all } g \in G, w \in W, \xi \in k \}.$$

Thus, if we define $\text{Mor}_G(G \times^P W, V)_j$ to be this last vector space, it follows that $\text{Mor}_G(G \times^P W, V) \cong \bigoplus_{j \geq 0} \text{Mor}_G(G \times^P W, V)_j$. Moreover, it's easy to see that in fact $\text{Mor}_G(G \times^P W, V)$ is a graded $k[G \times^P W]^G$ -module.

Similarly, again since P acts linearly on W , $k[W]^P \cong \bigoplus_{j \geq 0} k[W]_j^P$ and if

$$\text{Mor}_P(W, V)_j = \{ \psi \in \text{Mor}_P(W, V) \mid \psi(\xi w) = \xi^j \psi(w) \text{ for all } w \in W, \xi \in k \},$$

then $\text{Mor}_P(W, V) \cong \bigoplus_{j \geq 0} \text{Mor}_P(W, V)_j$ and therefore $\text{Mor}_P(W, V)$ is a graded $k[W]^P$ -module.

Let $\tau: W \rightarrow G \times^P W$ be the "inclusion" of the fibre over P defined by $\tau(w) = 1 * w$ for w in W , so τ is a morphism. Suppose ϕ is in $\text{Mor}_G(G \times^P W, V)$. Then, for p in P and w in W ,

$$p\phi\tau(w) = \phi(p * w) = \phi(1 * pw) = \phi\tau(pw),$$

so $\phi\tau$ is in $\text{Mor}_P(W, V)$, the $k[W]^P$ -module of all P -equivariant morphisms from W to V . We need the following version of Frobenius reciprocity.

Proposition 3.1. *The mapping $\phi \mapsto \phi\tau$ defines an isomorphism of graded vector spaces between $\text{Mor}_G(G \times^P W, V)$ and $\text{Mor}_P(W, V)$.*

Proof. First, suppose ϕ is in $\text{Mor}_G(G \times^P W, V)_j$. Then for w in W and ξ in k ,

$$\phi\tau(\xi w) = \phi(1 * \xi w) = \xi^j \phi(1 * w) = \xi^j \phi\tau(w),$$

so $\phi\tau$ is in $\text{Mor}_P(W, V)_j$.

It's straightforward to check that the mapping is a homomorphism of graded vector spaces. To show that it's injective, suppose ϕ_1 and ϕ_2 are in $\text{Mor}_G(G \times^P W, V)$ and $\phi_1\tau = \phi_2\tau$. Then for g in G and w in W ,

$$\begin{aligned} \phi_1(g * w) &= \phi_1(1 * w) \\ &= \phi_1\tau(w) \\ &= \phi_2\tau(w) \\ &= \phi_2(g * w), \end{aligned}$$

so $\phi_1 = \phi_2$.

To show the mapping is surjective, suppose $\tilde{\phi}$ is in $\text{Mor}_P(W, V)$. We define $\phi: G \times^P W \rightarrow V$ by $\phi(g * w) = g\tilde{\phi}(w)$. Then ϕ is easily seen to be a well-defined morphism and clearly $\phi\tau = \tilde{\phi}$. If g_1 is in G , then

$$\begin{aligned} \phi(g_1(g * w)) &= \phi(g_1g * w) \\ &= g_1g\tilde{\phi}(w) \\ &= g_1\phi(g * w), \end{aligned}$$

so ϕ is G -equivariant. This completes the proof of the proposition. □

Corollary 3.2. *The mapping $f \mapsto f\tau$ defines an isomorphism of graded k -algebras between $k[G \times^P W]^G$ and $k[W]^P$.*

Proof. This is just the preceding proposition when $V = k$ with the trivial action of G . □

Notice that if ϕ is in $\text{Mor}_G(G \times^P W, V)$ and f is in $k[G \times^P W]^G$, then $(f\phi) \circ \tau = (f \circ \tau)(\phi \circ \tau)$, so the isomorphism above intertwines the $k[G \times^P W]^G$ -module structure on $\text{Mor}_G(G \times^P W, V)$ and the $k[W]^P$ -module structure on $\text{Mor}_P(W, V)$.

Now suppose that W_1 is a subspace of W with the property that $pw - w$ is in W_1 for all p in P and w in W . Notice that W_1 is a P -stable subspace of W and that P acts trivially on W/W_1 . Let W_0 be a complement to W_1 in W . Then we may identify $k[W]$ and $k[W_0] \otimes k[W_1]$. If $k[W_0]$ is given the trivial P -action and P acts diagonally on $k[W_0] \otimes k[W_1]$, then this identification is P -equivariant.

Let $\theta: G \times^P W \rightarrow W_0$ by $\theta(g * (w_0 + w_1)) = w_0$. Notice that θ is well-defined since P acts trivially on W/W_1 . The comorphism, $\theta^\#: k[W_0] \rightarrow k[G \times^P W]$ defines the structure of a graded $k[W_0]$ -module on $k[G \times^P W]$.

Since P acts trivially on $k[W_0]$ in the decomposition $k[W] \cong k[W_0] \otimes k[W_1]$ we have G -equivariant isomorphisms of graded $k[W_0]$ -modules,

$$\begin{aligned} k[G \times^P W] &\cong (k[G] \otimes k[W_0] \otimes k[W_1])^P \\ &\cong k[W_0] \otimes (k[G] \otimes k[W_1])^P \\ &\cong k[W_0] \otimes k[G \times^P W_1], \end{aligned}$$

where G acts trivially on $k[W_0]$ and diagonally on $k[W_0] \otimes k[G \times^P W_1]$.

More generally, using the isomorphism in the last paragraph, we have isomorphisms of graded $k[W_0]$ -modules,

$$\begin{aligned} \text{Mor}_G(G \times^P W, V) &\cong \text{Hom}_G(V^*, k[G \times^P W]) \\ &\cong (V \otimes k[W_0] \otimes k[G \times^P W_1])^G \\ &\cong k[W_0] \otimes (V \otimes k[G \times^P W_1])^P \\ &\cong k[W_0] \otimes \text{Hom}_G(V^*, k[G \times^P W_1]). \end{aligned}$$

This proves the next lemma.

Lemma 3.3. *There is an isomorphism graded $k[W_0]$ -modules,*

$$\text{Mor}_G(G \times^P W, V) \cong k[W_0] \otimes \text{Hom}_G(V^*, k[G \times^P W_1]).$$

Proposition 3.4. *The comorphism, $\theta^\#: k[W_0] \rightarrow k[G \times^P W]$, is an injective homomorphism of graded k -algebras. Moreover, if PW_0 is a dense subvariety of W ,*

then $\theta^\#(k[W_0]) = k[G \times^P W]^G$. Thus, if PW_0 is dense in W , then $k[W_0]$ and $k[G \times^P W]^G$ are isomorphic, graded k -algebras.

Proof. It's easy to see that $\theta^\#$ is an injective homomorphism of graded k -algebras and that $\theta^\#(k[W_0]) \subseteq k[G \times^P W]^G$. Now suppose that PW_0 is dense in W . For f in $k[W_0]^G$, define $f_1: W_0 \rightarrow k$ by $f_1(w_0) = f(1 * w_0)$, so f_1 is in $k[W_0]$. Suppose g is in G , p is in P , and w_0 is in W_0 . Then,

$$\begin{aligned} f_1\theta(g * pw_0) &= f_1\theta(gp * w_0) \\ &= f_1(w_0) \\ &= f(1 * w_0) \\ &= f(gp * w_0) \\ &= f(g * pw_0). \end{aligned}$$

Therefore $f_1\theta$ and f agree on

$$G * PW_0 = \{ g * pw_0 \mid g \in G, p \in P, w_0 \in W_0 \}.$$

Since PW_0 is dense in W , $G * PW_0$ is dense in $G \times^P W$, and so $f_1\theta = f$. Therefore $\theta^\#(f_1) = f$ and hence $k[G \times^P W]^G \subseteq \theta^\#(k[W_0])$. This completes the proof of the proposition. \square

4. THE ACTION OF U ON $\mathfrak{z} + \mathfrak{u}$

For the rest of this paper, we return to the notation of the introduction: G is a connected, reductive, algebraic group; T is a maximal torus in G ; P is a parabolic subgroup of G with Levi decomposition $P = LU$; and \mathfrak{z} is the center of \mathfrak{l} . Also, let Φ denote the root system of $(\mathfrak{g}, \mathfrak{t})$ and let Φ_u be the subset of Φ consisting of the weights of \mathfrak{t} on \mathfrak{u} .

Lemma 4.1. *Suppose l is a semisimple element in \mathfrak{l} . Define $a_l: U \times Z_u(l) \rightarrow l + \mathfrak{u}$ by $a_l(u, z) = \text{Adu}(l + z)$. Then a_l is surjective with fibres isomorphic to $Z_U(l)$.*

Proof. To show that a_l is surjective, suppose n is in \mathfrak{u} . Let $l + n = (l + n)_s + (l + n)_n$ be the Jordan decomposition on $l + n$. Then Borho [1, §2] has shown that there is a u in U with $\text{Adu}(l) = (l + n)_s$. Set $z = \text{Adu}^{-1}((l + n)_n)$, so $\text{Adu}(z) = (l + n)_n$. Then clearly $\text{Adu}(l + z) = l + n$. Moreover, $[\text{Adu}(l), \text{Adu}(z)] = [(l + n)_s, (l + n)_n] = 0$. Thus, z is in $Z_u(l)$ and $a_l(u, z) = l + n$.

Suppose n is in \mathfrak{u} and consider $F = a_l^{-1}(l + n)$. Fix (u, z) in F . Since $\text{Adu}(l + z)$ has Jordan decomposition $\text{Adu}(l + z) = \text{Adu}(l) + \text{Adu}(z)$, if (u_1, z_1) is in $U \times Z_u(l)$, then (u_1, z_1) is in F if and only if $\text{Adu}_1(l) = \text{Adu}(l)$ and $\text{Adu}_1(z_1) = \text{Adu}(z)$. Thus $\rho: F \rightarrow uZ_U(l)$ by $\rho(u_1, z_1) = u_1$ is a well-defined morphism. If $\rho(u_1, z_1) = \rho(u_2, z_2)$, then $u_1 = u_2$ and so $\text{Adu}_1(z_1) = \text{Adu}_1(z_2)$, so $z_1 = z_2$ and hence ρ is injective. If v is in $Z_U(l)$, set $u_1 = uv$ and $z_1 = \text{Adv}^{-1}(l + z) - l$. It's straightforward to check that (u_1, z_1) is in F and so ρ is surjective. Finally, ρ is an isomorphism by Zariski's Main Theorem. \square

Corollary 4.2. *The P -saturation of $\mathfrak{z}_{\text{reg}}$, $\text{Ad}P(\mathfrak{z}_{\text{reg}})$, is $\mathfrak{z}_{\text{reg}} + \mathfrak{u}$, so $\text{Ad}P(\mathfrak{z}_{\text{reg}})$ is a dense open subvariety of $\mathfrak{z} + \mathfrak{u}$.*

Proof. Recall that $(\mathfrak{z}, \mathcal{A}^3)$ is an arrangement in \mathfrak{z} , so $\mathfrak{z}_{\text{reg}}$ is defined as in §2. It's easy to see that $\mathfrak{z}_{\text{reg}} = \{ t \in \mathfrak{t} \mid \alpha(t) \neq 0 \text{ for all } \alpha \in \Phi_u \}$ and hence that t is in $\mathfrak{z}_{\text{reg}}$ if and only if $Z_u(t) = 0$. If t is in $\mathfrak{z}_{\text{reg}}$, then it follows from Lemma 4.1 that $\text{Ad}P(t) = t + \mathfrak{u}$, so $\text{Ad}P(\mathfrak{z}_{\text{reg}}) = \mathfrak{z}_{\text{reg}} + \mathfrak{u}$. \square

For α in Φ let \mathfrak{g}_α denote the α -weight space in \mathfrak{g} and fix a non-zero root vector, e_α in \mathfrak{g}_α . Also, let U_α be the corresponding root subgroup in G and suppose $x_\alpha: k \rightarrow U_\alpha$ is a fixed isomorphism from k to U_α satisfying $\text{Ad}x_\alpha(\xi) = \exp(\text{ad}(\xi e_\alpha))$ for ξ in k , where ad is the adjoint representation of \mathfrak{g} .

For α and β in Φ_u , define $\alpha \sim \beta$ if $\ker \alpha|_{\mathfrak{z}} = \ker \beta|_{\mathfrak{z}}$. Clearly this is an equivalence relation on Φ_u . It's also clear that for α and β in Φ_u , $\alpha \sim \beta$ if and only if $\alpha|_{\mathfrak{z}}$ is a scalar multiple of $\beta|_{\mathfrak{z}}$. Let Ψ_1, \dots, Ψ_s be the equivalence classes for this relation on Φ_u .

Notice that each Ψ_i is a closed set of roots (that is, if α and β are in Ψ_i and $\alpha + \beta$ is a root, then $\alpha + \beta$ is in Ψ_i). It follows that if we define \mathfrak{u}_i to be the span of the root vectors e_α with α in Ψ_i and U_i to be the subgroup of U generated by the root subgroups, U_α , where α is in Ψ_i , then \mathfrak{u}_i is a subalgebra of \mathfrak{u} , U_i is a subgroup of U isomorphic to the product $\prod_{\alpha \in \Psi_i} U_\alpha$ (the product can be taken in any order), and $\text{Lie}(U_i) = \mathfrak{u}_i$. Moreover, if $U^i = \prod_{\beta \in \Phi_u \setminus \Psi_i} U_\beta$ (the product taken in some fixed order), then every element in U has a unique factorization $u = u^i u_i$ with u^i in U^i and u_i in U_i .

For $1 \leq i \leq s$, let H_i denote $\ker \alpha|_{\mathfrak{z}}$ for α in Ψ_i , so H_1, \dots, H_s are the distinct hyperplanes in \mathcal{A}^3 . Set $\mathfrak{z}^i = \{t \in \mathfrak{z} \mid \beta(t) \neq 0 \text{ for all } \beta \in \Phi_u \setminus \Psi_i\} = \mathfrak{z} \setminus \bigcup_{i \neq j} H_j$. Then \mathfrak{z}^i is an open subvariety of \mathfrak{z} containing $\mathfrak{z}_{\text{reg}}$. Finally, define $a: U \times (\mathfrak{z} + \mathfrak{u}) \rightarrow \mathfrak{z} + \mathfrak{u}$ by $a(u, t+n) = \text{Adu}(t+n)$ and let a_i denote the restriction of a to $U^i \times (\mathfrak{z}^i + \mathfrak{u}_i)$.

Proposition 4.3. *For $1 \leq i \leq s$, the morphism $a_i: U^i \times (\mathfrak{z}^i + \mathfrak{u}_i) \rightarrow \mathfrak{z} + \mathfrak{u}$ determines an isomorphism between $U^i \times (\mathfrak{z}^i + \mathfrak{u}_i)$ and $\mathfrak{z}^i + \mathfrak{u}$.*

Proof. We show that a_i is injective with image equal to $\mathfrak{z}^i + \mathfrak{u}$. Then by restricting the range we obtain a bijective, birational morphism of normal varieties between $U^i \times (\mathfrak{z}^i + \mathfrak{u}_i)$ and $\mathfrak{z}^i + \mathfrak{u}$ which must be an isomorphism by Zariski's Main Theorem.

To show that a_i is injective, suppose $\text{Adu}(t+n) = \text{Adu}_1(t_1+n_1)$ for u, u_1 in U^i , t, t_1 in \mathfrak{z}^i , and n, n_1 in \mathfrak{u}_i . Then clearly $t = t_1$ and $\text{Adu}_1^{-1}u(t+n) = t+n_1$. Just suppose $u_1^{-1}u$ is not in U_i and choose β in Φ_u with β not in Ψ_i , so that the height of β is minimal and such that $u_1^{-1}u = x_\beta(c)u'$ for some u' in U , with $c \neq 0$. We show that $\text{Ad}x_\beta(c)u'(t+n) = t - c\beta(t) + \sum_{\alpha \neq \beta} r_\alpha e_\alpha$. First, if e_α occurs with a non-zero coefficient in $\text{Adu}'(t)$, then $\alpha = \gamma_1 + \dots + \gamma_m$ with $\gamma_i \neq \beta$ for $1 \leq i \leq m$. If some γ_j are not in Ψ_i , then either $m = 1$ and $\alpha \neq \beta$, or $m > 1$ in which case the height of α is strictly greater than the height of β , so again $\alpha \neq \beta$. If every γ_j is in Ψ_i , then so is their sum, so again $\alpha \neq \beta$. It follows that $\text{Adu}'(t) = t + \sum_{\alpha \neq \beta} s_\alpha e_\alpha$ for some s_α 's in k . Thus $\text{Ad}x_\beta(c)u'(t) = t - c\beta(t) + \sum_{\alpha \neq \beta} s'_\alpha e_\alpha$ where the s'_α 's are in k . Next, if e_α occurs with a non-zero coefficient in $\text{Ad}x_\beta(c)u'(n)$, then again, either α is in Ψ_i in which case $\alpha \neq \beta$, or α is not in Ψ_i in which case the height of α is strictly greater than the height of β , so $\alpha \neq \beta$ in this case either. Thus, e_β does not appear in $\text{Ad}x_\beta(c)u'(n)$. It follows that $\text{Ad}x_\beta(c)u'(t+n) = t - c\beta(t) + \sum_{\alpha \neq \beta} r_\alpha e_\alpha$ as desired. Since $\beta \neq \alpha$, it must be that $c\beta(t) = 0$ and since t is in \mathfrak{z}^i , $\beta(t) \neq 0$, so $c = 0$, a contradiction. It follows that $u_1^{-1}u$ is in U_i and so it follows from the factorization $U = U^i U_i$ that $u = u_1$. Therefore $n = n_1$ also and so a_i is injective.

To show that the image of a_i is $\mathfrak{z}^i + \mathfrak{u}$, suppose that t is in \mathfrak{z}^i . If t is in H_i , then $Z_u(t) = \mathfrak{u}_i$ since t is not in H_j for $j \neq i$, so by Lemma 4.1 given n in \mathfrak{u} , there is a u in U and an n_1 in \mathfrak{u}_i so that $\text{Adu}(t+n_1) = t+n$. Write $u = u^i u_i$ with u^i in U^i and u_i in U_i . Then $\text{Adu}_i(t+n_1)$ is in $t+\mathfrak{u}_i$ (recall that Ψ_i is a closed set of roots), so say $\text{Adu}_i(t+n_1) = t+n'_1$ where n'_1 is in \mathfrak{u}_i . It follows that $t+n = \text{Adu}^i(t+n'_1)$

is in the image of a_i . If t is not in H_i , then t is in $\mathfrak{z}_{\text{reg}}$ and so again by Lemma 4.1, given n in \mathfrak{u} there is a u in U with $\text{Adu}(t) = t + n$. Factor $u = u^i u_i$ as above. Then $\text{Adu}_i(t)$ is in $t + \mathfrak{u}_i$, say $\text{Adu}_i(t) = t + n_1$. Then $t + n = \text{Adu}^i(t + n_1)$, and so $t + n$ is in the image of a_i . \square

For $1 \leq i \leq s$, fix a root, say α_i , in Ψ_i . This set of representatives will remain fixed for the rest of this paper.

Proposition 4.4. *Suppose $1 \leq i \leq s$ and u is in U_i . Then there is an element, n_1 , in \mathfrak{u}_i , so that for t in \mathfrak{z} , $\text{Adu}(t) = t + \alpha_i(t)n_1$.*

Proof. We can write $u = x_{\beta_n}(c_n) \cdots x_{\beta_1}(c_1)$ where the c_j 's are in k and β_1, \dots, β_n are in Ψ_i . We prove the result using induction on n . If $n = 1$, then $\text{Ad}x_{\beta_1}(c_1)(t) = t - \beta_1(t)c_1e_{\beta_1}$. Since β_1 is in Ψ_i , there is a rational number, r , so that $\beta_1|_{\mathfrak{z}} = r\alpha|_{\mathfrak{z}}$. Now we can take $n_1 = -rc_1e_{\beta_1}$.

Suppose that $n > 1$, $u = x_{\beta_n}(c_n) \cdots x_{\beta_1}(c_1)$, and

$$\text{Ad}(x_{\beta_{n-1}}(c_{n-1}) \cdots x_{\beta_1}(c_1))(t) = t + \alpha_i(t)n'_1$$

for some n'_1 in \mathfrak{u}_i and all t in \mathfrak{z} . Then

$$\begin{aligned} \text{Adu}(t) &= \text{Ad}x_{\beta_n}(c_n)(t + \alpha_i(t)n'_1) \\ &= t - \beta_n(t)c_n e_{\beta_n} + \alpha_i(t)\text{Ad}x_{\beta_n}(c_n)n'_1. \end{aligned}$$

Now β_n is in Ψ_i , so $\beta_n|_{\mathfrak{z}} = r\alpha|_{\mathfrak{z}}$ for some rational number, r , and $\text{Ad}x_{\beta_n}(c_n)n'_1 = n''_1$ for some n''_1 in \mathfrak{u}_i . The result follows by taking $n_1 = -c_n r e_{\beta_n} + n''_1$. \square

5. THE ISOMORPHISM

Let $\Phi_{\mathfrak{t}}$ denote the root system of $(\mathfrak{l}, \mathfrak{t})$, so Φ is the disjoint union of $\Phi_{\mathfrak{t}}$, $\Phi_{\mathfrak{u}}$, and $-\Phi_{\mathfrak{u}}$. Recall that $\mathfrak{z} \subseteq \mathfrak{t}$ and that $\mathfrak{z} = \bigcap_{\alpha \in \Phi_{\mathfrak{t}}} \ker \alpha|_{\mathfrak{z}}$. Also, for α in Φ , $\mathfrak{z} \subseteq \ker \alpha$ if and only if $\alpha \in \Phi_{\mathfrak{t}}$. Thus the hyperplanes, $\ker \alpha|_{\mathfrak{z}}$, for α in $\Phi_{\mathfrak{u}}$ determine a hyperplane arrangement in \mathfrak{z} , $(\mathfrak{z}, \mathcal{A}^{\mathfrak{z}})$. As in §2 we consider elements in $D(\mathcal{A}^{\mathfrak{z}})$ as morphisms from \mathfrak{z} to itself satisfying the equivalent conditions of Proposition 2.1.

For t in \mathfrak{z} , set $\Psi_t = \{\alpha \in \Phi_{\mathfrak{u}} \mid \alpha(t) = 0\}$. Then, with the notation of §2, $X_t = \bigcap_{\alpha \in \Psi_t} \ker \alpha|_{\mathfrak{z}}$. Recall that for α in Φ , the root subgroup of G corresponding to α is U_{α} . Then $Z_U(t) = \prod_{\alpha \in \Psi_t} U_{\alpha}$ and $Z_P(t) = LZ_U(t)$.

Let $\sigma: \mathfrak{z} \rightarrow G \times^P(\mathfrak{z} + \mathfrak{u})$ by $\sigma(t) = 1 * t$, so σ is a morphism. Suppose that $\phi: G \times^P(\mathfrak{z} + \mathfrak{u}) \rightarrow \mathfrak{g}$ is a G -equivariant morphism. Then $\phi\sigma: \mathfrak{z} \rightarrow \mathfrak{g}$ is a morphism and since ϕ is G -equivariant and the stabilizer in G of $\sigma(t)$ is $Z_P(t)$, it follows that $\phi\sigma(t)$ is in $\mathfrak{g}^{Z_P(t)}$ for t in \mathfrak{z} . Now $\mathfrak{g}^{Z_P(t)} \subseteq \mathfrak{g}^L = \mathfrak{z}$ and so

$$\mathfrak{g}^{Z_P(t)} = \{t_1 \in \mathfrak{z} \mid \alpha(t_1) = 0 \text{ for all } \alpha \in \Psi_t\} = X_t.$$

Therefore, $\phi\sigma(\mathfrak{z}) \subseteq \mathfrak{z}$ and for t in \mathfrak{z} , $\phi\sigma(t) \in X_t$, so by restricting the range of $\phi\sigma$ to \mathfrak{z} we obtain a morphism, $\Lambda(\phi)$, in $D(\mathcal{A}^{\mathfrak{z}})$.

The rest of this section is devoted to the proof of the following theorem.

Theorem 5.1. *The mapping $\Lambda: \text{Mor}_G(G \times^P(\mathfrak{z} + \mathfrak{u}), \mathfrak{g}) \rightarrow D(\mathcal{A}^{\mathfrak{z}})$ is an isomorphism of graded $k[\mathfrak{z}]$ -modules.*

It's well-known that P acts trivially on $(\mathfrak{z} + \mathfrak{u})/\mathfrak{u}$. Therefore, we can apply the results in §3 with $W = \mathfrak{z} + \mathfrak{u}$, $W_0 = \mathfrak{z}$, and $W_1 = \mathfrak{u}$, and conclude that $\text{Mor}_G(G \times^P(\mathfrak{z} + \mathfrak{u}), \mathfrak{g})$ has the structure of a graded $k[\mathfrak{z}]$ -module via the comorphism

of $\theta: G \times^P(\mathfrak{z} + \mathfrak{u}) \rightarrow \mathfrak{z}$. Moreover, by Corollary 4.2, $\text{Ad}P(\mathfrak{z})$ is dense in $\mathfrak{z} + \mathfrak{u}$, and so $k[\mathfrak{z}]$ and $k[G \times^P(\mathfrak{z} + \mathfrak{u})]^G$ are isomorphic graded k -algebras by Proposition 3.4.

It's straightforward to check that Λ is a homomorphism of $k[\mathfrak{z}]$ -modules. Moreover, if ϕ is in $\text{Mor}_G(G \times^P(\mathfrak{z} + \mathfrak{u}), \mathfrak{g})_j$ for some non-negative integer, j , then

$$\Lambda(\phi(\xi t)) = \phi(1 * \xi t) = \xi^j \Lambda(\phi)(t)$$

for t in \mathfrak{z} and ξ in k , so $\Lambda(\phi)$ is in $D(\mathcal{A}^3)_j$. Thus Λ is a homomorphism of graded $k[\mathfrak{z}]$ -modules.

To show that Λ is injective, suppose ϕ_1 and ϕ_2 are in $\text{Mor}_G(G \times^P(\mathfrak{z} + \mathfrak{u}), \mathfrak{g})$ with $\Lambda(\phi_1) = \Lambda(\phi_2)$. Define $G * (\mathfrak{z}_{\text{reg}} + \mathfrak{u}) = \{g * (t + n) \mid g \in G, t \in \mathfrak{z}_{\text{reg}}, n \in \mathfrak{u}\}$. Then if $g * (t + n)$ is in $G * (\mathfrak{z}_{\text{reg}} + \mathfrak{u})$, by Lemma 4.1 there is a u in U with $\text{Adu}(t) = t + n$ and so

$$\begin{aligned} \phi_1(g * (t + n)) &= \phi_1(gu * t) \\ &= \phi_1(1 * t) \\ &= \phi_1\sigma(t) \\ &= \phi_2\sigma(t) \\ &= \phi_2(gu * t) \\ &= \phi_2(g * (t + n)). \end{aligned}$$

Therefore, ϕ_1 and ϕ_2 agree on $G * (\mathfrak{z}_{\text{reg}} + \mathfrak{u})$. Since $\mathfrak{z}_{\text{reg}} + \mathfrak{u}$ is a dense subvariety of $\mathfrak{z} + \mathfrak{u}$, $G * (\mathfrak{z}_{\text{reg}} + \mathfrak{u})$ is a dense subvariety of $G \times^P(\mathfrak{z} + \mathfrak{u})$ and so $\phi_1 = \phi_2$.

To show that Λ is surjective we proceed as follows. Suppose η is in $D(\mathcal{A}^3)$. We first construct a P -equivariant morphism, $\tilde{\phi}: \mathfrak{z} + \mathfrak{u} \rightarrow \mathfrak{z} + \mathfrak{u}$ with $\tilde{\phi}|_{\mathfrak{z}} = \eta$. Extending the range of $\tilde{\phi}$ to all of \mathfrak{g} we can apply Proposition 3.1 and obtain a G -equivariant morphism, $\phi: G \times^P(\mathfrak{z} + \mathfrak{u}) \rightarrow \mathfrak{g}$, with $\phi(1 * (t + n)) = \tilde{\phi}(t + n)$ for all $t + n$ in $\mathfrak{z} + \mathfrak{u}$. Then clearly $\Lambda(\phi) = \eta$ and so Λ is surjective.

Now, to complete the proof of Theorem 5.1 it remains to construct a morphism, $\tilde{\phi}$, in $\text{Mor}_P(\mathfrak{z} + \mathfrak{u}, \mathfrak{z} + \mathfrak{u})$, given a morphism, η , in $D(\mathcal{A}^3)$. So for the rest of this section, fix η in $D(\mathcal{A}^3)$.

Let a_{reg} denote the restriction of $a: U \times (\mathfrak{z} + \mathfrak{u}) \rightarrow \mathfrak{z} + \mathfrak{u}$ to $U \times \mathfrak{z}_{\text{reg}}$. Then by Corollary 4.2, a_{reg} determines an isomorphism of varieties between $U \times \mathfrak{z}_{\text{reg}}$ and $\mathfrak{z}_{\text{reg}} + \mathfrak{u}$. Thus, if $1_U: U \rightarrow U$ is the identity morphism, then $a \circ (1_U \times \eta|_{\mathfrak{z}_{\text{reg}}}) \circ a_{\text{reg}}^{-1}: \mathfrak{z}_{\text{reg}} + \mathfrak{u} \rightarrow \mathfrak{z} + \mathfrak{u}$ is a morphism. Set $\tilde{\phi}_{\text{reg}} = a \circ (1_U \times \eta|_{\mathfrak{z}_{\text{reg}}}) \circ a_{\text{reg}}^{-1}$. Then $\tilde{\phi}_{\text{reg}}(\text{Adu}(t)) = \text{Adu}(\eta(t))$ for u in U and t in $\mathfrak{z}_{\text{reg}}$ so $\tilde{\phi}_{\text{reg}}|_{\mathfrak{z}_{\text{reg}}} = \eta|_{\mathfrak{z}_{\text{reg}}}$.

We next show that $\tilde{\phi}_{\text{reg}}$ is P -equivariant. Then, since $\mathfrak{z}_{\text{reg}} + \mathfrak{u}$ is dense in $\mathfrak{z} + \mathfrak{u}$, it will follow that if $\tilde{\phi}_{\text{reg}}$ extends to a morphism, $\tilde{\phi}: \mathfrak{z} + \mathfrak{u} \rightarrow \mathfrak{z} + \mathfrak{u}$, then $\tilde{\phi}$ is also P -equivariant. So suppose p is in P , t is in $\mathfrak{z}_{\text{reg}}$, and n is in \mathfrak{u} . Then there is a u in U so that $t + n = \text{Adu}(t)$. Write $pu = u_1l$ where u_1 is in U and l is in L . Then

$$\begin{aligned} \tilde{\phi}_{\text{reg}}(\text{Ad}p(t + n)) &= \tilde{\phi}_{\text{reg}}(\text{Ad}pu(t)) \\ &= \tilde{\phi}_{\text{reg}}(\text{Adu}_1(t)) \\ &= \text{Adu}_1(\eta(t)) \\ &= \text{Adu}_1l(\eta(t)) \\ &= \text{Ad}pu(\eta(t)) \\ &= \text{Ad}p(\tilde{\phi}_{\text{reg}}(t + n)). \end{aligned}$$

Therefore, $\tilde{\phi}_{\text{reg}}$ is P -equivariant.

Our strategy for showing that $\tilde{\phi}_{\text{reg}}$ extends to all of $\mathfrak{z} + \mathfrak{u}$ is to show that $\tilde{\phi}_{\text{reg}}$ extends to a morphism $\tilde{\phi}_i: \mathfrak{z}^i + \mathfrak{u} \rightarrow \mathfrak{z} + \mathfrak{u}$ for all α in $\Phi_{\mathfrak{u}}$ where as in §4, $\mathfrak{z}^i = \{t \in \mathfrak{z} \mid \beta(t) \neq 0 \text{ for all } \beta \in \Phi_{\mathfrak{u}} \setminus \Psi_i\}$.

Assuming this has been done and that $\tilde{\phi}_{\text{reg}}$ denotes the extension to $\bigcup_{i=1}^s (\mathfrak{z}^i + \mathfrak{u})$, we can complete the proof as follows. If f is in $k[\mathfrak{z} + \mathfrak{u}]$, then $f\tilde{\phi}_{\text{reg}}$ is a rational function on $\mathfrak{z} + \mathfrak{u}$ whose domain contains $\bigcup_{i=1}^s (\mathfrak{z}^i + \mathfrak{u})$. If $f\tilde{\phi}_{\text{reg}}$ is not a regular function on $\mathfrak{z} + \mathfrak{u}$, then the set of points at which it's not defined is a closed subset of $\mathfrak{z} + \mathfrak{u}$ with codimension 1. Now $\bigcup_{i=1}^s (\mathfrak{z}^i + \mathfrak{u})$ is contained in the domain of $f\tilde{\phi}_{\text{reg}}$ and so the set of points in $\mathfrak{z} + \mathfrak{u}$ at which $f\tilde{\phi}_{\text{reg}}$ is not defined is contained in $(\mathfrak{z} + \mathfrak{u}) \setminus \bigcup_{i=1}^s (\mathfrak{z}^i + \mathfrak{u})$. However

$$(\mathfrak{z} + \mathfrak{u}) \setminus \bigcup_{i=1}^s (\mathfrak{z}^i + \mathfrak{u}) = \left(\bigcup_{i \neq j} (H_i \cap H_j) \right) + \mathfrak{u}.$$

Thus, the set of points at which $f\tilde{\phi}_{\text{reg}}$ is not defined has codimension at least 2. Therefore $f\tilde{\phi}_{\text{reg}}$ must be defined on all of $\mathfrak{z} + \mathfrak{u}$ and so $f\tilde{\phi}_{\text{reg}}$ is in $k[\mathfrak{z} + \mathfrak{u}]$. It follows that $\tilde{\phi}_{\text{reg}}^\#(k[\mathfrak{z} + \mathfrak{u}]) \subseteq k[\mathfrak{z} + \mathfrak{u}]$, and so $\tilde{\phi}_{\text{reg}}$ extends to a morphism $\tilde{\phi}: \mathfrak{z} + \mathfrak{u} \rightarrow \mathfrak{z} + \mathfrak{u}$.

It remains to show that $\tilde{\phi}_{\text{reg}}$ can be extended to a morphism from $\mathfrak{z}^i + \mathfrak{u}$ to $\mathfrak{z} + \mathfrak{u}$ for $1 \leq i \leq s$. In order to show that $\tilde{\phi}_{\text{reg}}$ extends to $\mathfrak{z}^i + \mathfrak{u}$, we'll finally use the hypothesis that η is in $D(\mathcal{A}^{\mathfrak{z}})$. Since η is in $D(\mathcal{A}^{\mathfrak{z}})$, for α in $\Phi_{\mathfrak{u}}$, $\eta^\#(\alpha|_{\mathfrak{z}})$ is a $k[\mathfrak{z}]$ -multiple of $\alpha|_{\mathfrak{z}}$. Say $\eta^\#(\alpha_i|_{\mathfrak{z}}) = f_i\alpha_i|_{\mathfrak{z}}$ for $1 \leq i \leq s$. Define $\eta_i: \mathfrak{z}^i + \mathfrak{u}_i \rightarrow \mathfrak{z} + \mathfrak{u}$ by $\eta_i(t+n) = \eta(t) + f_i(t)n$, so η_i is a morphism. Recall that $a_i: U^i \times (\mathfrak{z}^i + \mathfrak{u}_i) \rightarrow \mathfrak{z} + \mathfrak{u}$ determines an isomorphism between $U^i \times (\mathfrak{z}^i + \mathfrak{u}_i)$ and $\mathfrak{z}^i + \mathfrak{u}$ by Proposition 4.3. Define $\tilde{\phi}_i = a \circ (1_U \times \eta_i) \circ a_i^{-1}: \mathfrak{z}^i + \mathfrak{u} \rightarrow \mathfrak{z} + \mathfrak{u}$.

Lemma 5.2. *The restriction of $\tilde{\phi}_i$ to $\mathfrak{z}_{\text{reg}} + \mathfrak{u}$ is equal to $\tilde{\phi}_{\text{reg}}$.*

Proof. Suppose $t+n$ is in $\mathfrak{z}_{\text{reg}} + \mathfrak{u}$. Then $t+n = \text{Adu}(t)$ for some u in U . Write $u = u^i u_i$ where u^i is in U^i and u_i is in U_i . It follows from Proposition 4.4 that there is an n_1 in \mathfrak{u}_i so that $\text{Adu}_i(t') = t' + \alpha_i(t')n_1$ for every t' in \mathfrak{z} . Then $t+n = \text{Adu}^i(t + \alpha_i(t)n_1)$ and so

$$\begin{aligned} \tilde{\phi}_i(t+n) &= \text{Adu}^i(\eta(t) + f_i(t)\alpha_i(t)n_1) \\ &= \text{Adu}^i(\eta(t) + \alpha_i(\eta(t))n_1) \\ &= \text{Adu}^i u_i(\eta(t)) \\ &= \text{Adu}(\eta(t)) \\ &= \tilde{\phi}_{\text{reg}}(t+n). \end{aligned}$$

This completes the proof of the lemma. □

It follows from the lemma that $\tilde{\phi}_{\text{reg}}$ extends to $\mathfrak{z}^i + \mathfrak{u}$ for α in $\Phi_{\mathfrak{u}}$, and so the proof of Theorem 5.1 is complete.

Corollary 5.3. *The graded $k[\mathfrak{z}]$ -modules, $\text{Hom}_G(\mathfrak{g}^*, k[G \times^P(\mathfrak{z} + \mathfrak{u})])$ and $D(\mathcal{A}^{\mathfrak{z}})$ are isomorphic.*

Proof. It is easily seen that the natural map between $\text{Hom}_G(\mathfrak{g}^*, k[G \times^P(\mathfrak{z} + \mathfrak{u})])$ and $\text{Mor}_G(G \times^P(\mathfrak{z} + \mathfrak{u}), \mathfrak{g})$ is an isomorphism of graded $k[\mathfrak{z}]$ -modules, and so if Γ is the

composition of Λ with this isomorphism, then

$$\Gamma: \text{Hom}_G(\mathfrak{g}^*, k[G \times^P(\mathfrak{z} + \mathfrak{u})]) \rightarrow D(\mathcal{A}^\mathfrak{z})$$

is an isomorphism of graded $k[\mathfrak{z}]$ -modules. □

6. CONCLUSIONS

As in §5, since P acts trivially on $(\mathfrak{z} + \mathfrak{u})/\mathfrak{u}$, we can take $W = \mathfrak{z} + \mathfrak{u}$, $W_0 = \mathfrak{z}$, and $W_1 = \mathfrak{u}$ in Lemma 3.3 and conclude that there is an isomorphism of graded $k[\mathfrak{z}]$ -modules, $k[\mathfrak{z}] \otimes \text{Hom}_G(\mathfrak{g}^*, k[G \times^P \mathfrak{u}]) \cong \text{Mor}_G(G \times^P(\mathfrak{z} + \mathfrak{u}), \mathfrak{g})$. Composing this isomorphism with the isomorphism of Theorem 5.1 and using that \mathfrak{g} affords a self-dual representation of G , we obtain an isomorphism of graded $k[\mathfrak{z}]$ -modules, $k[\mathfrak{z}] \otimes \text{Hom}_G(\mathfrak{g}, k[G \times^P \mathfrak{u}]) \cong D(\mathcal{A}^\mathfrak{z})$. It follows that $(\mathfrak{z}, \mathcal{A}^\mathfrak{z})$ is a free hyperplane arrangement. We can now prove the following result of Orlik and Terao [8].

Corollary 6.1. *If (V, \mathcal{A}) is a reflection arrangement arising from a finite Weyl group, then for X in the lattice of \mathcal{A} , the restricted arrangement, (X, \mathcal{A}^X) is free.*

Proof. Suppose Φ is a root system in a k -vector space V . Let \mathcal{A} be the arrangement of hyperplanes in V consisting of the hyperplanes orthogonal to the vectors in Φ . It's straightforward to check that if G is the adjoint group with root system $\Phi = \Phi(G, T)$, then every restriction of (V, \mathcal{A}) arises as $(\mathfrak{z}, \mathcal{A}^\mathfrak{z})$ for a suitable choice of P and L .

Alternately, if W is the Weyl group of (V, \mathcal{A}) , then W acts on the lattice of \mathcal{A} and it follows from standard results about root systems that if Π is a base of Φ , then every W -orbit in the lattice of \mathcal{A} contains a representative that's an intersection of the hyperplanes orthogonal to the vectors in a subset of Π (see [6]). Clearly, if X is in the lattice of \mathcal{A} and w is in W , then \mathcal{A}^X is free if and only if \mathcal{A}^{wX} is free. Now if G is the adjoint group with root system $\Phi = \Phi(G, T)$, B is a Borel subgroup of G containing T , P is a parabolic subgroup of G containing B , and L is the standard Levi factor in P , then $(\mathfrak{z}, \mathcal{A}^\mathfrak{z})$ is free. Since every orbit of W on the lattice of \mathcal{A} contains a representative of this form, the result follows. □

Finally, we show that the isomorphism of graded $k[\mathfrak{z}]$ -modules between $k[\mathfrak{z}] \otimes \text{Hom}_G(\mathfrak{g}, k[G \times^P \mathfrak{u}])$ and $D(\mathcal{A}^\mathfrak{z})$ can be used to prove the result of Sommers, Trapa, and Broer mentioned in the introduction.

Corollary 6.2. *The multiplicity of the adjoint representation of G in the graded representation $k[G \times^P \mathfrak{u}]$ is given by*

$$\sum_{j \geq 0} \dim \text{Hom}_G(\mathfrak{g}, k[G \times^P \mathfrak{u}]_j) q^j = q^{n_1^\mathfrak{z}} + \dots + q^{n_r^\mathfrak{z}}$$

where q is an indeterminate and $\{n_1^\mathfrak{z}, \dots, n_r^\mathfrak{z}\}$ is the multiset of exponents of $\mathcal{A}^\mathfrak{z}$.

Proof. As in §2, $\text{Hom}_G(\mathfrak{g}, k[G \times^P \mathfrak{u}])$ inherits a grading from the scalar action of k on \mathfrak{u} . Suppose $\{\phi_1, \dots, \phi_s\}$ is a homogeneous basis of $\text{Hom}_G(\mathfrak{g}, k[G \times^P \mathfrak{u}])$ with ϕ_i in $\text{Hom}_G(\mathfrak{g}, k[G \times^P \mathfrak{u}]_{d_i})$ for $1 \leq i \leq s$. Then clearly

$$\sum_{j \geq 0} \dim \text{Hom}_G(\mathfrak{g}, k[G \times^P \mathfrak{u}]_j) q^j = q^{d_1} + \dots + q^{d_s}.$$

On the other hand, the set $\{1 \otimes \phi_1, \dots, 1 \otimes \phi_s\}$ is a homogeneous basis of $k[\mathfrak{z}] \otimes \text{Hom}_G(\mathfrak{g}, k[G \times^P \mathfrak{u}])$ and so corresponds to a homogeneous basis of $D(\mathcal{A}^\mathfrak{z})$ under

the isomorphism $k[\mathfrak{J}] \otimes \text{Hom}_G(\mathfrak{g}, k[G \times^P \mathfrak{u}]) \cong D(\mathcal{A}^3)$. Therefore, the multisets $\{d_1, \dots, d_s\}$ and $\{n_1^3, \dots, n_r^3\}$ are equal. \square

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