ON MINUSCULE REPRESENTATIONS AND THE PRINCIPAL $SL_2$

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Abstract. We study the restriction of minuscule representations to the principal $SL_2$, and use this theory to identify an interesting test case for the Langlands philosophy of liftings.

In this paper, we review the theory of minuscule co-weights $\lambda$ for a simple adjoint group $G$ over $\mathbb{C}$, as presented by Deligne [D]. We then decompose the associated irreducible representation $V_\lambda$ of the dual group $\hat{G}$, when restricted to a principal $SL_2$. This decomposition is given by the action of a Lefschetz $SL_2$ on the cohomology of the flag variety $X = G/P$, where $P$ is the maximal parabolic subgroup of $G$ associated to the co-weight $\lambda$. We reinterpret a result of Vogan and Zuckerman [V-Z, Prop 6.19] to show that the cohomology of $X$ is mirrored by the bigraded cohomology of the $L$-packet of discrete series with infinitesimal character $\rho$, for a real form $G_0$ of $G$ with a Hermitian symmetric space.

We then focus our attention on those minuscule representations with a non-zero linear form $t : V \to \mathbb{C}$ fixed by the principal $SL_2$, such that the subgroup $\hat{H} \subset \hat{G}$ fixing $t$ acts irreducibly on the subspace $V_0 = \ker(t)$. We classify them in §10; since $\hat{H}$ turns out to be reductive, we have a decomposition

$$ V = C e + V_0 $$

where $e$ is fixed by $\hat{H}$, and satisfies $t(e) \neq 0$. We study $V$ as a representation of $\hat{H}$, and give an $\hat{H}$-algebra structure on $V$ with identity $e$.

The rest of the paper studies representations $\pi$ of $G$ which are lifted from $H$, in the sense of Langlands. We show this lifting is detected by linear forms on $\pi$ which are fixed by a certain subgroup $L$ of $G$. The subgroup $L$ descends to a subgroup $L_0 \to G_0$ over $\mathbb{R}$; both have Hermitian symmetric spaces $\mathcal{D}$ with $\dim_{\mathbb{C}}(\mathcal{D}_L) = \frac{1}{2} \dim_{\mathbb{C}}(\mathcal{D}_G)$. We hope this will provide cycle classes in the Shimura varieties associated to $G_0$, which will enable one to detect automorphic forms in cohomology which are lifted from $H$.

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1. Minuscule co-weights

Let $G$ be a simple algebraic group over $\mathbb{C}$, of adjoint type. Let $T \subset B \subset G$ be a maximal torus contained in a Borel subgroup, and let $\Delta$ be the corresponding set of simple roots for $T$. Then $\Delta$ gives a $\mathbb{Z}$-basis for $\text{Hom}(T,\mathbb{G}_m)$, so a co-weight $\lambda$ in $\text{Hom}(\mathbb{G}_m,T)$ is completely determined by the integers $\langle \lambda, \alpha \rangle$, for $\alpha$ in $\Delta$, which may be arbitrary. Let $P_+$ be the cone of dominant co-weights, where $\langle \lambda, \alpha \rangle \geq 0$ for all $\alpha \in \Delta$.

A co-weight $\lambda : \mathbb{G}_m \to T$ gives a $\mathbb{Z}$-grading $\mathfrak{g}_{\lambda}$ of $\mathfrak{g} = \text{Lie}(G)$, defined by
\[
\mathfrak{g}_{\lambda}(i) = \{ X \in \mathfrak{g} : \text{Ad}(\alpha)(X) = a^i \cdot X \}
\]
We say $\lambda$ is minuscule provided $\lambda \neq 0$ and the grading $\mathfrak{g}_{\lambda}$ satisfies $\mathfrak{g}_{\lambda}(i) = 0$ for $|i| \geq 2$. Thus
\[
\mathfrak{g} = \mathfrak{g}_{\lambda}(-1) + \mathfrak{g}_{\lambda}(0) + \mathfrak{g}_{\lambda}(1).
\]

The Weyl group $N_G(T)/T = W$ of $T$ acts on the set of minuscule co-weights, and the $W$-orbits are represented by the dominant minuscule co-weights. These have been classified.

**Proposition 1.2 (§D 1.2).** The element $\lambda$ is a dominant, minuscule co-weight if and only if there is a single simple root $\alpha$ with $\langle \lambda, \alpha \rangle = 1$, the root $\alpha$ has multiplicity 1 in the highest root $\beta$, and all other simple roots $\alpha'$ satisfy $\langle \lambda, \alpha' \rangle = 0$.

Thus, the $W$-orbits of minuscule co-weights correspond bijectively to simple roots $\alpha$ with multiplicity 1 in the highest root $\beta$. If $\lambda$ is minuscule and dominant, $\mathfrak{g}_{\lambda}(1)$ is the direct sum of the positive root spaces $\mathfrak{g}_{\gamma}$, where $\gamma$ is a positive root containing $\alpha$ with multiplicity 1. Hence the dimension $N$ of $\mathfrak{g}_{\lambda}(1)$ is given by the formula
\[
N = \dim \mathfrak{g}_{\lambda}(1) = \langle \lambda, 2\rho \rangle,
\]
where $\rho$ is half the sum of the positive roots.

The subgroup $W_{\lambda} \subset W$ fixing $\lambda$ is isomorphic to the Weyl group of $T$ in the subalgebra $\mathfrak{g}_{\lambda}(0)$, which has root basis $\Delta - \{\alpha\}$. We now tabulate the $W$-orbits of minuscule co-weights by listing the simple $\alpha$ occurring with multiplicity 1 in $\beta$ in the numeration of Bourbaki $B$. We also tabulate $N = \dim \mathfrak{g}_{\lambda}(1)$ and $(W : W_{\lambda})$; a simple comparison shows that $(W : W_{\lambda}) \geq N + 1$ in all cases; we will explain this inequality later.
Table 1.4.

<table>
<thead>
<tr>
<th></th>
<th></th>
<th>(W : W_\lambda)</th>
<th>N</th>
</tr>
</thead>
<tbody>
<tr>
<td>A_\ell</td>
<td>\alpha_k</td>
<td>\binom{\ell + 1}{k}</td>
<td>k(\ell + 1 - k)</td>
</tr>
<tr>
<td>1 \leq k \leq \ell</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>B_\ell</td>
<td>\alpha_1</td>
<td>2\ell</td>
<td>2\ell - 1</td>
</tr>
<tr>
<td>C_\ell</td>
<td>\alpha_\ell</td>
<td>2\ell</td>
<td>\ell(\ell+1)</td>
</tr>
<tr>
<td>D_\ell</td>
<td>\alpha_1, \alpha_\ell</td>
<td>2\ell - 2</td>
<td>\ell(\ell-1)</td>
</tr>
<tr>
<td>E_6</td>
<td>\alpha_1, \alpha_6</td>
<td>27</td>
<td>16</td>
</tr>
<tr>
<td>E_7</td>
<td>\alpha_1</td>
<td>56</td>
<td>27</td>
</tr>
</tbody>
</table>

2. The real form \( G_0 \)

We henceforth fix \( G \) and a dominant minuscule co-weight \( \lambda \). Let \( G_c \) be the compact real form for \( G \), so \( G = G_c(\mathbb{C}) \) and \( G_c(\mathbb{R}) \) is a maximal compact subgroup of \( G \). Let \( g \mapsto \overline{g} \) be the corresponding conjugation of \( G \).

Let \( T_c \subset G_c \) be a maximal torus over \( \mathbb{R} \). We have an identification of co-character groups

\[ \text{Hom}_{\text{cont}}(S^1, T_c(\mathbb{R})) = \text{Hom}_{\text{alg}}(\mathbb{G}_m, T). \]

We view \( \lambda \) as a homomorphism \( S^1 \to T_c(\mathbb{R}) \), and define

\[ \theta = \text{ad} \lambda(-1) \quad \text{in} \quad \text{Inn}(G). \]

Then \( \theta \) is a Cartan involution, which gives another descent \( G_0 \) of \( G \) to \( \mathbb{R} \). The group \( G_0 \) has real points

\[ G_0(\mathbb{R}) = \{ g \in G : \overline{g} = \theta(g) \}, \]

and a maximal compact subgroup \( K \) of \( G_0(\mathbb{R}) \) is given by

\[ K = \{ g \in G : g = \overline{g} \text{ and } g = \theta(g) \}, \]

\[ = G_0(\mathbb{R}) \cap G_c(\mathbb{R}). \]

The corresponding decomposition of the complex Lie algebra \( \mathfrak{g} \) under the action of \( K \) is given by \( \mathfrak{g} = \mathfrak{k} + \mathfrak{p} \), with

\[ \begin{cases} \mathfrak{k} = \text{Lie}(K) \otimes \mathbb{C} = \mathfrak{g}_\lambda(0) \\ \mathfrak{p} = \mathfrak{g}_\lambda(-1) + \mathfrak{g}_\lambda(1). \end{cases} \]

The torus \( \lambda(S^1) \) lies in the center of the connected component of \( K \), and the element \( \lambda(i) \) gives the symmetric space

\[ \mathcal{D} = G_0(\mathbb{R})/K \]

a complex structure, with

\[ N = \dim_{\mathbb{C}}(\mathcal{D}). \]

Proposition 2.4 ([D, 1.2]). The real Lie groups \( G_0(\mathbb{R}) \) and \( K \) have the same number of connected components, which is either 1 or 2. Moreover, the following are all equivalent:

1) \( G_0(\mathbb{R}) \) has 2 connected components.
2) The symmetric space \( \mathcal{D} \) is a tube domain.
3) The vertex of the Dynkin diagram of $G$ corresponding to the simple root $\alpha$ is fixed by the opposition involution of the diagram.

4) The subgroup $W_\lambda$ fixing $\lambda$ has a nontrivial normalizer in $W$, consisting of those $w$ with $w\lambda = \pm \lambda$.

In fact, the subgroup $W_c \subset W$ which normalizes $W_\lambda$ is precisely the normalizer of the compact torus $T_c(R)$ in $G_0(R)$. When $W_\lambda \neq W_c$, it is generated by $W_\lambda$ and the longest element $w_0$, which satisfies $w_0\lambda = -\lambda$.

As an example, let $G = SO_3$ and

$$\lambda(t) = \begin{pmatrix} t & 1 \\ 1 & t^{-1} \end{pmatrix}$$

Then $\theta$ is conjugation by

$$\lambda(-1) = \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}$$

and $G_0 = SO(1,2)$ has 2 connected components. We have $K \simeq O(2)$, $W_c = W$ has order 2 in this case, and $W_\lambda = 1$. The tube domain $D = G_0(R)/K$ is isomorphic to the upper half plane.

3. The Weyl Group (cf. [H])

The Weyl group $W$ is a Coxeter group, with generating reflections $s$ corresponding to the simple roots in $\Delta$. Recall that $\rho$ is half the sum of the positive roots and $W_\lambda \subset W$ is the subgroup fixing $\lambda$.

Proposition 3.1. Each coset $wW_\lambda$ of $W_\lambda$ in $W$ has a unique representative $y$ of minimal length. The length $d(y)$ of the minimal representative is given by the formula

$$d(y) = \langle \lambda, \rho \rangle - \langle w\lambda, \rho \rangle,$$

where $w$ is any element in the coset.

Proof. Let $R^\pm$ be the positive and negative roots, let $R^+_\lambda$ be the subsets of positive and negative roots which satisfy $\langle \lambda, \gamma \rangle = 0$. Then $R^+ - R^- \subset R^+_\lambda$ consists of the roots with $\langle \lambda, \gamma \rangle = 1$, and $R^- - R^+ \subset R^-\lambda$ consists of the roots with $\langle \lambda, \gamma \rangle = -1$. These sets are stable under the action of $W_\lambda$ on $R$. On the other hand, if $w \in W_\lambda$ stabilizes $R^+_\lambda$ (or $R^-\lambda$), then $w = 1$, as $W_\lambda$ is the Weyl group of the root system $R_\lambda = R^+_\lambda \cup R^-\lambda$.

Since the length $d(y)$ of $y$ in $W$ is given by

(3.2) $$d(y) = \# \{ \gamma \text{ in } R^+: y^{-1}(\gamma) \text{ is in } R^- \},$$

the set

(3.3) $$Y = \{ y \in W : y(R^-_\lambda) \subset R^+ \}$$

gives coset representatives for $W_\lambda$ of minimal length. Moreover, for $y \in Y$ the set $y^{-1}(R^+)$ contains $d(y)$ elements of $R^-_\lambda$, and hence $N - d(y)$ elements of $R^+_\lambda$. Hence,
if \( wW_\lambda = yW_\lambda \), we find
\[
\langle w\lambda, \rho \rangle = \langle y\lambda, \rho \rangle = \langle \lambda, y^{-1}\rho \rangle = \frac{1}{2}((N - d(y)) - d(y)) = \frac{1}{2}N - d(y).
\]
Since
\[
\langle \lambda, \rho \rangle = \frac{1}{2}N,
\]
we obtain the desired formula. \( \square \)

As an example of Proposition 3.1, the minimal representative of \( W_\lambda \) is \( y = 1 \), with \( d(y) = 0 \), and the minimal representative of \( s_\alpha W_\lambda \) is \( y = s_\alpha \), with \( d(y) = 1 \). If \( w_0 \) is the longest element in the Weyl group, then \( w_0(R^0) = R^0 \), so \( w_0^2 = 1 \), and \( w_0\rho = -\rho \). Hence
\[
\langle w_0\lambda, \rho \rangle = \langle \lambda, w_0^{-1}\rho \rangle = -\langle \lambda, \rho \rangle = -N/2.
\]
Consequently, the length of the minimal representative \( y \) of \( w_0W_\lambda \) is \( d(y) = N \). This is the maximal value of \( d \) on \( W/W_\lambda \), and we will soon see that \( d \) takes all integral values in the interval \([0, N]\).

Assume \( \lambda \) is fixed by the opposition involution \( -w_0 \), so \( w_0\lambda = -\lambda \). Then \( D \) is a tube domain, and \( W_\lambda \) has nontrivial normalizer \( W_c = \langle W_\lambda, w_0 \rangle \) in \( W \) by Proposition 2.4. The 2-group \( W_c/W_\lambda \) acts on the set \( W/W_\lambda \) by \( wW_\lambda \mapsto w w_0 W_\lambda \), and this action has no fixed points. Hence we get a fixed point-free action \( y \mapsto y^* \) on the set \( Y \), and find that
\[
d(y) + d(y^*) = N. \tag{3.4}
\]

4. The flag variety

Associated to the dominant minuscule co-weight \( \lambda \) is a maximal parabolic subgroup \( P \), which contains \( B \) and has Lie algebra
\[
\text{Lie}(P) = g_\lambda(0) + g_\lambda(1).
\]
The flag variety \( X = G/P \) is projective, of complex dimension \( N \).

The cohomology of \( X \) is all algebraic, so \( H^{2n+1}(X) = 0 \) for all \( n \geq 0 \). Let
\[
f_X(t) = \sum_{n \geq 0} \dim H^{2n}(X) \cdot t^n
\]
be the Poincaré polynomial of \( H^*(X) \). Then we have the following consequence of Chevalley-Bruhat theory, which also gives a convenient method of computing the values of the function \( d : W/W_\lambda \to \mathbb{Z} \).

**Proposition 4.3.** 1) We have \( f_X(t) = \sum_Y t^d(y) \).

2) If \( G \) is the split adjoint group over \( \mathbb{Z} \) with the same root datum as \( G \), and \( P \) is the standard parabolic corresponding to \( \lambda \), then
\[
f_X(q) = \#G(F)/P(F)
\]
for all finite fields \( F \), with \( q = \#F \).

3) The Euler characteristic of \( X \) is given by
\[
\chi = f_X(1) = \#(W : W_\lambda).
\]
Proof. We have the decomposition
\[ G = \bigcup_Y ByP, \]
where we have chosen a lifting of \( y \) from \( W \) to \( N_G(T) \). If \( U \) is the unipotent radical of \( B \), then \( B = UT \). Since \( y \) normalizes \( T \),
\[ UyP = ByP. \]
This gives a cell decomposition
\[ X = \bigcup_Y Uy/P \cap y^{-1}Uy \]
where the cell corresponding to \( y \) is an affine space of dimension \( d(y) \). This gives the first formula.

The formula for \( f_X(q) \) follows from the Bruhat decomposition, which can be used to prove the Weil conjectures for \( X \). Formula 3) for \( f_X(1) \) follows immediately from 1).

For example, let \( G = PSp_{2n} \) be of type \( C_n \). Then \( P \) is the Siegel parabolic subgroup, with Levi factor \( GL_n/\mu_2 \). From the orders of \( Sp_{2n}(q) \) and \( GL_n(q) \), we find that
\[
\#G(F)/P(F) = \frac{(q^2 - 1)(q^4 - 1) \ldots (q^{2n} - 1)}{(q - 1)(q^2 - 1) \ldots (q^n - 1)}
= (1 + q)(1 + q^2) \ldots (1 + q^n).
\]
Hence we find
\[
f_X(t) = (1 + t)(1 + t^2) \ldots (1 + t^n).
\]

The fact that \( X = G/P \) is a Kahler manifold imposes certain restrictions on its cohomology. For example, if \( \omega \) is a basis of \( H^2(X) \), then \( \omega^k \neq 0 \) in \( H^{2k}(X) \) for all \( 0 \leq k \leq N \). Hence we find that

\textbf{Corollary 4.5.} The function \( d : W/W_\Lambda \rightarrow \mathbb{Z} \) takes all integral values in \([0, N]\), and \( (W : W_\Lambda) \geq N + 1 \).

For \( 0 \leq k \leq N \), let
\[ m(k) = \# \{ y \in Y : d(y) = k \}. \]
We have seen that \( m(0) = m(1) = 1 \) in all cases. By Poincaré duality
\[
m(k) = m(N - k).
\]
Finally, the Lefschetz decomposition into primitive cohomology shows that
\[
m(k - 1) \leq m(k)
\]
whenever \( 2k \leq N \). Indeed, the representation of the Lefschetz \( SL_2 \) on \( H^*(G/P) \) has weights \( N - 2d(y) \) for the maximal torus \( \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \).
5. THE REPRESENTATION $V$ OF THE DUAL GROUP $\hat{G}$

Let $\hat{G}$ be the Langlands dual group of $G$, which is simply-connected of the dual root type. This group comes (in its construction) with subgroups $\hat{T} \subset \hat{B} \subset \hat{G}$, and an identification of the positive roots for $\hat{B}$ in $\text{Hom}(\hat{T}, G_m)$ with the positive co-roots for $\hat{B}$ in $\text{Hom}(G_m, T)$ (cf. [G]). Hence, the dominant co-weights for $\hat{T}$ give dominant weights for $\hat{B}$, which are the highest weights for $\hat{B}$ on irreducible representations of $\hat{G}$.

Let $V$ be the irreducible representation of $\hat{G}$, whose highest weight for $\hat{B}$ is the dominant, minuscule co-weight $\lambda$.

**Proposition 5.1.** The weights of $\hat{T}$ on $V$ consist of the elements in the $W$-orbit of $\lambda$. Each has multiplicity 1, so dim $V = (W : W_\lambda)$.

The central character $\chi$ of $V$ is given by the image of $\lambda$ in $\text{Hom}(\hat{T}, G_m)/\bigoplus_\Delta \mathbb{Z}\alpha^\vee$, and is nontrivial.

**Proof.** For $\mu$ and $\lambda$ dominant, we write $\mu \leq \lambda$ if $\lambda - \mu$ is a sum of positive co-roots. These are precisely the other dominant weights for $\hat{T}$ occurring in $V_\lambda$. When $\lambda$ is minuscule, $\mu \leq \lambda$ implies $\mu = \lambda$, so only the $W$-orbit of $\lambda$ occur as weights. Each has the same multiplicity as the highest weight, which is 1. Since $\mu = 0$ is dominant, $\lambda$ is not in the span of the co-roots, and $\chi \neq 1$.

This result gives another proof of the inequality of Corollary 4.5: $(W : W_\lambda) \geq N + 1$. Indeed, let $L$ be the unique line in $V_\lambda$ fixed by $\hat{B}$. The fixer of $L$ is the standard parabolic $\hat{P}$ dual to $P$. This gives an embedding of projective varieties:

$$\hat{G}/\hat{P} \hookrightarrow P(V_\lambda).$$

Since $\hat{G}/\hat{P}$ has dimension $N$, and $P(V_\lambda)$ has dimension $(W : W_\lambda) - 1$, this gives the desired inequality.

The real form $G_0$ defined in §2 has Langlands $L$-group

(5.2) $$L^G = \hat{G} \times \text{Gal}(C/R).$$

The action of $\text{Gal}(C/R)$ on $\hat{G}$ exchanges the irreducible representation $V$ with dominant weight $\lambda$ with the dual representation $V^*$ with dominant weight $-w_0\lambda$. Hence the sum $V + V^*$ always extends to a representation of $L^G$. The following is a simple consequence of Proposition 2.4.

**Proposition 5.3.** The following are equivalent:

1) We have $w_0\lambda = -\lambda$.
2) The symmetric space $D$ is a tube domain.
3) The representation $V$ is isomorphic to $V^*$.
4) The central character $\chi$ of $V$ satisfies $\chi^2 = 1$.
5) The representation $V$ of $\hat{G}$ extends to a representation of $L^G$.

6. THE PRINCIPAL $SL_2 \to \hat{G}$

The group $\hat{G}$ also comes equipped with a principal $\varphi : SL_2 \to \hat{G}$; see [G]. The co-character $G_m \to \hat{T}$ given by the restriction of $\varphi$ to the maximal torus $\begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix}$
of $SL_2$ is equal to $2\rho$ in $\text{Hom}(G_m, \hat{T}) = \text{Hom}(T, G_m)$. From this, and Proposition 5.1, we conclude the following:

**Proposition 6.1.** The restriction of the minuscule representation $V$ to the principal $SL_2$ in $G$ has weights

$$\bigoplus_{w/W_\lambda} t^{(w, 2\rho)}$$

for the maximal torus $\begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix}$ in $SL_2$.

On the other hand, by Proposition 3.1, we have

$$\langle w, 2\rho \rangle = \langle \lambda, 2\rho \rangle - 2d(y) = N - 2d(y)$$

where $d(y)$ is the length of the minimal representative $y$ in the coset $wW_\lambda$. Hence the weights for the principal $SL_2$ acting on $V$ are the integers

$$N - 2d(y), \quad y \in Y$$

in the interval $[-N, N]$. Since these are also the weights of the Lefschetz $SL_2$ acting on the cohomology $H^*(G/P)$ by §4, we obtain the following:

**Corollary 6.4.** The representation of the principal $SL_2$ of $\hat{G}$ on $V$ is isomorphic to the representation of the Lefschetz $SL_2$ on the cohomology of the flag variety $X = G/P$.

### 7. Examples

We now give several examples of the preceding theory, using the notation for roots and weights of $[B]$.

If $G$ is of type $A_\ell$ and $\alpha = \alpha_1$ we have $\lambda = e_1$. The flag variety $G/P$ is projective space $P^N$, with $N = \ell$, and the Poincaré polynomial is $1 + t + t^2 + \cdots + t^N$. The dual group $\hat{G}$ is $SL_{N+1}$, and $V$ is the standard representation. The restriction of $V$ to a principal $SL_2$ is irreducible, isomorphic to $S^N = \text{Sym}^N(C^2)$.

A similar result holds when $G$ is of type $B_\ell$, so $\alpha = \alpha_1$ and $\lambda = e_1$. Here $G/P$ is a quadric of dimension $N = 2\ell - 1$, with $P(t) = 1 + t + \cdots + t^N$ as before. The dual group $\hat{G} = \text{Sp}_{2\ell}$, the representation $V$ is the standard representation, and its restriction to the principal $SL_2$ is the irreducible representation $S^N$.

Next, suppose $G$ is of type $D_\ell$ and $\alpha = \alpha_1$, so $\lambda = e_1$. Then $G/P$ is a quadric of dimension $N = 2\ell - 2$, and we have $P(t) = 1 + t + \cdots + 2t^{\ell-1} + \cdots + t^N$. The dual group $\hat{G} = \text{Spin}_{2\ell}$, and $V$ is the standard representation of the quotient $SO_{2\ell}$. Its restriction to the principal $SL_2$ is a direct sum $S^N + S^0$, where $S^0$ is the trivial representation.

A more interesting case is when $G$ is of type $C_\ell$, so $\alpha = \alpha_\ell$ and $\lambda = \frac{e_1 + e_2 + \cdots + e_\ell}{2}$. Here $G/P$ is the Lagrangian Grassmanian of dimension $N = \frac{(\ell+1)}{2}$, and $P(t) = (1 + t)(1 + t^2) \cdots (1 + t^\ell)$ was calculated in (4.4). The dual group $\hat{G}$ is $\text{Spin}_{2\ell+1}$, and $V$ is the spin representation of dimension $2^\ell$. Its decomposition to a
principal $SL_2$ is given by §6, and we find the following representations, for $\ell \leq 6$:

$$
\begin{align*}
S^1 & \quad \ell = 1, \\
S^3 & \quad \ell = 2, \\
S^6 + S^0 & \quad \ell = 3, \\
S^{10} + S^4 & \quad \ell = 4, \\
S^{15} + S^9 + S^5 & \quad \ell = 5, \\
S^{21} + S^{15} + S^{11} + S^9 + S^3 & \quad \ell = 6.
\end{align*}
$$

As the last example, suppose $G$ is of type $E_6$. Then $G/P$ has dimension 16 and Poincaré polynomial

$$
P(t) = 1 + t + t^2 + t^3 + 2t^4 + 2t^5 + 2t^6 + 2t^7 + 3t^8 + 2t^9 + 2t^{10} + 2t^{11} + 2t^{12} + t^{13} + t^{14} + t^{15} + t^{16}.
$$

The representation $V$ has dimension 27, and its restriction to a principal $SL_2$ is the representation

$$
S^{16} + S^8 + S^0.
$$

**Proposition 7.3.** The representation $V$ of the principal $SL_2$ is irreducible, hence isomorphic to $S^N$, if and only if $G$ is of type $A_\ell$ or $B_\ell$ and $\alpha = \alpha_1$.

The representation $V$ of the principal $SL_2$ is isomorphic to $S^N + S^0$ if and only if $G$ is of type $D_\ell$ and $\alpha = \alpha_1$, or $G$ is of type $D_4$ and $\alpha = \alpha_3$ or $\alpha_4$, or $G$ is of type $C_3$ and $\alpha = \alpha_3$.

**Proof.** The condition $V = S^N$ as a representation of $SL_2$ is equivalent to the equality

$$
\dim V = (W : W_\lambda) = N + 1.
$$

The condition $V = S^N + S^0$ as a representation of $SL_2$ is equivalent to the equality

$$
\dim V = (W : W_\lambda) = N + 2.
$$

One obtains all the above cases by a consideration of the columns in Table 1.4. $\square$

### 8. Discrete series and a mirror theorem

Let $G_0$ be the real form of $G$ described in §2, and let $G_0^+(\mathbb{R})$ be the connected component of $G_0(\mathbb{R})$. The $L$-packet of discrete series representations $\pi^+$ of $G_0^+(\mathbb{R})$ with infinitesimal character the $W$-orbit of $\rho$ is in canonical bijection with the coset space $W_\lambda \setminus W$. Indeed, $W_\lambda$ is the compact Weyl group of the simply-connected algebraic cover $G_0^{sc}$ of $G_0$, and any discrete series for $G_0^{sc}(\mathbb{R})$ with infinitesimal character $\rho$ has trivial central character, so it descends to the quotient group $G_0^+(\mathbb{R})$. On the other hand, such discrete series for $G_0^{sc}(\mathbb{R})$ are parameterized by their Harish-Chandra parameters in $\text{Hom}(T_0^{sc}(\mathbb{R}), S^1)/W_\lambda$, which lie in the $W$-orbit of $\rho$. The coset $W_\lambda \rho$ corresponds to the holomorphic discrete series, and the coset $W_\lambda w_0 \rho = W_\lambda w_0^{-1} \rho$ corresponds to the anti-holomorphic discrete series.

**Proposition 8.1 ([V-Z] Prop. 6.19).** Assume the discrete series $\pi^+$ of $G_0^+(\mathbb{R})$ has Harish-Chandra parameter $W_\lambda w_0^{-1} \rho$. Then $\pi^+$ has bigraded cohomology

$$
H^{p,q}(\mathfrak{g}, K^+; \pi^+) \simeq \mathbb{C}
$$
for $p + q = N$ and $q = d(y)$, the length of the minimal representative of $wW_\lambda$. The cohomology of $\pi$ vanishes in all other bidegrees $(p', q')$.

**Proof.** The bigrading of the $(\mathfrak{g}, K^+)$ cohomology of any $\pi^+$ in the $L$-packet is discussed in [V-Z, (6.18)(a-c)]. The cohomology has dimension 1 for degree $N$, and dimension 0 otherwise, so we must have $p + q = N$.

On the other hand, Arthur (cf. [A, pp. 62–63]) interprets the calculation of [V-Z, Prop. 6.19] to obtain the formula

$$-\frac{1}{2}(p - q) = \langle \lambda, w^{-1}\rho \rangle = \langle w\lambda, \rho \rangle.$$

Since $\frac{1}{2}(p + q) = \langle \lambda, \rho \rangle$, we find that $q = \langle \lambda, \rho \rangle - \langle w\lambda, \rho \rangle = d(y)$, by Proposition 3.1.

If $G_0(\mathbb{R}) \neq G_0(\mathbb{R})^+$, the discrete series $\pi$ for $G_0(\mathbb{R})$ with infinitesimal character $\rho$ correspond to the coset space $W_c \backslash W$, where $W_c$ is the (nontrivial) normalizer of $W_\lambda$ in $W$. We find that the bigraded cohomology of $\pi$ with Harish-Chandra parameter $W_\lambda w^{-1}\rho$ is the direct sum of two lines of type $(p, q)$ and $(q, p)$, with $p + q = N$ and $q = d(y)$.

The 2-group $K/K^+$ acts on $H^N(\mathfrak{g}, K^+, \pi)$, switching the two lines. When $p = q = N/2$, there is a unique line in $H^{p,p}(\mathfrak{g}, K^+, \pi)$ fixed by $K/K^+$.

A suggestive way to restate the calculation of the bigraded cohomology is the following.

**Corollary 8.2.** The Hodge structure on the sum $H^N(G_0) = \bigoplus_\pi H^{*,*}(\mathfrak{g}, K^+, \pi)$ over the $L$-packet of discrete series for $G_0(\mathbb{R})$ with infinitesimal character $\rho$ mirrors the Hodge structure on $H^*(G/P)$. That is,

$$\dim H^{q,q}(G/P) = \dim H^{N-q,q}(G_0).$$

Indeed, both dimensions are equal to the number of classes $wW_\lambda$ in $W/W_\lambda$ with $d(w, W_\lambda) = q$.

9. **Discrete series for $SO(2, 2n)$**

Assume that $G$ is of type $D_{n+1}$ with $n \geq 2$, and that $\alpha = \alpha_1$. The group $G_0(\mathbb{R})$ is then isomorphic to $PSO(2, 2n) = SO(2, 2n)/(\pm 1)$, and $D$ is a tube domain of complex dimension $N = 2n$. There are $n + 1$ discrete series representations $\pi$ of $G_0(\mathbb{R})$ with infinitesimal character $\rho$. We will describe these as representations of $SO(2, 2n)$, with trivial central character, and will calculate their minimal $K^+$-types and Hodge cohomology.

Let $V$ be a 2-dimensional real vector space, with a positive definite quadratic form, and write $-V$ for the same space, with the negative form. For $k = 0, 1, \ldots, n$ define the quadratic space

$$W_k = V_0 + V_1 + \cdots + (-V_k) + \cdots + V_n,$$

so $SO(W_k) \simeq SO(2, 2n)$, a maximal compact torus $T_c$ in $SO(W_k)$ is given by $\prod_{i=0}^n SO(V_i)$, and a maximal compact, connected subgroup $K^+$ containing $T_c$ is given by $SO(V_k) \ltimes SO(V_k^\perp)$. If $e_i$ is a generator of $\text{Hom}(SO(V_i), S^1)$, then the character group of $T_c$ is $\bigoplus_{i=0}^n \mathbb{Z}e_i$, and the roots of $T_c$ on $\mathfrak{g}$ are the elements

$$\gamma_{ij} = \pm e_i \pm e_j \quad i \neq j.$$
The compact roots of $T_c$ on $k$ are those roots $\gamma_{ij}$ with $i \neq k + 1$ and $j \neq k + 1$, so the $(k + 1)$st coordinate of $\gamma$ is zero.

A set of positive roots is given by

$$R^+ = \{e_i \pm e_j : i < j\}.$$ 

This has root basis

$$\Delta = \{e_0 - e_1, e_1 - e_2, \ldots, e_{n-1} - e_n, e_{n-1} + e_n\}$$

and

$$\rho = (n, n - 1, n - 2, \ldots, 1, 0).$$

On the other hand, half the sum

$$\rho_c = (n - 1, n - 2, \ldots, n - k, 0, n - k - 1, k, \ldots, 1, 0).$$

At the two extremes, we find that

$$k = 0 \quad \rho_c = (0, n - 1, n - 2, \ldots, 1, 0),$$

$$k = n \quad \rho_c = (n - 1, n - 2, \ldots, 1, 0, 0).$$

The lowest $K^+$-type of a discrete series $\pi^+$ for $SO(2, 2n)^+$ with Harish-Chandra parameter $\lambda = \rho$ is given by Schmid’s formula:

$$\lambda + \rho - 2\rho_c = 2(\rho - \rho_c).$$

For the realizations $SO(2, 2n) \cong SO(W_k)$ above, we obtain $n + 1$ discrete series $\pi^+_k$ with minimal $K^+ \cong SO(2) \times SO(2n)$ type

$$\chi^{2(n-k)} \otimes (2, 2, 2, \ldots, 2, 0, 0 \ldots 0) \text{k times}$$

where $\chi$ is the fundamental character of $SO(2)$, giving the action on $p^+$. The irreducible representation of $SO(2n)$ with highest weight $2(e_1 + \cdots + e_k)$ appears with multiplicity 1 in $\operatorname{Sym}^2(\mathbf{C}^{2n})$, and the minimal $K^+$-type appears with multiplicity 1 in the representation $\wedge^k p_- \otimes \wedge^k p_+$. Hence the Hodge type of $\pi^+_k$ is $(2n - k, k)$.

Each discrete series $\pi_k$ of $SO(2, 2n)$ with infinitesimal character $\rho$ decomposes as $\pi_k = \pi^+_k + \pi^-_k$ when restricted to $SO(2, 2n)^+$ with $\pi^+_k$ as above, and $\pi^-_k$ its conjugate by $G_0(\mathbf{R})/G_0(\mathbf{R})^+$. The minimal $K^+$-type of $\pi^-_k$ is

$$\chi^{2(k-n)} \otimes (2, 2, 2, \ldots, 2, 0, 0 \ldots 0) \text{k times}$$

so $\pi^-_k$ has Hodge type $(k, 2n - k)$, and $\pi_k$ has Hodge type $(k, 2n - k) + (2n - k, k)$.

If we label the simple roots in the Dynkin diagram for $G$, white for non-compact roots, black for compact roots, then the discrete series $\pi_k$ of $SO(2, 2n)$ gives the labelled diagram below.

In the case $k = 0$, $\pi_k$ is the sum of holomorphic and anti-holomorphic discrete series, and is an admissible representation of the subgroup $SO(2) \subset K^+$. In the case $k = n$, $\pi_n$ is admissible for the subgroup $SO(2n) \subset K^+$, and has Hodge type $(n, n) + (n, n)$. 

10. A CLASSIFICATION THEOREM: \( V = C e + V_0 \)

We now return to the restriction of a minimal representation \( V \) of \( \hat{G} \) to a principal \( SL_2 \) in \( \hat{G} \). Since \( V \) will be fixed, we will replace the simply-connected group \( \hat{G} \) by its quotient which acts \textit{faithfully} on \( V \), and will henceforth use the symbol \( G \) for this subgroup of \( GL(V) \). The group \( G \) is therefore no longer necessarily of adjoint type. We have

\[
X_\bullet(T) = \mathbb{Z}\lambda + \bigoplus_{\text{co-roots}} \mathbb{Z}\alpha^\vee
\]

and \( \ell\lambda \) lies in the sublattice \( \bigoplus \mathbb{Z}\alpha^\vee \), with \( \ell \) the order of the (cyclic) center of \( \hat{G} \). Since \( \langle \alpha^\vee, \rho \rangle \) is an integer for all co-roots, we find that \( \rho \) is in \( X_\bullet(T) \) if and only if \( \langle \lambda, \rho \rangle \) is an integer. By (1.3) this occurs precisely when the integer \( N = \dim_{\mathbb{C}}(D) \) is even. Since the center \( \langle \pm 1 \rangle \) of a principal \( SL_2 \) in \( \hat{G} \) acts on \( V \) by the character \( (-1)^N \), we see that \( \rho \) is in \( X_\bullet(T) \) precisely when principal homomorphism \( SL_2 \to \hat{G} \) factors through the quotient group \( PGL_2 \).

\textbf{Proposition 10.2.} Assume that there is a non-zero linear form \( t: V \to \mathbb{C} \) which is fixed by the principal \( SL_2 \to \hat{G} \), and that the subgroup \( \hat{H} \) of \( \hat{G} \) fixing \( t \) acts \textit{irreducibly} on the hyperplane \( V_0 = \ker(t) \).

Then (up to the action of the outer automorphism group of the simply-connected cover of \( \hat{G} \)) the representation \( V \) is given by the following table:

<table>
<thead>
<tr>
<th>( \hat{G} )</th>
<th>( V )</th>
<th>( \hat{H} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( SL_{2n}/\mu_2 )</td>
<td>( C^{2n} )</td>
<td>( S_{p_{2n}}/\mu_2 )</td>
</tr>
<tr>
<td>( SO_{2n} )</td>
<td>( C^{2n} )</td>
<td>( SO_{2n-1} )</td>
</tr>
<tr>
<td>( E_6 )</td>
<td>( C^{27} )</td>
<td>( F_4 )</td>
</tr>
<tr>
<td>( Spin_7 )</td>
<td>( C^8 )</td>
<td>( G_2 )</td>
</tr>
</tbody>
</table>

\textbf{Proof.} By definition, \( \hat{H} \) contains the image of the principal \( SL_2 \) (which is isomorphic to \( PGL_2 \)). These subgroups of simple \( \hat{G} \) have been classified by de Siebenthal.
One has the chains:

\[ SL_2 \to SO_{2n+1} \to SL_{2n+1}, \]
\[ SL_2 \to Sp_{2n} \to SL_{2n}, \]
\[ SL_2 \to SO_{2n-1} \to SO_{2n}, \]
\[ SL_2 \to F_4 \to E_6, \]
\[ SL_2 \to G_2 \to \text{Spin}_7 \to SO_8, \]
\[ SL_2 \to G_2 \to SO_7 \to SL_7. \]

It is then a simple matter to check, for any \( V \), whether an \( \hat{H} \) containing the principal \( SL_2 \) can act irreducibly on \( V_0 \).

Beyond the examples given in Proposition 10.2, we have one semi-simple example with the same properties:

\[(10.3) \quad \hat{G} = SL_n^2/\Delta \mu_n \quad V = C^n \otimes (C^n)^* \quad \hat{H} = PGL_n.\]

In all cases, \( \hat{H} \) is a group of adjoint type.

**Proposition 10.4.** For the groups \( \hat{G} \) in Proposition 10.2, the center is cyclic of order \( \ell \geq 2 \). The integer \( \ell \) is the number of irreducible representations in the restriction of \( V \) to a principal \( SL_2 \).

The \( \hat{G} \)-invariants in the symmetric algebra on \( V^* \) form a polynomial algebra, on one generator \( d : V \to C \) of degree \( \ell \). The group \( \hat{G} \) has an open orbit on the projective space of lines in \( V \), with connected stabilizer \( \hat{H} \), consisting of the lines where \( d(v) \neq 0 \).

**Proof.** The first assertion is proved by an inspection of the following table. We derive the decomposition of \( V \) from §6.

<table>
<thead>
<tr>
<th>( \hat{G} )</th>
<th>( \ell = \text{order of center} )</th>
<th>\text{decomp. of } V</th>
</tr>
</thead>
<tbody>
<tr>
<td>( SL_2n/\mu_2 )</td>
<td>( n \geq 2 )</td>
<td>( S^{4n-4} + S^{4n-8} + \cdots + S^4 + S^0 )</td>
</tr>
<tr>
<td>( SO_{2n} )</td>
<td>2</td>
<td>( S^{2n-2} + S^0 )</td>
</tr>
<tr>
<td>( E_6 )</td>
<td>3</td>
<td>( S^{16} + S^8 + S^0 )</td>
</tr>
<tr>
<td>( \text{Spin}_7 )</td>
<td>2</td>
<td>( S^6 + S^0 )</td>
</tr>
<tr>
<td>( SL_n^2/\Delta \mu_n )</td>
<td>( n \geq 2 )</td>
<td>( S^{2n-2} + S^{2n-4} + \cdots + S^2 + S^0 )</td>
</tr>
</tbody>
</table>

The calculation of \( S^\bullet(V^*)\hat{G} \) follows from [S-K], which also identifies the connected component of the stabilizer with \( \hat{H} \). Note that the degree of any invariant is divisible by \( \ell \), as the center acts faithfully on \( V^* \).

**11. The representation \( V \) of \( \hat{H} \)**

Recall that \( \ell \geq 2 \) is the order of the cyclic center of \( \hat{G} \), tabulated in 10.5. Since the subgroup \( \hat{H} \subset \hat{G} \) fixing the linear form \( t : V \to C \) is reductive, we have a splitting of \( \hat{H} \)-modules

\[(11.1) \quad V = Ce + V_0 \]
with \( V_0 = \ker(t) \), and \( e \) a vector fixed by \( \hat{H} \) satisfying \( t(e) \neq 0 \). Once \( t \) has been chosen, we may normalize \( e \) by insisting that

\[
(11.2) \quad t(e) = \ell.
\]

**Proposition 11.3.** The representation \( V_0 \) of \( \hat{H} \) is orthogonal. Its weights consist of the short roots of \( \hat{H} \) and the zero weight. The zero weight space for \( \hat{H} \) in \( V \) has dimension \( \ell \), and \( V \) is a polar representation of \( \hat{H} \) of type \( A_{\ell-1} \): the \( \hat{H} \)-invariants in the symmetric algebra of \( V \simeq V^* \) form a polynomial algebra, with primitive generators in degrees \( 1, 2, 3, \ldots, \ell \).

**Proof.** The fact that \( V_0 \) is orthogonal, and its weights, are obtained from a consideration of the table in Proposition 10.2. Since

\[
\dim V = \ell + \# \{ \text{short roots of } \hat{H} \},
\]

this gives the dimension of the zero weight space.

Let \( \hat{S} \subset \hat{H} \) be a maximal torus, with normalizer \( \hat{N} \). The image of \( \hat{N}/\hat{S} \) in \( GL(V^{\hat{S}}) = GL_\ell \) is the symmetric group \( \Sigma_\ell \). The fact that \( V \) is polar follows from the tables in [D-K], which also gives an identification of algebras: \( S^*(V)^{\hat{H}} \simeq S^*(V^{\hat{S}})^{\hat{N}/\hat{S}} \). The latter algebra is generated by the elementary symmetric functions, of degrees \( 1, 2, 3, \ldots, \ell \).

**Note 11.4.** The integer \( \ell \) is also the number of distinct summands in the restriction of \( V \) to a principal \( SL_2 \). Since each summand is an orthogonal representation of \( SL_2 \), \( \ell = \dim V^S_0 \), where \( S_0 \subset SL_2 \) is a maximal torus. Hence \( V^S_0 = V^S \).

We will now define an \( \hat{H} \)-algebra structure on \( V \), with identity element \( e \), in a case by case manner. Although the multiplication law \( V \otimes V \rightarrow V \) is not in general associative, it is power associative, and for \( v \in V \) and \( k \geq 0 \) we can define \( v^k \) in \( V \) unambiguously. The primitive \( \hat{H} \)-invariants in \( S^*(V^*) \) can then be given by

\[
(11.5) \quad v \mapsto t(v^k) \quad 1 \leq k \leq \ell.
\]

In (11.5), \( t: V \rightarrow C \) is the \( \hat{H} \)-invariant linear form, normalized by the condition that

\[
t(e) = \ell.
\]

We will also identify the \( \hat{G} \)-invariant \( \ell \)-form \( \det: V \rightarrow C \), normalized by the condition that

\[
\det(e) = 1.
\]

The simplest case, when the algebra structure on \( V \) is associative, is when \( \hat{H} = PGL_n \) and \( V \) is the adjoint representation (of \( GL_n \)) on \( n \times n \) matrices. The algebra structure is matrix multiplication, \( e \) is the identity matrix, \( t \) is the trace, and \( \det \) is the determinant (which is invariant under the larger group \( \hat{G} = SL_n \times SL_n/\Delta \mu_n \) acting by \( v \mapsto AvB^{-1} \)).

Another algebra structure on \( V \), with the same powers \( v^k \), is given by the Jordan multiplication \( A \circ B = \frac{1}{4}(AB + BA) \). This algebra is isomorphic to the Jordan algebra of Hermitian symmetric \( n \times n \) matrices over the quadratic \( C \)-algebra \( C + C \), with involution \( (z, w) = (w, z) \).

The representation \( V \) has a similar Jordan algebra structure when \( \hat{H} = PSp_{2n} \) and when \( \hat{H} = F_4 \). In the first case, \( V \) is the algebra of Hermitian symmetric \( n \times n \)
matrices over the complex quaternion algebra $M_2(\mathbb{C})$, and in the second $V$ is the algebra of Hermitian symmetric $3 \times 3$ matrices over the complex octonion algebra.

When $\tilde{H} = SO_{2n-1}$, the representation $V = \mathbb{C}e + V_0$ has a Jordan multiplication given by the quadratic form $\langle , \rangle$ on $V$. We normalize this bilinear paring to satisfy $\langle e, e \rangle = 2$, so $\det(v) = \frac{\langle v, v \rangle}{2}$ is the $\tilde{G}$-invariant 2-form on $V$. The multiplication is defined, with $e$ as identity, by giving the product of two vectors $v, w$ in $V_0$:

$$v \cdot w = \frac{1}{2} \langle v, w \rangle e.$$  

Finally, when $\tilde{H} = G_2$, the representation $V$ of dimension 8 has the structure of an octonion algebra, with $t(v) = v + \bar{v}$ and $\det(v) = v \bar{v}$. In all cases but this one $\tilde{H}$ is the connected subgroup of $GL(V)$ preserving $\det$, and $\tilde{H}$ is the subgroup of $GL(V)$ preserving all the forms $t(v^k)$ for $1 \leq k \leq \ell$. In the octonionic case, the subgroup $SO_8 \subset GL_8$ preserves $\det$, and the subgroup $SO_7 \subset SO_8$ preserves $t(v)$ and $t(v^2)$.

In general, $\det: V \to \mathbb{C}$ is a polynomial in the $\tilde{H}$-invariants $t(v^k)$, given by the Newton formulae. The expression for $\ell! \cdot \det$ has integral coefficients; for example,

$$\begin{align*}
(11.6) \\
\begin{cases} \\
2 \ell! \cdot \det(v) = t(v)^2 - t(v^2) & \ell = 2 \\
6 \ell! \cdot \det(v) = t(v)^3 - 3t(v^2)t(v) + 2t(v^3) & \ell = 3.
\end{cases}
\end{align*}$$

12. Representations of $G$ lifted from $H$

We now describe the finite dimensional irreducible holomorphic representations $\pi$ of $G$ which are lifted from irreducible representations $\pi'$ of $H$. This notion of lifting is due to Langlands: the parameter of $\pi$, which is a homomorphism $\varphi: \mathbb{C}^* \to \tilde{G}$ up to conjugacy, should factor through a conjugate of $\tilde{H}$.

We can parameterize the finite dimensional irreducible holomorphic representations $\pi$ of $G$ by their highest weights $\omega$ for $B$. The weight $\omega$ is a positive, integral combination of the fundamental weights $\omega_i$ of the simply-connected cover of $G$, so we may write (using the numeration of $[B]$)

$$\omega = \sum_{i=1}^{\text{rank}(G)} b_i \omega_i \quad \text{for } b_i \geq 0.$$  

(12.1)

For $\omega$ to be a character of $G$, there are some congruences which must be satisfied by the coefficients $b_i$. (The group $G$ is not simply connected, as its dual $\tilde{G}$ acts faithfully on the minuscule representation $V$.)

Since

$$\text{rank}(G) = \text{rank}(H) + (\ell - 1),$$

(12.2)

there are $(\ell - 1)$ linear conditions on the coefficients $b_i$ which are necessary and sufficient for $\pi$ to be lifted from $\pi'$ of $H$. These conditions refine the congruences, and we tabulate them in Table 12.3 below.

When $G = SL_{2n}/\mu_n$, $SO_{2n}$, or $Sp_{2n}/\mu_2$ there are more classical descriptions of $\omega$ in the weight spaces $\mathbb{R}^{2n}$, $\mathbb{R}^n$, and $\mathbb{R}^3$, respectively. We describe, in this language, which representations are lifted from $H$. 
Table 12.3.

\[
\begin{array}{|c|c|c|c|}
\hline
G & H & \omega = \sum b_i \omega_i \text{ of } G & \omega \text{ lifted from } H \\
\hline
SL_{2n}/\mu_n & \text{Spin}_{2n+1} & \sum_{i=1}^{n-1} i(b_i - b_{2n-i}) \equiv 0(n) & b_i = b_{2n-i} \\
& & b_i = b_{2n-i} & 1 \leq i \leq n - 1 \\
SO_{2n} & Sp_{2n-2} & b_{n-1} - b_n \equiv 0(2) & b_{n-1} = b_n \\
E_6/\mu_3 & F_4 & (b_1 - b_3) + 2(b_2 - b_5) \equiv 0(3) & b_1 = b_6 \quad b_2 = b_5 \\
Sp_{6}/\mu_2 & G_2 & b_1 - b_3 \equiv 0(2) & b_1 = b_3 \\
SL_n \times SL_n' / \Delta \mu_n & SL_n & \sum_{i=1}^{n-1} i(b_i - b'_{n-1}) \equiv 0(n) & b_i = b'_{n-1} \\
& & & 1 \leq i \leq n - 1 \\
\hline
\end{array}
\]

For \( G = SL_{2n}/\mu_n \), a dominant weight \( \omega \) is a vector \((a_1, a_2, \ldots, a_{2n})\) in \( \mathbb{R}^{2n} \) with

\[
a_1 \geq a_2 \geq \cdots \geq a_{2n},
\]

\[
a_i \quad \text{ in } \frac{1}{2} \mathbb{Z} \quad 1 \leq i \leq 2n,
\]

\[
a_i \equiv a_j \pmod{\mathbb{Z}},
\]

\[
\sum a_i = 0.
\]

The representations lifted from \( \text{Spin}_{2n+1} \) give dominant weights \( \omega \) with

\[
a_i + a_{2n+1-i} = 0 \quad 1 \leq i \leq n.
\]

In particular, \( a_n \geq 0 \geq a_{n+1} \), as \( a_n + a_{n+1} = 0 \).

For \( G = SO_{2n} \), a dominant weight \( \omega \) is a vector \((a_1, \ldots, a_n)\) in \( \mathbb{Z}^n \) with

\[
a_1 \geq a_2 \geq \cdots \geq a_{n-1} \geq |a_n|.
\]

The representations lifted from \( Sp_{2n-2} \) satisfy \( a_n = 0 \).

Finally, for \( G = Sp_6/\mu_2 \), a dominant weight is given classically as a vector \((a_1, a_2, a_3)\) in \( \mathbb{Z}^3 \) with \( a_1 \geq a_2 \geq a_3 \geq 0 \) and \( a_1 \equiv a_2 + a_3 \pmod{2} \). The representations lifted from \( G_2 \) are those with \( a_1 = a_2 + a_3 \).

Define a connected, reductive subgroup \( L \) of \( G \) as follows:

- \( G = SL_{2n}/\mu_n \) \( L = SL_n^2 / \Delta \mu_n \) fixing a decomposition of the standard representation of \( SL_{2n} : \mathbb{C}^{2n} = \mathbb{C}^n + \mathbb{C}^n \), and having determinant 1 on each factor

- \( G = SO_{2n} \) \( L = SO_{n+1} \) fixing a non-degenerate subspace \( \mathbb{C}^{n-1} \) in the standard representation \( \mathbb{C}^{2n} \)

- \( G = E_6/\mu_3 \) \( L = SL_6/\mu_3 \) fixing the highest and lowest root spaces in the adjoint representation

- \( G = Sp_6/\mu_2 \) \( L = SL_3^2 / \Delta \mu_2 \) fixing a decomposition of the standard representation of \( Sp_6 : \mathbb{C}^6 = \mathbb{C}^2 + \mathbb{C}^2 + \mathbb{C}^2 \) into three non-degenerate, orthogonal subspaces

- \( G = SL_n^2 / \Delta \mu_n \) \( L = PGL_n \) fixing the identity matrix in the representation on \( M_n(\mathbb{C}) \)
Proposition 12.4. The finite dimensional irreducible representation π of G is lifted from H if and only if the space Hom_L(π, C) of L-invariant linear forms on π is non-zero. In this case, the dimension of the space of L-invariant linear forms is given by the following table:

<table>
<thead>
<tr>
<th>G</th>
<th>ω lifted from H</th>
<th>dim Hom_L(π, C)</th>
</tr>
</thead>
<tbody>
<tr>
<td>SL_{2n}/μ_n</td>
<td>b_1(ω_1 + ω_2) + b_2(ω_2 + ω_3) + ... + b_{n-1}(ω_{n-1} + ω_n) + b_nω_n</td>
<td>b_n + 1</td>
</tr>
<tr>
<td>SO_{2n}</td>
<td>b_1ω_1 + b_2ω_2 + ... + b_{n-2}ω_{n-2} + b_{n-1}(ω_{n-1} + ω_n) + b_nω_n</td>
<td>\prod_{1≤i&lt;j≤n-2} b_1b_2+...+b_{j-1}+j-i</td>
</tr>
<tr>
<td>E_6/μ_3</td>
<td>b_1(ω_1 + ω_6) + b_2(ω_2 + ω_3) + b_3(ω_3 + ω_5) + b_4ω_4</td>
<td>(b_2+1)(b_4+1)(b_2+b_4+2)</td>
</tr>
<tr>
<td>Sp_6/μ_2</td>
<td>b_1(ω_1 + ω_3) + b_2ω_2</td>
<td>b_2 + 1</td>
</tr>
<tr>
<td>SL_{n}/Δμ_n</td>
<td>V ⊗ V*</td>
<td>1</td>
</tr>
</tbody>
</table>

13. The proof of Proposition 12.4

The only easy case is when G = SL_{2n}/Δμ_n, so an irreducible π has the form V ⊗ V', where V and V' are irreducible representations of SL_n with inverse central characters. We have

\text{Hom}_L(π, C) = \text{Hom}_{SL_n}(V ⊗ V', C)

This space is non-zero if and only if V' ∼= V*, when it has dimension 1 by Schur’s lemma. These are exactly the π lifted from H.

When G = Sp_6/μ_2 and L = SL_{2n}/μ_n, the space Hom_L(π, C) was considered in [GS]. In the other cases, the subgroup L may be obtained as follows. Let G_R be the quasi-split inner form of G with non-trivial Galois action on the Dynkin diagram, and let K_R be a maximal compact subgroup of G_R. We have

<table>
<thead>
<tr>
<th>G</th>
<th>G_R</th>
<th>K_R</th>
</tr>
</thead>
<tbody>
<tr>
<td>SL_{2n}/μ_n</td>
<td>SU_{n,n}/μ_n</td>
<td>S(U_n × U_n)/μ_n</td>
</tr>
<tr>
<td>SO_{2n}</td>
<td>SO_{n+1,n-1}</td>
<td>S(O_{n+1} × O_{n-1})</td>
</tr>
<tr>
<td>E_6/μ_3</td>
<td>2E_6,4/μ_3</td>
<td>(SU_2 × SU_6/μ_3)/Δμ_2</td>
</tr>
</tbody>
</table>

Note that in each case we have a homomorphism

L ↪ K = complexification of K_R.

The image is a normal subgroup, and the connected component of the quotient is isomorphic to SO_2, SO_n-1, and SO_3, respectively.

There is a real parabolic P_R in G_R associated to the fixed vertices of the Galois action on the Dynkin diagram. The derived subgroup of a Levi factor of P_R is given in the diagram below.

Let B_R be the Borel subgroup of G_R contained in P_R, and let T_R be a Levi factor of B_R.

In the Cartan-Heegelsson theorem, one uses the Cartan decomposition G_R = K_R · B_R to show that K has an open orbit on the complex flag variety
G/B, with stabilizer the subgroup \( T^0 \) of \( T \) fixed by the Cartan involution. The representations \( \pi \) of \( G \) with \( \text{Hom}_K(\pi, \mathbb{C}) \neq 0 \) are those whose highest weight \( \chi \) is trivial on \( T^0 \), in which case \( \text{Hom}_K(\pi, \mathbb{C}) \) has dimension 1. This is proved in [G-W, 12.3], where the subgroup \( T \) is also calculated.

Similarly, one shows that the subgroup \( L \) of \( K \) has an open orbit on the \( G/P \) variety, with stabilizer the connected component of \( T \), which is a torus. The representations \( \pi \) of \( G \) with \( \text{Hom}_L(\pi, \mathbb{C}) \neq 0 \) are those whose highest weight \( \chi \) is trivial on \( (T^0)^0 \). We find that these, after a brief calculation, are those lifted from \( H \). The space \( \text{Hom}_L(\pi, \mathbb{C}) \) is isomorphic, as a representation of \( K/L \), to the irreducible representation of the Levi factor of \( P \) which has highest weight \( \chi \). This completes the proof.

14. The real form of \( L \)

We now descend the subgroup \( L \to G \) defined before Proposition 12.4 to a subgroup \( L_0 \to G_0 \) over \( \mathbb{R} \), by using minuscule co-weights. Let \( S \) be a maximal torus in \( L \), and let \( \lambda : \mathbb{G}_m \to T \) be a minuscule co-weight which occurs in the representation \( V \) of \( G \).

**Proposition 14.1.** There is an inclusion \( \alpha : L \to G \) mapping \( S \) into \( T \), and a minuscule co-weight \( \mu : \mathbb{G}_m \to S \) of \( L \), such that the following diagram commutes:

\[
\begin{array}{ccc}
\mathbb{G}_m & \xrightarrow{\mu} & S \\
\downarrow & & \downarrow \alpha \\
BG_\mu & \xrightarrow{\lambda} & T & \to & G \\
\end{array}
\]

**Proof.** If \( \alpha_0 : L \to G \) is any inclusion, the image of \( S \) is contained in a maximal torus \( T_0 \) of \( G \). Since \( T \) and \( T_0 \) are conjugate, we may conjugate \( \alpha_0 \) to an inclusion \( \alpha : L \to G \) mapping \( S \) into \( T \).

The co-character group \( X_\bullet(S) \) then injects into \( X_\bullet(T) \). To finish the proof, we must identify the image, and show that it intersects the \( W \)-orbit of \( \lambda \) in a single \( W_L \)-orbit of minuscule co-weights for \( L \). We check this case by case. For example, if \( G = E_6/\mu_3 \) and \( L = SL_6/\mu_3 \), the group \( X_\bullet(T) \) is the dual \( E_6^\vee \) of the \( E_6 \)-root lattice, and \( X_\bullet(S) \) is the subgroup orthogonal to a root \( \beta \). One checks, using the tables in Bourbaki [B], that precisely 15 of the 27 elements in the orbit \( W\lambda \) are orthogonal to each \( \beta \), and that these give a single \( W_\beta = W_{SL_6} \) orbit.
In each case, we tabulate the dimension of $T/S$, and the size of the $W_L$-orbit $W\lambda \cap X_{\bullet}(S) = W_L\mu$.

<table>
<thead>
<tr>
<th>$G$</th>
<th>$L$</th>
<th>$\dim(T/S)$</th>
<th>$#W\lambda$</th>
<th>$#W_L\mu$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$SL_{2n}/\mu_n$</td>
<td>$SL_{2n}/\mu_n$</td>
<td>1</td>
<td>$2n^2 - n$</td>
<td>$n^2$</td>
</tr>
<tr>
<td>$SO_{2n}$</td>
<td>$SO_{n+1}$</td>
<td>$\frac{n+1}{2}$ $n$ odd</td>
<td>$2n$</td>
<td>$n + 1$ $n$ odd</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$\frac{n}{2}$ $n$ even</td>
<td></td>
<td>$n$ $n$ even</td>
</tr>
<tr>
<td>$E_6/\mu_3$</td>
<td>$SL_6/\mu_3$</td>
<td>1</td>
<td>27</td>
<td>15</td>
</tr>
<tr>
<td>$Sp_6/\mu_2$</td>
<td>$SL_3/\mu_2$</td>
<td>0</td>
<td>8</td>
<td>8</td>
</tr>
<tr>
<td>$SL_n^2/\mu_n$</td>
<td>$PGL_n$</td>
<td>$n - 1$</td>
<td>$n^2$</td>
<td>$n$</td>
</tr>
</tbody>
</table>

Corollary 14.3. If $L_0$ is the real form of $L$ with Cartan involution $\theta = \text{ad } \mu(-1)$, then $L_0$ embeds as a subgroup of $G_0$ over $\mathbb{R}$. The symmetric space $D_L = L_0(\mathbb{R})/K_{L_0}$ has an invariant complex structure, and embeds analytically into $D$. Moreover,

$$\dim_C D_L = \frac{1}{2} \dim_C D.$$ 

The last inequality is checked, case by case. We tabulate $G_0$, $L_0$, $\dim D$, and $\dim D_L$ below.

<table>
<thead>
<tr>
<th>$G_0$</th>
<th>$L_0$</th>
<th>$\dim D$</th>
<th>$\dim D_L$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$SU_{2,2n-2}/\mu_n$</td>
<td>$SU_{1,n-1}/\mu_n$</td>
<td>$4n - 4$</td>
<td>$2n - 2$</td>
</tr>
<tr>
<td>$SO_{2,2n-2}$</td>
<td>$SO_{2,n-1}$</td>
<td>$2n - 2$</td>
<td>$n - 1$</td>
</tr>
<tr>
<td>$^2E_{6,2}/\mu_3$</td>
<td>$SU_{2,4}/\mu_3$</td>
<td>16</td>
<td>8</td>
</tr>
<tr>
<td>$Sp_6/\mu_2$</td>
<td>$SL_3/\mu_2$</td>
<td>6</td>
<td>3</td>
</tr>
<tr>
<td>$SU_{1,n-1}/\mu_m$</td>
<td>$PU_{1,n-1}$</td>
<td>$2n - 2$</td>
<td>$n - 1$</td>
</tr>
</tbody>
</table>

Since $\dim D_L = \frac{1}{2} \dim D$, this suggests the following problem. Let $G_\mathbb{Q}$ and $L_\mathbb{Q}$ be descents of $G_0$ and $L_0$ to $\mathbb{Q}$, with $L_\mathbb{Q} \hookrightarrow G_\mathbb{Q}$. This gives a morphism of Shimura varieties

$$S_L \to S_G$$

over $\mathbb{C}$, with $\dim(S_L) = \frac{1}{2} \dim S_G$. The algebraic cycles corresponding to $S_L$ contribute to the middle cohomology $H^{\dim S_G} (S_G, \mathbb{C})$. Can these Hodge classes detect the automorphic forms lifted from $H$?

15. THE GROUP $\hat{G}$ IN A LEVI FACTOR

Recall that the center $\mu_\ell$ of $\hat{G}$ is cyclic. Let

$$(15.1) \quad \hat{J} = G_m \times \hat{G}/\Delta \mu_\ell,$$

which is a group with connected center. We first observe that $\hat{J}$ is a Levi factor in a maximal parabolic subgroup $\hat{P}$ of a simple group of adjoint type $\hat{M}$. The minuscule
representation $V$ occurs as the action of $\hat{J}$ on the abelianization of the unipotent radical $\hat{U}$ of $\hat{P}$.

Recall that the maximal parabolic subgroups $\hat{P}$ of $\hat{M}$ are indexed, up to conjugacy, by the simple roots $\alpha$. We tabulate $\hat{M}$, the simple root $\alpha$ corresponding to $\hat{P}$, and the representation $U^{ab} = V$ below:

<table>
<thead>
<tr>
<th>$\hat{G}$</th>
<th>$\hat{M}$</th>
<th>$\alpha$ of $\hat{P}$</th>
<th>$V = U^{ab}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$SL_{2n}/\mu_2$</td>
<td>$PSO_{4n}$</td>
<td>$\alpha_{2n}$</td>
<td>$\mathbb{C}^{2n}$</td>
</tr>
<tr>
<td>$SO_{2n}$</td>
<td>$PSO_{2n+2}$</td>
<td>$\alpha_1$</td>
<td>$\mathbb{C}^{2n}$</td>
</tr>
<tr>
<td>$E_6$</td>
<td>$E_7$</td>
<td>$\alpha_7$</td>
<td>$\mathbb{C}^{27}$</td>
</tr>
<tr>
<td>Spin$_7$</td>
<td>$F_4$</td>
<td>$\alpha_4$</td>
<td>$\mathbb{C}^8$</td>
</tr>
<tr>
<td>$SL_n^2/\mu_n$</td>
<td>$PGL_{2n}$</td>
<td>$\alpha_n$</td>
<td>$\mathbb{C}^n \otimes \mathbb{C}^n$</td>
</tr>
</tbody>
</table>

**Proposition 15.2.** The centralizer of $\hat{H}$ in $\hat{M}$ is $SO_3$, and $\hat{H} \times SO_3$ is a dual reductive pair in $\hat{M}$.

This is checked case by case, and we list the pairs obtained below:

- $SO_3 \times PSp_{2n} \subset PSO_{4n}$,
- $SO_3 \times SO_{2n-1} \subset PSO_{2n+2}$,
- $SO_3 \times F_4 \subset E_7$,
- $SO_3 \times G_2 \subset F_4$,
- $SO_3 \times PGL_n \subset PGL_{2n}$.

**References**


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