REPRESENTATION THEORY
An Electronic Journal of the American Mathematical Society
Volume 4, Pages 330–341 (August 1, 2000)
S 1088-4165(00)00047-9

COMMUTATIVE QUANTUM CURRENT OPERATORS, SEMI-INFINITE CONSTRUCTION AND FUNCTIONAL MODELS

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Abstract. We construct the commutative current operator $\bar{x}^+(z)$ inside $U_q(\hat{\mathfrak{sl}}(2))$. With this operator and the condition of quantum integrability on the quantum currents of $U_q(\hat{\mathfrak{sl}}(2))$, we derive the quantization of the semi-infinite construction of integrable modules of $\hat{\mathfrak{sl}}(2)$ which has been previously obtained by means of the current operator $e(z)$ of $\hat{\mathfrak{sl}}(2)$. The quantization of the functional models for $\hat{\mathfrak{sl}}(2)$ is also given.

1. Introduction

In this paper, we fix the notation that z, w, z_i are commuting formal variables. Given a current operator

$$\bar{a}(z) = \sum_{\mathbb{Z}} \bar{a}(n) z^{-n},$$

if

$$[\bar{a}(z), \bar{a}(w)] = 0,$$

which is equivalent to the condition that all the components $\bar{a}(n)$ commute with each other, then we call the current operator $\bar{a}(z)$ a commutative current operator. Here, we also assume that the current operator $\bar{a}(z)$ always acts on a space F in a truncated way such that, for any element $v \in F$, there exists an integer m such that

$$\bar{a}(n)v = 0$$

if $n > m$. In this case, if a current operator $\bar{a}(z)$ is commutative, then $\bar{a}(z)^n = \bar{a}(z) \times \bar{a}(z) \cdots \times \bar{a}(z)$, for $n \in \mathbb{Z}_{>0}$, is a well defined current operator.

For any integrable highest weight module of $\hat{\mathfrak{sl}}(2)$ of level k, the commutative current operators $e(z)$ and $f(z)$ of $\hat{\mathfrak{sl}}(2)$ satisfy the following relations:

$$e(z)^{k+1} = f(z)^{k+1} = 0,$$

which we call the condition of integrability [LP]. For any integrable highest weight module of $\hat{\mathfrak{sl}}(2)$, there is a natural grading such that the grade of any homogeneous element is always larger or equal to zero and the action of $x(n)$ changes the grade of a homogeneous element by $-n$. This ensures that the current operators from $\hat{\mathfrak{sl}}(2)$ always act in a truncated way. For the case of quantum affine algebras, Drinfeld

Received by the editors April 17, 1998 and, in revised form, January 14, 2000.
2000 *Mathematics Subject Classification.* Primary 17B37.

presented a formulation of affine quantum groups with generators in the form of current operators [Dr3], which, for the case of $U_q(\hat{\mathfrak{sl}}(2))$, give us the quantized current operators corresponding to $e(z)$ and $f(z)$ of $\hat{\mathfrak{sl}}(2)$. In [DM], we derive the quantum integrability condition for $U_q(\hat{\mathfrak{sl}}(2))$. On any level k integrable highest weight module of $U_q(\hat{\mathfrak{sl}}(2))$, the matrix coefficients of $x^+(z_1)x^+(z_2)\cdots x^+(z_{k+1})$ are zero at $z_2/z_1 = z_3/z_2 = \cdots = z_{k+1}/z_k = q^2$, and those of $x^-(z_1)x^-(z_2)\cdots x^-(z_{k+1})$ are zero at $z_1/z_2 = z_2/z_3 = \cdots = z_k/z_{k+1} = q^2$, where $x^+(z)$ and $x^-(z)$ are the quantized current operators of $U_q(\hat{\mathfrak{sl}}(2))$ corresponding to $e(z)$ and $f(z)$ of $\hat{\mathfrak{sl}}(2)$ respectively. In the case of $\hat{\mathfrak{sl}}(2)$, the condition of integrability was used by Feigin and Stoyanovsky [FS1], [FS2] to construct a level k module from a semi-infinite tensor of the components of the current operator $e(z)$ of $\hat{\mathfrak{sl}}(2)$ and to use the function models to describe the dual spaces. With the condition of quantum integrability, still we cannot simply derive the quantization of the semi-infinite construction, because of the noncommutativity of the current operator $x^+(z)$, which is that

$$[x^+(z), x^+(w)] \neq 0.$$

Thus we have to modify the current operator $x^+(z)$ to "force" it to be a commutative current operator. We use the subalgebra coming from the Heisenberg algebra of $U_q(\hat{\mathfrak{sl}}(2))$ to construct a commutative current operator $\bar{x}^+(z) = \Sigma \bar{x}_i z^{-i}$ such that the condition

$$\bar{x}^+(z_1)\bar{x}^+(z_1q^2)\cdots\bar{x}^+(z_1q^{2k}) = 0$$

is satisfied as well. Then the quantization of the semi-infinite construction simply follows. Namely, the integrable modules of $U_q(\hat{\mathfrak{sl}}(2))$ can be identified with the space consisting of semi-infinite expressions $\bar{x}^+_{i_1}\cdots\bar{x}^+_{in}\cdots$, whose tails stabilize in a certain way and $\bar{x}^+_i$ acts by multiplication. Due to the introduction of the parameter q, we can describe the action of the operators explicitly, especially the action of the operator a_{-1} which corresponds to the operator h_{-1} of $\hat{\mathfrak{sl}}(2)$. As in the case of [FS2], the functional models for the dual spaces of the subspace generated by $\bar{x}^+(z)$ on the highest weight vector in any irreducible integrable module of level k are derived by using symmetric functions $f(t_1, \ldots, t_n)$, which are zero when $t_1 = t_2q^2 = \cdots = t_{k+1}q^{2k}$.

2. $U_q(\hat{\mathfrak{sl}}(n))$ AND COMMUTATIVE QUANTUM CURRENT OPERATORS

For the case of affine quantum groups, Drinfeld gave a realization of those algebras in the form of current operators [Dr3]. We will first present such a realization for the case of $U_q(\hat{\mathfrak{sl}}(n))$.

Let $A = (a_{ij})$ be the Cartan matrix of type A_{n-1}.

Definition 1. The algebra $U_q(\mathfrak{sl}_n)$ is an associative algebra with unit 1 and the generators $\varphi_i(-m)$, $\psi_i(m)$, $x^\pm_i(l)$, for $i = 1, \ldots, n-1$, $l \in \mathbb{Z}$ and $m \in \mathbb{Z}_{\geq 0}$ and a central element c. Let $x^\pm_i(z) = \sum_{l\in\mathbb{Z}} x^\pm_i(l)z^{-l}$, $\varphi_i(z) = \sum_{m\in\mathbb{Z}_{\leq 0}} \varphi_i(m)z^{-m}$ and $\psi_i(z) = \sum_{m\in\mathbb{Z}_{\geq 0}} \psi_i(m)z^{-m}$. In terms of the formal variables, the defining relations are

$$\begin{aligned}
&\varphi_i(0)\psi_i(0) = \psi_i(0)\varphi_i(0) = 1,\\
&\varphi_i(z)\varphi_j(w) = \varphi_j(w)\varphi_i(z),\\
&\psi_i(z)\psi_j(w) = \psi_j(w)\psi_i(z),\\
&\varphi_i(z)\psi_j(w)\varphi_i(z)^{-1}\psi_j(w)^{-1} = \frac{g_{ij}(\frac{z}{w}q^{-c})}{g_{ij}(\frac{z}{w}q^{c})},\\
&\varphi_i(z)x_j^{\pm}(w)\varphi_i(z)^{-1} = g_{ij}(\frac{z}{w}q^{\mp\frac{1}{2}c})^{\pm 1}x_j^{\pm}(w),\\
&\psi_i(z)x_j^{\pm}(w)\psi_i(z)^{-1} = g_{ij}(\frac{w}{z}q^{\mp\frac{1}{2}c})^{\mp 1}x_j^{\pm}(w),\\
&[x_i^+(z), x_j^-(w)] = \frac{\delta_{i,j}}{q-q^{-1}}\left\{\delta(\frac{z}{w}q^{-c})\psi_i(wq^{\frac{1}{2}c}) - \delta(\frac{z}{w}q^{c})\varphi_i(zq^{\frac{1}{2}c})\right\},\\
&(z - q^{\pm a_{ij}}w)x_i^{\pm}(z)x_j^{\pm}(w) = (q^{\pm a_{ij}}z - w)x_j^{\pm}(w)x_i^{\pm}(z),\\
&[x_i^{\pm}(z), x_j^{\pm}(w)] = 0 \quad \text{for } a_{ij} = 0,\\
&x_i^{\pm}(z_1)x_i^{\pm}(z_2)x_j^{\pm}(w) - (q+q^{-1})x_i^{\pm}(z_1)x_j^{\pm}(w)x_i^{\pm}(z_2)\\
&\quad + x_j^{\pm}(w)x_i^{\pm}(z_1)x_i^{\pm}(z_2) + \{z_1 \leftrightarrow z_2\} = 0, \quad \text{for } a_{ij} = -1
\end{aligned}$$

where

$$\delta(z) = \sum_{k\in\mathbb{Z}} z^k, \quad g_{ij}(z) = \frac{q^{a_{ij}}z - 1}{z - q^{a_{ij}}} \quad \text{about } z = 0.$$

We define a grading on this algebra such that $x_i^{\pm}(n)$, $\varphi_i(n)$ and $\psi_i(n)$ are of degree n. We also assume that q is a non-zero complex number, which is also not a root of unity; c always acts as a constant; and q^c is defined by the value of the analytic function $e^{x\log q}$ at $x = c$.

Clearly, we have that $x^+(z)$ is not a commutative current operator. In order to modify this operator, we have to rewrite the operators $\varphi_i(z)$ and $\psi_i(z)$ with new operators $a_{i,n}$ for $n \in \mathbb{Z}_{\neq 0}$. From now on, we assume that our current operators act on a highest weight module, where $\varphi_i(0)$ and $\psi_i(0)$, with a suitable weighted basis of the module, act as invertible and diagonal operators.

For the case of $U_q(\mathfrak{sl}_2)$, where we will write $x_1^{\pm}(z)$ as $x^{\pm}(z)$, $a_{1,n}$ as a_n, $\psi_1(z)$ as $\psi(z)$ and $\varphi_1(z)$ as $\varphi(z)$. Because $\psi(0)$ and $\varphi(0)$ are invertible and diagonal, the new operators are defined as

$$\begin{aligned}
-(q-q^{-1})\sum_{k>0} a_{-k}z^k &= \log(1 + (\varphi(z)\varphi(0)^{-1} - 1))\\
&= \sum_{n>0}(-\varphi(z)\varphi(0)^{-1} + 1)^n/n,\\
(q-q^{-1})\sum_{k<0} a_{-k}z^k &= \log(1 + (\psi(z)\psi(0)^{-1} - 1))\\
&= \sum_{n>0}(-\psi(z)\psi(0)^{-1} + 1)^n/n,
\end{aligned}$$

where the right side of the first formula is understood as an infinite series over $n > 0$. We also have that

$$\varphi(z) = \varphi(0)\exp[-(q-q^{-1})\sum_{k>0} a_{-k}z^k],$$
$$\psi(z) = \psi(0)\exp[(q-q^{-1})\sum_{k<0} a_{-k}z^k].$$

Proposition 2.

$$[a_k, a_l] = \delta_{k+l,0}(q^{2k}-q^{-2k})(q^c-q^{-c})/(k(q-q^{-1})^2),$$

$$[a_k, x^{\pm}(l)] = (q^{2k}-q^{-2k})q^{\mp|c|/2}x^{\pm}(k+l)/(k(q-q^{-1})),$$

where k, l are not zero.

Let $k^-(z)$ be a current operator in $U_q(\hat{\mathfrak{sl}}(2))$ such that

$$\text{(I)}\quad k^-(z) = 1 + \sum_{n>0} k^-(n)z^{-n},$$
$$\text{(II)}\quad k^-(z)x^+(w) = \frac{z-wq^2}{z-w}x^+(w)k^-(z),$$

where $k^-(n)$ are operators of degree n, $k^-(n)k^-(m) = k^-(m)k^-(n)$ and $\frac{z-wq^2}{z-w}$ is expanded about zero.

Let $\bar{x}^+(w) = x^+(w)k^-(w)$, then we have

Proposition 3.

$$(z-w)\bar{x}^+(z)\bar{x}^+(w) = (z-w)\bar{x}^+(w)\bar{x}^+(z).$$

The proof comes from the following calculation:

$$\begin{aligned}(z-w)\bar{x}^+(z)\bar{x}^+(w) &= (z-w)x^+(z)k^-(z)x^+(w)k^-(w)\\ &= (z-w)\frac{z-wq^2}{z-w}x^+(z)x^+(w)k^-(z)k^-(w)\\ &= (zq^2-w)x^+(w)x^+(z)k^-(w)k^-(z)\\ &= x^+(w)k^-(w)x^+(z)k^-(z)(zq^2-w)\left(\frac{w-zq^2}{w-z}\right)^{-1}\\ &= (z-w)\bar{x}^+(w)\bar{x}^+(z).\end{aligned}$$

Theorem 4.

$$\bar{x}^+(z)\bar{x}^+(w) = \bar{x}^+(w)\bar{x}^+(z).$$

Proof. Let V_k be a highest weight module of $U_q(\hat{\mathfrak{sl}}(2))$ and V_k^* be its restricted dual. Let $v \in V_k$ and $v^* \in V_k^*$. First, we have that $\langle v^*, x^+(z)x^+(w)v\rangle$ is a formal infinite series in z, w, z^{-1}, w^{-1}. However, we know that this infinite series converges in the complex domain $0 < |w| << |z|$ in $\mathbb{C}^2$, which we can extend to $\mathbb{C}^2$ as a single-valued holomorphic function through analytic continuation. Now, we treat $\langle v^*, x^+(z)x^+(w)v\rangle$ as this complex function. We will denote it by $F(z,w)_{v,v^*}$, which we call the correlation function. From the commutation relation between $x^+(z)$ and $x^+(w)$, we know that the function $F(z,w)_{v,v^*}$ is zero when $z = w$, thus $F(z,w)_{v,v^*}$

always has a factor $z-w$. This implies that $F(z,w)_{v,v^*}$ does not have a pole at $z=w$, thus

$$\bar{x}^+(z)\bar{x}^+(w)=\bar{x}^+(w)\bar{x}^+(z)$$

follows from

$$(z-w)\bar{x}^+(z)\bar{x}^+(w)=(z-w)\bar{x}^+(w)\bar{x}^+(z).$$

□

Proposition 5. *Let*

$$k^-(z)=\exp[(q-q^{-1})\sum_{n>0}-q^{n(2+c/2)}/(1+q^{2n})a_nz^{-n}].$$

Then $k^-(z)$ *satisfies* (I) *and* (II).

Now, we will denote $x^+(z)\exp[(q-q^{-1})\sum_{n>0}-q^{n(2+c/2)}/(1+q^{2n})a_nz^{-n}]$ by $\bar{x}^+(z)$ throughout this paper.

Let $k^+(z)$ be a current operator in $U_q(\hat{\mathfrak{sl}}(2))$ such that

$$\text{(I')}\quad k^+(z)=1+\Sigma_{n<0}k^+(n)z^{-n},$$
$$\text{(II')}\quad k^+(z)x^+(w)=\frac{z-w}{zq^2-w}x^+(w)k^+(z),$$

where $k^+(n)$ are operators of degree n, $k^+(n)k^+(m)=k^+(m)k^+(n)$ and $\frac{z-w}{zq^2-w}$ is expanded about zero. Let $\tilde{x}^+(w)=k^+(w)x^+(w)$. Then we have

Proposition 6.

$$(z-w)\tilde{x}^+(z)\tilde{x}^+(w)=(z-w)\tilde{x}^+(w)\tilde{x}^+(z).$$

The proof comes from the following calculation:

$$\begin{aligned}(z-w)\tilde{x}^+(z)\tilde{x}^+(w)&=(z-w)k^+(z)x^+(z)k^+(w)x^+(w)\\&=(z-w)\frac{z-wq^2}{z-w}k^+(z)k^+(w)x^+(z)x^+(w)\\&=(zq^2-w)k^+(z)k^+(w)x^+(w)x^+(z)\\&=k^+(w)x^+(w)k^+(z)x^+(z)(zq^2-w)\left(\frac{z-w}{zq^2-w}\right)\\&=(z-w)\tilde{x}^+(w)\tilde{x}^+(z).\end{aligned}$$

Theorem 7.

$$\tilde{x}^+(z)\tilde{x}^+(w)=\tilde{x}^+(w)\tilde{x}^+(z).$$

The proof is the same as that of Theorem 4 above.

Proposition 8. *Let*

$$k^+(z)=\exp[-(q-q^{-1})\sum_{n<0}-q^{n(2+c/2)}/(1+q^{2n})a_nz^{-n}].$$

Then $k^+(z)$ *satisfies the condition* (I') *and* (II').

Now, we will denote the operator

$$\exp[-(q-q^{-1})\sum_{n<0} -q^{n(2+c/2)}/(1+q^{2n})a_n w^{-n}]x^+(w)$$

by $\tilde{x}^+(w)$.

For the case of $U_q(\mathfrak{sl}_n)$, the new operators are defined as

$$\begin{aligned}
-(q-q^{-1})\sum_{k>0} a_{i,-k}z^k &= \log(1+(\varphi_i(z)\varphi_i(0)^{-1}-1))\\
&= \sum_{n>0}(-\varphi_i(z)\varphi_i(0)^{-1}+1)^n/n,\\
(q-q^{-1})\sum_{k<0} a_{i,-k}z^k &= \log(1+(\psi_i(z)\psi_i(0)^{-1}-1))\\
&= \sum_{n>0}(-\psi_i(z)\psi_i(0)^{-1}+1)^n/n.
\end{aligned}$$

We also have that

$$\varphi_i(z) = \varphi_i(0)\exp[-(q-q^{-1})\sum_{k>0} a_{i,-k}z^k],$$

$$\psi_i(z) = \psi_i(0)\exp[(q-q^{-1})\sum_{k<0} a_{i,-k}z^k].$$

Let

$$k_i^+(z) = \exp[-(q-q^{-1})\sum_{n<0} -q^{n(2+c/2)}/(1+q^{2n})a_{i,n}z^{-n}],$$

and

$$k_i^-(z) = \exp[(q-q^{-1})\sum_{n>0} -q^{n(2+c/2)}/(1+q^{2n})a_{i,n}z^{-n}].$$

Let

$$\bar{x}_i^+(z) = x_i^+(z)k_i^-(z),$$

and

$$\tilde{x}_i^+(z) = k_i^+(z)x_i^+(z).$$

Theorem 9.

$$\bar{x}_i^+(z)\bar{x}_i^+(w) = \bar{x}_i^+(w)\bar{x}_i^+(z),$$

$$\tilde{x}_i^+(z)\tilde{x}_i^+(w) = \tilde{x}_i^+(w)\tilde{x}_i^+(z).$$

It is obvious that both the set of current operators $\varphi_i(z)$, $\psi_i(z)$, $\tilde{x}_i^+(z)$ and $x_i^-(z)$ and the set of the current operators $\varphi_i(z)$, $\psi_i(z)$, $\bar{x}_i^+(z)$ and $x_i^-(z)$ generate the quantum affine algebra $U_q(\hat{\mathfrak{sl}}(n))$. The reformulation of the quantum affine algebra $U_q(\hat{\mathfrak{sl}}(n))$ with current operators $\varphi_i(z)$, $\psi_i(z)$, $\bar{x}_i^+(z)$ and $x_i^-(z)$ is the key for the quantized semi-infinite construction in the next section; namely, we need to use the kernel coming from the current operator $\bar{x}_i^+(z)$ to define the semi-infinite space. Now, we will restrict ourselves to the case of $U_q(\hat{\mathfrak{sl}}(2))$. The case for $U_q(\hat{\mathfrak{sl}}(n))$ can be dealt with in a similar way [FS1].

For the case of $U_q(\hat{\mathfrak{sl}}(2))$, the relations between $\psi(z)$, $\bar{x}^+(z)$ is the same as that of $\psi(z)$, $x^+(z)$, however, the rest are changed, which we write below.

Proposition 10.

$$\begin{aligned}
&\varphi(z)\bar{x}^+(w)\varphi(z)^{-1} = f_1(\frac{z}{w})g(\frac{z}{w}q^{-\frac{1}{2}c})\bar{x}^+(w),\\
&f_2(w/z)\bar{x}^+(z)x^-(w) - x^-(w)\bar{x}^+(z)\\
&\qquad = \frac{\delta_{i,j}}{q-q^{-1}}\left\{\delta(\frac{z}{w}q^{-c})\psi(wq^{\frac{1}{2}c})k^-(z) - \delta(\frac{z}{w}q^{c})\varphi(zq^{\frac{1}{2}c})k^-(z)\right\},\\
&\bar{x}^+(z)\bar{x}^+(w) = \bar{x}^+(w)\bar{x}^+(z),
\end{aligned}$$

where

$$f_1(\frac{z}{w}) = \left(\frac{(1-\frac{z}{w}q^2q^{c/2})(1-\frac{z}{w}q^{c/2})}{(1-\frac{z}{w}q^2q^{3c/2})(1-\frac{z}{w}q^{3c/2})}\right)^{-1}$$

and

$$f_2(w/z) = (q^2q^cw/z-1)(w/zq^c-1).$$

Similarly, one can write down the relations between $\varphi(z)$, $\psi(z)$, $\tilde{x}^+(z)$ and $x^-(z)$, which we omit here. In the next section, we will use $\bar{x}^+(z)$ instead of $x^+(z)$ as the current operator for our semi-infinite construction of representations of $U_q(\hat{\mathfrak{sl}}(2))$.

3. Quantum integrability condition and semi-infinite construction

The integrability condition of the current operator $e(z)$ induces the semi-infinite construction for the unquantized case. The quantum integrability condition was studied in [DM], which is stated as the following:

Theorem 11. *For any level $k \geq 1$ integrable highest weight module of $U_q(\hat{\mathfrak{sl}}(2))$, the correlation function of $x^+(z_1)x^+(z_2)\cdots x^+(z_k)x^+(z_{k+1})$ is zero if $z_2/z_1 = z_3/z_2 = \cdots = z_{k+1}/z_k = q^2$, the correlation function of $x^-(z_1)x^-(z_2)\cdots x^-(z_k)x^-(z_{k+1})$ is zero if $z_1/z_2 = z_2/z_3 = \cdots = z_k/z_{k+1} = q^2$.*

However, this condition cannot be directly used for the semi-infinite construction, because of the noncommutativity of the current operator $x^+(z)$ of $U_q(\hat{\mathfrak{sl}}(2))$. However, the theorem above implies:

Corollary 12. *For any level $k \geq 1$ integrable highest weight module of $U_q(\hat{\mathfrak{sl}}(2))$,*

$$\bar{x}^+(z_1)\bar{x}^+(z_2)\cdots\bar{x}^+(z_k)\bar{x}^+(z_{k+1}) = 0$$

if $z_2/z_1 = z_3/z_2 = \cdots = z_{k+1}/z_k = q^2$.

Proof. Let $\bar{F}(z_1,\dots,z_n)$ be the correlation function of a vector v in any level $k \geq 1$ integrable module of $U_q(\hat{\mathfrak{sl}}(2))$ and v^* in the dual space of this level k module, $\langle v^*, \bar{x}^+(z_1)\bar{x}^+(z_2)\cdots\bar{x}^+(z_k)\bar{x}^+(z_{k+1})v\rangle$. Then we have

$$\begin{aligned}
&\langle v^*, \bar{x}^+(z_1)\bar{x}^+(z_2)\cdots\bar{x}^+(z_k)\bar{x}^+(z_{k+1})v\rangle\\
&\quad = \langle v^*, \prod_{i<j}\frac{(z_i-z_jq^2)}{z_i-z_j}x^+(z_1)x^+(z_2)\cdots x^+(z_k)x^+(z_{k+1})\\
&\qquad\qquad \times k^+(z_1)k^+(z_2)\cdots k^+(z_k)k^+(z_{k+1})v\rangle.
\end{aligned}$$

Because $\langle v^*, x^+(z_1)x^+(z_2)\cdots x^+(z_k)x^+(z_{k+1})v_1\rangle$ for a vector v_1 is zero in any level $k \geq 1$ integrable module of $U_q(\hat{\mathfrak{sl}}(2))$ if $z_2/z_1 = z_3/z_2 = \cdots = z_{k+1}/z_k = q^2$, and the function $(\prod_{i<j}\frac{(z_i-z_jq^2)}{z_i-z_j})^{-1}$ is not zero if $z_2/z_1=z_3/z_2=\cdots=z_{k+1}/z_k=q^2$, we have

$$\bar{F}(z_1,\dots,z_n)=0.$$

With the preparation above, in this section, we will describe a quantized semi-infinite construction along the line of [FS1], [FS2]. Their starting point for the case of $\hat{\mathfrak{sl}}(2)$ is the integrability condition for level k integrable modules, namely, any level k highest weight module is an integrable module if and only if $e(z)^{k+1}$ is zero. □

Similarly, we can make the following claim:

Theorem 13. *Any level k module of $U_q(\hat{\mathfrak{sl}}(2))$ from the category of representations with highest weight is a sum of irreducible integrable representations if and only if $\bar{x}^+(z)\bar{x}^+(zq^2)\cdots\bar{x}^+(zq^{2k})$ is zero.*

Proof. The theorem above already gives the proof for half of the theorem. The other half comes from the fact that if we quotient by the relation $q=1$, the condition that $\bar{x}^+(z)\bar{x}^+(zq^2)\cdots x^+(zq^{2k})$ is zero simply degenerates into the condition that $e(z)^{k+1}$ is zero. Thus, it is integrable as a module of $\hat{\mathfrak{sl}}(2)$. From the theory of Lusztig, we know that all the integrable highest weight modules must come from the corresponding quantized module. Thus the module is also an integrable module when q is generic.

We will start our semi-infinite construction with the irreducible integrable module $V_{0,1}$ with the highest weight vector $v_{0,1}$ such that the weight of the highest weight vector is 0 and the central element c acts as 1.

Let $\bar{x}^+(z)=\Sigma\bar{x}^+_i z^{-i}$ and $U(\bar{x})$ be the subalgebra generated by $\bar{x}^+_i$. We denote by $U(\bar{x})^-$ the subalgebra generated by $\bar{x}^+_n, n\geq 0$ and by $U(\bar{x})^+$ the subalgebra generated by $\bar{x}^+_n, n<0$. Let $W=U(\bar{x})v_{0,1}$. Because $U(\bar{x})^+v_{0,1}=0$, we have that W is equivalent to $U(\bar{x})^+/Iv_{0,1}$, where I is an ideal. □

Lemma 14. *The ideal I is generated by $S^1_k=\sum_{i\leq k-i}\bar{x}_i\bar{x}_{k-i}(q^{2i}+q^{2k-2i})$, for $k<-1$.*

Proof. From the quantum integrability condition above, we know that the elements S^1_k for $k<-1$ are inside the ideal I. We will denote the ideal generated by those elements by I'. The proof follows from that fact that $U(\bar{x})^+/Iv_{0,1}$ has the same character as the case when we quotient by the relation $q-1=0$. Thus $I=I'$. □

Definition 15. $\bar{V}_{0,1}$ is a vector space with the basis of infinite monomials M of $x_{i_1}x_{i_2}\cdots x_{i_n}\cdots$, where $\{i_1,i_2,\dots\}$ is an infinite sequence of indices such that, for some n, i_n is odd and $i_{p+1}=i_p+2$, if $p>n$. Let $V_{0,1}$ be a quotient space of $\bar{V}_{0,1}$, the quotient is given by the following relations:

(1) $\bar{x}_i$ and $\bar{x}_j$ commute, if $i\neq j$.
(2) If an element $m\in\bar{V}_{0,1}$ contains a segment $\bar{x}_u\bar{x}_{2N+1}\bar{x}_{2N+3}\bar{x}_{2N+5}\cdots$ and $u>2N-1$, then $m=0$.
(3) The operator $S_k=\sum_{a+b=k,a\leq b}\bar{x}_a\bar{x}_b(q^{2b}+q^{2a})$ acts on $\bar{V}_{0,1}$ by $S_kv=0$ for $v\in\bar{V}_{0,1}$.

We define the action of $\bar{x}_i$ simply by multiplication. The action of a_i for $i > 0$ is given by

$$\begin{aligned} a_i\bar{x}_{i_1}\bar{x}_{i_2}\cdots \\ &= [a_i,\bar{x}_{i_1}]\bar{x}_{i_2}\cdots + \bar{x}_{i_1}[a_i,\bar{x}_{i_2}]\bar{x}_{i_3}\cdots \\ &\quad + \bar{x}_{i_1}\bar{x}_{i_2}\cdots[a_i,\bar{x}_{i_n}]\bar{x}_{i_{n+1}}\cdots . \end{aligned}$$

This is a finite expression. We define the action of a_0 by

$$a_0(\bar{x}_{2N+1}\bar{x}_{2N+3}\bar{x}_{2N+5}\cdots) = -2N\bar{x}_{2N+1}\bar{x}_{2N+3}\bar{x}_{2N+5}\cdots .$$

The action of a_{-1} is defined as

$$\begin{aligned} a_{-1}\bar{x}_1\bar{x}_3\bar{x}_5\cdots \\ &= \bar{x}_0\bar{x}_3\bar{x}_5\cdots + \bar{x}_1\bar{x}_2\bar{x}_5\cdots + \bar{x}_1\bar{x}_3\bar{x}_4\bar{x}_7\bar{x}_9\cdots + \bar{x}_1\bar{x}_3\bar{x}_5\bar{x}_6\bar{x}_9 + \cdots \\ &= \bar{x}_0\bar{x}_3\bar{x}_5\cdots - \frac{(q^6+1)}{(q^4+q^2)}\bar{x}_0\bar{x}_3\bar{x}_5\cdots + \frac{(q^6+1)}{(q^4+q^2)}\frac{(q^{10}+q^4)}{(q^6+q^8)}\bar{x}_0\bar{x}_3\bar{x}_5 + \cdots \\ &= \bar{x}_0\bar{x}_3\bar{x}_5\cdots\frac{1}{1+\frac{(q^6+1)}{(q^4+q^2)}}. \end{aligned}$$

Thus, it converges if $|\frac{(q^6+1)}{(q^4+q^2)}| < 1$.

We would like to define the action of a_{-2} as

$$\begin{aligned} a_{-2}\bar{x}_1\bar{x}_3\bar{x}_5\cdots \\ &= (q^2-q^{-2})(\bar{x}_{-1}\bar{x}_3\bar{x}_5\cdots + \bar{x}_1\bar{x}_1\bar{x}_5\cdots + \bar{x}_1\bar{x}_3\bar{x}_3\bar{x}_7\bar{x}_9\cdots + \bar{x}_1\bar{x}_3\bar{x}_5\bar{x}_5\bar{x}_9 + \cdots) \\ &= (q^2-q^{-2})(\bar{x}_{-1}\bar{x}_3\bar{x}_5\cdots - \left(\frac{(q^4+1)}{(2q)}\bar{x}_0\bar{x}_2\bar{x}_5\cdots + \frac{(q^6+q^{-2})}{(2q)}\bar{x}_{-1}\bar{x}_3\bar{x}_5\cdots\right) \\ &\quad - \left(\frac{(q^{10}+q^2)}{(q^6+q^6)}\bar{x}_1\bar{x}_1\bar{x}_5\bar{x}_7\cdots + \frac{(q^8+q^4)}{(q^6+q^6)}\bar{x}_1\bar{x}_2\bar{x}_4\bar{x}_7\cdots\right)\cdots . \end{aligned}$$

To use the relations (1), (2), (3) to reduce this expression to prove the convergence of the expression is very complicated. Similar problems appears in defining the action of $a_{-n}, n > 2$.

Thus, we will use the same trick played in [FS1]. Let $\bar{V}_{0,1}(r)$ be the subspace of $\bar{V}_{0,1}$, which consists of the elements $\bar{x}_{i_1}\cdots\bar{x}_{i_n}\cdots$ and $i_j > r$ for any j.

Lemma 16. *$\bar{V}_{0,1}(r)$ spans the whole space $V_{0,1}$.*

Proof. The proof is the same as that of Lemma 2.5.1 in [FS1] by using the relation (3) to express any element in $V_{0,1}$ with linear combination of elements in $V_{0,1}(r)$. □

For any element expressed in a linear combination of elements in $\bar{V}_{0,1}(r)$, we define the action of $x^-(k)$ from $x^-(z) = \sum x^-(k)z^{-k}$, for $k + r > 0$, as that of a_{-1} by using the commutation relations between $\bar{x}^+(z)$ and $\bar{x}^-(z)$. Because $k + r > 0$, we know that it is well defined. As in [FS1], this is a well defined action, namely, if we express an element in two different ways in $\bar{V}_{0,1}(r)$, the actions of $x^-(k)$ defined above coincide. Again, with the commutation relation between $\bar{x}^+(z)$ and $x^-(z)$, we can define the action of $a_n, n < -1$, because $\bar{x}^+(z)$ and $x^-(z)$ generate the whole algebra. Thus, we have

Theorem 17. *There exists an action of $U_q(\hat{\mathfrak{sl}}(2))$ on the space $\bar{V}_{0,1}$, such that $\bar{V}_{0,1}$ is equivalent to $V_{0,1}$ as a representation of $U_q(\hat{\mathfrak{sl}}(2))$ and the action of $\bar{x}_i$ acts by multiplication.*

Let $\bar{W}$ be the set of the elements $\bar{x}_{i_1}\cdots\bar{x}_{i_n}\cdots$ in $\bar{V}_{0,1}$, such that $i_{j+1}-i_j>1$.

Proposition 18. *$\bar{W}$ forms a basis of the space $\bar{V}_{0,1}$.*

The proof is the same as in [FS1] which gives the character of the representation.

Similarly, as in [FS2], a functional model for the description of $\bar{W}^*$, the dual space of $\bar{W}$, can be derived from the lemma above.

As a commutative algebra, $U(\bar{x})$ can be identified with the space $\mathbb{C}[t,t^{-1}]$. Let $U(\bar{x})^+=\bigoplus U(\bar{x})^+(n)$, where $U(\bar{x})^+(n)$ consists of the elements $\bar{x}^+_{i_1}\bar{x}^+_{i_2}\cdots\bar{x}^+_{i_n}$. We identify any element $\bar{x}^+_{i_1}\bar{x}^+_{i_2}\cdots\bar{x}^+_{i_n}$ in $U(\bar{x})^+(n)$ as $t_1^{i_1}\cdots t_n^{i_n}$, where t_i are variables. Similarly, we can express any element $\bar{x}^+_{i_1}\bar{x}^+_{i_2}\cdots\bar{x}^+_{i_m}$ in $\bar{W}$ as $t_1^{i_1}\cdots t_m^{i_m}$. Let $S^n(\Omega^1\mathbb{C})$ be the space spanned by the expressions $f(t_1,\dots,t_n)dt_1\cdots dt_n$, such that $f(t_1,\dots,t_n)$ is a symmetric function. $S^n(\Omega^1\mathbb{C})$ is also called the space of n particles. We can pair $S^n(\Omega^1\mathbb{C})$ with $U(\bar{x})^+(n)$ by

$$\begin{aligned}&\langle f(t_1,\dots,t_n)dt_1\cdots dt_n, t_1^{i_1}\cdots t_n^{i_n}\rangle\\ &\qquad=\mathrm{Residue}_{t_1=\cdots=t_n=0}(f(t_1,\dots,t_n)t_1^{i_1}\cdots t_n^{i_n}dt_1\cdots dt_n).\end{aligned}$$

Thus, $\bar{W}^*=\bigoplus\bar{W}^*\bigcap S^n(\Omega^1\mathbb{C})$.

Theorem 19.

$$\bar{W}^*\bigcap S^n(\Omega^1\mathbb{C})=\{f(t_1,\dots,t_n)dt_1\cdots dt_n : f=0,\quad \textit{if } t_1=t_2q^2\}.$$

Similarly, we can present the semi-infinite constructions for the higher level cases.

Let $V_{l,k}$ be the irreducible highest weight representation of $U_q(\hat{\mathfrak{sl}}(2))$ with the action of c as k, the highest weight as l times the fundamental weight and $v_{l,k}$ its highest weight vector. Let $W_{l,k}=U(\bar{x})v_{l,k}$. Because $U(\bar{x})^+v_{l,k}=0$, we have that $W_{l,k}$ is equivalent to $U(\bar{x})^+/I_{l,k}v_{l,k}$, where $I_{l,k}$ is an ideal.

Lemma 20. *The ideal $I_{l,k}$ is generated by*

$$S_i^{k+1}=\sum_{\sum a_i=-i,a_i\le a_j}\bar{x}_{a_1}\bar{x}_{a_2}\cdots\bar{x}_{a_{k+1}}(\sum_{\sigma\in S_{k+1}}(q^{\sum_{\sigma(i)=2,K+1}2(\sigma(i)-1)a_{\sigma i}})),$$

$i<-k$ and $\bar{x}_{-1}^{k-l+1}$, if $k-l+1>0$.

Definition 21. Let $\bar{V}_{l,k}$ be the space spanned by the elements of

$$\bar{x}_{i_1}\cdots\bar{x}_{i_n}\bar{x}^l_{2N}\bar{x}^{k-l}_{2N+1}\bar{x}^l_{2N+2}\bar{x}^{k-l}_{2N+3}\cdots,$$

such that

(1) $\bar{x}_i$ commutes with $\bar{x}_j$;
(2) if an element $m\in\bar{V}$ contains a part $\bar{x}_i\bar{x}^l_{2N}\bar{x}^{k-l}_{2N+1}\bar{x}^l_{2N+2}\bar{x}^{k-l}_{2N+3}\cdots$, $i>2N-1$ or $\bar{x}_i\bar{x}^{k-l}_{2N+1}\bar{x}^l_{2N+2}\bar{x}^{k-l}_{2N+3}\cdots$, $i>2N$, then $m=0$;
(3) the operator

$$S_k=\sum_{\sum a_i=n}\bar{x}_{a_1}\bar{x}_{a_2}\cdots\bar{x}_{a_{k+1}}(\sum_{\sigma\in S_{k+1}}(q^{\sum_{\sigma(i)=2,K+1}2(\sigma(i)-1)a_{\sigma i}}))$$

acts on $\bar{V}_{l,k}$ by $S_kv=0$ for $v\in\bar{V}_{l,k}$, where S_{k+1} is the permutation group on $k+1$ numbers.

Theorem 22. *On the space $\bar{V}_{l,k}$, there is an action of $U_q(\hat{\mathfrak{sl}}(2))$, such that the action of $\bar{x}^+(n)$ is given by comultiplication. This representation is the irreducible highest weight representation of $U_q(\hat{\mathfrak{sl}}(2))$ with the action of c as k and the highest weight is l times the fundamental weight.*

Let $\bar{W}_{(l,k)}$ be the set of the elements $\bar{x}_{i_1}\cdots\bar{x}_{i_n}\cdots$ in $\bar{V}_{l,k}$, such that $i_{j+k}-i_j>1$.

Proposition 23. *$\bar{W}_{(l,k)}$ forms a basis of the space $\bar{V}_{l,k}$.*

As in the case of $\bar{V}_{0,1}$, let $U(\bar{x})^+=\bigoplus U(\bar{x})^+(n)$, where $U(\bar{x})^+(n)$ consists of the elements $\bar{x}^+_{i_1}\bar{x}^+_{i_2}\cdots\bar{x}^+_{i_n}$. We identify any element $\bar{x}^+_{i_1}\bar{x}^+_{i_2}\cdots\bar{x}^+_{i_n}$ in $U(\bar{x})^+(n)$ as $t_1^{i_1}\cdots t_n^{i_n}$, where t_i are variables. Similarly, we can express any element $\bar{x}^+_{i_1}\bar{x}^+_{i_2}\cdots\bar{x}^+_{i_m}$ in $W_{l,k}$ as $t_1^{i_1}\cdots t_m^{i_m}$. Let $S^n(\Omega^1\mathbb{C})$ be the space of expressions $f(t_1,\ldots,t_n)dt_1\cdots dt_n$, such that $f(t_1,\ldots,t_n)$ is a symmetric function and different dt_i commute. $S^n(\Omega^1\mathbb{C})$ is also called the space of n particles. We can pair $S^n(\Omega^1\mathbb{C})$ with $U(\bar{x})^+(n)$ by

$$\begin{aligned}&\langle f(t_1,\ldots,t_n)dt_1\cdots dt_n, t_1^{i_1}\cdots t_n^{i_n}\rangle\\ &\qquad=\mathrm{Residue}_{t_1=\cdots=t_n=0}(f(t_1,\ldots,t_n)t_1^{i_1}\cdots t_n^{i_n}dt_1\cdots dt_n).\end{aligned}$$

Thus $\bar{W}^*_{l,k}=\bigoplus\bar{W}^*_{l,k}\bigcap S^n(\Omega^1\mathbb{C})$.

Theorem 24.

$$\begin{aligned}\bar{W}^*_{l,k}\bigcap S^n(\Omega^1\mathbb{C})=\{&f(t_1,\ldots,t_n)dt_1\cdots dt_n : f=0\\ &\textit{if } t_1=t_2q^2=\cdots=t_{k+1}q^{2(k)}\ \textit{or } t_1=\cdots t_{k-l+1}=0\}\end{aligned}$$

if $k-l+1>0$.

In Section 2, we define the operator $\bar{x}^+_i(z)$. It is clear that this can also be applied to other cases. Our next step is to extend the semi-infinite construction to the cases of $U_q(\hat{\mathfrak{g}})$, where $\mathfrak{g}$ is a simply-laced simple Lie algebra, for which we need to define the proper $\bar{x}^+_\alpha(z)$ associated to the roots of $\mathfrak{g}$. The simplest case $\mathfrak{g}=\mathfrak{sl}(3)$ will be the subject of a subsequent paper. On the other hand, this paper follows the algebraic theory developed in [FS1], [FS2], [FS3]. The semi-infinite constructions can be geometrically understood according to the structure of the corresponding infinite dimensional flag manifold and the infinite Schubert cells. The geometric interpretation of the quantized semi-infinite construction is still an open problem. This is related to another immediate problem to extend the explicit construction of the modular functors [FS2] to the quantized case. This should lead us toward the quantization of conformal field theory. It is also possible to extend such a construction to even more general cases. From the point of view of the functional realization of the dual space, one generalization is to substitute the condition $x_1=x_2q^2$, which is a generalization of the classical condition $x_1=x_2$, with more general conditions, for example, $x_1=x_2q_1=x_3q_2$. One would like to ask the following question: what kind of structures are behind the corresponding generalized spaces? We believe it is related to the recent work about the generalization of the quantum affine algebras [DI], where these kind of new conditions should be satisfied for the quantum current operators. We hope our construction can help us to understand the structures of those new algebras in [DI], for which we have not yet been able to give any concrete realization of the non-trivial integrable representations.

Acknowledgment

The authors thank T. Miwa for discussions. We thank R. Endelman for carefully reading though the paper. We would also like to thank the referee for advice.

References

[DM] J. Ding and T. Miwa *Zeros and poles of quantum current operators and the condition of quantum integrability,* q-alg/9608001,RIMS-1092.

[DI] J. Ding and K. Iohara *Generalization and deformation of the quantum affine algebras*, Rims-1090, q-alg/9608002.

[Dr1] V. G. Drinfeld, *Hopf algebra and the quantum Yang-Baxter Equation*, Dokl. Akad. Nauk. SSSR, **283**, 1985, 1060-1064. MR **87h:**58080

[Dr2] V.G. Drinfeld, *Quantum Groups*, ICM Proceedings, New York, Berkeley, 1986, 798-820. MR **89f:**17017

[Dr3] V. G. Drinfeld, *A new realization of Yangian and of quantum affine algebra*, Soviet Math. Doklady, **36**, 1988, 212-216. MR **88j:**17020

[FS1] B. L. Feigin and A. V. Stoyanovsky, *Quasi-particle models for the representations of Lie algebras and the geometry of the flag manifold*, RIMS-942.

[FS2] B. L. Feigin and A. V. Stoyanovsky, *Functional models of the representations of current algebras and the semi-infinite Schubert cells*, Funkts. Anal. Prilozhen., **28**, No. 1, 1994, 68-90. MR **95g:**17027

[FS3] B. L. Feigin and A. V. Stoyanovsky, *Realization of the modular functors in the space of differentials and geometric approximation of the moduli space of G-bundles*, Funkts. Anal. Prilozhen., **28**, No. 4, 1994, 42-65. MR **96k:**32039

[LP] J. Lepowsky and M. Primc, *Structure of standard modules for the affine Lie algebra* $A_1^{[1]}$, Contemp. Math. **45**, Amer. Math. Soc., Providence, 1985. MR **87g:**17021

[L] G. Lusztig *Quantum deformations of certain simple modules over enveloping algebras*, Adv. Math. **70**, 1988, 237-249. MR **89k:**17029

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