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THE CLOSURE DIAGRAMS FOR NILPOTENT ORBITS OF THE REAL FORMS E VI AND E VII OF E_7

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ABSTRACT. Let $\mathcal{O}_1$ and $\mathcal{O}_2$ be adjoint nilpotent orbits in a real semisimple Lie algebra. Write $\mathcal{O}_1 \geq \mathcal{O}_2$ if $\mathcal{O}_2$ is contained in the closure of $\mathcal{O}_1$. This defines a partial order on the set of such orbits, known as the closure ordering. We determine this order for the two noncompact nonsplit real forms of the simple complex Lie algebra E_7.

1. Introduction

The closure diagrams for adjoint nilpotent orbits in noncompact real forms of F_4 and G_2 were determined in [9] and for E_6 in [10]. In this paper we handle the two noncompact and nonsplit real forms of E_7.

By $\mathfrak{g}$ we denote a simple complex Lie algebra of type E_7, by $\mathfrak{g}_0$ a noncompact and nonsplit real form of $\mathfrak{g}$, and by G (respectively G_0) the adjoint group of $\mathfrak{g}$ (respectively $\mathfrak{g}_0$). As usual, let $\mathfrak{g}_0 = \mathfrak{k}_0 \oplus \mathfrak{p}_0$ be a Cartan decomposition of $\mathfrak{g}_0$, $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ its complexification, and θ the Cartan involution. Let σ be the complex conjugation of $\mathfrak{g}$ with respect to $\mathfrak{g}_0$, and let $\mathfrak{h}$ be a σ-stable Cartan subalgebra of $\mathfrak{k}$. Since $\mathfrak{g}_0$ is of inner type, $\mathfrak{h}$ is also a Cartan subalgebra of $\mathfrak{g}$.

Denote by $\mathcal{N}$ the nilpotent variety of $\mathfrak{g}$ and set

$$\mathcal{N}_{\mathbf{R}} = \mathcal{N} \cap \mathfrak{g}_0, \quad \mathcal{N}_1 = \mathcal{N} \cap \mathfrak{p}.$$

Let K^0 be the connected subgroup of G with Lie algebra $\mathfrak{k}$. It is known that the orbit spaces $\mathcal{N}_{\mathbf{R}}/G_0$ and $\mathcal{N}_1/K^0$, equipped with the quotient topologies, are homeomorphic and that the Kostant-Sekiguchi bijection is a homeomorphism $\mathcal{N}_{\mathbf{R}}/G_0 \to \mathcal{N}_1/K^0$ (see [6, 1]). We can think of the closure diagram for adjoint nilpotent orbits in $\mathfrak{g}_0$ as describing the topology of $\mathcal{N}_{\mathbf{R}}/G_0$ (or, equivalently, $\mathcal{N}_1/K^0$).

Our main results are depicted in Figures 2 and 6. In order to obtain these results, it was necessary to perform extensive and nontrivial computations. In addition to our own programs, we used heavily Maple [5] and, to a lesser extent, LiE [17].

2. Preliminaries

The closure diagram for adjoint nilpotent orbits in $\mathfrak{g}$ was determined by Mizuno [13] and verified later by Beynon and Spaltenstein [2]. We give this diagram in Figure 1 where each node represents a G-orbit in $\mathcal{N}$ and is labelled by the corresponding Bala-Carter symbol (see [6, 4]). This diagram is taken from [16] and is modified so that the orbits having the same dimension are positioned at the same

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level. Because of its length, the diagram is split into two pieces. The dimensions of the orbits are indicated on both sides of the figure. We remark that the closure diagram given in [4] for this case is incorrect: The line joining the nodes $D_6(a_2)$ and $D_5(a_1)+A_1$ is missing.

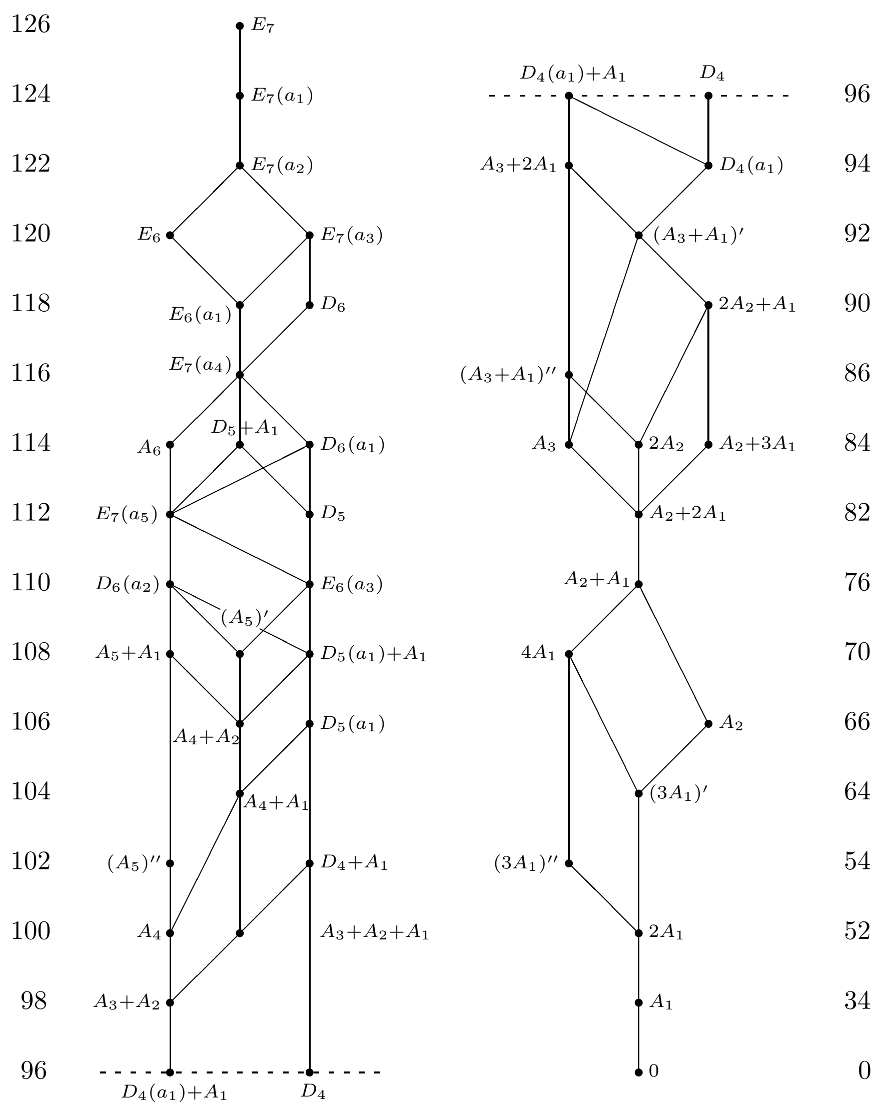

FIGURE 1. The closure diagram for E_7

There are 45 adjoint nilpotent orbits in $\mathfrak{g}$ (including the trivial orbit). The nonzero ones are listed in Table 1. The k-th orbit, i.e., the one that appears as the k-th entry in Table 1, will be denoted by $\mathcal{O}^k$. The second column of this table contains the Bala-Carter symbol of $\mathcal{O}^k$, and the third one gives the weighted Dynkin diagram of $\mathcal{O}^k$. The complex dimension of $\mathcal{O}^k$ is recorded in the last column.

TABLE 1. Nonzero nilpotent orbits in E_7

k		$\alpha_j(H)$	E V	E VI	E VII	dim
1	A_1	1000000	1	1	1,2	34
2	$2A_1$	0000010	2	2,3	3,4,5	52
3	$(3A_1)''$	0000002	3,4		6,7,8,9	54
4	$(3A_1)'$	0010000	5	4,5		64
5	A_2	2000000	6,7	6,7,8	10	66
6	$4A_1$	0100001	8,9			70
7	$A_2 + A_1$	1000010	10,11,12	9	11,12	76
8	$A_2 + 2A_1$	0001000	13,14,15	10,11		82
9	$A_2 + 3A_1$	0200000	16,17,18,19			84
10	A_3	2000010	20	12,13	13,14	84
11	$2A_2$	0000020	21	14,15	15	84
12	$(A_3 + A_1)''$	2000002	22,23		16,17,18,19	86
13	$2A_2 + A_1$	0010010	24	16		90
14	$(A_3 + A_1)'$	1001000	25	17,18		92
15	$D_4(a_1)$	0020000	26,27	19,20,21		94
16	$A_3 + 2A_1$	1000101	28,29			94
17	D_4	2020000	30	22,23		96
18	$D_4(a_1) + A_1$	0110001	31,32,33,34			96
19	$A_3 + A_2$	0001010	35,36,37	24		98
20	A_4	2000020	38,43	25,26	20	100
21	$A_3 + A_2 + A_1$	0000200	39,40,41,42			100
22	$(A_5)''$	2000022	44,45		21,22	102
23	$D_4 + A_1$	2110001	46,47			102
24	$A_4 + A_1$	1001010	48,49,50	27		104
25	$D_5(a_1)$	2001010	51,52,53	28		106
26	$A_4 + A_2$	0002000	54	29		106
27	$D_5(a_1) + A_1$	2000200	55,56,57,58			108
28	$(A_5)'$	1001020	59	30		108
29	$A_5 + A_1$	1001012	60,61			108
30	$E_6(a_3)$	0020020	62,63	31,32		110
31	$D_6(a_2)$	0110102	64,65			110
32	D_5	2020020	66	33,34		112
33	$E_7(a_5)$	0002002	67,68,69,70			112
34	A_6	0002020	71	35		114
35	$D_6(a_1)$	2110102	72,73			114
36	$D_5 + A_1$	2110110	74,75			114
37	$E_7(a_4)$	2002002	76,77,78,79			116
38	$E_6(a_1)$	2002020	80,81	36		118
39	D_6	2110122	82,83			118
40	E_6	2022020	84	37		120
41	$E_7(a_3)$	2002022	85,86,87,88			120
42	$E_7(a_2)$	2220202	89,90			122
43	$E_7(a_1)$	2220222	91,92			124
44	E_7	2222222	93,94			126

The nonzero G_0-orbits in $\mathcal{N}_{\mathbf{R}}$, or equivalently the nonzero K^0-orbits in $\mathcal{N}_1$, were classified in [7] (see also [6]). We shall keep the same numbering as in these two references for these orbits. The i-th nontrivial G_0-orbit in $\mathcal{N}_{\mathbf{R}}$ will be denoted by $\mathcal{O}_0^i$, and we denote by $\mathcal{O}_1^i$ the nontrivial K^0-orbit in $\mathcal{N}_1$ that corresponds to $\mathcal{O}_0^i$ under the Kostant-Sekiguchi bijection. In the fourth, fifth, and sixth columns of Table 1 we list the superscripts i of the orbits $\mathcal{O}_0^i$ (or, equivalently, $\mathcal{O}_1^i$) which are contained in $\mathcal{O}^k$. This depends on the type of the real form $\mathfrak{g}_0$ of $\mathfrak{g}$ (for the sake of completeness we have included also the split real form E V). For instance, if $k = 2$, then

$$\begin{array}{lll} \text{E V:} & \mathcal{O}^2 \cap \mathfrak{g}_0 = \mathcal{O}_0^2, & \mathcal{O}^2 \cap \mathfrak{p} = \mathcal{O}_1^2; \\ \text{E VI:} & \mathcal{O}^2 \cap \mathfrak{g}_0 = \mathcal{O}_0^2 \cup \mathcal{O}_0^3, & \mathcal{O}^2 \cap \mathfrak{p} = \mathcal{O}_1^2 \cup \mathcal{O}_1^3; \\ \text{E VII:} & \mathcal{O}^2 \cap \mathfrak{g}_0 = \mathcal{O}_0^3 \cup \mathcal{O}_0^4 \cup \mathcal{O}_0^5, & \mathcal{O}^2 \cap \mathfrak{p} = \mathcal{O}_1^3 \cup \mathcal{O}_1^4 \cup \mathcal{O}_1^5. \end{array}$$

Recall that a triple (E, H, F) in $\mathfrak{g}$ is called a *standard triple* if $[H, E] = 2E$, $[H, F] = -2F$, $[F, E] = H$ and E, H, F are nonzero. Such a triple is *normal* if also $H \in \mathfrak{k}$ and $E, F \in \mathfrak{p}$. We denote the root system of $(\mathfrak{g}, \mathfrak{h})$ by R, choose a system of positive roots $R^+ \subset R$ and a base $B = \{\alpha_i : 1 \le i \le 7\} \subset R^+$ of R. The simple roots $\alpha_i \in B$ are indexed as in [3].

Let us also introduce the subgroup $K = \{x \in G : \theta(x) = x\}$. Its identity component is the group K^0 defined above. In the case E VI we have $K = K^0$, while in the case E VII the group K is not connected and $K/K^0 = Z_2$. (By Z_k we denote a cyclic group of order k.)

We extend the enumeration of simple roots $\alpha_i,\ 1 \le i \le 7$, to the enumeration $\alpha_i,\ 1 \le i \le 63$, of R^+. It is the same as the one used in [8]. We have reproduced it in the Appendix. A negative root $-\alpha_i$ will be also written as α_{-i}. The coroot of α_i is denoted by $H_i \in \mathfrak{h}$. Note that $H_{-i} = -H_i$. For $\alpha \in R$ we let $\mathfrak{g}^\alpha$ be the root space of α. A nonzero element $X_\alpha \in \mathfrak{g}^\alpha$ is called a *root vector* of α. We assume that a root vector X_i is fixed for each root $\alpha_i,\ \pm i \in \{1, \ldots, 63\}$.

By adjoining the negative of the highest root, $\alpha_0 = -\alpha_{63} = \alpha_{-63}$ to B, we obtain the so-called extended base $B_e = B \cup \{\alpha_0\}$. Let R_0 be the root system of $(\mathfrak{k}, \mathfrak{h})$ where we view R_0 as a subsystem of R. We set $R_0^+ = R_0 \cap R^+$ and denote by B_0 the unique base of R_0 contained in R_0^+. It turns out that $B_0 \subset B_e$. Explicitly we have

$$\begin{array}{ll} \text{E V:} & B_0 = \{\alpha_0, \alpha_1, \alpha_3, \alpha_4, \alpha_5, \alpha_6, \alpha_7\}, \\ \text{E VI:} & B_0 = \{\alpha_0, \alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_7\}, \\ \text{E VII:} & B_0 = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6\}. \end{array}$$

Given a K^0-orbit $\mathcal{O}_1^i \subset \mathcal{N}_1$, we can choose a normal triple (E, H, F) such that $E \in \mathcal{O}_1^i,\ H \in \mathfrak{h}$, and $\alpha(H) \ge 0$ for all $\alpha \in B_0$. If $\mathfrak{g}_0$ is of type E V or E VI we set $B_0' = B_0$, while for type E VII we set $B_0' = B$. The integers $\alpha(H)$ for $\alpha \in B_0'$ determine uniquely H and, consequently, also the orbit $\mathcal{O}_1^i$.

In the case E VI we set, as in [7],

$$\beta_1 = \alpha_0,\ \beta_2 = \alpha_1,\ \beta_3 = \alpha_3,\ \beta_4 = \alpha_4,\ \beta_5 = \alpha_5,\ \beta_6 = \alpha_2,\ \beta_7 = \alpha_7;$$

and in the case E VII we set $\beta_i = \alpha_i,\ 1 \le i \le 7$.

The technique developed in [9] to find these closure diagrams is especially convenient for real forms of inner type and will be employed in this paper. There are two (up to isomorphism) noncompact and nonsplit real forms $\mathfrak{g}_0$ of $\mathfrak{g}$: E VI $= E_{7(-5)}$ and E VII $= E_{7(-25)}$. The closure diagrams for these two cases are given by Figures 2 and 6.

TABLE 2. Nonzero nilpotent orbits in $\mathfrak{p}$ ($\mathfrak{g}_0 = $ E VI)

k	i	$\beta_j(H^i)$	$E^i \in \mathcal{O}_1^i$	Type
1	1	000010 1	X_{-6}	A_1
2	2	010000 2	$(X_{-6})+(X_{-40})$	$2A_1$
2	3	000100 0	$(X_{-6})+(X_{-19})$	$2A_1$
4	4	000010 3	$(X_{45})+(X_{-12})+(X_{-35})$	$(3A_1)'$
4	5	010010 1	$(X_{-12})+(X_{-13})+(X_{-35})$	$(3A_1)'$
5	6	000000 4	$X_{13}+X_{-6}$	A_2
			$(X_{13})+(X_{45})+(X_{-12})+(X_{-35})$	$(4A_1)'$
5	7	000020 2	$(X_{45})+(X_{-12})+(X_{-13})+(X_{-35})$	$(4A_1)'$
5	8	020000 0	$X_{-6}+X_{-45}$	A_2
			$(X_{-6})+(X_{-19})+(X_{-40})+(X_{-41})$	$(4A_1)'$
7	9	110001 1	$(X_{-13}+X_{-40})+(X_{-27})$	A_2+A_1
8	10	200100 0	$(X_{-29}+X_{-34})+(X_{-18})+(X_{-30})$	A_2+2A_1
8	11	010100 2	$(X_{-13}+X_{-40})+(X_{54})+(X_{-27})$	A_2+2A_1
10	12	010020 4	$X_{45}+X_{-40}+X_{-13}$	A_3
10	13	000120 2	$X_{45}+X_{-19}+X_{-35}$	A_3
11	14	400000 0	$(X_{-6}+X_{-56})+(X_{-19}+X_{-50})$	$2A_2$
11	15	000200 0	$(X_{50}+X_{-18})+(X_{-29}+X_{-34})$	$2A_2$
13	16	010110 1	$(X_{54}+X_{-31})+(X_{-27}+X_{-30})+(X_{-35})$	$2A_2+A_1$
14	17	010030 1	$(X_{48}+X_{-19}+X_{-38})+(X_{-41})$	$(A_3+A_1)'$
14	18	010110 3	$(X_{48}+X_{-19}+X_{-38})+(X_{-40})$	$(A_3+A_1)'$
15	19	000040 0	$X_{40}+X_{-12}+X_{-35}+X_{-41}$	$D_4(a_1)$
			$(X_{40}+X_{-12}+X_{-41})+(X_{48})+(X_{-38})$	$(A_3+2A_1)'$
			$(X_{40}+X_{-12})+(X_{48}+X_{-25})+$ $(X_{-36}+X_{-38})$	$3A_2$
15	20	000200 4	$X_{41}+X_{-13}+X_{-35}+X_{-40}$	$D_4(a_1)$
			$(X_{41}+X_{-13}+X_{-40})+(X_{48})+(X_{-38})$	$(A_3+2A_1)'$
15	21	020020 2	$(X_{48}+X_{-19}+X_{-38})+(X_{-40})+(X_{-41})$	$(A_3+2A_1)'$
17	22	000040 8	$X_{19}+X_{-13}+X_{41}+X_{-40}$	D_4
17	23	020040 4	$X_{-19}+X_{45}+X_{-40}+X_{-41}$	D_4
19	24	201011 2	$(X_{56}+X_{-30}+X_{-42})+(X_{-31}+X_{-43})$	A_3+A_2
20	25	040000 4	$X_{-13}+X_{34}+X_{-27}+X_{-45}$	A_4
			$(X_{34}+X_{-13}+X_{-50})+(X_{56}+X_{-45}+X_{-33})$	$2A_3$
20	26	020200 0	$X_{-35}+X_{50}+X_{-27}+X_{-45}$	A_4
			$(X_{50}+X_{-35}+X_{-34})+(X_{56}+X_{-45}+X_{-33})$	$2A_3$
24	27	111110 1	$(X_{-40}+X_{-34}+X_{54}+X_{-41})+(X_{-38})$	A_4+A_1
25	28	201031 4	$X_{-30}+X_{45}+X_{-31}+X_{-40}+X_{-34}$	$D_5(a_1)$
26	29	004000 0	$(X_{-24}+X_{42}+X_{-33}+X_{-41})+(X_{51}+X_{-34})$	A_4+A_2
28	30	010310 3	$X_{48}+X_{-34}+X_{-40}+X_{50}+X_{-38}$	$(A_5)'$
30	31	020220 2	$(X_{48}+X_{-34}+X_{-40}+X_{50}+X_{-38})+(X_{-41})$	$(A_5+A_1)'$
30	32	000400 4	$X_{41}+X_{-35}+X_{50}+X_{-34}+X_{-25}+X_{-36}$	$E_6(a_3)$
			$(X_{41}+X_{-31}+X_{-38}+X_{50}+X_{-40})+(X_{48})$	$(A_5+A_1)'$
32	33	020240 4	$X_{-41}+X_{45}+X_{-40}+X_{50}+X_{-34}$	D_5
32	34	040040 8	$X_{-41}+X_{45}+X_{-19}+X_{34}+X_{-50}$	D_5
34	35	400400 0	$X_{-43}+X_{50}+X_{-35}+X_{-51}+X_{56}+X_{-44}$	A_6

TABLE 2. (continued)

k	i	$\beta_j(H^i)$	$E^i \in \mathcal{O}_1^i$	Type
38	36	040400 4	$X_{50}+X_{-38}+X_{44}+X_{-34}+X_{-41}+X_{-45}$	$E_6(a_1)$
			$X_{51}+X_{-34}+X_{44}+X_{-41}+X_{-43}$ $+X_{50}+X_{-42}$	A_7
40	37	040440 8	$X_{50}+X_{45}+X_{-43}+X_{-41}+X_{44}+X_{-34}$	E_6

For any integer j we define the subspaces

$$\mathfrak{g}_H(0,j) = \{X \in \mathfrak{k} : [H,X] = jX\},$$
$$\mathfrak{g}_H(1,j) = \{X \in \mathfrak{p} : [H,X] = jX\},$$

and for integers $i \geq 1$ we set

$$\mathfrak{p}_i(H) = \sum_{j \geq i} \mathfrak{g}_H(1,j).$$

By Q_H we denote the parabolic subgroup of K^0 with Lie algebra

$$\mathfrak{q}_H = \sum_{j \geq 0} \mathfrak{g}_H(0,j).$$

3. TYPE E VI

In this section $\mathfrak{g}_0$ is of type E VI. Hence $K = (\mathrm{Spin}_{12}/Z_2 \times \mathrm{SL}_2)/Z_2$, where Spin_{12}/Z_2 is the so-called semispin group (not isomorphic to SO_{12}). There are exactly 37 nontrivial K-orbits in $\mathcal{N}_1$ denoted by $\mathcal{O}_1^i$, $1 \leq i \leq 37$. We choose a normal triple (E^i, H^i, F^i) with $E^i \in \mathcal{O}_1^i$, $H^i \in \mathfrak{h}$, and such that $\beta_j(H^i) \geq 0$, $1 \leq j \leq 7$.

These 37 K-orbits are listed in Table 2. For each $i \in \{1, \ldots, 37\}$ we record in the first column the integer k such that $\mathcal{O}_1^i \subset \mathcal{O}^k$. The third column lists the integers $\beta_j(H^i)$, $1 \leq j \leq 7$. As $\mathfrak{k}$ is of type $D_6 + A_1$, and $\{\beta_i :\ 1 \leq i \leq 6\}$ is a base for this D_6, we separate the last integer $\beta_7(H^i)$ from the first six. The last two columns give a representative $E^i \in \mathcal{O}_1^i$ and its type (to be defined below). In some cases we give several representatives of different types.

A subalgebra of $\mathfrak{g}$ is called *regular* if it is normalized by a Cartan subalgebra of $\mathfrak{g}$. A regular subalgebra is *standard* if it is normalized by $\mathfrak{h}$. Of course, every regular subalgebra is G-conjugate to a standard one. Most of the time, two isomorphic regular semisimple subalgebras are G-conjugate but there are 6 exceptions (see [11]):

$$3A_1,\ 4A_1,\ A_3 + A_1,\ A_3 + 2A_1,\ A_5,\ A_5 + A_1.$$

In each of these cases, say X, there are two G-conjugacy classes: $(X)'$ and $(X)''$. A representative of $(3A_1)'$ is the subalgebra with simple roots $\{\alpha_3, \alpha_5, \alpha_7\}$, and for $(3A_1)''$ that with simple roots $\{\alpha_2, \alpha_5, \alpha_7\}$. A subalgebra of type $(A_3 + A_1)'$ respectively $(A_3 + A_1)''$ contains a regular subalgebra of type $(3A_1)'$ respectively $(3A_1)''$. Similarly, $(A_5)'$ respectively $(A_5)''$ contains $(3A_1)'$ respectively $(3A_1)''$. The regular subalgebras of types $(4A_1)''$, $(A_3 + 2A_1)''$, and $(A_5 + A_1)''$ are Levi subalgebras of $\mathfrak{g}$ while those of types $(4A_1)'$, $(A_3 + 2A_1)'$, and $(A_5 + A_1)'$ are not.

In most cases, the representative $E^i \in \mathcal{O}_1^i$ is the sum of root vectors for simple roots of a standard regular semisimple subalgebra and the type of E^i is, by definition, the type of that subalgebra (up to G-conjugacy). If this is not the case, then the type of E^i is the Bala-Carter symbol of the orbit $\mathcal{O}^k$ containing $\mathcal{O}_1^i$.

TABLE 3. Root spaces in $\mathfrak{p}_2(H^i)$

i	Indices of roots
1	$-6;$
2	$-6, -12, -18, -23, -24, -29, -35, -40;$
3	$-6, -12, -13, -19;$
4	$45, 48, 51, 54, 56, -12, -18, -23, -24, -27, -29, -33, -35, -38, -42;\ -6$
5	$-12, -13, -18, -23, -24, -29, -35;\ -6$
6	$13, 19, 25, 30, 31, 34, 36, 39, 41, 44, 45, 47, 48, 51, 54, 56, -6, -12, -18, -23, -24, -27, -29, -33, -35, -38, -40, -42, -43, -46, -50, -53;$
7	$45, 48, 51, 54, 56, -12, -13, -18, -23, -24, -27, -29, -33, -35, -38, -42;\ -6$
8	$-6, -12, -13, -18, -19, -23, -24, -25, -29, -30, -31, -35, -36, -40, -41, -45;$
9	$-13, -19, -23, -25, -27, -29, -31, -35, -40;\ -6, -12, -18, -24$
10	$-18, -23, -24, -25, -27, -29, -30, -31, -33, -34, -36, -39;\ -6, -12, -13, -19$
11	$54, 56, -13, -19, -27, -33, -35, -40;\ -18, -23, -24, -29, -6, -12$
12	$45, -13, -40;\ 48, 51, 54, 56, -27, -33, -38, -42, -12, -18, -23, -24, -29, -35, -6$
13	$45, 48, 51, -19, -35, -38, -42;\ 54, 56, -18, -23, -24, -27, -29, -33, -12, -13, -6$
14	$-6, -12, -13, -18, -19, -23, -24, -25, -27, -29, -30, -31, -33, -34, -35, -36, -38, -39, -40, -41, -42, -43, -44, -45, -46, -47, -48, -50, -51, -53, -54, -56;$
15	$50, 53, 54, 56, -18, -23, -24, -25, -27, -29, -30, -31, -33, -34, -36, -39;\ -6, -12, -13, -19$
16	$54, 56, -25, -27, -30, -31, -33, -35, -36;\ -18, -19, -23, -24, -29, -12, -13, -6$
17	$48, 51, 54, 56, -19, -25, -27, -30, -31, -33, -36, -38, -41, -42;\ -12, -18, -23, -24, -29, -35, -13, -6$
18	$48, 51, -19, -38, -40, -42;\ 54, 56, -13, -27, -33, -35, -18, -23, -24, -29, -12, -6$
19	$40, 43, 45, 46, 48, 50, 51, 53, 54, 56, -12, -18, -19, -23, -24, -25, -27, -29, -30, -31, -33, -34, -35, -36, -38, -39, -41, -42, -44, -47;\ -6, -13$
20	$41, 44, 45, 47, 48, 51, -13, -19, -35, -38, -40, -42, -43, -46;\ 54, 56, -18, -23, -24, -27, -29, -33, -6, -12$
21	$48, 51, 54, 56, -19, -25, -27, -30, -31, -33, -36, -38, -40, -41, -42;\ -12, -13, -18, -23, -24, -29, -35, -6$
22	$19, 25, 30, 31, 34, 36, 39, 41, 44, 47, -13, -40, -43, -46, -50, -53;\ 45, 48, 51, 54, 56, -12, -18, -23, -24, -27, -29, -33, -35, -38, -42, -6$
23	$45, -19, -25, -30, -31, -36, -40, -41;\ 48, 51, 54, 56, -27, -33, -38, -42, -12, -13, -18, -23, -24, -29, -35, -6$
24	$56, -30, -31, -34, -40, -42, -43;\ -19, -25, -29, -33, -35, -38, -13, -23, -24, -27, -12, -18, -6$

TABLE 3. (continued)

i	Indices of roots
25	$34, 39, 44, 47, 48, 51, 54, 56, -13, -19, -25, -27, -30, -31, -33, -36, -38, -41, -42, -43, -45, -46, -50, -53;\ -6, -12, -18, -23, -24, -29, -35, -40$
26	$50, 53, 54, 56, -27, -33, -34, -35, -39, -40, -41, -45;\ -18, -23, -24, -25, -29, -30, -31, -36, -6, -12, -13, -19$
27	$54, 56, -34, -38, -39, -40, -41;\ -27, -31, -33, -35, -36, -24, -25, -29, -30, -18, -19, -23, -12, -13, -6$
28	$45, 48, -30, -31, -34, -40, -43;\ 51, 54, -19, -25, 56, -42, -29, -33, -35, -38, -13, -23, -24, -27, -12, -18, -6$
29	$42, 46, 47, 50, 51, 53, 54, 56, -24, -27, -29, -31, -33, -34, -35, -36, -38, -39, -40, -41, -43, -44, -45, -48;\ -6, -12, -13, -18, -19, -23, -25, -30$
30	$48, 50, 51, 53, -34, -38, -39, -40, -42;\ -25, -30, -31, -35, -36, 54, 56, -27, -33, -18, -19, -23, -24, -29, -13, -12, -6$
31	$48, 50, 51, 53, -34, -38, -39, -40, -41, -42;\ 54, 56, -25, -27, -30, -31, -33, -35, -36, -18, -19, -23, -24, -29, -12, -13, -6$
32	$41, 44, 45, 47, 48, 50, 51, 53, -25, -30, -31, -34, -35, -36, -38, -39, -40, -42, -43, -46;\ 54, 56, -13, -18, -19, -23, -24, -27, -29, -33, -6, -12$
33	$45, 50, 53, -34, -39, -40, -41;\ 48, 51, -25, -30, -31, -36, -38, -42, 54, 56, -19, -27, -33, -35, -18, -23, -24, -29, -12, -13, -6$
34	$34, 39, 44, 45, 47, -19, -25, -30, -31, -36, -41, -43, -46, -50, -53;\ 48, 51, 54, 56, -13, -27, -33, -38, -40, -42, -12, -18, -23, -24, -29, -35, -6$
35	$50, 53, 54, 56, -35, -38, -40, -41, -42, -43, -44, -45, -46, -47, -48, -51;\ -18, -23, -24, -25, -27, -29, -30, -31, -33, -34, -36, -39, -6, -12, -13, -19$
36	$44, 47, 48, 50, 51, 53, -34, -38, -39, -41, -42, -43, -45, -46;\ 54, 56, -25, -27, -30, -31, -33, -35, -36, -40, -13, -18, -19, -23, -24, -29, -6, -12$
37	$44, 45, 47, 50, 53, -34, -39, -41, -43, -46;\ 48, 51, -25, -30, -31, -36, -38, -40, -42, 54, 56, -19, -27, -33, -35, -13, -18, -23, -24, -29, -12, -6$

In Table 3 we list, for each i, the indices k of the roots $\alpha = \alpha_k$ for which $\mathfrak{g}^\alpha \subset \mathfrak{p}_2(H^i)$. We list first those indices for which $\mathfrak{g}^\alpha \subset \mathfrak{g}_{H^i}(1,2)$ and separate them by a semi-colon from the indices for which $\mathfrak{g}^\alpha \subset \mathfrak{p}_3(H^i)$.

Theorem 3.1. *Let $\mathfrak{g}_0$ be of type* E VI. *Then the closure ordering of the nilpotent K-orbits in $\mathfrak{p}$ is as given in Figure* 2.

The horizontal dotted lines indicate that the K-orbits joined by these lines are contained in the same G-orbit. The numbers on the right-hand side of the diagram are the complex dimensions of the orbits on that level.

Proof. Let i, j be a pair of nodes in the diagram of Figure 2, with i above j, which are joined by a solid line. We prove that $\mathcal{O}_1^i > \mathcal{O}_1^j$ by showing that $\mathcal{O}_1^j$ meets $\mathfrak{p}_2(H^i)$ (see [9, Theorem 3.1]). In Table 4 we list all such pairs i, j and for each of them provide an element $E \in \mathfrak{p}_2(H^i) \cap \mathcal{O}_1^j$. We also indicate the type of E.

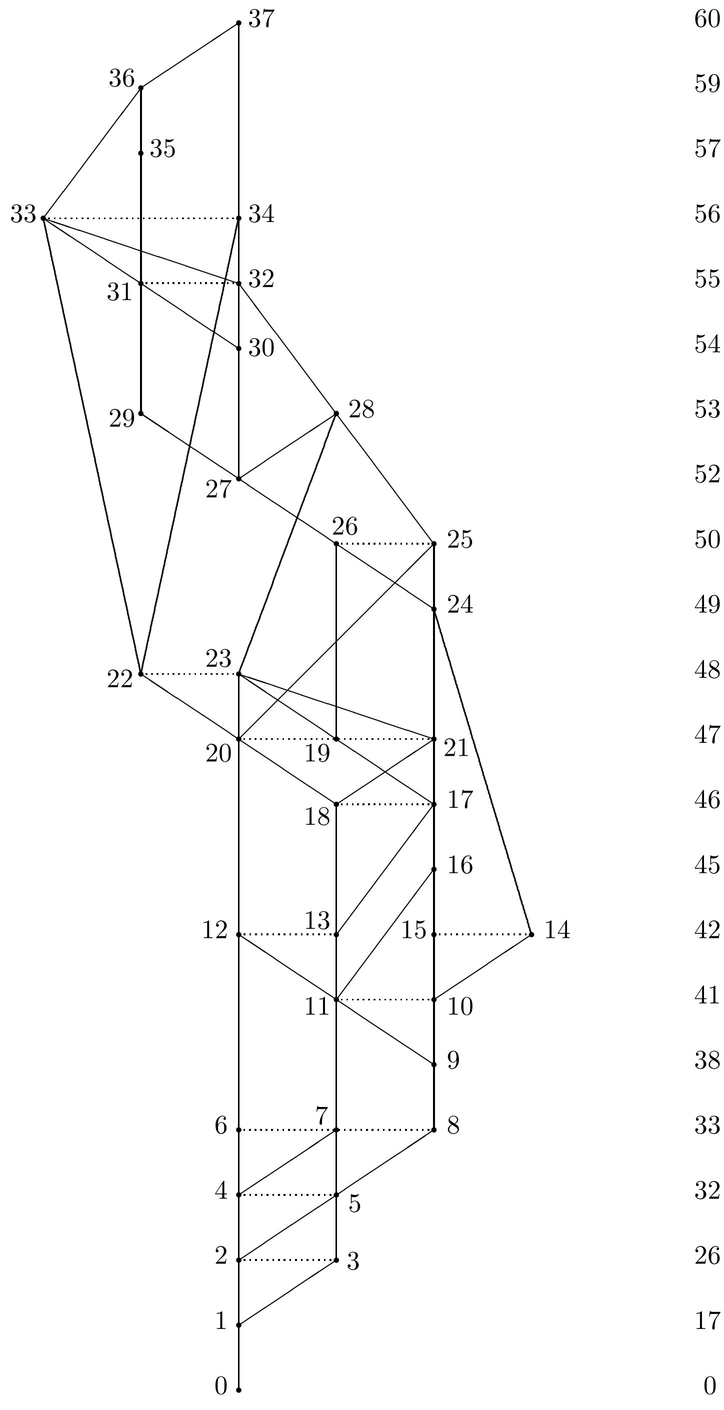

FIGURE 2. The closure diagram for E VI

TABLE 4. Elements $E \in \mathfrak{p}_2(H^i) \cap \mathcal{O}_1^j$

i	j	Type	E
37	36	A_7	$X_{51} + X_{-34} + X_{44} + X_{-41} + X_{-43} + X_{50} + X_{-42}$
37	34	D_5	$X_{-19} + X_{45} + X_{-41} + X_{44} + X_{-43}$
36	35	A_6	$X_{50} + X_{-43} + X_{-41} + X_{44} + X_{-34} + X_{51}$
36	33	D_5	$X_{50} + X_{-38} + X_{44} + X_{-34} + X_{-41}$
34	32	$(A_5 + A_1)'$	$(X_{47} + X_{-31} + X_{45} + X_{-30} + X_{-46}) + (X_{-35})$
34	22	D_4	$X_{45} + X_{-19} + X_{34} + X_{-50}$
33	31	$(A_5 + A_1)'$	$(X_{48} + X_{-34} + X_{-40} + X_{50} + X_{-38}) + (X_{-41})$
33	32	$E_6(a_3)$	$X_{-30} + X_{45} + X_{-31} + X_{-40} + X_{-39} + X_{53}$
33	22	D_4	$X_{-25} + X_{45} + X_{-40} + X_{-41}$
31,32	30	$(A_5)'$	$X_{48} + X_{-34} + X_{-40} + X_{50} + X_{-38}$
32	28	$D_5(a_1)$	$X_{-30} + X_{45} + X_{-31} + X_{-40} + X_{-34}$
31	29	$A_4 + A_2$	$(X_{-38} + X_{53} + X_{-40} + X_{-39}) + (X_{54} + X_{-41})$
29,30	27	$A_4 + A_1$	$(X_{-38} + X_{53} + X_{-40} + X_{-39}) + (X_{54})$
28	27	$A_4 + A_1$	$(X_{-30} + X_{45} + X_{-31} + X_{-43}) + (X_{-29})$
28	25	A_4	$X_{-30} + X_{48} + X_{-43} + X_{-31}$
28	23	D_4	$X_{-30} + X_{45} + X_{-31} + X_{-40}$
27	26	A_4	$X_{-40} + X_{-39} + X_{56} + X_{-41}$
26	24	$A_3 + A_2$	$(X_{56} + X_{-45} + X_{-33}) + (X_{50} + X_{-35})$
26	19	$D_4(a_1)$	$X_{56} + X_{-41} + X_{-39} + X_{-33}$
25	24	$A_3 + A_2$	$(X_{56} + X_{-30} + X_{-42}) + (X_{-31} + X_{-43})$
25	20	$D_4(a_1)$	$X_{48} + X_{-43} + X_{-19} + X_{-13}$
23,24	21	$(A_3 + 2A_1)'$	$(X_{56} + X_{-30} + X_{-42}) + (X_{-31}) + (X_{-40})$
24	14	$2A_2$	$(X_{-30} + X_{-42}) + (X_{-31} + X_{-43})$
23	20	$D_4(a_1)$	$X_{45} + X_{-40} + X_{-19} + X_{-13}$
23	19	$(A_3 + 2A_1)'$	$(X_{-25} + X_{45} + X_{-36}) + (X_{-12}) + (X_{-35})$
22	20	$(A_3 + 2A_1)'$	$(X_{47} + X_{-13} + X_{-46}) + (X_{48}) + (X_{-38})$
20,21	18	$(A_3 + A_1)'$	$(X_{48} + X_{-19} + X_{-38}) + (X_{-40})$
19,21	17	$(A_3 + A_1)'$	$(X_{48} + X_{-19} + X_{-38}) + (X_{-41})$
20	12	A_3	$X_{45} + X_{-40} + X_{-13}$
17,18	13	A_3	$X_{48} + X_{-19} + X_{-38}$
17	16	$2A_2 + A_1$	$(X_{54} + X_{-31}) + (X_{-27} + X_{-30}) + (X_{-35})$
16	15	$2A_2$	$(X_{54} + X_{-25}) + (X_{-27} + X_{-36})$
13,16	11	$A_2 + 2A_1$	$(X_{-19} + X_{-35}) + (X_{54}) + (X_{-27})$
14,15	10	$A_2 + 2A_1$	$(X_{-29} + X_{-34}) + (X_{-18}) + (X_{-30})$
12	11	$A_2 + 2A_1$	$(X_{-13} + X_{-40}) + (X_{54}) + (X_{-27})$
12	6	A_2	$X_{45} + X_{-40}$
11	9	$A_2 + A_1$	$(X_{-13} + X_{-40}) + (X_{-27})$
11	7	$(4A_1)'$	$(X_{56}) + (X_{-13}) + (X_{-33}) + (X_{-35})$
10	9	$A_2 + A_1$	$(X_{-25} + X_{-29}) + (X_{-27})$
9	8	A_2	$X_{-13} + X_{-40}$
7,8	5	$(3A_1)'$	$(X_{-12}) + (X_{-13}) + (X_{-35})$
6,7	4	$(3A_1)'$	$(X_{45}) + (X_{-12}) + (X_{-35})$
4,5	2	$2A_1$	$(X_{-12}) + (X_{-35})$
5	3	$2A_1$	$(X_{-12}) + (X_{-13})$
2,3	1	A_1	X_{-6}

The fact that $E \in \mathfrak{p}_2(H^i)$ can be verified by using Table 3. In most cases the verification of the claim that $E \in \mathcal{O}_1^j$ is straightforward. By using Table 3, one can easily determine whether or not E belongs to $\mathfrak{g}_{H^j}(1,2)$. Assume that it does and let k be such that $\mathcal{O}_1^j \subset \mathcal{O}^k$. The type of E shows that $E \in \mathcal{O}^k$. As $\mathfrak{g}_{H^j}(1,2) \cap \mathcal{O}^k \subset \mathcal{O}_1^j$, the claim follows. This argument is not applicable when $E \notin \mathfrak{g}_{H^j}(1,2)$, i.e., when (i,j) is one of the following pairs:

$$\begin{array}{lllll} (36,35), & (36,33), & (34,22), & (33,22), & (30,27), \\ (29,27), & (28,27), & (26,24), & (25,24). & \end{array}$$

In the case $(36,35)$, E is of type A_6 and, consequently, $E \in \mathcal{O}^{34}$ (see Table 1). The same table shows that $\mathcal{O}^{34} \cap \mathfrak{p} = \mathcal{O}_1^{35}$. As $E \in \mathfrak{p}$, we conclude that $E \in \mathcal{O}_1^{35}$. A similar argument can be applied to the pairs $(30,27)$, $(29,27)$, $(28,27)$, $(26,24)$, and $(25,24)$. The remaining three pairs $(36,33)$, $(34,22)$, and $(33,22)$ require a more elaborate argument.

Let us consider in detail the pair $(36,33)$. In this case

$$E = X_{50} + X_{-38} + X_{44} + X_{-34} + X_{-41}$$

is a standard principal nilpotent element of type D_5. Hence

$$E \in \mathcal{O}^{32} \cap \mathfrak{p} = \mathcal{O}_1^{33} \cup \mathcal{O}_1^{34}$$

and we have to show that in fact $E \in \mathcal{O}_1^{33}$. We do this by finding a normal triple (E,H,F) inside the standard regular simple subalgebra of type D_5 having $\{\alpha_{50}, \alpha_{-38}, \alpha_{44}, \alpha_{-34}, \alpha_{-41}\}$ as a base for its root system. The element H is given by

$$\begin{aligned} H &= 8H_{50} + 14H_{-38} + 18H_{44} + 10H_{-34} + 10H_{-41} \\ &= 2(H_1 + H_2 + H_4 - 4H_6 - H_7). \end{aligned}$$

We do not need to compute F but we remark that

$$F \in \langle X_{-50}, X_{38}, X_{-44}, X_{34}, X_{41} \rangle.$$

Next we compute

$$\alpha_m(H) = 4, 2, -4, 2, 6, -14, 4 \quad (1 \le m \le 7)$$

and deduce that

$$\beta_m(H) = -2, 4, -4, 2, 6, 2, 4 \quad (1 \le m \le 7).$$

Finally, by applying a suitable element w of the Weyl group of $(\mathfrak{k}, \mathfrak{h})$ to H we obtain the element $H' = w(H)$ such that

$$\beta_m(H') = 0, 2, 0, 2, 4, 0, 4 \quad (1 \le m \le 7).$$

By looking up Table 2, we conclude that indeed $E \in \mathcal{O}_1^{33}$.

The argument is similar in the other two cases. We only state that for the pair $(34,22)$ we have

$$\alpha_m(H) = 0, 0, 0, 4, -4, -6, 8 \quad (1 \le m \le 7),$$

and for $(33,22)$

$$\alpha_m(H) = 0, 4, 4, -4, 0, -6, 8 \quad (1 \le m \le 7).$$

TABLE 5. The integers $d_i(j,k)$ for the module $V(\omega_7)$

k	$d_0(j,k)$	$d_1(j,k)$
1	18 24	26 32
2	14 22 24	22 32
3	14 24	22 30 32
4	12 18 24	20 32
5	12 22 24	20 28 32
6	12 12 24	20 32
7	12 18 24	20 26 32
8	12 20 24	20 24 32
9	9 20 23 24	17 24 31 32
10	8 20 22 24	16 22 30 32
11	8 18 22 24	16 24 30 32
12	8 12 18 22 24	16 22 26 32
13	8 14 18 24	16 20 26 30 32
14	6 20 22 22 24	14 16 30 32
15	6 16 22 24	14 20 30 30 32
16	6 16 20 24	14 20 28 30 32
17	6 14 18 24	14 20 26 28 32
18	6 14 18 22 24	14 20 26 30 32
19	6 12 18 24	14 20 26 26 32
20	6 12 18 20 24	14 20 26 30 32
21	6 14 18 22 24	14 18 26 28 32
22	6 6 12 12 18 18 24	14 20 20 26 26 32
23	6 10 12 16 18 22 24	14 16 20 22 26 28 32
24	4 14 17 22 23 24	12 16 25 28 31 32
25	4 10 14 20 22 22 24	12 16 22 24 30 32
26	4 12 14 20 22 24	12 14 22 24 30 30 32
27	3 12 14 20 21 24	11 14 22 24 29 30 32
28	3 10 12 16 18 22 23 24	11 14 20 22 26 28 31 32
29	2 12 14 18 20 24	10 12 22 24 28 28 32
30	2 8 10 16 18 22 22 24	10 12 18 20 26 28 30 30 32
31	2 8 10 16 18 22 22 24	10 12 18 20 26 26 30 30 32
32	2 8 10 16 18 20 22 24	10 12 18 20 26 28 30 30 32
33	2 6 8 12 14 18 18 22 22 24	10 12 16 18 22 22 26 26 30 30 32
34	2 6 8 12 14 16 18 20 22 22 24	10 12 16 18 22 24 26 28 30 32
35	0 8 8 14 14 20 20 22 22 24	8 8 16 16 22 22 28 28 30 30 32
36	0 6 6 12 12 16 16 20 20 22 22 24	8 8 14 14 20 20 24 24 28 28 30 30 32
37	0 4 4 8 8 12 12 16 16 18 18 20 20 22 22 24	8 8 12 12 16 16 20 20 24 24 26 26 28 28 30 30 32

By inspection of Figure 2, we see that in order to complete the proof of the theorem we need to show that $\mathcal{O}_1^i \not> \mathcal{O}_1^j$ when (i,j) is one of the following pairs:

$$\begin{array}{lllll}(6,3), & (14,4), & (15,4), & (35,6), & (22,10),\\ (16,13), & (23,14), & (19,18), & (25,19), & (32,22),\\ (34,29). & & & & \end{array} \tag{3.1}$$

This assertion is valid for the pair $(16, 13)$ because $\mathcal{O}_1^{16} \subset \mathcal{O}^{13}$, $\mathcal{O}_1^{13} \subset \mathcal{O}^{10}$, and $\mathcal{O}^{13} \not\geq \mathcal{O}^{10}$ (see Table 1 and Figure 1). (Another proof for this case will be given below.)

Let V be the 56-dimensional simple $\mathfrak{g}$-module with highest weight ω_7 (one of the fundamental weights). It can be equipped with the $\mathbf{Z}_2$-grading such that $V = V_0 \oplus V_1$, $\dim V_0 = 24$, and $\dim V_1 = 32$. This means that

$$\mathfrak{k} \cdot V_i \subset V_i, \ \mathfrak{p} \cdot V_i \subset V_{1-i}$$

holds for $i = 0, 1$. We introduce the integers

$$d_i(j,k) = \dim V_i \cap \ker \rho(E^k)^j$$

where $i = 0, 1$; $j \geq 1$; $1 \leq k \leq 37$, and ρ is the representation of $\mathfrak{g}$ on V. They are easy to compute and are displayed in Table 5.

By applying [9, Theorem 4.1] and using Table 5, we see that $\mathcal{O}_1^i \not\geq \mathcal{O}_1^j$ when (i, j) is one of the following pairs:

$$\begin{array}{lllll} (6,3), & (14,4), & (24,6), & (9,7), & (14,11), \\ (16,13), & (21,14), & (22,14), & (24,19), & (24,20), \\ (22,21), & (32,22), & (29,25). & & \end{array} \tag{3.2}$$

In particular, this means that the pairs $(6, 3)$, $(14, 4)$, $(16, 13)$, and $(32, 22)$ from (3.1) have been taken care of.

The remaining seven pairs in (3.1) are handled by using the theory of prehomogeneous vector spaces [14, 15].

In order to determine the closure of an orbit $\mathcal{O}_1^k$ we shall employ the following recursive procedure. The centralizer $Z = Z_K(H^k)$ is a connected reductive subgroup of K which can be easily determined from the integers $\beta_i(H^k)$ given in Table 2. Furthermore, Z is a Levi factor of the parabolic subgroup Q_{H^k} of K. The centralizer of E^k in Z is reductive, and consequently, the PV $(Z, \mathfrak{g}_{H^k}(1,2))$ is regular [14]. Hence the singular set S of this PV is a union of irreducible conical hypersurfaces S_i defined by equations $f_i = 0$, where the f_i's are the basic relative invariants of this PV. One knows that the number of the basic relative invariants is $\leq m$, where m is the length of $\mathfrak{g}_{H^k}(1,2)$ as a Z-module [15]. In all cases below, m is actually equal to the number of the basic relative invariants. The pair $(Q_{H^k},\ \mathfrak{p}_2(H^k))$ is also a PV and its singular set is the union of the hypersurfaces $S_i + \mathfrak{p}_3(H^k)$. In most cases each of these hypersurfaces contains a dense open Q_{H^k}-orbit and we are able to identify to which K-orbits they belong. Then the closure of $\mathcal{O}_1^k$ is the union of $\mathcal{O}_1^k$ and the closures of K-orbits (of smaller dimension) which meet one of the hypersurfaces $S_i + \mathfrak{p}_3(H^k)$ in a dense open subset.

We start with the pair $(15, 4)$. The centralizer $Z = Z_K(H^{15}) = \mathrm{SL}_4 \cdot (\mathrm{SL}_2)^3 \cdot T_1$ has a 1-dimensional central torus T_1. The simple roots of this SL_4 are $\{\alpha_{-63}, \alpha_1, \alpha_3\}$ and those of the three SL_2 factors are $\{\alpha_2\}$, $\{\alpha_5\}$, and $\{\alpha_7\}$. The space $\mathfrak{g}_{H^{15}}(1,2)$ is a simple 16-dimensional Z-module on which the second factor SL_2 acts trivially. More precisely,

$$\mathfrak{g}_{H^{15}}(1,2) = V(4) \otimes V(2) \otimes V(1) \otimes V(2)$$

where $V(d)$ denotes the simple d-dimensional module for the corresponding simple factor of Z. For illustrative purposes we exhibit the weight diagram for this module in Figure 3 (all weights are simple). A vertex labelled i stands for the 1-dimensional weight space spanned by the root vector X_i. The action of the simple root vectors of the Lie algebra of Z is indicated by the arrows.

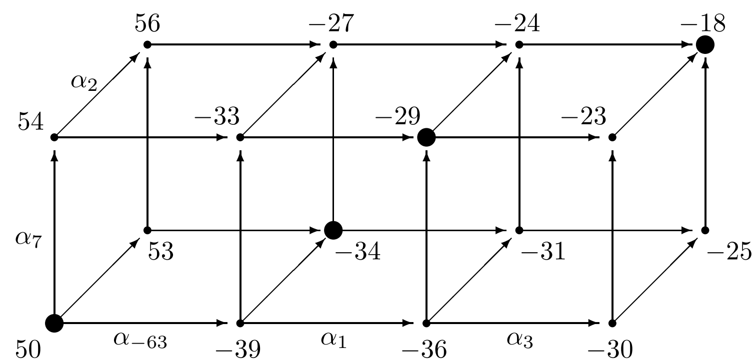

FIGURE 3. The weight diagram of $\mathfrak{g}_{H^{15}}(1,2)$

The singular set S of $(Z, \mathfrak{g}_{H^{15}}(1,2))$ is an irreducible hypersurface, while the singular set of $(Q_{H^{15}}, \mathfrak{p}_2(H^{15}))$ is $S + \mathfrak{p}_3(H^{15})$. The representative

$$E^{15} = (X_{50} + X_{-18}) + (X_{-29} + X_{-34})$$

from Table 2 is depicted in Figure 3 by the 4 enlarged nodes. This E^{15} is a generic element of both $\mathfrak{g}_{H^{15}}(1,2)$ and $\mathfrak{p}_2(H^{15})$. Now consider the element

$$X = (X_{-29} + X_{-34}) + (X_{-18}) + (X_{-30}).$$

A simple computation shows that the orbit $Q_{H^{15}} \cdot X$ has dimension 19. Since $\mathfrak{p}_2(H^{15})$ has dimension 20, it follows that this orbit is a dense open subset of the singular set $S + \mathfrak{p}_3(H^{15})$. As $X \in \mathcal{O}_1^{10}$, we conclude that

$$\overline{\mathcal{O}_1^{15}} = \mathcal{O}_1^{15} \cup \overline{\mathcal{O}_1^{10}}.$$

As $\mathcal{O}_1^{14} > \mathcal{O}_1^{10}$ and $\mathcal{O}_1^{14} \not> \mathcal{O}_1^{4}$ (see (3.2)), we infer that $\mathcal{O}_1^{10} \not> \mathcal{O}_1^{4}$, and consequently, $\mathcal{O}_1^{15} \not> \mathcal{O}_1^{4}$.

We proceed to the pair $(22, 10)$. Now

$$Z = Z_K(H^{22}) = \mathrm{SL}_6 \cdot T_2$$

where T_2 is the 2-dimensional central torus and the simple roots of SL_6 are $\{\alpha_{-63}, \alpha_1, \alpha_3, \alpha_4, \alpha_2\}$. The space $\mathfrak{g}_{H^{22}}(1,2)$ is a direct sum of two simple SL_6-modules: V_1 of dimension 15 with basis

$$\{X_{19}, X_{25}, X_{30}, X_{31}, X_{34}, X_{36}, X_{39}, X_{41}, X_{44}, X_{47}, X_{-40}, X_{-43}, X_{-46}, X_{-50}, X_{-53}\}$$

and the trivial module V_2 with basis $\{X_{-13}\}$. The weight diagram of $\mathfrak{g}_{H^{22}}(1,2)$ is exhibited in Figure 4. The enlarged nodes depict the representative

$$E^{22} = X_{19} + X_{-13} + X_{41} + X_{-40} \in \mathcal{O}_1^{22}$$

from Table 2. The singular set S of $(Z, \mathfrak{g}_{H^{22}}(1,2))$ is the union of two irreducible hypersurfaces S_1 and S_2. The singular set of $(Q_{H^{22}}, \mathfrak{p}_2(H^{22}))$ is the union of the hypersurfaces $S_1 + \mathfrak{p}_3(H^{22})$ and $S_2 + \mathfrak{p}_3(H^{22})$. If

$$X = (X_{19}) + (X_{41}) + (X_{-40}) + (X_{-6}),$$
$$Y = (X_{47} + X_{-13} + X_{-46}) + (X_{48}) + (X_{-38}),$$

then both orbits $Q_{H^{22}} \cdot X$ and $Q_{H^{22}} \cdot Y$ have dimension 31 while $\mathfrak{p}_2(H^{22})$ has dimension 32. As $X \in \mathcal{O}_1^6$ and $Y \in \mathcal{O}_1^{20}$, we conclude that the closures of these two orbits are the two irreducible hypersurfaces in $\mathfrak{p}_2(H^{22})$ mentioned above. As $\mathcal{O}_1^{20} > \mathcal{O}_1^6$, we conclude that

$$\overline{\mathcal{O}_1^{22}} = \mathcal{O}_1^{22} \cup \overline{\mathcal{O}_1^{20}}.$$

Hence the proof of $\mathcal{O}_1^{22} \not> \mathcal{O}_1^{10}$ is reduced to that of $\mathcal{O}_1^{20} \not> \mathcal{O}_1^{10}$.

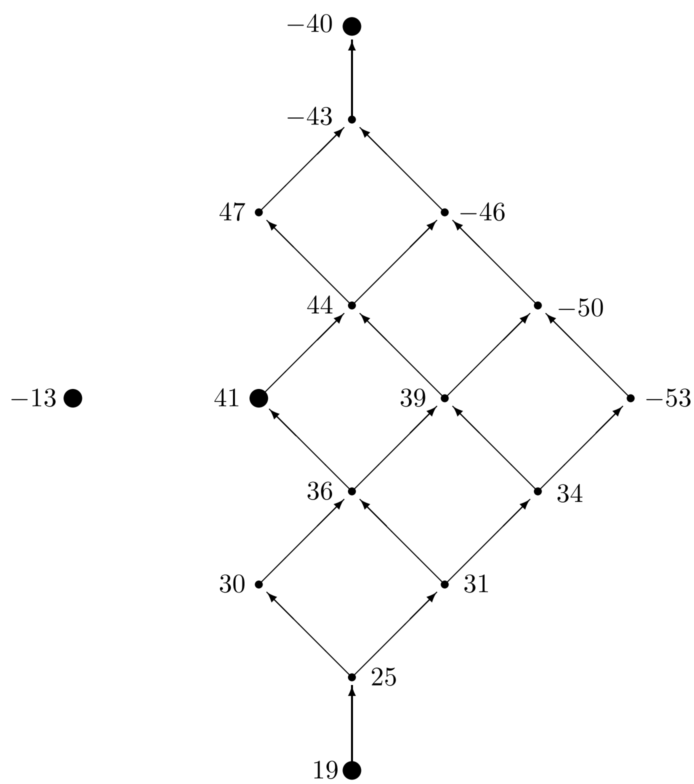

FIGURE 4. The weight diagram of $\mathfrak{g}_{H^{22}}(1, 2)$

We now repeat the above argument. The centralizer

$$Z = Z_K(H^{20}) = \mathrm{SL}_4 \cdot (\mathrm{SL}_2)^2 \cdot T_2$$

where T_2 is the 2-dimensional central torus and the simple roots of SL_4 are $\{\alpha_{-63}, \alpha_1, \alpha_3\}$ and those of the two factors SL_2 are $\{\alpha_2\}$ and $\{\alpha_5\}$. The first factor SL_2 acts trivially on $\mathfrak{g}_{H^{20}}(1, 2)$. This space is a direct sum of two simple Z-modules V_1 and V_2 with bases

$$V_1 : \quad \{X_{41}, X_{44}, X_{45}, X_{47}, X_{48}, X_{-35}, X_{-38}, X_{-40}, X_{-42}, X_{-43}, X_{-46}\},$$
$$V_2 : \quad \{X_{-13}, X_{-19}\}.$$

The singular set S of $(Z, \mathfrak{g}_{H^{20}}(1,2))$ is again a union of two irreducible hypersurfaces S_1 and S_2. The singular set of $(Q_{H^{20}}, \mathfrak{p}_2(H^{20}))$ is the union of the hypersurfaces $S_1 + \mathfrak{p}_3(H^{20})$ and $S_2 + \mathfrak{p}_3(H^{20})$. The dense open orbits in these two hypersurfaces are $Q_{H^{20}} \cdot X$ and $Q_{H^{20}} \cdot Y$ where now

$$X = X_{45} + X_{-40} + X_{-13} \in \mathcal{O}_1^{12},$$
$$Y = (X_{48} + X_{-19} + X_{-38}) + (X_{-40}) \in \mathcal{O}_1^{18}.$$

It follows that

$$\overline{\mathcal{O}_1^{20}} = \mathcal{O}_1^{20} \cup \overline{\mathcal{O}_1^{12}} \cup \overline{\mathcal{O}_1^{18}}.$$

Hence the proof of $\mathcal{O}_1^{22} \not\geq \mathcal{O}_1^{10}$ is now reduced to that for $\mathcal{O}_1^{12} \not\geq \mathcal{O}_1^{10}$ and $\mathcal{O}_1^{18} \not\geq \mathcal{O}_1^{10}$.

One can proceed in the same way to show that

$$\overline{\mathcal{O}_1^{12}} = \mathcal{O}_1^{12} \cup \overline{\mathcal{O}_1^{6}} \cup \overline{\mathcal{O}_1^{11}},$$
$$\overline{\mathcal{O}_1^{18}} = \mathcal{O}_1^{18} \cup \overline{\mathcal{O}_1^{13}},$$
$$\overline{\mathcal{O}_1^{13}} = \mathcal{O}_1^{13} \cup \overline{\mathcal{O}_1^{11}}.$$

These imply that $\mathcal{O}_1^{12} \not\geq \mathcal{O}_1^{10}$ and $\mathcal{O}_1^{18} \not\geq \mathcal{O}_1^{10}$. Hence we can conclude that $\mathcal{O}_1^{22} \not\geq \mathcal{O}_1^{10}$.

Next consider the pair $(23, 14)$. Then $Z = Z_K(H^{23}) = \mathrm{SL}_4 \cdot \mathrm{SL}_2 \cdot T_3$, where T_3 is the 3-dimensional central torus, SL_4 has simple roots $\{\alpha_3, \alpha_4, \alpha_2\}$, and the simple root of the SL_2-factor is α_{-63}. In this case $m = 3$. Let

$$X = (X_{-25} + X_{45} + X_{-36}) + (X_{-12}) + (X_{-35}),$$
$$X' = X_{45} + X_{-40} + X_{-19} + X_{-13},$$
$$X'' = (X_{56} + X_{-30} + X_{-42}) + (X_{-31}) + (X_{-40}).$$

The $Q_{H^{23}}$-orbits through these elements have each dimension 23 while $\mathfrak{p}_2(H^{23})$ has dimension 24. Since $X \in \mathcal{O}_1^{19}$, $X' \in \mathcal{O}_1^{20}$, and $X'' \in \mathcal{O}_1^{21}$, it follows that

$$\overline{\mathcal{O}_1^{23}} = \mathcal{O}_1^{23} \cup \overline{\mathcal{O}_1^{19}} \cup \overline{\mathcal{O}_1^{20}} \cup \overline{\mathcal{O}_1^{21}}.$$

By (3.2) we know that $\mathcal{O}_1^{19} \not\geq \mathcal{O}_1^{14}$ and $\mathcal{O}_1^{21} \not\geq \mathcal{O}_1^{14}$. We have already shown that $\mathcal{O}_1^{20} \not\geq \mathcal{O}_1^{10}$, and so $\mathcal{O}_1^{20} \not\geq \mathcal{O}_1^{14}$. We conclude that $\mathcal{O}_1^{23} \not\geq \mathcal{O}_1^{14}$.

The argument for the pair $(19, 18)$ is similar. Now $Z = Z_K(H^{19}) = \mathrm{SL}_5 \cdot \mathrm{SL}_2 \cdot T_1$ where T_1 is a central 1-dimensional torus, the simple roots of SL_5 are $\{\alpha_{-63}, \alpha_1, \alpha_3, \alpha_4, \alpha_2\}$, and SL_2 has the simple root α_7. The space $\mathfrak{g}_{H^{19}}(1,2)$ has dimension 30 and as a Z-module it is isomorphic to $V(15) \otimes V(2)$ where $V(15)$ is the second fundamental module of SL_5 and $V(2)$ the standard module of SL_2. The singular set S of $(Z, \mathfrak{g}_{H^{19}}(1,2))$ is an irreducible hypersurface. The singular set of $(Q_{H^{19}}, \mathfrak{p}_2(H^{19}))$ is $S + \mathfrak{p}_3(H^{19})$. If $E^{17} \in \mathcal{O}_1^{17}$ is the representative from Table 2, a computation shows that the orbit $Q_{H^{19}} \cdot E^{17}$ has dimension 31. Since $\mathfrak{p}_2(H^{19})$ has dimension 32, it follows that this orbit is a dense open subset of $S + \mathfrak{p}_2(H^{19})$. Hence

$$\overline{\mathcal{O}_1^{19}} = \mathcal{O}_1^{19} \cup \overline{\mathcal{O}_1^{17}},$$

and consequently, $\mathcal{O}_1^{19} \not\geq \mathcal{O}_1^{18}$.

Next we consider the pair $(34, 29)$. We have $Z = Z_K(H^{34}) = \mathrm{SL}_4 \cdot \mathrm{SL}_2 \cdot T_3$, where T_3 is the 3-dimensional central torus, SL_4 has simple roots $\{\alpha_3, \alpha_4, \alpha_2\}$, and SL_2

the simple root α_{-63}. In this case $m = 3$. The three $Q_{H^{34}}$-orbits through

$$X = X_{45} + X_{-19} + X_{34} + X_{-50},$$
$$X' = X_{47} + X_{-31} + X_{45} + X_{-30} + X_{-46},$$
$$X'' = (X_{34} + X_{-19} + X_{-50}) + (X_{56} + X_{-41} + X_{-33})$$

have each dimension 31. As $\mathfrak{p}_2(H^{34})$ has dimension 32 and

$$X \in \mathcal{O}_1^{22},\ X' \in \mathcal{O}_1^{32},\ X'' \in \mathcal{O}_1^{25},$$

we conclude that the closures of these three orbits are the three irreducible hypersurfaces in $\mathfrak{p}_2(H^{34})$ which form its singular set. Hence

$$\overline{\mathcal{O}_1^{34}} = \mathcal{O}_1^{34} \cup \overline{\mathcal{O}_1^{22}} \cup \overline{\mathcal{O}_1^{32}}$$

and so the proof of $\mathcal{O}_1^{34} \not\geq \mathcal{O}_1^{29}$ is reduced to that of $\mathcal{O}_1^{32} \not\geq \mathcal{O}_1^{29}$.

We now investigate the pair $(32, 29)$. Let $Z = Z_K(H^{32}) = \mathrm{SL}_4 \cdot (\mathrm{SL}_2)^2 \cdot T_2$, where T_2 is the central 2-dimensional torus, SL_4 has simple roots $\{\alpha_{-63}, \alpha_1, \alpha_3\}$ and the SL_2 factors have simple roots α_2 and α_5, respectively. In this case $m = 2$. Let

$$X = X_{48} + X_{-34} + X_{-40} + X_{50} + X_{-38},$$
$$Y = X_{-30} + X_{45} + X_{-31} + X_{-40} + X_{-34}.$$

Then the $Q_{H^{32}}$-orbits through X and Y have each dimension 31 while $\mathfrak{p}_2(H^{32})$ has dimension 32. Since $X \in \mathcal{O}_1^{30}$ and $Y \in \mathcal{O}_1^{28}$, we have

$$\overline{\mathcal{O}_1^{32}} = \mathcal{O}_1^{32} \cup \overline{\mathcal{O}_1^{28}} \cup \overline{\mathcal{O}_1^{30}}.$$

Hence the proof of $\mathcal{O}_1^{32} \not\geq \mathcal{O}_1^{29}$ reduces to that of $\mathcal{O}_1^{30} \not\geq \mathcal{O}_1^{29}$.

We now turn to the pair $(30, 29)$. Then $Z = Z_K(H^{30}) = (\mathrm{SL}_2)^3 \cdot T_4$, where T_4 is the 4-dimensional central torus, and the simple roots of the SL_2-factors are α_2, α_3, and α_{-63}. As Z-module, $\mathfrak{g}_{H^{30}}(1, 2)$ is a direct sum of three simple modules, V_1, V_2, V_3 with bases

$$V_1: \quad \{X_{50}, X_{53}, X_{-34}, X_{-39}\},$$
$$V_2: \quad \{X_{48}, X_{51}, X_{-38}, X_{-42}\},$$
$$V_3: \quad \{X_{-40}\}.$$

Let

$$X = (X_{-34} + X_{-40} + X_{50} + X_{-38}) + (X_{56}),$$
$$X' = (X_{-31} + X_{-38} + X_{50} + X_{-40}) + (X_{48}),$$
$$Y = X_{-34} + X_{48} + X_{-30} + X_{-38} + X_{50} + X_{-24}.$$

A computation shows that the three $Q_{H^{30}}$-orbits through X, X', and Y have each dimension 25. Although $X, X' \in \mathcal{O}_1^{27}$, the corresponding $Q_{H^{30}}$-orbits are different, as can be seen from the above expressions for X and X' and the module structure of $\mathfrak{g}_{H^{30}}(1, 2)$. Since the G-orbit through Y has dimension 94, we infer that

$$Y \in \mathcal{O}_1^{19} \cup \mathcal{O}_1^{20} \cup \mathcal{O}_1^{21}.$$

We conclude that

$$\overline{\mathcal{O}_1^{30}} \subset \mathcal{O}_1^{30} \cup \overline{\mathcal{O}^{24}}$$

and so $\mathcal{O}_1^{30} \not\geq \mathcal{O}_1^{29}$.

Next we consider the pair $(25, 19)$. Now $Z = Z_K(H^{25}) = \mathrm{Spin}_8 \cdot \mathrm{SL}_2 \cdot T_2$, where T_2 is the central 2-dimensional torus, the simple roots of Spin_8 are $\{\alpha_1, \alpha_4, \alpha_2, \alpha_5\}$, and the simple root of SL_2 is α_{-63}. In this case $m = 2$. Let

$$X = (X_{56} + X_{-45} + X_{-33}) + (X_{34} + X_{-13}),$$
$$Y = X_{48} + X_{-43} + X_{-19} + X_{-13}.$$

The $Q_{H^{25}}$-orbits through X and Y have each dimension 31, while $\mathfrak{p}_2(H^{25})$ has dimension 32. Since $X \in \mathcal{O}_1^{24}$ and $Y \in \mathcal{O}_1^{20}$, we obtain that

$$\overline{\mathcal{O}_1^{25}} = \mathcal{O}_1^{25} \cup \overline{\mathcal{O}_1^{20}} \cup \overline{\mathcal{O}_1^{24}}.$$

By (3.2), $\mathcal{O}_1^{24} \not\geq \mathcal{O}_1^{19}$ and so we can conclude that $\mathcal{O}_1^{25} \not\geq \mathcal{O}_1^{19}$.

The proof for the only remaining pair $(35, 6)$ is quite different.

As a K-module $\mathfrak{p}$ is irreducible of the form $\mathfrak{p} = V_1 \otimes V_2$ where V_1 is the half-spin module for Spin_{12}/Z_2 of dimension 32, and V_2 is the standard 2-dimensional simple module for SL_2 with basis $\{e_1, e_2\}$. We can identify V_1 as a Spin_{12}-module with the subspace of $\mathfrak{p}$ spanned by the root vectors X_i with

$$i \in \{13, 19, 25, 30, 31, 34, 36, 39, 41, 44, 45, 47, 48, 51, 54, 56, -6, -12, -18, \\ -23, -24, -27, -29, -33, -35, -38, -40, -42, -43, -46, -50, -53\}.$$

Then an explicit isomorphism of K-modules $\psi : V_1 \otimes V_2 \to \mathfrak{p}$ is given by

$$\psi(X \otimes e_1 + Y \otimes e_2) = X + [X_{-7}, Y].$$

If $V_1' = [X_{-7}, V_1]$, then the map $\varphi : V_1 \to V_1'$ given by $\varphi(X) = [X_{-7}, X]$ is an isomorphism of Spin_{12}-modules. Finally, let $\pi : \mathfrak{p} \to V_1$ be the projection with kernel V_1'.

The pair $(\mathrm{Spin}_{12} \cdot T_1, V_1)$ is a regular prehomogeneous vector space, where T_1 is the maximal torus of the SL_2-factor which leaves V_1 and V_1' invariant. Its singular set S is an irreducible quartic conical hypersurface [14]. The representative $E^6 \in \mathcal{O}_1^6$ (see Table 2) lies in V_1 but not in S (thus it is a generic element of V_1). This can be checked by using the explicit equation of S given in [14, 12]. On the other hand, the representative $E^{35} \in \mathcal{O}_1^{35}$ (see Table 2) can be written as $E^{35} = X + \varphi(Y)$ with $X = X_{-43} + X_{-35} + X_{56}$ and $Y \in \langle X_{54}, X_{-38}, X_{-46} \rangle$. If $g \in \mathrm{SL}_2$, then

$$g \cdot E^{35} = aX + bY + \varphi(cX + dY)$$

for some $a, b, c, d \in \mathbf{C}$ with $ad - bc = 1$. Hence $\pi(g \cdot E^{35}) = aX + bY \in V_3$ where

$$V_3 = \langle X_{-43}, X_{-35}, X_{56}, X_{54}, X_{-38}, X_{-46} \rangle.$$

The weight diagram of the Spin_{12}-module V_1 is shown in Figure 5. A node with label i represents the one-dimensional weight space spanned by the root vector X_i. The arrows indicate the action of the simple root vectors of the Lie algebra of Spin_{12}. Recall that we have introduced in Section 2 the basis $\{\beta_1, \ldots, \beta_6\}$ for the root system of Spin_{12}. The six enlarged nodes depict the basis vectors of the subspace V_3. Note that E^6 is the sum of the highest weight vector X_{-6} and the lowest weight vector X_{13}.

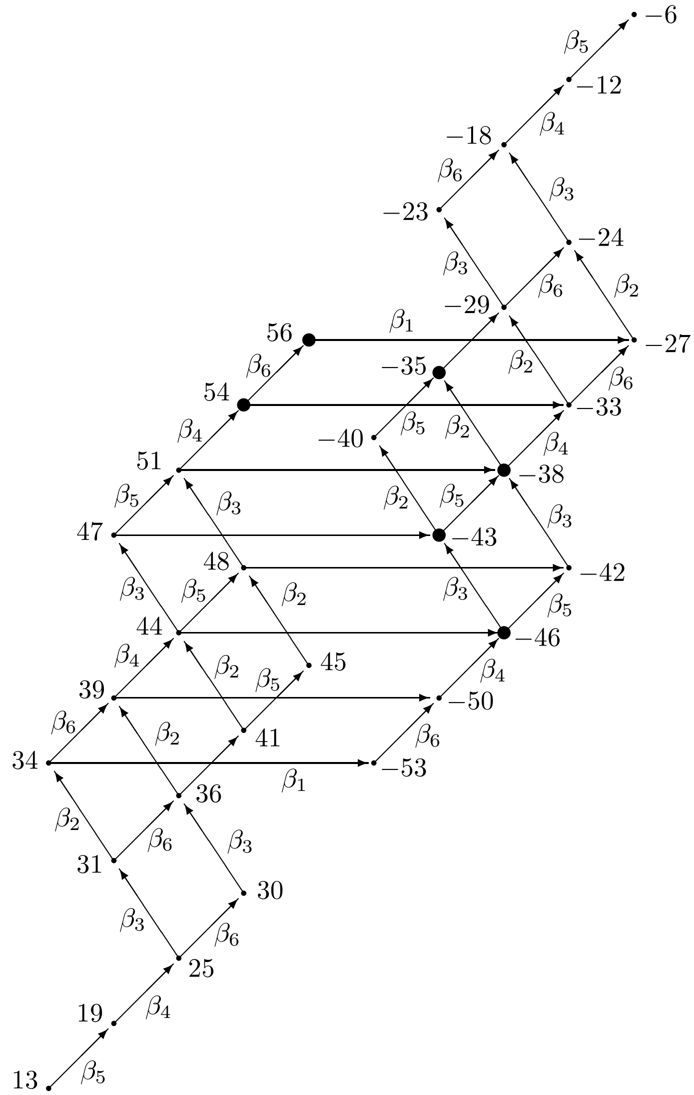

FIGURE 5. The weight diagram of the half-spin module V_1

By using the equation of S, one can easily verify that $V_3 \subset S$. Since

$$\pi(\mathrm{SL}_2 \cdot E^{35}) \subset V_3 \subset S$$

and π is Spin_{12}-equivariant, it follows that

$$\begin{aligned}\pi(\mathcal{O}_1^{35}) &= \pi(K \cdot E^{35}) = \pi(\mathrm{Spin}_{12}\mathrm{SL}_2 \cdot E^{35}) \\ &= \mathrm{Spin}_{12} \cdot \pi(\mathrm{SL}_2 \cdot E^{35}) \\ &\subset \mathrm{Spin}_{12} \cdot S = S.\end{aligned}$$

TABLE 6. Nonzero nilpotent orbits in $\mathfrak{p}$ ($\mathfrak{g}_0 = \mathrm{E\,VII}$)

k	i	$\alpha_j(H^i)$		E^i	Type of E^i
1	1	100000	0	X_{63}	A_1
1	2	000001	-2	X_{-7}	A_1
2	3	000001	0	$(X_{58}) + (X_{59})$	$2A_1$
2	4	100000	-2	$(X_{-30}) + (X_{-31})$	$2A_1$
2	5	100001	-2	$(X_{63}) + (X_{-7})$	$2A_1$
3	6	000000	2	$(X_7) + (X_{49}) + (X_{63})$	$(3A_1)''$
3	7	000000	-2	$(X_{-7}) + (X_{-49}) + (X_{-63})$	$(3A_1)''$
3	8	000002	-2	$(X_{49}) + (X_{63}) + (X_{-7})$	$(3A_1)''$
3	9	200000	-2	$(X_{63}) + (X_{-7}) + (X_{-49})$	$(3A_1)''$
5	10	020000	-2	$X_{56} + X_{-7}$	A_2
7	11	010010	-2	$(X_{56} + X_{-7}) + (X_{59})$	$A_2 + A_1$
7	12	011000	-3	$(X_{62} + X_{-31}) + (X_{-30})$	$A_2 + A_1$
10	13	300001	-2	$X_{34} + X_{-7} + X_{56}$	A_3
10	14	100003	-6	$X_{-34} + X_{63} + X_{-56}$	A_3
11	15	200002	-4	$(X_{52} + X_{-13}) + (X_{62} + X_{-45})$	$2A_2$
12	16	200002	-2	$(X_{34} + X_{-7} + X_{56}) + (X_{49})$	$(A_3 + A_1)''$
12	17	400000	-2	$(X_{34} + X_{-7} + X_{56}) + (X_{-49})$	$(A_3 + A_1)''$
12	18	000004	-6	$(X_{-34} + X_{63} + X_{-56}) + (X_{49})$	$(A_3 + A_1)''$
12	19	200002	-6	$(X_{-34} + X_{63} + X_{-56}) + (X_{-49})$	$(A_3 + A_1)''$
20	20	220002	-6	$X_{56} + X_{-30} + X_{52} + X_{-34}$	A_4
22	21	400004	-6	$X_{34} + X_{-13} + X_{49} + X_{-45} + X_{56}$	$(A_5)''$
22	22	400004	-10	$X_{-34} + X_{52} + X_{-49} + X_{62} + X_{-56}$	$(A_5)''$

Hence $\mathcal{O}_1^{35} \subset S + V_1'$, and consequently,

$$\overline{\mathcal{O}_1^{35}} \subset S + V_1'.$$

As $E^6 \in V_1 \setminus S$, we conclude that $E^6 \notin \overline{\mathcal{O}_1^{35}}$, i.e., $\mathcal{O}_1^{35} \not\geq \mathcal{O}_1^6$. □

We end this section with the following interesting observation.

Consider the G-orbits $\mathcal{O}^{28}$ and $\mathcal{O}^{26}$ with Bala-Carter labels A_5 and $A_4 + A_2$, respectively. By Table 1 we have

$$\mathfrak{p} \cap \mathcal{O}^{28} = \mathcal{O}_1^{30},\ \mathfrak{p} \cap \mathcal{O}^{26} = \mathcal{O}_1^{29}.$$

Our observation is that

$$\mathcal{O}_1^{29} \not\subset \overline{\mathcal{O}_1^{30}},$$

although,

$$\mathcal{O}^{26} \subset \overline{\mathcal{O}^{28}}.$$

4. TYPE E VII

In this section $\mathfrak{g}_0$ is the real form of type E VII of $\mathfrak{g}$, and so K has two connected components and $K^0 = (\mathrm{E}_6 \times \mathrm{GL}_1)/Z_3$. There are exactly 22 nonzero nilpotent K^0-orbits in $\mathfrak{p}$ denoted by $\mathcal{O}_1^i,\ 1 \leq i \leq 22$. We let (E^i, H^i, F^i) be a normal triple with $E^i \in \mathcal{O}_1^i,\ H^i \in \mathfrak{h}$ such that $\alpha_j(H^i) \geq 0$ for $1 \leq j \leq 6$. (Since the semisimple rank of K^0 is six, we cannot insist that $\alpha_7(H^i)$ be nonnegative.) These orbits are listed in Table 6. Its description is the same as for Table 2.

TABLE 7. Root spaces in $\mathfrak{p}_2(H^i)$

i	Indices of roots
1	63;
2	−7;
3	49, 52, 55, 57, 58, 59, 60, 61, 62, 63;
4	−7, −13, −19, −25, −30, −31, −36, −41, −45, −49;
5	63, −7;
6	7, 13, 19, 25, 30, 31, 34, 36, 39, 41, 44, 45, 47, 48, 49, 51, 52, 54, 55, 56, 57, 58, 59, 60, 61, 62, 63;
7	−7, −13, −19, −25, −30, −31, −34, −36, −39, −41, −44, −45, −47, −48, −49, −51, −52, −54, −55, −56, −57, −58, −59, −60, −61, −62, −63;
8	49, 52, 55, 57, 58, 59, 60, 61, 62, 63, −7;
9	63, −7, −13, −19, −25, −30, −31, −36, −41, −45, −49;
10	56, 58, 60, 61, 62, 63, −7, −13, −19, −25, −31, −34;
11	56, 58, 59, −7, −13; 60, 61, 62, 63
12	62, 63, −30, −31, −34; −7, −13, −19, −25
13	34, 39, 44, 47, 48, 51, 54, 56, −7; 52, 55, 57, 58, 59, 60, 61, 62, 63
14	63, −34, −39, −44, −47, −48, −51, −54, −56; −13, −19, −25, −30, −31, −36, −41, −45, −7
15	52, 55, 57, 58, 59, 60, 61, 62, −13, −19, −25, −30, −31, −36, −41, −45; 63, −7
16	34, 39, 44, 47, 48, 49, 51, 54, 56, −7; 52, 55, 57, 58, 59, 60, 61, 62, 63
17	34, 39, 44, 47, 48, 51, 52, 54, 55, 56, 57, 58, 59, 60, 61, 62, −7, −13, −19, −25, −30, −31, −36, −41, −45, −49; 63
18	49, 52, 55, 57, 58, 59, 60, 61, 62, 63, −13, −19, −25, −30, −31, −34, −36, −39, −41, −44, −45, −47, −48, −51, −54, −56; 7
19	63, −34, −39, −44, −47, −48, −49, −51, −54, −56; −13, −19, −25, −30, −31, −36, −41, −45, −7
20	52, 55, 56, 57, 59, −30, −34, −36, −41, −45; 58, 60, 61, 62, −13, −19, −25, −31, 63, −7
21	34, 39, 44, 47, 48, 49, 51, 54, 56, −13, −19, −25, −30, −31, −36, −41, −45; 52, 55, 57, 58, 59, 60, 61, 62, −7, 63
22	52, 55, 57, 58, 59, 60, 61, 62, −34, −39, −44, −47, −48, −49, −51, −54, −56; 63, −13, −19, −25, −30, −31, −36, −41, −45, −7

In Table 7, which is analogous to Table 3, we list the indices k of the roots $\alpha = \alpha_k$ for which $\mathfrak{g}^\alpha \subset \mathfrak{p}_2(H^i)$.

Theorem 4.1. *Let* $\mathfrak{g}_0$ *be of type* E VII*. Then the closure ordering of the nilpotent* K^0*-orbits in* $\mathfrak{p}$ *is as given in Figure* 6*. The group* $K/K^0 = Z_2$ *acts on the diagram as the reflection in the vertical axis of symmetry.*

The dotted lines join the K^0-orbits that are contained in the same G-orbit.

Proof. For each pair of nodes (i, j) in Figure 6 that are joined by a solid line, with i above j, we prove that $\mathcal{O}_1^i > \mathcal{O}_1^j$ by exhibiting an element $E \in \mathcal{O}_1^j$ that belongs to the subspace $\mathfrak{p}_2(H^i)$. These elements are given in Table 8.

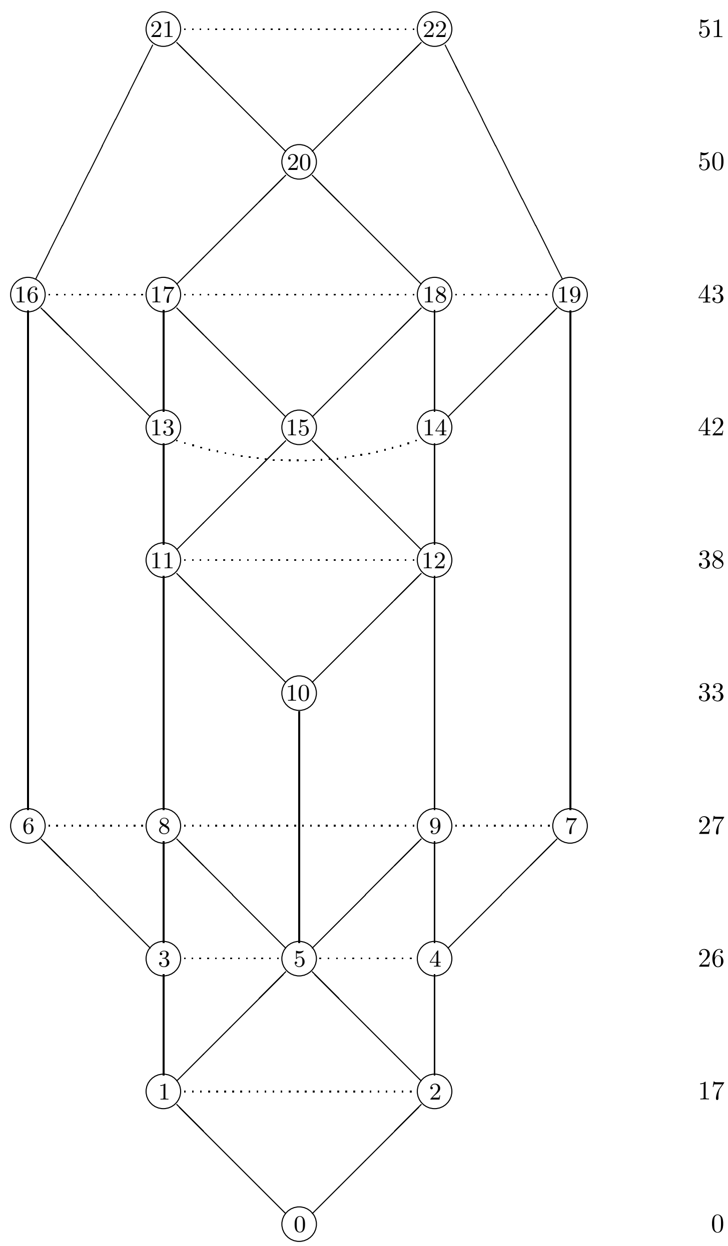

Figure 6. The closure diagram for E VII

By inspection of the diagram in Figure 6, we see that in order to complete the proof of the correctness of the diagram it suffices to prove that $\mathcal{O}_1^i \not\geq \mathcal{O}_1^j$ for the following pairs (i,j):

$$(1,7),\ (2,6),\ (19,3),\ (16,4),\ (22,6),\ (21,7),\ (18,13),\ (17,14). \tag{4.1}$$

Let V be the simple $\mathfrak{g}$-module with highest weight ω_7 (of dimension 56). It admits a Z_2-grading $V = V_0 \oplus V_1$ with both spaces V_0 and V_1 of dimension 28. This means that $\mathfrak{k} \cdot V_i \subset V_i$ and $\mathfrak{p} \cdot V_i \subset V_{1-i}$ for $i = 0, 1$. We compute the integers $d_i(j,k)$; $i = 0,1$; $j \geq 1$; for each k, $1 \leq k \leq 22$. They are given in Table 9. Since $d_0(1,7) > d_0(1,1)$, we conclude that $\mathcal{O}_1^7 \not\geq \mathcal{O}_1^1$. The argument is similar for the other pairs in (4.1).

The second assertion follows from the fact that an element of $K \setminus K^0$ induces an outer automorphism of the subalgebra E_6 of $\mathfrak{k}$. □

TABLE 8. Elements $E \in \mathfrak{p}_2(H^i) \cap \mathcal{O}_1^j$

i	j	Type	E
20	18	$(A_3+A_1)''$	$(X_{-30}+X_{52}+X_{-34})+(X_{62})$
20	17	$(A_3+A_1)''$	$(X_{55}+X_{-36}+X_{56})+(X_{-25})$
21	16	$(A_3+A_1)''$	$(X_{34}+X_{-7}+X_{56})+(X_{49})$
22	19	$(A_3+A_1)''$	$(X_{-34}+X_{63}+X_{-56})+(X_{-49})$
16,17	13	A_3	$X_{34}+X_{-7}+X_{56}$
18,19	14	A_3	$X_{-34}+X_{63}+X_{-56}$
17,18	15	$2A_2$	$(X_{52}+X_{-13})+(X_{62}+X_{-45})$
13	11	A_2+A_1	$(X_{56}+X_{-7})+(X_{59})$
14	12	A_2+A_1	$(X_{63}+X_{-34})+(X_{-30})$
15	11	A_2+A_1	$(X_{58}+X_{-13})+(X_{59})$
15	12	A_2+A_1	$(X_{62}+X_{-31})+(X_{-30})$
11	8	$(3A_1)''$	$(X_{58})+(X_{59})+(X_{-7})$
12	9	$(3A_1)''$	$(X_{63})+(X_{-30})+(X_{-31})$
16	6	$(3A_1)''$	$(X_{34})+(X_{49})+(X_{56})$
19	7	$(3A_1)''$	$(X_{-34})+(X_{-49})+(X_{-56})$
10	5	$2A_1$	$(X_{63})+(X_{-7})$
6,8	3	$2A_1$	$(X_{58})+(X_{59})$
7,9	4	$2A_1$	$(X_{-30})+(X_{-31})$
8,9	5	$2A_1$	$(X_{63})+(X_{-7})$
3,5	1	A_1	X_{63}
4,5	2	A_1	X_{-7}

TABLE 9. The integers $d_i(j,k)$ for the module $V(\omega_7)$

k	$d_0(j,k)$	$d_1(j,k)$
1	18 28	26 28
2	26 28	18 28
3	10 27 28	26 27 28
4	26 27 28	10 27 28
5	18 27 28	18 27 28
6	1 27 28	26 27 27 28
7	26 27 27 28	1 27 28
8	10 27 27 28	17 27 28
9	17 27 28	10 27 27 28
10	16 22 28	16 22 28
11	10 22 26 28	16 22 28
12	16 22 28	10 22 26 28
13	8 17 18 27 28	16 17 26 27 28
14	16 17 26 27 28	8 17 18 27 28
15	10 18 26 27 28	10 18 26 27 28
16	1 17 18 27 27 28	16 17 26 27 28
17	8 17 18 27 28	9 17 26 27 27 28
18	9 17 26 27 27 28	8 17 18 27 28
19	16 17 26 27 28	1 17 18 27 27 28
20	8 13 18 22 26 27 28	8 13 18 22 26 27 28
21	1 9 10 18 18 26 26 27 27 28	8 9 17 18 26 26 27 27 28
22	8 9 17 18 26 26 27 27 28	1 9 10 18 18 26 26 27 27 28

5. APPENDIX

In Table 10 we give our enumeration of the positive roots of E_7. We recall that the simple roots are chosen as in [3]. This enumeration is the same as in [8].

TABLE 10. Positive roots of E_7

i	α_i	i	α_i	i	α_i
1	1000000	22	0111100	43	1112210
2	0100000	23	0101110	44	1112111
3	0010000	24	0011110	45	0112211
4	0001000	25	0001111	46	1122210
5	0000100	26	1111100	47	1122111
6	0000010	27	1011110	48	1112211
7	0000001	28	0112100	49	0112221
8	1010000	29	0111110	50	1123210
9	0101000	30	0101111	51	1122211
10	0011000	31	0011111	52	1112221
11	0001100	32	1112100	53	1223210
12	0000110	33	1111110	54	1123211
13	0000011	34	1011111	55	1122221

Table 10. (continued)

i	α_i	i	α_i	i	α_i
14	1011000	35	0112110	56	1223211
15	0111000	36	0111111	57	1123221
16	0101100	37	1122100	58	1223221
17	0011100	38	1112110	59	1123321
18	0001110	39	1111111	60	1223321
19	0000111	40	0112210	61	1224321
20	1111000	41	0112111	62	1234321
21	1011100	42	1122110	63	2234321

We use this occasion to correct two misprints in that paper. First, on page 10 the representative E for the orbits 60,61 should be:

$$E = \pm[3X_7 + X_{28} + 2\sqrt{2}(X_{18} + X_{29}) + \sqrt{5}(X_{14} + X_{29})].$$

Second, the Weyl group mentioned at the bottom of page 7 should be $W(\mathfrak{k}^c, \mathfrak{h}^c)$.

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