REPRESENTATION THEORY
An Electronic Journal of the American Mathematical Society
Volume 5, Pages 111–128 (May 18, 2001)
S 1088-4165(01)00109-1

$U(\mathfrak{g})$-FINITE LOCALLY ANALYTIC REPRESENTATIONS

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WITH AN APPENDIX BY DIPENDRA PRASAD

Abstract. In this paper we continue our algebraic approach to the study of locally analytic representations of a p-adic Lie group G in vector spaces over a non-Archimedean complete field K. We characterize the smooth representations of Langlands theory which are contained in the new category. More generally, we completely determine the structure of the representations on which the universal enveloping algebra $U(\mathfrak{g})$ of the Lie algebra $\mathfrak{g}$ of G acts through a finite dimensional quotient. They are direct sums of tensor products of smooth and rational G-representations. Finally we analyze the reducible members of the principal series of the group $G = SL_2(\mathbb{Q}_p)$ in terms of such tensor products.

In this paper we continue the study of locally analytic representations of a p-adic Lie group G in vector spaces over a spherically complete non-Archimedean field K. In [ST], we began with an algebraic approach to this type of representation theory based on a duality functor that replaces locally analytic representations by certain topological modules over the algebra $D(G, K)$ of locally analytic distributions. As an application, we established the topological irreducibility of generic locally analytic principal series representations of $\mathbf{GL}_2(\mathbb{Q}_p)$ by proving the algebraic simplicity of the corresponding $D(\mathbf{GL}_2(\mathbb{Q}_p), K)$-modules.

In this paper we further exploit this algebraic point of view. We introduce a particular category of "analytic" $D(G, K)$-modules that lie in the image of the duality functor and therefore correspond to certain locally analytic representations. For compact groups G, these are finitely generated $D(G, K)$-modules that allow a (necessarily uniquely determined) Fréchet topology for which the $D(G, K)$-action is continuous. For more general groups, one tests analyticity by considering the action of $D(H, K)$ for a compact open subgroup H in G. The category of analytic modules has the nice property that any algebraic map between such modules is automatically continuous. The concept of analytic module is dual to the concept of strongly admissible G-representation introduced in [ST]. The actual definition can and will be given in a way that avoids any mention of a topology on the module.

Next, we study the modules dual to the traditional smooth representations of Langlands theory. We show that a smooth representation gives rise, under duality, to an analytic module precisely when it is "strongly admissible"; this is a condition on the multiplicities with which the irreducible representations of a compact open subgroup of G appear in the representation. In particular, if L is a finite extension of $\mathbb{Q}_p$ and G is the group of L-points of a connected reductive algebraic group over L, then any smooth representation of finite length is strongly admissible. This is basically a theorem of Harish-Chandra ([HC]), although we must use, in addition,

Received by the editors August 2, 2000 and, in revised form, September 25, 2000.
2000 *Mathematics Subject Classification.* Primary 17B15, 22D12, 22D15, 22D30, 22E50.

results of Vigneras ([Vig]) to deal with some complications arising from the fact that we do not assume that our coefficient field K is algebraically closed.

Given these foundational results, suppose that G is the group of L-points of a split, semisimple, and simply connected group over L. We completely determine the structure of analytic modules M that are $U(\mathfrak{g})$-finite, i.e., that are annihilated by a 2-sided ideal of finite codimension in the universal enveloping algebra $U(\mathfrak{g})$ of the Lie algebra $\mathfrak{g}$ of G. Such a module can be decomposed into a finite sum of modules of the form $E \otimes \mathrm{Hom}(V, K)$ where E is irreducible, finite dimensional, and algebraic, and V is smooth and strongly admissible. The dual representations $E^* \otimes V$ are irreducible—in fact, simple as $K[G]$ modules—if and only if V is irreducible. Some of the technical hypotheses on the group G in this section are consequences of the fact that our coefficient field is not algebraically closed.

We conclude the paper by studying the reducible members of the locally analytic principal series of $\mathbf{SL}_2(\mathbb{Q}_p)$. The corresponding modules contain a simple submodule such that the quotient is $U(\mathfrak{g})$-finite, and we use our methods to determine the structure of this quotient. In particular, we obtain the result that the topological length of the locally analytic principal series is at most three—a fact that is due to Morita ([Mor]) by a different method.

In the appendix by Dipendra Prasad, a global variant of the $U(\mathfrak{g})$-finite representations, called locally algebraic representations, is introduced and studied. This point of view allows us to simplify and generalize the argument for the irreducibility of tensor products.

1. Analytic modules

We fix fields $\mathbb{Q}_p \subseteq L \subseteq K$ such that $L/\mathbb{Q}_p$ is finite and K is spherically complete with respect to a non-Archimedean absolute value $|\ |$ extending the one on L. We let G be a d-dimensional locally L-analytic group and $D(G, K)$ be the corresponding K-algebra of K-valued distributions on G. Recall ([ST], 2.3) that $D(G, K)$ is an associative unital K-algebra with a natural locally convex topology in which the multiplication $*$ is separately continuous. Unless this topology is explicitly mentioned $D(G, K)$ is treated as an abstract algebra. In the following we want to single out a certain class of (unital left) $D(G, K)$-modules which seems to provide a convenient framework for the representation theory of G over K. Let M be a $D(G, K)$-module.

Definition. A K-linear form ℓ on M is called locally analytic if, for any $m \in M$, the linear form $\lambda \longmapsto \ell(\lambda m)$ on $D(G, K)$ is continuous.

Clearly the locally analytic linear forms on M form a vector subspace M' of the full K-linear dual M^* of M. We first consider the case of a compact group G. Then recall that $D(G, K)$ is a K-Fréchet algebra and as a locally convex K-vector space is reflexive ([ST], 1.1, 2.1, and 2.3).

Definition. Suppose G is compact; a $D(G, K)$-module M is called analytic if it is finitely generated and if, for any $0 \neq m \in M$, there is a locally analytic linear form ℓ on M such that $\ell(m) \neq 0$.

Proposition 1.1. *Suppose G is compact; for a finitely generated $D(G, K)$-module M the following assertions are equivalent:*

(i) *M is analytic;*

(ii) *M carries a Fréchet topology with respect to which it is a continuous $D(G,K)$-module.*

Proof. We first assume that (ii) holds true. Evidently, any continuous linear form on M then is locally analytic. Hence it follows from the Hahn-Banach theorem that M is analytic. Assume now vice versa that M is analytic. Choose an epimorphism $\alpha : D(G,K)^r \twoheadrightarrow M$ of $D(G,K)$-modules for some $r \geq 1$. Then the linear forms $\ell \circ \alpha$ for any $\ell \in M'$ are continuous and their simultaneous kernel coincides with the kernel of α. In particular, the kernel of α is closed in $D(G,K)^r$ so that the quotient topology via α on M has the required properties.

By the argument in the proof of [ST], 3.5, the above Fréchet topology on an analytic $D(G,K)$-module M is unique and therefore will be called the *canonical topology* of M. The continuous dual of M is M' and given the strong topology it is a vector space of compact type carrying a locally analytic G-representation ([ST], §§1 and 3); in particular, the canonical topology on M is reflexive. Again by [ST], 3.5 any $D(G,K)$-linear map between two analytic $D(G,K)$-modules is continuous in the canonical topologies.

Question. Is any $D(G,K)$-module of finite presentation analytic?

Example. As a consequence of [ST], 4.4, the answer is yes for the group $G = \mathbb{Z}_p$.

The above definition of an analytic $D(G,K)$-module for a compact group is extended to a general group G in the following way. Note first that for any compact open subgroup $H \subseteq G$ the algebra $D(H,K)$ is a subalgebra of $D(G,K)$ and that

$$D(G,K) = \bigoplus_{g \in G/H} \delta_g * D(H,K)$$

where δ_g denotes the Dirac distribution in $g \in G$.

Definition. A $D(G,K)$-module M is called analytic if it is analytic as a $D(H,K)$-module for any compact open subgroup $H \subseteq G$.

Lemma 1.2. *Fix a compact open subgroup $H \subseteq G$; a $D(G,K)$-module M is analytic if it is analytic as a $D(H,K)$-module.*

Proof. This follows easily from the fact that for any two compact open subgroups H and H' in G the intersection $H \cap H'$ is of finite index in H and in H'.

Suppose that M is an analytic $D(G,K)$-module. One easily checks that the canonical topology of M as a $D(H,K)$-module is independent of the choice of the compact open subgroup $H \subseteq G$, that the $D(G,K)$-action on M is separately continuous, and that M' is the continuous dual of M and equipped with the strong topology carries a locally analytic G-representation. Of course, any $D(G,K)$-linear map between two analytic $D(G,K)$-modules is continuous in the canonical topologies.

Definition. An analytic $D(G,K)$-module is called quasi-simple if it has no nonzero proper $D(G,K)$-submodules which are closed in the canonical topology.

An analytic $D(G,K)$-module M is trivially quasi-simple if it is (algebraically) simple. But, as a consequence of polarity, it is also quasi-simple (and usually not simple) if M' is a simple $D(G,K)$-module. For a noncompact G we will see examples of this later on. We don't know whether such examples also exist for compact groups.

2. Smooth G-representations

In this section we want to see how the smooth representation theory of G fits into our new framework. We recall that a smooth G-representation V (over K) is a K-vector space V with a linear G-action such that the stabilizer of each vector in V is open in G. (Traditionally one considers smooth G-representations in $\mathbb{C}$-vector spaces; but since the topology of the coefficient field plays absolutely no role in the definition, this makes a difference only in-so-far as we do not require K to be algebraically closed.) Moreover, a smooth G-representation V is called admissible if, for any compact open subgroup $H \subseteq G$, the vector subspace V^H of H-invariant vectors in V is finite dimensional. Finally, irreducibility of a smooth representation is always meant in the algebraic sense.

The unit element in G has a countable fundamental system of open compact neighborhoods. This implies that the finest locally convex topology on an admissible G-representation V is of compact type (being the countable locally convex inductive limit of the finite dimensional spaces V^H). Since the orbit maps $\rho_v(g) := gv$, for $v \in V$, are locally constant on G we see that any admissible G-representation V becomes a locally analytic G-representation on a vector space of compact type once we equip V with the finest locally convex topology; as such we denote it by V^c.

Let $\mathfrak{g}$ denote the Lie algebra of G and let $U(\mathfrak{g})$ be the universal enveloping algebra of $\mathfrak{g}$. The latter is naturally included in $D(G,K)$ ([ST], §2). The action of an $\mathfrak{x} \in \mathfrak{g}$ on a locally analytic G-representation W is given by

$$w \to \mathfrak{x}w := \frac{d}{dt}\exp(t\mathfrak{x})w|_{t=0} \tag{1}$$

where $\exp\colon \mathfrak{g} \dashrightarrow G$ denotes the exponential map defined locally around 0 ([ST], 3.2). In addition, Taylor's formula says that, for each fixed $w \in W$, there is a sufficiently small neighborhood U of 0 in $\mathfrak{g}$ such that, for $\mathfrak{x} \in U$, we have a convergent expansion

$$\exp(\mathfrak{x})w = \sum_{n=0}^{\infty} \frac{1}{n!}\mathfrak{x}^n w\,. \tag{2}$$

The formulas (1) and (2) together imply that the orbit maps ρ_w, for $w \in W$, are locally constant if and only if the $\mathfrak{g}$-action on W is trivial or equivalently if and only if the closed 2-sided ideal $I(\mathfrak{g})$ in $D(G,K)$ generated by $\mathfrak{g}$ annihilates W.

What can we say about the quotient algebra $D^\infty(G,K) := D(G,K)/I(\mathfrak{g})$?

$D(G,K)$ is the strong dual of the space $C^{an}(G,K)$ of K-valued locally analytic functions on G. Since the Dirac distributions generate a dense subspace in $D(G,K)$ ([ST], 3.1) the ideal $I(\mathfrak{g})$ is the orthogonal of the closed subspace in $C^{an}(G,K)$ which is the simultaneous kernel of all linear forms $\delta_g * \mathfrak{x} * \delta_h$ with $\mathfrak{x} \in \mathfrak{g}$ and $g, h \in G$. This is precisely the subspace of those functions in $C^{an}(G,K)$ which are annihilated by the action of $\mathfrak{g}$. And this in turn, by Taylor's formula, is the subspace $C^\infty(G,K)$ of all K-valued locally constant functions on G with the subspace topology. On the other hand, as a direct product of spaces of compact type the space $C^{an}(G,K)$ is reflexive. In this situation the strong dual of a closed subspace is the quotient of the strong dual by the orthogonal subspace ([B-TVS], IV.16, Cor.). In other words, we have

$$D^\infty(G,K) = C^\infty(G,K)'_b\,.$$

Moreover, if $H \subseteq G$ is a fixed compact open subgroup, then

$$C^\infty(G,K) = \prod_{g \in G/H} C^\infty(gH,K)$$

is the direct product of the spaces $C^\infty(gH,K)$ each of which is a locally convex inductive limit of finite dimensional spaces and hence carries the finest locally convex topology (compare [ST], 1.2.i). In particular, $C^\infty(H,K)$ is the inductive limit

$$C^\infty(H,K) = \varinjlim_N K[H/N]$$

of the algebraic group rings $K[H/N]$ with N running through the open normal subgroups of H.

All of this applies to V^c for any admissible G-representation V. In particular, V^c as well as its strong dual $(V^c)'_b$ are $D^\infty(G,K)$-modules. Clearly, $M := (V^c)'_b$ is an analytic $D(G,K)$-module if and only if M is finitely generated as a $D^\infty(H,K)$-module for some fixed but arbitrary choice of a compact open subgroup $H \subseteq G$. This condition can be expressed purely in terms of multiplicities as follows. Let $\widehat{H}$ denote the set of isomorphism classes of all irreducible smooth H-representations. Recall that any $\pi \in \widehat{H}$ is finite dimensional. We let

$$\mu(\pi) := \text{multiplicity of } \pi \text{ in } C^\infty(H,K)$$

so that we have

$$C^\infty(H,K) \cong \bigoplus_{\pi \in \widehat{H}} \mu(\pi) \cdot \pi$$

and

$$D^\infty(H,K) \cong \prod_{\pi \in \widehat{H}} (\pi^*)^{\times \mu(\pi)} \tag{3}$$

where π^* denotes the contragredient of π. Any smooth G-representation V is semisimple as an H-representation. Moreover, V is admissible if and only if the multiplicities

$$\mu(\pi, V) := \text{multiplicity of } \pi \text{ in } V$$

for any $\pi \in \widehat{H}$ are finite. We then have

$$V \cong \bigoplus_{\pi \in \widehat{H}} \mu(\pi, V) \cdot \pi$$

and

$$(V^c)'_b \cong \prod_{\pi \in \widehat{H}} (\pi^*)^{\times \mu(\pi, V)} \tag{4}$$

as $D^\infty(H,K)$-modules.

Definition. A smooth G-representation is called strongly admissible if there is a natural number m such that

$$\mu(\pi, V) \leq m \cdot \mu(\pi)$$

for any $\pi \in \widehat{H}$.

That the above definition does not depend on the particular choice of H can be seen as follows. Let $H_0 \subseteq H$ be a pair of compact open subgroups in G. For any $\pi \in \widehat{H}$ and $\sigma \in \widehat{H_0}$ let $\mu(\pi : \sigma)$ denote the multiplicity of σ in $\pi|H_0$. One easily checks that

$$[H : H_0] \cdot \mu(\sigma) = \sum_{\pi \in \widehat{H}} \mu(\pi : \sigma) \cdot \mu(\pi) \quad \text{and} \quad \mu(\pi) = \sum_{\sigma \in \widehat{H_0}} \mu(\pi : \sigma) \cdot \mu(\sigma)\,.$$

Assuming that $\mu(\pi, V) \leq m \cdot \mu(\pi)$, resp. $\mu(\sigma, V) \leq n \cdot \mu(\sigma)$, we compute

$$\begin{aligned} \mu(\sigma, V) &= \sum_{\pi \in \widehat{H}} \mu(\pi : \sigma) \cdot \mu(\pi, V) \\ &\leq m \cdot \sum_{\pi \in \widehat{H}} \mu(\pi : \sigma) \cdot \mu(\pi) \\ &= m \cdot [H : H_0] \cdot \mu(\sigma)\,, \end{aligned}$$

respectively,

$$\begin{aligned} \mu(\pi, V) &\leq \sum_{\sigma \in \widehat{H_0}} \mu(\pi : \sigma) \cdot \mu(\sigma, V) \\ &\leq n \cdot \sum_{\sigma \in \widehat{H_0}} \mu(\pi : \sigma) \cdot \mu(\sigma) \\ &= n \cdot \mu(\pi)\,. \end{aligned}$$

Proposition 2.1. *The functor $V \longmapsto (V^c)'_b$ is an (anti)equivalence of categories between the category of all strongly admissible G-representations and the category of all analytic $D(G, K)$-modules which are annihilated by $I(\mathfrak{g})$.*

Proof. Comparing (3) and (4) it is obvious that $(V^c)'_b$ is finitely generated as a $D^\infty(H, K)$-module if and only if V is strongly admissible. Hence the functor in question is well defined and fully faithful. Moreover, if M is an analytic $D(G, K)$-module annihilated by $I(\mathfrak{g})$, then we have a topological surjection $D^\infty(H, K)^r \twoheadrightarrow M$ for some $r \geq 1$. The dual embedding $M' \hookrightarrow C^\infty(H, K)^r$ shows that $V := M'_b$ carries the finest locally convex topology and therefore is a strongly admissible G-representation. By reflexivity we have $M = (V^c)'_b$ so that M lies in the image of our functor.

Proposition 2.2. *If G is the group of L-rational points of a connected reductive L-group $\mathbf{G}$, then any smooth G-representation of finite length is strongly admissible.*

Proof. Let C be a fixed algebraically closed field which contains K. We first want to reduce the assertion to the case where the coefficient field of the smooth representation is C. Denoting by $(.)_C$ the base extension functor from K to C we have

$$V_C \cong \bigoplus_{\pi \in \widehat{H}} \mu(\pi, V) \cdot \pi_C\,.$$

Let $\mathrm{Irr}_C(H)$ denote the set of isomorphism classes of all irreducible smooth H-representations over C. For each $\sigma \in \mathrm{Irr}_C(H)$ there is a unique $\pi(\sigma) \in \widehat{H}$ such that σ occurs in $\pi(\sigma)_C$. The theory of the Schur index tells us the following ([CR], (70:15)):

1) The Schur index $m_K(\sigma)$ of $\sigma \in \mathrm{Irr}_C(H)$ with respect to K only depends on $\pi(\sigma)$; we therefore put $m_K(\pi) := m_K(\sigma)$ if $\pi = \pi(\sigma)$.

2) For any $\pi \in \widehat{H}$ we have the decomposition

$$\pi_C \cong m_K(\pi) \cdot \bigoplus_{\pi(\sigma)=\pi} \sigma \,.$$

3) If $\pi = \pi(\sigma)$, then $\mu(\pi) \cdot m_K(\pi) = \dim_C \sigma$.

By using 2) our above decomposition of V_C becomes

$$V_C \cong \bigoplus_{\sigma \in \mathrm{Irr}_C(H)} \mu(\pi(\sigma), V) \cdot m_K(\pi(\sigma)) \cdot \sigma \,.$$

If we therefore show that there is an $m \in \mathbb{N}$ such that $\mu(\sigma, V_C) = \mu(\pi(\sigma), V) \cdot m_K(\pi(\sigma)) \leq m \cdot \dim_C \sigma$ for any $\sigma \in \mathrm{Irr}_C(H)$, then it follows from 3) that $\mu(\pi, V) \leq m \cdot \mu(\pi)$ for any $\pi \in \widehat{H}$. According to [Vig], II.4.3.c with V also V_C is of finite length. This reduces us to proving our assertion for smooth G-representation over some algebraically closed field C containing the field of complex numbers $\mathbb{C}$. We first look at the case when V is irreducible supercuspidal. By a character twist we may assume that the central character of V is of finite order. According to [Vig], II.4.9, the representation V is then the base extension to C of an irreducible supercuspidal G-representation over $\mathbb{C}$. For the latter our assertion is a theorem of Harish-Chandra ([HC], Cor. of Thm. 2), and it is obvious that the base extension between two algebraically closed fields respects our assertion. Since a general irreducible V is contained in a representation parabolically induced from a supercuspidal representation, it remains to show that parabolic induction respects strong admissibility. Let $P = P_L P_u$ be a parabolic subgroup of G with unipotent radical P_u and Levi factor P_L and let W be a strongly admissible smooth representation of P_L. We have to check that $V := \mathrm{Ind}_P^G(W)$ is again strongly admissible. Since V is known to be admissible ([Vig], II.2.1), we can do this by proving that the full linear dual V^* of V is finitely generated as an $D^\infty(H,K)$-module. Moreover, being completely free in the choice of the compact open subgroup H of G we may choose it in such a way that the Iwasawa decomposition $G = HP$ holds. Put $H_P := H \cap P$ and let H_L denote the image of H_P in P_L. As an H-representation we then have

$$\mathrm{Ind}_P^G(W) = \mathrm{Ind}_{H_P}^H(W|H_L) \,.$$

By assumption $(W|H_L)^*$ is a finitely generated $D^\infty(H_L,K)$-module. Therefore, it suffices to see that

$$\mathrm{Ind}_{H_P}^H(W|H_L)^* = D^\infty(H,K) \underset{D^\infty(H_P,K)}{\otimes} (W|H_L)^*$$

holds true. By semisimplicity this is an easy consequence of the analogous identity with $W|H_L$ replaced by $C^\infty(H_P,K)$.

As a consequence of these results we obtain that the functor $V \longmapsto (V^c)'_b$ induces a bijective correspondence between irreducible smooth G-representations and quasi-simple analytic $D(G,K)$-modules which are annihilated by $I(\mathfrak{g})$. It should be pointed out that $(V^c)'_b$ as a vector space is the full linear dual of V. The smooth linear forms, in general, form a proper $D^\infty(G,K)$-submodule of $(V^c)'_b$ so that the latter cannot be simple.

3. $U(\mathfrak{g})$-FINITE MODULES

In this section we let G be the group of L-rational points of a connected reductive split L-group $\mathbf{G}$. We want to understand more generally those analytic $D(G, K)$-modules M on which $U(\mathfrak{g})$ acts through a finite dimensional quotient. They will be called $U(\mathfrak{g})$-finite.

Let E be the underlying L-vector space of an irreducible L-rational algebraic representation of $\mathbf{G}$. For any $U(\mathfrak{g})$-finite analytic $D(G, K)$-module M we set

$$M^E := \mathrm{Hom}_{U(\mathfrak{g})}(E, M)\,.$$

$\mathrm{Hom}_L(E, M)$ and hence M^E as a closed vector subspace both inherit a natural Fréchet topology from M. The group G acts on M^E via the continuous K-linear endomorphisms

$$f \longmapsto {}^g f(x) := g(f(g^{-1}x)) \quad \text{for } g \in G \quad \text{and} \quad f \in M^E.$$

Moreover,

$$\begin{array}{ccc} E \times M^E & \longrightarrow & M \\ (x, f) & \longmapsto & f(x) \end{array}$$

is a continuous G-equivariant bilinear map.

Let $V := M'_b$ denote the strong dual of M as a locally analytic G-representation. In order to determine the topology on V we need the following result.

Proposition 3.1. *Let $J \subseteq U(\mathfrak{g})$ be a 2-sided ideal of finite codimension and let $H \subseteq G$ be a compact open subgroup; then the subspace topology on the subspace $C^{an}(H, K)^{J=0}$ of all vectors in $C^{an}(H, K)$ annihilated by J is the finest locally convex topology.*

Proof. Fix an ordered vector space basis for $\mathfrak{g}$, and an exponential map for G. This data, together with a choice of disk of sufficiently small radius s around the origin in $L^{\dim \mathfrak{g}}$, determines a "canonical chart of the second kind" on H. Let H_r be the family of standard compact open subgroups of H obtained from this canonical chart (see [Fea], 4.3.3). The Banach space of analytic functions on H_r is the standard Banach space $\mathcal{F}_{0,r}(K)$ of convergent series with coefficients in K on the disk of radius r for $0 < r \leq s$. Let

$$\mathcal{F}_r := \prod_{h \in H_r \backslash H} \mathcal{F}_{0,r}(K)\,.$$

Following the proof of [Fea], 3.3.4, we see that this Banach space is an analytic H_s-representation and

$$\varinjlim \mathcal{F}_r \xrightarrow{\sim} C^{an}(H, K)\,.$$

By [Fea], 4.7.3, there is a nondegenerate pairing between $U(\mathfrak{g})$ and the factor $\mathcal{F}_{0,r}$ of the product defining $\mathcal{F}_r$ corresponding to the trivial coset H_r. This pairing is given by evaluation at the identity element

$$\begin{array}{ccc} U(\mathfrak{g}) \times \mathcal{F}_{0,r} & \rightarrow & K \\ (\mathfrak{z}, f) & \mapsto & (\mathfrak{z}f)(1)\,. \end{array}$$

The ideal J is of finite codimension in $U(\mathfrak{g})$, and given the nondegeneracy of the pairing it follows that the space $\mathcal{F}_{0,r}^{J=0}$ is finite dimensional. Furthermore, because the $U(\mathfrak{g})$-action from the left commutes with the right translation action of H it follows immediately that $\mathcal{F}_r^{J=0}$ is finite dimensional. Then $C^{an}(H, K)^{J=0}$, being the direct limit of these finite dimensional spaces ([ST], 1.2.i), has the finest locally convex topology.

Since M is analytic we have a surjection $D(H,K)^m \twoheadrightarrow M$ of $D(H,K)$-modules for some $m \in \mathbb{N}$ and some (or any) compact open subgroup $H \subseteq G$. After dualizing, we obtain an injection $V \hookrightarrow C^{an}(H,K)^m$ which certainly is $U(\mathfrak{g})$-linear ([ST], 3.2). Moreover, by assumption there is a 2-sided ideal $J \subseteq U(\mathfrak{g})$ of finite codimension which annihilates M and hence V. Hence we actually have an injection $V \hookrightarrow (C^{an}(H,K)^{J=0})^m$. Applying Proposition 3.1 we now see that the topology on V necessarily is the finest locally convex one.

For general reasons $E \otimes_L V$ with G acting diagonally also is a locally analytic G-representation on a K-vector space of compact type ([Fea], 2.4.3 and [ST], 1.2.ii). For our particular V the topology on $E \otimes_L V$ is, according to the above discussion, the finest locally convex one. We let $E \otimes_{U(\mathfrak{g})} V$ denote the G-equivariant quotient of $E \otimes_L V$ by the (automatically closed) K-vector subspace generated by all vectors of the form $\mathfrak{x}x \otimes v + x \otimes \mathfrak{x}v$ for $\mathfrak{x} \in \mathfrak{g}$, $x \in E$, and $v \in V$. By [ST], 1.2.i this quotient is a locally analytic G-representation on a K-vector space of compact type whose topology is the finest locally convex one and whose strong dual evidently is M^E. In particular, both $E \otimes_{U(\mathfrak{g})} V$ and M^E are separately continuous $D(G,K)$-modules.

By continuity and [ST], 3.1, the above bilinear map $E \times M^E \longrightarrow M$ induces a continuous $D(G,K)$-module homomorphism

$$E \otimes_L M^E \longrightarrow M .$$

By construction the $\mathfrak{g}$-action on $E \otimes_{U(\mathfrak{g})} V$ derived from the G-action is trivial. Hence $I(\mathfrak{g})$ annihilates $E \otimes_{U(\mathfrak{g})} V$ and by duality also M^E. Provided that M^E is finitely generated as a $D(H,K)$-module for some compact open subgroup $H \subseteq G$, it follows from Proposition 2.1 that M^E is the dual of the strongly admissible G-representation $E \otimes_{U(\mathfrak{g})} V$.

Let $\widehat{\mathbf{G}}$ denote the set of isomorphism classes of all irreducible L-rational algebraic representations of $\mathbf{G}$. We have the continuous $D(G,K)$-module homomorphism

$$\bigoplus_{E \in \widehat{\mathbf{G}}} E \otimes_L M^E \longrightarrow M .$$

The direct sum on the left-hand side in fact is finite since the number of $E \in \widehat{\mathbf{G}}$ which are annihilated by a given 2-sided ideal of finite codimension in $U(\mathfrak{g})$ is finite.

Proposition 3.2. *Assume that* $\mathbf{G}$ *is split semisimple and simply connected; for any* $U(\mathfrak{g})$*-finite analytic* $D(G,K)$*-module* M *the natural map*

$$\bigoplus_{E \in \widehat{\mathbf{G}}} E \otimes_L M^E \xrightarrow{\cong} M$$

is an isomorphism of $D(G,K)$*-modules, each* M^E *is the linear dual of a strongly admissible* G*-representation over* K*, and* $M^E = 0$ *for all but finitely many* $E \in \widehat{\mathbf{G}}$.

Proof. We have already noted that the map in question is a homomorphism of $D(G,K)$-modules and that the direct sum on the left-hand side is finite. To establish the bijectivity we set $\mathfrak{g}_K := \mathfrak{g} \otimes_L K$ and we let $\widehat{\mathfrak{g}}$, resp. $\widehat{\mathfrak{g}_K}$, denote the set of isomorphism classes of all finite dimensional simple $\mathfrak{g}$-modules, resp. $\mathfrak{g}_K$-modules.

By assumption M is a $U(\mathfrak{g}_K)/J$-module for some 2-sided ideal $J \subseteq U(\mathfrak{g}_K)$ of finite codimension. Since $\mathfrak{g}$ is semisimple, the algebra $U(\mathfrak{g}_K)/J$ is semisimple ([Dix], 1.6.4). Hence M is a semisimple $\mathfrak{g}_K$-module and we have its isotypic decomposition

$$M = \bigoplus_{E \in \widehat{\mathfrak{g}_K}} M_E$$

(compare [Dix], 1.2.8). Moreover, since $\mathrm{End}_{U(\mathfrak{g}_K)}(E) = K$ ([Dix], 2.6.5 and 7.2.2(i)), we have the natural isomorphism

$$E \otimes_K \mathrm{Hom}_{U(\mathfrak{g}_K)}(E, M) \xrightarrow{\cong} M_E$$

for any $E \subseteq \widehat{\mathfrak{g}_K}$. Since the functor $E \longmapsto E \otimes_L K$ induces a bijection $\widehat{\mathfrak{g}} \xrightarrow{\sim} \widehat{\mathfrak{g}_K}$ (both sides are classified by the dominant weights) the above isotypic decomposition can be rewritten as a bijection

$$\bigoplus_{E \in \widehat{\mathfrak{g}}} E \otimes_L \mathrm{Hom}_{U(\mathfrak{g})}(E, M) \xrightarrow{\simeq} M\,.$$

But since $\mathbf{G}$ is assumed to be simply connected, we have, by derivation, the natural bijection $\widehat{\mathbf{G}} \xrightarrow{\sim} \widehat{\mathfrak{g}}$ so that the last bijection coincides with the isomorphism in the assertion.

With M also its direct summand, $E \otimes_L M^E$ is finitely generated as a $D(H, K)$-module for any compact open subgroup $H \subseteq G$. It follows that M^E is a finitely generated $D(H, K)$-module as well: Take finitely many tensors which generate $E \otimes_L M^E$; their M^E-components form a generating set for M^E. As explained above, M^E then is the linear dual of a strongly admissible G-representation.

Example. The assumptions on the group $\mathbf{G}$ in the above proposition cannot be weakened as the following example shows. Let $L = K := \mathbb{Q}_2$, $\mathbf{G} := \mathbf{PGL}_3$, and $\mathbf{G_o} := \mathbf{SL}_3$. Then $G_{\mathrm{o}} := \mathbf{SL}_3(\mathbb{Q}_2)$ is an open normal subgroup of index three in $G = \mathbf{PGL}_3(\mathbb{Q}_2)$. Let E_{o} denote the three-dimensional standard representation of G_{o} and let $M := \mathrm{Ind}_{G_{\mathrm{o}}}^{G}(E_{\mathrm{o}})$ be the induced G-representation (in the sense of abstract groups). It is clear that M is a $U(\mathfrak{g})$-finite analytic $D(G, K)$-module. One checks that as a G_{o}-representation, M is isomorphic to $E_{\mathrm{o}} \oplus E_{\mathrm{o}} \oplus E_{\mathrm{o}}$. Since $\widehat{\mathbf{G}}$ is a subset of $\widehat{\mathbf{G_o}} = \widehat{\mathfrak{g}}$ to which E_{o} does not belong, we see that $\mathrm{Hom}_{U(\mathfrak{g})}(E, M) = 0$ for any $E \in \widehat{\mathbf{G}}$.

For the sake of completeness we remark that vice versa any finite direct sum $E_1 \otimes_L \mathrm{Hom}_K(V_1, K) \oplus \ldots \oplus E_r \otimes_L \mathrm{Hom}_K(V_r, K)$ with $E_i \in \widehat{\mathbf{G}}$ and strongly admissible smooth G-representations V_i over K is a $U(\mathfrak{g})$-finite analytic $D(G, K)$-module. Apart from the finite generation which is contained in the subsequent lemma, this is clear.

Lemma 3.3. *Let $H \subseteq G$ be a compact open subgroup; for any finitely generated $D^\infty(H, K)$-module N and any $E \in \widehat{\mathbf{G}}$ the $D(H, K)$-module $E \otimes_L N$ is finitely generated.*

Proof. We begin with a general observation. Let $\mathcal{O}(\mathbf{G})$ denote the space of L-rational functions on G. Then the map

$$\begin{array}{ccc} \mathcal{O}(\mathbf{G}) \underset{L}{\otimes} C^\infty(H, K) & \longrightarrow & C^{an}(H, K) \\ (\psi, f) & \longmapsto & \psi | H \cdot f \end{array}$$

is injective. This can be seen as follows. Let $\sum_{j=1}^{m} \psi_j \otimes f_j$ be an element in the left-hand side such that $\sum_j \psi_j|H \cdot f_j = 0$. We may assume that $\psi_1, \ldots, \psi_m$ are linearly independent. Choose a disjoint covering $H = \dot{\bigcup}_{i=1}^{n} U_i$ by nonempty open subsets $U_i \subseteq H$ such that the restrictions $f_j|U_i$, for any $1 \leq j \leq m$ and $1 \leq i \leq n$, are constant. Since each U_i is Zariski dense in $\mathbf{G}$ (this can be deduced, e.g., from [DG], II.5.4.3 and II.6.2.1) it follows that $\psi_1|U_i, \ldots, \psi_m|U_i$ viewed in $C^{an}(U_i, K)$ still are linearly independent. Hence $f_j|U_i = 0$ for any i and j and therefore $f_j = 0$ for any j.

Coming back to our assertion it suffices, of course, to consider the case $N = D^\infty(H, K)$. On the other hand, if $J \subseteq U(\mathfrak{g})$ denotes the annihilator ideal of E^*, then we find some G-equivariant embedding $E^* \hookrightarrow \mathcal{O}(\mathbf{G})^{J=0}$. Combining this with the above map leads, using the Leibniz rule, to an H-equivariant embedding $E^* \otimes_L C^\infty(H, K) \hookrightarrow C^{an}(H, K)^{J=0} \subseteq C^{an}(H, K)$. As a consequence of Proposition 3.1 the topology induced by $C^{an}(H, K)$ on the left-hand side is the finest locally convex topology. By dualizing we therefore obtain a surjection $D(H, K) \twoheadrightarrow E \otimes_L D^\infty(H, K)$ of $D(H, K)$-modules.

We finally study the question when a $U(\mathfrak{g})$-finite analytic $D(G, K)$-module is quasi-simple.

Proposition 3.4. *If $E \in \widehat{\mathbf{G}}$ and V is an irreducible smooth G-representation over K, then $E \underset{L}{\otimes} V$ with the diagonal G-action is a simple module over the group ring $K[G]$.*

Proof. We show that each nonzero element $x \in E \otimes_L V$ generates $E \otimes_L V$ as a $K[G]$-module. But first we recall a few facts from rational representation theory (compare [Jan], II §§1 and 2). Fix a Borel subgroup $P \subseteq G$ and a maximal split torus $T \subseteq P$, and let N denote the unipotent radical of P.

1. The subspace E^N of N-invariants in E is one-dimensional and coincides with the weight space E_λ where λ is the highest weight of E (w.r.t. T and B).

2. If $e \in E_\mu$ has weight μ, then $Ne \subseteq e + \sum_{\mu < \nu} E_\nu$.

Fact 1 above holds true similarly on the level of Lie algebras. This shows that whenever $U_\circ \subseteq N$ is an open subgroup then

1′. $E^{U_\circ} = E^N = E_\lambda$ is one-dimensional.

Since E is also an irreducible module for the induced action of the Lie algebra of G, it follows that whenever $U \subseteq G$ is an open subgroup we have

3. $L[U] \cdot e = E$ for any nonzero $e \in E$.

Consider now a fixed nonzero element

$$x = e_1 \otimes v_1 + \ldots + e_r \otimes v_r$$

with $0 \neq e_i \in E$ and $0 \neq v_i \in V$. We may assume that each e_i is a weight vector. In order to show that $K[G] \cdot x = E \otimes_L V$ we may replace x when convenient by any other nonzero element in $K[G] \cdot x$. In a first step we will show that for this reason we may assume in fact that $r = 1$.

By the smoothness of V we find an open subgroup $U \subseteq G$ which fixes each of the vectors $v_1, \ldots, v_r$. Put $U_\circ := U \cap N$. If $U_\circ$ fixes x, we are immediately reduced to the case $r = 1$ since, by 1′, we then have $x \in (E \otimes_L V^U)^{U_\circ} = E^{U_\circ} \otimes_L V^U = E_\lambda \otimes_L V^U$. Otherwise, there is a $g \in U_\circ$ such that $gx - x \neq 0$ and we replace x by

$$gx - x = (ge_1 - e_1) \otimes v_1 + \ldots + (ge_r - e_r) \otimes v_r\,.$$

The point to note is that, by 2, each $ge_i - e_i$ lies in a sum of weight spaces where the occurring weights are strictly bigger than the weight of e_i. This means that one way or another after finitely many steps we have replaced x by a nonzero element in $E_\lambda \otimes_L V^U$ for which r can be assumed to be one.

Therefore, for the second step of the proof, let $x \in E \otimes_L V$ be an element of the form $x = e \otimes v$ with $0 \neq e \in E$ and $0 \neq v \in V$. Denote by $U \subseteq G$ the stabilizer of v. Using 3 and the irreducibility of V we obtain

$$K[G] \cdot x = K[G] \cdot ((L[U] \cdot e) \otimes v) = K[G] \cdot (E \otimes v) = E \otimes K[G] \cdot v = E \otimes V\,.$$

Corollary 3.5. *Assume that* $\mathbf{G}$ *is split semisimple and simply connected and let* M *be any* $U(\mathfrak{g})$*-finite analytic* $D(G,K)$*-module; then* M *is quasi-simple if and only if it is of the form* $M \cong E \otimes_L \operatorname{Hom}_K(V,K)$ *for some* $E \in \widehat{\mathbf{G}}$ *and some irreducible smooth* G*-representation* V *over* K*.*

Proof. If $E \in \widehat{\mathbf{G}}$ and V is irreducible smooth, then $E^* \otimes_L V$ is a simple $D(G,K)$-module by Proposition 3.4. Hence $E \otimes_L \operatorname{Hom}_K(V,K) = (E^* \otimes_L V)'$ is quasi-simple.

If, on the other hand, M is quasi-simple, then there is, by Proposition 3.2, an $E \in \widehat{\mathbf{G}}$ and a strongly admissible G-representation V such that $M = E \otimes_L \operatorname{Hom}_K(V,K)$. With M also $\operatorname{Hom}_K(V,K)$ is quasi-simple. Hence V is irreducible.

The results of this section have more or less obvious counterparts for G being a compact open subgroup in $\mathbf{G}(L)$. We leave precise formulations to the reader.

4. An example

In this last section we will analyze the reducible members of the locally analytic principal series of the group $\mathbf{SL}_2(\mathbb{Q}_p)$ and we will show that they contain tensor product representations of the kind considered in the last section.

Throughout this section let $G := \mathbf{SL}_2(\mathbb{Q}_p)$. Furthermore, let P denote the Borel subgroup of lower triangular matrices in G and T the subgroup of diagonal matrices. We actually will view T as a quotient of P. Assuming that K is contained in the completion of an algebraic closure of $\mathbb{Q}_p$ we fix a K-valued locally analytic character

$$\chi : T \to K^\times\,.$$

The corresponding principal series representation is

$$\operatorname{Ind}_P^G(\chi) := \{f \in C^{an}(G,K) : f(gp) = \chi(p^{-1})f(g) \text{ for any } g \in G, p \in P\}$$

with G acting by left translation. This is a locally analytic G-representation on a vector space of compact type and its strong dual

$$M_\chi := \operatorname{Ind}_P^G(\chi)'_b$$

is a $D(G,K)$-module which is finitely generated, e.g., as a $D(B,K)$-module where B is the Iwahori subgroup of G ([ST], §§5 and 6). By Proposition 1.1 the $D(G,K)$-module M_χ therefore is analytic.

The basic numerical invariant of the character χ which governs the irreducibility properties of $\operatorname{Ind}_P^G(\chi)$ is the number $c(\chi) \in K$ defined by the expansion

$$\chi\left(\begin{pmatrix} t^{-1} & 0 \\ 0 & t \end{pmatrix}\right) = \exp(c(\chi)\log(t))$$

for t sufficiently close to 1 in $\mathbb{Z}_p$. It is shown in [ST], 6.1 that M_χ is a simple $D(G,K)$-module if $c(\chi) \notin -\mathbb{N}_0$ ($\mathbb{N}_0$ is the non-negative integers). We therefore assume for the rest of this section that $m := -c(\chi) \in \mathbb{N}_0$. According to [ST], 6.2, we then have a nonzero homomorphism of $D(G,K)$-modules $M_{\chi'} \longrightarrow M_\chi$ where $\chi' := \epsilon^{m+1}\chi$ and $\epsilon\left(\begin{pmatrix} t^{-1} & 0 \\ 0 & t \end{pmatrix}\right) := t^2$ is the positive root of G with respect to P. Since $c(\chi') = m+2$, the module $M_{\chi'}$ is simple and the above map consequently is injective. It therefore remains to discuss the quotient module

$$M_\chi^{loc} := M_\chi / M_{\chi'}$$

which, of course, is finitely generated. On the other hand, the above map is exhibited in the proof of [ST], 6.2 as the dual I' of a G-equivariant continuous linear map

$$I : \mathrm{Ind}_P^G(\chi) \longrightarrow \mathrm{Ind}_P^G(\chi')$$

whose actual construction we will recall further below. By the argument in [ST], 3.5, the kernel of I again is a locally analytic G-representation on a vector space of compact type. We will see that I is a quotient map or equivalently that the image $I'(M_{\chi'})$ is closed in M_χ. The module M_χ^{loc} therefore is the continuous dual of the kernel of I and, in particular, is analytic.

Write $\chi = \chi_{alg} \cdot \chi_{lc}$ where $\chi_{alg}\left(\begin{pmatrix} t^{-1} & 0 \\ 0 & t \end{pmatrix}\right) := t^{-m}$ is a $\mathbb{Q}_p$-rational character and χ_{lc} is a K-valued locally constant character of T. The character χ_{alg} is dominant for the Borel subgroup P^- opposite to P; hence the algebraic induction $\mathrm{ind}_P^G(\chi_{alg})$ is the irreducible $\mathbb{Q}_p$-rational representation of highest weight χ_{alg} (w.r.t. P^-) of G (compare [Jan], II.2 and II.8.23). On the other hand, since the character χ_{lc} is locally constant, we may form the smooth induced G-representation

$$\mathrm{Ind}_{P,\infty}^G(\chi_{lc}) := \{f \in C^\infty(G,K) : f(gp) = \chi_{lc}(p^{-1})f(g) \text{ for any } g \in G, p \in P\}$$

over K with G acting by left translation. It is known ([Vig], II.5.13) to be a smooth G-representation of finite length which, by Proposition 2.2, implies that it is strongly admissible. There is the obvious G-equivariant linear map

$$\begin{array}{rccc} \tau : & \mathrm{ind}_P^G(\chi_{alg}) \underset{\mathbb{Q}_p}{\otimes} \mathrm{Ind}_{P,\infty}^G(\chi_{lc}) & \longrightarrow & \mathrm{Ind}_P^G(\chi) \\ & (\psi, f) & \longmapsto & \psi \cdot f\,. \end{array}$$

We claim that

$$(*) \qquad 0 \longrightarrow \mathrm{ind}_P^G(\chi_{alg}) \underset{\mathbb{Q}_p}{\otimes} \mathrm{Ind}_{P,\infty}^G(\chi_{lc}) \xrightarrow{\tau} \mathrm{Ind}_P^G(\chi) \xrightarrow{I} \mathrm{Ind}_P^G(\chi') \longrightarrow 0$$

is an exact sequence of locally convex K-vector spaces (where the left-hand term carries the finest locally convex topology). This means that it is exact as a sequence of vector spaces and that the maps involved are strict. By dualizing and observing that the rational representations of G are selfdual this leads to the following result.

Proposition 4.1. *If $c(\chi) \in -\mathbb{N}_0$, then the $D(G,K)$-module M_χ^{loc} is analytic and $U(\mathfrak{g})$-finite and is isomorphic to the tensor product of the $\mathbb{Q}_p$-rational G-representation $\mathrm{ind}_P^G(\chi_{alg})$ and the full K-linear dual of the smooth representation $\mathrm{Ind}_{P,\infty}^G(\chi_{lc})$ of finite length.*

We begin by recalling the construction of I from [ST], 6.2. The group G acts on $C^{an}(G, K)$ by left and right translations. Both actions derive into an action of the Lie algebra $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{Q}_p)$. Whereas the actions coming from left translation are denoted, as usual, by $f \mapsto gf$ for $g \in G$ and $f \mapsto \mathfrak{x}f$ for $\mathfrak{x} \in \mathfrak{g}$ we write $f \mapsto \mathfrak{x}_r f$ for the $\mathfrak{g}$-action derived from right translation. Then

$$I(f) = (\mathfrak{u}^-)_r^{1+m} f$$

where $\mathfrak{u}^- := \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \in \mathfrak{g}$.

Corresponding to the decomposition $G = BP \,\dot\cup\, BwP$ where $B \subseteq G$ is the Iwahori subgroup and $w := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, the sequence $(*)$ is the direct sum of the sequences

$$0 \longrightarrow \mathrm{ind}_P^G(\chi_{alg}) \underset{\mathbb{Q}_p}{\otimes} \mathrm{Ind}_{P,\infty}^{BP}(\chi_{lc}) \xrightarrow{\tau} \mathrm{Ind}_P^{BP}(\chi) \xrightarrow{I} \mathrm{Ind}_P^{BP}(\chi') \longrightarrow 0$$

and

$$0 \longrightarrow \mathrm{ind}_P^G(\chi_{alg}) \underset{\mathbb{Q}_p}{\otimes} \mathrm{Ind}_{P,\infty}^{BwP}(\chi_{lc}) \xrightarrow{\tau} \mathrm{Ind}_P^{BwP}(\chi) \xrightarrow{I} \mathrm{Ind}_P^{BwP}(\chi') \longrightarrow 0\,.$$

The superscripts BP and BwP indicate the subspaces of those functions in the induced representation which are supported in BP and BwP, respectively. Both these sequences can be computed explicitly as follows. Let U, resp. U^-, be the unipotent radical of P, resp. P^-, and define $U_\circ := U \cap B$ and $U_\circ^- := U^- \cap B$. Denoting by u, resp. u^-, the function on $U_\circ$, resp. $U_\circ^-$, which sends a matrix to its left lower, resp. right upper, entry we introduce the finite dimensional $\mathbb{Q}_p$-vector spaces $Pol^m(U_\circ)$ and $Pol^m(U_\circ^-)$ of polynomials of degree $\le m$ in u and u^-, respectively, with coefficients in $\mathbb{Q}_p$. By restricting, resp. translating by w and restricting, functions the above two sequences become isomorphic to

$$0 \longrightarrow Pol^m(U_\circ^-) \underset{\mathbb{Q}_p}{\otimes} C^\infty(U_\circ^-, K) \xrightarrow{\tau} C^{an}(U_\circ^-, K) \xrightarrow{(\frac{d}{du^-})^{1+m}} C^{an}(U_\circ^-, K) \longrightarrow 0$$

and

$$0 \longrightarrow Pol^m(U_\circ) \underset{\mathbb{Q}_p}{\otimes} C^\infty(U_\circ, K) \xrightarrow{\tau} C^{an}(U_\circ, K) \xrightarrow{(-\frac{d}{du})^{1+m}} C^{an}(U_\circ, K) \longrightarrow 0\,.$$

In these sequences the injectivity of the first map as well as the exactness in the middle are obvious. By Proposition 3.1 the subspace topology on the kernel of the second map is the finest locally convex topology. The surjectivity and strictness of the second map can either be checked directly or can be seen as a special case of the more general statement in [Fea], 2.5.4. This finishes the proof of the exactness of $(*)$.

The smooth G-representation $\mathrm{Ind}_{P,\infty}^G(\chi_{lc})$ is of length at most 2. More precisely, one has ([GGP], p. 173) that $\mathrm{Ind}_{P,\infty}^G(\chi_{lc})$ is irreducible except in the following cases:

A) $\chi_{lc} = 1$ is the trivial character. Then $\mathrm{Ind}_{P,\infty}^G(1)$ contains the one-dimensional trivial representation on the subspace of constant functions. The corresponding quotient is the so-called Steinberg representation which is irreducible.

B) χ_{lc} is the character $\chi_{lc}\left(\begin{pmatrix} t^{-1} & 0 \\ 0 & t \end{pmatrix}\right) = |t|^2$ where $|\ |$ denotes the normalized absolute value of $\mathbb{Q}_p$. Then $\mathrm{Ind}_{P,\infty}^G(\chi_{lc})$ contains the Steinberg representation and the corresponding quotient is the one dimensional trivial representation.

C) χ_{lc} is of the form $\chi_{lc}\left(\begin{pmatrix} t^{-1} & 0 \\ 0 & t \end{pmatrix}\right) = |t| \cdot \delta(t)$ for some non-trivial quadratic character $\delta : \mathbb{Q}_p^{\times} \longrightarrow K^{\times}$. Then $\mathrm{Ind}_{P,\infty}^G(\chi_{lc})$ either is irreducible (but not absolutely irreducible) or is the direct sum of two infinite dimensional non-equivalent irreducible G-representations.

If we combine this information with Proposition 4.1 and Corollary 3.5, we obtain a complete list of the quasi-simple constituents of M_χ^{loc} up to isomorphism. In particular, each of them is isomorphic to the tensor product of $\mathrm{ind}_P^G(\chi_{alg})$ and the full K-linear dual of one of the irreducible smooth representations in the above list. At this point it should be mentioned that the length of a Jordan-Hölder series for the kernel of I on $\mathrm{Ind}_P^G(\chi)$ was already determined in [Mor].

Appendix: Locally algebraic representations of p-adic groups

Let k be a non-Archimedean local field, and let $G = \underline{G}(k)$ be the k-rational points of a reductive algebraic group over k. It has been traditional as in the pioneering work of Jacquet and Langlands to study smooth representations of G over complex numbers. It seems likely that representations of G over vector spaces over k (or, extensions of k, such as $\overline{k}$, the algebraic closure of k, or k^c, the completion of $\overline{k}$) may be of interest too for number theoretic applications such as congruences of modular forms. Initial attempts were made by Morita (cf. [Mor]) who studied a certain class of representations of $SL_2(\mathbb{Q}_p)$. Recently Schneider and Teitelbaum have started a detailed program (cf. [ST] and this paper) of defining a good category of such representations which are essentially locally analytic. This involves rather delicate care with the topologies involved. In this note we study a much more restrictive class of representations which we call locally algebraic, defined as follows. In this note we will not differentiate between algebraic representations of $\underline{G}$ and of $G = \underline{G}(k)$ which we are allowed by the Zariski density of $\underline{G}(k)$ in $\underline{G}(\overline{k})$. We also note that an irreducible algebraic representation of $G = \underline{G}(k)$ remains irreducible when restricted to any open subgroup, and that the restriction map from irreducible algebraic representations of G to representations of an open subgroup is an injective map.

Definition. Let $\underline{G}$ be an algebraic group over a non-Archimedean local field k. Let V be a vector space over k', which is an extension of k. A representation π of $G = \underline{G}(k)$ on V is called locally algebraic if:

1. The restriction of π to any compact open subgroup K of G is a sum of finite dimensional irreducible representations of K.

2. For any vector v in V, there exists a compact open subgroup K_v in G, and a finite dimensional subspace U of V containing the vector v such that K_v leaves U invariant and operates on U via restriction to K_v of a finite dimensional algebraic representation of $\underline{G}$.

Examples. 1. Algebraic representations of $\underline{G}$.

2. The usual complex smooth representations of G.

Remark 1. It follows from the standard methods in the p-adic groups, using Lie algebras, that a finite dimensional analytic representation of a compact open subgroup of a semisimple simply connected group in characteristic 0 is indeed algebraic when restricted to an open subgroup of finite index. Therefore it is condition (1) in the definition of locally algebraic representations that is restrictive, and not condition (2) which may appear to be the main part of the condition. One may call representations which satisfy condition (i) of the definition above to be *locally finite dimensional*. From what we have just remarked, locally finite dimensional representation of a semisimple group in characteristic 0 is automatically locally algebraic. On the other hand, it is clear that tori, for instance, as ${\mathbb{Q}_p}^* = \mathbb{Z} \times {\mathbb{Z}_p}^*$, have many nonalgebraic representations. (A character on ${\mathbb{Z}_p}^*$ is, locally around the origin, of the form $x \to \exp(a \log x)$ where a belongs to a field extension of $\mathbb{Q}_p$, and hence is the restriction of a locally algebaric character of ${\mathbb{Q}_p}^*$ if and only if a belongs to $\mathbb{Z}$.)

Here is the main result of this note which classifies all the locally algebraic representations of G.

Theorem 1. 1. *Every irreducible locally algebraic representation π of G is the tensor product $\pi = \pi_1 \otimes \pi_2$ of an irreducible algebraic representation π_1 of G and of a smooth irreducible representation π_2 of G.*

2. *Conversely, the tensor product $\pi = \pi_1 \otimes \pi_2$ of an irreducible algebraic representation π_1 of G and of a smooth irreducible representation π_2 of G is an irreducible locally algebraic representation of G.*

Proof. By the definition of locally algebraic representation, there exists an algebraic representation π_1 of G, a compact open subgroup K_1 of G and a finite dimensional subspace U of π invariant under K_1 such that the action of K_1 on U is the restriction to K_1 of the representation π_1 of G. Clearly, we can assume that π_1 is an irreducible representation of G.

Define

$$\pi_2 = \varinjlim_K \mathrm{Hom}_K[\pi_1, \pi],$$

where the direct limit is taken over all the compact open subgroups K of G which have their common intersection as only $\{e\}$.

There is an action of G on π_2 defined in a natural way as follows. For $\phi \in \mathrm{Hom}_K[\pi_1, \pi]$, and $v_1 \in \pi_1$,

$$(g \cdot \phi)(v_1) = g\phi(g^{-1}v_1).$$

It is clear that if $\phi \in \mathrm{Hom}_K[\pi_1, \pi]$, then $g \cdot \phi \in \mathrm{Hom}_{gKg^{-1}}[\pi_1, \pi]$.

We now claim that the natural map,

$$\Phi : \pi_1 \otimes \varinjlim_K \mathrm{Hom}_K[\pi_1, \pi] \to \pi,$$

defined by mapping (v, ϕ) to $\phi(v)$ is a G-equivariant isomorphism. For this we note the following:

1. G-equivariance:

$$(gv, g \cdot \phi) \to (g \cdot \phi)(gv) = g\phi(g^{-1}gv) = g\phi(v).$$

2. Injection: Since the map

$$\pi_1 \otimes \mathrm{Hom}_K[\pi_1, \pi] \to \pi$$

is an injection for all K, so is the map Φ after taking the direct limit.

3. Surjection: The image of Φ being G-invariant must be all of π as π is irreducible.

This proves part (1) of the theorem. Now we prove that $\pi_1 \otimes \pi_2$ as in part (2) of the theorem is always irreducible.

Suppose that W is a G-invariant subspace of $\pi_1 \otimes \pi_2$. As in part (1) of the theorem, we can define $\lim_K \mathrm{Hom}_K[\pi_1, W]$ which will be a G-invariant subspace of

$$\lim_K \mathrm{Hom}_K[\pi_1, \pi_1 \otimes \pi_2] \cong \pi_2.$$

Since π_2 is irreducible, it follows that

$$\lim_K \mathrm{Hom}_K[\pi_1, W] = \pi_2.$$

Hence from the natural injections,

$$\pi_1 \otimes \lim_K \mathrm{Hom}_K[\pi_1, W] \hookrightarrow W \hookrightarrow \pi_1 \otimes \pi_2,$$

we find,

$$\pi_1 \otimes \pi_2 \hookrightarrow W \hookrightarrow \pi_1 \otimes \pi_2,$$

proving that $W = \pi_1 \otimes \pi_2$.

Remark 2. It follows from the theorem that there are no really interesting locally algebraic representations of p-adic groups. It could still happen that all locally analytic representations whose strong duals are quasi-simple analytic modules over $D(G,K)$ (as defined in the main part of this paper) are subquotients of the principal series representations obtained from locally finite dimensional representations of Levi subgroups (introduced in Remark 1), giving an analogue for analytic representations of p-adic groups of the famous subquotient theorem of Harish-Chandra for real groups.

Acknowledgement

This note was inspired by a lecture of P. Schneider at the Institut Henri Poincare on this subject where the author was visiting under a program of the Ministry of Education of France. The author thanks organizers of the special program on Automorphic forms at the IHP for the invitation to visit.

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