CLASSIFICATION OF ADMISSIBLE NILPOTENT ORBITS IN SIMPLE REAL LIE ALGEBRAS $E_{6(6)}$ AND $E_{6(-26)}$

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ABSTRACT. This paper completes the classification of admissible nilpotent orbits of the noncompact simple exceptional real Lie algebras. The author has previoulsy determined such orbits for exceptional real simple Lie algebras of inner type. Here he uses the same techniques, with some modifications, to classify the admissible nilpotent orbits of $E_{6(6)}$ and $E_{6(-26)}$ under their simply connected Lie groups.

INTRODUCTION

The admissible nilpotent orbits of real simple Lie algebras are now classified. For classical Lie algebras the classification was achieved by J. Schwartz [Sch] and T. Ohta [O] in 1987 and 1991 respectively. The author has, recently, determined such orbits for real exceptional Lie algebras of inner type [No1]. This paper completes the classification. The general approach is the following: the problem of determining admissible nilpotent orbits of a simple real Lie algebra $\mathfrak{g}$ with Lie group G is translated into that of classifying admissible nilpotent orbits of the complex symmetric space $\mathfrak{p}_{\mathbb{C}}$ attached to $\mathfrak{g}$. This is made possible by the Kostant-Sekiguchi bijection between real G nilpotent orbits on $\mathfrak{g}$ and complex $K_{\mathbb{C}}$-nilpotent orbits on $\mathfrak{p}_{\mathbb{C}}$, where $K_{\mathbb{C}}$ is a connected subgroup of $G_{\mathbb{C}}$, the complexification of G [Se], and by the fact that the bijection preserves admissibility of associated orbits [Sch]. We add that Monica Nevins [Ne] has determined such orbits for classical p-adic Lie groups. She has also computed admissible orbits of some simple real exceptional Lie groups using methods from p-adic groups. The study of these orbits is important because they seem to be good candidates for which a general method for quantization, as predicted by the Orbit method, could be established. The reader may consult [K], [A-K], [D], [V], [V1] for more information.

CLASSIFICATION ADMISSIBLE ORBITS OF $E_{6(6)}$ AND $E_{6(-26)}$

Because of Schwartz's results we will deal only with the admissibility of nilpotent orbits in symmetric spaces. For more details the reader may consult our paper on the admissibility of nilpotent orbits of exceptional real Lie algebras of inner type [No1].

Let $\mathfrak{g}$ be a real semisimple Lie algebra with adjoint group G and $\mathfrak{g}_{\mathbb{C}}$ its complexification. Also let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ be a Cartan decomposition of $\mathfrak{g}$. Finally, let θ be the

corresponding Cartan involution of $\mathfrak{g}$ and σ be the conjugation of $\mathfrak{g}_{\mathbb{C}}$ with regard to $\mathfrak{g}$. Then $\mathfrak{g}_{\mathbb{C}} = \mathfrak{k}_{\mathbb{C}} \oplus \mathfrak{p}_{\mathbb{C}}$ where $\mathfrak{k}_{\mathbb{C}}$ and $\mathfrak{p}_{\mathbb{C}}$ are obtained by complexifying $\mathfrak{k}$ and $\mathfrak{p}$ respectively. Denote by $K_{\mathbb{C}}$ the connected subgroup of the adjoint group $G_{\mathbb{C}}$ of $\mathfrak{g}_{\mathbb{C}}$, with Lie algebra $\mathfrak{k}_{\mathbb{C}}$.

A triple (x, e, f) in $\mathfrak{g}_{\mathbb{C}}$ is called a standard triple if $[x, e] = 2e$, $[x, f] = -2f$ and $[e, f] = x$. If $x \in \mathfrak{k}_{\mathbb{C}}$, e and $f \in \mathfrak{p}_{\mathbb{C}}$, then (x, e, f) is a normal triple. It is a result of Kostant and Rallis [K-R] that any nilpotent e of $\mathfrak{p}_{\mathbb{C}}$ can be embedded in a standard normal triple (x, e, f). Moreover, e is $K_{\mathbb{C}}$-conjugate to a nilpotent e' inside of a normal triple (x', e', f') with $\sigma(e') = f'$ [Se]. The triple (x', e', f') will be called a *Kostant-Sekiguchi* or KS-triple. From the representation theory of $\mathfrak{sl}_2$, $\mathfrak{g}_{\mathbb{C}}$ has the following eigenspace decomposition:

$$\mathfrak{g}_{\mathbb{C}} = \bigoplus_{j \in \mathbb{Z}} \mathfrak{g}_{\mathbb{C}}^{(j)} \qquad \text{where } \mathfrak{g}_{\mathbb{C}}^{(j)} = \{z \in \mathfrak{g}_{\mathbb{C}} | [x, z] = jz\}.$$

Similarly we have

$$\mathfrak{k}_{\mathbb{C}} = \bigoplus_{j \in \mathbb{Z}} \mathfrak{k}_{\mathbb{C}}^{(j)} \qquad \text{where } \mathfrak{k}_{\mathbb{C}}^{(j)} = \{z \in \mathfrak{k}_{\mathbb{C}} | [x, z] = jz\}$$

and

$$\mathfrak{p}_{\mathbb{C}} = \bigoplus_{j \in \mathbb{Z}} \mathfrak{p}_{\mathbb{C}}^{(j)} \qquad \text{where } \mathfrak{p}_{\mathbb{C}}^{(j)} = \{z \in \mathfrak{p}_{\mathbb{C}} | [x, z] = jz\}.$$

Moreover, the centralizers of e in $\mathfrak{k}_{\mathbb{C}}$ and $\mathfrak{p}_{\mathbb{C}}$ decompose as

$$\mathfrak{k}_{\mathbb{C}}^e = \bigoplus_{j \in \mathbb{Z}} (\mathfrak{k}_{\mathbb{C}}^e \cap \mathfrak{k}_{\mathbb{C}}^j) = \bigoplus_{j \geq 0} (\mathfrak{k}_{\mathbb{C}}^e \cap \mathfrak{k}_{\mathbb{C}}^j)$$

and

$$\mathfrak{p}_{\mathbb{C}}^e = \bigoplus_{j \in \mathbb{Z}} (\mathfrak{p}_{\mathbb{C}}^e \cap \mathfrak{p}_{\mathbb{C}}^j) = \bigoplus_{j \geq 0} (\mathfrak{p}_{\mathbb{C}}^e \cap \mathfrak{p}_{\mathbb{C}}^j).$$

Maintaining the above notation, let e be nonzero nilpotent in $\mathfrak{p}_{\mathbb{C}}$. Then $K_{\mathbb{C}}^e$, the centralizer of e in $K_{\mathbb{C}}$, acts as $\mathfrak{k}_{\mathbb{C}}/\mathfrak{k}_{\mathbb{C}}^e$. Define the character δ_e of $K_{\mathbb{C}}^e$ as

$$\delta_e(g) = (det(g|_{\mathfrak{k}_{\mathbb{C}}/\mathfrak{k}_{\mathbb{C}}^e}))^{-1} \qquad g \in K_{\mathbb{C}}^e.$$

Definitition [V1]. A representation χ of $K_{\mathbb{C}}^e$ is said to be admissible if its differential is half the differential of δ_e. The nilpotent e is admissible if $K_{\mathbb{C}}^e$ has at least one admissible representation.

Takuya Ohta [O] shows that δ_e acts trivially on the unipotent part of $K_{\mathbb{C}}^e$ and gives an explicit description of the differential of δ_e on the reductive centralizer $\mathfrak{k}_{\mathbb{C}}^{(x,e,f)}$ as follows:

Lemma 1 (Ohta). *Suppose that* $\mathfrak{t}_{\mathbb{C}}$ *is a toral subalgebra of* $\mathfrak{k}_{\mathbb{C}}$ *containing* x. *Then the centralizer* $\mathfrak{t}_{\mathbb{C}}^e$ *of* e *in* $\mathfrak{t}_{\mathbb{C}}$ *is a toral subalgebra of* $\mathfrak{t}_{\mathbb{C}}^{(x,e,f)}$ *and* $d\delta_e$ *on* $\mathfrak{t}_{\mathbb{C}}^e$ *is given by*

$$d\delta_e(t) = \sum_{i \geq 1} tr(ad(t)|_{\mathfrak{k}_{\mathbb{C}}^i}) - \sum_{i \geq 2} tr(ad(t)|_{\mathfrak{p}_{\mathbb{C}}^i}) = -\sum_{i \geq 1} tr(ad(t)|_{\mathfrak{p}_{\mathbb{C}}^i}) + \sum_{i \geq 2} tr(ad(t)|_{\mathfrak{k}_{\mathbb{C}}^i}).$$

We shall modify the above expression in order to define $d\delta_e$ in terms of imaginary roots only. Let $\mathfrak{h}$ be a fundamental Cartan subalgebra of $\mathfrak{g}$. Then $\mathfrak{h} = \mathfrak{t} \oplus \mathfrak{s}$, where

$\mathfrak{t}$ is a Cartan subalgebra of $\mathfrak{k}$ and $\mathfrak{s} \subseteq \mathfrak{p}$. We have the following decompositions in the root spaces of $\mathfrak{g}_\mathbb{C}$ generated by the roots of $\mathfrak{h}_\mathbb{C}$:

$$\mathfrak{k}_\mathbb{C} = \mathfrak{t}_\mathbb{C} \oplus \sum_{\alpha \text{ compact imaginary}} \mathbb{C}X_\alpha \oplus \sum_{(\alpha,\theta\alpha) \text{ complex pairs}} \mathbb{C}(X_\alpha + \theta(X_\alpha)),$$

$$\mathfrak{p}_\mathbb{C} = \mathfrak{s}_\mathbb{C} \oplus \sum_{\alpha \text{ noncompact imaginary}} \mathbb{C}X_\alpha \oplus \sum_{(\alpha,\theta\alpha) \text{ complex pairs}} \mathbb{C}(X_\alpha - \theta(X_\alpha)).$$

Here X_α is a nonzero vector of the root space $\mathfrak{g}_\mathbb{C}^\alpha$. An imaginary root α is compact (noncompact) if its root space $\mathfrak{g}_\mathbb{C}^\alpha$ lies in $\mathfrak{k}_\mathbb{C}$ ($\mathfrak{p}_\mathbb{C}$). See ([Kn]) for more details.

From the above expression we see that

$$2d\delta_e(t) = 2\{\sum_{i\geq2} tr(ad(t)|_{\mathfrak{k}_\mathbb{C}^i}) - \sum_{i\geq2} tr(ad(t)|_{\mathfrak{p}_\mathbb{C}^i})\} + tr(ad(t)|_{\mathfrak{k}_\mathbb{C}^1}) - tr(ad(t)|_{\mathfrak{p}_\mathbb{C}^1}).$$

It follows that the contribution of the complex pairs cancels out. Therefore we only need to evaluate imaginary roots on $t \in \mathfrak{t}_\mathbb{C}^e$. Define

$$\Phi_i := \{\alpha \text{ compact imaginary}: \alpha(x) = i\},$$

$$\Psi_i := \{\beta \text{ noncompact imaginary}: \beta(x) = i\}.$$

Then

$$2d\delta_e(t) = 2\{\sum_{i\geq2}(\sum_{\alpha\in\Phi_i} \alpha(t) - \sum_{\beta\in\Psi_i} \beta(t))\} + \sum_{\alpha\in\Phi_1} \alpha(t) - \sum_{\beta\in\Psi_1} \beta(t).$$

This is the version of $d\delta_e$ that we shall use below. We will use the following lemma to decide admissibility of nilpotent orbits.

Lemma 2 (Ohta). *Let $\mathfrak{t}_1$ be a Cartan subalgebra of $\mathfrak{k}_\mathbb{C}^{(x,e,f)}$ and T_1 the corresponding connected subgroup of $(K_\mathbb{C}^{(x,e,f)})_\circ$ the identity component of $K_\mathbb{C}^{(x,e,f)}$. Then e is admissible if and only if there exists a character, χ, of T_1 such that $\delta_e(g) = (\chi(g))^2$ for all $g \in T_1$.*

Our strategy will be to find a maximal torus $\mathfrak{t}_1$ in $\mathfrak{k}_\mathbb{C}^{(x,e,f)}$ and to compute $d\delta_e(t)$ where t is a generic element of $\mathfrak{t}_1$. If $\frac{1}{2}\delta_e$ is integral, that is, exponentiates to a character of T_1, then e is admissible.

Since all even nilpotents and all "noticed" nilpotents, that is, nilpotents e for which $\mathfrak{k}_\mathbb{C}^{(x,e,f)} = 0$ [No], are admissible ([Sch], [O]) we will study only non-even and non-noticed nilpotent orbits.

Let $\mathfrak{g}_\mathbb{C} = E_6$ and $\Delta = \{\alpha_1, \alpha_2, \ldots, \alpha_6\}$ the Bourbaki simple roots of $\mathfrak{g}_\mathbb{C}$. Define an involution θ on Δ as follows:

$$\theta(\alpha_1) = \alpha_6 \quad \theta(\alpha_2) = \alpha_2 \quad \theta(\alpha_3) = \alpha_5,$$
$$\theta(\alpha_4) = \alpha_4 \quad \theta(\alpha_5) = \alpha_3 \quad \theta(\alpha_6) = \alpha_1.$$

Furthermore, we require that:

$$\theta(X_{\alpha_1}) = X_{\alpha_6} \quad \theta(X_{\alpha_2}) = X_{\alpha_2} \quad \theta(X_{\alpha_3}) = X_{\alpha_5},$$
$$\theta(X_{\alpha_4}) = X_{\alpha_4} \quad \theta(X_{\alpha_5}) = X_{\alpha_3} \quad \theta(X_{\alpha_6}) = X_{\alpha_1}.$$

It is well known ([Kn]) that $\mathfrak{k}_\mathbb{C}$ is of type F_4 and $\mathfrak{p}_\mathbb{C}$ is the symmetric space associated to real form $E_{6(-26)}$ of E_6. Let $\mathfrak{h}_\mathbb{C}$ be the Cartan subalgebra of $\mathfrak{g}_\mathbb{C}$

associated with the root system generated by Δ. Then $\mathfrak{t}_{\mathbb{C}} = \mathfrak{h}_{\mathbb{C}} \cap \mathfrak{k}_{\mathbb{C}}$ is a Cartan subalgebra of $\mathfrak{k}_{\mathbb{C}}$. The simple roots of $(\mathfrak{k}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}})$ are

$$\beta_1 = \alpha_2 \quad \beta_2 = \alpha_4 \quad \beta_3 = \frac{\alpha_3 + \alpha_5}{2} \quad \beta_4 = \frac{\alpha_1 + \alpha_6}{2}.$$

From Djoković [D1] we know that $E_{6(-26)}$ has exactly two nilpotent orbits of which only the minimal one is not even. There are no noticed orbits.

Theorem 1. *The nilpotent orbits of $E_{6(-26)}$ are admissible.*

Proof. Since the trivial and the even maximal orbit are admissible, it suffices to show that the minimal orbit is also admissible. From Schwartz's theorem it is equivalent to show that the minimal orbit of $K_{\mathbb{C}}$ on $\mathfrak{p}_{\mathbb{C}}$ is admissible. First we compute the neutral element associated with the orbit. Using the label in [D1] we solve the following system of equations: $\beta_1(x) = 0$, $\beta_2(x) = 0$, $\beta_3(x) = 0$, $\beta_4(x) = 1$. We obtain $x = 2H_{\beta_1} + 4H_{\beta_2} + 3H_{\beta_3} + 2H_{\beta_4}$ where $H_{\beta_i} \in \mathfrak{t}_{\mathbb{C}}$ denotes the coroot of β_i. In the Δ basis $x = 2H_{\alpha_1} + 2H_{\alpha_2} + 3H_{\alpha_3} + 4H_{\alpha_4} + 3H_{\alpha_5} + 2H_{\alpha_6}$. The nilpotent is

$$e = X_{\alpha_1+\alpha_2+2\alpha_3+2\alpha_4+\alpha_5+\alpha_6} - \theta X_{\alpha_1+\alpha_2+2\alpha_3+2\alpha_4+\alpha_5+\alpha_6}$$

$$= X_{\alpha_1+\alpha_2+2\alpha_3+2\alpha_4+\alpha_5+\alpha_6} + \epsilon X_{\alpha_1+\alpha_2+\alpha_3+2\alpha_4+2\alpha_5+\alpha_6}$$

where $\epsilon = \pm 1$ in a given Chevalley basis and

$$f = X_{-(\alpha_1+\alpha_2+2\alpha_3+2\alpha_4+\alpha_5+\alpha_6)} + \epsilon X_{-(\alpha_1+\alpha_2+\alpha_3+2\alpha_4+2\alpha_5+\alpha_6)}.$$

We can find $\mathfrak{t}_{\mathbb{C}}^1 = \mathbb{C}H_{\alpha_2} \oplus \mathbb{C}H_{\alpha_4} \oplus \mathbb{C}(H_{\alpha_3} + H_{\alpha_5})$, a maximal torus of $\mathfrak{k}_{\mathbb{C}}^{(x,e,f)}$ which is of type B_3. To compute $2d\delta_e$ we use the appropriate subset of compact imaginary roots given in Table 1. There are no noncompact imaginary roots. A computation shows that $d\delta_e$ acts trivially on $\mathfrak{t}_{\mathbb{C}}^1$. Hence δ_e has a square root and e is admissible. $\square$

TABLE 1. Compact imaginary roots of $E_{6(-26)}$

1.	$\pm\alpha_2$
2.	$\pm\alpha_4$
3.	$\pm(\alpha_2 + \alpha_4)$
4.	$\pm(\alpha_3 + \alpha_4 + \alpha_5)$
5.	$\pm(\alpha_2 + \alpha_3 + \alpha_4 + \alpha_5)$
6.	$\pm(\alpha_2 + \alpha_3 + 2\alpha_4 + \alpha_5)$
7.	$\pm(\alpha_1 + \alpha_3 + \alpha_4 + \alpha_5 + \alpha_6)$
8.	$\pm(\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5 + \alpha_6)$
9.	$\pm(\alpha_1 + \alpha_2 + \alpha_3 + 2\alpha_4 + \alpha_5 + \alpha_6)$
10.	$\pm(\alpha_1 + \alpha_2 + 2\alpha_3 + 2\alpha_4 + 2\alpha_5 + \alpha_6)$
11.	$\pm(\alpha_1 + \alpha_2 + 2\alpha_3 + 3\alpha_4 + 2\alpha_5 + \alpha_6)$
12.	$\pm(\alpha_1 + 2\alpha_2 + 2\alpha_3 + 3\alpha_4 + 2\alpha_5 + \alpha_6)$

Maintaining the above notation, define $\beta_0 = \beta_1 + 2\beta_2 + 3\beta_3 + 2\beta_4$. Then roots $-\beta_0, \beta_4, \beta_3, \beta_2$ form a fundamental root system Δ' of type C_4. There is a unique involution θ' of $\mathfrak{g}_{\mathbb{C}}$ which is equivalent to θ except that $\theta'(X_{\alpha_2}) = -X_{\alpha_2}$ which makes α_2 a noncompact imaginary root. See Dojoković [D1] for more details. We shall refer to θ' as θ. Thus we have a new Cartan decomposition of $\mathfrak{g}$ with the new

Cartan involution θ. In this new decomposition $\mathfrak{k}_\mathbb{c}$ is of type C_4, $\mathfrak{p}_\mathbb{c}$ is the symmetric space associated with the split real form $E_{6(6)}$ of E_6 and Δ' generates a root system for $\mathfrak{k}_\mathbb{c}$. Furthermore, the new involution retains $\mathfrak{t}_\mathbb{c}$ as Cartan subalgebra of $\mathfrak{k}_\mathbb{c}$ (see [D1] page 199 for more details).

The following two tables contain the compact and noncompact imaginary roots of $E_{6(6)}$. Observe that θ define the same Vogan diagram given in [Kn] on page 361 where Proposition 6.104 allows us to decide which imaginary roots are compact or noncompact.

TABLE 2. Compact imaginary roots of $E_{6(6)}$

1.	$\pm\alpha_4$
2.	$\pm(\alpha_3 + \alpha_4 + \alpha_5)$
3.	$\pm(\alpha_1 + \alpha_3 + \alpha_4 + \alpha_5 + \alpha_6)$
4.	$\pm(\alpha_1 + 2\alpha_2 + 2\alpha_3 + 3\alpha_4 + 2\alpha_5 + \alpha_6)$

TABLE 3. Noncompact imaginary roots of $E_{6(6)}$

1.	$\pm\alpha_2$
2.	$\pm(\alpha_2 + \alpha_4)$
3.	$\pm(\alpha_2 + \alpha_3 + \alpha_4 + \alpha_5)$
4.	$\pm(\alpha_2 + \alpha_3 + 2\alpha_4 + \alpha_5)$
5.	$\pm(\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5 + \alpha_6)$
6.	$\pm(\alpha_1 + \alpha_2 + \alpha_3 + 2\alpha_4 + \alpha_5 + \alpha_6)$
7.	$\pm(\alpha_1 + \alpha_2 + 2\alpha_3 + 2\alpha_4 + 2\alpha_5 + \alpha_6)$
8.	$\pm(\alpha_1 + \alpha_2 + 2\alpha_3 + 3\alpha_4 + 2\alpha_5 + \alpha_6)$

Structure constants and the action of θ. We shall need a Chevalley Basis for E_6 where the action of θ on a root vector X_α is made precise. We shall sketch the construction of the Chevalley basis. For more information the reader may consult the papers of Djoković [D1] and Kurtzke [Ku]. Let $N = 36$ be the number of positive roots of E_6 in the root system Φ generated by Δ. Then θ is an involutory permutation of the roots in Φ. Here $\theta(H_\alpha) = H_{\theta(\alpha)}$ where H_α is the coroot of α for all $\alpha \in \Phi$. For any two nonzero root vectors X_α and X_β we require:

(i) $[X_\alpha, X_{-\alpha}] = H_\alpha$,

(ii) $[X_\alpha, X_\beta] = c_{\alpha+\beta}X_{\alpha+\beta}$, $c_{\alpha+\beta} = 0$ if $\alpha + \beta \notin \Phi$; otherwise, we have $c_{\alpha+\beta} = \epsilon(\alpha, \beta)(q + 1)$ where p and q are the largest non-negative integers such that $\beta + p\alpha$, $\beta - q\alpha \in \Phi$ and $\epsilon(\alpha, \beta) = \pm 1$. The H_α's and the X_α's form a basis of E_6. All $[X_\alpha, X_\beta]$ are determined up to $\epsilon(\alpha, \beta)$.

There is a natural ordering of the positive roots of E_6 which can be described as follows:

$$\alpha_i \precsim \alpha_j \quad \text{for} \quad 1 \le i < j \le 6.$$

For $\alpha = \sum_{i=1}^{6} k_i\alpha_i$ define $\text{height}(\alpha) = \sum_{i=1}^{6} k_i$. We can extend the previous ordering on the set of positive roots $\{\alpha_1, \alpha_2, \ldots, \alpha_{N-1}, \alpha_N\}$ by requiring that

$$\text{height}(\alpha_i) \le \text{height}(\alpha_j) \quad \text{for} \quad 1 \le i < j \le N.$$

As shown by J. F. Kurtzke [Ku], if α and β are positive roots, then we can choose:

(iii) $\epsilon(\alpha, \beta) = 1 \iff \alpha \precsim \beta$,

(iv) $\epsilon(\alpha, \beta) = -\epsilon(\beta, \gamma)$ whenever α, β, γ are consecutively linked in the Bourbaki Dynkin diagram [Bo].

We specify $\epsilon(\alpha_1, \alpha_3) = 1$. Then all other $\epsilon(\alpha, \beta)$ are uniquely determined by (iii) and (iv) (See [Ku], [D2]). We use the tables in Djoković [D2] to compute $c_{\alpha,\beta}$. Furthermore, we can define:

(v) $\theta X_\alpha = \xi(\alpha) X_{\theta(\alpha)}$.

Since $\xi(\alpha_i) = 1$ for $\alpha = 1, 3, 4, 5, 6$ and $\xi(\alpha_2) = -1$, we can compute all $\xi(\alpha)$'s using the fact that $\xi(-\alpha) = \xi(\alpha)$ and if $\alpha + \beta = \gamma$, then (see [D2])

$$(1) \qquad \xi(\gamma) c_{\alpha,\beta} = \xi(\alpha) \xi(\beta) c_{\theta(\alpha), \theta(\beta)}.$$

We can now proceed with classification.

Theorem 2. *All the non-even non-noticed nilpotent orbits of the algebra $E_{6(6)}$ under its simply connected group are admissible.*

Proof. See Table 4 below. $\qquad \square$

The admissible nonzero, non-even and non-noticed nilpotent orbits of $E_{6(6)}$ are listed in Table 4. The technique used is that of the proof of Theorem 1. Here we used subsets of the roots given in Tables 2 and 3 to compute $2d\delta_e$. Since the adjoint group of $\mathfrak{k}_\mathbb{C}$ is not simply connected, we used the weight space of $\mathfrak{k}_\mathbb{C}$ to search for a possible square root for δ_e when necessary. In table 4, for each orbit of $\mathfrak{p}_\mathbb{C}$ we give Djokovic's label and the normal triple (x, e, f); we do not need to write f down because we can deduce it from e by replacing every root vector X_β by $X_{-\beta}$. The maximal torus $\mathfrak{t}_\mathbb{C}^1$ and the value of $d\delta_e(z) = \sum_{i=1}^{r} a_i z_i$, where $r = \text{rank of } K_\mathbb{C}^{(x,e,f)}$, are also given. If e is not admissible under the adjoint group but is admissible under the simply connected cover, we say that e is **sc-admissible**.

The fundamental weights of $\mathfrak{k}_\mathbb{C}$ are $\lambda_1 = -\beta_0 + \beta_4 + \beta_3 + \beta_2/2$, $\lambda_2 = -\beta_0 + 2\beta_4 + 2\beta_3 + \beta_2$, $\lambda_3 = -\beta_0 + 2\beta_4 + 3\beta_3 + 3\beta_2/2$ and $\lambda_4 = -\beta_0 + 2\beta_4 + 3\beta_3 + 2\beta_2$. The character χ is the differential of a square root of δ_e.

Remark. The only time the Chevalley basis in Table 4 is needed in a non-trivial manner is in the computation associated with orbit #8. In that case the calculations which show that $[e, f] = x$ are more involved. We shall explain below.

The nilpotent orbit #8 is represented by

$$e = X_\alpha + X_\beta - \theta X_\alpha - \theta X_\beta + X_\gamma$$

where

$$\alpha = \alpha_1 + \alpha_3 + \alpha_4, \quad \beta = -\alpha_1 - \alpha_2 - \alpha_3 - \alpha_4, \quad \gamma = -\alpha_2 - \alpha_3 - 2\alpha_4 - \alpha_5.$$

Hence

$$\theta(\alpha) = \alpha_4 + \alpha_5 + \alpha_6, \quad \theta(\beta) = -\alpha_2 - \alpha_4 - \alpha_5 - \alpha_6, \quad \theta(\gamma) = \gamma.$$

TABLE 4.

1.	0001	$\mathfrak{k}_{\mathbb{C}}^{(x,e,f)} \simeq A_3$

$x = -H_{\alpha_2},$

$e = X_{-\alpha_2}$

$\mathfrak{t}_{\mathbb{C}}^1 = \mathbb{C}(H_{\alpha_1} + H_{\alpha_6}) \oplus \mathbb{C}(H_{\alpha_3} + H_{\alpha_5}) \oplus \mathbb{C}(H_{\alpha_2} + 2H_{\alpha_4})$

$d\delta_e(z) = 0$ **admissible** special

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2. 0100 $\mathfrak{k}_{\mathbb{C}}^{(x,e,f)} \simeq 3A_1$

$x = -H_{\alpha_2} - H_{\alpha_3} - H_{\alpha_4} - H_{\alpha_5},\ e = X_{-\alpha_2-\alpha_4-\alpha_5} - \theta X_{-\alpha_2-\alpha_4-\alpha_5}$

$\mathfrak{t}_{\mathbb{C}}^1 = \mathbb{C}H_{\alpha_4} \oplus \mathbb{C}(H_{\alpha_3} + H_{\alpha_5}) \oplus \mathbb{C}(H_{\alpha_1} + H_{\alpha_2} + H_{\alpha_6})$

$d\delta_e(z) = 0$ **admissible** special

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3. 1001 $\mathfrak{k}_{\mathbb{C}}^{(x,e,f)} \simeq A_1 + T1$

$x = -H_{\alpha_1} - 3H_{\alpha_2} - 2H_{\alpha_3} - 3H_{\alpha_4} - 2H_{\alpha_5} - H_{\alpha_6}$

$e = X_{-\alpha_2-\alpha_4-\alpha_5} - \theta X_{-\alpha_2-\alpha_4-\alpha_5} + X_{-\alpha_1-\alpha_2-\alpha_3-\alpha_4-\alpha_5-\alpha_6}$

$\mathfrak{t}_{\mathbb{C}}^1 = \mathbb{C}(H_{\alpha_1} + H_{\alpha_2} + 3H_{\alpha_4} + H_{\alpha_6}) \oplus \mathbb{C}(H_{\alpha_3} + H_{\alpha_5})$

$d\delta_e(z) = z_1$ **sc-admissible** $\chi = -\lambda_1$ Not special

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7. 0102 $\mathfrak{k}_{\mathbb{C}}^{(x,e,f)} \simeq A_1 + T1$

$x = -4H_{\alpha_2} - H_{\alpha_3} - 2H_{\alpha_4} - H_{\alpha_5}$

$e = \sqrt{3}(X_{\alpha_4+\alpha_5} - \theta X_{\alpha_4+\alpha_5}) + 2X_{-\alpha_2-\alpha_3-2\alpha_4-\alpha_5}$

$\mathfrak{t}_{\mathbb{C}}^1 = \mathbb{C}(H_{\alpha_1} + H_{\alpha_2} + 2H_{\alpha_4} + H_{\alpha_6}) \oplus \mathbb{C}(H_{\alpha_3} + H_{\alpha_5})$

$d\delta_e(z) = 0$ **admissible** special

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8. 0101 $\mathfrak{k}_{\mathbb{C}}^{(x,e,f)} \simeq A_1$

$x = -3H_{\alpha_2} - H_{\alpha_3} - 2H_{\alpha_4} - H_{\alpha_5}$

$e = X_{-\alpha_2-\alpha_4-\alpha_5-\alpha_6} - \theta X_{-\alpha_2-\alpha_4-\alpha_5-\alpha_6} + X_{-\alpha_2-\alpha_3-2\alpha_4-\alpha_5}$

$\quad + X_{\alpha_4+\alpha_5+\alpha_6} - \theta X_{\alpha_4+\alpha_5+\alpha_6}$

$\mathfrak{t}_{\mathbb{C}}^1 = \mathbb{C}(H_{\alpha_1} + H_{\alpha_2} + 2H_{\alpha_3} + 2H_{\alpha_4} + 2H_{\alpha_5} + H_{\alpha_6})$

$d\delta_e(z) = 0$ **admissible** special

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10. 1010 $\mathfrak{k}_{\mathbb{C}}^{(x,e,f)} \simeq T_1$

$x = -H_{\alpha_1} - 4H_{\alpha_2} - 2H_{\alpha_3} - 4H_{\alpha_4} - 2H_{\alpha_5} - H_{\alpha_6}$

$e = X_{\alpha_1+\alpha_3+\alpha_4+\alpha_5} - \theta X_{\alpha_1+\alpha_3+\alpha_4+\alpha_5}$

$\quad + \sqrt{2}(X_{-\alpha_2-\alpha_3-\alpha_4-\alpha_5} + X_{\alpha_1-\alpha_2-\alpha_3-2\alpha_4-\alpha_5-\alpha_6})$

$\mathfrak{t}_{\mathbb{C}}^1 = \mathbb{C}(H_{\alpha_1} - 2H_{\alpha_4} + H_{\alpha_6})$

$d\delta_e(z) = 0$ **admissible** special

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11. 1101 $\mathfrak{k}_{\mathbb{C}}^{(x,e,f)} \simeq T_1$

$x = -H_{\alpha_1} - 5H_{\alpha_2} - 3H_{\alpha_3} - 5H_{\alpha_4} - 3H_{\alpha_5} - H_{\alpha_6}$

$e = \sqrt{2}(X_{\alpha_1+\alpha_3+\alpha_4+\alpha_5} - \theta X_{\alpha_1+\alpha_3+\alpha_4+\alpha_5} + X_{-\alpha_1-\alpha_2-2\alpha_3-2\alpha_4-\alpha_5}$

$\quad - \theta X_{-\alpha_1-\alpha_2-2\alpha_3-2\alpha_4-\alpha_5}) + X_{\alpha_1-\alpha_2-\alpha_3-\alpha_4-\alpha_5-\alpha_6}$

$\mathfrak{t}_{\mathbb{C}}^1 = \mathbb{C}(H_{\alpha_1} + H_{\alpha_2} + H_{\alpha_3} + 5H_{\alpha_4} + H_{\alpha_5} + H_{\alpha_6})$

$d\delta_e(z) = 2z_1$ **admissible** Not special

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14. 1211 $\mathfrak{k}_{\mathbb{C}}^{(x,e,f)} \simeq T_1$

$x = -H_{\alpha_1} - 9H_{\alpha_2} - 4H_{\alpha_3} - 8H_{\alpha_4} - 4H_{\alpha_5} - H_{\alpha_6}$

$e = 2\sqrt{2}(X_{\alpha_1} - \theta X_{\alpha_1}) + \sqrt{5}(X_{\alpha_3+\alpha_4} - \theta X_{\alpha_3+\alpha_4})$

$\quad + 3X_{\alpha_1-\alpha_2-\alpha_3-2\alpha_4-\alpha_5-\alpha_6}$

$\mathfrak{t}_{\mathbb{C}}^1 = \mathbb{C}(H_{\alpha_1} + H_{\alpha_2} + 2H_{\alpha_3} + 2H_{\alpha_4} + 2H_{\alpha_5} + H_{\alpha_6})$

$d\delta_e(z) = z_1$ **sc-admissible** $\chi = \lambda_3$ Not special

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15. 1011 $\mathfrak{k}_{\mathbb{C}}^{(x,e,f)} \simeq T_1$

$x = -H_{\alpha_1} - 5H_{\alpha_2} - 2H_{\alpha_3} - 4H_{\alpha_4} - 2H_{\alpha_5} - H_{\alpha_6}$

$e = \sqrt{3}(X_{\alpha_1+\alpha_3+\alpha_4} - \theta X_{\alpha_1+\alpha_3+\alpha_4}) + 2X_{\alpha_1-\alpha_2-\alpha_3-2\alpha_4-\alpha_5-\alpha_6}$

$\quad + X_{-\alpha_2-\alpha_3-2\alpha_4-\alpha_5}$

$\mathfrak{t}_{\mathbb{C}}^1 = \mathbb{C}(H_{\alpha_1} + H_{\alpha_2} + 2H_{\alpha_3} + 2H_{\alpha_4} + 2H_{\alpha_5} + H_{\alpha_6})$

$d\delta_e(z) = z_1$ **sc-admissible** $\chi = \lambda_3$ Not special

Now

$$[e, f] = H_\alpha + H_\beta + H_{\theta(\alpha)} + H_{\theta(\beta)} + H_\gamma$$
$$+ [X_\alpha, -\theta(X_{-\beta})] + [X_\beta, -\theta(X_{-\alpha})]$$
$$+ [-\theta(X_\alpha), X_{-\beta}] + [-\theta(X_\beta), X_{-\alpha}].$$

All the other brackets are zero. Next we compute the action of θ on X_α and X_β. We know that $\theta X_\alpha = \xi(\alpha) X_{\theta(\alpha)}$. Hence we only need to compute $\xi(\alpha)$ but since $\alpha = (\alpha_1 + \alpha_3) + \alpha_4$ we can use formula (1) to find

$$\xi(\alpha) = \frac{\xi(\alpha_1 + \alpha_3)\xi(\alpha_4)c_{\alpha_5 + \alpha_6, \alpha_4}}{c_{\alpha_1 + \alpha_3, \alpha_4}} = \frac{\xi(\alpha_1)\xi(\alpha_3)\xi(\alpha_4)c_{\alpha_5, \alpha_6}c_{\alpha_5 + \alpha_6, \alpha_4}}{c_{\alpha_1 + \alpha_3, \alpha_4}} = -1.$$

Hence $\theta(X_\alpha) = -X_{\theta(\alpha)}$. Similary one shows that $\theta(X_\beta) = X_{\theta(\beta)}$. It follows that

$$[X_\alpha, -\theta(X_{-\beta})] + [X_\beta, -\theta(X_{-\alpha})] + [-\theta(X_\alpha), X_{-\beta}] + [-\theta(X_\beta), X_{-\alpha}]$$
$$= [X_\alpha, -X_{-\theta(\beta)})] + [X_\beta, +X_{\theta(-\alpha)}] + [X_{\theta(\alpha)}, X_{-\beta}] + [-X_{\theta(\beta)}, X_{-\alpha}]$$
$$= -X_\mu + X_\mu + X_{-\mu} - X_{-\mu} = 0,$$

where $\mu = \alpha_1 + \alpha_2 + \alpha_3 + 2\alpha_4 + \alpha_5 + \alpha_6$.

Thus we see that $[e, f] = x$.

Finally, as a corollary of Theorems 1 and 2 we have:

Corollary. *If G is a simply connected group of type $E_{6(6)}$ or $E_{6(-26)}$, then all the nilpotent orbits of G in $\mathfrak{g}$ are admissible.*

Acknowledgments

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