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COMPARISONS OF GENERAL LINEAR GROUPS AND THEIR METAPLECTIC COVERINGS II

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ABSTRACT. Let $\mathbf{A}$ be the adele ring of a number field containing the nth roots of unity, and let $\widetilde{\mathrm{GL}}(r,\mathbf{A})$ be an n-fold metaplectic covering of $\mathrm{GL}(r,\mathbf{A})$. Under an assumption on n, we prove identities between all of the terms in Arthur's invariant trace formulas for $\widetilde{\mathrm{GL}}(r,\mathbf{A})$ and $\mathrm{GL}(r,\mathbf{A})$. We then establish a correspondence between the automorphic representations of these groups.

1. Introduction

Correspondences of metaplectic covering groups have their origin in the work of Shimura ([35]). Shimura constructed a correspondence between modular forms of half-integral weight and cusp forms of even weight, which preserves L-functions. He suggested that this correspondence be studied further by using the representation-theoretic techniques developed by Jacquet and Langlands ([22]). This approach was explored (among others) by Flicker ([17]), who gave a complete description of the correspondence between the automorphic representations of an n-fold covering of GL(2) and the automorphic representations of GL(2). Flicker's correspondence was proved using the Selberg trace formula and followed Langlands' proof of base change for GL(2) ([29]). Trace formula methods were also exploited by Flicker, Kazhdan and Patterson ([18], [25]) in the proof of some additional correspondences between the automorphic representations of n-fold metaplectic coverings of $\mathrm{GL}(r)$, $r \geq 2$, and automorphic representations of $\mathrm{GL}(r)$. We prove correspondences of automorphic representations for the same groups under some assumptions on n, the order of the covering. Our approach is novel in that we use the invariant trace formula of Arthur ([8]) and follow Arthur and Clozel's proof of base change for $\mathrm{GL}(r)$ ([13]). We refer the reader to the introductions of [17], [24], [25] and [18] for the ramifications of metaplectic correspondences to number theory.

This paper is the sequel to another paper ([30]), in which the local metaplectic correspondence and the invariant trace formulas are described. We shall assume that the reader is familiar with this work and adopt its notation without further comment.

The topic of §§2–12 is the comparison of invariant trace formulas. This comparison is made under an assumption on the order of the covering. We first describe the content of these sections and then consider the assumption. Let $\mathbf{A}$ be the adele ring of a number field F. The conjectural invariant trace formula for $\widetilde{\mathrm{GL}}(r,\mathbf{A})$ is

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placed into the form

$$\sum_{M\in\mathcal{L}} |W_0^M||W_0^G|^{-1} \sum_{\gamma\in(M(F))_{F,S}/\mu_n^M} a^{\tilde{M}}(S,\gamma')I_M^{\mathcal{M}}(\gamma,\tilde{f})$$
$$= \sum_{M\in\mathcal{L}} |W_0^M||W_0^G|^{-1} \sum_{t\geq 0} \int_{\Pi(\tilde{M},t)} a^{\tilde{M}}(\tilde{\pi})I_{\tilde{M}}(\tilde{\pi},\tilde{f})d\tilde{\pi},$$

and a special version of the invariant trace formula for $\mathrm{GL}(r,\mathbf{A})$ is placed into the form

$$\sum_{M\in\mathcal{L}} |W_0^M||W_0^G|^{-1} \sum_{\gamma\in(M(F))_{F,S}/\mu_n^M} a^{M}(S,\gamma)I_M^{\Sigma}(\gamma,\tilde{f})$$
$$= \sum_{M\in\mathcal{L}} |W_0^M||W_0^G|^{-1} \sum_{t\geq 0} \int_{\Pi^{\Sigma}(M,t)} a^{M,\Sigma}(\tilde{\pi})I_M^{\Sigma}(\tilde{\pi},\tilde{f})d\tilde{\pi}.$$

The function $\tilde{f}$ in these equations is taken to be arbitrary in the global Hecke space of $\widetilde{\mathrm{GL}}(r,\mathbf{A})$. The interesting terms in these formulas are of two types: local and global. The local terms, which are each identified by an "I", are distributions defined in terms of weighted orbital integrals and weighted characters. The global terms, which are each distinguished by an "a", are constants which depend either on the automorphic nature of the representations involved, or the rational geometry of the groups.

The principal results of §§2–12 are Theorems A and B which match the global terms, $a^{\tilde{M}}(S,\gamma')$ with $a^M(S,\gamma)$, and $a^{\tilde{M}}(\tilde{\pi})$ with $a^{M,\Sigma}(\tilde{\pi})$; and the local distributions, $I_M^{\mathcal{M}}(\gamma)$ with $I_M^{\Sigma}(\gamma)$, and $I_{\tilde{M}}(\tilde{\pi})$ with $I_M^{\Sigma}(\tilde{\pi})$.

The structure of the proof of Theorems A and B follows II [13] very closely. There are however two notable deviations. The first is our use of the local invariant trace formula ([12]) in §§3 and 12. We assume that this formula holds for metaplectic coverings. The second is the use of strong approximation in §12. Otherwise, the reader familiar with [13] should have no difficulties in relating the ideas of §§2–12 to II [13]. To make this relation more transparent, the results of §§2–12, which have counterparts in II of [13], have references to their counterparts in parentheses immediately following their own numbering.

Let us now consider the assumption on the order of the covering. This is given as Assumption 1 in §2. Under this assumption n is relatively prime to the positive integers less than or equal to r and is also relatively prime to $i(1+2m)-1$, where $1 < i \leq r$, and $0 \leq m \leq n-1$ is a fixed integer which stems from the metaplectic covering. The reader may find it helpful to consider Assumption 1 with $m = 0$, in which case the assumption is greatly simplified, but the covering groups are still non-trivial. We list below the obstructions that are removed under this assumption. The list is given in increasing order of the author's perception of their difficulty.

Assumption 1 excludes even coverings. If n is even, the orbit map, the basic means of comparison, is defined only on a proper subset of $G(F_v)$ and contains a term defined in terms of K-theory. The former consequence would require the proof of an additional vanishing property for the geometric side of the trace formula of $\mathrm{GL}(r,\mathbf{A})$, or restrictions on $\tilde{f}$. The latter would make computations involving the orbit map more complicated.

Proposition 26.2 of [18] and the Appendix to [30] ensure that the local metaplectic correspondence commutes with parabolic induction under the assumption that

n is relatively prime to $i(1+2m)-1$, $1 < i \leq r$. It is expected that this assumption can be lifted, but to date there is no proof of this expectation.

Assumption 1 implies that $n \geq 3$. In this case the archimedean completions of F are all complex. Since metaplectic coverings of $\mathrm{GL}(r, \mathbf{C})$ are trivial, the representation theory at the archimedean valuations is very straightforward. Jordan canonical form may also be used to simplify the comparison at the archimedean valuations. In order to complete the comparison in the case $n = 2$, one would presumably have to prove identities of differential operators on $\widetilde{\mathrm{GL}}(r, \mathbf{R})$ parallel to those in [33].

The assumption that n is relatively prime to the positive integers less than or equal to r simplifies the comparison of γ and γ^n, and of their centralizers in $G(F_v)$. Many terms in the invariant trace formulas are expressed in terms of these objects (see §§3 and 4). Under the above assumption, the discrepancies in these terms may be described purely in terms of nth roots of unity.

As mentioned in the introduction of [30], the vanishing property for the geometric distributions of $\widetilde{\mathrm{GL}}(r, F_v)$ does not follow for general n. It does follow under Assumption 1 (Proposition 8.2, [30]). This obstacle might be circumvented by making restrictions on $\tilde{f}$, or by showing that the sum of the undesirable distributions vanishes.

Last, but certainly not least, the matching of weighted orbital integrals required for Lemma 3.4 and the approximation arguments of §12 is only proved in the case that n is relatively prime to certain integers which are included in Assumption 1 ([31]). Even under the assumption that the local trace formula ([12]) holds for metaplectic coverings, this matching is not immediate.

Theorem B entails some global metaplectic correspondences which are listed in Theorem 13.1. In broad terms, there is a correspondence between unitary automorphic representations of $\widetilde{\mathrm{GL}}(r, \mathbf{A})$ and unitary automorphic representations of Levi subgroups of $\mathrm{GL}(r, \mathbf{A})$. This correspondence preserves a character relation at almost every valuation of F. If a representation $\tilde{\pi}$ of $\widetilde{\mathrm{GL}}(r, \mathbf{A})$ corresponds as above to a cuspidal representation π of $\mathrm{GL}(r, \mathbf{A})$, then a character relation is preserved at every valuation of F and $\tilde{\pi}$ is the only representation of $\widetilde{\mathrm{GL}}(r, \mathbf{A})$ which corresponds to π. This implies the multiplicity one and strong multiplicity one properties for $\tilde{\pi}$.

Aside from the elimination of the rather vexing Assumption 1, there remains much more work to be done in obtaining a general metaplectic correspondence. The paper is concluded with two conjectures concerning the general correspondence and suggestions for their possible solutions.

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2. Statement of Theorem A

We adopt the notation of [30]. The global metaplectic covering,

$$\tilde{G}(\mathbf{A}) = \widetilde{\mathrm{GL}}(r, \mathbf{A}),$$

depends on three integral parameters, which are suppressed from the notation (see §2, [30]). The first is the rank, $r \geq 2$, of the general linear group. The second is the order, $n \geq 1$, of the metaplectic covering. The third is the degree, $0 \leq m \leq n-1$, of the "twist" of the underlying metaplectic two-cocycle defined by Matsumoto (p. 58, [18]). Any two metaplectic coverings associated to distinct triples are not isomorphic. Moreover, for any choice of the above three parameters, there exists a corresponding metaplectic covering. Henceforth, we assume that the following assumption holds on the parameters of $\tilde{G}(\mathbf{A})$.

Assumption 1. *The the order of the metaplectic covering n is relatively prime to $i(1+2m)-1$ and i for all $1 < i \leq r$.*

The relevance of Assumption 1 has been discussed in the introduction. The restrictions involving the fixed integer $0 \leq m \leq n-1$ are required for the arguments of the Appendix to [30], which ensure that parabolic induction commutes with the local metaplectic correspondence (cf. §26.2, [18] and §3.1, [30]). These restrictions are also needed for the geometric vanishing properties of §8 in [30].

The second main assumption that we work under is related to the trace formulas of Arthur. Recall that the centralizer of $\sigma \in G(F)$ in G is denoted by G_σ.

Assumption 2. *Suppose σ is a semisimple element of $G(F)$. Then the local trace formula of* [12] *and global trace formula of* [8] *are valid for $\widetilde{G_\sigma}$.*

Assumption 2 is rather expansive, but is not expected to be grave. We refer the reader to §1 and the beginning of §7 in [30] for more details on this matter.

In §9 of [30] we expressed the conjectural invariant trace formula for $\tilde{G}(\mathbf{A})$ as

$$\sum_{M \in \mathcal{L}} |W_0^M||W_0^G|^{-1} \sum_{\gamma \in (M(F))_{M,S}/\mu_n^M} a^{\tilde{M}}(S,\gamma') I_{\tilde{M}}^{\mathcal{M}}(\gamma, \tilde{f})$$
$$= \sum_t \sum_{M \in \mathcal{L}} |W_0^M||W_0^G|^{-1} \int_{\Pi(\tilde{M},t)} a^{\tilde{M}}(\tilde{\pi}) I_{\tilde{M}}(\tilde{\pi}, \tilde{f}) d\tilde{\pi},$$

where $\tilde{f}$ is a test function in the global Hecke space $\mathcal{H}(\tilde{G}(\mathbf{A}))$, and S is a sufficiently large finite set of valuations of F.

This function $\tilde{f} \in \mathcal{H}(\tilde{G}(\mathbf{A}))$ is taken to be a restricted tensor product of the form $\bigotimes_v \tilde{f}_v$. Each $\tilde{f}_v$ belongs to the local Hecke space $\mathcal{H}(\tilde{G}(F_v))$ and is associated to a function $\tilde{f}'_v$ defined on the tempered representations of $G(F_v)$. The function $\tilde{f}'_v$ is defined by

$$\tilde{f}'(\pi) = \begin{cases} \operatorname{tr}\left(\tilde{\pi}(\tilde{f})\right), & \text{if } \tilde{\pi}' = \pi \text{ for some } \tilde{\pi} \in \Pi_{\mathrm{temp}}(\tilde{G}(F_S)), \\ 0, & \text{otherwise.} \end{cases}$$

where $\tilde{\pi} \mapsto \tilde{\pi}'$ is the local metaplectic correspondence of Flicker and Kazhdan (Theorem 27.3, [18] and §3, [30]). In fact, $\tilde{f}'_v$ belongs to the Paley-Wiener space $\mathcal{I}(G(F_v))$ as defined in §3 of [30]. The distributions of Arthur's invariant trace formula pass to maps on Paley-Wiener spaces. We may therefore substitute

$$\tilde{f}' = \bigotimes_v \tilde{f}'_v \in \mathcal{I}(G(\mathbf{A}))$$

into the invariant trace formula for $G(\mathbf{A})$. With this substitution, the invariant trace formula for $G(\mathbf{A})$ is expressed in Proposition 9.2 of [30] as

$$\sum_{M \in \mathcal{L}} |W_0^M||W_0^G|^{-1} \sum_{\gamma \in (M(F))_{M,S}/\mu_n^M} a^M(S,\gamma) I_M^{\Sigma}(\gamma, \tilde{f})$$
$$= n^{-1} \sum_t \sum_{M \in \mathcal{L}} |W_0^M||W_0^G|^{-1} \int_{\Pi(M,t)} a^M(\pi) \hat{I}_M(\pi, \tilde{f}') d\pi.$$

The distribution $I_M^{\Sigma}(\gamma)$ on the left is defined by

$$I_M^{\Sigma}(\gamma, \tilde{f}) = \sum_{\eta \in \mu_n^M / \mu_n^G} \hat{I}_M(\eta\gamma, \tilde{f}'), \ \tilde{f} \in \mathcal{H}(\tilde{G}(F_S)),$$

where the sum is taken over a certain set of nth roots of unity in the center of $M(F)$ (§6, [30]). The forms of these two trace formulas and Theorem A in [13] motivate the first main theorem of our comparison.

Theorem A. *Under Assumptions 1 and 2, the following two assertions are true.*

(i) *Suppose that S is a finite set of valuations with the closure property. Then*

$$I_M^{\Sigma}(\gamma, \tilde{f}) = I_M^{\mathcal{M}}(\gamma, \tilde{f}), \ \gamma \in M_{\mathrm{comp}}(F_S), \ \tilde{f} \in \mathcal{H}(\tilde{G}(F_S)).$$

(ii) *Suppose $\gamma \in M(F)$. Then*

$$a^{\tilde{M}}(S, \gamma') = a^M(S, \gamma)$$

for any suitably large finite set S.

The closure property of assumption of Theorem A (i) is a technical property given in §3.1 of [30]. The closure property is satisfied if and only if S contains an archimedean valuation or S is comprised entirely of valuations which divide a fixed rational prime. The set $M_{\mathrm{comp}}(F_S)$ is a dense open subset of $M(F_S)$, defined in §4 of [30].

The proof of Theorem A will be completed in §12. Observe that by the splitting properties ((23) and Proposition 6.2, [30]), Theorem A (i) holds if and only if it holds in the case that S consists of a single valuation.

The proof of Theorem A consists of several induction arguments. Suppose σ is a semisimple element of $G(F)$. We make the induction hypothesis that Theorem A holds if G is replaced by the centralizer G_σ and $\dim(G_\sigma) < \dim(G)$.

The subgroup $\widetilde{G_\sigma}(\mathbf{A}) \subset \tilde{G}(\mathbf{A})$ is defined in terms of the same three parameters which determine $\tilde{G}(\mathbf{A})$. It should therefore not be too surprising that Assumption 1 pertains to $\widetilde{G_\sigma}(\mathbf{A})$ in the same way it pertains to $\tilde{G}(\mathbf{A})$. The relationship between the induction hypothesis and Assumption 1 is further explored in the Appendix.

Every Levi subgroup in $\mathcal{L}$ is of the form G_σ. Furthermore, it follows from Proposition 3, §2, II of [27] (Krasner's Lemma) and §1 of [25] that for any nonarchimedean valuation v and semisimple element $\sigma_1 \in G(F_v)$ there exists a semisimple element $\sigma \in G(F)$ such that $G_{\sigma_1}(F_v) = G_\sigma(F_v)$. This fact allows us to apply the induction hypothesis in the local nonarchimedean context.

Let us consider some immediate implications of our induction hypothesis. By combining the induction hypothesis with the descent properties, we obtain the following lemma.

Lemma 2.1. *Suppose M_1 and M are in $\mathcal{L}$ such that $M_1 \subsetneq M$. Suppose further that $\gamma \in M_1(F_S) \cap M_{\mathrm{comp}}(F_S)$ satisfies $M_{1,\gamma} = M_\gamma$. Then*

$$I_M^{\mathcal{M}}(\gamma, \tilde{f}) = I_M^{\Sigma}(\gamma, \tilde{f}), \ \tilde{f} \in \mathcal{H}(\tilde{G}(F_S)).$$

In particular, this equation holds for $\gamma \in M_1(F_S) \cap G_{\mathrm{oreg}}(F_S)$.

Proof. It follows from Jordan canonical form and exercises 33–34, Chapter 5 of [1] that $M_{1,\gamma} = M_\gamma$ if and only if $M_{1,\sigma} = M_\sigma$. We therefore have

$$M_{1,\sigma} \subset M_{1,\sigma^n} \subset M_{\sigma^n} = M_\sigma = M_{1,\sigma},$$

and so $M_{1,\sigma} = M_{1,\sigma^n}$. That is, $\gamma \in M_{1,\mathrm{comp}}(F_S)$. By the descent properties ((22), Corollary 6.1, [30]), we have

$$I_M^{\mathcal{M}}(\gamma, \tilde{f}) - I_M^{\Sigma}(\gamma, \tilde{f}) = \sum_{L \in \mathcal{L}(M_1)} d_{M_1}^G(M, L) \left(\hat{I}_{M_1}^{L,\mathcal{M}}(\gamma, \tilde{f}_{\tilde{L}}) - \hat{I}_{M_1}^{L,\Sigma}(\gamma, \tilde{f}_{\tilde{L}}) \right).$$

The induction hypothesis and the fact that $d_{M_1}^G(M, G) = 0$ (§7, [7]) imply that the right-hand side vanishes. □

Applying our induction hypothesis to the geometric sides of the trace formulas and also to expansions (28) and (29) of [30], we obtain the following lemma whose proof can be gleaned from the proof of Lemma 5.2, II of [13].

Lemma 2.2 (5.2)**.** *The distribution,*

$$\tilde{f} \mapsto I(\tilde{f}) - I^{\Sigma}(\tilde{f}'), \;\; \tilde{f} \in \mathcal{H}(\tilde{G}(F_S)),$$

is the sum of

$$\sum_{M \in \mathcal{L}, M \neq G} |W_0^M||W_0^G|^{-1} \sum_{\gamma \in (M(F))_{M,S}/\mu_n^M} a^M(S, \gamma) \left(I_M^{\mathcal{M}}(\gamma, \tilde{f}) - I_M^{\Sigma}(\gamma, \tilde{f}) \right)$$

and

$$\sum_{\delta \in A_G(F) \backslash \mu_n^G} \sum_{u \in (\mathcal{U}_G(F))_{G,S}} \left(a^{\tilde{G}}(S, u) - a^G(S, u) \right) I_G^{\mathcal{M}}(\delta u, \tilde{f}).$$

3. Comparison of $I_M^{\mathcal{M}}(\gamma, \tilde{f})$ and $I_M^{\Sigma}(\gamma, \tilde{f})$

Our aim here is to show that $I_M^{\Sigma}(\gamma, \tilde{f})$ is equal to $I_M^{\mathcal{M}}(\gamma, \tilde{f})$ under various circumstances. We begin by showing that Theorem A (i) holds if it holds at certain regular elements. In order to do this, we need to be able to compare the functions $r_M(\gamma, a)$, which appear in the definitions of our invariant distributions. The following two lemmas afford such a comparison.

Lemma 3.1. *Suppose that $\gamma \in M(F_v)$ is not $G(F_v)$-conjugate to any element $\gamma_0 \in L(F_v)$, such that $L \in \mathcal{L}$, $L \subsetneq M$ and $L_{\gamma_0} = M_{\gamma_0}$. Suppose further that $\gamma = \sigma u$ is the Jordan decomposition of γ in $M(F_v)$. Then there exists $\eta \in \mu_n^M$ such that $G_{\eta\gamma} = G_{\gamma^n}$ and $G_{\eta\sigma} = G_{\sigma^n}$. Moreover, $M_\sigma = M_{\sigma^n}$.*

Proof. We shall show that $\dim(G_{\eta\gamma}) = \dim(G_{\gamma^n})$ for some $\eta \in \mu_n^M$. The first assertion of the lemma then follows from the fact that centralizers in G are closed and connected. The remaining assertions shall follow easily from the proof of the first. Recall decomposition (4) of [30],

$$M = M(1) \times \cdots \times M(\ell) \cong \mathrm{GL}(r_1) \times \cdots \times \mathrm{GL}(r_\ell).$$

Identify $\gamma \in M(F_v)$ with $(\gamma_1, \ldots, \gamma_\ell)$, where $\gamma_i \in \mathrm{GL}(r_i, F_v)$, $1 \leq i \leq \ell$. As explained at the beginning of §1 of [25], there exist positive integers, k_i, r_{ij}, and commuting elements, $\sigma_{ij}, u_{ij} \in \mathrm{GL}(r_{ij}, F_v)$, for $1 \leq i \leq \ell$, $1 \leq j \leq k_i$, such that the elements $\sigma_{i1}, \ldots, \sigma_{ik_i}$ are semisimple, pairwise distinct and generate field extensions, $E_{i1}, \ldots, E_{ik_i}$, respectively; the elements $u_{i1}, \ldots, u_{ik_i}$ are unipotent and upper-triangular; and γ_i is $\mathrm{GL}(r_i, F_v)$-conjugate to the block diagonal matrix

$$\begin{pmatrix} \sigma_{i1}u_{i1} & & 0 \\ & \ddots & \\ 0 & & \sigma_{ik_i}u_{ik_i} \end{pmatrix} \in \mathrm{GL}(r_i, F_v) \cong M(i)(F_v). \tag{1}$$

This matrix determines a unique Levi subgroup,

$$L_i \cong \mathrm{GL}(r_{i1}) \times \cdots \times \mathrm{GL}(r_{ik_i}),$$

of $M(i) \cong \mathrm{GL}(r_i)$. In turn, we obtain a unique Levi subgroup $L \in \mathcal{L}$ such that $L \cong L_1 \times \cdots \times L_\ell$ and $L \subset M$. Suppose γ_0 is the element of $L(F_v)$ determined by (1). Then $L_{\gamma_0} = M_{\gamma_0}$. To see this, observe that for $1 \leq j_1 < j_2 \leq k_i$, the eigenvalues of σ_{ij_1} are pairwise distinct from the eigenvalues of σ_{ij_2}. This implies that $L_{i,\gamma_i} = M(i)_{\gamma_i}$ (exercise 33–34, Chapter 5, [1]). This last equality clearly implies $L_{\gamma_0} = M_{\gamma_0}$. By the hypothesis of the lemma we must have $L = M$. This means that $k_1 = \cdots = k_\ell = 1$ and that γ_i is $\mathrm{GL}(r_i, F_v)$-conjugate to $\sigma_i u_i$, where $\sigma_i = \sigma_{i1}$ and $u_i = u_{i1}$. An immediate consequence is that γ_i^n is $\mathrm{GL}(r_i, F_v)$-conjugate to $\sigma_i^n u_i^n$. The element σ_i^n generates a field F_i which lies between $E_i = E_{i1}$ and F_v. Assumption 1 implies that $E_i = F_i$. Indeed, after identifying σ_i with an element of E_i, we have $E_i = F_i(\sigma_i) = F_i((\sigma_i^n)^{1/n})$. According to Theorem 10 (b), VIII, §6 of [28], the index $[E_i : F_i]$ divides n. At the same time we have $[E_i : F_i] \leq r_i \leq r$ and so Assumption 1 implies that $[E_i : F_i] = 1$. It is immediate that the degree of the minimal polynomial of σ_i, namely $[E_i : F_v]$, is the same as that of σ_i^n. Furthermore, by applying Jordan canonical form (in $\mathrm{GL}(r_i/[E_i : F_v], E_i)$) and Lemma 4.1 of [30] to $\sigma_i^n u_i^n$, we find that it is $\mathrm{GL}(r_i, F_v)$-conjugate to $\sigma_i^n u_i$. Suppose that

$$\sigma_{j_1}^n \neq \sigma_{j_2}^n, \text{ whenever } \sigma_{j_1} \neq \sigma_{j_2},\ 1 \leq j_1 < j_2 \leq \ell. \tag{2}$$

Then the previous two observations imply that there is a degree-preserving bijection between the elementary divisors of γ and γ^n. Since the dimensions of G_γ and G_{γ^n} are determined by the degrees of the elementary divisors of γ and γ^n respectively (Theorem 5.15, [1]), we have $\dim(G_\gamma) = \dim(G_{\gamma^n})$. Now suppose that $\sigma_{j_1}^n = \sigma_{j_2}^n$, but $\sigma_{j_1} \neq \sigma_{j_2}$ for some $1 \leq j_1 < j_2 \leq \ell$. By regarding σ_{j_1} and σ_{j_2} as elements of $E_{j_1} (= E_{j_2})$, we find that $\sigma_{j_1} = \zeta \sigma_{j_2}$ for some $\zeta \in \mu_n$. This implies the existence of an element $\eta \in \mu_n^M$, such that if γ is replaced by $\eta\gamma$ in the earlier argument, then (2) holds. This proves the first assertion of the lemma. The second assertion is seen to follow easily from the above argument by taking $u = 1$. The final assertion of the lemma is a consequence of $E_i = F_i$. □

We already mentioned in the proof of Lemma 6.2 in [30] that $r_M(\gamma, a)$ is invariant under translation by $A_G(F_S)$ in the first variable. This justifies the appearance of the quotient in the index set of the sum in the following lemma.

Lemma 3.2. *Suppose $\gamma \in M(F_v)$ satisfies the hypotheses of Lemma 3.1 and $a \in A_{M,\mathrm{reg}}(F_v)$. Then*

$$\sum_{\eta \in \mu_n^M / \mu_n^G} r_M(\eta\gamma, a) = r_M(\gamma^n, a).$$

Proof. By Lemma 3.1, we may assume that γ satisfies $G_\gamma = G_{\gamma^n}$, $G_\sigma = G_{\sigma^n}$ and $M_\sigma = M_{\sigma^n}$. Suppose $\eta \in \mu_n^M$. We would like to use Lemma 8.2 of [9] to show that

$$r_M(\eta\gamma, a) = \begin{cases} r_M^{G_\eta}(\gamma, a), & \text{if } \mathfrak{a}_{G_\eta} = \mathfrak{a}_G, \\ 0, & \text{otherwise.} \end{cases} \tag{3}$$

In order to be able to invoke Lemma 8.2 of [9] we must verify the three conditions listed on p. 262 of [9]. To satisfy the first condition, we must have $\eta \in G(F)$. This is trivial. The second condition is satisfied if $\mathfrak{a}_{M_\eta} = \mathfrak{a}_M$. This is obvious as

$M_\eta = M$. Finally, we must verify that $G_{\eta\gamma}(F_v)$ is contained in $G_\gamma(F_v)$. This is simple to verify since,

$$G_{\eta\gamma}(F_v) \subset G_{(\eta\gamma)^n}(F_v) = G_{\gamma^n}(F_v) = G_\gamma(F_v).$$

We now have equation (3). A moments thought reveals that $\mathfrak{a}_{G_\eta} = \mathfrak{a}_G$ if and only if $\eta \in \mu_n^G$. Thus,

$$\sum_{\eta \in \mu_n^M/\mu_n^G} r_M(\eta\gamma, a) = r_M(\gamma, a).$$

Now $r_M(\gamma, a)$ is defined in the usual fashion (cf. §6, [2]) from the (G, M) family

$$r_P(\nu, \gamma, a) = r_P(\nu, \sigma u, a) = \prod_\beta \left|a^\beta - a^{-\beta}\right|^{\rho(\beta,u)\nu(\beta^\vee)/2}, \ P \in \mathcal{P}(M).$$

The product on the right is taken over the roots of $(P \cap G_\sigma, A_{M_\sigma})$, and $\rho(\beta, u)$ is a real number (defined in §3, [9]) which depends only on the conjugacy class of u in M_σ. Since u^n is conjugate to u in M_σ (Lemma 4.1, [30]) and

$$(P \cap G_\sigma, A_{M_\sigma}) = (P \cap G_{\sigma^n}, A_{M_{\sigma^n}}),$$

we have in turn that

$$r_P(\nu, \gamma^n, a) = r_P(\nu, \sigma^n u^n, a) = r_P(\nu, \sigma u, a) = r_P(\nu, \gamma, a)$$

and

$$r_M(\gamma^n, a) = r_M(\gamma, a).$$

This completes the lemma. □

Lemma 3.3 (3.6)**.** *Suppose that $\tilde{f}$ is a function in $\mathcal{H}(\tilde{G}(F_S))$ such that*

$$I_M^\Sigma(\gamma, \tilde{f}) = I_M^{\mathcal{M}}(\gamma, \tilde{f})$$

for every element $\gamma \in M(F_S) \cap G_{\mathrm{oreg}}(F_S)$. Then the same formula holds for any element $\gamma \in M_{\mathrm{comp}}(F_S)$.

Proof. By the splitting properties ((23) and Proposition 6.2, [30]), it suffices to prove the lemma in the case that S consists of a single valuation v. By Lemma 2.1, it suffices to prove the lemma under the assumption that $\gamma \in M_{\mathrm{comp}}(F_v)$ satisfies the hypotheses of Lemma 3.1. By following the argument of Lemma 3.6, II of [13], we may conclude that

$$I_M^\Sigma(\gamma, \tilde{f}) = I_M^{\mathcal{M}}(\gamma, \tilde{f}),$$

if the semisimple component σ of γ satisfies $G_\sigma = M_\sigma$. Now suppose that $a \in A_{M,\mathrm{reg}}(F_v)$ is close to the identity so that

$$I_M^\Sigma(a\gamma, \tilde{f}) = I_M^{\mathcal{M}}(a\gamma, \tilde{f})$$

holds. According to the definitions of our invariant distributions and Lemma 3.2, we have

$$\begin{aligned} I_M^\Sigma(\gamma, \tilde{f}) &= \lim_{a \to 1} \sum_{L \in \mathcal{L}(M)} \sum_{\eta \in \mu_n^M/\mu_n^L} r_M^L(\eta\gamma, a) I_L^\Sigma(a\eta\gamma, \tilde{f}) \\ &= \lim_{a \to 1} \sum_{L \in \mathcal{L}(M)} r_M^L(\gamma^n, a) I_L^{\mathcal{M}}(a\gamma, \tilde{f}) \\ &= I_M^{\mathcal{M}}(\gamma, \tilde{f}). \end{aligned}$$

□

Lemma 3.3 tells us that in order to prove Theorem A (i) it is enough to consider the case that γ belongs to $M(F_S) \cap G_{\mathrm{oreg}}(F_S)$. This is an important observation which we shall, often implicitly, make use of in the rest of this paper.

We now consider a very different comparison. At almost every place v, a function in $\mathcal{H}(\tilde{G}(\mathbf{A}))$ is of the form $\tilde{f}_v^0$ defined in (27) of [30]. It is therefore valuable to compare $I_M^{\mathcal{M}}(\gamma, \tilde{f}_v^0)$ with $I_M^{\Sigma}(\gamma, \tilde{f}_v^0)$.

Lemma 3.4. *Suppose v is a valuation such that $|n|_v = 1$. Then*

$$I_M^{\mathcal{M}}(\gamma, \tilde{f}_v^0) = I_M^{\Sigma}(\gamma, \tilde{f}_v^0), \ \gamma \in M_{\mathrm{comp}}(F_v).$$

Proof. By Lemma 3.3, it suffices to prove the lemma for $\gamma \in M(F_v) \cap G_{\mathrm{oreg}}(F_v)$. Obviously, $|n|_v = 1$ implies

$$I_M^{\mathcal{M}}(\gamma, \tilde{f}_v^0) = I_{\tilde{M}}(\gamma', \tilde{f}_v^0).$$

By Lemma 2.1 of [7] the term on the right is equal to the weighted orbital integral $J_{\tilde{M}}(\gamma', \tilde{f}_v^0)$ (cf. §2, [9]). Let $f_v^0 \in \mathcal{H}(G(F_v))$ be the characteristic function of K_v. It may easily be verified that $f_{v,G}^0 = (\tilde{f}_v^0)'$. Under Assumption 1 we may apply the Theorem of [31] to conclude that

$$J_{\tilde{M}}(\gamma', \tilde{f}_v^0) = \sum_{\eta \in \mu_n^M / \mu_n^G} J_M(\eta\gamma, f_v^0).$$

Again, by Lemma 2.1 of [7], the sum on the right is equal to

$$\sum_{\eta \in \mu_n^M / \mu_n^G} I_M(\eta\gamma, f_v^0) = I_M^{\Sigma}(\gamma, \tilde{f}_v^0).$$

□

We now turn to comparisons which are more obviously connected to Theorem A. Through the comparison of trace formulas in §12, we will prove Theorem A (i) in the special case that S contains $\{v : |n|_v \neq 1\}$. The purpose of the next theorem and its corollary is to show that the general case of Theorem A (i) follows from this special case.

Theorem 3.1 (6.1)**.** *In the special case that S contains $\{v : |n|_v \neq 1\}$, we suppose that*

$$I_L^{\mathcal{M}}(\gamma, \tilde{f}) = I_L^{\Sigma}(\gamma, \tilde{f}),$$

for any $\gamma \in L_{\mathrm{comp}}(F_S)$ and $L \in \mathcal{L}(M)$. Then there are unique constants

$$\varepsilon_L(S) = \varepsilon_L^G(S), \ L \in \mathcal{L}(M),$$

such that

$$I_M^{\mathcal{M}}(\gamma, \tilde{f}) = \sum_{L \in \mathcal{L}(M)} \hat{I}_M^{L,\Sigma}(\gamma, \varepsilon_L(S) \tilde{f}_{\tilde{L}}), \ \gamma \in M(F_S).$$

The constants have the descent property

$$\varepsilon_M(S) = \sum_{L \in \mathcal{L}(M_1)} d_{M_1}^G(M, L) \varepsilon_{M_1}^L(S), \ M_1 \subset M,$$

and the splitting property

$$\varepsilon_M(S) = \sum_{L_1, L_2 \in \mathcal{L}(M)} d_M^G(L_1, L_2) \varepsilon_M^{L_1}(S_1) \varepsilon_M^{L_2}(S_2), \ S = S_1 \cup S_2.$$

Proof. This theorem follows from the proof of Theorem 6.1, II of [13], with $I_M^{\mathcal{E}}$ replaced by $I_M^{\mathcal{M}}$ and I_M replaced by I_M^{Σ}. □

Corollary 3.1 (6.4)**.** *In the special case that S contains $\{v : |n|_v \neq 1\}$, we suppose that*

$$I_L^{\mathcal{M}}(\gamma, \tilde{f}) = I_L^{\Sigma}(\gamma, \tilde{f}),$$

for any $\gamma \in L_{\text{comp}}(F_S)$ and $L \in \mathcal{L}(M)$. Then

$$\varepsilon_M(S) = \begin{cases} 1, & M = G, \\ 0, & M \neq G \end{cases}$$

for any finite set S of valuations with the closure property.

Proof. Fix $M \subsetneq G$. By the induction hypothesis following Theorem A we have that $\varepsilon_M^L(S) = 0$, if $M \subsetneq L \subsetneq G$. It follows from the the descent property of Theorem 3.1 that $\varepsilon_M(S) = 0$ unless $M = M_0$. It follows from the splitting property of Theorem 3.1 that

$$\varepsilon_{M_0}(S) = \sum_{v \in S} \varepsilon_{M_0}(v).$$

The corollary therefore follows if we show that $\varepsilon_{M_0}(v) = 0$. Under Assumption 2, we assume that the invariant local trace formula of Arthur (Proposition 8.1, [12]) holds for $\tilde{G}(F_v)$ and we compare local trace formulas to show that $\varepsilon_{M_0}(v) = 0$. Our assumption is that the geometric side of the local trace formula ((8.5), [12]),

$$\sum_{M \in \mathcal{L}} |W_0^M||W_0^G|^{-1}(-1)^{\dim(A_M/A_G)} \int_{(\tilde{M}(F_v)_{\text{ell}})} I_{\tilde{M}}(\tilde{\gamma}, \tilde{f}_1 \times \tilde{f}_2) d\tilde{\gamma},$$

is equal to the spectral side ((8.6), [12]),

$$\sum_{M \in \mathcal{L}} |W_0^M||W_0^G|^{-1}(-1)^{\dim(A_M/A_G)}$$
$$\times \sum_{\tilde{\pi} \in \Pi_{\text{disc}}(\tilde{M})} a_{\text{disc}}^{\tilde{M}}(\tilde{\pi}) \int_{i\mathfrak{a}_M^*} r_{\tilde{M}}(\tilde{\pi}_\lambda, \tilde{P}) \text{tr}\left(\text{Ind}_{\tilde{P}}^{\tilde{G}}(\tilde{\pi}_\lambda, \tilde{f}_1 \times \tilde{f}_2)\right) d\lambda.$$

We shall make a few remarks concerning the notations. The set $(\tilde{M}(F_v)_{\text{ell}})$ is the set of $\tilde{M}(F_v)$-conjugacy classes of elements $\tilde{\gamma}$ in $M(F_v)$ such that $\mathbf{p}(\tilde{\gamma})$ is F_v-elliptic in $M(F_v)$. It is a measure space by virtue of an identification with a set of anisotropic tori. The invariant distribution $I_{\tilde{M}}(\tilde{\gamma}, \tilde{f}_1 \times \tilde{f}_2)$ is defined analogously to the distribution $I_{\tilde{M}}(\tilde{\gamma}, \tilde{f})$ of §7 in [30], the only differences being that $\tilde{f}_1$ and $\tilde{f}_2$ are Hecke functions on the same group and that $\tilde{f}_1$ is genuine, not antigenuine. The set $\Pi_{\text{disc}}(\tilde{M})$ is, in our case (Proposition 27, [18]), a subset of (equivalence classes of) representations of the form

$$\tilde{\pi} = \tilde{\pi}_1^\vee \otimes \tilde{\pi}_1, \ \tilde{\pi}_1 \in \Pi_{\text{temp}}(\tilde{M}(F_v)),$$

where $\tilde{\pi}_1^\vee$ denotes the contragredient of $\tilde{\pi}_1$. The term $r_{\tilde{M}}(\tilde{\pi}_\lambda, \tilde{P})$ is obtained from the (G, M) family of normalizing factors given in §5 of [30]. The reader is referred to [12] for a description of the remaining notations. Applying the local vanishing property (Lemma 8.1, [30]), making the change of variable from γ to γ', and making some obvious definitions, we find that the geometric side is equal to

$$\sum_{M\in\mathcal{L}} |W_0^M||W_0^G|^{-1}(-1)^{\dim(A_M/A_G)} \int_{(M(F_v)_{\mathrm{ell}})/\mu_n^M} |n|_v^r I_{\tilde{M}}(\gamma', \tilde{f}_1 \times \tilde{f}_2) d\gamma$$
$$= \sum_{M\in\mathcal{L}} |W_0^M||W_0^G|^{-1}(-1)^{\dim(A_M/A_G)} \int_{(M(F_v)_{\mathrm{ell}})/\mu_n^M} I_M^{\mathcal{M}}(\gamma, \tilde{f}_1 \times \tilde{f}_2) d\gamma.$$

The geometric side of the local trace formula for $G(F_v)$ at $\tilde{f}_1' \times \tilde{f}_2'$ is equal to

$$\sum_{M\in\mathcal{L}} |W_0^M||W_0^G|^{-1}(-1)^{\dim(A_M/A_G)} \int_{(M(F_v)_{\mathrm{ell}})} \hat{I}_M(\gamma, \tilde{f}_1' \times \tilde{f}_2') d\gamma$$
$$= n \sum_{M\in\mathcal{L}} |W_0^M||W_0^G|^{-1}(-1)^{\dim(A_M/A_G)} \int_{(M(F_v)_{\mathrm{ell}})/\mu_n^M} I_M^{\Sigma}(\gamma, \tilde{f}_1 \times \tilde{f}_2) d\gamma.$$

The comparison of the spectral side of the local trace formula of $G(F_v)$ follows in three steps. First, from a simple computation (cf. §8) and the definition of our normalizing factors (§5, [30]), it follows that

$$r_{\tilde{M}}(\tilde{\pi}_\lambda, \tilde{P}) = n^{\dim(A_M/A_G)} r_M(\tilde{\pi}'_{\lambda'}, P), \ P \in \mathcal{P}(M).$$

Second, the local metaplectic correspondence for tempered representations ((8), [30]) implies that

$$\mathrm{tr}\left(\mathrm{Ind}_{\tilde{P}}^{\tilde{G}}(\tilde{\pi}_\lambda, \tilde{f}_1 \times \tilde{f}_2)\right) = (\tilde{f}_1' \times \tilde{f}_2')\left(\mathrm{Ind}_P^G(\tilde{\pi}'_{\lambda'})\right).$$

Finally, we leave it to the interested reader to show that once the terms in the definition of $a_{\mathrm{disc}}^{\tilde{M}}(\tilde{\pi})$ and $a_{\mathrm{disc}}^M(\tilde{\pi}')$ are unraveled, they are easily seen to be equal. Taking the foregoing into consideration and noting that $d\lambda'$ equals $n^{\dim(A_M)}d\lambda$, we conclude that the spectral side of the local trace formula for $G(F_v)$ is n times the spectral side of the local trace formula for $\tilde{G}(F_v)$. The coefficient n cancels the n appearing in front of the geometric side of the trace formula for $G(F_v)$. Thus, taking the difference of our modified local trace formulas yields

$$\sum_{M\in\mathcal{L}} |W_0^M||W_0^G|^{-1}(-1)^{\dim(A_M/A_G)}$$
$$\times \int_{(M(F_v)_{\mathrm{ell}})/\mu_n^M} I_M^{\mathcal{M}}(\gamma, \tilde{f}_1 \times \tilde{f}_2) - I_M^{\Sigma}(\gamma, \tilde{f}_1 \times \tilde{f}_2) d\gamma = 0.$$

An application of the splitting properties of $I_M^{\mathcal{M}}(\gamma)$ and $I_M^{\Sigma}(\gamma)$ (which follow as in (23), [30] and Proposition 6.2, [30]) and the induction hypothesis to the integrand reduces this equation to

$$r^{-1}(-1)^{r-1} \int_{M_0(F_v)/\mu_n^M} 2|n|_v^r \varepsilon_{M_0}(v) \hat{I}_{\tilde{M}_0}^{\tilde{M}_0}(\gamma', \tilde{f}_{1,\tilde{M}_0}) \hat{I}_{\tilde{M}_0}^{\tilde{M}_0}(\gamma', \tilde{f}_{2,\tilde{M}_0}) d\gamma = 0,$$

which in turn implies

$$\int_{M_0(F_v)/\mu_n^M} \varepsilon_{M_0}(v) I_{\tilde{G}}(\gamma', \tilde{f}_1) I_{\tilde{G}}(\gamma', \tilde{f}_2) d\gamma = 0. \tag{4}$$

Choosing appropriate non-negative functions $\tilde{f}_1$, $\tilde{f}_2$ in this equation implies that $\varepsilon_{M_0}(v) = 0$ and the corollary is complete. □

4. Comparison of germs

We shall establish germ expansions for $I_M^{\mathcal{M}}(\gamma)$ and $I_M^{\Sigma}(\gamma)$ and then compare them. Before writing the germ expansion for $I_{\tilde{M}}(\tilde{\gamma})$ we extend the notion of (M,σ)-equivalence (cf. §2, [9]) to metaplectic coverings. Suppose that $\tilde{\sigma}$ is a semisimple element of $\tilde{M}(F_S)$, and ϕ_1, ϕ_2 are functions defined on an open subset Σ of $\tilde{M}_{\tilde{\sigma}}(F_S)$, whose closure contains an $\tilde{M}_{\tilde{\sigma}}$-invariant neighborhood of $\tilde{\sigma}$. We say ϕ_1 is $(\tilde{M},\tilde{\sigma})$-equivalent to ϕ_2 and write

$$\phi_1(\tilde{\gamma}) \overset{(\tilde{M},\tilde{\sigma})}{\sim} \phi_2(\tilde{\gamma}),\ \tilde{\gamma} \in \Sigma,$$

if there exists $\tilde{h} \in C_c^\infty(\tilde{M}(F_S))$ and a neighborhood U of $\tilde{\sigma}$ in $\tilde{M}(F_S)$ such that

$$\phi_1(\tilde{\gamma}) - \phi_2(\tilde{\gamma}) = I_{\tilde{M}}^{\tilde{M}}(\tilde{\gamma},\tilde{h}),\ \tilde{\gamma} \in \Sigma \cap U.$$

For the remainder of this section we assume that v is nonarchimedean and $\tilde{f} \in \mathcal{H}(\tilde{G}(F_v))$. Suppose that $\tilde{\gamma} \in \tilde{M}(F_v)$ such that $\mathbf{p}(\tilde{\gamma})$ has Jordan decomposition σu. Choose $\tilde{\sigma}$ such that $\mathbf{p}(\tilde{\sigma}) = \sigma$ and $\tilde{\gamma} = \tilde{\sigma} u$. The germ expansion of $I_{\tilde{M}}(\tilde{\gamma},\tilde{f})$ is the extension of (2.5) of [7] to metaplectic coverings and reads as

$$I_{\tilde{M}}(\tilde{\gamma},\tilde{f}) \overset{(\tilde{M},\tilde{\sigma})}{\sim} \sum_{L\in\mathcal{L}(M)} \sum_{\tilde{\delta}\in\tilde{\sigma}(\mathcal{U}_{L_\sigma}(F_v))} g_{\tilde{M}}^{\tilde{L}}(\tilde{\gamma},\tilde{\delta}) I_{\tilde{L}}(\tilde{\delta},\tilde{f}). \tag{5}$$

As in the germ expansions of invariant orbital integrals (§4, [30]), the germs $g_{\tilde{M}}^{\tilde{L}}(\tilde{\gamma},\tilde{\delta})$ are parameterized by classes in $(\mathcal{U}_{L_\sigma}(F_v))$ (by way of the orbit map $\mathbf{s}$). Therefore in comparing germ expansions at $\gamma = \sigma u \in M(F_v)$ and γ' it would be convenient to have a bijection between $(\mathcal{U}_{L_\sigma}(F_v))$ and $(\mathcal{U}_{L_{\sigma^n}}(F_v))$. The following lemma gives us such a bijection provided σ is F_v-elliptic in L.

Lemma 4.1. *Suppose $L \in \mathcal{L}(M)$ and σ is a semisimple element of $M(F_v)$ such that σ is F_v-elliptic in M and σ^n is F_v-elliptic in L. Then there exists an element η in μ_n^M such that $L_{\eta\sigma}(F_v) = L_{\sigma^n}(F_v)$. In particular, $\eta\sigma$ is F_v-elliptic in L.*

Proof. For the sake of convenience, we suppose that $L = G$. Recall decomposition (4) of [30],

$$M(F_v) = M(1)(F_v) \times \cdots \times M(\ell)(F_v) \cong \mathrm{GL}(r_1,F_v) \times \cdots \times \mathrm{GL}(r_\ell,F_v).$$

We identify $M(F_v)$ with this direct product of general linear groups. By assumption, σ is F_v-elliptic in $M(F_v)$. It is therefore $M(F_v)$-conjugate to

$$\begin{pmatrix} \begin{pmatrix} \sigma_1 & & 0 \\ & \ddots & \\ 0 & & \sigma_1 \end{pmatrix} & & 0 \\ & \ddots & \\ 0 & & \begin{pmatrix} \sigma_\ell & & 0 \\ & \ddots & \\ 0 & & \sigma_\ell \end{pmatrix} \end{pmatrix},$$

where $\sigma_i \in \mathrm{GL}(m_i,F_v)$ generates an extension F_i/F_v of degree m_i, m_i divides r_i, and σ_i appears r_i/m_i times (cf. §1, [25]). Since σ^n is F_v-elliptic in $G(F_v)$, it is

$M(F_v)$-conjugate to

$$\begin{pmatrix} \sigma_0 & & 0 \\ & \ddots & \\ 0 & & \sigma_0 \end{pmatrix},$$

where $\sigma_0 \in \mathrm{GL}(m_0, F_v)$ generates a field extension F_0 of degree m_0, m_0 divides r, and σ_0 appears r/m_0 times. The elements $\sigma_i \in \mathrm{GL}(m_i, F_v)$ may also be regarded as elements of the fields F_i, $0 \leq i \leq \ell$. From this perspective we have

$$F_i = F_v(\sigma_i) \supset F_v(\sigma_i^n) = F_v(\sigma_0) = F_0,\ 1 \leq i \leq \ell.$$

Obviously $[F_i : F_0] \leq r$ and, by Theorem 10 (b), VIII, §6 of [28], $[F_i : F_0]$ divides n, $1 \leq i \leq \ell$. As we are working under Assumption 1, we must have $F_0 = F_1 = \cdots = F_\ell$. As a result, $m_1 = \cdots = m_\ell = m_0$ and $\sigma_1^n = \cdots = \sigma_\ell^n = \sigma_0$. Since F_0 contains μ_n, it follows that there exist $\zeta_1, \ldots, \zeta_\ell \in \mu_n$ such that $\zeta_i \sigma_1 = \sigma_i$. The elements $\zeta_1, \ldots, \zeta_\ell$ determine an element $\eta \in \mu_n^M$ such that

$$\eta\sigma = \begin{pmatrix} \sigma_1 & & 0 \\ & \ddots & \\ 0 & & \sigma_1 \end{pmatrix}.$$

The lemma now follows from the fact that

$$G_{\eta\sigma}(F_v) \cong \mathrm{GL}(r/m_0, F_0) \cong G_{\sigma^n}(F_v).$$

□

In the hypothesis of Lemma 4.1 we take σ to be F_v-elliptic in $M(F_v)$. We shall see in a moment that this restriction does not pose a problem for the comparison of germs because the germs of (5) vanish for non-elliptic σ. We therefore assume for the rest of this section that σ is a semisimple element of $M(F_v)$ which is F_v-elliptic in $M(F_v)$.

Let us consider the ellipticity of σ^n. There exists a unique maximal Levi subgroup $L_0 \in \mathcal{L}(M)$ such that σ^n is F_v-elliptic in L_0. Indeed, if σ^n is F_v-elliptic in $L_1, L_2 \in \mathcal{L}(M)$, then the centralizer in G of the split torus $A_{L_1} \cap A_{L_2}$ is a Levi subgroup $L \in \mathcal{L}(M)$. It is then simple to verify that

$$A_L(F_v) = A_{L_1}(F_v) \cap A_{L_2}(F_v) = A_{L_{1,\sigma^n}}(F_v) \cap A_{L_{2,\sigma^n}}(F_v) = A_{L_{\sigma^n}}(F_v),$$

which is equivalent to σ^n being F_v-elliptic in L. By Lemma 4.1 we know that there exists $\eta_0 \in \mu_n^M$ such that $L_{0,\eta_0\sigma}(F_v) = L_{0,\sigma^n}(F_v)$. We shall assume that η_0 is the identity for the rest of this section. As a result, σ is F_v-elliptic in $L \in \mathcal{L}(M)$ if and only if σ^n is F_v-elliptic in L.

Lemma 4.2. *Suppose that $L \in \mathcal{L}(M)$, σ is F_v-elliptic in L, and $\eta \in \mu_n^M$. Then $\eta\sigma$ is F_v-elliptic in L if and only if $\eta \in \mu_n^L$.*

Proof. If $\eta \in \mu_n^L$, then $L_{\eta\sigma}(F_v) = L_\sigma(F_v)$, so $\eta\sigma$ is clearly seen to be F_v-elliptic in L. Conversely, suppose $\eta\sigma$ is F_v-elliptic in L and, for the sake of simplicity, that $L = G$. Since σ is F_v-elliptic in G, we may assume by rational canonical form that

$$\sigma = \begin{pmatrix} \sigma_0 & & 0 \\ & \ddots & \\ 0 & & \sigma_0 \end{pmatrix},$$

where $\sigma_0 \in \mathrm{GL}(m_0, F_v)$ generates a field extension F_0 of degree m_0 and σ_0 appears r/m_0 times. Consequently,

$$\eta\sigma = \begin{pmatrix} \eta_1\sigma_0 & & 0 \\ & \ddots & \\ 0 & & \eta_{r/m_0}\sigma_0 \end{pmatrix},$$

where the scalar matrices, $\eta_1, \ldots, \eta_{r/m_0}$, are projections of η into the $\mathrm{GL}(m_0, F_v)$-blocks. In order for $\eta\sigma$ to be F_v-elliptic in G, we must have $\eta_i\sigma_0 = \eta_j\sigma_0$ for $1 \le i, j \le r/m_0$. This implies that $\eta \in \mu_n^G$. □

We are now ready to give germ expansions for the local geometric terms of the trace formula.

Lemma 4.3. *Suppose $\gamma \in M_\sigma(F_v) \cap G_{\mathrm{oreg}}(F_v)$ and $\tilde{f} \in \mathcal{H}(\tilde{G}(F_v))$. Then there exist functions $\gamma \mapsto g_M^L(\gamma, \delta)$ such that*

$$I_M^\Sigma(\gamma, \tilde{f}) \overset{(M,\sigma)}{\sim} \sum_{L \in \mathcal{L}(M)} \sum_{\delta \in \sigma(\mathcal{U}_{L_\sigma}(F_v))} g_M^L(\gamma, \delta) I_L^\Sigma(\delta, \tilde{f}).$$

Proof. Suppose $f \in \mathcal{H}(G(F_v))$. If $\eta \in \mu_n^M$, then $\eta\sigma$ is a semisimple element and by (2.5) of [7], we have

$$I_M(\gamma, f) \overset{(M,\eta\sigma)}{\sim} \sum_{L \in \mathcal{L}(M)} \sum_{\delta \in (\eta\sigma\mathcal{U}_{L_{\eta\sigma}}(F_v))} g_M^L(\gamma, \delta) I_L(\delta, f),\ \gamma \in M_{\eta\sigma}(F_v) \cap G_{\mathrm{oreg}}(F_v).$$

As η lies in the center of $M(F_v)$, it is easy to see that $(M, \eta\sigma)$-equivalence of a function in γ is the same as (M, σ)-equivalence of a function in $\eta\gamma$. The above expansion may thus be written as

$$I_M(\eta\gamma, f) \overset{(M,\sigma)}{\sim} \sum_{L \in \mathcal{L}(M)} \sum_{\delta \in (\eta\sigma\mathcal{U}_{L_{\eta\sigma}}(F_v))} g_M^L(\eta\gamma, \delta) I_L(\delta, f),\ \gamma \in M_\sigma(F_v) \cap G_{\mathrm{oreg}}(F_v).$$

Consequently

$$\begin{aligned} I_M^\Sigma(\gamma, \tilde{f}) &= \sum_{\eta \in \mu_n^M/\mu_n^G} \hat{I}_M(\eta\gamma, \tilde{f}') \\ &\overset{(M,\sigma)}{\sim} \sum_{L \in \mathcal{L}(M)} \sum_{\eta \in \mu_n^M/\mu_n^G} \sum_{\delta \in \eta\sigma(\mathcal{U}_{L_{\eta\sigma}}(F_v))} g_M^L(\eta\gamma, \delta) \hat{I}_L(\delta, \tilde{f}'). \end{aligned}$$

Now if $\gamma = \sigma\gamma_1$ and $\delta = \eta\sigma u$, where $u \in \mathcal{U}_{L_{\eta\sigma}}(F_v)$, then by Lemma 9.2 of [7] we have

$$g_M^L(\eta\gamma, \delta) = \begin{cases} g_{M_\sigma}^{L_{\eta\sigma}}(\gamma_1, u), & \text{if } \eta\sigma \text{ is } F_v\text{-elliptic in } L, \\ 0, & \text{otherwise.} \end{cases}$$

If σ^n is not F_v-elliptic in L, then neither is $\eta\sigma$ for any $\eta \in \mu_n^M/\mu_n^G$ and $g_M^L(\eta\gamma, \delta) = 0$. On the other hand, if σ^n is F_v-elliptic in L, then, by our assumption on σ, Lemma

4.2 and Lemma 9.2 of [7],

$$\begin{aligned} g_M^L(\eta\gamma,\delta) &= \begin{cases} g_{M_\sigma}^{L_{\eta\sigma}}(\gamma_1,u), & \text{if } \eta\in\mu_n^L/\mu_n^G,\\ 0, & \text{otherwise}\end{cases}\\ &= \begin{cases} g_{M_\sigma}^{L_{\sigma}}(\gamma_1,u), & \text{if } \eta\in\mu_n^L/\mu_n^G,\\ 0, & \text{otherwise}\end{cases}\\ &= \begin{cases} g_{M}^{L}(\gamma,\delta), & \text{if } \eta\in\mu_n^L/\mu_n^G,\\ 0, & \text{otherwise.}\end{cases} \end{aligned}$$

Combining these last two observations, we find that

$$\begin{aligned} I_M^{\Sigma}(\gamma,\tilde f) &\overset{(M,\sigma)}{\sim} \sum_{L\in\mathcal{L}(M)}\sum_{\delta\in\sigma(\mathcal{U}_{L_\sigma}(F_v))} g_M^L(\gamma,\delta)\sum_{\eta\in\mu_n^L/\mu_n^G}\hat I_L(\eta\delta,\tilde f')\\ &= \sum_{L\in\mathcal{L}(M)}\sum_{\delta\in\sigma(\mathcal{U}_{L_\sigma}(F_v))} g_M^L(\gamma,\delta) I_L^{\Sigma}(\delta,\tilde f). \end{aligned}$$

□

The next lemma shows that there is a similar germ expansion for the distribution $I_M^{\mathcal{M}}(\gamma,f)$. We shall abuse notation slightly by identifying the index set $(\mathcal{U}_{L_\sigma}(F_v))$ below with $(\mathbf{s}(\mathcal{U}_{L_\sigma}(F_v)))$.

Lemma 4.4. *Suppose $\gamma\in M_\sigma(F_v)\cap G_{\mathrm{oreg}}(F_v)$ and $\tilde f\in\mathcal{H}(\tilde G(F_v))$. Then there exist functions $\gamma\mapsto g_{\tilde M}^{\tilde L}(\mathbf{s}(\gamma),\mathbf{s}(\delta))$ such that*

$$I_M^{\mathcal{M}}(\gamma,\tilde f) \overset{(M,\sigma)}{\sim} \sum_{L\in\mathcal{L}(M)}\sum_{\delta\in\sigma(\mathcal{U}_{L_\sigma}(F_v))} g_{\tilde M}^{\tilde L}(\mathbf{s}(\gamma),\mathbf{s}(\delta)) I_L^{\mathcal{M}}(\delta,\tilde f).$$

Proof. Expansion (2.5) of [7] translates into the metaplectic context as

$$I_{\tilde M}(\tilde\gamma,\tilde f) \overset{(\tilde M,\sigma')}{\sim} \sum_{L\in\mathcal{L}(M)}\sum_{\tilde\delta\in\sigma'(\mathcal{U}_{L_{\sigma^n}}(F_v))} g_{\tilde M}^{\tilde L}(\tilde\gamma,\tilde\delta) I_{\tilde L}(\tilde\delta,\tilde f)$$

for $\tilde\gamma\in\tilde M_{\sigma'}(F_v)\cap\tilde G_{\mathrm{reg}}(F_v)$. Lemma 9.2 of [9] translates as

$$g_{\tilde M}^{\tilde L}(\tilde\gamma,\tilde\delta) = \begin{cases} g_{\widetilde{M_{\sigma^n}}}^{\widetilde{L_{\sigma^n}}}(\gamma_1,\mathbf{s}(u)), & \text{if } \sigma^n \text{ is } F_v\text{-elliptic in } L,\\ 0, & \text{otherwise,}\end{cases}$$

where $\tilde\gamma=\sigma'\gamma_1$, $\tilde\delta=\sigma'\mathbf{s}(u)$ and $u\in\mathcal{U}_{L_{\sigma^n}}(F_v)$. If σ^n is F_v-elliptic in L, it follows from our assumptions on σ that $L_\sigma=L_{\sigma^n}$. Consequently, $\widetilde{L_\sigma}=\widetilde{L_{\sigma^n}}$ and the previous equation becomes

$$g_{\tilde M}^{\tilde L}(\tilde\gamma,\tilde\delta) = \begin{cases} g_{\widetilde{M_{\sigma}}}^{\widetilde{L_{\sigma}}}(\gamma_1,\mathbf{s}(u)), & \text{if } \sigma \text{ is } F_v\text{-elliptic in } L,\\ 0, & \text{otherwise.}\end{cases} \tag{6}$$

By taking these facts into consideration, we obtain the expansion,

$$I_{\tilde M}(\tilde\gamma,\tilde f) \overset{(\tilde M,\sigma')}{\sim} \sum_{L\in\mathcal{L}(M)}\sum_{\tilde\delta\in\sigma'(\mathcal{U}_{L_{\sigma}}(F_v))} g_{\tilde M}^{\tilde L}(\tilde\gamma,\tilde\delta) I_{\tilde L}(\tilde\delta,\tilde f),\ \tilde\gamma\in\tilde M_{\sigma'}(F_v)\cap\tilde G_{\mathrm{reg}}(F_v).$$

The local vanishing property, Lemma 8.3 of [30], tells us that $I_{\tilde L}(\tilde\delta,\tilde f)$ vanishes unless $\mathbf{p}(\tilde\delta)=\delta^n$ for some $\delta\in G(F_v)$. The set $\sigma(\mathcal{U}_{L_\sigma}(F_v))$ maps bijectively onto the set $\sigma'(\mathcal{U}_{L_\sigma}(F_v))$ under the orbit map. This can be deduced from Lemma 4.1 of [30] and the identity of cocycles (1) of [30]. Hence,

$$I_{\tilde{M}}(\tilde{\gamma}, \tilde{f}) \overset{(\tilde{M},\sigma')}{\sim} \sum_{L\in\mathcal{L}(M)} \sum_{\delta\in\sigma(\mathcal{U}_{L_\sigma}(F_v))} g_{\tilde{M}}^{\tilde{L}}(\tilde{\gamma}, \delta') I_{\tilde{L}}(\delta', \tilde{f}),\ \tilde{\gamma} \in \tilde{M}_{\sigma'}(F_v) \cap \tilde{G}_{\mathrm{reg}}(F_v).$$

Once again, by the local vanishing property, $I_{\tilde{M}}(\tilde{\gamma}, \tilde{f})$ vanishes unless $\mathbf{p}(\tilde{\gamma}) = \gamma^n$ for some $\gamma \in G(F_v) \cap G_{\mathrm{oreg}}(F_v)$. We claim that for each $L \in \mathcal{L}(M)$ and $\delta \in \sigma(\mathcal{U}_{L_\sigma}(F_v))$, the function $g_{\tilde{M}}^{\tilde{L}}(\tilde{\gamma}, \delta')$ has the same vanishing property. We may assume inductively that this is true for $L \neq G$. Fix $\delta_1 \in \sigma(\mathcal{U}_{L_\sigma}(F_v))$. There exists a function $\tilde{f}_1 \in \mathcal{H}(\tilde{G}(F_v))$ such that

$$I_{\tilde{G}}(\tilde{\delta}, \tilde{f}_1) = \left\{ \begin{array}{ll} 1, & \text{if } \tilde{\delta} = \delta_1', \\ 0, & \text{otherwise} \end{array} \right\},\ \tilde{\delta} \in \sigma'(\mathcal{U}_{G_\sigma}(F_v))$$

(cf. §3.3, [38]). In particular,

$$I_{\tilde{G}}(\delta', \tilde{f}_1) = \left\{ \begin{array}{ll} 1, & \text{if } \delta' = \delta_1', \\ 0, & \text{otherwise} \end{array} \right\},\ \delta \in \sigma(\mathcal{U}_{G_\sigma}(F_v)).$$

It is easily shown using Lemma 4.1 of [30] in this instance that $\delta' = \delta_1'$ if and only if $\delta = \delta_1$. Thus

$$I_{\tilde{G}}(\delta', \tilde{f}_1) = \left\{ \begin{array}{ll} 1, & \delta = \delta_1 \\ 0, & \text{otherwise} \end{array} \right\},\ \delta \in \sigma(\mathcal{U}_{G_\sigma}(F_v)).$$

If we substitute $\tilde{f}_1$ into our last germ expansion, the desired vanishing property for $g_{\tilde{M}}^{\tilde{G}}(\tilde{\gamma}, \delta')$ follows. Our germ expansion now has the form

$$I_{\tilde{M}}(\gamma', \tilde{f}) \overset{(\tilde{M},\sigma')}{\sim} \sum_{L\in\mathcal{L}(M)} \sum_{\delta\in\sigma(\mathcal{U}_{L_\sigma}(F_v))} g_{\tilde{M}}^{\tilde{L}}(\gamma', \delta') I_{\tilde{L}}(\delta', \tilde{f}),\ \gamma \in M(F_v) \cap G_{\mathrm{oreg}}(F_v).$$

As noted in §3.1 of [30], the orbital integral of any function in $C_c^\infty(\tilde{M}(F_v))$ is equal to the orbital integral of a function in $C_c^\infty(M(F_v))$. Therefore $(\tilde{M}, \sigma')$-equivalence may be taken to be (M, σ)-equivalence and we may write

$$I_M^{\mathcal{M}}(\gamma, \tilde{f}) \overset{(M,\sigma)}{\sim} |n|_v^{r/2} \sum_{L\in\mathcal{L}(M)} \sum_{\delta\in\sigma(\mathcal{U}_{L_\sigma}(F_v))} g_{\tilde{M}}^{\tilde{L}}(\gamma', \delta') I_{\tilde{L}}(\delta', \tilde{f}),\ \gamma \in M(F_v) \cap G_{\mathrm{oreg}}(F_v).$$

We now follow an argument on pp. 121–122 of [13]. Suppose σ is F_v-elliptic in L. Then by equation (6) we have

$$g_{\tilde{M}}^{\tilde{L}}(\gamma', \delta') = g_{\widetilde{M_\sigma}}^{\widetilde{L_\sigma}}(\gamma_1', u') = g_{\widetilde{M_\sigma}}^{\widetilde{L_\sigma}}(\mathbf{s}(\gamma_1)^n, \mathbf{s}(u)^n),$$

where $\gamma = \sigma\gamma_1$ and $\delta = \sigma u$. The metaplectic version of the homogeneity property of germs (Proposition 10.2, [9]) implies that $g_{\widetilde{M_\sigma}}^{\widetilde{L_\sigma}}(\mathbf{s}(\gamma_1)^n, \mathbf{s}(u)^n)$ is equal to the product of $|n|_v^{(\dim(L_{\sigma u})-r)/2}$ with

$$\sum_{L_1\in\mathcal{L}^L(M)} \sum_{u_1\in(\mathcal{U}_{L_{1,\sigma}}(F_v))} g_{\widetilde{M_\sigma}}^{\widetilde{L_{1,\sigma}}}(\mathbf{s}(\gamma_1), \mathbf{s}(u_1)) c_{L_{1,\sigma}}^{L_\sigma}(u_1, n)[u_1^{L_\sigma} : u],$$

where $c^{L_\sigma}_{L_{1,\sigma}}(u_1, n)$ is defined as in (20) of [30] and

$$[u_1^{L_\sigma} : u] = \begin{cases} 1, & \text{if } u = u_1^{L_\sigma}, \\ 0, & \text{otherwise.} \end{cases}$$

Suppose that $u = u_1^{L_\sigma}$ and set $\delta_1 = \sigma u_1$. Then

$$|n|_v^{r/2}|n|^{(\dim(L_{\sigma u})-r)/2} = |n|^{\dim(L_{1,\delta_1})/2} = \Lambda^{L_1}(\delta_1).$$

Furthermore, by Lemma 3.5, II of [13], we have

$$c^{L_{\sigma^n}}_{L_{1,\sigma^n}}(u_1, n) = c^{L_{\sigma^n}}_{L_{1,\sigma^n}}(u_1^n, n) = c^L_{L_1}(\sigma_1^n u, n) = c^L_{L_1}(\delta_1^n, n).$$

Again, by equation (6) we have

$$g^{\widetilde{L_{1,\sigma}}}_{\widetilde{M_\sigma}}(\mathbf{s}(\gamma_1), \mathbf{s}(u_1)) = g^{\tilde{L}_1}_{\tilde{M}}(\mathbf{s}(\gamma), \mathbf{s}(\delta_1)).$$

Putting these facts together, we see that

$$|n|_v^{r/2} \sum_{\delta \in \sigma(\mathcal{U}_{L_\sigma}(F_v))} g^{\tilde{L}}_{\tilde{M}}(\gamma', \delta') I_{\tilde{L}}(\delta', \tilde{f})$$

is (M, σ)-equivalent to

$$\sum_{L_1 \in \mathcal{L}^L(M)} \sum_{\delta_1 \in \sigma(\mathcal{U}_{L_{1,\sigma}}(F_v))} \Lambda^{L_1}(\delta_1) g^{\tilde{L}_1}_{\tilde{M}}(\mathbf{s}(\gamma), \mathbf{s}(\delta_1)) c^L_{L_1}(\delta_1^n, n) I_{\tilde{L}}((\delta_1^L)', \tilde{f}),$$

if σ is F_v-elliptic in L. If σ is not F_v-elliptic in L, then equation (6) and Lemma 3.5, II of [13] imply that both of these expressions vanish, and are thus still equal. In consequence, $I^{\mathcal{M}}_M(\gamma, \tilde{f})$ is (M, σ)-equivalent to the sum of the latter expression over $L \in \mathcal{L}(M)$. Interchanging the sums over L and L_1 and substituting (20) of [30] then completes the lemma. □

In the remainder of this section we compare the germ expansions of Lemma 4.3 and Lemma 4.4.

Lemma 4.5 (7.1)**.** *Suppose Theorem A (i) holds for G. Then for each $u \in \mathcal{U}_G(F_v)$ we have*

$$g^G_M(\gamma, u) \overset{(M,1)}{\sim} g^{\tilde{G}}_{\tilde{M}}(\mathbf{s}(\gamma), \mathbf{s}(u)), \ \gamma \in M(F_v) \cap G_{\mathrm{oreg}}(F_v).$$

Proof. We may assume by induction that

$$g^L_M(\gamma, u) \overset{(M,1)}{\sim} g^{\tilde{L}}_{\tilde{M}}(\mathbf{s}(\gamma), \mathbf{s}(u)), \ \gamma \in M(F_v) \cap G_{\mathrm{oreg}}(F_v),$$

for all $L \in \mathcal{L}(M)$ such that $L \neq G$. We may equate the germ expansions of Lemmas 4.3 and 4.4 since we are assuming Theorem A holds. Together with the induction assumption, this yields

$$\sum_{u \in (\mathcal{U}_G(F_v))} \left(g^{\tilde{G}}_{\tilde{M}}(\mathbf{s}(\gamma), \mathbf{s}(u)) - g^G_M(\gamma, u) \right) \Lambda^G(u) I_{\tilde{G}}(u', \tilde{f}) \overset{(M,1)}{\sim} 0,$$

for $\gamma \in M(F_v) \cap G_{\mathrm{oreg}}(F_v)$. As in Lemma 4.4, for a a fixed element $u_1 \in (\mathcal{U}_G(F_v))$, we may choose $\tilde{f}_1 \in \mathcal{H}(\tilde{G}(F_v))$ such that

$$I_{\tilde{G}}(u, \tilde{f}_1) = \left\{ \begin{array}{ll} 1, & u = u_1, \\ 0, & \text{otherwise} \end{array} \right\}, \ u \in (\mathcal{U}_G(F_v)).$$

The lemma now follows by replacing $\tilde{f}$ with $\tilde{f}_1$ in the last $(M, 1)$-equivalence. □

Lemma 4.6 (7.2)**.** *Suppose* $\gamma \in M(F_v) \cap G_{\rm oreg}(F_v)$. *Then*

$$g_{\tilde{M}}^{\tilde{G}}(\mathbf{s}(\gamma), 1) \overset{(M,1)}{\sim} g_M^G(\gamma, 1).$$

Proof. Fix a supercuspidal representation $\tilde{\pi} \in \Pi_{\rm temp}(\tilde{G}(F_v))$ and let $\tilde{f}$ be a matrix coefficient of $\tilde{\pi}$. Although $\tilde{f}$ only has compact support modulo the center of $\tilde{G}(F_v)$, the distributions $I_M(\gamma)$ and $I_M^\Sigma(\gamma)$ are still defined at $\tilde{f}$. Indeed, these distributions are defined on the space $\mathcal{H}_{\rm ac}(\tilde{G}(F_v))$ of "almost compact" Hecke functions (cf. §11, [10]). Furthermore, it is clear from §27.3 of [18] that there exists a matrix coefficient f of $\tilde{\pi}'$ such that

$$\Lambda^G(\gamma) I_{\tilde{G}}(\gamma', \tilde{f}) = I_G(\gamma, f),\ \gamma \in G_{\rm oreg}(F_v).$$

The local trace formula ((9.3), [12]) implies the equations,

$$nI_{\tilde{M}}(\gamma', \tilde{f}) = (-1)^{\dim(A_M)} |D^G(\gamma^n)|_v^{1/2} \Theta_{\tilde{\pi}}(\gamma'),$$

$$I_M(\gamma, f) = (-1)^{\dim(A_M)} |D^G(\gamma)|_v^{1/2} \Theta_{\tilde{\pi}'}(\gamma),$$

for $\gamma \in M(F_v) \cap G_{\rm oreg}(F_v)$ which are F_v-elliptic in M (the coefficient n in the former equation arises from an invariant sum over $\mathbf{i}(\mu_n)$). It is immediate from these equations that we also have

$$I_M^{\mathcal{M}}(\gamma, \tilde{f}) = n^{-1}|n|_v^{r/2}(-1)^{\dim(A_M)} |D^G(\gamma^n)|_v^{1/2} \Theta_{\tilde{\pi}}(\gamma'),$$

$$I_M^\Sigma(\gamma, \tilde{f}) = (-1)^{\dim(A_M)} \sum_{\eta \in \mu_n^M/\mu_n^G} |D^G(\eta\gamma)|_v^{1/2} \Theta_{\tilde{\pi}'}(\eta\gamma).$$

According to character relation (6) of [30] the right-hand sides of these equations are equal. If $\gamma \in M(F_v) \cap G_{\rm oreg}(F_v)$ is not F_v-elliptic in M, then

$$I_M^{\mathcal{M}}(\gamma, \tilde{f}) = 0 = I_M^\Sigma(\gamma, \tilde{f}),$$

by [6]. Applying Lemma 3.3 we obtain

$$I_M^{\mathcal{M}}(1, \tilde{f}) = I_M^\Sigma(1, \tilde{f}).$$

Suppose $u \in M(F_v)$ is unipotent and $u \neq 1$. Then u can be represented by an induced unipotent conjugacy class u_1^M, where $u_1 \in (\mathcal{U}_{M_1}(F_v))$, $M_1 \in \mathcal{L}$ and $M_1 \subsetneqq M$. Expansion (20) in [30] and the descent formula of Corollary 8.2 in [7] (extended to metaplectic coverings) imply that

$$\begin{aligned} I_M^{\mathcal{M}}(u, \tilde{f}) &= \Lambda^M(u) \sum_{L \in \mathcal{L}(M)} c_M^L(u^n, n) I_{\tilde{L}}((u_1^L)', \tilde{f}) \\ &= \Lambda^M(u) \sum_{L \in \mathcal{L}(M)} c_M^L(u^n, n) \sum_{L_1 \in \mathcal{L}(M_1)} d_{M_1}^G(L, L_1) \hat{I}_{\tilde{M}_1}^{\tilde{L_1}}(u_1', \tilde{f}_{\tilde{L_1}}). \end{aligned}$$

Since $\tilde{f}$ is a supercusp form on $\tilde{G}$, we have $\tilde{f}_{\tilde{L_1}} = 0$ for any $L_1 \in \mathcal{L}$ such that $L_1 \subsetneqq G$. It follows that the right-hand side of this equation vanishes. Using a similar, but simpler argument we see that $I_M^\Sigma(u, \tilde{f})$ also vanishes. Collecting our results, we find that many of the terms in the respective germ expansions in Lemma 4.3 and Lemma 4.4 of $I_M^\Sigma(\gamma, \tilde{f})$ and $I_M^{\mathcal{M}}(\gamma, \tilde{f})$ about the identity disappear. Assuming inductively that the lemma holds for G replaced by $L \in \mathcal{L}$ with $L \subsetneq G$, we find that the difference of the remaining terms is

$$\begin{aligned} & g_{\tilde{M}}^{\tilde{G}}(\mathbf{s}(\gamma),1) I_G^{\mathcal{M}}(1,\tilde{f}) - g_M^G(\gamma,1) I_G^{\Sigma}(\gamma,\tilde{f}) \\ = \quad & \left(g_{\tilde{M}}^{\tilde{G}}(\mathbf{s}(\gamma),1) - g_M^G(\gamma,1) \right) |n|_v^{r^2/2} \tilde{f}(1) \\ \overset{(M,1)}{\sim} \quad & 0. \end{aligned}$$

The lemma now follows from the fact that $\tilde{f}(1) \neq 0$. □

We now define a subspace $\mathcal{H}(\tilde{G}(F_S))^0$ of $\mathcal{H}(\tilde{G}(F_S))$ upon which our distributions shall be easier to compare. Set $\mathcal{H}(\tilde{G}(F_S))^0$ to be the subspace of $\mathcal{H}(\tilde{G}(F_S))$ generated by functions,

$$\tilde{f} = \prod_{v \in S} \tilde{f}_v, \ \ \tilde{f}_v \in \mathcal{H}(G(F_v)),$$

such that for each nonarchimedean valuation $v \in S$ the orbital integral of $\tilde{f}_v$ vanishes at any element of the form

$$(\delta_v u_v, \zeta), \ \ \delta \in A_G(F_v), \ \ u_v \in \mathcal{U}_G(F_v), \ \ u_v \neq 1, \ \ \zeta \in \mu_n.$$

Proposition 4.1 (7.3)**.** *Suppose σ is as above and $\tilde{f} \in \mathcal{H}(\tilde{G}(F_v))^0$. Then*

$$I_M^{\mathcal{M}}(\gamma,\tilde{f}) \overset{(M,\sigma)}{\sim} \sum_{L \in \mathcal{L}(M)} \sum_{\delta \in (\mathcal{U}_{L_\sigma}(F_v))} g_M^L(\gamma,\delta) I_L^{\mathcal{M}}(\delta,\tilde{f}), \ \ \gamma \in M_\sigma(F_v) \cap G_{\mathrm{oreg}}(F_v).$$

Proof. Suppose $\delta = \sigma u$, where $u \in \mathcal{U}_{L_\sigma}(F_v)$, and $\gamma = \sigma\gamma_1$, where $\gamma_1 \in M_\sigma(F_v)$. Then according to Lemma 9.2 of [9] and the proof of Lemma 4.4,

$$g_M^L(\gamma,\delta) = \begin{cases} g_{M_\sigma}^{L_\sigma}(\gamma_1,u), & \text{if } \sigma \text{ is } F_v\text{-elliptic in } L, \\ 0, & \text{otherwise,} \end{cases}$$

and

$$g_{\tilde{M}}^{\tilde{L}}(\mathbf{s}(\gamma),\mathbf{s}(\delta)) = \begin{cases} g_{\widetilde{M_\sigma}}^{\widetilde{L_\sigma}}(\mathbf{s}(\gamma_1),\mathbf{s}(u)), & \text{if } \sigma \text{ is } F_v\text{-elliptic in } L, \\ 0, & \text{otherwise.} \end{cases}$$

By the germ expansion of Lemma 4.4, it suffices to show that

$$g_{M_\sigma}^{L_\sigma}(\gamma_1,u) \overset{(M_\sigma,1)}{\sim} g_{\widetilde{M_\sigma}}^{\widetilde{L_\sigma}}(\mathbf{s}(\gamma_1),\mathbf{s}(u)),$$

if σ is F_v-elliptic in L, and $L \in \mathcal{L}(M)$. Suppose that σ is F_v-elliptic in L and that $L_\sigma \neq G$. Then the induction assumption following Theorem A holds for L_σ and we may apply Lemma 4.5, with L_σ in place of G, to obtain

$$g_{M_\sigma}^{L_\sigma}(\gamma_1,u) \overset{(M_\sigma,1)}{\sim} g_{\widetilde{M_\sigma}}^{\widetilde{L_\sigma}}(\mathbf{s}(\gamma_1),\mathbf{s}(u)).$$

By Lemma 2.1 of [9], it follows that $(M_\sigma,1)$-equivalence of these germs as functions of γ_1 is that same as (M,σ)-equivalence as functions of γ. That is,

$$g_M^L(\gamma,\delta) = g_{M_\sigma}^{L_\sigma}(\gamma_1,u) = g_{\widetilde{M_\sigma}}^{\widetilde{L_\sigma}}(\mathbf{s}(\gamma_1),\mathbf{s}(u)) = g_{\tilde{M}}^{\tilde{L}}(\mathbf{s}(\gamma),\mathbf{s}(\delta)).$$

The remaining possibility is that $\sigma \in A_G(F_v)$ and $L = G$. Suppose this is the case. Then, by our assumptions on σ and Lemma 4.5 of [30],

$$\sum_{\delta \in \sigma(\mathcal{U}_G(F_v))} g_{\tilde{M}}^{\tilde{G}}(\mathbf{s}(\gamma),\mathbf{s}(\delta)) I_G^{\mathcal{M}}(\delta,\tilde{f}) = \sum_{\delta \in \sigma(\mathcal{U}_G(F_v))} g_{\tilde{M}}^{\tilde{G}}(\mathbf{s}(\gamma),\mathbf{s}(\delta)) \Lambda^G(\delta) I_{\tilde{G}}(\delta,\tilde{f}).$$

Since $\tilde{f} \in \mathcal{H}(\tilde{G}(F_v))^0$, we see that $I_{\tilde{G}}(\delta, \tilde{f})$ vanishes in the above sum unless $\delta = \sigma$. Since σ is central,

$$g_{\tilde{M}}^{\tilde{G}}(\mathbf{s}(\gamma), \mathbf{s}(\sigma)) = g_{\tilde{M}}^{\tilde{G}}(\mathbf{s}(\gamma_1), 1) = g_M^G(\gamma_1, 1) = g_M^G(\gamma, \sigma)$$

by Lemma 9.2 of [9] and Lemma 4.6. Hence,

$$\begin{aligned} \sum_{\delta \in \sigma(\mathcal{U}_G(F_v))} g_{\tilde{M}}^{\tilde{G}}(\mathbf{s}(\gamma), \mathbf{s}(\delta)) I_G^{\mathcal{M}}(\delta, \tilde{f}) &= g_{\tilde{M}}^{\tilde{G}}(\mathbf{s}(\gamma), \mathbf{s}(\sigma)) I_G^{\mathcal{M}}(\sigma, \tilde{f}) \\ &\overset{(M,\sigma)}{\sim} g_M^G(\gamma, \sigma) I_G^{\mathcal{M}}(\sigma, \tilde{f}) \\ &= \sum_{\delta \in \sigma(\mathcal{U}_G(F_v))} g_M^G(\gamma, \delta) I_G^{\mathcal{M}}(\delta, \tilde{f}) \end{aligned}$$

and the lemma is complete. □

5. The distributions $I_{\tilde{M}}(\tilde{\pi}, X)$ and $I_M^{\Sigma}(\tilde{\pi}, X)$

Leaving the geometric side of the trace formula behind, we examine the spectral side. The spectral side contains invariant distributions,

$$I_{\tilde{M}}(\tilde{\pi}, X, \tilde{f}),\ \tilde{\pi} \in \Pi(\tilde{M}(F_S)),\ X \in \mathfrak{a}_{M,S},\ \tilde{f} \in \mathcal{H}(\tilde{G}(F_S)),$$

which are introduced in §3 of [7] in the case of the trivial covering. We assume that the reader is to some degree familiar with this introduction and describe some of it below in the context of non-trivial coverings.

Given $\tilde{\rho} \in \Sigma(\tilde{M}(F_S))$, we define

$$I_M^{G,\Sigma}(\tilde{\rho}, X, \tilde{f}) = n^{-\dim(A_G)} \hat{I}_M^G(\tilde{\rho}', X', \tilde{f}') = n^{-1} \hat{I}_M(\tilde{\rho}', X', \tilde{f}'),$$

for all $X \in \mathfrak{a}_{M,S}$ and $\tilde{f} \in \mathcal{H}(\tilde{G}(F_S))$. The coefficient n^{-1} appears in this definition in order absorb the same coefficient appearing in $I^{\Sigma}(\tilde{f}) = n^{-1} I(\tilde{f}')$. If $L \in \mathcal{L}(M)$ and $\lambda \in \mathfrak{a}_{M,S}^*$ is in general position, then the induced representation $\tilde{\rho}_\lambda^{\tilde{L}}$ belongs to $\Sigma(\tilde{L}(F_S))$. When $\tilde{\rho}_\lambda^{\tilde{L}}$ appears as an argument of $I_{\tilde{L}}(\cdot)$ or $I_L^{\Sigma}(\cdot)$, we will often suppress the superscript $\tilde{L}$. For $\tilde{\pi} \in \Pi(\tilde{M}(F_S))$, we define $I_M^{\Sigma}(\tilde{\pi}, X, \tilde{f})$ as

$$(7) \qquad \sum_P \omega_P \sum_L \sum_{\tilde{\rho}} \int_{\varepsilon_P + i\mathfrak{a}_{M,S}^*/i\mathfrak{a}_{L,S}^*} r_{\tilde{M}}^{\tilde{L}}(\tilde{\pi}_\lambda, \tilde{\rho}_\lambda) I_L^{\Sigma}(\tilde{\rho}_\lambda, h_L(X), \tilde{f}) e^{-\lambda(X)} d\lambda,$$

where P, L and $\tilde{\rho}$ are summed over $\mathcal{P}(M)$, $\mathcal{L}(M)$ and $\Sigma(\tilde{M}(F_S))$ respectively. For the definitions of $h_L(X)$ and ω_P see §6 of [10] and §3 of [7]. The definition of $r_{\tilde{M}}^{\tilde{L}}(\tilde{\pi}_\lambda, \tilde{\rho}_\lambda)$ follows §6 of [10].

As on p. 127 of [13], we identify a representation $\tilde{\pi}$ in $\Pi(\tilde{M}(F_S)^1)$ with the orbit $\{\tilde{\pi}_\lambda : \lambda \in \mathfrak{a}_{M,\mathbf{C}}^*\}$ in $\Pi(\tilde{M}(F_S))$, if $\tilde{\pi}$ is not unitary. If $\tilde{\pi}$ belongs to $\Pi_{\text{unit}}(\tilde{M}(F_S)^1)$, then we identify it with the orbit $\{\tilde{\pi}_\lambda : \lambda \in i\mathfrak{a}_M^*\}$ in $\Pi_{\text{unit}}(\tilde{M}(F_S))$. We make similar identifications for representations in $\Pi(\tilde{M}(\mathbf{A})^1)$ and $\Pi_{\text{unit}}(\tilde{M}(\mathbf{A})^1)$. If $\tilde{\pi} \in \Pi_{\text{unit}}(\tilde{M}(F_S)^1)$, set

$$I_{\tilde{M}}(\pi, \tilde{f}) = I_{\tilde{M}}(\tilde{\pi}_\lambda, 0, \tilde{f}),$$

and

$$I_M^{\Sigma}(\tilde{\pi}, \tilde{f}) = I_M^{\Sigma}(\tilde{\pi}_\lambda, 0, \tilde{f}),$$

for any $\lambda \in i\mathfrak{a}_M^*$. It may be verified that these definitions are well-defined. Both of these definitions are independent of S, if S is large, and therefore may be extended to representations in $\Pi_{\text{unit}}(\tilde{M}(\mathbf{A})^1)$. In complete analogy with the expressions of the geometric side of the trace formula, we shall compare $I_{\tilde{M}}(\tilde{\pi}, \tilde{f})$ with the term $I_M^{\Sigma}(\tilde{\pi}, \tilde{f})$ occurring in the spectral sides of the trace formulas.

We may draw parallels between the local geometric and the local spectral terms of the trace formulas. In order to compare the local geometric terms of the trace formulas for G and $\tilde{G}$, we use the orbit map. One might surmise that the analogous transfer map for the local spectral terms might be given by the local metaplectic correspondence (11) of [30]. Unfortunately, this map does not intrinsically relate the characters of the representations to each other. In what follows, we define certain constants which relate representations in $\Pi(\tilde{M}(F_S))$ to representations in $\Pi(M(F_S))$ in a fashion that is compatible with their characters.

By the Langlands quotient theorem and §5 of [10] there exist constants $\Delta(\tilde{\pi}, \tilde{\rho})$ and $\Gamma(\tilde{\rho}, \tilde{\pi})$ for arbitrary $\tilde{\rho} \in \Sigma(\tilde{M}(F_S))$ and $\tilde{\pi} \in \Pi(\tilde{M}(F_S))$, such that

$$\text{tr}(\tilde{\rho}) = \sum_{\tilde{\pi} \in \Pi(\tilde{M}(F_S))} \Gamma(\tilde{\rho}, \tilde{\pi}) \text{tr}(\tilde{\pi}),$$

and

$$\text{tr}(\tilde{\pi}) = \sum_{\tilde{\rho} \in \Sigma(\tilde{M}(F_S))} \Delta(\tilde{\pi}, \tilde{\rho}) \text{tr}(\tilde{\rho}).$$

For $\pi \in \Pi(M(F_S))$ and $\tilde{\rho} \in \Sigma(\tilde{M}(F_S))$ define

$$\Delta(\pi, \tilde{\rho}) = \Delta(\pi, \tilde{\rho}').$$

Given $\tilde{\pi} \in \Pi(\tilde{M}(F_S))$, we set

$$\delta(\pi, \tilde{\pi}) = \sum_{\tilde{\rho} \in \Sigma(\tilde{M}(F_S))} \Delta(\pi, \tilde{\rho}) \Gamma(\tilde{\rho}, \tilde{\pi}).$$

As can be seen from the next proposition, the map of virtual characters,

$$\Theta_\pi \mapsto \sum_{\tilde{\pi} \in \Pi(\tilde{M}(F_S))} \delta(\pi, \tilde{\pi}) \Theta_{\tilde{\pi}}, \tag{8}$$

is the transfer map which allows us to compare characters of representations.

Proposition 5.1 (8.2)**.** *For any $\tilde{f} \in \mathcal{H}(\tilde{M}(F_S))$ and matching $f \in \mathcal{H}(M(F_S))$ we have*

$$\text{tr}\,(\pi(f)) = \sum_{\tilde{\pi} \in \Pi(\tilde{M}(F_S))} \delta(\pi, \tilde{\pi}) \text{tr}(\tilde{\pi}(\tilde{f})), \ \pi \in \Pi(M(F_S)).$$

Proof. According to the character relations satisfied by the local metaplectic correspondence

$$\begin{aligned}
\mathrm{tr}\,(\pi(f)) &= \sum_{\rho\in\Sigma(M(F_S))} \Delta(\pi,\rho)\mathrm{tr}\,(\rho(f)) \\
&= \sum_{\tilde\rho\in\Sigma(\tilde M(F_S))} \Delta(\pi,\tilde\rho')\mathrm{tr}(\tilde\rho(\tilde f)) \\
&= \sum_{\tilde\rho\in\Sigma(\tilde M(F_S))}\sum_{\tilde\pi\in\Pi(\tilde M(F_S))} \Delta(\pi,\tilde\rho')\Gamma(\tilde\rho,\tilde\pi)\mathrm{tr}(\tilde\pi(\tilde f)) \\
&= \sum_{\tilde\pi\in\Pi(\tilde M(F_S))} \delta(\pi,\tilde\pi)\mathrm{tr}(\tilde\pi(\tilde f)).
\end{aligned}$$

□

Corollary 5.1 (8.3)**.** *Suppose S consists of a single valuation v, for which $|n|_v = 1$, and that $\tilde\pi\in\Pi(\tilde M(F_v))$ is an unramified representation. Then for any $\pi\in\Pi(M(F_v))$*

$$\delta(\pi,\tilde\pi) = \begin{cases} 1, & \text{if } \pi = \tilde\pi', \\ 0, & \text{otherwise.}\end{cases}$$

Proof. Take $\tilde f$ to be an arbitrary function in $\mathcal{H}(M(F_v))$ which is bi-invariant under $\mathbf{s}\,(K_v\cap M(F_v))$. Theorem 16 of [18] tells us that $\mathrm{tr}\,(\tilde\pi'(f)) = \mathrm{tr}(\tilde\pi(\tilde f))$ for any $f\in\mathcal{H}(M(F_v))$ matching $\tilde f$. The corollary now follows from Proposition 5.1 and the linear independence of characters. □

This corollary allows us to define a map

$$\delta(\pi,\tilde\pi) = \prod_v \delta(\pi_v,\tilde\pi_v)$$

for adelic representations $\tilde\pi = \bigotimes_v \tilde\pi_v \in \Pi(\tilde M(\mathbf{A}))$ and $\pi = \bigotimes_v \pi_v\in\Pi(M(\mathbf{A}))$. All of the above formulas remain valid in the adelic context. Given $\tilde\pi\in\Pi(\tilde M(\mathbf{A})^1)$ and $\pi\in\Pi(M(\mathbf{A})^1)$, set

$$\delta(\pi,\tilde\pi) = \sum_{\lambda\in\mathfrak{a}^*_{M,\mathbf{C}}} \delta(\pi_{\lambda_1},\tilde\pi_\lambda)$$

for arbitrary $\lambda_1\in\mathfrak{a}^*_{M,\mathbf{C}}$. This definition may be verified to be well-defined.

6. Statement of Theorem B

Now that we have a spectral transfer map (8), we can begin to compare the spectral sides of the trace formulas. The spectral side of the (conjectural) trace formula for $\tilde G$ is of the form

$$I(\tilde f) = \sum_{t\geq 0} I_t(\tilde f),$$

where

$$I_t(\tilde f) = \sum_{M\in\mathcal{L}} |W_0^M||W_0^G|^{-1}\int_{\Pi(\tilde M,t)} a^{\tilde M}(\tilde\pi) I_{\tilde M}(\tilde\pi,\tilde f)d\tilde\pi. \tag{9}$$

A description of the terms occurring in this formula may be extrapolated from §4 of [8]. The definition of $\Pi(\tilde M,t)$ is given here (cf. §4 of [8]), as we shall often consider it. Let $M_1 \in \mathcal{L}$ and t be a positive real number. We are obliged to first define two other sets, $\Pi(\tilde{M}(\mathbf{A})^1, t)$ and $\Pi_{\text{disc}}(\tilde{M}_1, t)$, before we define $\Pi(\tilde{M}, t)$. Given a representation $\tilde{\pi} \in \Pi(\tilde{M}(\mathbf{A})^1)$, let $\nu_{\tilde{\pi}}$ be the infinitesimal character of the archimedean factor of $\tilde{\pi}$. The set $\Pi(\tilde{M}(\mathbf{A})^1, t)$ is defined to be the set of (equivalence classes) of representations $\tilde{\pi} \in \Pi(\tilde{M}(\mathbf{A})^1)$ such that $\|\text{Im}(\nu_{\tilde{\pi}})\| = t$. We write $\Pi_{\text{disc}}(\tilde{M}_1, t)$ for the subset of $\Pi_{\text{unit}}(\tilde{M}(\mathbf{A})^1, t)$ consisting of irreducible constituents of induced representations

$$\tilde{\sigma}_\lambda^{\tilde{M}_1},\ L \in \mathcal{L}^{M_1},\ \tilde{\sigma} \in \Pi_{\text{unit}}(\tilde{L}(\mathbf{A})^1, t),\ \lambda \in i\mathfrak{a}_L^*/i\mathfrak{a}_{M_1}^*,$$

in which $\tilde{\sigma}_\lambda$ satisfies the following two conditions:

1. $a_{\text{disc}}^{\tilde{L}}(\tilde{\sigma}) \neq 0$ (see (4.4), [8]).
2. There is an element $s \in W^{M_1}(\mathfrak{a}_L)_{\text{reg}}$ such that $s\tilde{\sigma}_\lambda = \tilde{\sigma}_\lambda$.

Then $\Pi(\tilde{M}, t)$ is defined as the disjoint union over $M_1 \in \mathcal{L}^M$ of the sets

$$\Pi_{\tilde{M}_1}(\tilde{M}, t) = \{\tilde{\pi} = \tilde{\pi}_{1,\lambda} : \tilde{\pi}_1 \in \Pi_{\text{disc}}(\tilde{M}_1, t),\ \lambda \in i\mathfrak{a}_{M_1}^*/i\mathfrak{a}_M^*\}.$$

The measure in (9) is given by

$$d\tilde{\pi} = d\tilde{\pi}_{1,\lambda} = |W_0^{M_1}||W_0^M|^{-1}d\lambda.$$

Let us again consider §2, where we examined the geometric sides of the trace formulas. In Proposition 9.1 of [30], the geometric side of the trace formula for $\tilde{G}(\mathbf{A})$ was expressed in a manner that was compatible with the orbit map. We shall follow suit by expressing the spectral side of the trace formula for $\tilde{G}(\mathbf{A})$ in a manner that is compatible with the spectral transfer map (8). This will be carried out in §9. For the the time being, we set up the appropriate grouping of representations for the global datum $a^{\tilde{M}}(\tilde{\pi})$. In other words, we define the global datum $a^{M,\Sigma}(\tilde{\pi})$, which should equal the global datum $a^{\tilde{M}}(\tilde{\pi})$, occurring in the trace formula for $\tilde{G}(\mathbf{A})$. This is similar in spirit to the grouping of the local geometric terms according to μ_n^M in §6 of [30]. We first define $a_{\text{disc}}^{M_1,\Sigma}(\tilde{\pi}_1)$ for $M_1 \in \mathcal{L}^M$. Set

$$a_{\text{disc}}^{M_1,\Sigma}(\tilde{\pi}_1) = \sum_{\pi \in \Pi(M_1(\mathbf{A})^1)} a_{\text{disc}}^{M_1}(\pi)\delta(\pi, \tilde{\pi}_1),\ \tilde{\pi}_1 \in \Pi(\tilde{M}_1(\mathbf{A})^1).$$

This sum may be shown to be finite using the arguments of Lemma 9.1, II of [13]. Given $\tilde{\pi} = \tilde{\pi}_{1,\lambda}$, where $\lambda \in \mathfrak{a}_{M_1,\mathbf{C}}^*/\mathfrak{a}_{M,\mathbf{C}}^*$, we set

$$a^{M,\Sigma}(\tilde{\pi}) = a_{\text{disc}}^{M_1,\Sigma}(\tilde{\pi}_1) r_{\tilde{M}_1}^{\tilde{M}}(\tilde{\pi}_{1,\lambda}).$$

The number $r_{\tilde{M}_1}^{\tilde{M}}(\pi_{1,\lambda})$ is not defined for arbitrary $\tilde{\pi}_1 \in \Pi(\tilde{M}_1(\mathbf{A})^1)$, and so the definition of $a^{M,\Sigma}(\tilde{\pi})$ is not valid as it now stands. The obstacle stems from the fact that $r_{\tilde{M}_1}^{\tilde{M}}(\tilde{\pi}_{1,\lambda})$ is derived from the adelic version of the normalizing factors of intertwining operators (§5, [30] and §8). As such, it is defined as an infinite product over the valuations of F and might not converge. One expects such normalizing factors to converge and have analytic continuation for automorphic representations. This is borne out by the theory of Eisenstein series (§4, [10]). In order to rectify the above definition, we make the following induction hypothesis. We assume that for any $M_1 \in \mathcal{L}$ with $M_1 \neq G$, that

$$a_{\text{disc}}^{M_1,\Sigma}(\tilde{\pi}_1) = a_{\text{disc}}^{M_1}(\tilde{\pi}_1),\ \tilde{\pi}_1 \in \Pi(\tilde{M}_1(\mathbf{A})^1).$$

In this case $a_{\mathrm{disc}}^{M_1,\Sigma}(\tilde{\pi}_1)$ vanishes unless $\tilde{\pi}_1$ belongs to $\Pi_{\mathrm{disc}}(\tilde{M}_1,t)$. If $M_1 \in \mathcal{L}^M$, $\tilde{\pi}_1 \in \Pi_{\mathrm{disc}}(\tilde{M}_1,t)$ and $\lambda \in i\mathfrak{a}^*_{M_1,\mathbf{C}}/i\mathfrak{a}^*_{M,\mathbf{C}}$, then $r^{\tilde{M}}_{\tilde{M}_1}(\tilde{\pi}_{1,\lambda})$ is defined and the above definition of $a^{M,\Sigma}(\tilde{\pi}_{1,\lambda})$ makes sense.

The global datum $a^{M,\Sigma}(\tilde{\pi}_1)$, suggests the definitions of sets of representations are made along the same lines as the definitions of $\Pi_{\mathrm{disc}}(\tilde{M}_1,t)$, $\Pi_{\tilde{M}_1}(\tilde{M},t)$ and $\Pi(\tilde{M},t)$ above. We define the sets of (equivalence classes of) representations $\Pi^{\Sigma}_{\mathrm{disc}}(\tilde{M}_1,t)$, $\Pi^{\Sigma}_{\tilde{M}_1}(\tilde{M},t)$ and $\Pi^{\Sigma}(\tilde{M},t)$ as above, except that $a^{\tilde{M}}_{\mathrm{disc}}$ is replaced with $a^{M,\Sigma}_{\mathrm{disc}}$.

We are now in the position to state the spectral analogue of Theorem A.

Theorem B. *Under Assumptions 1 and 2 the following assertions are true.*

(i) *Suppose that S is a finite set of valuations with the closure property. Then*

$$I^{\Sigma}_M(\tilde{\pi},\tilde{f}) = I_{\tilde{M}}(\tilde{\pi},\tilde{f}),\ \tilde{\pi} \in \Pi_{\mathrm{unit}}(\tilde{M}(\mathbf{A})^1),\ \tilde{f} \in \mathcal{H}(\tilde{G}(F_S)).$$

(ii) *For any given*

$$\tilde{\pi} = \tilde{\pi}_{1,\lambda},\ \tilde{\pi}_1 \in \Pi(\tilde{M}_1(\mathbf{A})^1),\ \lambda \in \mathfrak{a}^*_{M_1,\mathbf{C}}/\mathfrak{a}^*_{M,\mathbf{C}},$$

we have

$$a^{M,\Sigma}(\tilde{\pi}) = a^{\tilde{M}}(\tilde{\pi}).$$

The proof of this theorem will be completed at the end of §12.

7. Comparison of $I_{\tilde{M}}(\tilde{\pi},X,\tilde{f})$ and $\hat{I}^{\Sigma}_M(\tilde{\pi},X,\tilde{f})$

The purpose of this section is to show that Theorem A (i) implies Theorem B (i). We achieve this using the maps $\theta^{\tilde{L}}_{\tilde{M}}$ and ${}^c\theta^{\tilde{L}}_{\tilde{M}}$ defined as in §4 of [7]. These maps are defined on $\tilde{\mathcal{H}}_{\mathrm{ac}}(\tilde{L}(F_S))$ and take values in $\tilde{\mathcal{I}}_{\mathrm{ac}}(\tilde{M}(F_S))$ for every pair of Levi subgroups $M \subset L$ in $\mathcal{L}$. The spaces $\tilde{\mathcal{H}}_{\mathrm{ac}}(\tilde{L}(F_S))$ and $\tilde{\mathcal{I}}_{\mathrm{ac}}(\tilde{M}(F_S))$ contain $\mathcal{H}(\tilde{L}(F_S))$ and $\mathcal{I}(\tilde{M}(F_S))$ respectively and are defined as in §11 of [10]. The map (13) of [30] extends in an obvious manner from a map on $\mathcal{H}(\tilde{L}(F_S))$ to a map on $\tilde{\mathcal{H}}_{\mathrm{ac}}(\tilde{L}(F_S))$. The above maps satisfy the following properties:

$$\sum_{L\in\mathcal{L}(M)} \hat{\theta}^{\tilde{L}}_{\tilde{M}}({}^c\theta_{\tilde{L}}(\tilde{f})) = \sum_{L\in\mathcal{L}(M)} {}^c\hat{\theta}^{\tilde{L}}_{\tilde{M}}(\theta_{\tilde{L}}(\tilde{f})) = 0,$$

$$I_{\tilde{M}}(\tilde{\gamma},\tilde{f}) = \sum_{L\in\mathcal{L}(M)} {}^c\hat{I}^{\tilde{L}}_{\tilde{M}}(\tilde{\gamma},\theta_{\tilde{L}}(\tilde{f})), \tag{10}$$

and

$${}^cI_{\tilde{M}}(\tilde{\gamma},\tilde{f}) = \sum_{L\in\mathcal{L}(M)} \hat{I}^{\tilde{L}}_{\tilde{M}}(\tilde{\gamma},{}^c\theta_{\tilde{L}}(\tilde{f})), \tag{11}$$

for $\tilde{\gamma} \in \tilde{M}(F_S)$ and $\tilde{f} \in \tilde{\mathcal{H}}_{\mathrm{ac}}(\tilde{G}(F_S))$. For the definition of ${}^cI_{\tilde{M}}(\tilde{\gamma})$ see §4 of [7]. Set

$${}^cI^{\mathcal{M}}_M(\gamma,\tilde{f}) = \Lambda^M(\gamma)\,{}^cI_{\tilde{M}}(\gamma',\tilde{f}),\ \gamma \in M(F_S)\cap G_{\mathrm{oreg}}(F_S).$$

The analogue of property (11) for ${}^cI^{\mathcal{M}}_M(\gamma)$ is then seen to be

$$\begin{aligned} {}^cI^{\mathcal{M}}_M(\gamma,\tilde{f}) &= \sum_{L\in\mathcal{L}(M)} \Lambda^M(\gamma)\hat{I}^{\tilde{L}}_{\tilde{M}}(\gamma',{}^c\theta_{\tilde{L}}(\tilde{f})) \\ &= \sum_{L\in\mathcal{L}(M)} \hat{I}^{L,\mathcal{M}}_M(\gamma,{}^c\theta_{\tilde{L}}(\tilde{f})),\ \gamma \in M(F_S)\cap G_{\mathrm{oreg}}(F_S). \end{aligned} \tag{12}$$

After a similar computation we may conclude that the analogue of property (10) is

$$I_M^{\mathcal{M}}(\gamma,\tilde{f}) = \sum_{L\in\mathcal{L}(M)} {}^c\hat{I}_M^{L,\mathcal{M}}(\gamma,\theta_{\tilde{L}}(\tilde{f})),\ \gamma\in M(F_S)\cap G_{\mathrm{oreg}}(F_S). \tag{13}$$

Properties (10) and (11) may also be adapted to the distributions of the form $I_M^{\Sigma}(\gamma)$. We mimic the arguments of §6 in [30] to arrive at the equation

$${}^c\hat{I}_M(\eta\gamma,\tilde{f}') = {}^c\hat{I}_M(\gamma,\tilde{f}'),\ \tilde{f}\in\tilde{\mathcal{H}}_{\mathrm{ac}}(\tilde{G}(F_S)),\, \gamma\in M(F_S),\, \eta\in\mu_n^G.$$

Thus, imitating the definition of $I_M^{\Sigma}(\gamma)$, we set

$${}^cI_M^{\Sigma}(\gamma,\tilde{f}) = \sum_{\eta\in\mu_n^M/\mu_n^G} {}^c\hat{I}_M(\eta\gamma,\tilde{f}'),\ \gamma\in M(F_S),\ \tilde{f}\in\tilde{\mathcal{H}}_{\mathrm{ac}}(\tilde{G}(F_S)).$$

The analogue of property (11) for ${}^cI_M^{\Sigma}(\gamma)$ is then seen to be

$$\begin{aligned}
{}^cI_M^{\Sigma}(\gamma,\tilde{f}) &= \sum_{\eta\in\mu_n^M/\mu_n^G}\sum_{L\in\mathcal{L}(M)} \hat{I}_M^L(\eta\gamma,\,{}^c\hat{\theta}_L(\tilde{f}'))\\
&= \sum_{L\in\mathcal{L}(M)}\sum_{\eta\in\mu_n^M/\mu_n^L}\sum_{\eta_1\in\mu_n^L/\mu_n^G} \hat{I}_M^L(\eta_1\eta\gamma,\,{}^c\hat{\theta}_L(\tilde{f}'))\\
&= \sum_{L\in\mathcal{L}(M)}\sum_{\eta\in\mu_n^M/\mu_n^L} \hat{I}_M^L(\eta\gamma,\sum_{\eta_1\in\mu_n^L/\mu_n^G} {}^{\eta_1 c}\hat{\theta}_L(\tilde{f}')).
\end{aligned} \tag{14}$$

Here, ${}^{\eta_1 c}\hat{\theta}_M(\tilde{f}')$ is defined as $({}^{\eta_1}h)_M$ where $h\in\tilde{\mathcal{H}}_{\mathrm{ac}}(M(F_S))$ such that $h_M = {}^c\hat{\theta}_M(\tilde{f}')$ (Lemma 4.1, [7]), and ${}^{\eta_1}h$ is the left-translate of h by $\eta_1\in\mu_n^L$.

Theorem 7.1 (10.2)**.** *Fix an element $M\in\mathcal{L}$ and assume that*

$$I_L^{\mathcal{M}}(\gamma,\tilde{f}) = I_L^{\Sigma}(\gamma,\tilde{f}),\ \tilde{f}\in\mathcal{H}(\tilde{G}(F_S)),$$

for each $L\in\mathcal{L}(M)$ and $\gamma\in L_{\mathrm{comp}}(F_S)$. Then for any $\tilde{f}\in\mathcal{H}_{\mathrm{ac}}(\tilde{G}(F_S))$ and $X\in\mathfrak{a}_{M,S}$, we have

$${}^c\theta_{\tilde{M}}(\tilde{f})' = \sum_{\eta\in\mu_n^M/\mu_n^G} {}^{\eta c}\hat{\theta}_M(\tilde{f}'),$$

$$\theta_{\tilde{M}}(\tilde{f})' = \sum_{\eta\in\mu_n^M/\mu_n^G} {}^{\eta}\hat{\theta}_M(\tilde{f}'),$$

$$I_{\tilde{M}}(\tilde{\rho},X,\tilde{f}) = I_M^{\Sigma}(\tilde{\rho},X,\tilde{f}),\ \tilde{\rho}\in\Sigma(\tilde{M}(F_S)),$$

and

$$I_{\tilde{M}}(\tilde{\pi},X,\tilde{f}) = I_M^{\Sigma}(\tilde{\pi},X,\tilde{f}),\ \tilde{\pi}\in\Pi(\tilde{M}(F_S)).$$

Proof. We shall not prove this theorem in its entirety as its proof is almost identical to the proof of Theorem 10.2, II of [13]. We shall, however, provide a proof of the first and third assertions. We begin with the first assertion. We assume inductively that the theorem holds if G is replaced by $L_1\in\mathcal{L}$ such that $L_1\subsetneqq G$. We also make the inductive assumption that the theorem holds if M is replaced by $L_1\in\mathcal{L}(M)$ such that $L_1\supsetneqq M$. We may assume that $\tilde{f}\in\mathcal{H}(\tilde{G}(F_S))$ as the restriction of any function in $\tilde{\mathcal{H}}_{\mathrm{ac}}(\tilde{G}(F_S))$ to

$$\tilde{G}(F_S) = \{\tilde{\gamma}\in\tilde{G}(F_S) : H_G(\mathbf{p}(\tilde{\gamma})) = Z\},\ Z\in\mathfrak{a}_{G,S},$$

lies in $\mathcal{H}(\tilde{G}(F_S))$. Suppose $\gamma \in M(F_S) \cap G_{\mathrm{oreg}}(F_S)$ and consider the expression

$$\begin{aligned} &{}^cI_M^{\mathcal{M}}(\gamma, \tilde{f}) - {}^cI_M^{\Sigma}(\gamma, \tilde{f}) \\ &= \sum_{L\in\mathcal{L}(M)} \hat{I}_M^{L,\mathcal{M}}(\gamma, {}^c\theta_{\tilde{L}}(\tilde{f})) - \sum_{\eta\in\mu_n^M/\mu_n^L} \hat{I}_M^L(\eta\gamma, \sum_{\eta_1\in\mu_n^L/\mu_n^G} {}^{\eta_1 c}\hat{\theta}_L(\tilde{f}')). \end{aligned}$$

Under the assumption of the theorem and the above induction hypotheses this difference reduces to

$$\begin{aligned} &\hat{I}_M^{M,\mathcal{M}}(\gamma, \theta_{\tilde{M}}(\tilde{f})) - \hat{I}_M^M(\gamma, \sum_{\eta\in\mu_n^M/\mu_n^G} {}^{\eta c}\hat{\theta}_M(\tilde{f}')) \\ &+ \sum_{\{L\in\mathcal{L}(M): L\neq M\}} \hat{I}_M^{L,\mathcal{M}}(\gamma, {}^c\theta_{\tilde{L}}(\tilde{f})) - \hat{I}_M^{\Sigma}(\gamma, {}^c\theta_{\tilde{L}}(\tilde{f})) \\ &= \hat{I}_M^M(\gamma, {}^c\theta_{\tilde{M}}(\tilde{f})' - \sum_{\eta\in\mu_n^M/\mu_n^G} {}^{\eta c}\hat{\theta}_M(\tilde{f}')). \end{aligned}$$

By Lemma 4.4 of [7], ${}^cI_M^{\mathcal{M}}(\cdot, \tilde{f}) - {}^cI_M^{\Sigma}(\cdot, \tilde{f})$ is a function of compact support in the space $M(F_S)$-conjugacy classes of $M(F_S)$. Given $X \in \mathfrak{a}_{M,S}$ and $\gamma \in M(F_S) \cap G_{\mathrm{oreg}}(F_S)$ such that $H_M(\gamma) = X$, then $\hat{I}_M^M(\gamma, \theta_{\tilde{M}}(\tilde{f})' - \sum_{\eta\in\mu_n^M/\mu_n^G} {}^{\eta c}\hat{\theta}_M(\tilde{f}'))$ is the orbital integral of a function defined on

$$M(F_S)^X = \{\gamma_1 \in M(F_S) : H_M(\gamma_1) = X\}.$$

The tempered characters of this function are

$${}^c\theta_{\tilde{M}}(\tilde{f})'(\pi, X) - \sum_{\eta\in\mu_n^M/\mu_n^G} {}^{\eta c}\hat{\theta}_M(\tilde{f}', \pi, X),\ \pi \in \Pi_{\mathrm{temp}}(M(F_S)).$$

Since $\hat{I}_M^M(\gamma, {}^c\theta_{\tilde{M}}(\tilde{f})') - \sum_{\eta\in\mu_n^M/\mu_n^G} {}^{\eta c}\hat{\theta}_M(\tilde{f}'))$ has compact support in γ, and H_M maps $M(F_S) \cap G_{\mathrm{oreg}}(F_S)$ onto $\mathfrak{a}_{M,S}$, this difference has compact support in $X \in \mathfrak{a}_{M,S}$. We shall combine this fact with the classical Paley-Wiener theorem in order to deform contours of integration later in the proof. We proceed by showing that $\sum_{\eta\in\mu_n^M/\mu_n^G} {}^{\eta c}\hat{\theta}_M(\tilde{f}', \pi, X)$ vanishes if $\pi \in \Pi_{\mathrm{temp}}(M(F_S))$ is not metic, that is if π is not in the image of (10) in [30]. It follows easily from the definitions that for $\pi \in \Pi_{\mathrm{temp}}(M(F_S))$,

$$\sum_{\eta\in\mu_n^M/\mu_n^G} {}^{\eta c}\hat{\theta}_M(\tilde{f}', \pi, X) = \begin{cases} n^{\dim(A_M/A_G)}\, {}^c\hat{\theta}_M(\tilde{f}', \pi, X), & \text{if } \pi \text{ is } \mu_n^M\text{-invariant}, \\ 0, & \text{otherwise.} \end{cases}$$

If π is metic, then it is μ_n^M-invariant. Suppose therefore that π is not metic but is μ_n^M-invariant. Define

$${}^c\hat{I}_M(\rho, X, \tilde{f}') = \sum_{L\in\mathcal{L}(M)} \hat{I}_M^L(\rho, X, {}^c\hat{\theta}_L(\tilde{f}')),\ \rho \in \Sigma(M(F_S)),\ X \in \mathfrak{a}_{M,S}.$$

According to Lemma 5.2 of [7] and Proposition 5.4 of [7] there exist constants ω_P and a meromorphic function,

$${}^c\hat{\theta}_M(\tilde{f}', \rho_\lambda) = \int_{\mathfrak{a}_{M,S}} {}^c\hat{\theta}_M(\tilde{f}', \rho_\lambda, X) e^{\lambda(X)} dX,\ \lambda \in \mathfrak{a}_{M,\mathbf{C}}^*,$$

such that

$$(15) \qquad {}^c\hat{I}_M(\rho, X, \tilde{f}') = \lim_\beta \sum_{P\in\mathcal{P}(M)} \omega_P \int_{\varepsilon_P + i\mathfrak{a}_{M,S}^*} \hat{\beta}(\lambda)\, {}^c\hat{\theta}_M(\tilde{f}', \rho_\lambda) e^{-\lambda(X)} d\lambda.$$

Here β is a test function in $C_c^\infty(\mathfrak{a}_{M,S})$ which approaches Dirac measure at the origin and $X \in \mathfrak{a}_{M,S}$ is any point at which the left-hand side is smooth. We take $\rho = \pi_\nu$, where $\nu \in \mathfrak{a}_M^*$ is in general position. We can then deform the contour of integration in the above integral so that

$$\begin{aligned} {}^c\hat{I}_M(\pi_\nu, X, \tilde{f}') &= \lim_\beta \sum_{P \in \mathcal{P}(M)} \omega_P \int_{i\mathfrak{a}_{M,S}^*} \hat{\beta}(\lambda)\, {}^c\hat{\theta}_M(\tilde{f}', \pi_{\nu+\lambda}) e^{-\lambda(X)} d\lambda \\ &= e^{\nu(X)} \lim_\beta \sum_{P \in \mathcal{P}(M)} \omega_P \int_{\nu + i\mathfrak{a}_{M,S}^*} \hat{\beta}(\lambda - \nu)\, {}^c\hat{\theta}_M(\tilde{f}', \pi_\lambda) e^{-\lambda(X)} d\lambda. \end{aligned}$$

Notice that the function

$$\lambda \mapsto \hat{\beta}(\lambda - \nu),$$

is the Fourier transform of the function,

$$X \mapsto e^{-\nu(X)}\beta(X),$$

which approaches Dirac measure at the origin if β does. We may therefore replace $\hat{\beta}(\lambda - \nu)$ with $\hat{\beta}(\lambda)$ in the previous equation to obtain

$$e^{-\nu(X)}\, {}^c\hat{I}_M(\pi_\nu, X, \tilde{f}') = \lim_\beta \sum_{P \in \mathcal{P}(M)} \omega_P \int_{\nu + i\mathfrak{a}_{M,S}^*} \hat{\beta}(\lambda)\, {}^c\hat{\theta}_M(\tilde{f}', \pi_\lambda) e^{-\lambda(X)} d\lambda.$$

By definition

$$\lambda \mapsto {}^c\hat{\theta}_M(\tilde{f}', \pi_\lambda) \tag{16}$$

the Fourier transform of the function

$$X \mapsto {}^c\hat{\theta}_M(\tilde{f}', \pi, X).$$

By our assumptions on π we have

$$X \mapsto {}^c\theta_{\tilde{M}}(\tilde{f})'(\pi, X) - \sum_{\eta \in \mu_n^M/\mu_n^G} {}^{\eta c}\hat{\theta}_M(\tilde{f}', \pi, X) = n^{\dim(A_M/A_G)}\, {}^c\hat{\theta}_M(\tilde{f}', \pi, X).$$

This function has compact support in $\mathfrak{a}_{M,S}$. It follows that (16) is the Fourier transform of a function of compact support and is therefore entire. We may consequently deform the contour of integration in the previous integral to $i\mathfrak{a}_{M,S}^*$. This implies that $e^{-\nu(X)}\, {}^c\hat{I}(\pi_\nu, X, \tilde{f}')$ is independent of $\nu \in \mathfrak{a}_M^*$, for ν in general position, and by a comment on p. 143 of [13] it follows that $e^{-\nu(X)}\, {}^c\hat{I}(\pi_\nu, X, \tilde{f}')$ is independent of $\nu \in \mathfrak{a}_M^*$ without restriction. According to Lemma 4.5 of [7] we have

$$\sum_{P \in \mathcal{P}(M)} \omega_P(X) e^{-\nu_P(X)}\, {}^c\hat{I}_M(\pi_{\nu_P}, X, \tilde{f}') = 0.$$

Here, $\omega_P(X) = \mathrm{vol}(\mathfrak{a}_P \cap B_X)\mathrm{vol}(B_X)^{-1}$, where B_X is a small ball in $\mathfrak{a}_M$ centered at X and ν_P is a point in the chamber $(\mathfrak{a}_P^*)^+$ which is far from the walls. It is obvious from this equation and the independence in ν just mentioned that ${}^c\hat{I}_M(\pi_\nu, X, \tilde{f}') = 0$. Setting $\nu = 0$ and applying Lemma 4.7 of [7] we obtain

$${}^c\hat{\theta}_M(\tilde{f}', \pi, X) = {}^c\hat{I}_M(\pi, X, \tilde{f}') = 0.$$

We have just proved that $\sum_{\eta\in\mu_n^M/\mu_n^G} {}^{\eta c}\hat{\theta}_M(\tilde{f}',\pi,X)$ vanishes if π is not metic. Now suppose that $\tilde{\pi}\in\Pi_{\mathrm{temp}}(\tilde{M}(F_S))$. Define

$$\begin{aligned} {}^cI^{\Sigma}_M(\tilde{\rho},X,\tilde{f}) &= n^{-\dim(A_G)}\,{}^c\hat{I}_M(\tilde{\rho}',X',\tilde{f}') \\ &= \sum_{L\in\mathcal{L}(M)} n^{-\dim(A_G)}\hat{I}^L_M(\tilde{\rho}',X',\,{}^c\hat{\theta}_L(\tilde{f}')) \\ &= \sum_{L\in\mathcal{L}(M)} n^{-\dim(A_L)}\hat{I}^L_M(\tilde{\rho}',X',\sum_{\eta\in\mu_n^L/\mu_n^G}{}^{\eta c}\hat{\theta}_L(\tilde{f}')), \end{aligned}$$

for $\tilde{\rho}\in\Sigma(\tilde{M}(F_S))$ and $X\in\mathfrak{a}_{M,S}$. Equation (15) allows us to write ${}^c\hat{I}^{\Sigma}_M(\tilde{\rho},X,\tilde{f})$ as

$$\begin{aligned} & n^{-\dim(A_G)}\lim_{\beta}\sum_{P\in\mathcal{P}(M)}\omega_P\int_{\varepsilon_P+i\mathfrak{a}^*_{M,S}}\hat{\beta}(\lambda')\,{}^c\hat{\theta}_M(\tilde{f}',(\tilde{\rho}_\lambda)')e^{-\lambda'(X')}d\lambda' \\ &= \lim_{\beta}\sum_{P\in\mathcal{P}(M)}\omega_P\int_{\varepsilon_P+i\mathfrak{a}^*_{M,S}}\hat{\beta}(\lambda)n^{\dim(A_M/A_G)}\,{}^c\hat{\theta}_M(\tilde{f}',(\tilde{\rho}_\lambda)')e^{-\lambda(X)}d\lambda. \end{aligned}$$

Defining

$${}^c\hat{I}_{\tilde{M}}(\tilde{\rho},X,\tilde{f})=\sum_{L\in\mathcal{L}(M)}\hat{I}^{\tilde{L}}_{\tilde{M}}(\tilde{\rho},X,\,{}^c\theta_{\tilde{L}}(\tilde{f})),\ \tilde{\rho}\in\Sigma(\tilde{M}(F_S)),\ X\in\mathfrak{a}_{M,S},$$

we follow the earlier argument to show that ${}^cI_{\tilde{M}}(\tilde{\pi}_\nu,X,\tilde{f})-{}^cI^{\Sigma}_M(\tilde{\pi}_\nu,X,\tilde{f})$ is equal to

$$\begin{aligned} &\lim_{\beta}\sum_{P\in\mathcal{P}(M)}\omega_P \\ &\times\int_{i\mathfrak{a}^*_{M,S}}\hat{\beta}(\lambda)\left({}^c\theta_{\tilde{M}}(\tilde{f})'((\tilde{\pi}_{\nu+\lambda})')-n^{\dim(A_M/A_G)}\,{}^c\hat{\theta}_M(\tilde{f}',(\tilde{\pi}_{\nu+\lambda})')\right)e^{-\lambda(X)}d\lambda, \end{aligned}$$

and that this expression vanishes. It now follows as before that

$${}^c\theta_{\tilde{M}}(\tilde{f},\tilde{\pi},X)-n^{-\dim(A_G)}\hat{\theta}_M(\tilde{f}',\tilde{\pi}',X')=0.$$

This equation may be rewritten as

$$n^{-\dim(A_M)}\,{}^c\theta_{\tilde{M}}(\tilde{f})'(\tilde{\pi}',X')-n^{-\dim(A_G)}\,{}^c\hat{\theta}_M(\tilde{f}',\tilde{\pi}',X')=0,$$

and the first assertion follows. The first assertion implies that

$${}^c\theta_{\tilde{M}}(\tilde{f})'((\tilde{\pi}_\lambda)')=n^{\dim(A_M/A_G)}\,{}^c\hat{\theta}_M(\tilde{f}',(\tilde{\pi}_\lambda)'),\ \tilde{\pi}\in\Pi_{\mathrm{temp}}(\tilde{M}(F_S)),\ \lambda\in\mathfrak{a}^*_{M,\mathbf{C}}.$$

By analytic continuation, this formula holds if $\tilde{\pi}$ is replaced by a standard representation. Recalling (15), this implies that

$${}^cI_{\tilde{M}}(\tilde{\rho},X,\tilde{f})={}^cI^{\Sigma}_M(\tilde{\rho},X,\tilde{f}),\ \tilde{\rho}\in\Sigma(\tilde{M}(F_S)).$$

By the induction assumption and the first assertion we have

$$\begin{aligned} 0 &= {}^cI_{\tilde{M}}(\tilde{\rho},X,\tilde{f})-{}^cI^{\Sigma}_M(\tilde{\rho},X,\tilde{f}) \\ &= \sum_{L\in\mathcal{L}(M)}\hat{I}^{\tilde{L}}_{\tilde{M}}(\tilde{\rho},X,\,{}^c\theta_{\tilde{L}}(\tilde{f}))-\hat{I}^{L,\Sigma}_M(\tilde{\rho},X,\,{}^c\theta_{\tilde{L}}(\tilde{f})) \\ &= I_{\tilde{M}}(\tilde{\rho},X,\tilde{f})-I^{\Sigma}_M(\tilde{\rho},X,\tilde{f}). \end{aligned}$$

This is the third assertion. □

With Theorem 7.1 in place, the proof of of Theorem B (i) follows *mutatis mutandis* from the argument on p. 145 of [13]. We include it here for the sake of continuity. We wish to show that

$$I_M^{\Sigma}(\tilde{\pi}, 0, \tilde{f}) = I_{\tilde{M}}(\tilde{\pi}, 0, \tilde{f}), \ \tilde{\pi} \in \Pi_{\text{unit}}(\tilde{M}(F_S)).$$

The distribution on the left is defined by equation (7), while the distribution on the right is defined by

$$\sum_P \omega_P \sum_L \sum_{\tilde{\rho}} \int_{\varepsilon_P + i\mathfrak{a}^*_{\tilde{M},S}/i\mathfrak{a}^*_{\tilde{L},S}} r_{\tilde{M}}^{\tilde{L}}(\tilde{\pi}_\lambda, \tilde{\rho}_\lambda) I_{\tilde{L}}(\tilde{\rho}_\lambda, h_{\tilde{L}}(X), \tilde{f}) e^{-\lambda(X)} d\lambda,$$

where P, L and $\tilde{\rho}$ are summed over $\mathcal{P}(M)$, $\mathcal{L}(M)$ and $\Sigma(\tilde{M}(F_S))$ respectively. The number $r_{\tilde{M}}^{\tilde{L}}(\tilde{\pi}_\lambda, \tilde{\rho}_\lambda)$ vanishes if $\Delta(\tilde{\pi}, \tilde{\rho})$ vanishes ((6.4), [10]). Comparing the expansions of these two distributions, it is clear that it suffices to prove

$$I_L^{\Sigma}(\tilde{\rho}_\lambda, h_L(X), \tilde{f}) = I_{\tilde{L}}(\tilde{\rho}_\lambda, h_L(X), \tilde{f}), \tag{17}$$

for all $L \in \mathcal{L}(M)$, $X \in \mathfrak{a}_M$, $\lambda \in \mathfrak{a}^*_{M,\mathbf{C}}$ with small real part, and $\tilde{\rho} \in \Sigma(\tilde{M}(F_S))$ such that $\Delta(\tilde{\pi}, \tilde{\rho}) \neq 0$. By using the splitting property for these distributions (Proposition 9.4, [7]) it suffices to prove (17) for $S = \{v\}$. Suppose then that $\tilde{\rho} \in \Sigma(\tilde{M}(F_v))$ and $\Delta(\tilde{\pi}, \tilde{\rho}) \neq 0$. Then the central character of $\tilde{\rho}$ must be unitary. From the definition of standard representations, it follows that $\tilde{\rho}$ is either tempered or induced from a proper parabolic subgroup of M. Suppose first that ρ is tempered. Then by the proof of Lemma 3.1 of [7], we have

$$\begin{aligned} I_L^{\Sigma}(\tilde{\rho}_\lambda, h_L(X), \tilde{f}) &= \begin{cases} 0, & L \neq G, \\ \int_{i\mathfrak{a}^*_{G,v}} \operatorname{tr}\left(\tilde{\rho}_{\lambda+\nu}(\tilde{f})\right) e^{-\nu(h_G(X))} d\nu, & L = G \end{cases} \\ &= I_{\tilde{L}}(\tilde{\rho}_\lambda, h_L(X), \tilde{f}). \end{aligned}$$

Now suppose that $\tilde{\rho} = \tilde{\rho}_1^{\tilde{M}}$, where $\tilde{\rho}_1 \in \Sigma(\tilde{M}_1(F_v))$ and M_1 is a proper Levi subgroup of M. We apply the descent property (Corollary 8.5, [7]) to (17) and find that it suffices to show

$$\hat{I}_{M_1}^{L_1,\Sigma}(\tilde{\rho}_{1,\lambda}, X_1, \tilde{f}_{\tilde{L}_1}) = \hat{I}_{\tilde{M}_1}^{\tilde{L_1}}(\tilde{\rho}_{1,\lambda}, X_1, \tilde{f}_{\tilde{L}_1}), \tag{18}$$

for $X_1 \in \mathfrak{a}_{M_1}$ and $L_1 \in \mathcal{L}(M_1)$ with $L_1 \neq G$. The induction hypothesis of following Theorem A allows us to apply Theorem 7.1, with G replaced by L_1, in order to obtain (18). The proof is now complete. □

8. More on normalizing factors

This section is devoted to the construction of a few additional (G, M) families which are required for the comparison of the spectral sides of the trace formulas. We shall return to the actual comparison of the trace formulas in the following section.

Take $\tilde{\pi} = \bigotimes_v \tilde{\pi}_v$ and $\pi = \bigotimes_v \pi_v$ to be representations in $\Pi(\tilde{M}(\mathbf{A}))$ and $\Pi(M(\mathbf{A}))$ respectively, and suppose that

$$\delta(\pi, \tilde{\pi}) = \prod_v \delta(\pi_v, \tilde{\pi}_v)$$

does not vanish (§5). Define

$$\tilde{r}_{\tilde{P}_1|\tilde{P}_2}(\pi_v, \tilde{\pi}_v) = r_{P_1|P_2}(\pi_v)^{-1} r_{\tilde{P}_1|\tilde{P}_2}(\tilde{\pi}_v), \ P_1, P_2 \in \mathcal{P}(M).$$

If $|n|_v = 1$ and both $\tilde{\pi}_v$ and π_v are unramified representations, then $\tilde{\pi}'_v = \pi_v$ by Corollary 5.1. The normalization of §5 in [30] then implies that

$$\tilde{r}_{\tilde{P}_1|\tilde{P}_2}(\pi_v, \tilde{\pi}_v) = r_{P_1|P_2}(\pi_v)^{-1} r_{\tilde{P}_1|\tilde{P}_2}(\tilde{\pi}_v) = r_{P_1|P_2}(\pi_v)^{-1} r_{P_1|P_2}(\pi_v) = 1.$$

This fact allows us to define the infinite product

$$\tilde{r}_{\tilde{P}_1|\tilde{P}_2}(\pi_{\lambda'}, \tilde{\pi}_\lambda) = \prod_v \tilde{r}_{P_1\tilde{|}P_2}(\pi_{v,\lambda'}, \tilde{\pi}_{v,\lambda}),\ P_1, P_2 \in \mathcal{P}(M),\ \lambda \in \mathfrak{a}^*_{M,\mathbf{C}}.$$

Given $P_1 \in \mathcal{P}(M)$ we define a (G, M) family by

$$\tilde{r}_{\tilde{P}}(\nu, \pi_{\lambda'}, \tilde{\pi}_\lambda, P_1) = \tilde{r}_{\tilde{P}|\tilde{P}_1}(\pi_{\lambda'}, \tilde{\pi}_\lambda)^{-1} \tilde{r}_{\tilde{P}|\tilde{P}_1}(\pi_{\lambda'+\nu'}, \tilde{\pi}_{\lambda+\nu}),$$

where $P \in \mathcal{P}(M)$ and $\nu \in i\mathfrak{a}^*_M$.

Lemma 8.1 (11.3)**.** (a) *Take $\tilde{\pi}$ and π as above. Then for each $L \in \mathcal{L}(M)$, $\tilde{r}^{\tilde{L}}_{\tilde{M}}(\pi_{\lambda'}, \tilde{\pi}_\lambda)$ is independent of P_1 and is also a rational function of the variables $\{\lambda(\alpha^\vee), q_v^{-\lambda(\alpha^\vee)}\}_{v\in S}$, where α runs over the roots of (G, A_M), q_v is the order of the residue field of F_v and S is a finite set of valuations outside of which $\tilde{\pi}$ and π are unramified.*

(b) *Suppose, in addition, that $\tilde{\pi} \in \Pi_{\mathrm{disc}}(\tilde{M}, t)$ and $\pi \in \Pi_{\mathrm{disc}}(M, t)$. Then*

$$\tilde{r}_{\tilde{P}_1|\tilde{P}_2}(\pi_{\lambda'}, \tilde{\pi}_\lambda) = r_{P_1|P_2}(\pi_{\lambda'})^{-1} r_{\tilde{P}_1|\tilde{P}_2}(\tilde{\pi}_\lambda),\ P_1, P_2 \in \mathcal{P}(M).$$

In particular, for each $L \in \mathcal{L}(M)$, the function $\tilde{r}^{\tilde{L}}_{\tilde{M}}(\pi_{\lambda'}, \tilde{\pi}_\lambda)$ is regular for $\lambda \in i\mathfrak{a}^*_M$. Moreover,

$$r^{\tilde{L}}_{\tilde{M}}(\tilde{\pi}_\lambda) = \sum_{L_1 \in \mathcal{L}^L(M)} n^{\dim(A_M/A_{L_1})} r^{L_1}_M(\pi_{\lambda'}) \tilde{r}^{\tilde{L}}_{\tilde{L}_1}(\pi_{\lambda'}, \tilde{\pi}_\lambda).$$

Proof. (a) of the lemma follows from the computations on p. 149 of [13]. Under the hypotheses of part (b), $r_{\tilde{P}_1|\tilde{P}_2}(\tilde{\pi}_\lambda)$ and $r_{P_1|P_2}(\pi_{\lambda'})$ are regular functions in $\lambda \in i\mathfrak{a}^*_M$ (§6, [3]). Thus, if one unravels the definition of $\tilde{r}_{\tilde{P}_1|\tilde{P}_2}(\pi_{\lambda'}, \tilde{\pi}_\lambda)$, one obtains the first equality and the regularity on $i\mathfrak{a}^*_M$. The last equality follows from an application of Lemma 6.5 in [2] to

$$r_{\tilde{P}}(\nu, \tilde{\pi}_\lambda, \tilde{P}_1) = \tilde{r}_{\tilde{P}}(\nu, \pi_{\lambda'}, \tilde{\pi}_\lambda, \tilde{P}_1) r_P(\nu', \pi_{\lambda'}, P_1).$$

□

We may define further (G, M) families along the same lines as $\tilde{r}_{\tilde{M}}(\pi_{\lambda'}, \tilde{\pi}_\lambda)$. If we replace $\tilde{\pi}$ in the above discussion with some $\tilde{\rho} \in \Sigma(\tilde{M}(\mathbf{A}))$ such that $\Delta(\pi, \tilde{\rho}) \neq 0$, we obtain the (G, M) family

$$\tilde{r}_{\tilde{P}}(\nu, \pi_\lambda, \tilde{\rho}_\lambda, \tilde{P}_1) = \tilde{r}_{\tilde{P}|\tilde{P}_1}(\pi_{\lambda'}, \tilde{\rho}_\lambda)^{-1} \tilde{r}_{\tilde{P}|\tilde{P}_1}(\pi_{\lambda'+\nu'}, \tilde{\rho}_{\lambda+\nu}).$$

We define yet another (G, M) family for representations $\pi_1, \pi_2 \in \Pi(M(\mathbf{A}))$ such that $\delta(\pi_i, \tilde{\pi}) \neq 0$ for $i = 1, 2$. Set

$$\tilde{r}_{\tilde{P}}(\nu, \pi_{1,\lambda'}, \pi_{2,\lambda'}, \tilde{P}_1) = \tilde{r}_{\tilde{P}}(\nu, \pi_{1,\lambda'}, \tilde{\pi}, \tilde{P}_1) \tilde{r}_{\tilde{P}}(\nu, \pi_{2,\lambda'}, \tilde{\pi}_\lambda, \tilde{P}_1)^{-1}.$$

This (G, M) family is independent of $\tilde{\pi}$. Lemma 6.5 of [2] applied to this last (G, M) family yields

$$\tilde{r}^{\tilde{L}}_{\tilde{M}}(\pi_{1,\lambda'}, \tilde{\pi}_\lambda) = \sum_{L_1 \in \mathcal{L}^L(M)} \tilde{r}^{\tilde{L_1}}_{\tilde{M}}(\pi_{1,\lambda'}, \pi_{2,\lambda'}) \tilde{r}^{\tilde{L}}_{\tilde{L}_1}(\pi_{2,\lambda'}, \tilde{\pi}_\lambda). \tag{19}$$

For arbitrary $\pi \in \Pi(M(\mathbf{A}))$, $\tilde{\rho} = \bigotimes_v \tilde{\rho}_v \in \Sigma(\tilde{M}(\mathbf{A}))$ and $\tilde{\pi} \in \Pi(\tilde{M}(\mathbf{A}))$. Set

$$r_{\tilde{P}}(\tilde{\pi}_\lambda, \tilde{\rho}_\lambda) = \Delta(\tilde{\pi}, \tilde{\rho})\tilde{r}_{\tilde{P}}(\tilde{\pi}_\lambda, \tilde{\rho}_\lambda),$$

$$r_{\tilde{P}}(\nu, \pi_{\lambda'}, \tilde{\pi}_\lambda) = \delta(\pi, \tilde{\pi})\tilde{r}_{\tilde{P}}(\nu, \pi_{\lambda'}, \tilde{\pi}_\lambda),$$

and

$$r_{\tilde{P}}(\nu, \pi_{\lambda'}, \tilde{\rho}_\lambda) = \Delta(\pi, \tilde{\rho})\tilde{r}_{\tilde{P}}(\nu, \pi_{\lambda'}, \tilde{\rho}_\lambda).$$

In the following lemma we compare the final (G, M) family in this list to another, keeping in mind the normalization in §5 of [30].

Lemma 8.2 (11.4)**.** *For each $L \in \mathcal{L}(M)$ we have*

$$r^{\tilde{L}}_{\tilde{M}}(\pi_{\lambda'}, \tilde{\rho}_\lambda) = n^{\dim(A_M/A_L)} r^L_M(\pi_{\lambda'}, \tilde{\rho}'_{\lambda'}).$$

Proof. According to Lemma 6.2 of [2], $r^{\tilde{L}}_{\tilde{M}}(\pi_{\lambda'}, \tilde{\rho}_\lambda)$ is equal to

$$\Delta(\pi, \tilde{\rho}) \lim_{\nu \to 0} \sum_{P \in \mathcal{P}^L(M)} \theta^L_P(\nu)^{-1} r^L_{P|P_1}(\pi_{\lambda'}) r^L_{P|P_1}(\pi_{\lambda'+\nu'})^{-1} r^{\tilde{L}}_{\tilde{P}|\tilde{P}_1}(\tilde{\rho}_\lambda)^{-1} r^{\tilde{L}}_{\tilde{P}|\tilde{P}_1}(\tilde{\rho}_{\lambda+\nu}),$$

where

$$\theta^L_P(\nu) = \operatorname{vol}\left(\mathfrak{a}^L_M/\mathbf{Z}(\Delta^L_P)^\vee\right)^{-1} \prod_{\alpha \in \Delta^L_P} \nu(\alpha^\vee),$$

and Δ^L_P are the simple roots of (P, A_M). (We apologize to the reader for the similarity in notation of θ^L_P with the maps of §7.) Making the change of variables from ν to ν' in the limit and noting that $\theta^L_P(n^{-1}\nu) = n^{-\dim(A_M/A_L)}\theta^L_P$, this expression can be rewritten as the product of $\Delta(\pi, \tilde{\rho}')n^{\dim(A_M/A_L)}$ with

$$\lim_{\nu \to 0} \sum_{P \in \mathcal{P}^L(M)} \theta^L_P(\nu)^{-1} r^L_{P|P_1}(\pi_{\lambda'}) r^L_{P|P_1}(\pi_{\lambda'+\nu})^{-1} r^L_{P|P_1}(\tilde{\rho}'_{\lambda'})^{-1} r^L_{P|P_1}(\tilde{\rho}'_{\lambda'+\nu}).$$

This is by definition equal to $n^{\dim(A_M/A_L)} r^L_M(\pi_{\lambda'}, \tilde{\rho}'_{\lambda'})$. □

The other (G, M) families defined in this section satisfy versions of Lemma 8.2 as well. These versions are proven similarly.

9. A formula for I^Σ_t

The object of this section is to express the spectral side

$$\sum_{t \geq 0} I^\Sigma_t(\tilde{f}) = \sum_{t \geq 0} n^{-1} I_t(\tilde{f}')$$

in a manner that is compatible with the spectral expansion (8). This amounts to expressing the spectral side for $G(\mathbf{A})$ in terms of the global datum $a^{M,\Sigma}$ and the set of representations $\Pi^\Sigma(M, t)$.

The integrals in $I^\Sigma_t(\tilde{f})$ are taken over spaces of representations which, on the face of it, are not metic. We need the following spectral vanishing property to ensure that these non-metic representations do not appear in $I^\Sigma(\tilde{f})$.

Proposition 9.1. *Suppose $\rho \in \Sigma(M(F_S))$. Then*

$$\hat{I}_M(\rho, X, \tilde{f}') = 0, \ \ \tilde{f} \in \mathcal{H}(\tilde{G}(F_S)),$$

unless ρ is metic.

Proof. Recall that ρ is metic if there exists $\tilde{f}_1 \in \mathcal{H}(\tilde{M}(F_S))$ such that $\tilde{f}_1'(\rho) \neq 0$. Therefore, the proposition follows in the case $M = G$. We assume inductively that the proposition holds if G is replaced by $L \in \mathcal{L}$ with $L \subsetneqq G$. By the splitting property, Proposition 9.4 [7] (and the remark immediately following it), it suffices to prove the proposition for $S = \{v\}$. Suppose that ρ is not metic. We may write $\rho = \pi_\nu^M$, where $M_1 \in \mathcal{L}^M$, $\pi \in \Pi_{\mathrm{temp}}(M_1(F_v))$ and $\nu \in \mathfrak{a}_{M_1}^*$. Since the local metaplectic correspondence commutes with induction, it is clear that π_ν is not metic. According to Corollary 8.5 of [7], the Fourier transform,

$$\int_{i\mathfrak{a}_{M_1,v}^*/i\mathfrak{a}_{M,v}^*} \hat{I}_M(\rho_\lambda, X, \tilde{f}')e^{-\lambda(X_1)}d\lambda,$$

(where the projection $h_M(X_1)$ of $X_1 \in \mathfrak{a}_{M_1,v}$ onto $\mathfrak{a}_{M,v}$ is $X \in \mathfrak{a}_{M,v}$,) is equal to

$$\sum_{L\in\mathcal{L}(M_1)} d_{M_1}^G(M,L)\hat{I}_{M_1}^L(\pi_\nu, X, \tilde{f}').$$

If $M_1 \subsetneqq M$, then $d_{M_1}^G(M,G) = 0$ and the right-hand side vanishes by the induction assumption. Taking the inverse Fourier transform then implies that $\hat{I}_M(\rho_\lambda, X, \tilde{f}')$ vanishes as well. We may therefore suppose that $M_1 = M$, in which case $\rho = \pi_\nu$. If v is nonarchimedean, the infinitesimal character of π_ν is given by a finite set of cuspidal pairs $(\rho_1, M_1), \dots, (\rho_s, M_s)$ where $M_1, \dots, M_s \in \mathcal{L}^M$ and $\rho_1 \in \Sigma(M_1(F_v)), \dots, \rho_s \in \Sigma(M_s(F_v))$. If v is an archimedean valuation, then the infinitesimal character of π_ν also determines a set of cuspidal pairs $(\rho_1, M_0), \dots, (\rho_s, M_0)$ as above. It follows from the remark immediately preceding Theorem 27.3 of [18] and Theorem 26 of [18] in the nonarchimedean case; and §3.2 of [30] in the archimedean case, that these supercuspidal (and hence elliptic) representations are not metic. The discussion of §7 in [11] tells us that for $\nu \in \mathfrak{a}_M^*$ in general position we may express $\hat{I}_M(\pi_\nu, X, \tilde{f}')e^{-\nu(X)}$ as

$$\sum_{\{L_1, L\in\mathcal{L}(M):L_1\supset L\supsetneqq M\}} \int_{\nu_L + i\mathfrak{a}_{L,v}^*/i\mathfrak{a}_{M_1,v}^*} \sum_{i=1}^{t}\sum_{j=1}^{s} \left(\Delta_{ij}(\pi)\Phi\right)(\lambda_i, \rho_j)d\lambda.$$

Here, $\Delta_{ij}(\pi)$ is a differential operator on $\mathfrak{a}_{M,\mathbf{C}}^* \times \mathfrak{a}_{M_j,\mathbf{C}}^*$ and $\Phi(\lambda_i, \rho_j)$ is equal to the product of

$$\lim_{\nu_1\to 0} \sum_{Q\in\mathcal{P}^{L_1}(L)} \tilde{r}_{Q|Q_0}(\pi_{\lambda_i+\lambda}^L, \rho_{j,\lambda})^{-1}\tilde{r}_{Q|Q_0}(\pi_{\lambda_i+\lambda+\nu_1}^L, \rho_{j,\lambda+\nu_1})\theta_Q^{L_1}(\nu_1)^{-1}$$

with $\hat{I}_{L_1}(\rho_{j,\lambda}^L, h_{L_1}(X), \tilde{f}')$. It is obvious from this equation that $\hat{I}_M(\pi_\nu, X, \tilde{f}')$ vanishes if

$$\hat{I}_{L_1}(\rho_{j,\lambda}^L, h_{L_1}(X), \tilde{f}') = 0,\ L_1 \supset L \supsetneqq M.$$

Since $L_1 \supsetneqq M$ and $\rho_{j,\lambda}$ is not metic, the earlier descent argument shows that this is indeed true. We have thus succeeded in showing that $\hat{I}_M(\pi_\nu, X, \tilde{f}') = 0$ if $\nu \in \mathfrak{a}_M^*$ is in general position. However, the mean value property (Lemma 3.2, [7]) expresses the value of $\hat{I}_M(\pi_\nu, X, \tilde{f}')$ at any $\nu \in \mathfrak{a}_M^*$ in terms of nearby points. In consequence, $\hat{I}_M(\pi_\nu, X, \tilde{f}')$ vanishes for all $\nu \in \mathfrak{a}_M^*$. □

The following lemma gives an expansion of the local distributions occurring in $I_t^\Sigma(\tilde{f})$ in terms of the distributions $I_M^\Sigma(\tilde{\pi})$.

Lemma 9.1 (12.1)**.** *Suppose that* $\pi \in \Pi_{\mathrm{unit}}(M(\mathbf{A})^1)$ *and* $\tilde{f} \in \mathcal{H}(\tilde{G}(F_S))$. *Then* $n^{-1}\hat{I}_M(\pi, \tilde{f}')$ *is equal to*

$$\sum_{L\in\mathcal{L}(M)} \sum_{\tilde{\pi}\in\Pi(\tilde{M}(\mathbf{A})^1)} \int_{\varepsilon_M+i\mathfrak{a}_M^*/i\mathfrak{a}_L^*} r_{\tilde{M}}^{\tilde{L}}(\pi_{\lambda'}, \tilde{\pi}_\lambda) I_L^{\Sigma}(\tilde{\pi}_\lambda, \tilde{f}) d\lambda. \tag{20}$$

Proof. This proof is almost identical to the proof of Lemma 12.1, II of [13]. It is included so that the reader may feel a sense of continuity. Any statements which seem unjustified may be compared to the analogous statements in Lemma 12.1, where the details are given. To begin, relabel the summation index L in (20) by L_1. We then replace $I_{L_1}^{\Sigma}(\tilde{\pi}_\lambda, \tilde{f})$ with expression (7),

$$\sum_{Q\in\mathcal{P}(L_1)} \omega_Q \sum_{L\in\mathcal{L}(L_1)} \sum_{\tilde{\rho}\in\Pi(\tilde{M}(\mathbf{A})^1)} \int_{\varepsilon_Q+i\mathfrak{a}_{L_1}^*/i\mathfrak{a}_L^*} r_{\tilde{L}_1}^{\tilde{L}}(\tilde{\pi}_{\lambda+\mu}, \tilde{\rho}_{\lambda+\mu}) I_L^{\Sigma}(\tilde{\rho}_{\lambda+\mu}, \tilde{f}) d\mu.$$

We deform the contour of integration in μ so that (20) is equal to the sum over $L_1, L \in \mathcal{L}(M)$, with $L_1 \subset L$, of

$$\sum_{\tilde{\pi}\in\Pi(\tilde{M}(\mathbf{A})^1)} \sum_{\tilde{\rho}\in\Sigma(\tilde{M}(\mathbf{A})^1)} \int_{\varepsilon_M+i\mathfrak{a}_M^*/i\mathfrak{a}_L^*} r_{\tilde{M}}^{\tilde{L}}(\pi_{\lambda'}, \tilde{\pi}_\lambda) r_{\tilde{L}_1}^{\tilde{L}}(\tilde{\pi}_\lambda, \tilde{\rho}_\lambda) I_L^{\Sigma}(\tilde{\rho}_\lambda, \tilde{f}) d\lambda.$$

Taking the sums over L_1 and $\tilde{\pi}$ inside the integral we find that

$$\begin{aligned}
&\sum_{\tilde{\pi}\in\Pi(\tilde{M}(\mathbf{A})^1)} \sum_{L_1\in\mathcal{L}^L(M)} r_{\tilde{M}}^{\tilde{L}_1}(\pi_{\lambda'}, \tilde{\pi}_\lambda) r_{\tilde{L}_1}^{\tilde{L}}(\tilde{\pi}_\lambda, \tilde{\rho}_\lambda) \\
&= \tilde{r}_{\tilde{M}}^{\tilde{L}}(\pi_{\lambda'}, \tilde{\rho}_\lambda) \sum_{\tilde{\pi}\in\Pi(\tilde{M}(\mathbf{A})^1)} \delta(\pi, \tilde{\pi}) \Delta(\tilde{\pi}, \tilde{\rho}) \\
&= \tilde{r}_{\tilde{M}}^{\tilde{L}}(\pi_{\lambda'}, \tilde{\rho}_\lambda) \sum_{\tilde{\rho}_1\in\Sigma(\tilde{M}(\mathbf{A})^1)} \Delta(\pi, \tilde{\rho}_1) \sum_{\tilde{\pi}\in\Pi(\tilde{M}(\mathbf{A})^1)} \Gamma(\tilde{\rho}_1, \tilde{\pi}) \Delta(\tilde{\pi}, \tilde{\rho}) \\
&= \tilde{r}_{\tilde{M}}^{\tilde{L}}(\pi_{\lambda'}, \tilde{\rho}_\lambda) \Delta(\pi, \tilde{\rho}) \\
&= r_{\tilde{M}}^{\tilde{L}}(\pi_{\lambda'}, \tilde{\rho}_\lambda).
\end{aligned}$$

Expansion (20) is therefore equal to

$$\sum_{L\in\mathcal{L}(M)} \sum_{\tilde{\rho}\in\Sigma(\tilde{M}(\mathbf{A})^1)} \int_{\varepsilon_M+i\mathfrak{a}_M^*/i\mathfrak{a}_L^*} r_{\tilde{M}}^{\tilde{L}}(\pi_{\lambda'}, \tilde{\rho}_\lambda) I_L^{\Sigma}(\tilde{\rho}_\lambda, \tilde{f}) d\lambda. \tag{21}$$

By Lemma 8.2 and the spectral vanishing property, Proposition 9.1, we have

$$\begin{aligned}
&\sum_{\tilde{\rho}\in\Sigma(\tilde{M}(\mathbf{A})^1)} r_{\tilde{M}}^{\tilde{L}}(\pi_{\lambda'}, \tilde{\rho}_\lambda) I_L^{\Sigma}(\tilde{\rho}_\lambda, \tilde{f}) \\
&\quad= n^{\dim(A_M/A_L)} \sum_{\tilde{\rho}\in\Sigma(M(\mathbf{A})^1)} r_M^L(\pi_{\lambda'}, \tilde{\rho}'_{\lambda'}) \left(n^{-1}\hat{I}_L(\tilde{\rho}'_{\lambda'}, \tilde{f}')\right) \\
&\quad= n^{\dim(A_M/A_L)} \sum_{\rho\in\Sigma(M(\mathbf{A})^1)} r_M^L(\pi_{\lambda'}, \rho_{\lambda'}) \left(n^{-1}\hat{I}_L(\rho_{\lambda'}, \tilde{f}')\right).
\end{aligned}$$

Substituting back into (21) and noting that $d\lambda' = n^{\dim(A_M/A_L)} d\lambda$, we obtain

$$\sum_{L\in\mathcal{L}(M)} \sum_{\rho\in\Sigma(M(\mathbf{A})^1)} \int_{\varepsilon_M+i\mathfrak{a}_M^*/i\mathfrak{a}_L^*} r_M^L(\pi_\lambda, \rho_\lambda) \left(n^{-1}\hat{I}_L(\rho_\lambda, \tilde{f}')\right) d\lambda.$$

Since π is unitary, this expression is equal to $n^{-1}\hat{I}_M(\pi, \tilde{f}')$. $\square$

Proposition 9.2 (12.2). *Suppose that* $t \geq 0$ *and* $\tilde{f} \in \mathcal{H}(G(\mathbf{A}))$. *Then*

$$I_t^{\Sigma}(\tilde{f}) = \sum_{M \in \mathcal{L}} |W_0^M||W_0^G|^{-1} \int_{\Pi^{\Sigma}(M,t)} a^{M,\Sigma}(\tilde{\pi}) I_M^{\Sigma}(\tilde{\pi}, \tilde{f}) d\pi. \tag{22}$$

Proof. From §5 (cf. (9)) we know that $I_t^{\Sigma}(\tilde{f})$ equals

$$\sum_{\{M_1, M \in \mathcal{L} : M \supset M_1\}} |W_0^{M_1}||W_0^G|^{-1} \times \sum_{\pi \in \Pi_{\mathrm{disc}}(M_1,t)} \int_{i\mathfrak{a}_{M_1}^*/i\mathfrak{a}_M^*} a_{\mathrm{disc}}^M(\pi) r_{M_1}^M(\pi_{\lambda'}) n^{-1} \hat{I}_M(\pi_{\lambda'}, \tilde{f}') d\lambda'.$$

Substituting expression (20), deforming the contour of integration appropriately, and noting that $d\lambda' = n^{\dim(A_{M_1}/A_M)} d\lambda$, we find that $I_t^{\Sigma}(\tilde{f})$ equals

$$\sum_{\{M_1, M, L \in \mathcal{L} : M_1 \subset M \subset L\}} \sum_{\tilde{\pi}_1 \in \Pi(\tilde{M}_1(\mathbf{A})^1, t)} |W_0^{M_1}||W_0^G|^{-1} n^{\dim(A_{M_1}/A_M)}$$

$$\times \int_{\varepsilon_M + i\mathfrak{a}_{M_1}^*/i\mathfrak{a}_L^*} \sum_{\pi \in \Pi_{\mathrm{disc}}(M_1,t)} a_{\mathrm{disc}}^{M_1}(\pi) r_{M_1}^M(\pi_{\lambda'}) r_{\tilde{M}}^{\tilde{L}}(\pi_{\lambda'}, \tilde{\pi}_{1,\lambda}) I_L^{\Sigma}(\tilde{\pi}_{1,\lambda}, f) d\lambda. \tag{23}$$

The term $r_{\tilde{M}}^{\tilde{L}}(\pi_{\lambda'}, \tilde{\pi}_{1,\lambda})$ in the above sum vanishes unless $\delta(\pi, \tilde{\pi}_1) \neq 0$. Fix some $\pi_1 \in \Pi_{\mathrm{disc}}(M_1, t)$ such that $\delta(\pi_1, \tilde{\pi}_1) \neq 0$. Then for any other $\pi \in \Pi_{\mathrm{disc}}(M_1, t)$ with $\delta(\pi, \tilde{\pi}_1) \neq 0$ we may write

$$\begin{aligned} n^{\dim(A_{M_1}/A_M)} r_{M_1}^M(\pi_{\lambda'}) &= \sum_{\{L_1 : M_1 \subset L_1 \subset M\}} n^{\dim(A_{M_1}/A_{L_1})} r_{M_1}^{L_1}(\pi_{1,\lambda'}) n^{\dim(A_{L_1}/A_M)} \tilde{r}_{L_1}^M(\pi_{1,\lambda'}, \pi_{\lambda'}) \\ &= \sum_{\{L_1 : M_1 \subset L_1 \subset M\}} n^{\dim(A_{M_1}/A_{L_1})} r_{M_1}^{L_1}(\pi_{1,\lambda'}) \tilde{r}_{\tilde{L}_1}^{\tilde{M}}(\pi_{1,\lambda'}, \pi_{\lambda'}). \end{aligned}$$

We substitute this expression into (23) and deform the contour of integration from $\varepsilon_M + i\mathfrak{a}_{M_1}^*/i\mathfrak{a}_L^*$ to $\varepsilon_{L_1} + i\mathfrak{a}_{M_1}^*/i\mathfrak{a}_L^*$ for some small regular point ε_{L_1} in $\mathfrak{a}_{L_1}^*$. We then bring the sum over M inside the integral. Notice that

$$\sum_{\{M : L_1 \subset M \subset L\}} \tilde{r}_{\tilde{L}_1}^{\tilde{M}}(\pi_{1,\lambda'}, \pi_{\lambda'}) r_{\tilde{M}}^{\tilde{L}}(\pi_{\lambda'}, \tilde{\pi}_{1,\lambda}) = \delta(\pi, \tilde{\pi}_1) \tilde{r}_{\tilde{L}_1}^{\tilde{L}}(\pi_{1,\lambda'}, \tilde{\pi}_{1,\lambda})$$

by equation (19). Observe also that

$$\sum_{\pi \in \Pi_{\mathrm{disc}}(M_1,t)} a_{\mathrm{disc}}^{M_1}(\pi) \delta(\pi, \tilde{\pi}_1) = a_{\mathrm{disc}}^{M_1,\Sigma}(\tilde{\pi}_1).$$

Now (23) may be written as

$$\sum_{\{M_1, L \in \mathcal{L} : M_1 \subset L\}} |W_0^{M_1}||W_0^G|^{-1} \sum_{\tilde{\pi}_1 \in \Pi(\tilde{M}_1(\mathbf{A})^1, t)} \sum_{\{L_1 : M_1 \subset L_1 \subset L\}} n^{\dim(A_{M_1}/A_{L_1})}$$

$$\times \int_{\varepsilon_{L_1} + i\mathfrak{a}_{M_1}^*/i\mathfrak{a}_L^*} a_{\mathrm{disc}}^{M_1,\Sigma}(\tilde{\pi}_1) r_{M_1}^{L_1}(\pi_{1,\lambda'}) \tilde{r}_{\tilde{L}_1}^{\tilde{L}}(\pi_{1,\lambda'}, \tilde{\pi}_{1,\lambda}) I_L^{\Sigma}(\tilde{\pi}_{1,\lambda}, \tilde{f}) d\lambda. \tag{24}$$

The summand indexed by $M_1 = G$ in (24) reduces to

$$\sum_{\tilde{\pi}_1 \in \Pi^{\Sigma}_{\mathrm{disc}}(\tilde{G},t)} a^{\Sigma}_{\mathrm{disc}}(\tilde{\pi}_1) I^{\Sigma}_{G}(\tilde{\pi}_1, \tilde{f}). \tag{25}$$

If $M_1 \neq G$, then the induction hypothesis stated after the definition of $a^{M_1,\Sigma}_{\mathrm{disc}}$ implies that

$$a^{M_1,\Sigma}_{\mathrm{disc}}(\tilde{\pi}_1) = a^{\tilde{M}_1}_{\mathrm{disc}}(\tilde{\pi}_1),\ \tilde{\pi}_1 \in \Pi(\tilde{M}_1(\mathbf{A})^1, t).$$

It is immediate from this equation that $a^{M_1,\Sigma}_{\mathrm{disc}}(\tilde{\pi}_1)$ vanishes unless $\tilde{\pi}_1 \in \Pi_{\mathrm{disc}}(\tilde{M}_1, t)$. Thus, by a variant of Lemma 8.1, the integrand of (24) is analytic for λ near $i\mathfrak{a}^*_{M_1}$, and we may deform the contour of integration from $\varepsilon_{L_1} + i\mathfrak{a}^*_{M_1}/i\mathfrak{a}^*_L$ to $i\mathfrak{a}^*_{M_1}/i\mathfrak{a}^*_L$. This allows us to take the sum over L_1 inside the integral. It is a simple exercise in (G,M) families (cf. proof of Lemma 8.1 (b)) to show that

$$\sum_{\{L_1 : M_1 \subset L_1 \subset L\}} n^{\dim(A_{M_1}/A_{L_1})} r^{L_1}_{M_1}(\pi_{1,\lambda'}) \tilde{r}^{\tilde{L}}_{\tilde{L}_1}(\pi_{1,\lambda'}, \tilde{\pi}_{1,\lambda}) = r^{\tilde{L}}_{\tilde{M}_1}(\tilde{\pi}_{1,\lambda}).$$

Therefore the summand indexed by $M_1 \neq G$ in (24) is equal to

$$|W^{M_1}_0||W^G_0|^{-1} \sum_{L \in \mathcal{L}(M_1)} \sum_{\tilde{\pi}_1 \in \Pi_{\mathrm{disc}}(\tilde{M}_1,t)} \int_{i\mathfrak{a}^*_{M_1}/i\mathfrak{a}^*_L} a^{M_1,\Sigma}_{\mathrm{disc}}(\tilde{\pi}_1) r^{\tilde{L}}_{\tilde{M}_1}(\tilde{\pi}_{1,\lambda}) I^{\Sigma}_{L}(\tilde{\pi}_{1,\lambda}, \tilde{f}) d\lambda.$$

Combining this expression with (25) we obtain

$$I^{\Sigma}_t(\tilde{f}) = \sum_{L \in \mathcal{L}} |W^L_0||W^G_0|^{-1} \int_{\Pi^{\Sigma}(\tilde{L},t)} a^{L,\Sigma}(\tilde{\pi}) I^{\Sigma}_L(\tilde{\pi}, \tilde{f}) d\tilde{\pi}.$$

□

We can now apply Proposition 9.2 to obtain a striking comparison between the spectral sides of the trace formulas.

Lemma 9.2 (12.3)**.** *Suppose that* $t \geq 0$ *and* $\tilde{f} \in \mathcal{H}(\tilde{G}(\mathbf{A}))$. *Then*

$$I^{\Sigma}_t(\tilde{f}) - I_t(\tilde{f}) = \sum_{\tilde{\pi} \in \Pi(\tilde{G}(\mathbf{A})^1, t)} \left(a^{\Sigma}_{\mathrm{disc}}(\tilde{\pi}) - a_{\mathrm{disc}}(\tilde{\pi})\right) \operatorname{tr}(\tilde{\pi}(\tilde{f}^1)),$$

where $\tilde{f}^1$ *is the restriction of* $\tilde{f}$ *to* $\tilde{G}(\mathbf{A})^1$.

Proof. This proof is almost identical to Lemma 12.3, II of [13] and is included solely for the sake of continuity. Consider the difference of (22) and

$$I_t(\tilde{f}) = \sum_{M \in \mathcal{L}} |W^M_0||W^G_0|^{-1} \int_{\Pi(\tilde{M},t)} a^M(\tilde{\pi}) I_{\tilde{M}}(\tilde{\pi}, \tilde{f}) d\tilde{\pi}.$$

If $M_1 \in \mathcal{L}$ and $M_1 \subsetneqq M \subset G$, then the induction hypothesis of §6 implies that

$$a^{M,\Sigma}(\tilde{\pi}) = a^{\tilde{M}}(\tilde{\pi}),\ \tilde{\pi} \in \Pi_{\tilde{M}_1}(\tilde{M}, t),$$

and $\Pi^{\Sigma}_{\tilde{M}_1}(\tilde{M}, t) = \Pi_{\tilde{M}_1}(\tilde{M}, t)$. If $\tilde{\pi}$ is not unitary, both $a^{M,\Sigma}(\tilde{\pi})$ and $a^M(\tilde{\pi})$ vanish. When $\tilde{\pi}$ is unitary, we know from the discussion at the end of §7 that

$$I^{\Sigma}_M(\tilde{\pi}, \tilde{f}) = I_{\tilde{M}}(\tilde{\pi}, \tilde{f}).$$

Therefore, the only terms which remain in the difference, $I_t^{\Sigma}(\tilde f) - I_t(\tilde f)$, are those parametrized by $M_1 = M = G$. The lemma now follows from

$$I_G^{\Sigma}(\tilde\pi, \tilde f) = \mathrm{tr}(\tilde\pi(\tilde f^1)) = I_{\tilde G}(\tilde\pi, \tilde f),\ \tilde\pi \in \Pi_{\mathrm{disc}}(\tilde G, t).$$

□

10. The map ε_M

Having simplified the comparison of the spectral sides, we do the same for the geometric sides. We lighten the burden of this task by adding yet another induction hypothesis. From this point on, fix $M \in \mathcal{L}$ such that $M \neq G$. The additional induction hypothesis is that

$$I_L^{\mathcal{M}}(\gamma, \tilde f) = I_L^{\Sigma}(\gamma, \tilde f),\ \gamma \in L_{\mathrm{comp}}(F_S),\ \tilde f \in \mathcal{H}(\tilde G(F_S)),$$

where $L \in \mathcal{L}(M)$ with $L \neq M$.

The following proposition brings us closer to the proof of Theorem A (i) in that it expresses the difference of our invariant distributions in terms of orbital integrals, which are easier to manage.

Proposition 10.1 (13.2). *There exists a unique map*

$$\varepsilon_M : \mathcal{H}(\tilde G(F_S))^0 \to \mathcal{I}_{\mathrm{ac}}(M(F_S)),$$

such that

$$I_M^{\mathcal{M}}(\gamma, \tilde f) - I_M^{\Sigma}(\gamma, \tilde f) = \hat I_M^M(\gamma, \varepsilon_M(\tilde f)),\ \gamma \in M_{\mathrm{comp}}(F_S),\ \tilde f \in \mathcal{H}(\tilde G(F_S))^0.$$

The map satisfies the splitting property

$$\varepsilon_M(\tilde f) = \varepsilon_M(\tilde f_1)\tilde f'_{2,M} + \tilde f'_{1,M}\varepsilon_M(\tilde f_2),$$

where S is the disjoint union of S_1 and S_2 (each satisfying the closure property) and $\tilde f = \tilde f_1 \tilde f_2$ is a corresponding decomposition.

Proof. If ε_M satisfies the first property of the proposition, then the earlier splitting properties ((23) and Proposition 6.2, [30]) imply that the splitting property for ε_M. Therefore, we only need to show that the map ε_M exists. In other words, we must show that

$$\varepsilon_M(\gamma, \tilde f) = I_M^{\mathcal{M}}(\gamma, \tilde f) - I_M^{\Sigma}(\gamma, \tilde f),\ \gamma \in M_{\mathrm{comp}}(F_S),$$

is an orbital integral at γ of a function in $\mathcal{H}_{\mathrm{ac}}(M(F_S))$. The splitting properties ((23) and Proposition 6.2, [30]) allow us to restrict our proof to the case that S is comprised of a single valuation v. Also, Lemma 3.3 allows us to restrict to the case that $\gamma \in M(F_v) \cap G_{\mathrm{oreg}}(F_v)$. Suppose first that v is archimedean. By our assumptions on F, this means that $F_v = \mathbf{C}$. We may therefore apply Jordan canonical form to conclude that every element of $M(\mathbf{C}) \cap G_{\mathrm{oreg}}(\mathbf{C})$ is $G(\mathbf{C})$-conjugate to an element in $M_0(\mathbf{C})$. If $M \supsetneq M_0$, then Lemma 2.1 implies that $\varepsilon_M(\gamma, \tilde f)$ vanishes. In other words, if $M \supsetneq M_0$, then

$$\varepsilon_M(\tilde f) = 0,\ \tilde f \in \mathcal{H}(\tilde G(\mathbf{C})). \tag{26}$$

We therefore need only consider the case $M = M_0$. We proceed by showing that, for fixed $\tilde f \in \mathcal{H}(\tilde G(\mathbf{C}))$, the function,

$$\gamma \mapsto \varepsilon_{M_0}(\gamma, \tilde f),\ \gamma \in M_0(\mathbf{C}) \cap G_{\mathrm{oreg}}(\mathbf{C}),$$

extends to a smooth function of $M_0(\mathbf{C})$. Suppose γ_0 is an arbitrary element of $M_0(\mathbf{C})$. Then, by the definition of $I^{\mathcal{M}}_{M_0}(\gamma_0,\tilde{f}) - I^{\Sigma}_{M_0}(\gamma_0,\tilde{f})$ (§§6 and 7, [30]) and Lemma 3.2, the limit,

$$\lim_{\{a\to 1: a\in A_{M_0,\mathrm{reg}}(\mathbf{C})\}} \sum_{L\in\mathcal{L}} r^L_{M_0}(\gamma_0^n, a)\left(I^{\mathcal{M}}_L(a\gamma_0,\tilde{f}) - I^{\Sigma}_L(a\gamma_0,\tilde{f})\right) = \lim_{\{a\to 1: a\in A_{M_0,\mathrm{reg}}(\mathbf{C})\}} \varepsilon_{M_0}(a\gamma_0,\tilde{f}),$$

exists. In consequence, $\varepsilon_{M_0}(\cdot,\tilde{f})$ extends to a continuous function on $M_0(\mathbf{C})$. To see that this function is smooth, suppose that z is a differential operator in $\mathcal{Z}_{\mathbf{C}}$, the center of the universal enveloping algebra of the complexified Lie algebra of $G(\mathbf{C})$. This operator passes to a differential operator on $\tilde{G}(\mathbf{C}) \cong G(\mathbf{C}) \times \mu_n$ by virtue of its action on $G(\mathbf{C})$. It follows from Theorem 1 of [33] and results on p. 9 of [33] that

$$\begin{aligned}\varepsilon_{M_0}(\gamma_0, z\tilde{f}) &= \lim_{\{a\to 1: a\in A_{M_0,\mathrm{reg}}(\mathbf{C})\}} \varepsilon_{M_0}(a\gamma_0, z\tilde{f}) \\ &= \lim_{\{a\to 1: a\in A_{M_0,\mathrm{reg}}(\mathbf{C})\}} z'_{M_0}\varepsilon_{M_0}(a\gamma_0,\tilde{f}).\end{aligned}$$

Here, z_{M_0} is the image of z under the Harish-Chandra isomorphism and

$$z_{M_0} \mapsto z'_{M_0}$$

is an algebra automorphism (§3, [33]). We can choose a set of generators, $z_1,\dots,z_{2r}$, of $\mathcal{Z}_{\mathbf{C}}$ and apply a well-known argument involving the fundamental theorem of calculus to show that $z'_{i,M_0}\varepsilon_{M_0}(\gamma_0,\tilde{f})$, $1\le i\le 2r$ exists and is equal to

$$\lim_{\{a\to 1: a\in A_{M_0,\mathrm{reg}}(\mathbf{C})\}} z'_{i,M_0}\varepsilon_{M_0}(a\gamma_0,\tilde{f}) = \varepsilon_{M_0}(\gamma_0, z_i\tilde{f}).$$

A simple induction argument on the number of generators occurring in z then implies that $z'_{M_0}\varepsilon_{M_0}(\gamma_0,\tilde{f})$ exists and

$$z'_{M_0}\varepsilon_{M_0}(\gamma_0,\tilde{f}) = \varepsilon_{M_0}(\gamma_0, z\tilde{f}). \tag{27}$$

This proves that $\varepsilon_{M_0}(\cdot,\tilde{f})$ extends to a smooth function on $M_0(\mathbf{C})$. We continue by showing that the function,

$$\tilde{\varepsilon}_{M_0}(\tilde{f},\pi,X) = \int_{M_0(\mathbf{C})^X} \varepsilon_{M_0}(\gamma,\tilde{f})\pi(\gamma)d\gamma,\ X\in\mathfrak{a}_{M_0},\ \pi\in\Pi_{\mathrm{temp}}(M_0(\mathbf{C})),$$

where

$$M_0(\mathbf{C})^X = \{\gamma\in M_0(\mathbf{C}) : H_{M_0}(\gamma) = X\},$$

is a Schwartz function of X. To see this, set

$${}^c\varepsilon_{M_0}(\gamma,\tilde{f}) = {}^cI^{\mathcal{M}}_{M_0}(\gamma,\tilde{f}) - {}^cI^{\Sigma}_{M_0}(\gamma,\tilde{f}),\ \gamma\in M_0(\mathbf{C})\cap G_{\mathrm{oreg}}(\mathbf{C}),$$

and recall that by expansions (13) and (14) and the first assertion of Theorem 7.1 we have

$$\varepsilon_{M_0}(\gamma,\tilde{f}) = {}^c\varepsilon_{M_0}(\gamma,\tilde{f}) + \hat{I}^{M_0}_{M_0}\left(\gamma, \sum_{\eta\in\mu_n^{M_0}/\mu_n^G} {}^{\eta c}\hat{\theta}_{M_0}(\tilde{f}') - {}^c\theta_{\tilde{M}_0}(\tilde{f})'\right),$$

for any $\gamma \in M_0 \cap G_{\mathrm{oreg}}(\mathbf{C})$. Given $X \in \mathfrak{a}_{M_0}$ and $\pi \in \Pi_{\mathrm{temp}}(M_0(\mathbf{C}))$, the function $\tilde{\varepsilon}_{M_0}(\tilde{f}, \pi, X)$ is equal to the sum of

$$\int_{M_0(\mathbf{C})^X} {}^c\varepsilon_{M_0}(\gamma, \tilde{f})\pi(\gamma)d\gamma,$$

and

$$\sum_{\eta \in \mu_n^{M_0}/\mu_n^G} {}^{\eta c}\hat{\theta}_{M_0}(\tilde{f}', \pi, X) - {}^c\theta_{\tilde{M}_0}(\tilde{f})'(\pi, X).$$

The former summand vanishes for X outside of a fixed compact subset (Lemma 4.4, [7]). Corollary 5.3 of [7] tells us that any invariant differential operator on $\mathfrak{a}_{M_0}$ applied to the latter summand yields a rapidly decreasing function in $X \in \mathfrak{a}_{M_0}$. Consequently the same property holds for $\varepsilon_{M_0}(\tilde{f}, \pi, X)$. The smoothness of $\tilde{\varepsilon}_{M_0}(\tilde{f}, \pi, X)$ in $X \in \mathfrak{a}_{M_0}$ follows from the smoothness of $\varepsilon_{M_0}(\gamma, \tilde{f})$ in $\gamma \in M_0(\mathbf{C})$, hence $\varepsilon_{M_0}(\tilde{f}, \pi, X)$ is a Schwartz function on $\mathfrak{a}_{M_0}$. Now if $\tilde{\varepsilon}_{M_0}(\tilde{f})$ belongs to $\mathcal{I}_{\mathrm{ac}}(M_0(\mathbf{C}))$, then we may define $\varepsilon_{M_0}(\tilde{f})$ to be $\tilde{\varepsilon}_{M_0}(\tilde{f})$ and the proposition is proven at the archimedean valuations. It is easy to see that $\tilde{\varepsilon}_{M_0}(\tilde{f})$ is almost compact in the sense of §11 in [10] as M_0 is a minimal Levi subgroup. The only requirement that $\tilde{\varepsilon}_{M_0}(\tilde{f})$ does not obviously satisfy for it to be in $\mathcal{I}_{\mathrm{ac}}(\tilde{M}_0(\mathbf{C}))$ is the $K_v \cap M_0(\mathbf{C})$-finiteness requirement. This is equivalent to showing that there exists a finite set Γ_{M_0} of (equivalence classes of) irreducible admissible representations of $K_v \cap M_0(\mathbf{C})$ such that

$$\int_{M_0(\mathbf{C})} \varepsilon_{M_0}(\gamma, \tilde{f})\pi(\gamma)d\gamma = \int_{\mathfrak{a}_{M_0}} \tilde{\varepsilon}_{M_0}(\tilde{f}, \pi, X)dX$$

vanishes unless the restriction of $\pi \in \Pi_{\mathrm{temp}}(M_0(\mathbf{C}))$ to $K_v \cap M_0(\mathbf{C})$ contains a representation in Γ_{M_0}. Since $\tilde{f}$ is $\tilde{K}_v$-finite, there exists a finite set $\tilde{\Gamma}$ of (equivalence classes of) irreducible admissible representations of $\tilde{K}_v$ such that $\tilde{f}_{\tilde{M}_0}(\tilde{\pi})$, $\tilde{\pi} \in \Pi_{\mathrm{temp}}(\tilde{M}_0(\mathbf{C}))$, vanishes unless the restriction of $\tilde{\pi}^{\tilde{G}}$ to $\tilde{K}_v$ contains a representation in $\tilde{\Gamma}$. Clearly, $\tilde{\Gamma}$ determines a finite set of (equivalence classes of) irreducible admissible representations Γ of K_v such that $\tilde{f}'_{M_0}(\pi)$, $\pi \in \Pi_{\mathrm{temp}}(M_0(\mathbf{C}))$, vanishes unless the restriction of π^G to K_v contains a representation in Γ. Let Γ_{M_0} be the finite set of irreducible constituents of restrictions of representations in Γ to $K_v \cap M_0(\mathbf{C})$. It is a straightforward consequence of equation (27) that

$$\tilde{f} \mapsto \int_{\mathfrak{a}_{M_0}} \tilde{\varepsilon}_{M_0}(\tilde{f}, \tilde{\pi}', X)dX, \ \tilde{\pi} \in \Pi_{\mathrm{temp}}(\tilde{M}_0(\mathbf{C})),$$

is an invariant eigendistribution of $\mathcal{Z}_{\mathbf{C}}$. It then follows from Harish-Chandra's work ([19]) that there exists a smooth function c on $\Pi_{\mathrm{temp}}(\tilde{M}_0(\mathbf{C}))$ such that

$$(28) \qquad \int_{\mathfrak{a}_{M_0}} \tilde{\varepsilon}_{M_0}(\tilde{f}, \tilde{\pi}', X)dX = c(\tilde{\pi})\tilde{f}_{\tilde{M}_0}(\tilde{\pi}) = c(\tilde{\pi})\tilde{f}'_{M_0}(\tilde{\pi}'), \ \tilde{\pi} \in \Pi_{\mathrm{temp}}(\tilde{M}_0(\mathbf{C})).$$

Moreover, if $\pi \in \Pi_{\mathrm{temp}}(M_0(\mathbf{C}))$ is not metic, then it is not $\mu_n^{M_0}$-invariant (Lemma 3.1, [30]). Since $\varepsilon_{M_0}(\cdot, \tilde{f})$ is $\mu_n^{M_0}$-invariant, we must have

$$\int_{M_0(\mathbf{C})} \varepsilon_{M_0}(\gamma, \tilde{f})\pi(\gamma)d\gamma = 0.$$

This vanishing property together with equation (28) imply that Γ_{M_0} satisfies the $K_v \cap M_0(\mathbf{C})$-finiteness requirement for $\tilde{\varepsilon}_{M_0}(\tilde{f})$. This completes the archimedean case of the proposition.

Now suppose v is nonarchimedean. For $\gamma \in M(F_v) \cap G_{\mathrm{oreg}}(F_v)$ define

$$^c\varepsilon_M(\gamma, \tilde{f}) = {}^cI_M^{\mathcal{M}}(\gamma, \tilde{f}) - {}^cI_M^{\Sigma}(\gamma, \tilde{f}), \ \tilde{f} \in \mathcal{H}(\tilde{G}(F_v))^0,$$

(cf. §7). By Lemma 4.4 of [7], this function has compact support as a function of $M(F_v)$-conjugacy classes of $M(F_v)$. It is related to $\varepsilon_M(\gamma, \tilde{f})$ by

$$\varepsilon_M(\gamma, \tilde{f}) - {}^c\varepsilon_M(\gamma, \tilde{f}) \quad = \quad \hat{I}_M^M(\gamma, \theta_{\tilde{M}}(\tilde{f})' - \sum_{\eta \in \mu_n^M/\mu_n^G} {}^\eta\hat{\theta}_M(\tilde{f}')).$$

To see this, recall that by the induction assumption after Theorem A, Theorem 7.1, equations (10) and (11) and their analogues for $I_M^{\Sigma}(\gamma)$, we have

$$\begin{aligned} &\varepsilon_M(\gamma, \tilde{f}) - {}^c\varepsilon_M(\gamma, \tilde{f}) \\ &= \sum_{\{L \in \mathcal{L}(M) : L \subsetneq G\}} {}^c\hat{I}_M^{L,\mathcal{M}}(\gamma, \theta_{\tilde{L}}(\tilde{f})) - \sum_{\eta_1 \in \mu_n^M/\mu_n^L} \hat{I}_M^L\left(\eta_1\gamma, \sum_{\eta \in \mu_n^M/\mu_n^G} {}^\eta\hat{\theta}_L(\tilde{f}')\right) \\ &= \sum_{\{L \in \mathcal{L}(M) : M \subsetneq L \subsetneq G\}} {}^c\hat{I}_M^{L,\mathcal{M}}(\gamma, \theta_{\tilde{L}}(\tilde{f})) - {}^c\hat{I}_M^{L,\Sigma}(\gamma, \theta_{\tilde{L}}(\tilde{f})) \\ &+ \hat{I}_M^M\left(\gamma, \theta_{\tilde{M}}(\tilde{f})' - \sum_{\eta \in \mu_n^M/\mu_n^G} {}^\eta\hat{\theta}_M(\tilde{f}')\right) \\ &= \hat{I}_M^M\left(\gamma, \theta_{\tilde{M}}(\tilde{f})' - \sum_{\eta \in \mu_n^M/\mu_n^G} {}^\eta\hat{\theta}_M(\tilde{f}')\right). \end{aligned}$$

Since $\theta_{\tilde{M}}(\tilde{f})' - \sum_{\eta \in \mu_n^M/\mu_n^G} {}^\eta\hat{\theta}_M(\tilde{f}')$ belongs to $\mathcal{I}_{\mathrm{ac}}(M(F_v))$ (§4, [7]), this equation implies that ${}^c\varepsilon_M(\gamma, \tilde{f})$ is an orbital integral of a function in $\mathcal{H}_{\mathrm{ac}}(M(F_v))$ if and only if $\varepsilon_M(\gamma, \tilde{f})$ is as well. More generally, given a semisimple element $\sigma \in M(F_v)$, it implies that

$$^c\varepsilon_M(\gamma, \tilde{f}) \overset{(M,\sigma)}{\sim} 0, \ \gamma \in M_\sigma(F_v) \cap G_{\mathrm{oreg}}(F_v)$$

if and only if

$$\varepsilon_M(\gamma, \tilde{f}) \overset{(M,\sigma)}{\sim} 0, \ \gamma \in M_\sigma(F_v) \cap G_{\mathrm{oreg}}(F_v).$$

We shall complete the proposition by showing that

$$\varepsilon_M(\gamma, \tilde{f}) \overset{(M,\sigma)}{\sim} 0, \ \gamma \in M_\sigma(F_v) \cap G_{\mathrm{oreg}}(F_v)$$

and then showing that this implies that ${}^c\varepsilon_M(\gamma, \tilde{f})$ is an orbital integral of a function in $\mathcal{H}_{\mathrm{ac}}(M(F_v))$. If σ is not F_v-elliptic in M, that is, if $\mathfrak{a}_{M_\sigma} \neq \mathfrak{a}_M$, then there is a proper Levi subgroup $M_1 \in \mathcal{L}^M$ of M such that $M_\sigma(F_v) \subset M_1(F_v)$. Lemma 2.1 then implies that

$$\varepsilon_M(\gamma, \tilde{f}) \overset{(M,\sigma)}{\sim} 0, \ \gamma \in M_\sigma(F_v) \cap G_{\mathrm{oreg}}(F_v).$$

Suppose that σ is F_v-elliptic in M. We may assume that σ is F_v-elliptic in $L \in \mathcal{L}(M)$ if and only if σ^n is F_v-elliptic in L. Indeed, we know by Lemma 4.1 that a translate of σ by an element in μ_n^M satisfies this property and we also know that

$$\varepsilon_M(\eta\gamma, \tilde{f}) = \varepsilon_M(\gamma, \tilde{f}),\ \eta \in \mu_n^M.$$

We may therefore apply Lemma 4.3 and Proposition 4.1 to see that $\varepsilon_M(\gamma, \tilde{f})$ is (M, σ)-equivalent to

$$\sum_{L \in \mathcal{L}(M)} \sum_{\delta \in \sigma(\mathcal{U}_{L_\sigma}(F_v))} g_M^L(\gamma, \delta) \left(I_L^{\mathcal{M}}(\delta, \tilde{f}) - I_L^{\Sigma}(\delta, \tilde{f}) \right),\ \gamma \in M_\sigma(F_v) \cap G_{\mathrm{oreg}}(F_v).$$

Since $g_M^M(\gamma, \delta) = 0$, we may apply the induction hypothesis of this section to conclude that the right-hand side vanishes. From our earlier remark we now have

$${}^c\varepsilon_M(\gamma, \tilde{f}) \overset{(M,\sigma)}{\sim} \varepsilon_M(\gamma, \tilde{f}) \overset{(M,\sigma)}{\sim} 0,\ \gamma \in M_\sigma(F_v) \cap G_{\mathrm{oreg}}(F_v),$$

for any semisimple element $\sigma \in M(F_v)$. We use a partition of unity argument on the compact support of ${}^c\varepsilon_M(\gamma, \tilde{f})$ in the space of $M(F_v)$-conjugacy classes to conclude that ${}^c\varepsilon_M(\gamma, \tilde{f})$ is in fact an orbital integral. □

Suppose v is a valuation for which $|n|_v = 1$. Then it is an immediate consequence of Lemma 3.4 that $\varepsilon_M(\tilde{f}_v^0) = 0$. It therefore follows from the splitting property of Proposition 10.1 that the map ε_M extends to a map on $\mathcal{H}(\tilde{G}(\mathbf{A}))^0$.

11. Comparison for $\tilde{f} \in \mathcal{H}(\tilde{G}(\mathbf{A}), M)$

We now give a sketch of the proof that $I(\tilde{f}) = I^{\Sigma}(\tilde{f})$ for $\tilde{f}$ in a certain subspace $\mathcal{H}(\tilde{G}(\mathbf{A}), M)$ of $\mathcal{H}(\tilde{G}(\mathbf{A}))$. The train of reasoning in this section is based entirely on §§15 and 16, II of [13]. We shall outline the arguments found there and leave the confirmation of the details to the reader.

Recall that $M \in \mathcal{L}$ was fixed in the previous section. Let $\mathcal{H}(\tilde{G}(\mathbf{A}), M)$ be the subspace of $\mathcal{H}(\tilde{G}(\mathbf{A}))$ spanned by functions

$$\tilde{f} = \prod_v \tilde{f}_v,\ \tilde{f}_v \in \mathcal{H}(\tilde{G}(F_v)),$$

which satisfy the following property. For two nonarchimedean places v_1 and v_2, which are not in $\{v : |n|_v \neq 1\}$,

$$\tilde{f}_{v_i, \tilde{L}} = 0,\quad L \in \mathcal{L},\ i = 1, 2,$$

unless L contains a conjugate of M. If S contains $\{v : |n|_v \neq 1\}$ and at least two other nonarchimedean places, we define $\mathcal{H}(\tilde{G}(F_S), M)$ in the same way.

The proof of Lemma 13.1, II of [13] may be imitated to obtain the following lemma.

Lemma 11.1 (13.1)**.** *For $\tilde{f} \in \mathcal{H}(\tilde{G}(\mathbf{A}), M)$, the distribution*

$$I(\tilde{f}) - I^{\Sigma}(\tilde{f})$$

equals the sum of

$$|W(\mathfrak{a}_M)|^{-1} \sum_{\gamma \in (M(F))_{M,S}/\mu_n^M} a^M(S, \gamma) \left(I_M^{\mathcal{M}}(\gamma, \tilde{f}) - I_M^{\Sigma}(\gamma, \tilde{f}) \right)$$

and

$$\sum_{\delta\in A_G(F)\backslash\mu_n^G}\ \sum_{u\in(\mathcal{U}_G(F))_{G,S}} \Big(a^{\tilde{M}}(S,u')-a(S,u)\Big)\, I_G^{\mathcal{M}}(\delta u,\tilde{f}).$$

Proof. The lemma follows from the splitting properties and the properties of $\tilde{f}$. See Lemma 13.1, II of [13] for details. □

Let $\mathcal{H}(\tilde{G}(\mathbf{A}),M)^0$ be the space of functions $\tilde{f}$ in

$$\mathcal{H}(G(\mathbf{A}),M)\cap\mathcal{H}(G(\mathbf{A}))^0$$

which satisfy one additional condition, namely, that $\tilde{f}$ vanishes at any element in $\tilde{G}(\mathbf{A})$ whose component at each nonarchimedean place v belongs to $A_G(F_v)$. This ensures that $\tilde{f}\in\mathcal{H}(\tilde{G}(\mathbf{A}),M)^0$ vanishes at

$$(\delta u,\zeta),\ \delta\in A_G(F),\ u\in\mathcal{U}_G(F),\ \zeta\in\mu_n.$$

Lemma 11.2 (15.1)**.** *Suppose that $\tilde{f}\in\mathcal{H}(\tilde{G}(\mathbf{A}),M)^0$. Then*

$$I(\tilde{f})-I^{\Sigma}(\tilde{f})=n^{\dim(A_M)}|W(\mathfrak{a}_M)|^{-1}\hat{I}^M(\varepsilon_M(\tilde{f})),$$

where I^M is the analogue for M of $I=I^G$.

Proof. By the properties of $\tilde{f}$ Lemma 11.1 and Proposition 10.1, we see that $I(\tilde{f})-I^{\Sigma}(\tilde{f})$ is equal to

$$n^{\dim(A_M)}|W(\mathfrak{a}_M)|^{-1}\sum_{\gamma\in(M(F))_{M,S}} a^M(S,\gamma)\hat{I}_M^M(\gamma,\varepsilon_M(\tilde{f})),$$

for a large set of valuations S. By the descent property (Corollary 8.3, [7]) and Lemma 2.1 it follows that

$$\hat{I}_{M_1}^{M_1}(\gamma,\varepsilon_M(\tilde{f})_{M_1})=\hat{I}_M^M(\gamma,\varepsilon_M(\tilde{f}))=0,\ \gamma\in M_1(F_S)\cap G_{\mathrm{oreg}}(F_S),$$

for any $M_1\in\mathcal{L}$ such that $M_1\subsetneqq M$. Therefore, for such M_1, $\varepsilon_M(\tilde{f})_{M_1}=0$. Combining this with the splitting property (Proposition 9.1, [7]) applied to $I_{M_1}^M(\gamma)$, we find that

$$\hat{I}_{M_1}^M(\gamma,\varepsilon_M(f))=0,\ \gamma\in M_1(F)\cap G_{\mathrm{oreg}}(F),$$

for any $M_1\in\mathcal{L}^M$ such that $M_1\subsetneqq M$. Thus by the geometric side of the trace formula for M is

$$\begin{aligned}
n^{\dim(A_M)}|W(\mathfrak{a}_M)|^{-1}&\sum_{\gamma\in(M(F))_{M,S}} a^M(S,\gamma)\hat{I}_M^M(\gamma,\varepsilon_M(\tilde{f}))\\
&=n^{\dim(A_M)}|W(\mathfrak{a}_M)|^{-1}\sum_{M_1\in\mathcal{L}^M}|W_0^{M_1}||W_0^M|^{-1}\\
&\quad\times\sum_{\gamma\in(M_1(F))_{M_1,S}} a^{M_1}(S,\gamma)\hat{I}_{M_1}^M(\gamma,\varepsilon_M(\tilde{f}))\\
&=n^{\dim(A_M)}|W(\mathfrak{a}_M)|^{-1}\hat{I}^M(\varepsilon_M(\tilde{f})).
\end{aligned}$$

□

We now give an outline of the method of separation by infinitesimal characters, which is found in §15, II of [13]. Let S_∞ denote the set of archimedean (in our case complex) valuations of F. Then $G(F_{S_\infty})$ may be regarded as a real Lie group. Let $\mathfrak{h}_{\mathbf{C}}$ denote the standard Cartan subalgebra of its complexified Lie algebra. Let $\mathfrak{h}$ be the real form of $\mathfrak{h}_{\mathbf{C}}$ associated to the split real form of $G(F_{S_\infty})$. Then $\mathfrak{h}$ contains $\mathfrak{a}_{M_0}$ as a vector space and in turn contains all vector spaces of the form $\mathfrak{a}_L$, $L \in \mathcal{L}$. Let $\mathfrak{h}^1$ be the orthogonal complement of $\mathfrak{a}_G$ in $\mathfrak{h}$. We recall the theory of multipliers ([4]). Let α belong to $\mathcal{E}(\mathfrak{h}^1)^W$, the convolution algebra of compactly supported, distributions on $\mathfrak{h}^1$, which are invariant under the complex Weyl group W of $G(F_{S_\infty})$. Then there is an action, $\tilde{f} \mapsto \tilde{f}_\alpha$, on $\mathcal{H}(\tilde{G}(\mathbf{A}))$ such that

$$\tilde{f}_{\alpha,\tilde{M}}(\tilde{\pi}) = \hat{\alpha}(\nu_{\tilde{\pi}})\tilde{f}_{\tilde{M}}(\tilde{\pi}),\ \tilde{\pi} \in \Pi(\tilde{M}(\mathbf{A})).$$

As usual, $\nu_{\tilde{\pi}}$ is taken to be the infinitesimal character of the archimedean factor of $\tilde{\pi}$. This action of $\mathcal{E}(\mathfrak{h}^1)^W$ on $\mathcal{H}(\tilde{G}(\mathbf{A}))$ affects only the archimedean factor of $\tilde{f}$, and is supported on characters. The map,

$$\alpha \mapsto \alpha',\ \alpha \in \mathcal{E}(\mathfrak{h}^1)^W,$$

given by

$$\alpha'(\nu) = n^{-\dim(\mathfrak{h}^1)}\alpha(n^{-1}\nu), \nu \in \mathfrak{h}^1, \tag{29}$$

is compatible with the map (13) of [30]. More precisely, given $\alpha \in \mathcal{E}(\mathfrak{h}^1)^W$ and $\tilde{\pi} \in \Pi_{\mathrm{temp}}(\tilde{G}(F_S))$, we have $\nu_{\tilde{\pi}} = \nu_{n^{-1}\tilde{\pi}'}$ (§3.1, [30]) and in turn

$$(\tilde{f}_\alpha)'(\tilde{\pi}') = \hat{\alpha}(n^{-1}\nu_{\tilde{\pi}'})\tilde{f}_{\tilde{G}}(\tilde{\pi}) = \hat{\alpha}'(\nu_{\tilde{\pi}'})\tilde{f}'(\tilde{\pi}') = \tilde{f}'_{\alpha'}(\tilde{\pi}').$$

This map is also compatible with the map ε_M of Proposition 10.1.

Lemma 11.3 (14.4)**.** *Suppose $\tilde{f} \in \mathcal{H}(\tilde{G}(\mathbf{A}))$, and $\alpha \in \mathcal{E}(\mathfrak{h}^1)^W$. Then $\varepsilon_M(\tilde{f}_\alpha) = \varepsilon_M(f)_{\alpha'}$.*

Proof. Suppose $\tilde{f} = \prod_v \tilde{f}_v \in \mathcal{H}(\tilde{G}(\mathbf{A}))$ and suppose S contains all valuations at which $\tilde{f}_v \neq \tilde{f}_v^0$. If v is nonarchimedean, then α acts trivially on $\tilde{f}_v$, that is, $\tilde{f}_{v,\alpha} = \tilde{f}_v$. If v is complex, it follows from equations (26) and (28) that $\varepsilon_M(\tilde{f}_{v,\alpha}) = \varepsilon_M(\tilde{f})_{\alpha'}$. Repeated applications of the splitting property of Proposition 10.1 yield

$$\begin{aligned}
\varepsilon_M(\tilde{f}_\alpha) &= \sum_{v_1 \in S} \varepsilon_M(f_{v_1,\alpha}) \prod_{w \in S - v_1} (\tilde{f}_{w,\alpha})'_M \prod_{v \notin S} (\tilde{f}_v^0)'_M \\
&= \sum_{v_1 \in S} \varepsilon_M(f_{v_1})_{\alpha'} \prod_{w \in S - v_1} ((\tilde{f}_w)'_{\alpha'})_M \prod_{v \notin S} (\tilde{f}_v^0)'_M \\
&= \varepsilon_M(\tilde{f})_{\alpha'}.
\end{aligned}$$

□

Let $\mathfrak{h}_u^*$ be the set of points ν in $\mathfrak{h}_{\mathbf{C}}^*/i\mathfrak{a}_G^*$ such that the complex conjugate of ν with respect to $\mathfrak{h}^*$ is equal to $-s\nu$ for some element $s \in W$ of order two. The archimedean infinitesimal character of $\nu_{\tilde{\pi}}$ associated to any $\tilde{\pi} \in \Pi_{\mathrm{unit}}(\tilde{G}(\mathbf{A})^1)$ belongs to $\mathfrak{h}_u^*$. Given $\nu_1 \in \mathfrak{h}_u^*$ define

$$\Pi_{\nu_1}(\tilde{G}(\mathbf{A})^1) = \{\tilde{\pi} \in \Pi(\tilde{G}(\mathbf{A})^1) : \nu_{\tilde{\pi}} = s\nu_1 \text{ for some } s \in W\}.$$

We can use multipliers to prove the following lemma.

Lemma 11.4 (15.4)**.** *For each* $\tilde{f} \in \mathcal{H}(\tilde{G}(\mathbf{A}), M)^0$ *and* $\nu_1 \in \mathfrak{h}_u^*$*, we have*

$$\sum_{\tilde{\pi} \in \Pi_{\nu_1}(\tilde{G}(\mathbf{A})^1)} \left(a_{\mathrm{disc}}(\tilde{\pi}) - a^{\Sigma}_{\mathrm{disc}}(\tilde{\pi})\right) \mathrm{tr}\left(\tilde{\pi}(\tilde{f}^1)\right) = 0.$$

As the proof of Lemma 11.4 is almost identical to that of Lemma 15.4, II of [13], we shall only sketch the proof. Given $\tilde{f} \in \mathcal{H}(\tilde{G}(\mathbf{A}), M)^0$, $T \geq 0$ and $\nu_1 \in \mathfrak{h}_u^*$ we choose $\alpha_1 \in \mathcal{E}(\mathfrak{h}^1)^W$ as in Lemma 15.2, II of [13]. The multiplier α_1 has the property that $0 \leq \hat{\alpha_1}(\nu_{\tilde{\pi}}) \leq 1$ for any $\tilde{\pi} \in \Pi_{\mathrm{unit}}(\tilde{G}(\mathbf{A}))$ whose K-type is the same as that of $\tilde{f}$. Furthermore, the inverse image of 1 under $\hat{\alpha_1}$ is $\{s\nu_1 : s \in W\}$. Following Corollaries 14.2 and 14.3, II of [13], we find that $\varepsilon_M(\tilde{f})$ is a moderate function in $\mathcal{I}_{\mathrm{ac}}(M(\mathbf{A}))$ in the sense of §6 of [8]. We may then apply Corollary 6.5 of [8] (cf. (15.5), II of [13]) to obtain

$$\left| \sum_{t \leq T} I_t(\tilde{f}_{\alpha_1^m}) - I_t^{\Sigma}(\tilde{f}_{\alpha_1^m}) - n^{\dim(A_M)} |W(\mathfrak{a}_M)|^{-1} \hat{I}_t^M(\varepsilon_M(\tilde{f})_{(\alpha_1')^m}) \right| \leq Ce^{-kNm}$$

for some positive constants C, k and N. Thus

$$\sum_{t \leq T} I_t(\tilde{f}_{\alpha_1^m}) - I_t^{\Sigma}(\tilde{f}_{\alpha_1^m}) \tag{30}$$

approaches

$$\sum_{t \leq T} n^{\dim(A_M)} |W(\mathfrak{a}_M)|^{-1} \hat{I}_t^M(\varepsilon_M(\tilde{f})_{(\alpha_1')^m}) \tag{31}$$

as m approaches infinity. According to Lemma 15.3, II of [13] we may write (31) as

$$\sum_{t \leq T} n^{\dim(A_M)} |W(\mathfrak{a}_M)|^{-1} \sum_{\pi \in \Pi_{\mathrm{disc}}(M,t)} a^M_{\mathrm{disc}}(\pi) \int_{i\mathfrak{a}_M^*/i\mathfrak{a}_G^*} \varepsilon_M(\tilde{f}^1, \pi, \lambda) \hat{\alpha}_1'(\nu_\pi + \lambda)^m d\lambda,$$

for some Schwartz function

$$\lambda \mapsto \varepsilon_M(\tilde{f}^1, \pi, \lambda), \ \lambda \in i\mathfrak{a}_M^*/i\mathfrak{a}_G^*.$$

The multiplier α_1 was chosen so that

$$0 \leq \hat{\alpha}_1'(\nu_\pi + \lambda) < 1,$$

for all but finitely many $\lambda \in i\mathfrak{a}_M^*/i\mathfrak{a}_G^*$ in the above integral. Thus, by the dominated convergence theorem, the integral approaches zero as m approaches infinity. Expression (30) therefore also approaches zero as m approaches infinity. Moreover, by Lemma 9.2 it is equal to

$$\sum_{t \leq T} \sum_{\tilde{\pi} \in \Pi(\tilde{G}(\mathbf{A})^1, t)} (a_{\mathrm{disc}}(\tilde{\pi}) - a^{\Sigma}_{\mathrm{disc}}(\tilde{\pi})) \mathrm{tr}\left(\tilde{\pi}(\tilde{f}^1)\right) \hat{\alpha}_1(\nu_{\tilde{\pi}})^m.$$

By our choice of α_1, the limit of this expression as m approaches infinity is the expression in Lemma 11.4.

In §16, II of [13] it is shown, using the Plancherel formula, how to extend Lemma 11.4 to functions in $\mathcal{H}(\tilde{G}(\mathbf{A}), M)$. We shall not repeat the arguments here. We merely translate the final result of that section into the following proposition.

Proposition 11.1 (16.2)**.** *For any* $\tilde{f} \in \mathcal{H}(\tilde{G}(\mathbf{A}), M)$*, we have*

$$I(\tilde{f}) = I^{\Sigma}(\tilde{f}).$$

Proof. By Lemma 9.2 and the extension of Lemma 11.4 to $\mathcal{H}(\tilde{G}(\mathbf{A}), M)$, the difference $I_t(\tilde{f}) - I_t^{\Sigma}(\tilde{f})$, $t \geq 0$, is equal to

$$\sum_{\{\nu_1:\|\mathrm{Im}(\nu_1)\|=t\}} \sum_{\tilde{\pi}\in\Pi_{\nu_1}(\tilde{G}(\mathbf{A})^1)} \left(a_{\mathrm{disc}}(\tilde{\pi}) - a^{\Sigma}_{\mathrm{disc}}(\tilde{\pi})\right) \mathrm{tr}\left(\tilde{\pi}(\tilde{f}^1)\right) = 0.$$

It is then obvious that

$$I(\tilde{f}) = \sum_t I_t(\tilde{f}) = \sum_t I_t^{\Sigma}(\tilde{f}) = I^{\Sigma}(\tilde{f}).$$

□

12. Completion of the proofs of Theorems A and B

We first prove Theorem A (i). The structure of the proof is as follows. We begin by showing that for any valuation v of F and each $\gamma_v \in M(F_v) \cap G_{\mathrm{oreg}}(F_v)$ there exists a constant $\varepsilon_M(\gamma_v)$ such that

$$I_M^{\mathcal{M}}(\gamma_v, \tilde{f}) - I_M^{\Sigma}(\gamma_v, \tilde{f}) = \varepsilon_M(\gamma_v) I_G^{\mathcal{M}}(\gamma_v, \tilde{f}),\ \tilde{f} \in \mathcal{H}(\tilde{G}(F_v)).$$

This part of the proof follows §17, II of [13] almost exactly. It then follows from the splitting properties that for any S_0 containing $\{v : |n|_v \neq 1\}$ and

$$\gamma = \prod_{v\in S_0} \gamma_v \in M(F_{S_0}) \cap G_{\mathrm{oreg}}(F_{S_0}),$$

we have

$$I_M^{\mathcal{M}}(\gamma, \tilde{f}) - I_M^{\Sigma}(\gamma, \tilde{f}) = \left(\sum_{v\in S_0} \varepsilon_M(\gamma_v)\right) I_G^{\mathcal{M}}(\gamma, \tilde{f}),\ \tilde{f} \in \mathcal{H}(\tilde{G}(F_{S_0})). \tag{32}$$

Using the strong approximation theorem, we show that $\sum_{v\in S_0} \varepsilon_M(\gamma_v)$ vanishes unless $M = M_0$.[1] The induction hypothesis of §10 and Corollary 3.1 then imply that $\varepsilon_M(\gamma_v) = 0$ unless $M = M_0$. We then take care of the case $M = M_0$ by using the local trace formula as in the proof of Corollary 3.1.

Suppose $\tilde{f} \in \mathcal{H}(\tilde{G}(\mathbf{A}), M)$. According to Lemma 11.1 and Proposition 11.1 the sum of

$$|W(\mathfrak{a}_M)|^{-1} \sum_{\gamma\in(M(F))_{M,S}/\mu_n^M} a^M(S,\gamma) \left(I_M^{\mathcal{M}}(\gamma, \tilde{f}) - I_M^{\Sigma}(\gamma, \tilde{f})\right) \tag{33}$$

and

$$\sum_{\delta\in A_G(F)\backslash\mu_n^G} \sum_{u\in(\mathcal{U}_G(F))_{G,S}} \left(a^{\mathcal{M}}(S,u) - a(S,u)\right) I_G^{\mathcal{M}}(\delta u, \tilde{f}) \tag{34}$$

vanishes. Recall that S is a large finite set of valuations which depends on $\tilde{f}$ and contains $\{v : |n|_v \neq 1\}$. We may assume that $S = \{v_1, \ldots, v_k\}$, where v_1 and v_2 are any two distinct valuations satisfying

$$|n|_{v_1} = |n|_{v_2} = 1.$$

[1] I am indebted to J. Arthur for providing the underlying ideas for this portion of the proof.

Take γ to be an element in $M(F) \cap G_{\mathrm{oreg}}(F)$ such that γ is F_{v_i}-elliptic in $M(F_{v_i})$, $i = 1, 2$. Choose $\tilde{f}_i \in \mathcal{H}(\tilde{G}(F_{v_i}))$ such that it is supported on a very small neighborhood of $\gamma' \mathbf{i}(\mu_n)$ in $\tilde{G}(F_{v_i})$ and

$$\hat{I}_M^{M,\mathcal{M}}(\gamma, \tilde{f}_{i,\tilde{M}}) = I_G^{\mathcal{M}}(\gamma, \tilde{f}_i) = 1, \ i = 1, 2.$$

For the remaining $3 \leq i \leq k$, let $\tilde{f}_i \in \mathcal{H}(\tilde{G}(F_{v_i}))$ be arbitrary and set $\tilde{f} = \tilde{f}_1 \cdots \tilde{f}_k$. This function lies in $\mathcal{H}(\tilde{G}(F_S), M)$ by the construction of $\tilde{f}_1$ and $\tilde{f}_2$. If we shrink the support of $\tilde{f}_1$ and $\tilde{f}_2$ around $\gamma' \mathbf{i}(\mu_n)$, the set S is not affected. Bearing in mind that $\tilde{M}(F)$ is a discrete subgroup of $\tilde{M}(\mathbf{A})$, we may then assume that the support of $\tilde{f}_1$ and $\tilde{f}_2$ is so small that (34) vanishes and the only contribution to the sum in (33) comes from $\tilde{G}(F_S)$-conjugates of γ'. The distributions in (33) are constant on $\tilde{G}(F_S)$-conjugacy classes. By Theorem 8.2 of [5] we have

$$a^M(S, \gamma) = \mathrm{vol}(M_\gamma(F) \backslash M_\gamma(\mathbf{A})^1),$$

for S large enough (in a sense depending only on γ). Consequently,

$$I_M^{\mathcal{M}}(\gamma, \tilde{f}) - I_M^{\Sigma}(\gamma, \tilde{f}) = 0. \tag{35}$$

We apply the splitting properties ((23) and Proposition 6.2, [30]) and the induction hypothesis of §2 to this equation repeatedly in order to obtain

$$\sum_{i=1}^{k} \left(I_M^{\mathcal{M}}(\gamma, \tilde{f}_i) - I_M^{\Sigma}(\gamma, \tilde{f}_i) \right) \prod_{j \neq i} \hat{I}_M^{M,\mathcal{M}}(\gamma, \tilde{f}_{j,\tilde{M}}) = 0. \tag{36}$$

Suppose $3 \leq i \leq k$. Choose $\gamma \in M(F)$ as in (36), $\tilde{f}_i \in \mathcal{H}(\tilde{G}(F_{v_i}))$ such that

$$\hat{I}_M^{M,\mathcal{M}}(\gamma, \tilde{f}_{i,\tilde{M}}) = I_G^{\mathcal{M}}(\gamma, \tilde{f}_i) = 0,$$

and the remaining $\tilde{f}_j \in \mathcal{H}(\tilde{G}(F_{v_j}))$, $j \neq i$, such that

$$\hat{I}_M^{M,\mathcal{M}}(\gamma, \tilde{f}_{j,\tilde{M}}) \neq 0.$$

In this case, the left-hand side of (36) is a nonzero multiple of

$$I_M^{\mathcal{M}}(\gamma, \tilde{f}_i) - I_M^{\Sigma}(\gamma, \tilde{f}_i).$$

Consequently, this distribution vanishes for any $\tilde{f}_i \in \mathcal{H}(\tilde{G}(F_{v_i}))$ such that $I_G^{\mathcal{M}}(\gamma, \tilde{f}_i)$ vanishes. This implies that there exists a constant $\varepsilon_M(\gamma)$ such that

$$I_M^{\mathcal{M}}(\gamma, \tilde{f}_i) - I_M^{\Sigma}(\gamma, \tilde{f}_i) = \varepsilon_M(\gamma) I_G^{\mathcal{M}}(\gamma, \tilde{f}_i), \tag{37}$$

for any $\tilde{f}_i \in \mathcal{H}(\tilde{G}(F_{v_i}))$. Observe that if $M \supsetneqq M_0$ and $F_{v_i} = \mathbf{C}$, then the left-hand side of this equation vanishes by Jordan canonical form and Lemma 2.1. This implies that $\varepsilon_M(\gamma) = 0$ at the archimedean places whenever $M \supsetneqq M_0$.

Let $V_i = \{v_i, v_1, v_2\}$. Suppose that v_i is nonarchimedean. The set

$$\left\{ \prod_{v \in V_i} \gamma_v \in M(F_{V_i}) \cap G_{\mathrm{oreg}}(F_{V_i}) : \gamma_{v_j} \text{ is } F_{v_j}\text{-elliptic in } M(F_{v_j}),\ j = 1, 2 \right\}$$

is open in $M(F_{V_i})$. Since $M(F)$ is dense in $M(F_{V_i})$, it is also dense in the above open subset. It follows that we may approximate any $\gamma_i \in M(F_{v_i})$ by an element γ occurring in (37). Since all of the distributions in (37) are smooth on $M(F_{v_i}) \cap$

$G_{\text{oreg}}(F_{v_i})$, the function ε_M also extends to a smooth function on this space and we have

$$(38)\qquad I_M^{\mathcal{M}}(\gamma_i,\tilde{f}_i)-I_M^{\Sigma}(\gamma_i,\tilde{f}_i)=\varepsilon_M(\gamma_i)I_G^{\mathcal{M}}(\gamma_i,\tilde{f}_i),\ \gamma_i\in M(F_{v_i})\cap G_{\text{oreg}}(F_{v_i}).$$

Now suppose that $F_{v_i}=\mathbf{C}$. If $M\supsetneqq M_0$, then, as we have already remarked, equation (38) holds with $\varepsilon_M(\gamma_i)=0$. Otherwise, it is obvious that one can approximate $\gamma_i\in M_0(\mathbf{C})\cap G_{\text{oreg}}(\mathbf{C})$ by an element $\gamma\in M(F)\cap G_{\text{oreg}}(F)$ which is F_{v_j}-elliptic in $M_0(F_{v_j})$, for $j=1,2$. After all, the latter condition is satisfied for any element in $M_0(F_{v_j})$, $j=1,2$. As a result, equation (38) holds for the archimedean places as well. Finally, notice that since our choice of v_1 and v_2 was arbitrary as long as $|n|_{v_1}=|n|_{v_2}=1$, equation (38) holds for any $1\le i\le k$.

Suppose $M\supsetneqq M_0$ and $S_0\supset\{v:|n|_v\neq 1\}$. We may assume that $S\supset S_0$. By approximating elements of $M(F_S)$ with elements of $M(F)$ as above, repeated use of the splitting properties ((23) and Proposition 6.2, [30]) and the induction hypothesis of §2, we obtain equation (32) from (35). We would like to show that $\sum_{v\in S_0}\varepsilon_M(\gamma_v)$ vanishes and then apply Corollary 3.1.

Suppose that $v\notin S_0$ and γ_v lies in $K_v\cap M(F_v)\cap G_{\text{oreg}}(F_v)$. It is obvious that $I_G^{\mathcal{M}}(\gamma_v,\tilde{f}_v^0)\neq 0$ (cf. (27), [30]). However, according to Lemma 3.4 the left-hand side of equation (38) vanishes. Therefore,

$$(39)\qquad \varepsilon_M(\gamma_v)=0,\ \gamma_v\in K_v\cap M(F_v)\cap G_{\text{oreg}}(F_v),\ v\notin S_0.$$

Recall decomposition (4) of [30],

$$M=\prod_{i=1}^{\ell}M(i)\cong\prod_{i=1}^{\ell}\mathrm{GL}(r_i).$$

We consider the subgroup $\prod_{i=1}^{\ell}\mathrm{SL}(r_i)$ of $\prod_{i=1}^{\ell}\mathrm{GL}(r_i)$ and identify it with its image in M using the above isomorphism. We shall show that

$$(40)\qquad \sum_{v\in S_0}\varepsilon_M(\gamma_v)=0,\ \gamma_0=\prod_{v\in S_0}\gamma_v\in\prod_{i=1}^{\ell}\mathrm{SL}(r_i,F_{S_0})\cap G_{\text{oreg}}(F_{S_0}).$$

One should keep in mind that the summands parameterized by archimedean valuations all vanish, as we are assuming that $M\supsetneqq M_0$. We may suppose that v_1,v_2 are as earlier and $v_1,v_2\notin S_0$. For $i=1,2$ we choose γ_{v_i} to be F_{v_i}-elliptic elements in $M(F_{v_i})$ which also lie in $\prod_{j=1}^{\ell}\mathrm{SL}(r_j,F_{v_i})$, K_{v_i} and $G_{\text{oreg}}(F_{v_i})$. To see that this is possible, consider a finite Galois extension E of F_{v_i}. Let x_1 be an element of E such that $F_{v_i}(x_1^n)=E$. Choose a non-trivial element σ in $\mathrm{Gal}(E/F_{v_i})$ and set $x=x_1/\sigma(x_1)$. Then $|x|_E=1$, $N_{E/F_{v_i}}(x)=1$ and $F(x^n)=E$. These properties of x can be translated respectively into the above context as $\gamma_{v_i}\in K_{v_i}$, $\gamma_{v_i}\in\prod_{i=1}^{\ell}\mathrm{SL}(r_i,F_{v_i})$ and $\gamma_{v_i}\in G_{\text{oreg}}(F_{v_i})$ being F_{v_i}-elliptic in $M(F_{v_i})$. Now suppose $\gamma_0\in\prod_{i=1}^{\ell}\mathrm{SL}(r_i,F_{S_0})\cap G_{\text{oreg}}(F_{S_0})$. The the strong approximation theorem ([26]) allows us to choose $\gamma\in\prod_{i=1}^{\ell}\mathrm{SL}(r_i,F)\cap G_{\text{oreg}}(F)$ such that γ is close to γ_0 at the nonarchimedean valuations in S_0, γ is close to γ_{v_i} in $M(F_{v_i})$ for $i=1,2$, and $\gamma\in K_v$ for any $v\notin S_0\cup\{v_1,v_2\}$. Choose $\tilde{f}_i\in\mathcal{H}(\tilde{G}(F_{v_i}))$, $1\le i\le k$, such that $\tilde{f}_1,\tilde{f}_2$ are supported on very small neighborhoods of $\gamma'\mathbf{i}(\mu_n)$ and $\tilde{f}_i=\tilde{f}_{v_i}^0$ if $v_i\notin S_0$. It then follows from equation (36), equation (39) and our earlier observations that equation (40) holds.

Suppose $\delta = \prod_{v\in S_0} \delta_v \in M(F_{S_0}) \cap G_{\text{oreg}}(F_{S_0})$ and

$$\gamma_0 = \prod_{v\in S_0} \gamma_v \in \prod_{i=1}^{\ell} \mathrm{SL}(r_i, F_{S_0}) \cap G_{\text{oreg}}(F_{S_0})$$

such that $\delta\gamma_0 \in M(F_{S_0}) \cap G_{\text{oreg}}(F_{S_0})$. Our next step is to show that

$$\sum_{v\in S_0} \varepsilon_M(\delta_v\gamma_v) = \sum_{v\in S_0} \varepsilon_M(\delta_v). \tag{41}$$

It is easily seen that we may choose γ_{v_i}, $i = 1, 2$, as in the previous paragraph, but with the additional requirement that $\gamma_{v_i}^2$ satisfies the same conditions as γ_{v_i}. Weak approximation allows us to choose $\gamma_1 \in M(F) \cap G_{\text{oreg}}(F)$ such that γ_1 is close to γ_{v_i} in $M(F_{v_i})$ for $i = 1, 2$, and γ_1 is close to δ at the nonarchimedean valuations of S_0. Let V be the finite set of valuations $\{v \notin S_0 : \gamma_1 \notin K_v\}$. Choosing S large enough and $\tilde{f}_1, \ldots, \tilde{f}_k$ appropriately in equation (36), we may conclude that

$$0 = \sum_{v\in S_0\cup V} \varepsilon_M(\gamma_1) = \sum_{v\in S_0} \varepsilon_M(\delta_v) + \sum_{v\in V} \varepsilon_M(\gamma_1). \tag{42}$$

(In this equation we are abusing notation as $\varepsilon_M(\gamma_1)$ depends on $v \in S_0 \cup V$.) As earlier, by strong approximation, we may choose $\gamma_2 \in \prod_{i=1}^{\ell} \mathrm{SL}(r_i, F) \cap G_{\text{oreg}}(F)$ such that it is close to γ_{v_i} in $M(F_{v_i})$ for $i = 1, 2$; it is close to γ_0 at the nonarchimedean valuations in S_0; it is close to the identity in $M(F_V)$; and it lies in K_v for $v \notin S_0 \cup V$. We may assume that the product $\gamma_1\gamma_2 \in M(F)$ is also in $G_{\text{oreg}}(F)$. By construction, $\gamma_1\gamma_2$ is close to $\gamma_{v_i}^2$ in $M(F_{v_i})$ for $i = 1, 2$; close to $\delta\gamma_0$ at the nonarchimedean valuations of S_0; and close to γ_1 in $M(F_V)$. Furthermore, $\gamma_1\gamma_2$ lies in K_v for valuations $v \notin S_0 \cup V$. Again, a judicious choice of S and $\tilde{f}_1, \ldots, \tilde{f}_k$ in equation (36) implies that

$$0 = \sum_{v\in S_0\cup V} \varepsilon_M(\gamma_1\gamma_2) = \sum_{v\in S_0} \varepsilon_M(\delta_v\gamma_v) + \sum_{v\in V} \varepsilon_M(\gamma_1).$$

Comparing this equation with equation (42), we find that (41) is true.

It is a simple exercise to show that any $\delta_0 = \prod_{v\in S_0} \delta_{0,v} \in M(F_{S_0}) \cap G_{\text{oreg}}(F_{S_0})$ can be written as a product $\delta\gamma_0$, where $\delta \in M_0(F_{S_0}) \cap G_{\text{oreg}}(F_{S_0})$ and γ_0 is as above. As $M \supsetneq M_0$, equation (41) and Lemma 2.1 imply that

$$\sum_{v\in S_0} \varepsilon_M(\delta_{0,v}) = \sum_{v\in S_0} \varepsilon_M(\delta_v) = 0.$$

By equation (32) and Corollary 3.1 this constitutes a proof of Theorem A (i) for $M \supsetneq M_0$.

To take care of the case $M = M_0$, we must show that

$$\varepsilon_{M_0}(\gamma) = 0,\ \gamma \in M_0(F_v) \cap G_{\text{oreg}}(F_v).$$

Bearing in mind the induction hypothesis of §10, we may argue as in the proof of Corollary 3.1 (cf. equation (4)) to conclude that

$$\int_{M_0(F_v)/\mu_n^M} \varepsilon_{M_0}(\gamma) I_{\tilde{G}}(\gamma', \tilde{f}_1) I_{\tilde{G}}(\gamma', \tilde{f}_2) d\gamma = 0,$$

for any genuine Hecke function $\tilde{f}_1$ on $\tilde{G}(F_v)$ and $\tilde{f}_2 \in \mathcal{H}(\tilde{G}(F_v))$. Fix $\gamma_1 \in M_0(F_v) \cap G_{\text{oreg}}(F_v)$. Since $\mathcal{H}(\tilde{G}(F_v))$ is dense in $C_c^\infty(\tilde{G}(F_v))$, we may let $\tilde{f}_2$ approach the antigenuine analogue of Dirac measure on $\tilde{G}(F_v)$ at γ_1'. The Weyl integration formula then implies that the function

$$\tilde{\gamma} \mapsto |D^G(\mathbf{p}(\tilde{\gamma}))|^{1/2} I_{\tilde{G}}(\tilde{\gamma}, \tilde{f}_2),\ \tilde{\gamma} \in \tilde{M}_0(F_v),$$

approaches the antigenuine analogue of Dirac measure on $\tilde{M}_0(F_v)$ at γ_1'. It then follows from our equation that

$$\varepsilon_{M_0}(\gamma_1)|D^G(\gamma_1^n)|^{-1/2} I_{\tilde{G}}(\gamma_1', \tilde{f}_1) = 0,$$

and in turn that $\varepsilon_{M_0}(\gamma_1) = 0$. This concludes the proof of Theorem A (i).

All that remains to be done now is to prove Theorem A (ii) and Theorem B.

Proof of Theorem A (ii). We wish to show that

$$a^{\tilde{M}}(S, \gamma') = a^M(S, \gamma),\ \gamma \in M(F).$$

Suppose first that $\gamma \in M(F)$ has Jordan decomposition $\gamma = \sigma u$, where the semisimple element σ is not in $A_G(F)$ if $M = G$. Then

$$\dim(M_\sigma) < \dim(G),$$

so we may apply the induction hypothesis of §2 to decompositions (28) and (29) of [30] and the lemma follows. On the other hand, if $M = G$ and $\sigma \in A_G(F)$, then

$$a^{\tilde{G}}(S, \gamma') = a^{\tilde{G}}(S, u')$$

and

$$a^G(S, \gamma) = a^G(S, u),$$

by (28) and (29) of [30] respectively. It follows from Theorem A (i) and Lemma 11.1, where we may now take $M = M_0$, that

$$\sum_{\sigma \in A_G(F)} \sum_{u \in (\mathcal{U}_G(F))} \left(a^{\tilde{G}}(S, u') - a^G(S, u)\right) I_G^{\mathcal{M}}(\sigma u, \tilde{f}) = 0,$$

for any $\tilde{f} \in \mathcal{H}(\tilde{G}(\mathbf{A}))$. We may choose $\tilde{f} \in \mathcal{H}(\tilde{G}(\mathbf{A}))$ above so that for a fixed element $u_1 \in \mathcal{U}_G(F)$, we have

$$I_G^{\mathcal{M}}(\sigma u, \tilde{f}) = \begin{cases} 1, & \text{if } \sigma = 1 \text{ and } u = u_1, \\ 0, & \text{otherwise} \end{cases}$$

(§3.3, [38]). This clearly implies that $a^{\tilde{G}}(S, u_1') = a^G(S, u_1)$. □

It has already been shown in §7 that Theorem B (i) follows from Theorem A (i). This leaves us with a single proof to be completed.

Proof of Theorem B (ii). By the induction hypothesis of §6, we need only show that

$$a^{G,\Sigma}_{\mathrm{disc}}(\tilde{\pi}) = a^{\tilde{G}}_{\mathrm{disc}}(\tilde{\pi}),\ \tilde{\pi} \in \Pi(\tilde{G}(\mathbf{A})^1).$$

Let ν_1 be the infinitesimal character of the archimedean factor of some fixed $\tilde{\pi} \in \Pi(\tilde{G}(\mathbf{A})^1)$, and let K_1 be a compact open subgroup of $\prod_{v \notin S_\infty} K_v$ such that $\tilde{\pi}$ is bi-$\mathbf{s}(K_1)$-invariant. Let $\Pi_{\nu_1, K_1}(\tilde{G}(\mathbf{A})^1)$ be the set of bi-$\mathbf{s}(K_1)$-invariant representations in $\Pi(\tilde{G}(\mathbf{A})^1)$ with infinitesimal character ν_1. In the process of proving Proposition 11.1 (cf. (16.6), II of [13]), one obtains

$$\sum_{\tilde{\pi} \in \Pi_{\nu_1, K_1}(\tilde{G}(\mathbf{A})^1)} \left(a^{G,\Sigma}_{\mathrm{disc}}(\tilde{\pi}) - a^{\tilde{G}}_{\mathrm{disc}}(\tilde{\pi})\right) \operatorname{tr}\left(\tilde{\pi}(\tilde{f}^1)\right) = 0,$$

for any $\tilde{f} \in \mathcal{H}(\tilde{G}(\mathbf{A})^1)$ which is bi-$\mathbf{s}(K_1)$-invariant. This sum is finite by Lemma 4.2 of [8], and the linear forms,

$$\tilde{f} \mapsto \mathrm{tr}\left(\tilde{\pi}(\tilde{f}^1)\right), \ \tilde{\pi} \in \Pi_{\nu_1, K_1}(\tilde{G}(\mathbf{A})^1),$$

on the space of bi-$\mathbf{s}(K_1)$-invariant functions in $\mathcal{H}(\tilde{G}(\mathbf{A}))$ are linearly independent. The result follows. □

13. A global correspondence

We derive a global correspondence from a special case of Theorem B (ii). The global datum $a^{\tilde{G}}_{\mathrm{disc}}(\tilde{\pi})$ is defined by the equation (cf. (4.3), [8] and (9.2), II, [13]),

$$\sum_{\tilde{\pi} \in \Pi(\tilde{G}(\mathbf{A})^1, t)} a^{\tilde{G}}_{\mathrm{disc}}(\tilde{\pi}) I_{\tilde{G}}(\tilde{\pi}, \tilde{f})$$
$$= \sum_{M \in \mathcal{L}} |W_0^M||W_0^G|^{-1} \sum_{s \in W(\mathfrak{a}_M)_{\mathrm{reg}}} |\det(s-1)_{\mathfrak{a}_M^G}|^{-1} \mathrm{tr}\left(\tilde{M}(s,0)\rho_{\tilde{Q},t}(0,\tilde{f})\right). \tag{43}$$

Here, $\tilde{f} \in \mathcal{H}(\tilde{G}(\mathbf{A}))$, Q is any element of $\mathcal{P}(M)$ and $\rho_{\tilde{Q},t}(0)$ is the induced representation of $\tilde{G}(\mathbf{A})^1$ obtained from the genuine subrepresentation of $\tilde{M}(\mathbf{A})^1$ on $L^2(\mathbf{s_0}(M(F))\backslash\tilde{M}(\mathbf{A})^1)$ which decomposes into a discrete sum of elements in $\Pi(\tilde{M}(\mathbf{A})^1, t)$. The term $\tilde{M}(s,0)$ is the global intertwining operator associated to an element in

$$W(\mathfrak{a}_M)_{\mathrm{reg}} = \{s \in W(\mathfrak{a}_M) : \det(s-1)_{\mathfrak{a}_M^G} \neq 0\}.$$

By Theorem B (ii) and the definitions of §§5–7, the right-hand side of (43) is equal to

$$\begin{aligned}
&\sum_{\tilde{\pi} \in \Pi(\tilde{G}(\mathbf{A})^1, t)} a^{G,\Sigma}_{\mathrm{disc}}(\tilde{\pi}) I^{\Sigma}_G(\tilde{\pi}, \tilde{f}) \\
= &\sum_{\pi \in \Pi(G(\mathbf{A})^1, nt)} a^G_{\mathrm{disc}}(\pi) n^{-1} \hat{I}_G(\pi, \tilde{f}') \\
= \ & n^{-1} \sum_{M \in \mathcal{L}} |W_0^M||W_0^G|^{-1} \sum_{s \in W(\mathfrak{a}_M)_{\mathrm{reg}}} |\det(s-1)_{\mathfrak{a}_M^G}|^{-1} \mathrm{tr}\left(M(s,0)\rho_{Q,nt}(0,f)\right),
\end{aligned}$$

where $f \in \mathcal{H}(G(\mathbf{A}))$ is a function whose "Fourier transform" f_G ((13), [30]) is equal to $\tilde{f}' \in \mathcal{I}(G(\mathbf{A}))$. (This uses the trace Paley-Wiener theorems of [16] and [15].) We would like to convert this identity of representations on $\tilde{G}(\mathbf{A})^1$ and $G(\mathbf{A})^1$ to an identity of representations on $\tilde{G}(\mathbf{A})$ and $G(\mathbf{A})$. We may embed the connected component $A_G(\mathbf{R})^0$ of $A_G(\mathbf{R})$ diagonally into $\prod_{v \in S_\infty} A_G(F_v)$. The map,

$$H_M : A_G(\mathbf{R})^0 \to \mathfrak{a}_G,$$

is an isomorphism which allows us to pull back the Haar measure on $\mathfrak{a}_G$ to $A_G(\mathbf{R})^0$. Given $\lambda \in i\mathfrak{a}_G^*$, we define

$$I_{\mathrm{disc},t,\lambda}(\tilde{f}) = \int_{A_G(\mathbf{R})^0} \sum_{\tilde{\pi} \in \Pi(\tilde{G}(\mathbf{A})^1, t)} a^G_{\mathrm{disc}}(\tilde{\pi}) I_{\tilde{G}}(\tilde{\pi}, \tilde{f}_{a'}) e^{\lambda(H_G(\mathbf{p}(a')))} da',$$

where

$$\tilde{f}_{a'}(\tilde{\gamma}) = \tilde{f}(a'\tilde{\gamma}), \ \tilde{\gamma} \in \tilde{G}(\mathbf{A}), \ a \in A_G(\mathbf{R})^0.$$

Since $\tilde{f}'_{a'} = \tilde{f}'_a$, $\lambda(H_G(\mathbf{p}(a'))) = \lambda'(H_G(a))$ and $da' \circ \mathbf{s} = nda$, our earlier identity becomes

$$I_{\mathrm{disc},t,\lambda}(\tilde{f}) = I_{\mathrm{disc},nt,\lambda'}(f). \tag{44}$$

This identity may be rewritten as

$$\sum_{M\in\mathcal{L}} |W_0^M||W_0^G|^{-1} \sum_{s\in W(\mathfrak{a}_M)_{\mathrm{reg}}} |\det(s-1)_{\mathfrak{a}_M^G}|^{-1} \mathrm{tr}\left(\tilde{M}(s,0)\rho_{\tilde{Q},t,\lambda}(0,\tilde{f})\right)$$
$$= \sum_{M\in\mathcal{L}} |W_0^M||W_0^G|^{-1} \sum_{s\in W(\mathfrak{a}_M)_{\mathrm{reg}}} |\det(s-1)_{\mathfrak{a}_M^G}|^{-1} \mathrm{tr}\left(M(s,0)\rho_{Q,nt,\lambda'}(0,f)\right).$$

Here

$$\rho_{\tilde{Q},t,\lambda}(0) = \bigoplus_{\tilde{\pi}\in\Pi_{\mathrm{disc}}(\tilde{M}(\mathbf{A}),t,\lambda)} \mathrm{Ind}_{\tilde{Q}}^{\tilde{G}}\tilde{\pi},$$

where $\Pi_{\mathrm{disc}}(\tilde{M}(\mathbf{A}),t,\lambda)$ is the set of irreducible, λ-equivariant and genuine subrepresentations of $\tilde{M}(\mathbf{A})$ on $L^2(\mathbf{s_0}(M(F))\backslash\tilde{M}(\mathbf{A}))$ whose infinitesimal characters' imaginary parts have norm equal to $t \geq 0$. Since $\tilde{M}(s,0)$ intertwines $\rho_{\tilde{Q},t,\lambda}(0)$ with itself, Schur's lemma implies that there exist complex numbers $c_{s,\tilde{\pi}}$ such that

$$\mathrm{tr}\left(\tilde{M}(s,0)\rho_{\tilde{Q},t,\lambda}(0,\tilde{f})\right) = \sum_{\tilde{\pi}\in\Pi_{\mathrm{disc}}(\tilde{M}(\mathbf{A}),t,\lambda)} c_{s,\tilde{\pi}}\mathrm{tr}\left((\mathrm{Ind}_{\tilde{Q}}^{\tilde{G}}\tilde{\pi})(\tilde{f})\right).$$

Set

$$c_{\tilde{\pi}} = |W_0^M||W_0^G|^{-1} \sum_{s\in W(\mathfrak{a}_M)_{\mathrm{reg}}} |\det(s-1)_{\mathfrak{a}_M^G}|^{-1} c_{s,\tilde{\pi}},\ \tilde{\pi}\in\Pi_{\mathrm{disc}}(\tilde{M}(\mathbf{A}),t,\lambda).$$

Identity (44) now has the form

$$\sum_{M\in\mathcal{L}} \sum_{\tilde{\pi}\in\Pi_{\mathrm{disc}}(\tilde{M}(\mathbf{A}),t,\lambda)} c_{\tilde{\pi}}\mathrm{tr}\left(\tilde{\pi}^{\tilde{G}}(\tilde{f})\right) - \sum_{\pi\in\Pi_{\mathrm{disc}}(M(\mathbf{A}),nt,\lambda')} c_{\pi}\mathrm{tr}\left(\pi^G(f)\right) = 0. \tag{45}$$

Given a valuation v such that $|n|_v = 1$, let $\mathcal{H}(\tilde{G}(F_v),K_v)$ be the subset of $\mathcal{H}(\tilde{G}(F_v))$ of $\mathbf{s}(K_v)$-bi-invariant functions. Suppose $\tilde{\pi} = \bigotimes_v \tilde{\pi}_v \in \Pi(\tilde{G}(\mathbf{A}))$ and $\pi = \bigotimes_v \pi_v \in \Pi(G(\mathbf{A}))$. We say that $\tilde{\pi}$ corresponds (or lifts) weakly to π if

$$\mathrm{tr}\left(\tilde{\pi}_v(\tilde{f}_v)\right) = \mathrm{tr}\left(\pi_v(f_v)\right),$$

for any matching functions $\tilde{f}_v \in \mathcal{H}(\tilde{G}(F_v),K_v)$, $f_v \in \mathcal{H}(G(F_v),K_v)$ and almost every valuation v of F. We say that $\tilde{\pi}$ corresponds (or lifts) to π if

$$\mathrm{tr}\left(\tilde{\pi}_v(\tilde{f}_v)\right) = \mathrm{tr}\left(\pi_v(f_v)\right),$$

for any matching functions $\tilde{f}_v \in \mathcal{H}(\tilde{G}(F_v))$, $f_v \in \mathcal{H}(G(F_v))$ and every valuation v of F. We say that π is metic if π_v is metic (§2) for all valuations v and that $f = \prod_v f_v \in \mathcal{H}(G(\mathbf{A}))$ matches $\tilde{f} = \prod_v \tilde{f}_v \in \mathcal{H}(\tilde{G}(\mathbf{A}))$ if f_v matches $\tilde{f}_v$ for all valuations v.

Theorem 13.1. *Under Assumptions 1 and 2, the following assertions are true.*

(i) *Suppose $t \geq 0$, $\lambda \in i\mathfrak{a}_G^*$ and $\tilde{\pi}_0 = \bigotimes_v \tilde{\pi}_{0,v}$ belongs to $\Pi_{\mathrm{disc}}(\tilde{G}(\mathbf{A}),t,\lambda)$. Then there exist a unique Levi subgroup $L \in \mathcal{L}$ and a unique representation $\pi_0 \in \Pi_{\mathrm{disc}}(L(\mathbf{A}),nt,\lambda')$ such that π_0 is metic and $\tilde{\pi}_0$ corresponds weakly to π_0^G.*

(ii) *Suppose* $t \geq 0$, $\lambda \in i\mathfrak{a}_G^*$ *and* $\pi_0 = \pi_{0,v} \in \Pi_{\mathrm{disc}}(G(\mathbf{A}), nt, \lambda')$ *is metic. Then there exists* $\tilde{\pi}_0 \in \Pi_{\mathrm{disc}}(\tilde{G}(\mathbf{A}), t, \lambda)$ *such that* $\tilde{\pi}_0$ *corresponds weakly to* π_0. *Moreover, if* $\tilde{\pi} \in \Pi_{\mathrm{disc}}(\tilde{M}(\mathbf{A}), t, \lambda)$ *and* $\tilde{\pi}^{\tilde{G}}$ *corresponds weakly to* π_0, *then* $M = G$.

(iii) *Suppose the representation* π_0 *of* (ii) *is cuspidal. Then there exists a unique representation* $\tilde{\pi}_0 \in \Pi_{\mathrm{disc}}(\tilde{G}(\mathbf{A}), t, \lambda)$ *which corresponds to* π_0. *Moreover, if* $\tilde{\pi}_1 \in \Pi_{\mathrm{disc}}(\tilde{M}(\mathbf{A}), t, \lambda)$ *and* $\tilde{\pi}_1^{\tilde{G}}$ *corresponds weakly to* π_0, *then* $\tilde{\pi}_1 = \tilde{\pi}_0$.

Proof of (i). We assume inductively that (i) holds if G is replaced by a proper Levi subgroup in $\mathcal{L}$. Let V be the finite set of valuations at which $\tilde{\pi}_0$ is ramified. Suppose $w \notin V$. By §§11, 16, 17 of [18], we can assign to each unramified representation in $\Pi(\tilde{G}(F_w))$ a Satake parameter. Recall that a Satake parameter corresponds to an element of $(\mathbf{C}^r)^{S_r}$, the set of equivalence classes of $\mathbf{C}^r$ under permutation. Suppose $\tilde{f} = \prod_v \tilde{f}_v \in \mathcal{H}(\tilde{G}(\mathbf{A}))$ such that $\tilde{f}_w \in \mathcal{H}(\tilde{G}(F_w), K_w)$, $f = \prod_v f_v \in \mathcal{H}(G_v)$ matches $\tilde{f}$, and $t_w \in (\mathbf{C}^r)^{S_r}$. Set

$$a(t_w) = \sum_{M \in \mathcal{L}} \left(\sum_{\tilde{\pi}} c_{\tilde{\pi}} \prod_{v \neq w} \mathrm{tr}\left(\tilde{\pi}_v^{\tilde{G}}(\tilde{f}_v)\right) - \sum_{\pi} c_{\pi} \prod_{v \neq w} \mathrm{tr}\left(\pi_v^G(f_v)\right) \right),$$

where the first sum is parametrized by the representations $\tilde{\pi} = \bigotimes_v \tilde{\pi}_v$ belonging to $\Pi_{\mathrm{disc}}(\tilde{M}(\mathbf{A}), t, \lambda)$ such that $\tilde{\pi}_w$ has Satake parameter t_w and the second sum is parametrized by the representations $\pi = \bigotimes_v \tilde{\pi}_v$ belonging to $\Pi_{\mathrm{disc}}(M(\mathbf{A}), nt, \lambda')$ such that π_w has Satake parameter nt_w. By Theorem 19 of [18] $f_w \in \mathcal{H}(G(F_w), K_w)$ and so equation (45) implies that

$$\sum_{t_w \in (\mathbf{C}^r)^{S_r} - \{0\}} a(t_w) \tilde{f}_w^{\vee}(t_w) = 0,$$

where $\tilde{f}_w^{\vee}$ is the Satake transform (§11, [18]) of $\tilde{f}_w$. It follows from the Satake isomorphism that

$$a(t_w) = 0, \ t_w \in (\mathbf{C}^r)^{S_r} - \{0\}.$$

Suppose $t_{\tilde{\pi}_0, w}$ is the Satake parameter of $\pi_{0,w}$. We can combine the earlier argument with an induction argument on the number of valuations outside of V to conclude that

$$(46) \quad \sum_{M \in \mathcal{L}} \sum_{\tilde{\pi} \in \Pi(\tilde{M}, \tilde{\pi}_0, V)} c_{\tilde{\pi}} \prod_{v \in V} \mathrm{tr}\left(\tilde{\pi}_v^{\tilde{G}}(\tilde{f}_v)\right) = \sum_{M \in \mathcal{L}} \sum_{\pi \in \Pi(M, \tilde{\pi}_0, V)} c_{\pi} \prod_{v \in V} \mathrm{tr}\left(\pi_v^G(f_v)\right).$$

Here, $\Pi(\tilde{M}, \tilde{\pi}_0, V)$ is the set of representations,

$$\tilde{\pi} = \bigotimes_v \tilde{\pi}_v \in \Pi_{\mathrm{disc}}(\tilde{M}(\mathbf{A}), t, \lambda),$$

which satisfy

$$\mathrm{tr}\left(\tilde{\pi}_w^{\tilde{G}}(\tilde{h})\right) = \tilde{h}^{\vee}(t_{\tilde{\pi}_0, w}), \ \tilde{h} \in \mathcal{H}(\tilde{G}(F_w), K_w), \ w \notin V,$$

and $\Pi(M, \tilde{\pi}_0, V)$ is the set of representations,

$$\pi = \bigotimes_v \pi_v \in \Pi_{\mathrm{disc}}(M(\mathbf{A}), nt, \lambda'),$$

which satisfy

$$\mathrm{tr}\left(\pi_w^G(h)\right) = h^{\vee}(nt_{\tilde{\pi}_0, w}), \ h \in \mathcal{H}(G(F_w), K_w), \ w \notin V.$$

Of course, $\tilde{\pi}_0$ is an element of set $\Pi(\tilde{G}, \tilde{\pi}_0, V)$ and $c_{\tilde{\pi}_0} = 1$. If $\Pi(\tilde{M}, \tilde{\pi}_0, V)$ is empty for every $M \neq G$, then Lemma 16.1.1 of [22], applied to equation (46), implies that $\Pi(M, \tilde{\pi}_0, V)$ is not empty for some $M \in \mathcal{L}$. If $\Pi(\tilde{M}, \tilde{\pi}_0, V)$ is not empty for some $M \neq G$, then our induction assumption implies that $\Pi(M, \tilde{\pi}, V_1)$ is not empty for some finite set $V_1 \supset V$. In any case, we have shown the existence of π_0 as in the theorem. The uniqueness of this representation follows from the strong multiplicity one property for the cuspidal representations of general linear groups (Theorem 4.4, [23]) and the construction of the discrete spectrum of general linear groups in terms of the cuspidal representations ([34]).

Proof of (ii). Suppose V is the finite set of valuations at which π_0 is ramified. Given $w \notin V$, set $t_{\pi_0,w}$ to be the Satake parameter of $\pi_{0,w}$. Given a finite set $V_1 \supset V$, define $\Pi(\tilde{M}, \pi_0, V_1)$ to be the set of representations

$$\tilde{\pi} = \bigotimes_v \tilde{\pi}_v \in \Pi_{\mathrm{disc}}(\tilde{M}(\mathbf{A}), t, \lambda)$$

which satisfy

$$\mathrm{tr}\left(\tilde{\pi}_w^{\tilde{G}}(\tilde{h})\right) = \tilde{h}^\vee(nt_{\pi_0,w}),\ \tilde{h} \in \mathcal{H}(\tilde{G}(F_w), K_w),\ w \notin V.$$

Arguing as in (i) and applying the strong multiplicity property for general linear groups, we obtain

$$\prod_{v \in V_1} \mathrm{tr}\left(\pi_{0,v}^G(f_v)\right) = \sum_{M \in \mathcal{L}} \sum_{\tilde{\pi} \in \Pi(\tilde{M}, \pi_0, V_1)} c_{\tilde{\pi}} \prod_{v \in V_1} \mathrm{tr}\left(\tilde{\pi}_v^{\tilde{G}}(\tilde{f}_v)\right). \tag{47}$$

Since the left-hand side is not zero, there exists a representation $\tilde{\pi}_0$ belonging to $\Pi_{\mathrm{disc}}(\tilde{M}(\mathbf{A}), t, \lambda)$ such that $\tilde{\pi}^{\tilde{G}}$ corresponds weakly to π_0. Furthermore, any $\tilde{\pi} \in \Pi_{\mathrm{disc}}(\tilde{M}(\mathbf{A}), t, \lambda)$ such that $\tilde{\pi}^{\tilde{G}}$ corresponds weakly to π_0 occurs as a summand on the right-hand side for some $V_1 \supset V$. By (i) (with G replaced by M) there exist $M_1 \in \mathcal{L}^M$ and $\pi_1 \in \Pi_{\mathrm{disc}}(M_1, nt, \lambda')$ such that $\tilde{\pi}_0$ corresponds weakly to π_1^M. In particular, $\pi_{1,v}$ and $\pi_{0,v}$ have the same Satake parameters at almost every valuation v. By the strong multiplicity one property mentioned earlier, it follows that $\pi_1 = \pi_0$ and in turn that $M_1 = M = G$.

Proof of (iii). According to Theorem 27.3 of [18], for every valuation v of F, there exists a unitary representation $\tilde{\pi}_{0,v} \in \Pi(\tilde{G}(F_v))$ such that

$$\mathrm{tr}\left(\tilde{\pi}_{0,v}(\tilde{f}_v)\right) = \mathrm{tr}\left(\pi_{0,v}(f_v)\right),$$

for any matching functions $\tilde{f}_v \in \mathcal{H}(\tilde{G}(F_v))$ and $f_v \in \mathcal{H}(G(F_v))$. Together with the second assertion of (ii) this allows us to rewrite equation (47) as

$$\prod_{v \in V_1} \mathrm{tr}\left(\tilde{\pi}_{0,v}(\tilde{f})\right) = \sum_{\tilde{\pi} \in \Pi(\tilde{G}, \pi_0, V_1)} \prod_{v \in V_1} \mathrm{tr}\left(\tilde{\pi}_v(\tilde{f})\right).$$

It follows from Lemma 16.1.1, applied to this equation, that there exists a unique representation $\tilde{\pi}_0 \in \Pi(\tilde{G}, \pi_0, V_1)$ which corresponds to π_0. The second assertion of (iii) follows from the second assertion of (ii) and the fact that the finite set $V_1 \supset V$ is arbitrary. □

Theorem 13.1 (iii) yields the multiplicity one and strong multiplicity one property for the representations of $\tilde{G}(\mathbf{A})$ which lift to cuspidal representations of $G(\mathbf{A})$. We expect the same properties to hold for representations of $\tilde{G}(\mathbf{A})$ which lift to any representation in the discrete spectrum of $G(\mathbf{A})$. One could attempt to prove this by showing that the local metaplectic correspondence preserves character identities for the local factors of representations in the residual spectrum of $G(\mathbf{A})$ and then argue as in Theorem 13.1 (iii). Let us describe the residual spectrum of $G(\mathbf{A})$ in more detail so that we can state these character identities as a precise conjecture.

Recall decomposition (4) of [30],

$$M = M(1) \times \cdots \times M(\ell) \cong \mathrm{GL}(r_1) \times \cdots \mathrm{GL}(r_\ell).$$

Suppose $r_1 = \cdots = r_\ell$, so that $r = \ell r_1$. Suppose further that $\pi = \bigotimes_v \pi_v$ is a unitary, cuspidal and metic representation of $\mathrm{GL}(r_1, \mathbf{A})$. Let $\bigotimes_{i=1}^{\ell} \pi |\det(\cdot)|^{(\ell(2i-1))/2}$ be the representation of $M(\mathbf{A})$ given by

$$\begin{pmatrix} \gamma_1 & & 0 \\ & \ddots & \\ 0 & & \gamma_\ell \end{pmatrix} \mapsto \prod_{i=1}^{\ell} \pi(\gamma_i) |\det(\gamma_i)|^{\frac{\ell-(2i-1)}{2}}, \ \gamma_1, \ldots, \gamma_\ell \in \mathrm{GL}(r_1, \mathbf{A}),$$

where $|\cdot| = \bigotimes_v |\cdot|_v$, and let $P \in \mathcal{P}(M)$ be the unique parabolic subgroup containing the group of upper-triangular matrices. The induced representation $\mathrm{Ind}_P^G \left(\bigotimes_{i=1}^{\ell} \pi |\det(\cdot)|^{(\ell-(2i-1))/2} \right)$ has a unique irreducible quotient which belongs to the discrete spectrum of $G(\mathbf{A})$ (§2, [21]). Conversely, any discrete representation of $G(\mathrm{A})$ is of the above form ([34]).

Conjecture 1. *Suppose that v is a nonarchimedean valuation of F and $\pi_v \in \Pi(\mathrm{GL}(r_1, F_v))$ and $P \in \mathcal{P}(M)$ are as above. Suppose further that π_v is metic and that π_1 is the unique irreducible quotient of $\mathrm{Ind}_P^G \left(\pi_v |\det(\cdot)|_v^{(\ell-(2i-1))/2} \right)$. Then there exists $\tilde{\pi}_v \in \Pi(\widetilde{\mathrm{GL}}(r_1, F_v))$ such that*

$$\mathrm{Ind}_{\tilde{P}}^{\tilde{G}} \left(\tilde{\pi}_v |\det \circ \mathbf{p}(\cdot)|_v^{\frac{\ell-(2i-1)}{2}} \right)$$

has a unique irreducible quotient $\tilde{\pi}_1$ which satisfies

$$\mathrm{tr}\left(\tilde{\pi}_1(\tilde{f}) \right) = \mathrm{tr}\left(\pi_1(f) \right)$$

for any matching functions $\tilde{f} \in \mathcal{H}(\tilde{G}(F_v))$ and $f \in \mathcal{H}(G(F_v))$.

This conjecture can be shown to hold in the case that π_v is supercuspidal by using an argument in the proof of Theorem 29.1 in [18]. Conjecture 1 would follow from a character identity of unitary representations if one could generalize the work of Bernstein, Tadić and Zelevinsky ([14], [39], [37]) to $\tilde{G}(F_v)$ (see also [20]).

Taking the case $r = 2$ into consideration ([17]), we make a second conjecture.

Conjecture 2. *Suppose $t \geq 0$, $\lambda \in i\mathfrak{a}_G^*$ and $\tilde{\pi}$ belongs to $\Pi_{\mathrm{disc}}(\tilde{G}(\mathbf{A}), t, \lambda)$. Then there exists a unique metic representation $\pi \in \Pi_{\mathrm{disc}}(G(\mathbf{A}), nt, \lambda')$ such that $\tilde{\pi}$ corresponds to π. Moreover, if $\tilde{\pi}_1$ is a representation in $\Pi_{\mathrm{disc}}(\tilde{M}(\mathbf{A}), t, \lambda)$ such that $\tilde{\pi}_1^{\tilde{G}}$ corresponds weakly to π, then $\tilde{\pi}_1 = \tilde{\pi}$. This correspondence maps cuspidal representations to cuspidal representations and residual representations to residual representations.*

This conjecture holds true given the general strong multiplicity one property, which follows from Conjecture 1, and given a characterization of the residual spectrum of $\tilde{G}(\mathbf{A})$ as in [34]. We refer the reader to [36] for a discussion of the residual spectrum of $\tilde{G}(\mathbf{A})$. The final assertion of Conjecture 2 is false if Assumption 1 is not assumed. This can be seen in [17] and §29 of [18].

14. Appendix

Suppose F_v is nonarchimedean and σ is a semisimple element of $G(F_v)$ such that $\dim(G_\sigma) < \dim(G)$. It is implicit in the induction hypothesis of §2 that the results of the proof of Theorem A apply to $G_\sigma(F_v)$ just as they do for $G(F_v)$. In particular, the results depending on the fixed integer $0 \leq m \leq n-1$ must hold for $G_\sigma(F_v)$ just as they do for $G(F_v)$. The purpose of this appendix is to convince the reader that these particular results do indeed hold for $G_\sigma(F_v)$ under Assumption 1.

There are two such results and both occur in the preparatory paper [30]. The first result pertains to the vanishing of some local geometric terms in the trace formula. This is the local geometric vanishing property of Lemma 8.3 in [30]. The other result depending on m occurs in the Appendix (Lemma 10.1, [30]) and is used to show that the local metaplectic correspondence commutes with parabolic induction. In both of these results the integer m appears purely by way of the commutator computation of Proposition 0.1.5 in [24]. As $\widetilde{G_\sigma}(F_v)$ is a subgroup of $\tilde{G}(F_v)$, the analogous commutator computation in $\widetilde{G_\sigma}(F_v)$ is identical to the one in the ambient group $\tilde{G}(F_v)$. This observation should be sufficient to convince the reader that Lemma 8.3 of [30] follows for $G_\sigma(F_v)$ exactly as it does for $G(F_v)$.

We shall reproduce Lemma 10.1 of [30] here in the context of $G_\sigma(F_v)$. In §1 of [25] it is shown that there are integers, $1 \leq a_1, \dots, a_k \leq r$, and field extensions, $E_1, \dots, E_k$, of F_v such that $G_\sigma(F_v)$ is isomorphic to $\mathrm{GL}(a_1, E_1) \times \cdots \times \mathrm{GL}(a_k, E_k)$. Suppose L is a Levi subgroup of of G_σ which is defined over F. Then there exist integers $m_0 = 0 < m_1 < \cdots < m_k$ and $b_1, \dots, b_{m_k}$ such that $\sum_{j=m_{i-1}+1}^{m_i} b_j = a_i$, for $1 \leq i \leq k$, and $L(F_v)$ is isomorphic to

$$\prod_{j=1}^{m_1} \mathrm{GL}(b_j, E_1) \times \cdots \times \prod_{j=m_{k-1}+1}^{m_k} \mathrm{GL}(b_j, E_k).$$

Suppose $1 \leq i \leq k$, $m_{i-1} + 1 \leq j < m_i$ and set $L(j)(F_v)$ to be the subgroup of $L(F_v)$ which corresponds to $\mathrm{GL}(b_j, E_i)$ in this isomorphism.

Let $(\cdot, \cdot)_{F_v} : F_v^\times \times F_v^\times \to \mu_n$ be the nth Hilbert symbol on F_v and let B be a maximal subgroup of $F_v^\times$ with respect to the property that $(x_1, x_2)_{F_v} = 1$ for all $x_1, x_2 \in B$. Set

$$\tilde{L}^B(j)(F_v) = \{\tilde{\gamma} \in \tilde{L}(j)(F_v) : \det(\mathbf{p}(\tilde{\gamma})) \in B\},$$

where the determinant above is taken with respect to the ambient group $G(F_v)$. It is a simple matter to check that $\tilde{L}^B(j)(F_v)$ is a normal subgroup of finite index in $\tilde{L}(j)(F_v)$.

Suppose $\tilde{\pi}$ is a genuine irreducible admissible representation of $\tilde{L}(j)(F_v)$. Its restriction to $\tilde{L}^B(j)(F_v)$ is the sum of conjugates of some irreducible representation $\tilde{\rho}$ of $\tilde{L}^B(j)(F_v)$. In other words, we have

$$\tilde{\pi}|_{\tilde{L}^B(j)} = \sum_\gamma \tilde{\rho}^\gamma,$$

where the sum runs over representatives γ of cosets in $\tilde{L}(j)(F_v)/\tilde{L}^B(j)(F_v)$ and

$$\tilde{\rho}^\gamma(\gamma_1) = \tilde{\rho}(\gamma\gamma_1\gamma^{-1}), \ \gamma_1 \in \tilde{L}^B(j)(F_v).$$

Lemma 14.1. *Suppose that γ is a representative as above. Then $\tilde{\rho}$ is not equivalent to $\tilde{\rho}^\gamma$ unless $L(j)(F_v) \cong F_v^\times$ or $\gamma \in \tilde{L}^B(j)(F_v)$.*

Proof. If $L(j)(F_v) \cong F_v^\times$, then $\tilde{L}(j)(F_v) \cong F_v^\times \times \mu_n$. In particular, $\tilde{L}(j)(F_v)$ is abelian and $\tilde{\rho} = \tilde{\rho}^\gamma$. Suppose that $L(j)(F_v)$ is not isomorphic to $F_v^\times$. By using the Iwasawa decomposition, it is easy to see that representatives of the quotient $\tilde{L}(j)(F_v)/\tilde{L}^B(j)(F_v)$ may be taken to be diagonal matrices in $\mathrm{GL}(b_j, E_i)$. Let γ be such a representative corresponding to the diagonal element

$$\begin{pmatrix} \gamma_1 & & 0 \\ & \ddots & \\ 0 & & \gamma_{b_j} \end{pmatrix} \in \mathrm{GL}(b_j, E_i),$$

and suppose that $\tilde{\rho}^\gamma$ is equivalent to $\tilde{\rho}$. In other words, suppose that there exists a linear isomorphism T such that

$$T \circ \tilde{\rho}^\gamma(\tilde{\gamma}) = \tilde{\rho}(\tilde{\gamma}) \circ T, \ \tilde{\gamma} \in \tilde{L}^B(j)(F_v).$$

Suppose $x \in B$ and choose $\tilde{\gamma} \in \tilde{L}^B(j)(F_v)$ such that $\mathbf{p}(\tilde{\gamma})$ corresponds to the scalar matrix

$$\begin{pmatrix} x & & 0 \\ & \ddots & \\ 0 & & x \end{pmatrix} \in \mathrm{GL}(b_j, F_v).$$

Let $(\cdot, \cdot)_{E_i}$ be the nth Hilbert symbol of E_i. By Proposition 0.1.5 of [24] and the properties of the Hilbert symbol, we have

$$\begin{aligned} \tilde{\rho}(\tilde{\gamma}) &= T \circ \tilde{\rho}^\gamma(\tilde{\gamma}) \circ T^{-1} \\ &= T \circ \tilde{\rho}(\gamma\tilde{\gamma}\gamma^{-1}) \circ T^{-1} \\ &= \left((\det(\gamma), \det(\mathbf{p}(\tilde{\gamma})))_{F_v}^{1+2m} / \prod_{t=1}^{b_j} (\gamma_t, x)_{E_i} \right) T \circ \tilde{\rho}(\tilde{\gamma}) \circ T^{-1} \\ &= \big((\det(\gamma), \det(\mathbf{p}(\tilde{\gamma})))_{F_v}^{1+2m} / (N_{E_i/F_v}(\gamma_1 \cdots \gamma_{b_j}), x)_{F_v} \big) \, T \circ \tilde{\rho}(\tilde{\gamma}) \circ T^{-1} \\ &= \big((\det(\gamma), \det(\mathbf{p}(\tilde{\gamma})))_{F_v}^{1+2m} / (\det(\gamma), x)_{F_v} \big) \, T \circ \tilde{\rho}(\tilde{\gamma}) \circ T^{-1}. \end{aligned}$$

It may be verified by following 0.1.1 of [24] that $\tilde{\gamma}$ is in the center of $\tilde{L}^B(j)(F_v)$ and so, by Schur's lemma, $\tilde{\rho}(\tilde{\gamma})$ is a nonzero scalar operator. Consequently the above identity reduces to

$$(\det(\gamma), x)_{F_v}^{b_j(1+2m)-1} = 1.$$

As n and $b_j(1+2m)-1$ are relatively prime by Assumption 1, we have $(\det(\gamma), x)_{F_v} = 1$. The element $x \in B$ was chosen arbitrarily, so this means that $\gamma \in \tilde{L}^B(j)(F_v)$. □

This lemma can now be employed in showing how the genuine representations of $\tilde{L}(F_v)$ are related to those of $\tilde{L}(j)(F_v)$ as in §26.2 of [18] and the Appendix of [30].

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