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SPHERICAL FUNCTIONS OF THE SYMMETRIC SPACE $G(\mathbb{F}_{q^2})/G(\mathbb{F}_q)$

ANTHONY HENDERSON

ABSTRACT. We apply Lusztig's theory of character sheaves to the problem of calculating the spherical functions of $G(\mathbb{F}_{q^2})/G(\mathbb{F}_q)$, where G is a connected reductive algebraic group. We obtain the solution for generic spherical functions for any G, and for all spherical functions when $G = GL_n$. The proof includes a result about convolution of character sheaves and its interaction with the associated two-sided cells.

0. INTRODUCTION

Let Γ be a finite group and $\tau : \Gamma \to \Gamma$ a nontrivial group involution. Let Γ^τ be the fixed-point subgroup. Then the quotient set Γ/Γ^τ is called a *finite symmetric space*. For any function $f : \Gamma \to \overline{\mathbb{Q}}$, we write $\mathrm{Ave}(f)$ for the function on Γ/Γ^τ given by averaging over the coset

$$\mathrm{Ave}(f)(\gamma\Gamma^\tau) = \frac{1}{|\Gamma^\tau|} \sum_{\gamma' \in \Gamma^\tau} f(\gamma\gamma').$$

The functions $\mathrm{Ave}(\chi)$ where χ is an irreducible constituent of $\mathrm{Ind}_{\Gamma^\tau}^{\Gamma}(1)$ are called the *spherical functions* of the symmetric space Γ/Γ^τ. For example, when $\Gamma = \Gamma' \times \Gamma'$ and τ is the involution interchanging the factors, the spherical functions are (up to scalar multiple) the irreducible characters of Γ'.

Let G be a connected reductive algebraic group defined over the finite field $\mathbb{F}_q$. Take $\Gamma = G(\mathbb{F}_{q^2})$ and τ the restriction of the Frobenius map on G, so that $\Gamma^\tau = G(\mathbb{F}_q)$. This paper is concerned with the following problem.

Problem 0.1. Calculate the spherical functions of $G(\mathbb{F}_{q^2})/G(\mathbb{F}_q)$.

The analogous problem of calculating the irreducible characters of $G(\mathbb{F}_q)$ has a long history. The first significant progress was the solution of the case $G = GL_n$ (with the split $\mathbb{F}_q$-structure) by Green in [7]. For general groups, Lusztig in [12], building on joint work with Deligne in [4], provided a classification of irreducible characters, and results relating their values to those of certain basic functions $\mathrm{tr}(-, R_T^\theta)$. Work of Springer and Kazhdan provided an alternative geometric definition of these basic functions, which enabled Shoji to define algorithms for their computation. This in turn was generalized and expanded in Lusztig's crucial work [13] on *character sheaves*. Later work by Lusztig and Shoji among others has all

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but completed the calculation of the irreducible characters in full generality (see [18] for more details).

Problem 0.1 has not been so extensively studied. The most general results are contained in Lusztig's paper [15]: he classifies the spherical functions for all G, and computes their value at the identity coset. In the case $G = GL_n$ (with either split or non-split $\mathbb{F}_q$-structure) this had already been done in [6]. Given this classification, and the above-mentioned results on the character table of $G(\mathbb{F}_{q^2})$, the spherical functions can in principle be calculated directly from the definition. However, this quickly becomes impractical. To be more precise about Problem 0.1, we could replace "Calculate" by "Find an effective algorithm for computing", where "effective" means "not significantly worse than the algorithms for calculating irreducible characters of $G(\mathbb{F}_q)$".

The aim of this paper is to apply the theory of character sheaves to Problem 0.1. We will come well short of a complete solution, but our results are enough to solve Problem 0.1 in the case $G = GL_n$ (with either $\mathbb{F}_q$-structure; see Theorem 6.7 and Theorem 7.5), and for general groups in the case of generic spherical functions (see Theorem 5.7; here we need an assumption on characteristic, Assumption 4.7).

There are two main ideas involved. The first is to derive analogues for $\tilde{F}$ of Lusztig's results for F, where $F : G \to G$ is the Frobenius map with respect to $\mathbb{F}_q$ and $\tilde{F} : G \to G$ is the map $g \mapsto F(g)^{-1}$. As we will see below, the symmetric space G^{F^2}/G^F in question can be identified with the fixed-point set $G^{\tilde{F}}$ of $\tilde{F}$. So just as Lusztig considers the characteristic functions of F-stable character sheaves, which are functions on G^F closely related to the irreducible characters, we consider the characteristic functions of $\tilde{F}$-stable character sheaves, which are functions on $G^{\tilde{F}}$ somewhat more loosely related to the spherical functions. The second idea, which is necessary to make this relationship precise in the case $G = GL_n$, is a certain fact about convolution of character sheaves and its interaction with the associated two-sided cells (see below, Theorem 3.7). This fact was stated (in greater generality) by Grojnowski in the inspiring document [8], and our proof follows his sketch.

There is a strong similarity between the results proved in Sections 6 and 7 below for the symmetric spaces $GL_n(\mathbb{F}_{q^2})/GL_n(\mathbb{F}_q)$, $GL_n(\mathbb{F}_{q^2})/U_n(\mathbb{F}_{q^2})$ and the results of [1], which amount to a computation of the spherical functions of the symmetric space $GL_{2n}(\mathbb{F}_q)/Sp_{2n}(\mathbb{F}_q)$. Grojnowski noticed that the results of [1] can also be approached via character sheaves—specifically, the theory of character sheaves on symmetric spaces initiated by Ginzburg in [5]. I will expand on this in [9].

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Section 1 contains definitions and conventions concerning equivariant derived categories, perverse sheaves, $\mathbb{F}_q$-structures, and algebraic groups over a finite field. Section 2 concerns spherical functions and the map $\tilde{F}$. Section 3 introduces character sheaves and convolution thereof, and concludes with the crucial Theorem 3.7 and its corollaries. In Section 4 we define basic functions for $G^{\tilde{F}}$ analogous to the usual ones for G^F, and state some analogous results. Section 5 applies these results to solve Problem 0.1 in the case of generic spherical functions. Sections 6 and 7 present the solution of Problem 0.1 for the symmetric spaces $GL_n(\mathbb{F}_{q^2})/GL_n(\mathbb{F}_q)$ and $GL_n(\mathbb{F}_{q^2})/U_n(\mathbb{F}_{q^2})$ respectively; these sections are more explicitly combinatorial than the rest of the paper. (The case of $GL_2(\mathbb{F}_{q^2})/GL_2(\mathbb{F}_q)$ is worked out in full in Example 6.10.) Section 8 is devoted to the proof of Theorem 3.7, and Sections 9 and 10 to the proofs of some results stated in Section 4.

1. Equivariant Derived Categories and Perverse Sheaves

Let $\mathbb{F}_q$ be a finite field of characteristic p and cardinality q. Let k be an algebraic closure of $\mathbb{F}_q$. All varieties considered will be over k. When the variety structure is evident, we will identify a variety with its set of k-points.

Let l be a prime invertible in k. If X is a variety, we write $\mathbf{D}(X)$ for the bounded derived category of constructible $\overline{\mathbb{Q}_l}$-sheaves of finite rank on X, as defined in [2, 2.2.18]. Objects of $\mathbf{D}(X)$ are referred to as *complexes*. The cohomology sheaves of a complex K are written $\mathcal{H}^i K$. If $f : X \to Y$ is a morphism of varieties (henceforth simply called a *map*), we have the functors $f_! : \mathbf{D}(X) \to \mathbf{D}(Y)$ and $f^* : \mathbf{D}(Y) \to \mathbf{D}(X)$ as in [2], both of which commute with the shift operations $[m]$. We will need various well-known properties of these functors (such as base change).

Let $\mathbf{M}(X)$ denote the abelian full subcategory of $\mathbf{D}(X)$ whose objects are the *perverse sheaves*. Write ${}^pH^i : \mathbf{D}(X) \to \mathbf{M}(X)$, $i \in \mathbb{Z}$, for the perverse cohomology functors determined by the t-structure of which $\mathbf{M}(X)$ is the heart. If $\L$ is a local system on a nonsingular subvariety U of X, write $IC(\overline{U}, \L)[\dim U] \in \mathbf{M}(X)$ for its (shifted) *intersection cohomology extension*.

If G is an algebraic group acting on the variety X, write $\mathbf{D}^G(X)$ for the *G-equivariant derived category* in the sense of Bernstein and Lunts (see [3, 4.3]), a triangulated category with t-structure, and $\mathbf{M}^G(X)$ for its heart, the category of *G-equivariant perverse sheaves*. We have forgetful functors $\mathbf{D}^G(X) \to \mathbf{D}(X)$ and $\mathbf{M}^G(X) \to \mathbf{M}(X)$, which respect the t-structures, and are equivalences when $G = \{1\}$. Any complex K in the essential image of $\mathbf{D}^G(X) \to \mathbf{D}(X)$ is *weakly G-equivariant* in the sense that

$$a^*K \cong \overline{\mathbb{Q}_l} \boxtimes K \text{ in } \mathbf{D}(G \times X),$$

where $a : G \times X \to X$ is the action. If G is connected, then $\mathbf{M}^G(X) \to \mathbf{M}(X)$ is fully faithful. If in addition G has finitely many orbits on X, then it follows from [2, 4.3.1] that every simple $A \in \mathbf{M}^G(X)$ is of the form $IC(\overline{\O}, \L)[\dim \O]$ for some G-orbit $\O$ and simple G-equivariant local system $\L$ on $\O$.

If G also acts on Y, and $f : X \to Y$ is a G-equivariant map, there are functors $f_! : \mathbf{D}^G(X) \to \mathbf{D}^G(Y)$ and $f^* : \mathbf{D}^G(Y) \to \mathbf{D}^G(X)$ which lift their non-equivariant analogues (and have similar properties). As in [3, 2.2.5], we have that if $f : X \to Y$ is a principal G-bundle, then $f^* : \mathbf{D}^G(Y) \to \mathbf{D}^G(X)$ factors through an equivalence $\mathbf{D}(Y) \xrightarrow{\sim} \mathbf{D}^G(X)$. We write the inverse of this equivalence as $f_\flat : \mathbf{D}^G(X) \xrightarrow{\sim} \mathbf{D}(Y)$.

If T is a torus acting on X and $\L$ is a tame rank-one local system on T, we can define in an analogous way a category $\mathbf{D}_{\L}(X)$, the *(T, L)-covariant derived category*, which if $\L = \overline{\mathbb{Q}_l}$ is equivalent to $\mathbf{D}^T(X)$. Choose n such that $\L^{\otimes n} \cong \overline{\mathbb{Q}_l}$, and let $T(n)$ be the same as T but with the action $(t, x) \mapsto t^n.x$ on X, and trivial action on T, so that the action $a : T \times X \to X$ is $T(n)$-equivariant. Then $\mathbf{D}_{\L}(X)$ is defined as the full subcategory of $\mathbf{D}^{T(n)}(X)$ consisting of complexes K such that

$$a^*K \cong \L \boxtimes K \text{ in } \mathbf{D}^{T(n)}(T \times X).$$

Results about $\mathbf{D}_{\mathbb{L}}(X)$ can be deduced from results in [3]. In particular, $\mathbf{D}_{\mathbb{L}}(X)$ inherits a t-structure whose heart is the category $\mathbf{M}_{\mathbb{L}}(X)$ of (T, L)-*covariant perverse sheaves*, and we have forgetful functors $\mathbf{D}_{\mathbb{L}}(X) \to \mathbf{D}(X)$ and $\mathbf{M}_{\mathbb{L}}(X) \to \mathbf{M}(X)$ of which the latter is fully faithful. If an algebraic group G and a torus T have commuting actions on X, one can also define categories $\mathbf{D}_{\mathbb{L}}^G(X)$ and $\mathbf{M}_{\mathbb{L}}^G(X)$ relating to the above categories in obvious ways.

Suppose that the variety X is defined over $\mathbb{F}_q$, with Frobenius map $F : X \to X$, and $K \in \mathbf{D}(X)$ is equipped with an $\mathbb{F}_q$-structure. (If we are considering more than one $\mathbb{F}_q$-structure on X, we will speak of an $\mathbb{F}_q$-structure on K *relative to* F to specify the $\mathbb{F}_q$-structure with Frobenius map F.) We write ϕ_K^F for the induced isomorphism $F^*K \xrightarrow{\sim} K$, and define the *characteristic function* $\chi_K^F : X^F \to \overline{\mathbb{Q}_l}$ by

$$\chi_K^F(x) = \sum_i (-1)^i \mathrm{tr}(\mathcal{H}_x^i \phi_K^F, \mathcal{H}_x^i K).$$

More generally, for any isomorphism $\phi : F^*K \xrightarrow{\sim} K$ we define $\chi_{K,\phi}$ in the same way. If the action of G on X is defined over $\mathbb{F}_q$, and $K \in \mathbf{D}^G(X)$ has an $\mathbb{F}_q$-structure, then χ_K^F is a G^F-equivariant function on X^F.

We refer to [2, §5] for the notions of *pure* and *mixed* $\mathbb{F}_q$-structures, and their relations with the functors defined above. We will need the following result.

Lemma 1.1. *Let X be a variety with Frobenius map F. Let K be a complex on X with a mixed $\mathbb{F}_q$-structure. Let $\mathcal{A}_K$ be the set of isomorphism classes of simple constituents of ${}^pH^iK$ for some i. Let $\mathcal{A}_K^F$ be the set of isomorphism classes which are stable under F^*. Let $\dot{\mathcal{A}}_K^F \subseteq \dot{\mathcal{A}}_K$ be sets of representatives for the isomorphism classes in $\mathcal{A}_K^F \subseteq \mathcal{A}_K$, and choose an $\mathbb{F}_q$-structure for each $A \in \dot{\mathcal{A}}_K^F$. Then χ_K^F is a linear combination of $\{\chi_A^F \mid A \in \dot{\mathcal{A}}_K^F\}$.*

Proof. Since $\chi_K^F = \sum (-1)^i \chi_{{}^pH^iK}^F$, we can assume that K is perverse. Let X_0 be the $\mathbb{F}_q$-scheme giving rise to the $\mathbb{F}_q$-structure on X, and K_0 the perverse sheaf on X_0 giving rise to the $\mathbb{F}_q$-structure on K. By [2, 5.3.5], we have a filtration of K_0 by weights so that the ith piece of the filtration $(K_0)_i$ is pure of weight i. Let K_i be the complex on X obtained from $(K_0)_i$ by base change; then clearly $\chi_K^F = \sum \chi_{K_i}^F$. So we may assume that there is only one piece in the filtration, i.e. that K_0 is pure. Then by [2, 5.3.8], K is a semisimple perverse sheaf, so

$$K \cong \bigoplus_{A \in \dot{\mathcal{A}}_K} A \otimes \mathrm{Hom}_{\mathbf{M}(X)}(A, K).$$

Now for each $A \in \dot{\mathcal{A}}_K^F$, let $\sigma_A : \mathrm{Hom}_{\mathbf{M}(X)}(A, K) \to \mathrm{Hom}_{\mathbf{M}(X)}(A, K)$ be the map $v \mapsto \phi_K^F \circ F^*(v) \circ (\phi_A^F)^{-1}$. Then as in [13, (10.4.2)], we find that

$$\chi_K^F = \sum_{A \in \dot{\mathcal{A}}_K^F} \mathrm{tr}(\sigma_A, \mathrm{Hom}_{\mathbf{M}(X)}(A, K))\ \chi_A^F,$$

which proves the result. □

2. Spherical Functions

For any finite set X, write $\mathcal{F}(X)$ for the $\overline{\mathbb{Q}_l}$-vector space of functions $X \to \overline{\mathbb{Q}_l}$. If Γ is a group acting on X, write $\mathcal{F}^\Gamma(X)$ for the subspace of Γ-invariant functions. Fix once and for all a field involution $\overline{} : \overline{\mathbb{Q}_l} \to \overline{\mathbb{Q}_l}$ which sends every root of unity to its inverse. We define a Hermitian inner product $\langle \cdot, \cdot \rangle$ on $\mathcal{F}^\Gamma(X)$ by

$$\langle f, f' \rangle = \frac{1}{|X|} \sum_{x \in X} f(x) \overline{f'(x)}.$$

In particular, if Γ is a finite group, we will write $\mathcal{F}^\Gamma(\Gamma)$ for the space of $\overline{\mathbb{Q}_l}$-valued class functions on Γ, and $\widehat{\Gamma}$ for the set of irreducible characters, an orthonormal basis of $\mathcal{F}^\Gamma(\Gamma)$.

Let G be a connected reductive algebraic group over k, defined over $\mathbb{F}_q$, with Frobenius map $F : G \to G$. Write $\tilde{F}$ for the map $g \mapsto F(g)^{-1}$. So any F-stable subgroup H of G is also $\tilde{F}$-stable, and $\tilde{F}$ is the Frobenius map for some $\mathbb{F}_q$-structure of H as a variety (though not as a group, unless H is abelian). Note that $\tilde{F}^2 = F^2$. By Lang's Theorem,

$$G^{\tilde{F}} = \{gF(g)^{-1} \mid g \in G^{F^2}\}.$$

So we can identify G^{F^2}/G^F with $G^{\tilde{F}}$ via $gG^F \mapsto gF(g)^{-1}$; under this identification the left multiplication action of G^F on G^{F^2}/G^F becomes its conjugation action on $G^{\tilde{F}}$. We now restate the definitions of the introduction in these terms. For $f \in \mathcal{F}^{G^{F^2}}(G^{F^2})$, define $\mathrm{Ave}(f) \in \mathcal{F}^{G^F}(G^{\tilde{F}})$ by

$$\mathrm{Ave}(f)(g) = \frac{1}{|G^F|} \sum_{\substack{g_1 \in G^{F^2} \\ g_1 F(g_1)^{-1} = g}} f(g_1).$$

Definition 2.1. The functions $\mathrm{Ave}(\chi) \in \mathcal{F}^{G^F}(G^{\tilde{F}})$, as χ runs over the irreducible constituents of $\mathrm{Ind}_{G^F}^{G^{F^2}}(1)$, are called the *spherical functions* of G^{F^2}/G^F.

Henceforth we interpret Problem 0.1 according to this definition. The condition on χ is motivated by the following fact:

Proposition 2.2. *For* $\chi \in \widehat{G^{F^2}}$,

1. $\mathrm{Ave}(\chi)(1) = \langle \chi, \mathrm{Ind}_{G^F}^{G^{F^2}}(1) \rangle$, *and*
2. $\langle \chi, \mathrm{Ind}_{G^F}^{G^{F^2}}(1) \rangle = 0 \Leftrightarrow \mathrm{Ave}(\chi) = 0$.

Proof. (1) is by Frobenius reciprocity, and (2) is well known. □

Lemma 2.3. *For* $\chi, \chi' \in \widehat{G^{F^2}}$,

$$\langle \mathrm{Ave}(\chi), \mathrm{Ave}(\chi') \rangle = \begin{cases} \frac{\langle \chi, \mathrm{Ind}_{G^F}^{G^{F^2}}(1) \rangle}{\chi(1)}, & \text{if } \chi = \chi', \\ 0, & \text{if } \chi \neq \chi'. \end{cases}$$

Proof. This is a simple calculation using the well-known formula for the convolution of two irreducible characters. □

Remark 2.4. Some authors (see for instance [1]) use the terminology of Hecke algebras (or double coset algebras) instead of our terminology of spherical functions. The *character table* of the Hecke algebra $\mathcal{H}(G^{F^2}, G^F)$ has entries

$$\frac{|G^F g G^F|}{|G^F|} \mathrm{Ave}(\chi)(gF(g)^{-1})$$

where g runs over a set of representatives for $G^F \setminus G^{F^2}/G^F$, and χ runs over the irreducible constituents of $\mathrm{Ind}_{G^F}^{G^{F^2}}(1)$. This normalization of the values of the spherical functions makes them algebraic integers.

Let G_{ss}, G_{rss}, G_{uni} be the subvarieties of semisimple, regular semisimple, and unipotent elements of G. These subvarieties are all F-stable and $\tilde{F}$-stable. Let $\mathcal{B}$ denote the variety of Borel subgroups of G. For any maximal torus T of G, write $W(T)$ for the Weyl group $N_G(T)/T$.

3. Convolution of Character Sheaves

Let $\widehat{G}$ denote the set of isomorphism classes of *character sheaves* on G, as defined by Lusztig in [13, 2.10]. These are certain G-equivariant simple perverse sheaves on G, which we will view as objects of $\mathbf{M}^G(G)$. In a slight abuse of notation, we write $A \in \widehat{G}$ to mean "A is a character sheaf".

If a character sheaf A is F^2-stable, i.e. $(F^2)^*A \cong A$, then its characteristic function $\chi_A^{F^2}$ is well defined up to a scalar depending on the choice of $\mathbb{F}_{q^2}$-structure on A. Now Lusztig in [13] proved that $\{\chi_A^{F^2} \,|\, (F^2)^*A \cong A\}$ is an orthogonal basis of $\mathcal{F}^{G^{F^2}}(G^{F^2})$, and made an explicit conjecture as to the transition matrix between this basis and the basis of irreducible characters, which has now been verified almost completely (see [18]). So calculating the functions $\mathrm{Ave}(\chi_A^{F^2})$ would be a large step towards calculating the spherical functions.

Problem 3.1. Calculate (up to scalar multiple) the functions $\mathrm{Ave}(\chi_A^{F^2})$ for all F^2-stable character sheaves A on G.

Let $m : G \times G \to G$ be the multiplication map $(g_1, g_2) \mapsto g_1 g_2$. This map is G-equivariant for the action of G by conjugation on all factors G. There is an $\mathbb{F}_q$-structure on the variety $G \times G$ with Frobenius map

$$\tilde{F} : (g_1, g_2) \mapsto (F(g_2)^{-1}, F(g_1)^{-1}),$$

and m is $\tilde{F}$-equivariant. Note that $(G \times G)^{\tilde{F}} = \{(g, F(g)^{-1}) \,|\, g \in G^{F^2}\}$.

Lemma 3.2. *If $A \in \widehat{G}$ has an $\mathbb{F}_{q^2}$-structure, then $A \boxtimes \tilde{F}^*A$ is a simple perverse sheaf on $G \times G$ with a natural $\mathbb{F}_q$-structure relative to $\tilde{F}$. For any $g \in G^{F^2}$,*

$$\chi_A^{F^2}(g) = \chi_{A \boxtimes \tilde{F}^*A}^{\tilde{F}}(g, F(g)^{-1}).$$

Proof. The first statement is obvious. The second follows from the fact that $\phi_{A \boxtimes \tilde{F}^*A}^{\tilde{F}}$ is just $\phi_A^{F^2} \boxtimes \mathrm{id} : (F^2)^*A \boxtimes \tilde{F}^*A \xrightarrow{\sim} A \boxtimes \tilde{F}^*A$ under the obvious identification of $\tilde{F}^*(A \boxtimes \tilde{F}^*A)$ with $(F^2)^*A \boxtimes \tilde{F}^*A$. □

Corollary 3.3. *If $A \in \widehat{G}$ has an $\mathbb{F}_{q^2}$-structure,* $\mathrm{Ave}(\chi_A^{F^2}) = |G^F|^{-1}\chi_{m_!(A \boxtimes \tilde{F}^*A)}^{\tilde{F}}$, *where $m_!(A \boxtimes \tilde{F}^*A) \in \mathbf{D}^G(G)$ has the induced $\mathbb{F}_q$-structure, relative to $\tilde{F}$.*

Proof. By the definition of Ave and Lemma 3.2,

$$\mathrm{Ave}(\chi_A^{F^2})(g) = \frac{1}{|G^F|} \sum_{\substack{x \in (G \times G)^{\tilde{F}} \\ m(x) = g}} \chi_{A \boxtimes \tilde{F}^*A}^{\tilde{F}}(x),$$

to which we apply the Grothendieck Trace Formula. □

So Problem 3.1 is equivalent to calculating the characteristic functions of the complexes $m_!(A \boxtimes \tilde{F}^*A) \in \mathbf{D}^G(G)$, relative to $\tilde{F}$.

Remark 3.4. The operation $(A_1, A_2) \mapsto m_!(A_1 \boxtimes A_2)$ is called *convolution*, because it corresponds to convolution of class functions on G^F if one uses the ordinary Frobenius maps $F \times F$ on $G \times G$ and F on G.

Let $\mathcal{I}$ be the set of pairs $(T, Ł)$ of a maximal torus T of G and an isomorphism class of tame local systems $Ł$ of rank 1 on T. Then G acts naturally on $\mathcal{I}$ by $g.(T, Ł) = (\mathrm{Ad}(g)T, \mathrm{Ad}(g^{-1})^*Ł)$. Let

$$W(T)_Ł = \{w \in W(T) \,|\, w^*Ł \cong Ł\}.$$

For $(T, Ł) \in \mathcal{I}$, let $\widehat{G}_{(T,Ł)}$ denote the set of (isomorphism classes of) *character sheaves of central character* $(T, Ł)$, as defined by Lusztig (who uses the notation $\widehat{G}_Ł$; see [13, 2.10]). So if $(T, Ł)$ and $(T', Ł')$ are in the same G-orbit, $\widehat{G}_{(T,Ł)}$ and $\widehat{G}_{(T',Ł')}$ are equal, and otherwise they are disjoint; also these sets exhaust $\widehat{G}$. (In other words, central character of a character sheaf is uniquely defined up to G-conjugacy.) Let $\mathbf{D}^G(G)_{(T,Ł)}$ be the full subcategory of $\mathbf{D}^G(G)$ consisting of complexes K such that all simple constituents of all ${}^pH^iK$ are in $\widehat{G}_{(T,Ł)}$.

Lemma 3.5. *If* $A \in \widehat{G}_{(T,Ł)}$*, then* $\tilde{F}^*A \in \widehat{G}_{(F^{-1}T,\tilde{F}^*Ł)}$*,* $(F^2)^*A \in \widehat{G}_{(F^{-2}T,(F^2)^*Ł)}$*.*

Proof. This follows immediately from the definition in [13]. □

Lemma 3.6. *If a G-orbit in $\mathcal{I}$ contains both $(T, Ł)$ and $(F^{-1}T, \tilde{F}^*Ł)$ for some $(T, Ł)$, then it contains a pair $(T', Ł')$ which is $\tilde{F}$-stable, i.e. $FT' = T'$, $\tilde{F}^*Ł' \cong Ł'$. The same is true for F^2.*

Proof. This follows from Lang's Theorem for F and for F^2. □

Now in [13, §16], Lusztig defines, for each $(T, Ł) \in \mathcal{I}$, a partition of $W(T)_Ł$ into subsets called *two-sided cells*, and a partial order $\leqslant$ (which he writes $\leqslant_{LR}$) on the set of two-sided cells. He also gives a way of associating a unique two-sided cell $\mathcal{C}(A)$ of $W(T)_Ł$ to each character sheaf A of central character $(T, Ł)$. For $\mathcal{C}$ a two-sided cell of $W(T)_Ł$, let $\widehat{G}^{\leqslant\mathcal{C}}_{(T,Ł)}$ be the subset of $\widehat{G}_{(T,Ł)}$ consisting of (isomorphism classes of) character sheaves A such that $\mathcal{C}(A) \leqslant \mathcal{C}$. Similarly define $\mathbf{D}^G(G)^{\leqslant\mathcal{C}}_{(T,Ł)}$. We will also use the obvious notations $\widehat{G}^{\leqslant\mathcal{C},\mathcal{C}'}_{(T,Ł)}$ and $\mathbf{D}^G(G)^{\leqslant\mathcal{C},\mathcal{C}'}_{(T,Ł)}$.

The following crucial result will be proved in §8. As is explained there, it is part of a more general result due to Grojnowski.

Theorem 3.7. *Let* $A_1 \in \widehat{G}_{(T_1,Ł_1)}$*,* $A_2 \in \widehat{G}_{(T_2,Ł_2)}$*.*

1. *If $(T_1, Ł_1)$ and $(T_2, Ł_2)$ are not in the same G-orbit, $m_!(A_1 \boxtimes A_2) \cong 0$.*
2. *If $(T_1, Ł_1) = (T_2, Ł_2) = (T, Ł)$, and $A_1 \in \widehat{G}^{\leqslant\mathcal{C}_1}_{(T,Ł)}$, $A_2 \in \widehat{G}^{\leqslant\mathcal{C}_2}_{(T,Ł)}$, then*

$$m_!(A_1 \boxtimes A_2) \in \mathbf{D}^G(G)^{\leqslant\mathcal{C}_1,\mathcal{C}_2}_{(T,Ł)}.$$

Specializing to the case of interest to us and using Lemma 3.6, we get:

Corollary 3.8. *Let* $A \in \widehat{G}_{(T,Ł)}$*.*

1. *If $(T, Ł)$ is not G-conjugate to an $\tilde{F}$-stable pair, $m_!(A \boxtimes \tilde{F}^*A) \cong 0$.*
2. *If $(T, Ł)$ is $\tilde{F}$-stable, then $m_!(A \boxtimes \tilde{F}^*A) \in \mathbf{D}^G(G)^{\leqslant\mathcal{C}(A),\mathcal{C}(\tilde{F}^*A)}_{(T,Ł)}$.*

Corollary 3.9. *Let $(T, Ł)$ be $\tilde{F}$-stable, and let $A \in \widehat{G}_{(T,Ł)}$ be F^2-stable. Then $\chi^{\tilde{F}}_{m_!(A \boxtimes \tilde{F}^* A)}$ is a linear combination of $\chi^{\tilde{F}}_{A'}$ where A' runs over $\tilde{F}$-stable character sheaves of central character $(T, Ł)$ such that $\mathcal{C}(A') \leqslant \mathcal{C}(A)$, $\mathcal{C}(A') \leqslant \mathcal{C}(\tilde{F}^* A)$.*

Proof. We can assume the $\mathbb{F}_{q^2}$-structure on A is such that it is pure. Then the $\mathbb{F}_q$-structure on $m_!(A \boxtimes \tilde{F}^* A)$ is mixed, so we can apply Lemma 1.1. □

4. Basic Functions

Corollaries 3.3 and 3.9 reduce Problem 3.1 to the following two problems.

Problem 4.1. Calculate (up to scalar multiple) the characteristic functions $\chi^{\tilde{F}}_A$ of all $\tilde{F}$-stable character sheaves A on G.

Problem 4.2. Find the coefficients in the linear combination of Corollary 3.9.

Problem 4.1 is analogous to the problem of calculating the characteristic functions χ^F_A for F-stable character sheaves A, which was solved by Lusztig in [13]. In this paper we will only prove analogues of the "principal series" results of [13].

Let $(T, Ł) \in \mathcal{I}$. Following Lusztig, [13, 8.1], we associate to $(T, Ł)$ a G-equivariant semisimple perverse sheaf $K_{(T,Ł)}$ on G. Consider the diagram

$$T \xleftarrow{\alpha} \{(g, xT) \in G_{\mathrm{rss}} \times G/T \,|\, x^{-1}gx \in T\} \xrightarrow{\pi} G_{\mathrm{rss}}$$

where π is the first projection and $\alpha(g, xT) = x^{-1}gx$. We let

$$K_{(T,Ł)} = IC(G, \pi_! \alpha^* Ł)[\dim G].$$

Up to isomorphism, $K_{(T,Ł)}$ depends only on the G-orbit of $(T, Ł)$.

Using the stalk at the identity to normalize, we can define an action of $W(T)_Ł$ on Ł (compatible with the action of $W(T)_Ł$ on T). The above diagram is $W(T)$-equivariant (where $W(T)$ acts trivially on G_{rss}), so this induces an action of $W(T)_Ł$ on $K_{(T,Ł)}$ by automorphisms in $\mathbf{M}^G(G)$. Since π is a Galois covering with group $W(T)$, we in fact have an isomorphism $\overline{\mathbb{Q}}_l W(T)_Ł \xrightarrow{\sim} \mathrm{End}_{\mathbf{M}^G(G)}(K_{(T,Ł)})$. So

$$K_{(T,Ł)} \cong \bigoplus_{E \in \widehat{W(T)}_Ł} E \otimes K_{(T,Ł,E)}$$

as perverse sheaves with a $W(T)_Ł$-action, where $K_{(T,Ł,E)}$ is a G-equivariant simple perverse sheaf whose restriction to G_{rss} is a simple local system shifted.

Proposition 4.3. $K_{(T,Ł,E)} \in \widehat{G}_{(T,Ł)}$ *for all* $E \in \widehat{W(T)}_Ł$.

Proof. This follows directly from the definition in [13], since $K_{(T,Ł)} \cong K^Ł_1$ in Lusztig's notation (see [13, (8.2.3)]). □

Remark 4.4. A character sheaf which is isomorphic to some $K_{(T,Ł,E)}$ is said to be *in the principal series*. If $G = GL_n$, all character sheaves are in the principal series.

Assume in addition that $(T, Ł)$ is $\tilde{F}$-stable. Then there is a unique $\mathbb{F}_q$-structure on Ł, relative to $\tilde{F}$, such that $\chi^{\tilde{F}}_Ł$ is a character of the group $T^{\tilde{F}}$. The above diagram is $\tilde{F}$-equivariant if we define $\tilde{F}(g, xT) = (F(g)^{-1}, F(x)T)$. So we get an induced $\mathbb{F}_q$-structure on $K_{(T,Ł)}$, relative to $\tilde{F}$.

Definition 4.5. For $\tilde{F}$-stable $(T,\mathcal{L}) \in \mathcal{I}$, the *basic function* on $G^{\tilde{F}}$ associated to $(T,\mathcal{L})$ is the characteristic function $\chi^{\tilde{F}}_{K_{(T,\mathcal{L})}} \in \mathcal{F}^{G^F}(G^{\tilde{F}})$.

If $g \in G^F$, the isomorphism $K_{(T,\mathcal{L})} \cong K_{(gTg^{-1},\mathrm{Ad}(g^{-1})^*\mathcal{L})}$ respects the $\mathbb{F}_q$-structures, so $\chi^{\tilde{F}}_{K_{(T,\mathcal{L})}}$ depends only on the G^F-orbit of $(T,\mathcal{L})$.

Remark 4.6. These basic functions on $G^{\tilde{F}}$ are analogous to the basic functions $\chi^F_{K_{(T,\mathcal{L})}} \in \mathcal{F}^{G^F}(G^F)$ for F-stable $(T,\mathcal{L})$, which play a vital role in the character theory of G^F (see for instance [13], [11], [14], and [18]).

We define the *Green function* $\tilde{Q}^G_T \in \mathcal{F}^{G^F}(G^{\tilde{F}}_{\mathrm{uni}})$ as follows. The G^F-conjugacy class of T corresponds to an F-conjugacy class in the canonical Weyl group W, namely the one containing the relative position of B and FB for any Borel subgroup B containing T. If w is in this class, let

$$\tilde{Q}^G_T(u) = \sum_i (-1)^i \mathrm{tr}(F^* \circ w^{-1}, H^i(\mathcal{B}_u, \overline{\mathbb{Q}_l})),$$

where $\mathcal{B}_u$ denotes the Springer fibre and we use Lusztig's definition of the Springer representation of W on $H^i(\mathcal{B}_u, \overline{\mathbb{Q}_l})$. (Notice that $u \in G^{\tilde{F}}_{\mathrm{uni}} \Rightarrow \mathcal{B}_u$ is F-stable.) We want to compare $\tilde{Q}^G_T$ to the usual Green function Q^G_T defined by the same formula

$$Q^G_T(u) = \sum_i (-1)^i \mathrm{tr}(F^* \circ w^{-1}, H^i(\mathcal{B}_u, \overline{\mathbb{Q}_l})),$$

but for u in G^F_{uni} rather than $G^{\tilde{F}}_{\mathrm{uni}}$. For this we will use the Lie algebra $\mathfrak{g}$ of G. Let F on $\mathfrak{g}$ be the induced Frobenius map, and define $\tilde{F}$ on $\mathfrak{g}$ by $\tilde{F}(x) = -F(x)$. Let $\mathfrak{g}_{\mathrm{nil}}$ be the subvariety of nilpotent elements of $\mathfrak{g}$. We need to assume the following.

Assumption 4.7. There are bijections $L : G^F \setminus G^F_{\mathrm{uni}} \xrightarrow{\sim} G^F \setminus \mathfrak{g}^F_{\mathrm{nil}}$ and $\tilde{L} : G^F \setminus G^{\tilde{F}}_{\mathrm{uni}} \xrightarrow{\sim} G^F \setminus \mathfrak{g}^{\tilde{F}}_{\mathrm{nil}}$ such that for all $u \in G^F_{\mathrm{uni}}$ (or $G^{\tilde{F}}_{\mathrm{uni}}$) there is some $n \in L(G^F.u)$ (or $\tilde{L}(G^F.u)$) for which $\mathcal{B}_u = \mathcal{B}_n$ as subsets of $\mathcal{B}$.

If p is sufficiently large, we have a G-equivariant isomorphism $\log : G_{\mathrm{uni}} \xrightarrow{\sim} \mathfrak{g}_{\mathrm{nil}}$ which commutes with F and $\tilde{F}$, and satisfies $\mathcal{B}_{\log(u)} = \mathcal{B}_u$. This induces bijections L and $\tilde{L}$ as required. So if p is sufficiently large, Assumption 4.7 holds.

Now if $\alpha \in k^\times$ is any element for which $\alpha^q = -\alpha$, multiplication by α defines a bijection $G^F \setminus \mathfrak{g}^F_{\mathrm{nil}} \xrightarrow{\sim} G^F \setminus \mathfrak{g}^{\tilde{F}}_{\mathrm{nil}}$, and clearly $\mathcal{B}_{\alpha n} = \mathcal{B}_n$. So Assumption 4.7 implies that there is a (not necessarily unique) bijection $\Psi : G^F \setminus G^F_{\mathrm{uni}} \xrightarrow{\sim} G^F \setminus G^{\tilde{F}}_{\mathrm{uni}}$ such that for any $u \in G^F_{\mathrm{uni}}$ there is some $u' \in \Psi(G^F.u)$ with $\mathcal{B}_{u'} = \mathcal{B}_u$. Clearly:

Lemma 4.8. *If $u \in G^F_{\mathrm{uni}}$ and $u' \in G^{\tilde{F}}_{\mathrm{uni}}$ are in orbits which correspond under such a bijection Ψ, then $Q^G_T(u) = \tilde{Q}^G_T(u')$ for all F-stable T.*

Consequently the algorithms which compute the values of Q^G_T also compute the values of $\tilde{Q}^G_T$, if Assumption 4.7 holds.

Now let $g \in G^{\tilde{F}}$ have Jordan decomposition su, with $s \in G^{\tilde{F}}_{\mathrm{ss}}$ and $u \in G^{\tilde{F}}_{\mathrm{uni}}$. Then $Z^\circ_G(s)$ is another group of the same kind as G. We have the following result analogous to [13, Theorem 8.5], to be proved in §9 below:

Theorem 4.9 (Character Formula). *With notations as above,*

$$\chi^{\tilde{F}}_{K_{(T,\mathcal{L})}}(g) = \frac{(-1)^{\dim T}}{|Z^\circ_G(s)^F|} \sum_{\substack{x \in G^F \\ x^{-1}sx \in T}} \chi^{\tilde{F}}_{\mathcal{L}}(x^{-1}sx) \tilde{Q}^{Z^\circ_G(s)}_{xTx^{-1}}(u).$$

As a consequence of the Character Formula and the algorithm for computing Green functions, the basic functions $\chi^{\tilde{F}}_{K_{(T,\mathcal{L})}}$ are explicitly computable if Assumption 4.7 holds. We also know the inner products of the basic functions, thanks to the following analogue of [13, Theorem 9.2], to be proved in §10 below:

Theorem 4.10 (Inner Product Formula). *For any $\tilde{F}$-stable $(T, \mathcal{L})$, $(T', \mathcal{L}')$,*

$$\langle \chi^{\tilde{F}}_{K_{(T,\mathcal{L})}}, \chi^{\tilde{F}}_{K_{(T',\mathcal{L}')}} \rangle = \begin{cases} \frac{|G^F||T^{\tilde{F}}|}{|G^{\tilde{F}}||T^F|} |W(T)^F_{\mathcal{L}}|, & \text{if } (T,\mathcal{L}),\ (T',\mathcal{L}') \text{ are } G^F\text{-conjugate}, \\ 0, & \text{if they are not.} \end{cases}$$

5. Generic Spherical Functions

Definition 5.1. We say that $(T, \mathcal{L}) \in \mathcal{I}$ is *generic* if $W(T)_{\mathcal{L}} = \{1\}$.

Lemma 5.2. *If $(T, \mathcal{L})$ is generic, then $\widehat{G}_{(T,\mathcal{L})}$ has only one element, namely the isomorphism class of $K_{(T,\mathcal{L})}$.*

Proof. This follows immediately from the definitions in [13]. □

Now in the generic case, the relationship between characteristic functions of character sheaves and irreducible characters is particularly simple:

Theorem 5.3. *If $(T, \mathcal{L})$ is F^2-stable and generic, then*

$$\chi_{(T,\mathcal{L})} := (-1)^{\mathrm{rank}_{\mathbb{F}_{q^2}}(T) + \mathrm{rank}_{\mathbb{F}_{q^2}}(G) + \dim G} \chi^{F^2}_{K_{(T,\mathcal{L})}}$$

is an irreducible character of G^{F^2}. Moreover $\chi_{(T,\mathcal{L})} = \chi_{(T',\mathcal{L}')}$ if and only if $(T, \mathcal{L})$ and $(T', \mathcal{L}')$ are G^{F^2}-conjugate.

Proof. In the notation of [4], $\chi^{F^2}_{K_{(T,\mathcal{L})}}$ is the character of $(-1)^{\dim G} R^\theta_T$ for $\theta = \chi^{F^2}_{\mathcal{L}}$, as is proved in [14, Lemma 7.2] and [11, Lemma 2.4.1]. Since $(T, \mathcal{L})$ is generic, θ is in general position, so $\chi_{(T,\mathcal{L})}$ is an irreducible character by [4, Proposition 7.4]. The second statement follows from [4, Theorem 6.8]. □

Remark 5.4. The $\chi_{(T,\mathcal{L})}$ defined here are the *regular semisimple* characters of G^{F^2}.

Proposition 5.5. *If $(T, \mathcal{L})$ is F^2-stable and generic,*

$$\langle \chi_{(T,\mathcal{L})}, \mathrm{Ind}^{G^{F^2}}_{G^F}(1) \rangle = \begin{cases} 1, & \text{if } (T,\mathcal{L}) \text{ is } G^{F^2}\text{-conjugate to an } \tilde{F}\text{-stable pair}, \\ 0, & \text{otherwise.} \end{cases}$$

Proof. This is a special case of [15, Theorem 1.4]. □

Definition 5.6. The *generic spherical functions* of G^{F^2}/G^F are the functions $\mathrm{Ave}(\chi_{(T,\mathcal{L})})$ where $(T, \mathcal{L})$ is an $\tilde{F}$-stable generic pair.

Theorem 5.7. *Let $(T, \mathcal{L})$ be F^2-stable and generic.*

1. *If $(T, \L)$ is not G^{F^2}-conjugate to an $\tilde{F}$-stable pair, then*

$$\mathrm{Ave}(\chi_{(T,\L)}) = \mathrm{Ave}(\chi^{F^2}_{K_{(T,\L)}}) = 0.$$

2. *If $(T, \L)$ is $\tilde{F}$-stable, then*

$$\mathrm{Ave}(\chi_{(T,\L)}) = (-1)^{\mathrm{rank}_{\mathbb{F}_{q^2}}(T)+\mathrm{rank}_{\mathbb{F}_{q^2}}(G)+\dim G}\mathrm{Ave}(\chi^{F^2}_{K_{(T,\L)}}) = \frac{\chi^{\tilde{F}}_{K_{(T,\L)}}}{\chi^{\tilde{F}}_{K_{(T,\L)}}(1)}.$$

Proof. In case (1), $\mathrm{Ave}(\chi_{(T,\L)}) = 0$ by (2) of Proposition 2.2 and Proposition 5.5. In case (2), $\mathrm{Ave}(\chi^{F^2}_{K_{(T,\L)}})$ is a scalar multiple of $\chi^{\tilde{F}}_{m_!(K_{(T,\L)}\boxtimes\tilde{F}^*K_{(T,\L)})}$ by Corollary 3.3, which is a scalar multiple of $\chi^{\tilde{F}}_{K_{(T,\L)}}$ by Corollary 3.9, and also $\mathrm{Ave}(\chi_{(T,\L)})(1) = 1$ by (1) of Proposition 2.2 and Proposition 5.5. □

Assuming 4.7, this Theorem solves Problems 0.1 and 3.1 in the generic case. Note that the idea of applying a result like Theorem 3.7 to the averages of regular semisimple characters was mentioned (in a more general setting) in [8, §5].

6. The Symmetric Space $GL_n(\mathbb{F}_{q^2})/GL_n(\mathbb{F}_q)$

Let V be a vector space of dimension n over k, defined over $\mathbb{F}_q$ with Frobenius map F. In this section we consider the case when $G = GL(V)$ with the induced $\mathbb{F}_q$-structure and Frobenius map. Thus $G^F \cong GL_n(\mathbb{F}_q)$ and $G^{F^2} \cong GL_n(\mathbb{F}_{q^2})$.

We first recall Green's results on the characters of G^F. As in [16, Chapter IV], we consider the system of maps $\widehat{\mathbb{F}^\times_{q^e}} \to \widehat{\mathbb{F}^\times_{q^{e'}}}$ for $e \,|\, e'$ (the transpose of the norm map), and its limit $L = \mathrm{colim}\, \widehat{\mathbb{F}^\times_{q^e}}$. Let σ denote the q-th power map on both $k^\times$ and L, so that $(k^\times)^{\sigma^e} \cong \mathbb{F}^\times_{q^e}$, $L^{\sigma^e} \cong \widehat{\mathbb{F}^\times_{q^e}}$ for all $e \geq 1$. Write $\langle\cdot,\cdot\rangle^{\sigma^e} : (k^\times)^{\sigma^e} \times L^{\sigma^e} \to \overline{\mathbb{Q}}^\times$ for the canonical pairing. For $\alpha \in k^\times$, we write m_α for $|\langle\sigma\rangle.\alpha|$; in other words, the smallest $e \geq 1$ such that $\alpha \in \mathbb{F}^\times_{q^e}$. Similarly define m_ξ for $\xi \in L$.

Let $\mathcal{P}_n$ be the set of collections of partitions $\underline{\lambda} = (\lambda_\alpha)_{\alpha\in k^\times}$, almost all zero, such that $\sum_{\alpha\in k^\times} |\lambda_\alpha| = n$. Let $\mathcal{P}^\sigma_n$ be the subset of $\mathcal{P}_n$ consisting of all $\underline{\lambda}$ such that $\lambda_{\sigma(\alpha)} = \lambda_\alpha$ for all α. As in [16, IV.§2], we define a bijection between $\mathcal{P}^\sigma_n$ and the set of conjugacy classes in G^F, so that if $g \in G^F$ in the class corresponding to $\underline{\lambda}$ has Jordan decomposition su, then:

1. the multiplicity of $\alpha \in k^\times$ as an eigenvalue of s is $|\lambda_\alpha|$;
2. consequently,

$$Z_{G^F}(s) \cong \prod_{\alpha\in\langle\sigma\rangle\backslash k^\times} GL_{|\lambda_\alpha|}(\mathbb{F}_{q^{m_\alpha}});$$

3. under this isomorphism, the Jordan type of u is $\prod_{\alpha\in\langle\sigma\rangle\backslash k^\times} \lambda_\alpha$.

For $\underline{\lambda}, \underline{\mu} \in \mathcal{P}^\sigma_n$, we write $|\underline{\lambda}| = |\underline{\mu}|$ to mean that $|\lambda_\alpha| = |\mu_\alpha|$ for all α.

Dually, let $\widehat{\mathcal{P}}_n$ be the set of collections of partitions $\underline{\nu} = (\nu_\xi)_{\xi\in L}$, almost all zero, such that $\sum_{\xi\in L} |\nu_\xi| = n$. Let $\widehat{\mathcal{P}}^\sigma_n$ be the subset of $\widehat{\mathcal{P}}_n$ of all $\underline{\nu}$ such that $\nu_{\sigma(\xi)} = \nu_\xi$ for all ξ. We can define a bijection between $\widehat{\mathcal{P}}^\sigma_n$ and the set of G^F-conjugacy classes of F-stable $(T, \L) \in \mathcal{I}$, so that if $(T, \L)$ is in the class corresponding to $\underline{\nu}$:

1. there is an isomorphism

$$T \cong \prod_{\xi\in\langle\sigma\rangle\backslash L} \prod_{j=1}^{\ell(\nu_\xi)} \underbrace{k^\times \times \cdots \times k^\times}_{m_\xi(\nu_\xi)_j \text{ factors}},$$

under which F on T corresponds to cyclic permutation of each group of factors $k^\times$, composed with σ;

2. consequently,

$$T^F \cong \prod_{\xi\in\langle\sigma\rangle\backslash L} \prod_{j=1}^{\ell(\nu_\xi)} (k^\times)^{\sigma^{m_\xi(\nu_\xi)_j}};$$

3. under this isomorphism, $\chi_{Ł}^F$ corresponds to

$$\prod_{\xi\in\langle\sigma\rangle\backslash L} \prod_{j=1}^{\ell(\nu_\xi)} \langle\cdot,\xi\rangle^{\sigma^{m_\xi(\nu_\xi)_j}},$$

for some choice of representatives $\{\xi\}$ for $\langle\sigma\rangle\setminus L$.

For $\underline{\nu},\underline{\rho}\in\widehat{\mathcal{P}}_n^\sigma$, we write $|\underline{\nu}|=|\underline{\rho}|$ to mean that $|\nu_\xi|=|\rho_\xi|$ for all ξ.

For $\underline{\nu}\in\widehat{\mathcal{P}}_n^\sigma$, let $B_{\underline{\nu}}^F$ be the (signed) basic function $(-1)^n\chi^F_{K_{(T,Ł)}}$ for F-stable $(T,Ł)$ in the corresponding class. This can be described combinatorially as follows. Let $\Lambda_{\underline{\nu}}^\sigma$ be a set of representatives for the orbits of $\langle\sigma\rangle$ (acting on the first factor) on $\{(\xi,j)\in L\times\mathbb{Z}\,|\,1\le j\le\ell(\nu_\xi)\}$. Let $\Phi_{\underline{\nu}}^\sigma$ be the set of maps $\phi:\Lambda_{\underline{\nu}}^\sigma\to k^\times$ such that $m_{\phi(\xi,j)}\,|\,m_\xi(\nu_\xi)_j$ for all $(\xi,j)\in\Lambda_{\underline{\nu}}^\sigma$. For any $\phi\in\Phi_{\underline{\nu}}^\sigma$, define $\underline{\lambda}(\phi)=(\lambda(\phi)_\alpha)\in\mathcal{P}_n^\sigma$ by letting $\lambda(\phi)_\alpha$ be the partition whose parts are all $\frac{m_\xi(\nu_\xi)_j}{m_\alpha}$ for $(\xi,j)\in\phi^{-1}(\langle\sigma\rangle.\alpha)$.

Proposition 6.1. *If $g\in G^F$ is in the conjugacy class corresponding to $\underline{\lambda}\in\mathcal{P}_n^\sigma$, then*

$$B_{\underline{\nu}}^F(g)=\sum_{\substack{\phi\in\Phi_{\underline{\nu}}^\sigma\\ |\underline{\lambda}(\phi)|=|\underline{\lambda}|}} \prod_{(\xi,j)\in\Lambda_{\underline{\nu}}^\sigma} \langle\phi(\xi,j),\xi\rangle^{\sigma^{m_\xi(\nu_\xi)_j}} \prod_{\alpha\in\langle\sigma\rangle\backslash k^\times} Q^{\lambda_\alpha}_{\lambda(\phi)_\alpha}(q^{m_\alpha})$$

where $Q^{\lambda_\alpha}_{\lambda(\phi)_\alpha}$ is the Green polynomial (*see* [16, III.§7]).

Proof. Let $g=su$ be the Jordan decomposition. Let $(T,Ł)$ be an F-stable pair in the class corresponding to $\underline{\nu}$. Since $Z_G(s)$ is connected, the Character Formula [13, Theorem 8.5] in this case becomes

$$B_{\underline{\nu}}^F(g)=\sum_{\substack{s'\in T^F\\ s'\sim s\text{ in }G^F}} \chi_{Ł}^F(s')Q_T^{Z_G(s')}(u'),$$

where u' denotes xux^{-1} for any $x\in G^F$ such that $s'=xsx^{-1}$, $Q_T^{Z_G(s')}(u')$ being independent of the choice. Using the above description of $\chi_{Ł}^F$ and the well-known identification of the geometrically-defined Green functions with the appropriate Green polynomials, we get the result. Note that the right-hand side clearly does not depend on the choice of $\Lambda_{\underline{\nu}}^\sigma$. □

So to compute the character table of G^F, it suffices to express the irreducible characters as linear combinations of these functions $B_{\underline{\nu}}^F$.

Theorem 6.2. *For $\underline{\rho} \in \widehat{\mathcal{P}}_n^\sigma$,*

$$\chi^{\underline{\rho}} := (-1)^{n+\sum_{\xi\in\langle\sigma\rangle\backslash L}|\rho_\xi|} \sum_{\substack{\underline{\nu}\in\widehat{\mathcal{P}}_n^\sigma \\ |\underline{\nu}|=|\underline{\rho}|}} \left(\prod_{\xi\in\langle\sigma\rangle\backslash L} (z_{\nu_\xi})^{-1} \chi_{\nu_\xi}^{\rho_\xi} \right) B_{\underline{\nu}}^F$$

is an irreducible character of G^F, and all irreducible characters arise in this way for unique $\underline{\rho} \in \widehat{\mathcal{P}}_n^\sigma$. (Here $\chi_{\nu_\xi}^{\rho_\xi}$, z_{ν_ξ} are defined as in [16, Chapter I].*)*

Proof. This is a restatement of the main results of [7] and [16, Chapter IV]; see also [1, Theorem 1.2.10]. Note that Macdonald's parameters in $\widehat{\mathcal{P}}_n^\sigma$ differ from those of [1] by transposing all partitions; we are following the convention of [1]. □

We can also interpret $\chi^{\underline{\rho}}$ geometrically. Let $(T, Ł)$ be an F-stable pair in the class corresponding to $((1^{|\rho_\xi|}))_{\xi\in L} \in \widehat{\mathcal{P}}_n^\sigma$. Such pairs are *maximally split* in the sense of [4, Definition 5.2.5]; for example, $(T, \overline{\mathbb{Q}_l})$ is such a pair if and only if T is a split torus. We have an isomorphism

$$W(T)_Ł \cong \prod_{\xi\in\langle\sigma\rangle\backslash L} \underbrace{S_{|\rho_\xi|}\times\cdots\times S_{|\rho_\xi|}}_{m_\xi \text{ factors}},$$

where F corresponds to cyclic permutation of each group of factors $S_{|\rho_\xi|}$, so that

$$W(T)_Ł^F \cong \prod_{\xi\in\langle\sigma\rangle\backslash L} S_{|\rho_\xi|}.$$

Let E be the F-stable irreducible representation of $W(T)_Ł$ labelled by the partition ρ_ξ on the factor corresponding to ξ.

Proposition 6.3. *For this $(T, Ł, E)$,*

$$\chi^{\underline{\rho}} = (-1)^{\sum_{\xi\in\langle\sigma\rangle\backslash L}|\rho_\xi|} \chi_{K_{(T,Ł,E)},\phi_{(T,Ł,E)}},$$

where $\phi_{(T,Ł,E)} : F^*K_{(T,Ł,E)} \xrightarrow{\sim} K_{(T,Ł,E)}$ is induced by the obvious isomorphism $F^*E \xrightarrow{\sim} E$ of $W(T)_Ł$-modules.

Proof. Let $\sigma_E : E \xrightarrow{\sim} E$ be the composition of this isomorphism $F^*E \xrightarrow{\sim} E$ with the canonical isomorphism $E \xrightarrow{\sim} F^*E$ of vector spaces. Then as a special case of [13, (10.4.5)], we have

$$\chi_{K_{(T,Ł,E)},\phi_{(T,Ł,E)}} = \frac{1}{|W(T)_Ł|} \sum_{w\in W(T)_Ł} \mathrm{tr}(w\circ\sigma_E^{-1}, E)\chi_{K_{(T_w,Ł_w)}}^F$$

where $(T_w, Ł_w)$ is an F-stable pair obtained from $(T, Ł)$ by twisting by w. Now there is a surjection $\psi : W(T)_Ł \to W(T)_Ł^F$, defined on the factor corresponding to ξ by the map $(w_1, \ldots, w_{m_\xi}) \mapsto w_{m_\xi}\ldots w_1$, with the following properties:

1. every fibre of ψ has the same number of elements;
2. w, w' F-conjugate in $W(T)_Ł \Leftrightarrow \psi(w)$, $\psi(w')$ conjugate in $W(T)_Ł^F$;
3. if $\psi(w)$ is in the conjugacy class corresponding to $\underline{\nu} \in \widehat{\mathcal{P}}_n^\sigma$, where $|\underline{\nu}| = |\underline{\rho}|$, then $(T_w, Ł_w)$ is in the class corresponding to $\underline{\nu}$, and

$$\mathrm{tr}(w\circ\sigma_E^{-1}, E) = \prod_{\xi\in\langle\sigma\rangle\backslash L} \chi_{\nu_\xi}^{\rho_\xi}.$$

The result follows. ◻

By the classification of character sheaves in [13, §18], every F-stable character sheaf is isomorphic to some $K_{(T,Ł,E)}$ as above. So Proposition 6.3 says that the relationship between characteristic functions of character sheaves and irreducible characters is as close as one could hope.

All of the above remains true, of course, when q is replaced by q^2, F by F^2, and σ by σ^2 throughout. We will use in this case the analogous notations $\mathcal{P}_n^{\sigma^2}$, $\widehat{\mathcal{P}}_n^{\sigma^2}$, $\chi^{\underline{\rho}}$ for $\underline{\rho} \in \widehat{\mathcal{P}}_n^{\sigma^2}$, etc. We will not actually need the above results for q itself, except as helpful analogies for the following results about $\tilde{F}$.

Let $\tilde{\sigma}$ denote the $(-q)$-th power map on both $k^\times$ and L. So $\tilde{\sigma}^2 = \sigma^2$. For $e \geq 1$, let $\langle \cdot, \cdot \rangle^{\tilde{\sigma}^e} : (k^\times)^{\tilde{\sigma}^e} \times L^{\tilde{\sigma}^e} \to \overline{\mathbb{Q}}^\times$ be the canonical pairing (the same as $\langle \cdot, \cdot \rangle^{\sigma^e}$ if e is even). Define $\tilde{m}_\alpha$, $\tilde{m}_\xi$, $\mathcal{P}_n^{\tilde{\sigma}}$, $\widehat{\mathcal{P}}_n^{\tilde{\sigma}}$ exactly as m_α, m_ξ, $\mathcal{P}_n^\sigma$, $\widehat{\mathcal{P}}_n^\sigma$ were defined, but with $\tilde{\sigma}$ instead of σ. Clearly the conjugacy class of G^{F^2} corresponding to $\underline{\lambda} \in \mathcal{P}_n^{\sigma^2}$ meets $G^{\tilde{F}}$ in a unique G^F-orbit if $\underline{\lambda} \in \mathcal{P}_n^{\tilde{\sigma}}$, and does not meet $G^{\tilde{F}}$ otherwise. So the G^F-orbits on $G^{\tilde{F}}$ are in bijection with $\mathcal{P}_n^{\tilde{\sigma}}$. Similarly there is a bijection between G^F-conjugacy classes of $\tilde{F}$-stable pairs $(T, Ł)$ and $\widehat{\mathcal{P}}_n^{\tilde{\sigma}}$ so that if $(T, Ł)$ is in the class corresponding to $\underline{\nu}$:

1. there is an isomorphism
$$T \cong \prod_{\xi \in \langle \tilde{\sigma} \rangle \setminus L} \prod_{j=1}^{\ell(\nu_\xi)} \underbrace{k^\times \times \cdots \times k^\times}_{\tilde{m}_\xi(\nu_\xi)_j \text{ factors}},$$
under which $\tilde{F}$ on T corresponds to cyclic permutation of each group of factors $k^\times$, composed with $\tilde{\sigma}$;
2. consequently,
$$T^{\tilde{F}} \cong \prod_{\xi \in \langle \tilde{\sigma} \rangle \setminus L} \prod_{j=1}^{\ell(\nu_\xi)} (k^\times)^{\tilde{\sigma}^{\tilde{m}_\xi(\nu_\xi)_j}};$$
3. under this isomorphism, $\chi_Ł^{\tilde{F}}$ corresponds to
$$\prod_{\xi \in \langle \tilde{\sigma} \rangle \setminus L} \prod_{j=1}^{\ell(\nu_\xi)} \langle \cdot, \xi \rangle^{\tilde{\sigma}^{\tilde{m}_\xi(\nu_\xi)_j}},$$
for some choice of representatives $\{\xi\}$ for $\langle \tilde{\sigma} \rangle \setminus L$.

Moreover we have the following description of the induced character $\operatorname{Ind}_{G^F}^{G^{F^2}}(1)$:

Theorem 6.4. *For $\underline{\rho} \in \widehat{\mathcal{P}}_n^{\sigma^2}$,*
$$\langle \chi^{\underline{\rho}}, \operatorname{Ind}_{G^F}^{G^{F^2}}(1) \rangle = \begin{cases} 1, & \text{if } \underline{\rho} \in \widehat{\mathcal{P}}_n^{\tilde{\sigma}}, \\ 0, & \text{otherwise.} \end{cases}$$

Proof. See [6, Theorem 3.6], or [15, Theorem 1.4]. ◻

Corollary 6.5.
1. *If $\underline{\rho} \in \widehat{\mathcal{P}}_n^{\sigma^2} \setminus \widehat{\mathcal{P}}_n^{\tilde{\sigma}}$, then* $\operatorname{Ave}(\chi^{\underline{\rho}}) = 0$.
2. *If $\underline{\rho} \in \widehat{\mathcal{P}}_n^{\tilde{\sigma}}$,* $\operatorname{Ave}(\chi^{\underline{\rho}})(1) = 1$.
3. $\{\operatorname{Ave}(\chi^{\underline{\rho}}) \,|\, \underline{\rho} \in \widehat{\mathcal{P}}_n^{\tilde{\sigma}}\}$ *is an orthogonal basis of* $\mathcal{F}^{G^F}(G^{\tilde{F}})$.

Proof. (1) and (2) follow from Proposition 2.2. For (3), different $\mathrm{Ave}(\chi^{\underline{\rho}})$ are orthogonal by Lemma 2.3, and $\dim \mathcal{F}^{G^F}(G^{\tilde{F}}) = |\mathcal{P}_n^{\tilde{\sigma}}| = |\widehat{\mathcal{P}}_n^{\tilde{\sigma}}|$. □

Our aim is to compute the functions $\mathrm{Ave}(\chi^{\underline{\rho}})$ for $\underline{\rho} \in \widehat{\mathcal{P}}_n^{\tilde{\sigma}}$.

For $\underline{\nu} \in \widehat{\mathcal{P}}_n^{\tilde{\sigma}}$, let $B_{\underline{\nu}}^{\tilde{F}}$ be the (signed) basic function $(-1)^n \chi^{\tilde{F}}_{K_{(T,\L)}}$ for $\tilde{F}$-stable $(T,\L)$ in the corresponding class. We have a combinatorial description of $B_{\underline{\nu}}^{\tilde{F}}$ analogous to the above description of $B_{\underline{\nu}}^{F}$. Let $\Lambda_{\underline{\nu}}^{\tilde{\sigma}}$ be a set of representatives for the orbits of $\langle\tilde{\sigma}\rangle$ (acting on the first factor) on $\{(\xi, j) \in L \times \mathbb{Z} | 1 \leq j \leq \ell(\nu_\xi)\}$. Let $\Phi_{\underline{\nu}}^{\tilde{\sigma}}$ be the set of maps $\phi : \Lambda_{\underline{\nu}}^{\tilde{\sigma}} \to k^\times$ such that $\tilde{m}_{\phi(\xi,j)} \mid \tilde{m}_\xi(\nu_\xi)_j$ for all $(\xi, j) \in \Lambda_{\underline{\nu}}^{\tilde{\sigma}}$. For any $\phi \in \Phi_{\underline{\nu}}^{\tilde{\sigma}}$, define $\underline{\mu}(\phi) = (\mu(\phi)_\alpha) \in \mathcal{P}_n^{\tilde{\sigma}}$ by letting $\mu(\phi)_\alpha$ be the partition whose parts are all $\frac{\tilde{m}_\xi(\nu_\xi)_j}{\tilde{m}_\alpha}$ for $(\xi, j) \in \phi^{-1}(\langle\tilde{\sigma}\rangle.\alpha)$.

Proposition 6.6. *If $g \in G^{\tilde{F}}$ is in the G^F-orbit corresponding to $\underline{\mu} \in \mathcal{P}_n^{\tilde{\sigma}}$, then*

$$B_{\underline{\nu}}^{\tilde{F}}(g) = \sum_{\substack{\phi \in \Phi_{\underline{\nu}}^{\tilde{\sigma}} \\ |\underline{\mu}(\phi)| = |\underline{\mu}|}} \prod_{(\xi,j) \in \Lambda_{\underline{\nu}}^{\tilde{\sigma}}} \langle \phi(\xi, j), \xi \rangle^{\tilde{\sigma}^{\tilde{m}_\xi(\nu_\xi)_j}} \prod_{\alpha \in \langle\tilde{\sigma}\rangle \backslash k^\times} Q^{\mu_\alpha}_{\mu(\phi)_\alpha}(q^{\tilde{m}_\alpha}).$$

Proof. This is obtained in a similar way to Proposition 6.1, using the Character Formula 4.9. We can identify the Green functions for $\tilde{F}$ with the appropriate Green polynomials without invoking Lemma 4.8 (and without any assumption on p). The key point is that for $u \in G^{\tilde{F}}_{\mathrm{uni}}$, F^* acts by the scalar q^i on $H^{2i}(\mathcal{B}_u, \overline{\mathbb{Q}_l})$; this is well-known for $u \in G^F_{\mathrm{uni}}$, and the same arguments apply. □

So it suffices to express $\mathrm{Ave}(\chi^{\underline{\rho}})$ as a linear combination of these $B_{\underline{\nu}}^{\tilde{F}}$.

Let r and t be indeterminates. For any $d \geq 1$, it is known that there are unique rational functions $\Omega^\lambda_\mu(r,t) \in \mathbb{Q}(r,t)$ for all partitions λ, μ of d satisfying:

1. For all λ, ν there are some $a_{\lambda\nu} \in \mathbb{Q}(r,t)$ such that
$$\Omega^\lambda_\mu(r,t) = \chi^\lambda_\mu + \sum_{\nu > \lambda} a_{\lambda\nu} \chi^\nu_\mu$$
for all μ.
2. For all $\lambda \neq \nu$,
$$\sum_\mu z_\mu^{-1} \Omega^\lambda_\mu(r,t) \Omega^\nu_\mu(r,t) \prod_{j=1}^{\ell(\mu)} \frac{1 - r^{\mu_j}}{1 - t^{\mu_j}} = 0.$$

In the notation of [16, VI.§4], $(\Omega^\lambda_\mu(r,t))$ is the transition matrix between the two bases $\{P_{\lambda'}(x; r, t)\}$ and $\{\epsilon_\mu z_\mu^{-1} p_\mu(x)\}$ of symmetric functions with two parameters. In the notation of [19],

$$\Omega^\lambda_\mu(r,t) = \epsilon_\mu c_\mu(t) Y^{\lambda'}_\mu(r,t).$$

Note that if $r = t$, $\Omega^\lambda_\mu(r,t) = \chi^\lambda_\mu$ for all λ, μ. In this paper, we will only ever use $\Omega^\lambda_\mu(r,t)$ in cases where $r = t$ or $r = -t$. It is easy to see that all specializations $\Omega^\lambda_\mu(r_0, t_0)$ in any field of characteristic 0 make sense, provided r_0 and t_0 are not roots of unity and $\sum_\mu z_\mu^{-1} (\chi^\lambda_\mu)^2 \prod_{j=1}^{\ell(\mu)} \frac{1 - r_0^{\mu_j}}{1 - t_0^{\mu_j}} \neq 0$ for all λ, which for us will always be the case. For $\underline{\rho} \in \widehat{\mathcal{P}}_n^{\tilde{\sigma}}$, let

$$f_{\underline{\rho}}(t) = (1-t)(1-t^2)\cdots(1-t^n) \times \prod_{\xi\in\langle\tilde{\sigma}\rangle\setminus L} (-t)^{\tilde{m}_\xi n(\rho_\xi)} \prod_{s\in\rho_\xi} (1-(-t)^{\tilde{m}_\xi \ell(s)} t^{\tilde{m}_\xi(a(s)+1)})^{-1} \in \mathbb{Q}(t).$$

Here $\prod_{s\in\rho_\xi}$ is a product over boxes $s=(i,j)$ in the Young diagram of the partition ρ_ξ, and $a(s) = (\rho_\xi)_i - j$ and $\ell(s) = (\rho_\xi)'_j - i$ are the arm-length and leg-length of the box. The solution of Problem 0.1 in this case is completed by:

Theorem 6.7. *For all* $\underline{\rho} \in \widehat{\mathcal{P}}_n^{\tilde{\sigma}}$,

$$\mathrm{Ave}(\chi^{\underline{\rho}}) = (f_{\underline{\rho}}(q))^{-1} \sum_{\substack{\underline{\nu}\in\widehat{\mathcal{P}}_n^{\tilde{\sigma}} \\ |\underline{\nu}|=|\underline{\rho}|}} \left(\prod_{\xi\in\langle\tilde{\sigma}\rangle\setminus L} (z_{\nu_\xi})^{-1} \Omega_{\nu_\xi}^{\rho_\xi}((-q)^{\tilde{m}_\xi}, q^{\tilde{m}_\xi}) \right) B_{\underline{\nu}}^{\tilde{F}}.$$

Proof. For all $\underline{\rho} \in \widehat{\mathcal{P}}_n^{\tilde{\sigma}}$, define

$$A_{\underline{\rho}} = \sum_{\substack{\underline{\nu}\in\widehat{\mathcal{P}}_n^{\tilde{\sigma}} \\ |\underline{\nu}|=|\underline{\rho}|}} \left(\prod_{\xi\in\langle\tilde{\sigma}\rangle\setminus L} (z_{\nu_\xi})^{-1} \Omega_{\nu_\xi}^{\rho_\xi}((-q)^{\tilde{m}_\xi}, q^{\tilde{m}_\xi}) \right) B_{\underline{\nu}}^{\tilde{F}}.$$

Using Proposition 6.6 and the formulas

$$Q_\lambda^{(1^n)}(t) = (1-t)(1-t^2)\cdots(1-t^n) \prod_{j=1}^{\ell(\lambda)} (1-t^{\lambda_j})^{-1},$$

(6.1)

$$\sum_{|\mu|=|\lambda|} (z_\mu)^{-1} \Omega_\mu^\lambda(r,t) \prod_{j=1}^{\ell(\mu)} (1-t^{\mu_j})^{-1} = r^{n(\lambda)} \prod_{s\in\lambda} (1-r^{l(s)} t^{a(s)+1})^{-1},$$

the second of which follows from [16, VI, §6] (cf. [1, Theorem 6.4.5]), we get $A_{\underline{\rho}}(1) = f_{\underline{\rho}}(q)$. Since $\mathrm{Ave}(\chi^{\underline{\rho}})(1) = 1$ by (2) of Corollary 6.5, it suffices to prove that $\mathrm{Ave}(\chi^{\underline{\rho}})$ is a scalar multiple of $A_{\underline{\rho}}$. By the Inner Product Formula, different $B_{\underline{\nu}}^{\tilde{F}}$ are orthogonal, and

$$\langle B_{\underline{\nu}}^{\tilde{F}}, B_{\underline{\nu}}^{\tilde{F}} \rangle = \prod_{i=1}^n \frac{q^i-1}{q^i+1} \prod_{\xi\in\langle\tilde{\sigma}\rangle\setminus L} z_{\nu_\xi} \prod_{j=1}^{\ell(\nu_\xi)} \frac{q^{\tilde{m}_\xi(\nu_\xi)_j} - (-1)^{\tilde{m}_\xi(\nu_\xi)_j}}{q^{\tilde{m}_\xi(\nu_\xi)_j} - 1}.$$

So for $\underline{\rho} \neq \underline{\tau} \in \widehat{\mathcal{P}}_n^{\tilde{\sigma}}$,

$$\langle A_{\underline{\rho}}, A_{\underline{\tau}} \rangle = (-1)^n \prod_{i=1}^n \frac{q^i-1}{q^i+1} \times \sum_{\substack{\underline{\nu}\in\widehat{\mathcal{P}}_n^{\tilde{\sigma}} \\ |\underline{\nu}|=|\underline{\rho}|=|\underline{\tau}|}} \prod_{\xi\in\langle\tilde{\sigma}\rangle\setminus L} (z_{\nu_\xi})^{-1} \Omega_{\nu_\xi}^{\rho_\xi}((-q)^{\tilde{m}_\xi}, q^{\tilde{m}_\xi}) \Omega_{\nu_\xi}^{\tau_\xi}((-q)^{\tilde{m}_\xi}, q^{\tilde{m}_\xi}) \prod_{j=1}^{\ell(\nu_\xi)} \frac{1-(-q)^{\tilde{m}_\xi(\nu_\xi)_j}}{1-q^{\tilde{m}_\xi(\nu_\xi)_j}},$$

which vanishes by (2) of the definition of $\Omega_\mu^\lambda(r,t)$. That is, different $A_{\underline{\rho}}$ are orthogonal, as are different $\mathrm{Ave}(\chi^{\underline{\rho}})$ by (3) of Corollary 6.5. So to prove that $\mathrm{Ave}(\chi^{\underline{\rho}})$ is a scalar multiple of $A_{\underline{\rho}}$, it suffices to prove that the transition matrix from the basis

$\{\mathrm{Ave}(\chi^{\underline{\rho}})\}$ to the set $\{A_{\underline{\rho}}\}$ is triangular with respect to some total order on $\widehat{\mathcal{P}}_n^{\tilde{\sigma}}$. For $\underline{\tau} \in \widehat{\mathcal{P}}_n^{\tilde{\sigma}}$, let

$$C_{\underline{\tau}} = \sum_{\substack{\underline{\nu}\in\widehat{\mathcal{P}}_n^{\tilde{\sigma}}\\ |\underline{\nu}|=|\underline{\tau}|}} \left(\prod_{\xi\in\langle\tilde{\sigma}\rangle\setminus L} (z_{\nu_\xi})^{-1} \chi_{\nu_\xi}^{\tau_\xi} \right) B_{\underline{\nu}}^{\tilde{F}}.$$

Then by (1) of the definition of $\Omega_\mu^\lambda(r,t)$,

$$A_{\underline{\rho}} = \sum_{\substack{\underline{\tau}\in\widehat{\mathcal{P}}_n^{\tilde{\sigma}}\\ |\underline{\tau}|=|\underline{\rho}|\\ \tau_\xi\geq\rho_\xi,\ \forall\xi}} a_{\underline{\rho},\underline{\tau}} C_{\underline{\tau}}$$

for some $a_{\underline{\rho},\underline{\tau}} \in \mathbb{Q}$. So it would suffice to prove the same triangularity property for $\mathrm{Ave}(\chi^{\underline{\rho}})$, namely that

$$\mathrm{Ave}(\chi^{\underline{\rho}}) = \sum_{\substack{\underline{\tau}\in\widehat{\mathcal{P}}_n^{\tilde{\sigma}}\\ |\underline{\tau}|=|\underline{\rho}|\\ \tau_\xi\geq\rho_\xi,\ \forall\xi}} b_{\underline{\rho},\underline{\tau}} C_{\underline{\tau}} \tag{6.2}$$

for some $b_{\underline{\rho},\underline{\tau}} \in \overline{\mathbb{Q}_l}$. Let $(T, \text{Ł})$ be an $\tilde{F}$-stable pair in the class corresponding to $((1^{|\rho_\xi|}))_{\xi\in L} \in \widehat{\mathcal{P}}_n^{\tilde{\sigma}}$. Then there is an isomorphism

$$W(T)_{\text{Ł}} \cong \prod_{\xi\in\langle\tilde{\sigma}\rangle\setminus L} \underbrace{S_{|\rho_\xi|} \times \cdots \times S_{|\rho_\xi|}}_{\tilde{m}_\xi \text{ factors}},$$

where F corresponds to cyclic permutation of each group of factors $S_{|\rho_\xi|}$, so that

$$W(T)_{\text{Ł}}^F \cong \prod_{\xi\in\langle\tilde{\sigma}\rangle\setminus L} S_{|\rho_\xi|}.$$

For $\underline{\tau} \in \widehat{\mathcal{P}}_n^{\tilde{\sigma}}$ such that $|\underline{\tau}| = |\underline{\rho}|$, let $E_{\underline{\tau}}$ be the F-stable irreducible representation of $W(T)_{\text{Ł}}$ labelled by the partition τ_ξ on the factor corresponding to ξ. We have an isomorphism $\tilde{\varphi}_{(T,\text{Ł},E_{\underline{\tau}})} : \tilde{F}^* K_{(T,\text{Ł},E_{\underline{\tau}})} \xrightarrow{\sim} K_{(T,\text{Ł},E_{\underline{\tau}})}$ induced by the obvious isomorphism $F^*E_{\underline{\tau}} \xrightarrow{\sim} E_{\underline{\tau}}$ of $W(T)_{\text{Ł}}$-modules. Exactly as with Proposition 6.3,

$$\chi_{K_{(T,\text{Ł},E_{\underline{\tau}})},\tilde{\varphi}_{(T,\text{Ł},E_{\underline{\tau}})}} = (-1)^n C_{\underline{\tau}}.$$

On the other hand, $\chi^{\underline{\rho}}$ is, up to sign, $\chi_{K_{(T,\text{Ł},E_{\underline{\rho}})},\phi_{(T,\text{Ł},E_{\underline{\rho}})}}$, where

$$\phi_{(T,\text{Ł},E_{\underline{\rho}})} : (F^2)^* K_{(T,\text{Ł},E_{\underline{\rho}})} \xrightarrow{\sim} K_{(T,\text{Ł},E_{\underline{\rho}})}$$

is as defined in Proposition 6.3, so by Corollary 3.3, $\mathrm{Ave}(\chi^{\underline{\rho}})$ is a scalar multiple of $\chi^F_{m_!(K_{(T,\text{Ł},E_{\underline{\rho}})}\boxtimes\tilde{F}^*K_{(T,\text{Ł},E_{\underline{\rho}})})}$. Now if we take $A = K_{(T,\text{Ł},E_{\underline{\rho}})}$ in Corollary 3.9, the character sheaves A' which arise are precisely the $K_{(T,\text{Ł},E_{\underline{\tau}})}$ for $\underline{\tau} \in \widehat{\mathcal{P}}_n^{\tilde{\sigma}}$ such that $|\underline{\tau}| = |\underline{\rho}|$ and $\tau_\xi \geq \rho_\xi$, $\forall\xi$. (This follows from the classification in [13, §18].) So (6.2) is exactly the content of Corollary 3.9 in this case. □

Remark 6.8. The appearance of Macdonald's symmetric functions with two parameters above is analogous to that in [1, §6].

The following special case is worth mentioning.

Corollary 6.9. *Let $\underline{\rho} \in \widehat{\mathcal{P}}_n^{\tilde{\sigma}}$ be such that $\rho_\xi = (|\rho_\xi|)$ for all ξ, or equivalently that $\chi^{\underline{\rho}}$ is a semisimple character of G^{F^2}. Then*

$$\mathrm{Ave}(\chi^{\underline{\rho}}) = \frac{\prod_{\xi \in \langle\tilde{\sigma}\rangle\backslash L}(1-q^{\tilde{m}_\xi})(1-q^{2\tilde{m}_\xi})\cdots(1-q^{|\rho_\xi|\tilde{m}_\xi})}{(1-q)(1-q^2)\cdots(1-q^n)} \times \sum_{\substack{\underline{\nu}\in\widehat{\mathcal{P}}_n^{\tilde{\sigma}}\\ |\underline{\nu}|=|\underline{\rho}|}} \left(\prod_{\xi\in\langle\tilde{\sigma}\rangle\backslash L} (z_{\nu_\xi})^{-1} \right) B_{\underline{\nu}}^{\tilde{F}}.$$

Proof. By (1) of the definition of $\Omega_\mu^\lambda(r,t)$,

$$\Omega_\mu^{(d)}(r,t) = \chi_\mu^{(d)} = 1.$$

So the result follows from Theorem 6.7. □

Example 6.10. Let us work out what Theorem 6.7 says in the case when $n=2$. We will use an exponential notation for elements of $\mathcal{P}_2^{\tilde{\sigma}}$ and $\widehat{\mathcal{P}}_2^{\tilde{\sigma}}$, so that for instance $\alpha^{(2)}$ means the function in $\mathcal{P}_2^{\tilde{\sigma}}$ assigning the partition (2) to $\alpha \in (k^\times)^{\tilde{\sigma}}$ and (0) to every other $\alpha' \in k^\times$, and $\xi_1^{(1)}\xi_2^{(1)}$ or simply $\xi_1\xi_2$ means the function in $\widehat{\mathcal{P}}_2^{\tilde{\sigma}}$ assigning the partition (1) to both of $\xi_1 \neq \xi_2 \in L^{\tilde{\sigma}}$, and (0) to every other $\xi \in L$.

The elements of $\mathcal{P}_2^{\tilde{\sigma}}$ correspond to G^F-orbits in $G^{\tilde{F}}$ as follows:

- $\alpha^{(1^2)}$, for $\alpha \in (k^\times)^{\tilde{\sigma}}$, corresponds to the orbit of the central element $\alpha.1$.
- $\alpha^{(2)}$, for $\alpha \in (k^\times)^{\tilde{\sigma}}$, corresponds to the orbit of non-semisimple elements with semisimple part $\alpha.1$.
- $\alpha_1\alpha_2$, for $\alpha_1 \neq \alpha_2 \in (k^\times)^{\tilde{\sigma}}$, corresponds to the orbit of ($\tilde{F}$-split) semisimple elements with eigenvalues α_1 and α_2.
- $\alpha\tilde{\sigma}(\alpha)$, for $\alpha \in (k^\times)^{\tilde{\sigma}^2} \setminus (k^\times)^{\tilde{\sigma}}$, corresponds to the orbit of (non-$\tilde{F}$-split) semisimple elements with eigenvalues α and $\tilde{\sigma}(\alpha)$.

These four classes of orbits contain, respectively, $q+1$ orbits of size 1, $q+1$ orbits of size q^2-1, $\frac{q^2+q}{2}$ orbits of size q^2+q, and $\frac{q^2-q-2}{2}$ orbits of size q^2-q, for a total of $q(q+1)(q^2+1)$ elements of $G^{\tilde{F}}$.

The list of elements of $\widehat{\mathcal{P}}_2^{\tilde{\sigma}}$ is similar. The basic functions, as computed by Proposition 6.6, are as follows. (The entry in the row labelled by $\underline{\nu} \in \widehat{\mathcal{P}}_2^{\tilde{\sigma}}$ and the column labelled by $\underline{\mu} \in \mathcal{P}_2^{\tilde{\sigma}}$ is $B_{\underline{\nu}}^{\tilde{F}}(g)$ for $g \in G^{\tilde{F}}$ in the orbit corresponding to $\underline{\mu}$.)

	$\alpha^{(1^2)}$	$\alpha^{(2)}$	$\alpha_1\alpha_2$	$\alpha\tilde{\sigma}(\alpha)$
$\xi^{(1^2)}$	$(1+q)(\langle\alpha,\xi\rangle^{\tilde{\sigma}})^2$	$(\langle\alpha,\xi\rangle^{\tilde{\sigma}})^2$	$2\langle\alpha_1,\xi\rangle^{\tilde{\sigma}}\langle\alpha_2,\xi\rangle^{\tilde{\sigma}}$	0
$\xi^{(2)}$	$(1-q)\langle\alpha,\xi\rangle^{\tilde{\sigma}^2}$	$\langle\alpha,\xi\rangle^{\tilde{\sigma}^2}$	0	$2\langle\alpha,\xi\rangle^{\tilde{\sigma}^2}$
$\xi_1\xi_2$	$(1+q)\langle\alpha,\xi_1\rangle^{\tilde{\sigma}}\langle\alpha,\xi_2\rangle^{\tilde{\sigma}}$	$\langle\alpha,\xi_1\rangle^{\tilde{\sigma}}\langle\alpha,\xi_2\rangle^{\tilde{\sigma}}$	$\langle\alpha_1,\xi_1\rangle^{\tilde{\sigma}}\langle\alpha_2,\xi_2\rangle^{\tilde{\sigma}}$ $+\langle\alpha_1,\xi_2\rangle^{\tilde{\sigma}}\langle\alpha_2,\xi_1\rangle^{\tilde{\sigma}}$	0
$\xi\tilde{\sigma}(\xi)$	$(1-q)\langle\alpha,\xi\rangle^{\tilde{\sigma}^2}$	$\langle\alpha,\xi\rangle^{\tilde{\sigma}^2}$	0	$\langle\alpha,\xi\rangle^{\tilde{\sigma}^2}$ $+\langle\tilde{\sigma}(\alpha),\xi\rangle^{\tilde{\sigma}^2}$

The elements of $\widehat{\mathcal{P}}_2^{\tilde{\sigma}}$ correspond to irreducible constituents of $\mathrm{Ind}_{G^F}^{G^{F^2}}(1)$ as follows:

- $\chi^{\xi^{(2)}}$, for $\xi \in L^{\tilde{\sigma}}$, is the degree-one character $\langle \det(\cdot), \xi\rangle^{\tilde{\sigma}^2}$.
- $\chi^{\xi^{(1^2)}}$, for $\xi \in L^{\tilde{\sigma}}$, is $\mathrm{St}.\chi^{\xi^{(2)}}$, where St is the Steinberg character of degree q.
- $\chi^{\xi_1\xi_2}$, for $\xi_1 \neq \xi_2 \in L^{\tilde{\sigma}}$, is a principal series character of degree $q+1$.
- $\chi^{\xi\tilde{\sigma}(\xi)}$, for $\xi \in L^{\tilde{\sigma}^2} \setminus L^{\tilde{\sigma}}$, is a discrete series character of degree $q-1$.

The averages of the semisimple characters can be computed using Corollary 6.9:

$$\begin{aligned}\mathrm{Ave}(\chi^{\xi^{(2)}}) &= \frac{1}{2}B^{\tilde{F}}_{\xi^{(1^2)}} + \frac{1}{2}B^{\tilde{F}}_{\xi^{(2)}},\\ \mathrm{Ave}(\chi^{\xi_1\xi_2}) &= \frac{1}{1+q}B^{\tilde{F}}_{\xi_1\xi_2},\\ \mathrm{Ave}(\chi^{\xi\tilde{\sigma}(\xi)}) &= \frac{1}{1-q}B^{\tilde{F}}_{\xi\tilde{\sigma}(\xi)}.\end{aligned}$$

To compute $\mathrm{Ave}(\chi^{\xi^{(1^2)}})$ we first observe that

$$\begin{aligned}\Omega^{(1^2)}_{(1^2)}(r,t) &= \frac{(1+r)(1-t)}{1-rt},\\ \Omega^{(1^2)}_{(2)}(r,t) &= -\frac{(1-r)(1+t)}{1-rt},\\ f_{\xi^{(1^2)}}(t) &= -t\frac{1-t^2}{1+t^2}.\end{aligned}$$

So by Theorem 6.7,

$$\begin{aligned}\mathrm{Ave}(\chi^{\xi^{(1^2)}}) &= -\frac{1+q^2}{q(1-q^2)}\left[\frac{1}{2}\frac{(1-q)^2}{1+q^2}B^{\tilde{F}}_{\xi^{(1^2)}} - \frac{1}{2}\frac{(1+q)^2}{1+q^2}B^{\tilde{F}}_{\xi^{(2)}}\right]\\ &= -\frac{1-q}{2q(1+q)}B^{\tilde{F}}_{\xi^{(1^2)}} + \frac{1+q}{2q(1-q)}B^{\tilde{F}}_{\xi^{(2)}}.\end{aligned}$$

Thus the values of the spherical functions are (for simplicity we write $\langle\cdot,\cdot\rangle$ for $\langle\cdot,\cdot\rangle^{\tilde{\sigma}}$):

	$\alpha^{(1^2)}$	$\alpha^{(2)}$	$\alpha_1\alpha_2$	$\alpha\tilde{\sigma}(\alpha)$
$\mathrm{Ave}(\chi^{\xi^{(2)}})$	$\langle\alpha^2,\xi\rangle$	$\langle\alpha^2,\xi\rangle$	$\langle\alpha_1\alpha_2,\xi\rangle$	$\langle\alpha\tilde{\sigma}(\alpha),\xi\rangle$
$\mathrm{Ave}(\chi^{\xi^{(1^2)}})$	$\langle\alpha^2,\xi\rangle$	$\frac{2}{1-q^2}\langle\alpha^2,\xi\rangle$	$-\frac{1-q}{q(1+q)}\langle\alpha_1\alpha_2,\xi\rangle$	$\frac{1+q}{q(1-q)}\langle\alpha\tilde{\sigma}(\alpha),\xi\rangle$
$\mathrm{Ave}(\chi^{\xi_1\xi_2})$	$\langle\alpha,\xi_1\xi_2\rangle$	$\frac{1}{1+q}\langle\alpha,\xi_1\xi_2\rangle$	$\frac{1}{1+q}(\langle\alpha_1,\xi_1\rangle\langle\alpha_2,\xi_2\rangle$ $+\langle\alpha_1,\xi_2\rangle\langle\alpha_2,\xi_1\rangle)$	0
$\mathrm{Ave}(\chi^{\xi\tilde{\sigma}(\xi)})$	$\langle\alpha,\xi\tilde{\sigma}(\xi)\rangle$	$\frac{1}{1-q}\langle\alpha,\xi\tilde{\sigma}(\xi)\rangle$	0	$\frac{1}{1-q}(\langle\alpha,\xi\rangle^{\tilde{\sigma}^2}$ $+\langle\tilde{\sigma}(\alpha),\xi\rangle^{\tilde{\sigma}^2})$

It is an interesting exercise to calculate these spherical functions directly from the definition.

Recall from Remark 2.4 that to convert these values into the character table of the double coset algebra $\mathcal{H}(GL_2(\mathbb{F}_{q^2}), GL_2(\mathbb{F}_q))$, each column should be multiplied by the size of the corresponding G^F-orbit on $G^{\tilde{F}}$; the entries then become algebraic integers as expected. Another interesting point is that since $\mathrm{Ave}(\chi^{1^{(1^2)}})$ is a linear combination of both the constant function 1 and the characteristic function of the character sheaf A_{St} corresponding to the Steinberg character, both $\overline{\mathbb{Q}_l}[4]$ and A_{St} must occur as simple perverse constituents of $m_!(A_{\mathrm{St}} \boxtimes A_{\mathrm{St}})$, which is not obvious from the characteristic functions with respect to F.

7. The Symmetric Space $GL_n(\mathbb{F}_{q^2})/U_n(\mathbb{F}_{q^2})$

Keep the notations of the previous section, except that the Frobenius map F on G is no longer that induced by F on V, but its composition with some outer involution of G which commutes with it. Thus F^2 is the same as in the previous section, so we still have $G^{F^2} \cong GL_n(\mathbb{F}_{q^2})$, but now $G^F \cong U_n(\mathbb{F}_{q^2})$.

As shown in [6, Lemma 1.3], the conjugacy class of G^{F^2} corresponding to $\underline{\lambda} \in \mathcal{P}_n^{\sigma^2}$ meets $G^{\tilde{F}}$ in a unique G^F-orbit if $\underline{\lambda} \in \mathcal{P}_n^{\sigma}$, and does not meet $G^{\tilde{F}}$ otherwise. So the G^F-orbits on $G^{\tilde{F}}$ are in bijection with $\mathcal{P}_n^{\sigma}$. (By contrast, the conjugacy classes in G^F are naturally in bijection with $\mathcal{P}_n^{\tilde{\sigma}}$, so that the situation of the previous section has been reversed.) Similarly there is a bijection between G^F-conjugacy classes of $\tilde{F}$-stable pairs $(T, Ł)$ and $\widehat{\mathcal{P}}_n^{\sigma}$ so that if $(T, Ł)$ is in the class corresponding to $\underline{\nu}$:

1. there is an isomorphism
$$T \cong \prod_{\xi \in \langle\sigma\rangle \setminus L} \prod_{j=1}^{\ell(\nu_\xi)} \underbrace{k^\times \times \cdots \times k^\times}_{m_\xi(\nu_\xi)_j \text{ factors}},$$
under which $\tilde{F}$ on T corresponds to cyclic permutation of each group of factors $k^\times$, composed with σ;
2. consequently,
$$T^{\tilde{F}} \cong \prod_{\xi \in \langle\sigma\rangle \setminus L} \prod_{j=1}^{\ell(\nu_\xi)} (k^\times)^{\sigma^{m_\xi(\nu_\xi)_j}};$$
3. under this isomorphism, $\chi_{Ł}^{\tilde{F}}$ corresponds to
$$\prod_{\xi \in \langle\sigma\rangle \setminus L} \prod_{j=1}^{\ell(\nu_\xi)} \langle \cdot, \xi \rangle^{\sigma^{m_\xi(\nu_\xi)_j}},$$
for some choice of representatives $\{\xi\}$ for $\langle\sigma\rangle \setminus L$.

For instance, if T_w is an F-stable maximal torus obtained by twisting a maximally split torus by an element w of the canonical Weyl group $W \cong S_n$, then $(T_w, \overline{\mathbb{Q}_l})$ corresponds to the partition giving the cycle type of ww_0, where $w_0 \in W$ is the longest element. So the maximal tori corresponding to the partition (1^n) are now the minimally split tori, not the maximally split tori as in the previous section.

Moreover we have the following description of the induced character $\mathrm{Ind}_{G^F}^{G^{F^2}}(1)$:

Theorem 7.1. *For* $\underline{\rho} \in \widehat{\mathcal{P}}_n^{\sigma^2}$,
$$\langle \chi^{\underline{\rho}}, \mathrm{Ind}_{G^F}^{G^{F^2}}(1) \rangle = \begin{cases} 1, & \textit{if } \underline{\rho} \in \widehat{\mathcal{P}}_n^{\sigma}, \\ 0, & \textit{otherwise.} \end{cases}$$

Proof. See [6, Theorems 2.1, 2.4], or [15, Theorem 1.4]. ∎

Corollary 7.2. 1. *If* $\underline{\rho} \in \widehat{\mathcal{P}}_n^{\sigma^2} \setminus \widehat{\mathcal{P}}_n^{\sigma}$, *then* $\mathrm{Ave}(\chi^{\underline{\rho}}) = 0$.

2. *If* $\underline{\rho} \in \widehat{\mathcal{P}}_n^\sigma$, $\mathrm{Ave}(\chi^{\underline{\rho}})(1) = 1$.
3. $\{\mathrm{Ave}(\chi^{\underline{\rho}}) \,|\, \underline{\rho} \in \widehat{\mathcal{P}}_n^\sigma\}$ *is an orthogonal basis of* $\mathcal{F}^{G^F}(G^{\tilde{F}})$.

Proof. This is deduced exactly as in the previous section. □

For $\underline{\nu} \in \widehat{\mathcal{P}}_n^\sigma$, let $B_{\underline{\nu}}^{\tilde{F}}$ be the (signed) basic function $(-1)^n \chi_{K_{(T,\text{Ł})}}^{\tilde{F}}$ for $\tilde{F}$-stable $(T, \text{Ł})$ in the corresponding class. As in the previous section, we have a combinatorial description of $B_{\underline{\nu}}^{\tilde{F}}$.

Proposition 7.3. *If* $g \in G^{\tilde{F}}$ *is in the* G^F*-orbit corresponding to* $\underline{\mu} \in \mathcal{P}_n^\sigma$, *then*

$$B_{\underline{\nu}}^{\tilde{F}}(g) = \sum_{\substack{\phi \in \Phi_{\underline{\nu}}^\sigma \\ |\underline{\mu}(\phi)| = |\underline{\mu}|}} \prod_{(\xi,j) \in \Lambda_{\underline{\nu}}^\sigma} \langle \phi(\xi, j), \xi \rangle^{\sigma^{m_\xi(\nu_\xi)_j}} \prod_{\alpha \in \langle\sigma\rangle \backslash k^\times} Q_{\mu(\phi)_\alpha}^{\mu_\alpha}((-q)^{m_\alpha}).$$

Proof. Again, this follows from the Character Formula 4.9, once we identify the Green functions for $\tilde{F}$ with the appropriate Green polynomials of $-q$. For this we have to know that for $u \in G_{\text{uni}}^{\tilde{F}}$, $F^* \circ w_0$ acts by the scalar $(-q)^i$ on $H^{2i}(\mathcal{B}_u, \overline{\mathbb{Q}_l})$. This statement for $u \in G_{\text{uni}}^F$ is Lemma 3.2 of [10], and the same arguments apply here. (The assumption that q is sufficiently large is no longer necessary, by results of Shoji in [18].) □

Remark 7.4. The values of the basic functions on $G^{\tilde{F}}$ here are the same as those of the basic functions on G^F in the previous section, with q replaced by $-q$.

So as in the previous section, to solve Problem 0.1 it suffices to express $\mathrm{Ave}(\chi^{\underline{\rho}})$ for $\underline{\rho} \in \widehat{\mathcal{P}}_n^\sigma$ as a linear combination of the $B_{\underline{\nu}}^{\tilde{F}}$. Define

$$\begin{aligned} g_{\underline{\rho}}(t) &= (1-t)(1-t^2)\cdots(1-t^n) \\ &\quad \times \prod_{\xi \in \langle\sigma\rangle \backslash L} (-t)^{m_\xi n(\rho_\xi)} \prod_{s \in \rho_\xi} (1 - (-t)^{m_\xi \ell(s)} t^{m_\xi(a(s)+1)})^{-1} \in \mathbb{Q}(t), \end{aligned}$$

where the notation is the same as before Theorem 6.7. Then the solution of Problem 0.1 in this case is completed by:

Theorem 7.5. *For all* $\underline{\rho} \in \widehat{\mathcal{P}}_n^\sigma$,

$$\mathrm{Ave}(\chi^{\underline{\rho}}) = (g_{\underline{\rho}}(-q))^{-1} \sum_{\substack{\underline{\nu} \in \widehat{\mathcal{P}}_n^\sigma \\ |\underline{\nu}| = |\underline{\rho}|}} \left(\prod_{\xi \in \langle\sigma\rangle \backslash L} (z_{\nu_\xi})^{-1} \Omega_{\nu_\xi}^{\rho_\xi}(q^{m_\xi}, (-q)^{m_\xi}) \right) B_{\underline{\nu}}^{\tilde{F}}.$$

Proof. The proof is entirely analogous to the proof of Theorem 6.7. □

Corollary 7.6. *Let* $\underline{\rho} \in \widehat{\mathcal{P}}_n^\sigma$ *be such that* $\rho_\xi = (|\rho_\xi|)$ *for all* ξ. *Then*

$$\begin{aligned} \mathrm{Ave}(\chi^{\underline{\rho}}) &= \frac{\prod_{\xi \in \langle\sigma\rangle \backslash L} (1-(-q)^{m_\xi})(1-(-q)^{2m_\xi})\cdots(1-(-q)^{|\rho_\xi| m_\xi})}{(1+q)(1-q^2)\cdots(1-(-q)^n)} \\ &\quad \times \sum_{\substack{\underline{\nu} \in \widehat{\mathcal{P}}_n^\sigma \\ |\underline{\nu}| = |\underline{\rho}|}} \left(\prod_{\xi \in \langle\sigma\rangle \backslash L} (z_{\nu_\xi})^{-1} \right) B_{\underline{\nu}}^{\tilde{F}}. \end{aligned}$$

Proof. This is deduced as in the previous section. □

8. Proof of Theorem 3.7

In this section we are not concerned with our given $\mathbb{F}_q$-structures, so all maximal tori are equivalent. So we fix a maximal torus T and speak only of character sheaves of central character Ł (determined up to $W(T)$-conjugacy) where Ł is a tame rank-one local system on T. We drop T from the notation and write W, $\widehat{G}_{Ł}$, $\mathbf{D}^G(G)_{Ł}$, $\widehat{G}_{Ł}^{\leqslant\mathcal{C}}$, $\mathbf{D}^G(G)_{Ł}^{\leqslant\mathcal{C}}$, etc. We write $W'_{Ł}$ for $W(T)_{Ł}$ (following [13], where the notation $W_{Ł}$ is reserved for a certain subgroup of $W'_{Ł}$).

It is most convenient to reformulate Theorem 3.7 in terms of the map

$$\tilde{m} : G \times G \to G : (g_1, g_2) \mapsto g_1^{-1} g_2.$$

Since pull-back through the inverse map interchanges $\widehat{G}_{Ł}^{\leqslant\mathcal{C}}$ and $\widehat{G}_{Ł^{-1}}^{\leqslant\mathcal{C}}$ (using the obvious identification of $W'_{Ł}$ with $W'_{Ł^{-1}}$), Theorem 3.7 is equivalent to the following:

Theorem 8.1. *Let $A \in \widehat{G}_{Ł}$, $A' \in \widehat{G}_{Ł'}$.*

1. *If $Ł^{-1}$, $Ł'$ are not in the same W-orbit, then $\tilde{m}_!(A \boxtimes A') \cong 0$.*
2. *If $Ł' = Ł^{-1}$ and $A \in \widehat{G}_{Ł}^{\leqslant\mathcal{C}}$, $A' \in \widehat{G}_{Ł^{-1}}^{\leqslant\mathcal{C}'}$, then $\tilde{m}_!(A \boxtimes A') \in \mathbf{D}^G(G)_{Ł^{-1}}^{\leqslant\mathcal{C},\mathcal{C}'}$.*

The rest of this section will be devoted to the proof of this theorem. The arguments below are all present, if sometimes sketchily, in Grojnowski's thesis [8]. Theorem 8.1 itself is a special case of the proposition and corollary stated in [8, §5].

Fix a Borel subgroup B of G containing T with unipotent radical U. Identify $\mathcal{B}$ with G/B and write $\tilde{\mathcal{B}}$ for G/U. Then T acts freely on $\tilde{\mathcal{B}}$ by right multiplication, and the obvious map $q : \tilde{\mathcal{B}} \to \mathcal{B}$ is a principal T-bundle. Also $T \times T$ acts on $\tilde{\mathcal{B}} \times \tilde{\mathcal{B}}$ in the obvious way, and so on for more factors. If Ł and Ł$'$ are tame local systems of rank one on T, then $Ł \boxtimes Ł'$ is a tame local system of rank one on the torus $T \times T$. As in §1, define $\mathbf{D}_{Ł}(\tilde{\mathcal{B}})$, $\mathbf{D}_{Ł\boxtimes Ł'}(\tilde{\mathcal{B}} \times \tilde{\mathcal{B}})$, and so on for more factors. If $K \in \mathbf{D}_{Ł}(\tilde{\mathcal{B}})$ and $K' \in \mathbf{D}_{Ł'}(\tilde{\mathcal{B}})$, then $K \boxtimes K' \in \mathbf{D}_{Ł\boxtimes Ł'}(\tilde{\mathcal{B}} \times \tilde{\mathcal{B}})$, etc. Clearly the interchange map $i : \tilde{\mathcal{B}} \times \tilde{\mathcal{B}} \to \tilde{\mathcal{B}} \times \tilde{\mathcal{B}}$ induces an equivalence $i^* : \mathbf{D}_{Ł\boxtimes Ł'}(\tilde{\mathcal{B}} \times \tilde{\mathcal{B}}) \to \mathbf{D}_{Ł'\boxtimes Ł}(\tilde{\mathcal{B}} \times \tilde{\mathcal{B}})$.

Let $p_{12}, p_{23}, p_{13} : \tilde{\mathcal{B}} \times \tilde{\mathcal{B}} \times \tilde{\mathcal{B}} \to \tilde{\mathcal{B}} \times \tilde{\mathcal{B}}$ be the three projections. Factor p_{13} as $\widetilde{p_{13}}(\mathrm{id} \times q \times \mathrm{id})$, where $\widetilde{p_{13}} : \tilde{\mathcal{B}} \times \mathcal{B} \times \tilde{\mathcal{B}} \to \tilde{\mathcal{B}} \times \tilde{\mathcal{B}}$. Let $K \in \mathbf{D}_{Ł_1\boxtimes Ł_2}(\tilde{\mathcal{B}} \times \tilde{\mathcal{B}})$, $K' \in \mathbf{D}_{Ł_2^{-1}\boxtimes Ł_3}(\tilde{\mathcal{B}} \times \tilde{\mathcal{B}})$. Form $p_{12}^* K \otimes p_{23}^* K' \in \mathbf{D}_{Ł_1\boxtimes\overline{\mathbb{Q}_l}\boxtimes Ł_3}(\tilde{\mathcal{B}} \times \tilde{\mathcal{B}} \times \tilde{\mathcal{B}})$ and

$$(\mathrm{id} \times q \times \mathrm{id})_\flat (p_{12}^* K \otimes p_{23}^* K') \in \mathbf{D}_{Ł_1\boxtimes Ł_3}(\tilde{\mathcal{B}} \times \mathcal{B} \times \tilde{\mathcal{B}}).$$

We write $K \star K'$ for the *convolution*

$$(\widetilde{p_{13}})_!(\mathrm{id} \times q \times \mathrm{id})_\flat (p_{12}^* K \otimes p_{23}^* K')[-\dim \tilde{\mathcal{B}} - \dim T] \in \mathbf{D}_{Ł_1\boxtimes Ł_3}(\tilde{\mathcal{B}} \times \tilde{\mathcal{B}}).$$

(The shift makes convolution commute with Verdier duality.)

Now G acts diagonally on $\tilde{\mathcal{B}} \times \tilde{\mathcal{B}}$ by $g.(g_1U, g_2U) = (gg_1U, gg_2U)$ and this action commutes with that of $T \times T$. Define $\mathbf{D}^G_{Ł\boxtimes Ł'}(\tilde{\mathcal{B}} \times \tilde{\mathcal{B}})$ as in §1. It is clear that convolution lifts uniquely to these categories. In particular, we get a functor

$$\star : \mathbf{D}^G_{Ł\boxtimes Ł^{-1}}(\tilde{\mathcal{B}} \times \tilde{\mathcal{B}}) \times \mathbf{D}^G_{Ł\boxtimes Ł^{-1}}(\tilde{\mathcal{B}} \times \tilde{\mathcal{B}}) \to \mathbf{D}^G_{Ł\boxtimes Ł^{-1}}(\tilde{\mathcal{B}} \times \tilde{\mathcal{B}}).$$

Proposition 8.2. *The simple objects in $\mathbf{M}^G_{Ł\boxtimes Ł'}(\tilde{\mathcal{B}} \times \tilde{\mathcal{B}})$ are indexed by*

$$\{w \in W \mid (w^{-1})^* Ł' \cong Ł^{-1}\}.$$

In particular, $\mathbf{D}^G_{Ł\boxtimes Ł'}(\tilde{\mathcal{B}} \times \tilde{\mathcal{B}}) \cong \{0\}$ if $Ł'$ and $Ł^{-1}$ are not in the same W-orbit.

Proof. Let A be a G-equivariant $(T \times T, Ł \boxtimes Ł')$-covariant simple perverse sheaf on $\tilde{\mathcal{B}} \times \tilde{\mathcal{B}}$. Then supp A is a $(G \times T \times T)$-invariant irreducible closed subvariety of $\tilde{\mathcal{B}} \times \tilde{\mathcal{B}}$, hence of the form $\overline{\widetilde{\mathcal{O}_w}}$ for some $w \in W$, where

$$\widetilde{\mathcal{O}_w} = \{(g_1 U, g_2 U) \,|\, g_1^{-1} g_2 \in BwB\}.$$

Thus $A = IC(\overline{\widetilde{\mathcal{O}_w}}, \mathcal{E})[\dim \widetilde{\mathcal{O}_w}]$ for some G-equivariant $(T \times T, Ł \boxtimes Ł')$-covariant simple local system $\mathcal{E}$ on $\widetilde{\mathcal{O}_w}$. Since $\widetilde{\mathcal{O}_w}$ is a homogeneous space for $G \times T \times T$, and the stabilizer in $G \times T \times T$ of $(U, \dot{w}U) \in \widetilde{\mathcal{O}_w}$ is

$$\{(tu, t, {}^{w^{-1}}t) \,|\, t \in T, u \in U \cap {}^w U\}$$

which is connected, such a local system can only exist if

$$(Ł \boxtimes Ł')|_{\{(t, {}^{w^{-1}}t)\}} \cong \overline{\mathbb{Q}_l}, \text{ i.e. } (w^{-1})^* Ł' \cong Ł^{-1},$$

and is unique up to isomorphism if this is true. □

If $w \in W'_Ł$, let $A_w^Ł$ be the simple object of $\mathbf{M}^G_{Ł \boxtimes Ł^{-1}}(\tilde{\mathcal{B}} \times \tilde{\mathcal{B}})$ associated to w by this proposition. Clearly $i^* A_w^Ł \cong A_{w^{-1}}^{Ł^{-1}}$. It is easy to see that

$$A_1^Ł \star A_w^Ł \cong A_w^Ł \star A_1^Ł \cong A_w^Ł$$

for all $w \in W'_Ł$. General convolutions $A_w^Ł \star A_{w'}^Ł$ will be described in Theorem 8.4 below. For $\mathcal{C}$ a two-sided cell of $W'_Ł$, let $\mathbf{D}^G_{Ł \boxtimes Ł^{-1}}(\tilde{\mathcal{B}} \times \tilde{\mathcal{B}})^{\leqslant \mathcal{C}}$ denote the full subcategory of $\mathbf{D}^G_{Ł \boxtimes Ł^{-1}}(\tilde{\mathcal{B}} \times \tilde{\mathcal{B}})$ consisting of complexes K such that every simple constituent of every ${}^p H^i K$ is isomorphic to some $A_w^Ł$ with $w \leqslant \mathcal{C}$. Since two-sided cells are stable under inversion, $i^* \mathbf{D}^G_{Ł \boxtimes Ł^{-1}}(\tilde{\mathcal{B}} \times \tilde{\mathcal{B}})^{\leqslant \mathcal{C}} = \mathbf{D}^G_{Ł^{-1} \boxtimes Ł}(\tilde{\mathcal{B}} \times \tilde{\mathcal{B}})^{\leqslant \mathcal{C}}$.

Now let $u^{1/2}$ be an indeterminate, and write $\mathcal{A} = \mathbb{Z}[u^{1/2}, u^{-1/2}]$. In [13, 6.1], Lusztig defines an associative $\mathcal{A}$-algebra $\mathcal{H}'_Ł$, the *Hecke algebra* associated to $W'_Ł$ (see also [17, 3.3]). Let $\{C_w^Ł | w \in W'_Ł\}$ denote its second Kazhdan-Lusztig basis (written $\{C'_w\}$ in [13, (12.9.2)]). Let $(\quad)^t : \mathcal{H}'_Ł \to \mathcal{H}'_{Ł^{-1}}$ be the unique anti-algebra isomorphism satisfying $(C_w^Ł)^t = C_{w^{-1}}^{Ł^{-1}}$ for all $w \in W'_Ł = W'_{Ł^{-1}}$. For $\mathcal{C}$ a two-sided cell of $W'_Ł$, let $(\mathcal{H}'_Ł)^{\leqslant \mathcal{C}}$ be the $\mathcal{A}$-submodule of $\mathcal{H}'_Ł$ spanned by $\{C_w^Ł | w \leqslant \mathcal{C}\}$, a two-sided ideal of $\mathcal{H}'_Ł$. It is evident that $((\mathcal{H}'_Ł)^{\leqslant \mathcal{C}})^t = (\mathcal{H}'_{Ł^{-1}})^{\leqslant \mathcal{C}}$.

Suppose we are given some $\mathbb{F}_{q^e}$-structure on G (not necessarily related to the $\mathbb{F}_q$-structures in other sections of this paper) such that T is $\mathbb{F}_{q^e}$-split, and $Ł$ is defined over $\mathbb{F}_{q^e}$ and pure of weight 0. Then $\tilde{\mathcal{B}}$ inherits an $\mathbb{F}_{q^e}$-structure, and every $A_w^Ł$ can be given an $\mathbb{F}_{q^e}$-structure which is also pure of weight 0. We have a mixed analogue $\mathbf{D}^{G,\text{mixed}}_{Ł \boxtimes Ł^{-1}}(\tilde{\mathcal{B}} \times \tilde{\mathcal{B}})$ of $\mathbf{D}^G_{Ł \boxtimes Ł^{-1}}(\tilde{\mathcal{B}} \times \tilde{\mathcal{B}})$. For $K \in \mathbf{D}^{G,\text{mixed}}_{Ł \boxtimes Ł^{-1}}(\tilde{\mathcal{B}} \times \tilde{\mathcal{B}})$, let ${}^p H^i_j K$ denote the pure-of-weight-j constituent of the weight filtration on ${}^p H^i K$. Define

$$\chi_u(K) = \sum_{i,j} (-1)^{i+j} [{}^p H^i_j K] u^{j/2} \in \mathcal{H}'_Ł,$$

where the Grothendieck group $K_0 \mathbf{M}^G_{Ł \boxtimes Ł^{-1}}(\tilde{\mathcal{B}} \times \tilde{\mathcal{B}})$ is embedded in $\mathcal{H}'_Ł$ by identifying $[A_w^Ł]$ with $C_w^Ł$. This definition is inspired by [13, 6.3]; as noted there, we have:

Lemma 8.3. *If* (K_1, K_2, K_3) *is a distinguished triangle in* $\mathbf{D}^{G,\mathrm{mixed}}_{\L\boxtimes\L^{-1}}(\tilde{\mathcal{B}}\times\tilde{\mathcal{B}})$*, then* $\chi_u(K_2) = \chi_u(K_1) + \chi_u(K_3)$*.*

If K is pure of weight 0, then ${}^pH^iK$ is pure of weight i, so

$$\chi_u(K) = \sum_i [{}^pH^iK]u^{i/2}. \tag{8.1}$$

It is clear that

$$K \in \mathbf{D}^{G,\mathrm{mixed}}_{\L\boxtimes\L^{-1}}(\tilde{\mathcal{B}}\times\tilde{\mathcal{B}})^{\leqslant\mathcal{C}} \Leftrightarrow \chi_u(K) \in (\mathcal{H}'_{\L})^{\leqslant\mathcal{C}}.$$

In extending the definition of convolution to $\mathbf{D}^{G,\mathrm{mixed}}_{\L\boxtimes\L^{-1}}(\tilde{\mathcal{B}}\times\tilde{\mathcal{B}})$, we add a Tate twist, so that

$$K \star K' = (\widetilde{p_{13}})_!(\mathrm{id}\times q\times\mathrm{id})_\flat(p_{12}^*K\otimes p_{23}^*K')[-\dim\tilde{\mathcal{B}} - \dim T](-\frac{\dim\tilde{\mathcal{B}}}{2} - \frac{\dim T}{2}).$$

Then since $\widetilde{p_{13}}$ is proper, if K and K' are pure of weight 0, $K\star K'$ is also.

Theorem 8.4. *If* $K, K' \in \mathbf{D}^{G,\mathrm{mixed}}_{\L\boxtimes\L^{-1}}(\tilde{\mathcal{B}}\times\tilde{\mathcal{B}})$*, then* $\chi_u(K\star K') = \chi_u(K)\chi_u(K')$*. Also* $\chi_u(K)^t = \chi_u(i^*K)$ *where* $i : \tilde{\mathcal{B}}\times\tilde{\mathcal{B}} \to \tilde{\mathcal{B}}\times\tilde{\mathcal{B}}$ *is the interchange map.*

Proof. The second assertion follows trivially from the definitions. For the first, it suffices, by Lemma 8.3, to prove that

$$\chi_u(A^{\L}_w \star A^{\L}_{w'}) = \chi_u(A^{\L}_w)\chi_u(A^{\L}_{w'}).$$

By (8.1), this is equivalent to proving that

$$\sum_i [{}^pH^i(A^{\L}_w\star A^{\L}_{w'})]u^{i/2} = C^{\L}_w C^{\L}_{w'}.$$

This result is due to Mars and Springer—it is equivalent to the special case $x, y \in W'_\eta$ of [17, Corollary 4.2.5]. To translate between our notation and theirs, note that

$$\mathbf{D}^G(\tilde{\mathcal{B}}\times\tilde{\mathcal{B}}) \cong \mathbf{D}^{G\times U\times U}(G\times G) \cong \mathbf{D}^{U\times U}(G)$$

where $U\times U$ acts on G by two-sided translation. It is easy to check that their notion of convolution coincides with ours after this change of viewpoint. (Note that their indeterminate t corresponds to our $u^{1/2}$.) □

Corollary 8.5. *If* $K \in \mathbf{D}^{G,\mathrm{mixed}}_{\L\boxtimes\L^{-1}}(\tilde{\mathcal{B}}\times\tilde{\mathcal{B}})^{\leqslant\mathcal{C}}$*,* $K' \in \mathbf{D}^{G,\mathrm{mixed}}_{\L\boxtimes\L^{-1}}(\tilde{\mathcal{B}}\times\tilde{\mathcal{B}})^{\leqslant\mathcal{C}'}$*, then*

$$K\star K' \in \mathbf{D}^{G,\mathrm{mixed}}_{\L\boxtimes\L^{-1}}(\tilde{\mathcal{B}}\times\tilde{\mathcal{B}})^{\leqslant\mathcal{C},\mathcal{C}'}.$$

Moreover if $y \leqslant \mathcal{C}, \mathcal{C}'$*, there exist some* $w \leqslant \mathcal{C}$ *and* $w' \leqslant \mathcal{C}'$ *so that* $A^{\L}_y$ *is a simple constituent of some* ${}^pH^i(A^{\L}_w \star A^{\L}_{w'})$*.*

Proof. By Theorem 8.4, we need only show that

$$(\mathcal{H}'_{\L})^{\leqslant\mathcal{C}}(\mathcal{H}'_{\L})^{\leqslant\mathcal{C}'} = (\mathcal{H}'_{\L})^{\leqslant\mathcal{C},\mathcal{C}'}.$$

But this follows from the fact that $\mathcal{H}'_{\L}\otimes_{\mathcal{A}}\mathbb{Q}(u^{1/2})$ is a semisimple algebra whose simple modules fall into families parametrized by the two-sided cells of $W'_{\L}$, in such a way that $(\mathcal{H}'_{\L})^{\leqslant\mathcal{C}}\otimes_{\mathcal{A}}\mathbb{Q}(u^{1/2})$ is the sum of all the simple two-sided ideals corresponding to cells $\mathcal{C}'\leqslant\mathcal{C}$. □

Now consider the diagram

$$\tilde{\mathcal{B}} \times \tilde{\mathcal{B}} \times \tilde{\mathcal{B}} \times \tilde{\mathcal{B}} \xleftarrow{\alpha} G \times \tilde{\mathcal{B}} \times \tilde{\mathcal{B}} \times \tilde{\mathcal{B}} \times \tilde{\mathcal{B}} \xrightarrow{\beta} \tilde{\mathcal{B}} \times \tilde{\mathcal{B}} \times \tilde{\mathcal{B}} \times \tilde{\mathcal{B}}$$

where $\alpha(g, g_1U, g_2U, g_3U, g_4U) = (gg_1U, g_2U, gg_3U, g_4U)$ and β is the projection. Clearly we get a functor

$$\beta_!\alpha^* : \mathbf{D}^{G[12]\times G[34]}_{Ł_1\boxtimes Ł_2\boxtimes Ł_3\boxtimes Ł_4}(\tilde{\mathcal{B}} \times \tilde{\mathcal{B}} \times \tilde{\mathcal{B}} \times \tilde{\mathcal{B}}) \to \mathbf{D}^{G[13]\times G[24]}_{Ł_1\boxtimes Ł_2\boxtimes Ł_3\boxtimes Ł_4}(\tilde{\mathcal{B}} \times \tilde{\mathcal{B}} \times \tilde{\mathcal{B}} \times \tilde{\mathcal{B}})$$

where $G[12]$ denotes G acting simultaneously on the first and second factors, and so on. We will need to consider two embeddings:

$$\mathbf{D}^{G}_{Ł_1\boxtimes Ł_2}(\tilde{\mathcal{B}} \times \tilde{\mathcal{B}}) \times \mathbf{D}^{G}_{Ł_3\boxtimes Ł_4}(\tilde{\mathcal{B}} \times \tilde{\mathcal{B}}) \hookrightarrow \mathbf{D}^{G[12]\times G[34]}_{Ł_1\boxtimes Ł_2\boxtimes Ł_3\boxtimes Ł_4}(\tilde{\mathcal{B}} \times \tilde{\mathcal{B}} \times \tilde{\mathcal{B}} \times \tilde{\mathcal{B}}),$$

which we will write $(K, K') \mapsto K \boxtimes K'$, and

$$\mathbf{D}^{G}_{Ł_1\boxtimes Ł_3}(\tilde{\mathcal{B}} \times \tilde{\mathcal{B}}) \times \mathbf{D}^{G}_{Ł_2\boxtimes Ł_4}(\tilde{\mathcal{B}} \times \tilde{\mathcal{B}}) \hookrightarrow \mathbf{D}^{G[13]\times G[24]}_{Ł_1\boxtimes Ł_2\boxtimes Ł_3\boxtimes Ł_4}(\tilde{\mathcal{B}} \times \tilde{\mathcal{B}} \times \tilde{\mathcal{B}} \times \tilde{\mathcal{B}}),$$

which we will write $(K, K') \mapsto K \overset{\leftrightarrow}{\boxtimes} K'$. Since $G \times T \times T$ has finitely many orbits on $\tilde{\mathcal{B}} \times \tilde{\mathcal{B}}$, every simple object in $\mathbf{M}^{G[13]\times G[24]}_{Ł_1\boxtimes Ł_2\boxtimes Ł_3\boxtimes Ł_4}(\tilde{\mathcal{B}} \times \tilde{\mathcal{B}} \times \tilde{\mathcal{B}} \times \tilde{\mathcal{B}})$ is isomorphic to $A \overset{\leftrightarrow}{\boxtimes} A'$ for some simple $A \in \mathbf{M}^{G}_{Ł_1\boxtimes Ł_3}(\tilde{\mathcal{B}} \times \tilde{\mathcal{B}})$, $A' \in \mathbf{M}^{G}_{Ł_2\boxtimes Ł_4}(\tilde{\mathcal{B}} \times \tilde{\mathcal{B}})$.

Lemma 8.6. *If $Ł'$ and $Ł^{-1}$ are not in the same W-orbit, then $\beta_!\alpha^* = 0$ on $\mathbf{D}^{G[12]\times G[34]}_{Ł\boxtimes Ł^{-1}\boxtimes Ł'\boxtimes(Ł')^{-1}}(\tilde{\mathcal{B}} \times \tilde{\mathcal{B}} \times \tilde{\mathcal{B}} \times \tilde{\mathcal{B}})$.*

Proof. It suffices to note that

$$\mathbf{D}^{G[13]\times G[24]}_{Ł\boxtimes Ł^{-1}\boxtimes Ł'\boxtimes(Ł')^{-1}}(\tilde{\mathcal{B}} \times \tilde{\mathcal{B}} \times \tilde{\mathcal{B}} \times \tilde{\mathcal{B}}) \cong \{0\},$$

by the previous remark and Proposition 8.2. ☐

If $\mathcal{C}$, $\mathcal{C}'$ are two-sided cells of $W'_{Ł} = W'_{Ł^{-1}}$, write

$$\mathbf{D}^{G[13]\times G[24]}_{Ł\boxtimes Ł^{-1}\boxtimes Ł^{-1}\boxtimes Ł}(\tilde{\mathcal{B}} \times \tilde{\mathcal{B}} \times \tilde{\mathcal{B}} \times \tilde{\mathcal{B}})^{\leqslant \mathcal{C}\times\mathcal{C}'}$$

for the subcategory of $\mathbf{D}^{G[13]\times G[24]}_{Ł\boxtimes Ł^{-1}\boxtimes Ł^{-1}\boxtimes Ł}(\tilde{\mathcal{B}} \times \tilde{\mathcal{B}} \times \tilde{\mathcal{B}} \times \tilde{\mathcal{B}})$ consisting of complexes K such that every simple perverse constituent of every ${}^pH^iK$ is isomorphic to $A^{Ł}_{w} \overset{\leftrightarrow}{\boxtimes} A^{Ł^{-1}}_{w'}$ for some $w \leqslant \mathcal{C}$, $w' \leqslant \mathcal{C}'$.

Suppose that we have an $\mathbb{F}_{q^e}$-structure on G as above. Via the identification

$$K_0\mathbf{M}^{G[13]\times G[24]}_{Ł\boxtimes Ł^{-1}\boxtimes Ł^{-1}\boxtimes Ł}(\tilde{\mathcal{B}} \times \tilde{\mathcal{B}} \times \tilde{\mathcal{B}} \times \tilde{\mathcal{B}}) \cong K_0\mathbf{M}^{G}_{Ł\boxtimes Ł^{-1}}(\tilde{\mathcal{B}} \times \tilde{\mathcal{B}}) \otimes K_0\mathbf{M}^{G}_{Ł^{-1}\boxtimes Ł}(\tilde{\mathcal{B}} \times \tilde{\mathcal{B}}),$$

define $\chi_u(K) \in \mathcal{H}'_{Ł} \otimes \mathcal{H}'_{Ł^{-1}}$ for any $K \in \mathbf{D}^{G[13]\times G[24],\mathrm{mixed}}_{Ł\boxtimes Ł^{-1}\boxtimes Ł^{-1}\boxtimes Ł}(\tilde{\mathcal{B}} \times \tilde{\mathcal{B}} \times \tilde{\mathcal{B}} \times \tilde{\mathcal{B}})$. Clearly

$$K \in \mathbf{D}^{G[13]\times G[24],\mathrm{mixed}}_{Ł\boxtimes Ł^{-1}\boxtimes Ł^{-1}\boxtimes Ł}(\tilde{\mathcal{B}} \times \tilde{\mathcal{B}} \times \tilde{\mathcal{B}} \times \tilde{\mathcal{B}})^{\leqslant \mathcal{C}\times\mathcal{C}'} \Leftrightarrow \chi_u(K) \in (\mathcal{H}'_{Ł})^{\leqslant \mathcal{C}} \otimes (\mathcal{H}'_{Ł^{-1}})^{\leqslant \mathcal{C}'}.$$

Proposition 8.7. *If* $x = \chi_u(\beta_!\alpha^*(A_1^{Ł} \boxtimes A_1^{Ł^{-1}})) \in \mathcal{H}'_{Ł} \otimes \mathcal{H}'_{Ł^{-1}}$, *then*

$$\begin{aligned}\chi_u(\beta_!\alpha^*(K \boxtimes K')) &= (\chi_u(K) \otimes 1)x(1 \otimes \chi_u(K')) \\ &= (1 \otimes \chi_u(K)^t)x(\chi_u(K')^t \otimes 1)\end{aligned}$$

for all $K \in \mathbf{D}^{G,\mathrm{mixed}}_{Ł\boxtimes Ł^{-1}}(\tilde{\mathcal{B}} \times \tilde{\mathcal{B}})$, $K' \in \mathbf{D}^{G,\mathrm{mixed}}_{Ł^{-1}\boxtimes Ł}(\tilde{\mathcal{B}} \times \tilde{\mathcal{B}})$.

Proof. Define a functor

$$\begin{aligned}\Phi : \mathbf{D}_{Ł\boxtimes Ł^{-1}}(\tilde{\mathcal{B}} \times \tilde{\mathcal{B}}) \times \mathbf{D}_{Ł\boxtimes Ł^{-1}\boxtimes Ł'\boxtimes(Ł')^{-1}}(\tilde{\mathcal{B}} \times \tilde{\mathcal{B}} \times \tilde{\mathcal{B}} \times \tilde{\mathcal{B}}) \\ \to \mathbf{D}_{Ł\boxtimes Ł^{-1}\boxtimes Ł'\boxtimes(Ł')^{-1}}(\tilde{\mathcal{B}} \times \tilde{\mathcal{B}} \times \tilde{\mathcal{B}} \times \tilde{\mathcal{B}})\end{aligned}$$

in the same way as convolution, using only the first two factors of $\tilde{\mathcal{B}} \times \tilde{\mathcal{B}} \times \tilde{\mathcal{B}} \times \tilde{\mathcal{B}}$. In the case $Ł' = Ł^{-1}$, it is easy to see that this lifts uniquely to a functor

$$\begin{aligned}\Phi : \mathbf{D}^{G,\mathrm{mixed}}_{Ł\boxtimes Ł^{-1}}(\tilde{\mathcal{B}} \times \tilde{\mathcal{B}}) \times \mathbf{D}^{G[13]\times G[24],\mathrm{mixed}}_{Ł\boxtimes Ł^{-1}\boxtimes Ł^{-1}\boxtimes Ł}(\tilde{\mathcal{B}} \times \tilde{\mathcal{B}} \times \tilde{\mathcal{B}} \times \tilde{\mathcal{B}}) \\ \to \mathbf{D}^{G[13]\times G[24],\mathrm{mixed}}_{Ł\boxtimes Ł^{-1}\boxtimes Ł^{-1}\boxtimes Ł}(\tilde{\mathcal{B}} \times \tilde{\mathcal{B}} \times \tilde{\mathcal{B}} \times \tilde{\mathcal{B}}),\end{aligned}$$

and by Theorem 8.4,

$$\chi_u(\Phi(K, K')) = (\chi_u(K) \otimes 1)\chi_u(K'),$$

for any $K \in \mathbf{D}^{G,\mathrm{mixed}}_{Ł\boxtimes Ł^{-1}}(\tilde{\mathcal{B}}\times\tilde{\mathcal{B}})$ and $K' \in \mathbf{D}^{G[13]\times G[24],\mathrm{mixed}}_{Ł\boxtimes Ł^{-1}\boxtimes Ł^{-1}\boxtimes Ł}(\tilde{\mathcal{B}}\times\tilde{\mathcal{B}}\times\tilde{\mathcal{B}}\times\tilde{\mathcal{B}})$. Easy base-change arguments show that for $K_1, K_2 \in \mathbf{D}^{G,\mathrm{mixed}}_{Ł\boxtimes Ł^{-1}}(\tilde{\mathcal{B}} \times \tilde{\mathcal{B}})$, $K_3 \in \mathbf{D}^{G,\mathrm{mixed}}_{Ł^{-1}\boxtimes Ł}(\tilde{\mathcal{B}} \times \tilde{\mathcal{B}})$,

$$\beta_!\alpha^*((K_1 \star K_2) \boxtimes K_3) \cong \Phi(K_1, \beta_!\alpha^*(K_2 \boxtimes K_3)).$$

Applying χ_u, we get that

$$\chi_u(\beta_!\alpha^*((K_1 \star K_2) \boxtimes K_3)) = (\chi_u(K_1) \otimes 1)\chi_u(\beta_!\alpha^*(K_2 \boxtimes K_3)).$$

An exactly analogous argument shows that

$$\chi_u(\beta_!\alpha^*(K_2 \boxtimes (K_3 \star K_4)) = \chi_u(\beta_!\alpha^*(K_2 \boxtimes K_3))(1 \otimes \chi_u(K_4)).$$

Combining these formulas, we get

$$\begin{aligned}&\chi_u(\beta_!\alpha^*((K_1 \star K_2) \boxtimes (K_3 \star K_4))) \\ &\quad = (\chi_u(K_1) \otimes 1)\chi_u(\beta_!\alpha^*(K_2 \boxtimes K_3))(1 \otimes \chi_u(K_4)).\end{aligned}$$

Setting $K_2 = A_1^{Ł}$, $K_3 = A_1^{Ł^{-1}}$, we get the first equality in the proposition. The second must also hold by symmetry (consider reversing the order of all factors $\tilde{\mathcal{B}}$, and using the second statement of Theorem 8.4). □

Corollary 8.8. *If* $K \in \mathbf{D}^{G,\mathrm{mixed}}_{Ł\boxtimes Ł^{-1}}(\tilde{\mathcal{B}} \times \tilde{\mathcal{B}})^{\leqslant \mathcal{C}}$, $K' \in \mathbf{D}^{G,\mathrm{mixed}}_{Ł^{-1}\boxtimes Ł}(\tilde{\mathcal{B}} \times \tilde{\mathcal{B}})^{\leqslant \mathcal{C}'}$, *then*

$$\beta_!\alpha^*(K \boxtimes K') \in \mathbf{D}^{G[13]\times G[24],\mathrm{mixed}}_{Ł\boxtimes Ł^{-1}\boxtimes Ł^{-1}\boxtimes Ł}(\tilde{\mathcal{B}} \times \tilde{\mathcal{B}} \times \tilde{\mathcal{B}} \times \tilde{\mathcal{B}})^{\leqslant \mathcal{C}\times\mathcal{C}, \mathcal{C}'\times\mathcal{C}'}.$$

Proof. Thanks to Proposition 8.7, this follows from

$$\begin{aligned}&((\mathcal{H}'_{Ł})^{\leqslant \mathcal{C}} \otimes 1)(\mathcal{H}'_{Ł} \otimes \mathcal{H}'_{Ł^{-1}})(1 \otimes (\mathcal{H}'_{Ł^{-1}})^{\leqslant \mathcal{C}'}) \\ &\qquad \cap (1 \otimes (\mathcal{H}'_{Ł^{-1}})^{\leqslant \mathcal{C}})(\mathcal{H}'_{Ł} \otimes \mathcal{H}'_{Ł^{-1}})((\mathcal{H}'_{Ł})^{\leqslant \mathcal{C}'} \otimes 1) \\ &\qquad\qquad \subseteq (\mathcal{H}'_{Ł})^{\leqslant \mathcal{C},\mathcal{C}'} \otimes (\mathcal{H}'_{Ł^{-1}})^{\leqslant \mathcal{C},\mathcal{C}'}.\end{aligned}$$

□

Now we must relate all this to character sheaves. Consider the diagram

$$\tilde{\mathcal{B}} \times \tilde{\mathcal{B}} \xleftarrow{r} G \times \tilde{\mathcal{B}} \xrightarrow{\mathrm{id}\times q} G \times \mathcal{B} \xrightarrow{p} G$$

where $p : G \times \mathcal{B} \to G$ is the first projection and $r(g, g'U) = (gg'U, g'U)$. Following Ginzburg (see [5]), define a functor $\mathrm{Ch} : \mathbf{D}_{\L\boxtimes\L^{-1}}(\tilde{\mathcal{B}} \times \tilde{\mathcal{B}}) \to \mathbf{D}(G)$ by

$$\mathrm{Ch}(K) = p_!(\mathrm{id} \times q)_\flat r^* K[\dim U - \dim T].$$

(This shift ensures that Ch commutes with Verdier duality, since p is proper.) Since it is defined by a G-equivariant diagram, Ch lifts uniquely to a functor

$$\mathrm{Ch} : \mathbf{D}^G_{\L\boxtimes\L^{-1}}(\tilde{\mathcal{B}} \times \tilde{\mathcal{B}}) \to \mathbf{D}^G(G).$$

We also need the functor $\mathrm{Ch}\boxtimes\mathrm{Ch} : \mathbf{D}_{\L\boxtimes\L^{-1}\boxtimes\L'\boxtimes(\L')^{-1}}(\tilde{\mathcal{B}}\times\tilde{\mathcal{B}}\times\tilde{\mathcal{B}}\times\tilde{\mathcal{B}}) \to \mathbf{D}(G\times G)$ defined in the obvious way. This lifts uniquely to functors

$$\mathrm{Ch}\boxtimes\mathrm{Ch} : \mathbf{D}^{G[12]\times G[34]}_{\L\boxtimes\L^{-1}\boxtimes\L'\boxtimes(\L')^{-1}}(\tilde{\mathcal{B}}\times\tilde{\mathcal{B}}\times\tilde{\mathcal{B}}\times\tilde{\mathcal{B}}) \to \mathbf{D}^{G[1]\times G[2]}(G\times G),$$
$$\mathrm{Ch}\boxtimes\mathrm{Ch} : \mathbf{D}^{G[13]\times G[24]}_{\L\boxtimes\L^{-1}\boxtimes\L'\boxtimes(\L')^{-1}}(\tilde{\mathcal{B}}\times\tilde{\mathcal{B}}\times\tilde{\mathcal{B}}\times\tilde{\mathcal{B}}) \to \mathbf{D}^{G[12]\times G[12]}(G\times G),$$

where $G[1] \times G[2]$ denotes $G \times G$ acting on itself by

$$(g_1, g_2).(g_1', g_2') = (g_1 g_1' g_1^{-1}, g_2 g_2' g_2^{-1}),$$

and $G[12] \times G[12]$ denotes $G \times G$ acting on itself by

$$(g_1, g_2).(g_1', g_2') = (g_1 g_1' g_2^{-1}, g_1 g_2' g_2^{-1}).$$

Proposition 8.9. *For a G-equivariant simple perverse sheaf A on G, the following conditions are equivalent (in (4) and (7) we have an $\mathbb{F}_{q^e}$-structure on G as above):*

1. *$A \in \widehat{G}^{\leqslant\mathcal{C}}_{\L}$.*
2. *A is a constituent of some ${}^pH^i\mathrm{Ch}(A^{\L}_w)$ for $w \leqslant \mathcal{C}$.*
3. *A is a constituent of some ${}^pH^i\mathrm{Ch}(K)$ for some $K \in \mathbf{D}^G_{\L\boxtimes\L^{-1}}(\tilde{\mathcal{B}} \times \tilde{\mathcal{B}})^{\leqslant\mathcal{C}}$.*
4. *A is a constituent of some ${}^pH^i\mathrm{Ch}(K)$ for some $K \in \mathbf{D}^{G,\mathrm{mixed}}_{\L\boxtimes\L^{-1}}(\tilde{\mathcal{B}} \times \tilde{\mathcal{B}})^{\leqslant\mathcal{C}}$.*
5. *$\tilde{m}^*A[\dim G]$ is a constituent of some ${}^pH^i(\mathrm{Ch}\boxtimes\mathrm{Ch})(A^{\L^{-1}}_w \overset{\leftrightarrow}{\boxtimes} A^{\L}_{w'})$ for $w, w' \leqslant \mathcal{C}$.*
6. *$\tilde{m}^*A[\dim G]$ is a constituent of some ${}^pH^i(\mathrm{Ch}\boxtimes\mathrm{Ch})(K)$ for some*
$$K \in \mathbf{D}^{G[13]\times G[24]}_{\L^{-1}\boxtimes\L\boxtimes\L\boxtimes\L^{-1}}(\tilde{\mathcal{B}}\times\tilde{\mathcal{B}}\times\tilde{\mathcal{B}}\times\tilde{\mathcal{B}})^{\leqslant\mathcal{C}\times\mathcal{C}}.$$
7. *$\tilde{m}^*A[\dim G]$ is a constituent of some ${}^pH^i(\mathrm{Ch}\boxtimes\mathrm{Ch})(K)$ for some*
$$K \in \mathbf{D}^{G[13]\times G[24],\mathrm{mixed}}_{\L^{-1}\boxtimes\L\boxtimes\L\boxtimes\L^{-1}}(\tilde{\mathcal{B}}\times\tilde{\mathcal{B}}\times\tilde{\mathcal{B}}\times\tilde{\mathcal{B}})^{\leqslant\mathcal{C}\times\mathcal{C}}.$$

Proof. The equivalences (2) $\Leftrightarrow$ (3), (2) $\Leftrightarrow$ (4), (5) $\Leftrightarrow$ (6), and (5) $\Leftrightarrow$ (7) follow by general principles. For $w \in W'_{\L}$, it is easy to see that (up to shift) $\mathrm{Ch}(A^{\L}_w) \cong \overline{K}^{\L}_w$ in Lusztig's notation. So without the information on cells, (1) $\Leftrightarrow$ (2) is simply [13, Proposition 12.7]. The version involving cells is easy to deduce from the results of [13, §16]. In view of Corollary 8.5, (2) $\Leftrightarrow$ (5) follows from the formula

$$(\mathrm{Ch}\boxtimes\mathrm{Ch})(K \overset{\leftrightarrow}{\boxtimes} K') \cong \tilde{m}^*\mathrm{Ch}(i^*K \star K') \otimes H^\bullet_c(T, \overline{\mathbb{Q}_l})[\dim G]$$

in $\mathbf{D}^{G[12]\times G[12]}(G\times G)$, for $K\in\mathbf{D}^G_{\text{Ł}\boxtimes\text{Ł}^{-1}}(\tilde{\mathcal{B}}\times\tilde{\mathcal{B}})$, $K'\in\mathbf{D}^G_{\text{Ł}^{-1}\boxtimes\text{Ł}}(\tilde{\mathcal{B}}\times\tilde{\mathcal{B}})$. To prove this formula, factor the map $r\times r$ used in defining $\mathrm{Ch}\boxtimes\mathrm{Ch}$ as $\alpha\gamma$, where

$$\gamma : G\times\tilde{\mathcal{B}}\times G\times\tilde{\mathcal{B}}\to G\times\tilde{\mathcal{B}}\times\tilde{\mathcal{B}}\times\tilde{\mathcal{B}}\times\tilde{\mathcal{B}}$$
$$: (g_1, g_1'U, g_2, g_2'U)\mapsto(g_1, g_1'U, g_1'U, g_1^{-1}g_2g_2'U, g_2'U)$$

and α is as above. Since K is G-equivariant,

$$\alpha^*(K\overset{\leftrightarrow}{\boxtimes}K')\cong\beta^*(K\overset{\leftrightarrow}{\boxtimes}K')$$

where β is as above. So

$$(\mathrm{Ch}\boxtimes\mathrm{Ch})(K\overset{\leftrightarrow}{\boxtimes}K')\cong(p\times p)_!(\mathrm{id}\times q\times\mathrm{id}\times q)_\flat\gamma^*\beta^*(K\overset{\leftrightarrow}{\boxtimes}K')[2\dim U-2\dim T].$$

Now straightforward base-change arguments show that

$$(p\times p)_!(\mathrm{id}\times q\times\mathrm{id}\times q)_\flat\gamma^*\beta^*\cong\tilde{m}^*\tilde{p}_!(\mathrm{id}\times q\times q)_\flat(\delta\tilde{r})^*,$$

where in the diagram

$$\tilde{\mathcal{B}}\times\tilde{\mathcal{B}}\times\tilde{\mathcal{B}}\times\tilde{\mathcal{B}}\xleftarrow{\delta}\tilde{\mathcal{B}}\times\tilde{\mathcal{B}}\times\tilde{\mathcal{B}}\xleftarrow{\tilde{r}}G\times\tilde{\mathcal{B}}\times\tilde{\mathcal{B}}\xrightarrow{\mathrm{id}\times q\times q}G\times\mathcal{B}\times\mathcal{B}\xrightarrow{\tilde{p}}G$$

we have $\tilde{p}$ is the first projection, $\tilde{r}(g, g_1'U, g_2'U)=(g_1'U, gg_2'U, g_2'U)$, and δ is the map which repeats the first factor. Again, it is easy to see that

$$\tilde{p}_!(\mathrm{id}\times q\times q)_\flat\tilde{r}^*\cong p_!(\mathrm{id}\times q)_\flat r^*(p_{23})_!,$$

so it suffices to prove that

$$(p_{23})_!\delta^*(K\overset{\leftrightarrow}{\boxtimes}K')\cong(i^*K\star K')\otimes H_c^\bullet(T,\overline{\mathbb{Q}_l})[\dim\tilde{\mathcal{B}}+\dim T].$$

But from the definition of convolution,

$$(i^*K\star K')[\dim\tilde{\mathcal{B}}+\dim T]\cong(\widetilde{p_{23}})_!(q\times\mathrm{id}\times\mathrm{id})_\flat\delta^*(K\overset{\leftrightarrow}{\boxtimes}K').$$

It only remains to note that $q_!\overline{\mathbb{Q}_l}\cong H_c^\bullet(T,\overline{\mathbb{Q}_l})$, because of the Cartesian square

$$\begin{array}{ccc} G\times T & \xrightarrow{\tilde{q}} & G \\ \downarrow & & \downarrow \\ \tilde{\mathcal{B}} & \xrightarrow{q} & \mathcal{B} \end{array}$$

where $\tilde{q}$ is the projection, and the vertical maps are $(g,t)\mapsto gt^{-1}U$ and $g\mapsto gB$. □

Proposition 8.10. *There is an isomorphism of functors*

$$\tilde{m}^*\tilde{m}_!(\mathrm{Ch}\boxtimes\mathrm{Ch})\cong(\mathrm{Ch}\boxtimes\mathrm{Ch})\beta_!\alpha^* :$$
$$\mathbf{D}^{G[12]\times G[34]}_{Ł\boxtimes Ł^{-1}\boxtimes Ł'\boxtimes(Ł')^{-1}}(\tilde{\mathcal{B}}\times\tilde{\mathcal{B}}\times\tilde{\mathcal{B}}\times\tilde{\mathcal{B}})\to\mathbf{D}^{G[12]\times G[12]}(G\times G).$$

Proof. The first step is to consider the Cartesian square

$$\begin{array}{ccc} G\times G\times G & \xrightarrow{\hat{\alpha}} & G\times G \\ {\scriptstyle\hat{\beta}}\downarrow & & {\scriptstyle\tilde{m}}\downarrow \\ G\times G & \xrightarrow{\tilde{m}} & G \end{array}$$

where $\hat{\alpha}(g, g_1, g_2) = (gg_1, gg_2)$ and $\hat{\beta}(g, g_1, g_2) = (g_1, g_2)$. By base-change, we have an isomorphism of functors

$$\tilde{m}^* \tilde{m}_! \cong \hat{\beta}_! \hat{\alpha}^* : \mathbf{D}^{G[1]\times G[2]}(G \times G) \to \mathbf{D}^{G[12]\times G[12]}(G \times G).$$

The proof that $\hat{\beta}_! \hat{\alpha}^*(\mathrm{Ch} \boxtimes \mathrm{Ch}) \cong (\mathrm{Ch} \boxtimes \mathrm{Ch})\beta_! \alpha^*$ is then entirely routine. □

We can now prove Theorem 8.1. First suppose we are in the situation of (1). By (1)⇒(3) in Proposition 8.9, we can find some $K \in \mathbf{D}^G_{Ł\boxtimes Ł^{-1}}(\tilde{\mathcal{B}} \times \tilde{\mathcal{B}})$ and some $K' \in \mathbf{D}^G_{Ł'\boxtimes(Ł')^{-1}}(\tilde{\mathcal{B}} \times \tilde{\mathcal{B}})$ such that A is a simple constituent of some ${}^pH^i\mathrm{Ch}(K)$, and A' is a simple constituent of some ${}^pH^i\mathrm{Ch}(K')$. It then suffices to prove that $\tilde{m}_!(\mathrm{Ch}(K) \boxtimes \mathrm{Ch}(K')) \cong 0$. But by Proposition 8.10,

$$\begin{aligned}\tilde{m}^*\tilde{m}_!(\mathrm{Ch}(K) \boxtimes \mathrm{Ch}(K')) &\cong (\mathrm{Ch} \boxtimes \mathrm{Ch})(\beta_! \alpha^*(K \boxtimes K')) \\ &\cong 0, \text{ by Lemma 8.6.}\end{aligned}$$

Now suppose we are in the situation of (2) of Theorem 8.1. By (1)⇒(4) in Proposition 8.9, we can choose $K \in \mathbf{D}^{G,\mathrm{mixed}}_{Ł\boxtimes Ł^{-1}}(\tilde{\mathcal{B}} \times \tilde{\mathcal{B}})^{\leqslant \mathcal{C}}$ and $K' \in \mathbf{D}^{G,\mathrm{mixed}}_{Ł^{-1}\boxtimes Ł}(\tilde{\mathcal{B}} \times \tilde{\mathcal{B}})^{\leqslant \mathcal{C}'}$ such that A is a simple constituent of some ${}^pH^i\mathrm{Ch}(K)$, and A' is a simple constituent of some ${}^pH^i\mathrm{Ch}(K')$. It then suffices to prove that

$$\tilde{m}_!(\mathrm{Ch}(K) \boxtimes \mathrm{Ch}(K')) \in \mathbf{D}^{G,\mathrm{mixed}}(G)^{\leqslant \mathcal{C},\mathcal{C}'}_{Ł^{-1}}.$$

Now by Proposition 8.10 (or more precisely its mixed analogue),

$$\tilde{m}^*\tilde{m}_!(\mathrm{Ch}(K) \boxtimes \mathrm{Ch}(K')) \cong (\mathrm{Ch} \boxtimes \mathrm{Ch})(\beta_! \alpha^*(K \boxtimes K'))$$

in $\mathbf{D}^{G[12]\times G[12],\mathrm{mixed}}(G \times G)$, and

$$\beta_! \alpha^*(K \boxtimes K') \in \mathbf{D}^{G[13]\times G[24],\mathrm{mixed}}_{Ł\boxtimes Ł^{-1}\boxtimes Ł^{-1}\boxtimes Ł}(\tilde{\mathcal{B}} \times \tilde{\mathcal{B}} \times \tilde{\mathcal{B}} \times \tilde{\mathcal{B}})^{\leqslant \mathcal{C}\times\mathcal{C},\mathcal{C}'\times\mathcal{C}'}$$

by Corollary 8.8. Using (7)⇒(1) in Proposition 8.9, we are done.

9. Proof of the Character Formula

This section imitates the "principal series" case of [13, §8]. We first describe an alternative construction of $K_{(T,Ł)}$. Let B be any Borel subgroup of G containing T, and let $\pi_B : B \to T$ be the canonical projection. Consider the diagram

$$T \xleftarrow{\hat{\alpha}} \{(g, xB) \in G \times G/B | x^{-1}gx \in B\} \xrightarrow{\psi} G$$

where ψ is the first projection and $\hat{\alpha}(g, xB) = \pi_B(x^{-1}gx)$. We let

$$\mathrm{Ind}^G_{T\subset B}(Ł[\dim T]) = \psi_! \hat{\alpha}^* Ł[\dim G] \in \mathbf{D}^G(G).$$

Theorem 9.1. 1. *For every B, we have a canonical isomorphism*

$$\mathrm{Ind}^G_{T\subset B}(Ł[\dim T]) \xrightarrow{\sim} K_{(T,Ł)}.$$

2. *The natural isomorphism $\tilde{F}^*\mathrm{Ind}^G_{T\subset FB}(Ł[\dim T]) \xrightarrow{\sim} \mathrm{Ind}^G_{T\subset B}(Ł[\dim T])$ induced by $\phi^{\tilde{F}}_{Ł} : \tilde{F}^*Ł \xrightarrow{\sim} Ł$ becomes $\phi^{\tilde{F}}_{K_{(T,Ł)}} : \tilde{F}^*K_{(T,Ł)} \xrightarrow{\sim} K_{(T,Ł)}$ under the identifications of* (1).

Proof. Part (1) follows from the fact that the map ψ is small and proper, and (2) is obvious from the construction. See [13, 8.2] for more details. □

Corollary 9.2. *For $u \in G^{\tilde{F}}_{\mathrm{uni}}$, $\chi^{\tilde{F}}_{K_{(T,Ł)}}(u) = (-1)^{\dim G}\tilde{Q}^G_T(u)$.*

Proof. As in the proof of [13, (8.3.2)], we deduce from (2) of Theorem 9.1 that the left-hand side does not depend on Ł. In the case $\text{Ł} = \overline{\mathbb{Q}_l}$, (2) of Theorem 9.1 also implies that for all $g \in G^F$,

$$\chi^{\tilde{F}}_{K_{(T,\overline{\mathbb{Q}_l})}}(g) = \sum_i (-1)^{i+\dim G}\mathrm{tr}(F^* \circ w^{-1}, H^i(\mathcal{B}_g, \overline{\mathbb{Q}_l})),$$

where w is as in the definition of the Green function. So we are done. □

We can now prove the Character Formula, Theorem 4.9.

Proof. Let $\mathcal{U}$ be an $\tilde{F}$-stable open subvariety of $Z^\circ_G(s)$ containing u such that for all Borel subgroups B containing T,

$$g \in s\mathcal{U}, x \in G, x^{-1}gx \in B \Rightarrow x^{-1}sx \in B,$$

and hence

$$g \in s\mathcal{U}, x \in G, x^{-1}gx \in T \Rightarrow x^{-1}sx \in T.$$

(Such a $\mathcal{U}$ certainly exists: take the $\mathcal{U}$ defined in [13] and intersect with $\mathcal{U}^{-1}$.) Now fix a Borel subgroup B containing T. Let $Q = \{x \in G | x^{-1}sx \in T\}$. Then Q/T is a disjoint union of $Z^\circ_G(s)$-orbits $\varnothing$. Moreover $QB = \{x \in G | x^{-1}sx \in B\}$, and QB/B is a disjoint union of $Z^\circ_G(s)$-orbits

$$\hat{\varnothing} = \{xB \in G/B | xT \in \varnothing\}.$$

Thus by definition of $\mathcal{U}$, the variety

$$\{(g, xB) \in s\mathcal{U} \times G/B | x^{-1}gx \in B\}$$

is a disjoint union of its subvarieties

$$Y_{\mathcal{U},\hat{\varnothing}} = \{(g, xB) \in s\mathcal{U} \times \hat{\varnothing} | x^{-1}gx \in B\}$$

where $\varnothing$ runs over the above set of orbits. Consequently

$$\mathrm{Ind}^G_{T\subset B}(\text{Ł}[\dim T])|_{s\mathcal{U}} \cong \bigoplus_{\varnothing} (\psi_{\mathcal{U},\hat{\varnothing}})_!(\hat{\alpha}_{\mathcal{U},\hat{\varnothing}})^*\text{Ł}[\dim G],$$

where $\psi_{\mathcal{U},\hat{\varnothing}}$ and $\hat{\alpha}_{\mathcal{U},\hat{\varnothing}}$ are the restrictions of ψ and $\hat{\alpha}$ to $Y_{\mathcal{U},\hat{\varnothing}}$. Now choose for each $\varnothing$ a representative $x_\varnothing \in G$ such that $x_{F\varnothing} = F(x_\varnothing)$. Let $B_\varnothing = x_\varnothing B x_\varnothing^{-1} \cap Z^\circ_G(s)$, a Borel subgroup of $Z^\circ_G(s)$ containing the maximal torus $T_\varnothing = x_\varnothing T x_\varnothing^{-1}$. Then we have a commutative diagram:

$$\begin{array}{ccccc}
T & \xleftarrow{\hat{\alpha}_{\mathcal{U},\hat{\varnothing}}} & \{(g, xB) \in s\mathcal{U} \times \hat{\varnothing} | x^{-1}gx \in B\} & \xrightarrow{\psi_{\mathcal{U},\hat{\varnothing}}} & s\mathcal{U} \\
{\scriptstyle\gamma}\Big\uparrow & & {\scriptstyle\delta}\Big\uparrow & & {\scriptstyle\epsilon}\Big\uparrow \\
T_\varnothing & \xleftarrow{\hat{\alpha}'_{\mathcal{U},\varnothing}} & \{(g, xB_\varnothing) \in \mathcal{U} \times Z^\circ_G(s)/B_\varnothing | x^{-1}gx \in B_\varnothing\} & \xrightarrow{\psi'_{\mathcal{U},\varnothing}} & \mathcal{U}
\end{array}$$

where $\hat{\alpha}'_{\mathcal{U},\varnothing}$ and $\psi'_{\mathcal{U},\varnothing}$ are defined analogously to $\hat{\alpha}$ and ψ but for the group $Z^\circ_G(s)$ instead of G, $\gamma(t) = x_\varnothing^{-1} st x_\varnothing$, $\delta(g, xB_\varnothing) = (sg, xx_\varnothing B)$, and $\epsilon(g) = sg$. It is easy to see that δ is an isomorphism, and ϵ is obviously an isomorphism. So if $\text{Ł}_\varnothing$ denotes the local system $\gamma^*\text{Ł}$ on $T_\varnothing$, we have

$$\epsilon^*(\mathrm{Ind}^G_{T\subset B}(\text{Ł}[\dim T])|_{s\mathcal{U}})[-(\dim G - \dim Z^\circ_G(s))] \cong \bigoplus_{\varnothing} \mathrm{Ind}^{Z^\circ_G(s)}_{T_\varnothing \subset B_\varnothing}(\text{Ł}_\varnothing[\dim T_\varnothing])|_{\mathcal{U}}.$$

By Theorem 9.1, this is equivalent to

$$\epsilon^*(K_{(T,\text{Ł})}|_{s\mathcal{U}})[-(\dim G - \dim Z_G^\circ(s))] \cong \bigoplus_{\varnothing} K_{(T_\varnothing,\text{Ł}_\varnothing)}|_{\mathcal{U}}$$

where $K_{(T_\varnothing,\text{Ł}_\varnothing)}$ means the perverse sheaf on $Z_G^\circ(s)$. The crucial question is why this isomorphism respects the $\mathbb{F}_q$-structures. The reason is that this isomorphism is the result of applying the IC functor to an isomorphism

$$\epsilon^*(K_{(T,\text{Ł})}|_{s\mathcal{U}\cap G_{\text{rss}}})[-(\dim G - \dim Z_G^\circ(s))] \cong \bigoplus_{\varnothing} K_{(T_\varnothing,\text{Ł}_\varnothing)}|_{\mathcal{U}\cap s^{-1}G_{\text{rss}}}$$

obtained by a similar procedure to the above, using the first construction of $K_{(T,\text{Ł})}$, which results from an $\tilde{F}$-stable diagram. (Note that the required action of $\tilde{F}$ on $\{\varnothing\}$ is simply that of F.)

Once we know that this isomorphism respects $\mathbb{F}_q$-structures, we consider the stalk at $u \in \mathcal{U}$ and find:

$$\begin{aligned}
\chi^{\tilde{F}}_{K_{(T,\text{Ł})}}(su) &= \sum_{\substack{\varnothing \\ F\varnothing=\varnothing}} \chi^{\tilde{F}}_{K_{(T_\varnothing,\text{Ł}_\varnothing)}}(u) \\
&= \sum_{\substack{\varnothing \\ F\varnothing=\varnothing}} (-1)^{\dim Z_G^\circ(s)} \chi^{\tilde{F}}_{\text{Ł}_\varnothing}(1) \tilde{Q}^{Z_G^\circ(s)}_{T_\varnothing}(u) \\
&= \sum_{\substack{\varnothing \\ F\varnothing=\varnothing}} (-1)^{\dim T} \chi^{\tilde{F}}_{\text{Ł}}(x_\varnothing s x_\varnothing^{-1}) \tilde{Q}^{Z_G^\circ(s)}_{T_\varnothing}(u) \\
&= \frac{(-1)^{\dim T}}{|Z_G^\circ(s)^F|} \sum_{\substack{x\in G^F \\ x^{-1}sx\in T}} \chi^{\tilde{F}}_{\text{Ł}}(x^{-1}sx) \tilde{Q}^{Z_G^\circ(s)}_{xTx^{-1}}(u),
\end{aligned}$$

as required. □

10. Proof of the Inner Product Formula

This section imitates the "principal series" case of [13, §9].

Lemma 10.1. *If $(T, Ł)$ is $\tilde{F}$-stable, then so is $(T, Ł^{-1})$, and* $\chi^{\tilde{F}}_{K_{(T,Ł^{-1})}} = \overline{\chi^{\tilde{F}}_{K_{(T,Ł)}}}$.

Proof. This follows easily from the fact that $\chi^{\tilde{F}}_{\text{Ł}^{-1}}(t) = \chi^{\tilde{F}}_{\text{Ł}}(t^{-1}) = \overline{\chi^{\tilde{F}}_{\text{Ł}}(t)}$. □

We first prove the "geometric" part of Theorem 4.10.

Theorem 10.2. *If $(T, Ł)$, $(T', Ł')$ are not G-conjugate,* $\langle \chi^{\tilde{F}}_{K_{(T,Ł)}}, \chi^{\tilde{F}}_{K_{(T',Ł')}} \rangle = 0$.

Proof. By Lemma 10.1 and the Grothendieck Trace Formula, it is enough to show

$$H^i_c(G, K_{(T,\text{Ł})} \otimes K_{(T',(\text{Ł}')^{-1})}) = 0,\ \forall i.$$

By Theorem 9.1, this is equivalent to

$$H^i_c(G, \operatorname{Ind}^G_{T\subset B}(\text{Ł}[\dim T]) \otimes \operatorname{Ind}^G_{T'\subset B'}((\text{Ł}')^{-1}[\dim T'])) = 0,\ \forall i.$$

From the definition of $\operatorname{Ind}^G_{T\subset B}$ it is easy to see that this will follow from

$$H^i_c(Y, \hat{\alpha}^*\text{Ł} \boxtimes (\hat{\alpha}')^*(\text{Ł}')^{-1}) = 0,\ \forall i,$$

where

$$Y = \{(g, xB, x'B') \in G \times G/B \times G/B' \mid x^{-1}gx \in B, (x')^{-1}gx' \in B'\}.$$

Now there are only finitely many G-orbits $\emptyset$ on $G/B \times G/B'$, so it suffices to prove this vanishing for the preimage of each. But for each $\emptyset$, the map

$$\{(g, xB, x'B') \in Y \mid (xB, x'B') \in \emptyset\} \to \emptyset$$

is a fibre bundle, so by the Leray spectral sequence it suffices to prove the vanishing for one fibre of it. We consider the fibre over $(B, x'B')$ where $x'T'(x')^{-1} = T$. That is, we must prove that for any such x',

$$H_c^i(B \cap x'B'(x')^{-1}, \pi_B^*(Ł \otimes \mathrm{Ad}((x')^{-1})^*(Ł')^{-1})) = 0, \ \forall i.$$

Now $\pi_B : B \cap x'B'(x')^{-1} \to T$ is an affine space bundle, so this will follow from

$$H_c^i(T, Ł \otimes \mathrm{Ad}((x')^{-1})^*(Ł')^{-1}) = 0, \ \forall i.$$

But since $(T, Ł)$ and $(T', Ł')$ are not G-conjugate, $Ł \otimes \mathrm{Ad}((x')^{-1})^*(Ł')^{-1}$ is non-trivial, and it is well known that all cohomology groups of a torus with coefficients in a nontrivial local system vanish. ☐

Now take $Ł' = \overline{\mathbb{Q}_l}$, and assume that $Ł|_{Z(G)} \cong \overline{\mathbb{Q}_l}$ but $Ł$ is non-trivial, which can always be arranged unless G is a torus. Applying Theorem 10.2 and using the Character Formula, we find that the following expression vanishes:

$$(10.1) \quad \sum_{s \in G_{\mathrm{ss}}^{\tilde{F}}} \frac{1}{|Z_G^\circ(s)^F|^2} \sum_{\substack{x, x' \in G^F \\ x^{-1}sx \in T \\ (x')^{-1}sx' \in T'}} \chi_Ł^{\tilde{F}}(x^{-1}sx) \sum_{u \in Z_G^\circ(s)_{\mathrm{uni}}^{\tilde{F}}} \tilde{Q}_{xTx^{-1}}^{Z_G^\circ(s)}(u) \tilde{Q}_{x'T'(x')^{-1}}^{Z_G^\circ(s)}(u).$$

We can now prove the analogue of [13, Theorem 9.3]:

Theorem 10.3. *For any $\tilde{F}$-stable T, T',*

$$\frac{1}{|G^F|} \sum_{u \in G_{\mathrm{uni}}^{\tilde{F}}} \tilde{Q}_T^G(u) \tilde{Q}_{T'}^G(u) = \frac{|N_G(T, T')^F|}{|T^F||(T')^F|}.$$

Proof. If G is a torus this theorem is trivial. By induction on dim G, we may assume the Theorem known when G is replaced by $Z_G^\circ(s)$ for any $s \in G_{\mathrm{ss}}^{\tilde{F}} - Z(G)^{\tilde{F}}$. Since the terms where $s \in Z(G)^{\tilde{F}}$ contribute $|Z(G)^{\tilde{F}}| \sum_{u \in G_{\mathrm{uni}}^{\tilde{F}}} \tilde{Q}_T^G(u) \tilde{Q}_{T'}^G(u)$ to (10.1), it is enough to prove that (10.1) still vanishes when the left-hand side of the theorem is replaced by the right-hand side throughout. But

$$\begin{aligned}
&\sum_{s\in G_{\mathrm{ss}}^{\tilde F}} \frac{1}{|Z_G^\circ(s)^F|} \sum_{\substack{x,x'\in G^F\\ x^{-1}sx\in T\\ (x')^{-1}sx'\in T'}} \chi_{Ł}^{\tilde F}(x^{-1}sx)\frac{|N_{Z_G^\circ(s)}(xTx^{-1},x'T'(x')^{-1})^F|}{|(xTx^{-1})^F||(x'T'(x')^{-1})^F|}\\
&= \frac{1}{|T^F||(T')^F|} \sum_{\substack{s\in G_{\mathrm{ss}}^{\tilde F}\\ x,x'\in G^F\\ s\in xTx^{-1}\cap x'T'(x')^{-1}\\ n\in Z_G^\circ(s)^F\\ x'T'(x')^{-1}=nxTx^{-1}n^{-1}}} \frac{1}{|Z_G^\circ(s)^F|}\chi_{Ł}^{\tilde F}(x^{-1}sx)\\
&= \frac{|G^F||N_G(T,T')^F|}{|T^F||(T')^F|}\sum_{t\in T^{\tilde F}}\chi_{Ł}^{\tilde F}(t)=0,
\end{aligned}$$

by assumption on Ł. □

The full Inner Product Formula, Theorem 4.10, is deduced as follows:

Proof. Using Lemma 10.1, the Character Formula, and Theorem 10.3,

$$\begin{aligned}
|G^{\tilde F}|\mathrm{LHS} &= \sum_{g\in G^{\tilde F}} \chi^{\tilde F}_{K_{(T,Ł)}}(g)\chi^{\tilde F}_{K_{(T',(Ł')^{-1})}}(g)\\
&= \sum_{\substack{s\in G_{\mathrm{ss}}^{\tilde F}\\ x,x'\in G^F\\ x^{-1}sx\in T\\ (x')^{-1}sx'\in T'}} \frac{1}{|Z_G^\circ(s)^F|}\chi_{Ł}^{\tilde F}(x^{-1}sx)\chi_{Ł'}^{\tilde F}((x')^{-1}s^{-1}x')\frac{|N_{Z_G^\circ(s)}(xTx^{-1},x'T'(x')^{-1})^F|}{|(xTx^{-1})^F||(x'T'(x')^{-1})^F|}\\
&= \frac{1}{|T^F||(T')^F|}\sum_{\substack{s\in G_{\mathrm{ss}}^{\tilde F}\\ x,x'\in G^F\\ s\in xTx^{-1}\cap x'T'(x')^{-1}\\ \tilde n\in Z_G^\circ(s)^F\\ x'T'(x')^{-1}=\tilde n xTx^{-1}\tilde n^{-1}}} \frac{1}{|Z_G^\circ(s)^F|}\chi_{Ł}^{\tilde F}(x^{-1}sx)\chi_{Ł'}^{\tilde F}((x')^{-1}s^{-1}x')\\
&= \frac{1}{|T^F||(T')^F|}\sum_{\substack{s\in G_{\mathrm{ss}}^{\tilde F}\\ x\in G^F\\ s\in xTx^{-1}\\ \tilde n\in Z_G^\circ(s)^F\\ n\in N_G(T,T')^F}} \frac{1}{|Z_G^\circ(s)^F|}\chi_{Ł}^{\tilde F}(x^{-1}sx)\chi_{Ł'}^{\tilde F}(nx^{-1}s^{-1}xn^{-1})\\
&= \frac{|G^F|}{|T^F||(T')^F|}\sum_{\substack{t\in T^{\tilde F}\\ n\in N_G(T,T')^F}}\chi_{Ł}^{\tilde F}(t)\chi^{\tilde F}_{\mathrm{Ad}(n)^*(Ł')}(t^{-1})\\
&= \frac{|G^F||T^{\tilde F}|}{|T^F||(T')^F|}|\{n\in N_G(T,T')|\mathrm{Ad}(n)^*Ł'\cong Ł\}^F|\\
&= |G^{\tilde F}|\mathrm{RHS}.
\end{aligned}$$

□

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Department of Mathematics, Massachusetts Institute of Technology, Cambridge, Massachusetts 02139

Current address: School of Mathematics and Statistics, University of Sydney, NSW 2006, Australia

E-mail address: anthonyh@maths.usyd.edu.au