

## TWINING CHARACTER FORMULA OF KAC-WAKIMOTO TYPE FOR AFFINE LIE ALGEBRAS

SATOSHI NAITO

ABSTRACT. We prove a formula of Kac-Wakimoto type for the twining characters of irreducible highest weight modules of symmetric, noncritical, integrally dominant highest weights over affine Lie algebras. This formula describes the twining character in terms of the subgroup of the integral Weyl group consisting of elements which commute with the Dynkin diagram automorphism. The main tools in our proof are the (Jantzen) translation functor and the existence result of a certain local composition series which is stable under the Dynkin diagram automorphism.

### 1. INTRODUCTION

In [FSS] and [FRS], they introduced the notion of twining characters of certain highest weight modules over (generalized) Kac-Moody algebras, corresponding to Dynkin diagram automorphisms. Moreover, they gave formulas for the twining characters of Verma modules of arbitrary symmetric highest weights and irreducible highest weight modules of symmetric, dominant integral highest weights (see Theorems 4.4 and 4.5). The aim of this paper is to give a formula of Kac-Wakimoto type for the twining characters of irreducible highest weight modules over affine Lie algebras of symmetric, noncritical, integrally dominant highest weights (including symmetric, dominant integral ones). We should note that our method of proof is quite different from that in [FSS] and [FRS], since an irreducible highest weight module is not integrable if its highest weight is not dominant integral.

Let us explain our formula more precisely. Let  $\mathfrak{g} := \mathfrak{g}(A)$  be an affine Lie algebra over  $\mathbb{C}$  with  $A = (a_{ij})_{i,j \in I}$  the Cartan matrix,  $\mathfrak{h}$  the Cartan subalgebra,  $\Pi = \{\alpha_i\}_{i \in I} \subset \mathfrak{h}^* := \text{Hom}_{\mathbb{C}}(\mathfrak{h}, \mathbb{C})$  the set of simple roots,  $\Pi^\vee = \{h_i\}_{i \in I} \subset \mathfrak{h}$  the set of simple coroots, and  $W = \langle r_i \mid i \in I \rangle \subset GL(\mathfrak{h}^*)$  the Weyl group, where  $r_i$  is a simple reflection. For a real root  $\alpha = w(\alpha_i) \in \Delta^{re} := W \cdot \Pi$  with  $w \in W$  and  $i \in I$ , the dual real root  $h_\alpha$  of  $\alpha$  is defined to be an element  $h_\alpha := w(h_i) \in \sum_{i \in I} \mathbb{Z}h_i$ . For  $\lambda \in \mathfrak{h}^*$ , we set

$$\Delta(\lambda) := \{\alpha \in \Delta^{re} \mid \lambda(h_\alpha) \in \mathbb{Z}\},$$

$$W(\lambda) := \langle r_\alpha \mid \alpha \in \Delta(\lambda) \rangle \subset W,$$

---

Received by the editors December 19, 2000 and, in revised form, February 17, 2002.  
2000 *Mathematics Subject Classification*. Primary 17B67; Secondary 17B10, 17B40.  
*Key words and phrases*. Affine Lie algebra, character, diagram automorphism.

where  $r_\alpha \in W$  is the reflection of  $\mathfrak{h}^*$  corresponding to  $\alpha$ . Furthermore, let  $\Pi(\lambda)$  be the set of simple roots for the positive system  $\Delta(\lambda)_+ := \Delta(\lambda) \cap \Delta_+$ .

For  $\lambda \in \mathfrak{h}^*$ , we denote by  $M(\lambda)$  the Verma module of highest weight  $\lambda$ , and by  $L(\lambda)$  the irreducible highest weight module of highest weight  $\lambda$  over  $\mathfrak{g}$ . Let  $\omega : I \rightarrow I$  be a bijection such that

$$a_{\omega(i),\omega(j)} = a_{ij} \quad \text{for } i, j \in I$$

(such a bijection  $\omega$  is called a (Dynkin) diagram automorphism). We choose and fix a set of representatives  $\widehat{I}$  of the  $\omega$ -orbits in  $I$ , and denote by  $N_i$  the number of elements in the  $\omega$ -orbit of  $i \in I$ . This  $\omega$  induces an automorphism (also called a diagram automorphism by abuse of notation)  $\omega : \mathfrak{g} \rightarrow \mathfrak{g}$  of the affine Lie algebra  $\mathfrak{g}$ , which stabilizes  $\mathfrak{h}$ . We denote the dual map of the restriction of  $\omega$  to  $\mathfrak{h}$  by  $\omega^* : \mathfrak{h}^* \rightarrow \mathfrak{h}^*$ . In addition, we define the following subgroup of  $W$

$$\widetilde{W} := \{w \in W \mid \omega^*w = w\omega^*\}.$$

Let  $\lambda \in \mathfrak{h}^*$  be a symmetric weight (i.e.,  $\omega^*(\lambda) = \lambda$ ). Then the diagram automorphism  $\omega$  induces on the highest weight modules  $V(\lambda) = M(\lambda), L(\lambda)$  unique linear automorphisms

$$\tau_\omega : V(\lambda) \rightarrow V(\lambda)$$

such that

$$\tau_\omega(xv) = \omega^{-1}(x)\tau_\omega(v) \quad \text{for } x \in \mathfrak{g}, v \in V(\lambda)$$

and the restriction of  $\tau_\omega$  to the highest weight space  $V(\lambda)_\lambda$  is the identity. Now the twining character  $\text{ch}^\omega(V(\lambda))$  is defined to be the formal sum

$$\text{ch}^\omega(V(\lambda)) := \sum_{\substack{\chi \in \mathfrak{h}^* \\ \omega^*(\chi) = \chi}} \text{Tr}(\tau_\omega|_{V(\lambda)_\chi}) e(\chi),$$

where  $V(\lambda)_\chi$  is the weight space corresponding to  $\chi \in \mathfrak{h}^*$ .

Our main result in this paper is the following theorem.

**Theorem.** *Let  $\lambda$  be an element of  $\mathfrak{h}^*$  such that  $\omega^*(\lambda) = \lambda$ ,  $(\lambda + \rho)(c) \neq 0$ , and  $(\lambda + \rho)(h_\alpha) > 0$  for all  $\alpha \in \Delta(\lambda) \cap \Delta_+$ . Here  $\rho \in \mathfrak{h}^*$  is a fixed element such that  $\rho(h_i) = 1$  for all  $i \in I$  and  $\omega^*(\rho) = \rho$ , and  $c \in \mathfrak{h}$  is the canonical central element.*

(a) *When  $W(\lambda) \cap \widetilde{W} = \{1\}$ , we have*

$$\begin{aligned} \text{ch}^\omega(L(\lambda)) &= \text{ch}^\omega(M(\lambda)) \\ &= e(\lambda) \cdot \left( \sum_{w \in \widetilde{W}} (-1)^{\widehat{\ell}(w)} e(w(\rho) - \rho) \right)^{-1}, \end{aligned}$$

where  $\widehat{\ell} : \widetilde{W} \rightarrow \mathbb{Z}$  denotes the length function of the Coxeter group  $\widetilde{W}$ .

(b) *When  $W(\lambda) \cap \widetilde{W} \neq \{1\}$ , we further assume that for each  $\alpha \in \Pi(\lambda)$ , the integer  $\lambda(h_\alpha)$  is a multiple of the greatest common divisor of the integers  $\sum_{k=0}^{N_i-1} l_{\omega^k(i)}^\alpha$ ,*

$i \in \widehat{I}$ , where  $h_\alpha = \sum_{i \in I} l_i^\alpha h_i \in \sum_{i \in I} \mathbb{Z}_{\geq 0} h_i$ . In this case, we have

$$\begin{aligned} \text{ch}^\omega(L(\lambda)) &= \sum_{w \in W(\lambda) \cap \widetilde{W}} (-1)^{\widehat{\ell}_\lambda(w)} \text{ch}^\omega(M(w(\lambda + \rho) - \rho)) \\ &= \frac{\sum_{w \in W(\lambda) \cap \widetilde{W}} (-1)^{\widehat{\ell}_\lambda(w)} e(w(\lambda + \rho))}{\sum_{w \in \widetilde{W}} (-1)^{\widehat{\ell}(w)} e(w(\rho))}, \end{aligned}$$

where  $\widehat{\ell}_\lambda : W(\lambda) \cap \widetilde{W} \rightarrow \mathbb{Z}$  denotes the length function of the Coxeter group  $W(\lambda) \cap \widetilde{W}$ .

Since the additional assumption on  $\lambda$  in part (b) of the theorem is made for some technical reasons, it seems very likely that this assumption can be removed. However, for our proof of the theorem, it is essential (see §6.2).

This paper is organized as follows. In §2, we recall some basic facts about affine Lie algebras from [Ka]. In §3, we review the notion of orbit Lie algebras from [FRS] and [FSS]. In §4, we recall the definition of twining characters and main results of [FSS] or [FRS]. In §5, we introduce the (Jantzen) translation functor and show some of its important properties. In §6, we show the existence of a certain  $\tau_\omega$ -stable local composition series. Making essential use of this, we finally prove our main result (Theorem) stated above.

**Acknowledgments.** I express my sincere thanks to Professor Kiyokazu Suto for many valuable discussions and helpful suggestions, which lead to the existence result of a certain  $\tau_\omega$ -stable local composition series (which is named a Suto filtration after him). I also profitted greatly from discussions via e-mail with Professors Seok-Jin Kang, Jae-Hoon Kwon, and Christoph Schweigert.

## 2. AFFINE LIE ALGEBRAS

We recall the definition and some basic properties of affine Lie algebras from [Ka].

**2.1. Generalized Kac-Moody algebras of at most affine type.** Though our interest is focused upon the twining characters for affine Lie algebras, it is convenient for later use to explain the notion of generalized Kac-Moody algebras (GKM algebras) of at most affine type. Here we follow the notation of [Ka] (see also [B]).

Let  $I = \{1, 2, \dots, n\}$  be a finite index set, and let  $A = (a_{ij})_{i,j \in I}$  be an  $n \times n$  real matrix (called a GGCM) satisfying:

- (C1) either  $a_{ii} = 2$  or  $a_{ii} \leq 0$  for  $i \in I$ ;
- (C2)  $a_{ij} \leq 0$  for  $i, j \in I$  if  $i \neq j$ , and  $a_{ij} \in \mathbb{Z}$  for  $j \neq i$  if  $a_{ii} = 2$ ;
- (C3)  $a_{ij} = 0$  if and only if  $a_{ji} = 0$ .

We assume that after reordering the indices, the matrix  $A$  decomposes into a direct sum of generalized Cartan matrices (GCMs) of finite type, GCMs of affine type, and the  $1 \times 1$  zero matrices. We call such a matrix  $A$  a GGCM of at most affine type. Let  $\mathfrak{g} = \mathfrak{g}(A)$  be the generalized Kac-Moody algebra (GKM algebra) over  $\mathbb{C}$  associated to a GGCM  $A = (a_{ij})_{i,j \in I}$  of at most affine type, with  $\mathfrak{h}$  the Cartan subalgebra,  $\{e_i, f_i\}_{i \in I}$  the Chevalley generators,  $\Pi = \{\alpha_i\}_{i \in I} \subset \mathfrak{h}^* := \text{Hom}_{\mathbb{C}}(\mathfrak{h}, \mathbb{C})$

the set of simple roots, and  $\Pi^\vee = \{h_i\}_{i \in I} \subset \mathfrak{h}$  the set of simple coroots. We have a root space decomposition of  $\mathfrak{g}$  with respect to the Cartan subalgebra  $\mathfrak{h}$ :  $\mathfrak{g} = \left(\bigoplus_{\alpha \in \Delta_-} \mathfrak{g}_\alpha\right) \oplus \mathfrak{h} \oplus \left(\bigoplus_{\alpha \in \Delta_+} \mathfrak{g}_\alpha\right)$ , where  $\Delta_+ \subset Q_+ := \sum_{i \in I} \mathbb{Z}_{\geq 0} \alpha_i$  is the set of positive roots,  $\Delta_- = -\Delta_+$  is the set of negative roots, and  $\mathfrak{g}_\alpha$  is the root space corresponding to a root  $\alpha \in \Delta = \Delta_- \sqcup \Delta_+$ . Note that  $\mathfrak{g}_{\alpha_i} = \mathbb{C}e_i$ ,  $\mathfrak{g}_{-\alpha_i} = \mathbb{C}f_i$  for  $i \in I$ . We call  $Q := \sum_{i \in I} \mathbb{Z} \alpha_i \subset \mathfrak{h}^*$  the root lattice, and  $Q^\vee := \sum_{i \in I} \mathbb{Z} h_i \subset \mathfrak{h}$  the coroot lattice of  $\mathfrak{g}$ .

We set  $I^{re} := \{i \in I \mid a_{ii} = 2\}$ ,  $I^{im} := \{i \in I \mid a_{ii} = 0\}$ , and call  $\Pi^{re} := \{\alpha_i \in \Pi \mid i \in I^{re}\}$  the set of real simple roots,  $\Pi^{im} := \{\alpha_i \in \Pi \mid i \in I^{im}\}$  the set of imaginary simple roots. Then the Weyl group  $W$  of the GKM algebra  $\mathfrak{g}$  is defined to be  $W = \langle r_i \mid i \in I^{re} \rangle \subset GL(\mathfrak{h}^*)$ , where  $r_i$  is a simple reflection. Note that  $W$  is a Coxeter group with the canonical generator system  $\{r_i \mid i \in I^{re}\}$ . We denote by  $\ell : W \rightarrow \mathbb{Z}$  the length function of  $W$ . We call  $\Delta^{re} := W \cdot \Pi^{re}$  the set of real roots, and  $\Delta^{im} := \Delta \setminus \Delta^{re}$  the set of imaginary roots. For a real root  $\alpha = w(\alpha_i)$  with  $w \in W$  and  $i \in I^{re}$ , the dual real root  $h_\alpha$  of  $\alpha$  is given by  $h_\alpha = w(h_i) \in Q^\vee$ .

**2.2. Invariant bilinear forms.** A GGCM  $A = (a_{ij})_{i,j \in I}$  of at most affine type is clearly symmetrizable, i.e., there exist an invertible diagonal matrix  $D = \text{diag}(\varepsilon_i)_{i \in I}$  and a symmetric matrix  $B = (b_{ij})_{i,j \in I}$  such that  $A = DB$ . Note that all the  $\varepsilon_i$  can be taken to be positive rational numbers, and all the  $b_{ij}$  to be rational numbers since  $A = (a_{ij})_{i,j \in I}$  is an integral matrix. Thus, there exists a nondegenerate, symmetric, invariant bilinear form  $(\cdot|\cdot)$  on  $\mathfrak{g} = \mathfrak{g}(A)$ . The restriction of this form  $(\cdot|\cdot)$  to  $\mathfrak{h}$  is again nondegenerate, so that it induces (through a linear isomorphism  $\nu : \mathfrak{h} \rightarrow \mathfrak{h}^*$  defined by  $\nu(h)(h') = (h|h')$  for  $h, h' \in \mathfrak{h}$ ) on  $\mathfrak{h}^*$  a nondegenerate, symmetric,  $W$ -invariant bilinear form, which is also denoted by  $(\cdot|\cdot)$ . We note that

$$(\alpha_i|\alpha_j) = b_{ij} = a_{ij}/\varepsilon_i \quad \text{for } i, j \in I, \quad \lambda(h_i) = \varepsilon_i(\lambda|\alpha_i) \quad \text{for } \lambda \in \mathfrak{h}^*, \quad i \in I,$$

$$\lambda(h_\alpha) = 2(\lambda|\alpha)/(\alpha|\alpha) \quad \text{for } \lambda \in \mathfrak{h}^*, \quad \alpha \in \Delta^{re}.$$

*Remark 2.2.1.* A root  $\alpha$  is an imaginary root if and only if  $(\alpha|\alpha) \leq 0$ , while for a real root  $\alpha = w(\alpha_i)$  with  $w \in W$  and  $i \in I^{re}$ , we have  $(\alpha|\alpha) = (\alpha_i|\alpha_i) = 2/\varepsilon_i > 0$ .

**2.3. Affine Lie algebras.** Let us assume in this subsection that the matrix  $A = (a_{ij})_{i,j \in I}$  is a GCM of affine type. The Kac-Moody algebra  $\mathfrak{g} = \mathfrak{g}(A)$  is called an affine Lie algebra.

Let  $c = \sum_{i \in I} a_i^\vee h_i \in Q^\vee$  (with the  $a_i^\vee, i \in I$ , relatively prime positive integers) be the canonical central element spanning the (one-dimensional) center. We know that the restriction of the bilinear form  $(\cdot|\cdot)$  on  $\mathfrak{h}^*$  to  $\sum_{i \in I} \mathbb{R} \alpha_i$  is positive-semidefinite with one-dimensional radical. Let  $\delta = \sum_{i \in I} a_i \alpha_i \in Q$  (with the  $a_i, i \in I$ , relatively prime positive integers) be the null root spanning the radical. We remark that the set  $\Delta_+^{im} := \Delta^{im} \cap \Delta_+$  of positive imaginary roots is equal to  $\mathbb{Z}_{\geq 1} \delta$ , and  $w(\delta) = \delta$  for all  $w \in W$ . Note that  $\nu(c) = q\delta$  for some positive rational number  $q \in \mathbb{Q}$ .

### 3. ORBIT LIE ALGEBRAS

We review the notion of orbit Lie algebras mainly from [FRS] and [FSS]. However, since we need to deal with decomposable GCMs, there are some additional considerations. See also [N1] for the “transposed” version of orbit Lie algebras, which were called folding subalgebras. In this section, we assume that the matrix  $A = (a_{ij})_{i,j \in I}$  decomposes, after reordering the indices, into a direct sum of GCMs of finite type and those of affine type.

**3.1. Diagram automorphisms.** A bijection  $\omega : I \rightarrow I$  such that

$$(3.1.1) \quad a_{\omega(i),\omega(j)} = a_{ij} \quad \text{for } i, j \in I$$

is called a (Dynkin) diagram automorphism. This induces an automorphism of the Dynkin diagram  $S(A)$  of the GCM  $A$  as a graph. Since the graph  $S(A)$  is not necessarily connected, we have the following decomposition of  $S(A)$  into connected components:  $S(A) = \bigsqcup_{l=1}^m S(A(l))$ , where for each  $l$ ,  $1 \leq l \leq m$ , the subgraph  $S(A(l))$  of  $S(A)$  is a connected component corresponding to the subset  $I(l)$  of the index set  $I$ . Note that by assumption the submatrix  $A(l) := (a_{ij})_{i,j \in I(l)}$  is a GCM of finite type or affine type for  $1 \leq l \leq m$ . We set  $K := \{1, \dots, m\}$ . It is easy to see that the diagram automorphism  $\omega : I \rightarrow I$  maps a connected component  $S(A(l))$  to another (or the same) connected component, say  $S(A(\dot{\omega}(l)))$ , for  $l \in K$ . Thus  $\omega$  induces a bijection  $\dot{\omega} : K \rightarrow K$ . It is obvious that the Dynkin diagram  $S(A(\dot{\omega}(l)))$  is isomorphic to the Dynkin diagram  $S(A(l))$  as a graph for  $l \in K$ .

Let  $N$  be the order of  $\omega : I \rightarrow I$ , and  $N_i$  the number of elements in the  $\omega$ -orbit of  $i \in I$  in  $I$ . It is clear that the restriction of  $\omega$  to each  $\omega$ -orbit of  $i \in I$  is a cyclic permutation of order  $N_i$ . Similarly, let  $M$  be the order of  $\dot{\omega} : K \rightarrow K$ , and  $M_l$  the number of elements in the  $\dot{\omega}$ -orbit of  $l \in K$  in  $K$ . Then the restriction of  $\dot{\omega}$  to each  $\dot{\omega}$ -orbit of  $l \in K$  is a cyclic permutation of order  $M_l$ , and hence the restriction of  $\omega^{M_l}$  to  $I(l)$  induces an automorphism of the Dynkin diagram  $S(A(l))$  as a graph for  $l \in K$ .

Since the matrix  $A = (a_{ij})_{i,j \in I}$  is symmetrizable, we have a decomposition  $A = DB$  with  $D = \text{diag}(\varepsilon_i)_{i \in I}$  as in §2.2. We immediately obtain the following:

**Lemma 3.1.1.** *For each  $l \in K$ , there exists a positive rational number  $R_l$  such that  $\varepsilon_{\omega(i)} = R_l \varepsilon_i$  for all  $i \in I(l)$ .*

*Proof.* Since  $A = DB$ , we have  $a_{ij} = \varepsilon_i b_{ij}$  for  $i, j \in I$ . Hence, by condition (3.1.1), we have  $\varepsilon_{\omega(i)} b_{\omega(i),\omega(j)} = \varepsilon_i b_{ij}$  and  $\varepsilon_{\omega(j)} b_{\omega(j),\omega(i)} = \varepsilon_j b_{ji}$  for  $i, j \in I$ . So we obtain

$$\varepsilon_i \varepsilon_{\omega(j)} b_{ij} b_{\omega(j),\omega(i)} = \varepsilon_j \varepsilon_{\omega(i)} b_{ji} b_{\omega(i),\omega(j)}.$$

Note that if  $a_{ij} = a_{\omega(i),\omega(j)} \neq 0$ , then  $b_{ij} = b_{ji} \neq 0$  and  $b_{\omega(i),\omega(j)} = b_{\omega(j),\omega(i)} \neq 0$ . Thus we have  $\varepsilon_i^{-1} \cdot \varepsilon_{\omega(i)} = \varepsilon_j^{-1} \cdot \varepsilon_{\omega(j)}$  if  $a_{ij} \neq 0$ . Recall that the subgraph  $S(A(l))$  of the Dynkin diagram  $S(A)$  corresponding to the subset  $I(l)$  of  $I$  is connected. Therefore there exists a positive rational number  $R_l$  such that  $\varepsilon_i^{-1} \cdot \varepsilon_{\omega(i)} = R_l$  for all  $i \in I(l)$ .  $\square$

*Remark 3.1.2.* For  $l \in K$  with  $M_l = 1$ , it follows that  $R_l = 1$  since  $\prod_{i \in I(l)} \varepsilon_{\omega(i)} = R_l^{\sharp(I(l))} \cdot \left( \prod_{i \in I(l)} \varepsilon_i \right)$  implies  $R_l = 1$ .

By Lemma 3.1.1, we can write  $D = D_1 D_2$ , where  $D_1 = \text{diag}(\varepsilon'_i)_{i \in I}$  such that  $\varepsilon'_{\omega(i)} = \varepsilon'_i$  for all  $i \in I$  and every  $\varepsilon'_i$  is a positive rational number, and where  $D_2$  is a diagonal matrix with all the diagonal entries positive rational numbers. Let us set  $D' := D_1$  and  $B' := D_2 B$ . By taking these new matrices  $D'$  and  $B'$  if necessary, we may (and will henceforth) assume that  $\varepsilon_{\omega(i)} = \varepsilon_i$  for all  $i \in I$  in the decomposition  $A = DB$  with  $D = \text{diag}(\varepsilon_i)_{i \in I}$ . Then it follows that  $b_{\omega(i),\omega(j)} = b_{ij}$  for all  $i, j \in I$  since  $a_{ij} = \varepsilon_i b_{ij}$ . Thus we obtain the following:

**Lemma 3.1.3.** *Let  $\omega : I \rightarrow I$  be a diagram automorphism. Then the bilinear form  $(\cdot | \cdot)$  on  $\mathfrak{g} = \mathfrak{g}(A)$  can be defined in such a way that the following hold:*

$$(1) \quad (h_{\omega(i)} | h_{\omega(j)}) = (h_i | h_j) \quad \text{for } i, j \in I;$$

$$(2) (\alpha_{\omega(i)}|\alpha_{\omega(j)}) = (\alpha_i|\alpha_j) \text{ for } i, j \in I.$$

As in [FSS, §3.2] the diagram automorphism  $\omega : I \rightarrow I$  induces an automorphism  $\omega$  of the Lie algebra  $\mathfrak{g}$  of order  $N$  such that

$$\begin{cases} \omega(e_i) = e_{\omega(i)} & \text{for } i \in I, \\ \omega(f_i) = f_{\omega(i)} & \text{for } i \in I, \\ \omega(h_i) = h_{\omega(i)} & \text{for } i \in I, \end{cases}$$

and  $(\omega(h)|\omega(h')) = (h|h')$  for  $h, h' \in \mathfrak{h}$ . This automorphism  $\omega : \mathfrak{g} \rightarrow \mathfrak{g}$  is also called a diagram automorphism by abuse of notation. The restriction of the automorphism  $\omega$  to  $\mathfrak{h}$  induces a dual map  $\omega^* : \mathfrak{h}^* \rightarrow \mathfrak{h}^*$  defined by:  $\omega^*(\lambda)(h) = \lambda(\omega(h))$  for  $\lambda \in \mathfrak{h}^*$  and  $h \in \mathfrak{h}$ .

**Lemma 3.1.4.** *The following hold for the dual map  $\omega^* : \mathfrak{h}^* \rightarrow \mathfrak{h}^*$ :*

- (1)  $\nu \circ \omega = (\omega^*)^{-1} \circ \nu$ . Hence we have  $(\omega^*(\lambda)|\omega^*(\mu)) = (\lambda|\mu)$  for  $\lambda, \mu \in \mathfrak{h}^*$ ;
- (2)  $\omega^*(\alpha_i) = \alpha_{\omega^{-1}(i)}$  for  $i \in I$ , and hence  $\omega^*(Q_+) = Q_+$ . Furthermore, we have  $(\omega^*)^{-1}r_i\omega^* = r_{\omega(i)}$  for  $i \in I$ ;
- (3)  $\omega(\mathfrak{g}_\alpha) = \mathfrak{g}_{(\omega^*)^{-1}(\alpha)}$  for  $\alpha \in \mathfrak{h}^*$ , and hence  $\omega^*(\Delta) = \Delta$ . In addition,  $\omega^*(\Delta^{re}) = \Delta^{re}$  and  $\omega^*(\Delta_+) = \Delta_+$ ;
- (4) for  $\alpha \in \Delta^{re}$ , we have  $h_{\omega^*(\alpha)} = \omega^{-1}(h_\alpha)$  and  $(\omega^*)^{-1}r_\alpha\omega^* = r_{(\omega^*)^{-1}(\alpha)}$ .

*Proof.* We will show only part (4). Recall from [Ka, Prop. 5.1 d)] that for  $\alpha \in \Delta^{re}$ , we have  $h_\alpha = 2\nu^{-1}(\alpha)/(\alpha|\alpha)$ . Since  $(\omega^*(\alpha)|\omega^*(\alpha)) = (\alpha|\alpha) > 0$ ,  $\omega^*(\alpha) \in \Delta$  is a real root. Hence we have by part (1) of the lemma,

$$h_{\omega^*(\alpha)} = 2\nu^{-1}(\omega^*(\alpha))/(\omega^*(\alpha)|\omega^*(\alpha)) = 2\omega^{-1}(\nu^{-1}(\alpha))/(\alpha|\alpha) = \omega^{-1}(h_\alpha).$$

Recall that for  $\lambda \in \mathfrak{h}^*$ ,  $r_\alpha(\lambda) = \lambda - (2(\lambda|\alpha)/(\alpha|\alpha))\alpha$ . Hence, for  $\lambda \in \mathfrak{h}^*$ , we have

$$((\omega^*)^{-1}r_\alpha\omega^*)(\lambda) = \lambda - (2(\omega^*(\lambda)|\alpha)/(\alpha|\alpha))(\omega^*)^{-1}(\alpha).$$

On the other hand, we have

$$\begin{aligned} r_{(\omega^*)^{-1}(\alpha)}(\lambda) &= \lambda - (2(\lambda|(\omega^*)^{-1}(\alpha))/((\omega^*)^{-1}(\alpha)|(\omega^*)^{-1}(\alpha)))(\omega^*)^{-1}(\alpha) \\ &= \lambda - (2(\omega^*(\lambda)|\alpha)/(\alpha|\alpha))(\omega^*)^{-1}(\alpha). \end{aligned}$$

This completes the proof. □

It follows from Lemma 3.1.4 (3) that  $\omega(\mathfrak{n}_+) = \mathfrak{n}_+$  and  $\omega(\mathfrak{n}_-) = \mathfrak{n}_-$ , where  $\mathfrak{n}_+ := \bigoplus_{\alpha \in \Delta_+} \mathfrak{g}_\alpha$  and  $\mathfrak{n}_- := \bigoplus_{\alpha \in \Delta_-} \mathfrak{g}_\alpha$ .

**3.2. Orbit Lie algebras.** For each  $i \in I$ , set

$$s_i := \begin{cases} 2 / \sum_{k=0}^{N_i-1} a_{i,\omega^k(i)} & \text{if } \sum_{k=0}^{N_i-1} a_{i,\omega^k(i)} > 0, \\ 1 & \text{if } \sum_{k=0}^{N_i-1} a_{i,\omega^k(i)} \leq 0, \end{cases}$$

and define

$$\widehat{a}_{ij} := s_j \sum_{k=0}^{N_j-1} a_{i,\omega^k(j)} \quad \text{for } i, j \in I.$$

We choose a set of representatives  $\widehat{I}$  of the  $\omega$ -orbits in  $I$ , and define  $\widehat{A} := (\widehat{a}_{ij})_{i,j \in \widehat{I}}$ , which does not depend on the choice of representatives of the  $\omega$ -orbits. Then, by [FRS, Lem. 2.1] (see also [N1, Props. 4.2 and 4.3]), we have the following:

**Proposition 3.2.1.** *The matrix  $\widehat{A} = (\widehat{a}_{ij})_{i,j \in \widehat{I}}$  is a symmetrizable GGCM. In fact, if we set  $\widehat{D} := \text{diag}(\widehat{\varepsilon}_i)_{i \in \widehat{I}}$  for  $i \in \widehat{I}$  with  $\widehat{\varepsilon}_i := \varepsilon_i s_i^{-1} N_i^{-1}$ , then the matrix  $(\widehat{D})^{-1} \widehat{A}$  is symmetric.*

Let  $\widehat{K}$  be a set of representatives of the  $\omega$ -orbits in  $K$ . For each  $l \in \widehat{K}$ , the restriction of  $\omega^{M_l}$  to  $I(l)$  induces an automorphism of the Dynkin diagram  $S(A(l))$  as a graph, as indicated in §3.1. So, for  $l \in \widehat{K}$ , we choose a set of representatives  $\widehat{I}(l)$  of the  $\omega^{M_l}$ -orbits in  $I(l)$ . Then, as  $\widehat{I}$ , we can take the set  $\bigsqcup_{l \in \widehat{K}} \widehat{I}(l)$ . Furthermore, the matrix  $\widehat{A}$  decomposes, after reordering the indices, into the direct sum  $\bigoplus_{l \in \widehat{K}} \widehat{A}(l)$ , where  $\widehat{A}(l) := (\widehat{a}_{ij})_{i,j \in \widehat{I}(l)}$ . Since the subgraphs  $S(A(l))$ ,  $l \in K$ , are connected components of the Dynkin diagram  $S(A)$ , we have  $\sum_{k=0}^{N_j-1} a_{i, \omega^k(j)} = \sum_{k=0}^{N_j/M_l-1} a_{i, (\omega^{M_l})^k(j)}$  for  $i, j \in \widehat{I}(l)$ .

Now recall from a remark in [FRS, §2] (cf. [N1, Lem. 4.3]) that if  $\sum_{k=0}^{N_i-1} a_{i, \omega^k(i)} > 0$  for some  $i \in I$ , then there are only two possibilities:

**Case 1.**  $\sum_{k=0}^{N_i-1} a_{i, \omega^k(i)} = 1$ . In this case,  $N_i$  is even, and the Dynkin diagram corresponding to the  $\omega$ -orbit of  $i$  is of type  $A_2 \times \cdots \times A_2$  ( $N_i/2$  times). In fact, we have  $a_{i, \omega^{N_i/2}(i)} = -1$ , and  $a_{i, \omega^k(i)} = 0$  for other  $1 \leq k \leq N_i - 1$ ,  $k \neq N_i/2$ .

**Case 2.**  $\sum_{k=0}^{N_i-1} a_{i, \omega^k(i)} = 2$ . In this case, the Dynkin diagram corresponding to the  $\omega$ -orbit of  $i$  is totally disconnected, i.e., of type  $A_1 \times \cdots \times A_1$  ( $N_i$  times).

Also, recall from [FSS, §2.4] that if the GCM  $A = (a_{ij})_{i,j \in I}$  is of finite or affine type, then the diagram automorphism  $\omega : I \rightarrow I$  satisfies the condition:

$$\sum_{k=0}^{N_i-1} a_{i, \omega^k(i)} > 0 \quad \text{for all } i \in I,$$

except for the case where the Dynkin diagram  $S(A)$  is of type  $A_{n-1}^{(1)}$  with  $n \geq 2$  and  $\omega$  is a cyclic permutation of  $I$  of order  $n$ . In this case, we have  $\widehat{I} = \{i_0\}$ ,  $N_{i_0} = n$ , and  $\widehat{a}_{i_0, i_0} = \sum_{k=0}^{n-1} a_{i_0, \omega^k(i_0)} = 0$  for each  $i_0 \in I$ . Thus  $\widehat{A}$  is the  $1 \times 1$  zero matrix. Except for this case, the matrix  $\widehat{A}$  is a GCM of finite (resp. affine) type if  $A$  is a GCM of finite (resp. affine) type (see [FSS, §2.3] and also [N1, Cor. 4.1]).

We set

$$\check{A} := (\check{a}_{ij})_{i,j \in \check{I}}, \quad \text{where } \check{I} := \{i \in \widehat{I} \mid \sum_{k=0}^{N_i-1} a_{i, \omega^k(i)} > 0\}.$$

Putting all the facts above together, we obtain the following:

**Proposition 3.2.2.** *Let the notation be as above. The submatrix  $\check{A}$  of the GGCM  $\widehat{A}$  is a GCM, which decomposes, after reordering the indices, into a direct sum of GCMs of finite type and those of affine type. Moreover,  $\check{A}$  is a GGCM of at most affine type. More precisely, for  $l \in \widehat{K}$ , the matrix  $\widehat{A}(l)$  is the  $1 \times 1$  zero matrix if and only if the Dynkin diagram  $S(A(l))$  is of type  $A_{p-1}^{(1)}$  for some  $p \geq 2$  and the restriction of  $\omega^{M_l}$  to  $I(l)$  is a cyclic permutation of order  $p$ . Except for this case, the matrix  $\widehat{A}(l)$  is a GCM of finite (resp. affine) type if  $A(l)$  is a GCM of finite*

(resp. affine) type. In particular, the set  $\check{I} = (\hat{I})^{re}$  is the disjoint union of  $\hat{I}(l)$  for  $l \in \hat{K}$  such that  $\hat{A}(l)$  is not the  $1 \times 1$  zero matrix.

Let  $\hat{\mathfrak{g}} := \mathfrak{g}(\hat{A})$  be the GKM algebra over  $\mathbb{C}$  associated to the GGCM  $\hat{A} = (\hat{a}_{ij})_{i,j \in \hat{I}}$  with  $\hat{\mathfrak{h}}$  the Cartan subalgebra,  $\{\hat{e}_i, \hat{f}_i\}_{i \in \hat{I}}$  the Chevalley generators,  $\hat{\Pi} = \{\hat{\alpha}_i\}_{i \in \hat{I}} \subset \hat{\mathfrak{h}}^*$  the set of simple roots,  $\hat{\Pi}^\vee = \{\hat{h}_i\}_{i \in \hat{I}} \subset \hat{\mathfrak{h}}$  the set of simple coroots, and  $\hat{W} = \langle \hat{r}_i \mid i \in \check{I} \rangle \subset GL(\hat{\mathfrak{h}}^*)$  the Weyl group. We denote the nondegenerate, symmetric, invariant bilinear form on  $\hat{\mathfrak{g}}$  by the same symbol  $(\cdot | \cdot)$  as for  $\mathfrak{g}$  (because of (3.2.2) and (3.2.3) below).

We define the Lie algebra  $\check{\mathfrak{g}}$  to be the Lie subalgebra of  $\hat{\mathfrak{g}}$  generated by  $\hat{\mathfrak{h}}$  and  $\hat{e}_i, \hat{f}_i$  with  $i \in \check{I}$ . This Lie algebra  $\check{\mathfrak{g}}$  can be thought of as a Kac-Moody algebra associated to the GCM  $\check{A} = (\check{a}_{ij})_{i,j \in \check{I}}$ , though the Cartan subalgebra  $\hat{\mathfrak{h}}$  may not be “minimal”.

**Definition 3.2.3** ([FRS, Def. 2.1]). The Lie algebra  $\hat{\mathfrak{g}}$  is called the orbit Lie algebra associated to the diagram automorphism  $\omega$  of  $\mathfrak{g}$ . The Lie algebra  $\check{\mathfrak{g}}$  is also called the orbit Lie algebra.

*Remark 3.2.4.* A table of all diagram automorphisms  $\omega : I \rightarrow I$  and the (Dynkin diagrams of) corresponding orbit Lie algebras  $\hat{\mathfrak{g}}$  for all finite-dimensional simple Lie algebras and affine Lie algebras can be found, for example, in [FSS, §2.4 and §2.5].

We set

$$\mathfrak{h}^0 := \{h \in \mathfrak{h} \mid \omega(h) = h\}, \quad (\mathfrak{h}^*)^0 := \{\lambda \in \mathfrak{h}^* \mid \omega^*(\lambda) = \lambda\}.$$

Then we can identify  $(\mathfrak{h}^*)^0$  with  $(\mathfrak{h}^0)^* := \text{Hom}_{\mathbb{C}}(\mathfrak{h}^0, \mathbb{C})$  in a natural way. We call an element of  $(\mathfrak{h}^*)^0$  a symmetric weight. We know from [FSS, §3.3] that there exists a linear isomorphism  $P_\omega : \mathfrak{h}^0 \rightarrow \hat{\mathfrak{h}}$  such that

$$(3.2.1) \quad P_\omega\left(\sum_{k=0}^{N_i-1} h_{\omega^k(i)}\right) = N_i \hat{h}_i \quad \text{for } i \in \hat{I},$$

$$(3.2.2) \quad (P_\omega(h) | P_\omega(h')) = (h | h') \quad \text{for } h, h' \in \mathfrak{h}^0.$$

This linear isomorphism  $P_\omega : \mathfrak{h}^0 \rightarrow \hat{\mathfrak{h}}$  induces a dual map  $P_\omega^* : \hat{\mathfrak{h}}^* \rightarrow (\mathfrak{h}^0)^* \cong (\mathfrak{h}^*)^0$  defined by:  $P_\omega^*(\hat{\lambda})(h) = \hat{\lambda}(P_\omega(h))$  for  $\hat{\lambda} \in \hat{\mathfrak{h}}^*, h \in \mathfrak{h}^0$ . Note that

$$(3.2.3) \quad (P_\omega^*(\hat{\lambda}) | P_\omega^*(\hat{\mu})) = (\hat{\lambda} | \hat{\mu}) \quad \text{for } \hat{\lambda}, \hat{\mu} \in \hat{\mathfrak{h}}^*,$$

$$(3.2.4) \quad P_\omega^*(\hat{\alpha}_i) = s_i \beta_i \quad \text{for } i \in \hat{I}, \quad \text{where } \beta_i := \sum_{k=0}^{N_i-1} \alpha_{\omega^k(i)} \in (\mathfrak{h}^*)^0.$$

We take and fix an element  $\rho \in \mathfrak{h}^*$  (called a Weyl vector) such that  $\rho(h_i) = 1$  for all  $i \in I$ . Replacing  $\rho$  above by  $(1/N) \sum_{k=0}^{N-1} (\omega^*)^k(\rho) \in \mathfrak{h}^*$  if necessary, we may (and will henceforth) assume that  $\omega^*(\rho) = \rho$ . We now define a shifted action (called the dot-action) of the Weyl group  $W$  on  $\mathfrak{h}^*$  by

$$(3.2.5) \quad w \circ \lambda := w(\lambda + \rho) - \rho \quad \text{for } \lambda \in \mathfrak{h}^*.$$

**3.3. Weyl groups.** We define the following subgroup of  $W$

$$\widetilde{W} := \{w \in W \mid \omega^* w = w \omega^*\}.$$

It is obvious that the group  $\widetilde{W}$  stabilizes the subspace  $(\mathfrak{h}^*)^0$  of  $\mathfrak{h}^*$ . For  $i \in I$  with  $\sum_{k=0}^{N_i-1} a_{i,\omega^k(i)} > 0$ , we set

$$(3.3.1) \quad w_i := \begin{cases} \prod_{k=0}^{N_i/2-1} r_{\omega^k(i)} r_{\omega^{k+N_i/2}(i)} r_{\omega^k(i)} & \text{if } \sum_{k=0}^{N_i-1} a_{i,\omega^k(i)} = 1, \\ \prod_{k=0}^{N_i-1} r_{\omega^k(i)} & \text{if } \sum_{k=0}^{N_i-1} a_{i,\omega^k(i)} = 2. \end{cases}$$

We know from [FSS, §5.1] that  $w_{\omega(i)} = w_i$ ,  $w_i^2 = 1$ , and  $(\omega^*)^{-1} w_i \omega^* = w_i$ , i.e.,  $w_i \in \widetilde{W}$ . In addition, we know the following:

**Proposition 3.3.1** ([FRS, Prop. 3.3]). *The group  $\widetilde{W}$  is generated by the  $w_i$ 's for  $i \in \check{I}$ . Moreover,  $\widetilde{W}$  acts on  $(\mathfrak{h}^*)^0$  faithfully, and the Weyl group  $\widehat{W} = \langle \widehat{r}_i \mid i \in \check{I} \rangle$  of the orbit Lie algebra  $\widehat{\mathfrak{g}}$  is isomorphic to the group  $\widetilde{W}$  restricted to  $(\mathfrak{h}^*)^0$ , which is denoted by  $\widetilde{W}|_{(\mathfrak{h}^*)^0}$ . Namely, we have an isomorphism of groups  $\Theta : \widehat{W} \rightarrow \widetilde{W}|_{(\mathfrak{h}^*)^0}$ , where  $\Theta(\widehat{r}_i) = w_i|_{(\mathfrak{h}^*)^0}$  for  $i \in \check{I}$ . In fact,  $\Theta(\widehat{w}) = P_\omega^* \circ \widehat{w} \circ (P_\omega^*)^{-1}$  for  $\widehat{w} \in \widehat{W}$ .*

By Proposition 3.3.1, we get the following commutative diagram for each  $\widehat{w} \in \widehat{W}$ :

$$\begin{array}{ccc} \widehat{\mathfrak{h}}^* & \xrightarrow{P_\omega^*} & (\mathfrak{h}^*)^0 \\ \widehat{w} \downarrow & & \downarrow \Theta(\widehat{w}) \\ \widehat{\mathfrak{h}}^* & \xrightarrow{P_\omega^*} & (\mathfrak{h}^*)^0. \end{array}$$

Consequently, there exists an isomorphism of groups  $\Theta : \widehat{W} \rightarrow \widetilde{W}$  such that  $\Theta(\widehat{r}_i) = w_i$  for  $i \in \check{I}$ . Since the Weyl group  $\widehat{W}$  is a Coxeter group with the canonical generator system  $\{\widehat{r}_i \mid i \in \check{I}\}$ , the group  $\widetilde{W}$  is also a Coxeter group with the canonical generator system  $\{w_i \mid i \in \check{I}\}$ . We denote the length function of  $\widetilde{W}$  by  $\widehat{\ell} : \widehat{W} \rightarrow \mathbb{Z}$ .

*Remark 3.3.2.* From Proposition 3.3.1, we see that  $\widetilde{W} = \{1\}$  if and only if  $\check{I} = \emptyset$ . In particular,  $\widetilde{W} = \{1\}$  when the Dynkin diagram  $S(A)$  is of type  $A_{n-1}^{(1)}$  and  $\omega : I \rightarrow I$  is a cyclic permutation of  $I$  of order  $n$ .

#### 4. TWINING CHARACTERS

From now on, we assume that the matrix  $A = (a_{ij})_{i,j \in I}$  is a GCM of affine type. Let  $\omega : I \rightarrow I$  be a diagram automorphism.

We recall the definition of the twining character of a certain highest weight  $\mathfrak{g}$ -module of a symmetric highest weight, following [FRS] and [FSS]. However, since no comment about the “normalization” of the map  $\tau_\omega$  is given in [FRS] or [FSS], we have to give additional comments.

Let  $(\pi, V)$  be a  $\mathfrak{g}$ -module, i.e., let  $\pi : \mathfrak{g} \rightarrow \mathfrak{gl}(V)$  be a representation on the vector space  $V$ . We define a new  $\mathfrak{g}$ -module  $(\pi^\omega, V)$  of  $\mathfrak{g}$  by:  $\pi^\omega(x)v = \pi(\omega(x))v$  for  $x \in \mathfrak{g}$ ,  $v \in V$ . If we take a highest weight  $\mathfrak{g}$ -module  $(\pi_\lambda, V(\lambda))$  of highest weight  $\lambda \in \mathfrak{h}^*$  with  $v_\lambda \in V(\lambda)$  the (canonical) highest weight vector, then the module

$(\pi_\lambda^\omega, V(\lambda))$  is a highest weight  $\mathfrak{g}$ -module of highest weight  $\omega^*(\lambda) \in \mathfrak{h}^*$  with  $v_\lambda$  a highest weight vector since  $\omega(\mathfrak{n}_+) = \mathfrak{n}_+$  and  $\omega(\mathfrak{n}_-) = \mathfrak{n}_-$ .

Throughout this paper, as a highest weight  $\mathfrak{g}$ -module  $V(\lambda)$  of highest weight  $\lambda$ , we will consider only two kinds of modules: the Verma module  $M(\lambda)$  of highest weight  $\lambda \in \mathfrak{h}^*$  and the irreducible highest weight module  $L(\lambda)$  of highest weight  $\lambda$ . It is known that  $L(\lambda)$  is the quotient module  $M(\lambda)/J(\lambda)$ , where  $J(\lambda)$  is the unique maximal proper submodule of  $M(\lambda)$ .

It is easy to see that the module  $(\pi_\lambda^\omega, M(\lambda))$  is torsion free as a  $U(\mathfrak{n}_-)$ -module, and the module  $(\pi_\lambda^\omega, L(\lambda))$  is an irreducible  $\mathfrak{g}$ -module. Therefore, in both cases where  $V(\lambda) = M(\lambda)$  and  $L(\lambda)$ ,  $(\pi_\lambda^\omega, V(\lambda))$  and  $(\pi_{\omega^*(\lambda)}, V(\omega^*(\lambda)))$  are isomorphic as  $\mathfrak{g}$ -modules. In other words, there exists a linear isomorphism

$$(4.1) \quad \tau_\omega : V(\lambda) \rightarrow V(\omega^*(\lambda))$$

satisfying  $\tau_\omega(\pi_\lambda^\omega(x)v) = \pi_{\omega^*(\lambda)}(x)\tau_\omega(v)$  for  $x \in \mathfrak{g}$ ,  $v \in V(\lambda)$ , or equivalently,  $\tau_\omega(\pi_\lambda(x)v) = \pi_{\omega^*(\lambda)}(\omega^{-1}(x))\tau_\omega(v)$  for  $x \in \mathfrak{g}$ ,  $v \in V(\lambda)$ . Now we assume that  $\lambda \in \mathfrak{h}^*$  is a symmetric weight, i.e.,  $\omega^*(\lambda) = \lambda$ . Then there exists a linear automorphism

$$(4.2) \quad \tau_\omega : V(\lambda) \rightarrow V(\lambda)$$

such that  $\tau_\omega(\pi_\lambda(x)v) = \pi_\lambda(\omega^{-1}(x))\tau_\omega(v)$  for  $x \in \mathfrak{g}$ ,  $v \in V(\lambda)$ . We usually write  $xv$  to denote  $\pi_\lambda(x)v$  for  $x \in \mathfrak{g}$ ,  $v \in V(\lambda)$ .

*Remark 4.1.* Let  $\lambda \in (\mathfrak{h}^*)^0$ , and let  $f : V(\lambda) \rightarrow V(\lambda)$  be a linear endomorphism such that  $f(xv) = \omega^{-1}(x)f(v)$  for  $x \in \mathfrak{g}$ ,  $v \in V(\lambda)$ . Then we have  $f(V(\lambda)_\mu) \subset V(\lambda)_{\omega^*(\mu)}$  for  $\mu \in \mathfrak{h}^*$ , where  $V(\lambda)_\chi$  is the weight space corresponding to  $\chi \in \mathfrak{h}^*$ . If, in addition, the linear endomorphism  $f$  is bijective, then the equality holds, i.e.,  $f(V(\lambda)_\mu) = V(\lambda)_{\omega^*(\mu)}$  for  $\mu \in \mathfrak{h}^*$ .

Since  $V(\lambda)_\lambda = \mathbb{C}v_\lambda$  and  $\omega^*(\lambda) = \lambda$ , it follows that  $\tau_\omega(v_\lambda) \in \mathbb{C}v_\lambda$  by Remark 4.1. Hence we have  $\tau_\omega(v_\lambda) = cv_\lambda$  for some  $c \in \mathbb{C} \setminus \{0\}$ . We should note that  $c \in \mathbb{C} \setminus \{0\}$  is not necessarily equal to 1.

**Lemma 4.2.** *Let  $\lambda \in (\mathfrak{h}^*)^0$ . Then we have*

$$\{f \in \text{End}_{\mathbb{C}}(V(\lambda)) \mid f(xv) = \omega^{-1}(x)f(v) \text{ for } x \in \mathfrak{g}, v \in V(\lambda)\} = \mathbb{C}\tau_\omega.$$

*Proof.* Let  $f \in \text{End}_{\mathbb{C}}(V(\lambda))$  be such that  $f(xv) = \omega^{-1}(x)f(v)$  for  $x \in \mathfrak{g}$ ,  $v \in V(\lambda)$ . Then, by Remark 4.1,  $f(v_\lambda) = zv_\lambda$  for some  $z \in \mathbb{C}$ . Since  $V(\lambda)$  is a highest weight  $\mathfrak{g}$ -module, we have  $V(\lambda) = U(\mathfrak{g})v_\lambda$ , where  $U(\mathfrak{g})$  denotes the universal enveloping algebra of  $\mathfrak{g}$ . Thus we have

$$f(xv_\lambda) = \omega^{-1}(x)f(v_\lambda) = z\omega^{-1}(x)v_\lambda \text{ for all } x \in U(\mathfrak{g}),$$

where  $\omega$  above is a unique algebra automorphism  $\omega : U(\mathfrak{g}) \rightarrow U(\mathfrak{g})$  extending the diagram automorphism  $\omega : \mathfrak{g} \rightarrow \mathfrak{g}$ . Hence we obtain

$$f(xv_\lambda) = (z/c) \cdot c\omega^{-1}(x)v_\lambda = (z/c)\tau_\omega(xv_\lambda) \text{ for all } x \in U(\mathfrak{g}),$$

which implies that  $f = (z/c)\tau_\omega$ . □

By Remark 4.1 and Lemma 4.2, we see that there exists a unique linear automorphism  $f : V(\lambda) \rightarrow V(\lambda)$  such that  $f(xv) = \omega^{-1}(x)f(v)$  for  $x \in \mathfrak{g}$ ,  $v \in V(\lambda)$  and  $f(v) = v$  for all  $v \in V(\lambda)_\lambda$ . Because  $\tau_\omega$  in (4.2) is a nonzero scalar multiple of this linear automorphism  $f$ , we may (and will henceforth) assume that  $\tau_\omega = f$ , i.e.,  $\tau_\omega(v) = v$  for all  $v \in V(\lambda)_\lambda$ .

We are ready to give the definition of twining characters.

**Definition 4.3** ([FRS, Def. 2.3]). Let  $\lambda \in (\mathfrak{h}^*)^0$ , and let  $V(\lambda)$  be either the Verma module  $M(\lambda)$  or the irreducible highest weight module  $L(\lambda)$ . Then the twining character  $\text{ch}^\omega(V(\lambda))$  of  $V(\lambda)$  is defined to be the formal sum

$$\text{ch}^\omega(V(\lambda)) := \sum_{\chi \in (\mathfrak{h}^*)^0} \text{Tr}(\tau_\omega|_{V(\lambda)_\chi}) e(\chi).$$

For the twining character  $\text{ch}^\omega(V(\lambda))$  of  $V(\lambda) = M(\lambda), L(\lambda)$  of highest weight  $\lambda \in (\mathfrak{h}^*)^0$ , we know the following theorems.

**Theorem 4.4** ([FRS, Th. 3.1]). *Let  $\lambda \in (\mathfrak{h}^*)^0$ . Then we have*

$$\text{ch}^\omega(M(\lambda)) = e(\lambda) \cdot \left( \sum_{w \in \widetilde{W}} \widehat{\varepsilon}(w) e(w(\rho) - \rho) \right)^{-1},$$

where  $\widehat{\varepsilon}(w) := (-1)^{\widehat{\ell}(w)} = \det(w|_{(\mathfrak{h}^*)^0})$  for  $w \in \widetilde{W}$ .

Let  $P_+ := \{\Lambda \in \mathfrak{h}^* \mid \Lambda(h_i) \in \mathbb{Z}_{\geq 0} \text{ for all } i \in I\}$  be the set of dominant integral weights.

**Theorem 4.5** ([FRS, Prop. 3.5 and Th. 3.1]). *Let  $\Lambda \in P_+$  be a dominant integral weight such that  $\omega^*(\Lambda) = \Lambda$ . Then, for every  $w \in \widetilde{W}$ , we have*

$$w(\text{ch}^\omega(L(\Lambda))) = \text{ch}^\omega(L(\Lambda)).$$

Moreover, with the same notation as Theorem 4.4, we have

$$\text{ch}^\omega(L(\Lambda)) = \frac{\sum_{w \in \widetilde{W}} \widehat{\varepsilon}(w) e(w(\Lambda + \rho))}{\sum_{w \in \widetilde{W}} \widehat{\varepsilon}(w) e(w(\rho))}.$$

## 5. TRANSLATION FUNCTORS

We show some important properties of the translation functor concerning the twining characters for the affine Lie algebra  $\mathfrak{g} = \mathfrak{g}(A)$ .

**5.1. Some categories.** Let us begin by recalling the definition of the category  $\mathcal{O}$  from [Ka, Ch. 9]. Its objects are  $\mathfrak{g}$ -modules  $V$  which satisfy the following:

- (1) the module  $V$  admits a weight space decomposition  $V = \bigoplus_{\chi \in \mathfrak{h}^*} V_\chi$  with finite-dimensional weight spaces  $V_\chi$ ;
- (2) there exist finitely many elements  $\chi_1, \dots, \chi_s \in \mathfrak{h}^*$  such that the set  $P(V)$  of all weights of  $V$  is contained in a union  $\bigcup_{i=1}^s (\chi_i - Q_+)$ .

The morphisms in  $\mathcal{O}$  are  $\mathfrak{g}$ -module homomorphisms. For a  $\mathfrak{g}$ -module  $V$  in  $\mathcal{O}$  and  $\mu \in \mathfrak{h}^*$ , we denote by  $[V : L(\mu)]$  the multiplicity of  $L(\mu)$  in  $V$ . We note that  $[V : L(\mu)] > 0$  if and only if  $L(\mu)$  is an irreducible subquotient of  $V$ . We define the character  $\text{ch} V$  of the module  $V$  in  $\mathcal{O}$  to be the formal sum  $\text{ch} V = \sum_{\chi \in \mathfrak{h}^*} (\dim_{\mathbb{C}} V_\chi) e(\chi)$ . Then we have (see [Ka, Ch. 9])  $\text{ch} V = \sum_{\mu \in \mathfrak{h}^*} [V : L(\mu)] \text{ch} L(\mu)$ .

For a nonempty subset  $S$  of  $\mathfrak{h}^*$ , we denote by  $\mathcal{O}\{S\}$  the full subcategory of  $\mathcal{O}$  consisting of  $\mathfrak{g}$ -modules  $V$  such that  $[V : L(\mu)] > 0$  for  $\mu \in \mathfrak{h}^*$  implies  $\mu \in S$ , i.e.,

$$\text{ch} V = \sum_{\mu \in S} c_\mu \text{ch} L(\mu), \quad c_\mu \in \mathbb{Z}_{\geq 0}.$$

It is clear that submodules, quotient modules, and finite direct sums of modules in  $\mathcal{O}\{S\}$  are again in  $\mathcal{O}\{S\}$ .

Here we need some notation for integral subroot systems. For  $\lambda \in \mathfrak{h}^*$ , we set

$$\Delta(\lambda) := \{\alpha \in \Delta^{re} \mid \lambda(h_\alpha) \in \mathbb{Z}\}.$$

Then the set  $\Delta(\lambda)$  is a “subroot system” of  $\Delta^{re}$ , i.e.,  $r_\alpha(\beta) \in \Delta(\lambda)$  for  $\alpha, \beta \in \Delta(\lambda) \subset \Delta^{re}$  (we refer to [MP, Ch. 5] and [KT1, §2.2] for details about subroot systems). Moreover, we set

$$\Delta(\lambda)_+ := \Delta(\lambda) \cap \Delta_+, \quad W(\lambda) := \langle r_\alpha \mid \alpha \in \Delta(\lambda) \rangle \subset W,$$

and define  $\Pi(\lambda)$  to be the set of all  $\alpha \in \Delta(\lambda)_+$  which cannot be written as a sum of two or more elements from  $\Delta(\lambda)_+$ . From [MP, Ch. 5], [KT1, §2.2], and [KT2, Lem. 2.3], we know the following:

- (1) The set  $\Pi(\lambda)$  is a (not necessarily linearly independent) finite set, which we write as:  $\Pi(\lambda) = \{\phi_j\}_{j \in J}$  for some finite index set  $J$ .
- (2)  $\Delta(\lambda) = \{w(\alpha) \mid w \in W(\lambda), \alpha \in \Pi(\lambda)\}$ .
- (3)  $\Delta(\lambda)_+ \subset \sum_{\alpha \in \Pi(\lambda)} \mathbb{Z}_{\geq 0} \alpha$ .
- (4) The matrix  $A(\lambda) := (2(\beta|\alpha)/(\alpha|\alpha))_{\alpha, \beta \in \Pi(\lambda)}$  is a GCM. In particular, we have  $(\alpha|\beta) \leq 0$  for  $\alpha \neq \beta \in \Pi(\lambda)$ .
- (5) The group  $W(\lambda)$  is a Coxeter group with the canonical generator system  $\{r_\alpha \mid \alpha \in \Pi(\lambda)\}$ . Moreover,  $W(\lambda)$  is isomorphic to the Weyl group  $W^\lambda = \langle r_j^\lambda \mid j \in J \rangle \subset GL((\mathfrak{h}^\lambda)^*)$  of the Kac-Moody algebra  $\mathfrak{g}^\lambda := \mathfrak{g}(A(\lambda))$  associated to the GCM  $A(\lambda)$  with Cartan subalgebra  $\mathfrak{h}^\lambda$ .

We denote the length function of the Coxeter group  $W(\lambda) \cong W^\lambda$  by  $\ell_\lambda : W(\lambda) \rightarrow \mathbb{Z}_{\geq 0}$ .

We write  $\Pi(\lambda) = \{\phi_j\}_{j \in J}$  with  $J$  a finite index set as above. Let  $\Pi^\lambda = \{\beta_j\}_{j \in J} \subset (\mathfrak{h}^\lambda)^*$  be the set of simple roots, and  $(\Pi^\lambda)^\vee = \{\beta_j^\vee\}_{j \in J} \subset \mathfrak{h}^\lambda$  the set of simple coroots of  $\mathfrak{g}^\lambda$ . We set

$$Q^\lambda := \sum_{j \in J} \mathbb{Z} \beta_j \subset (\mathfrak{h}^\lambda)^*, \quad (Q^\lambda)^\vee := \sum_{j \in J} \mathbb{Z} \beta_j^\vee \subset \mathfrak{h}^\lambda,$$

$$Q(\lambda) := \sum_{j \in J} \mathbb{Z} \phi_j \subset \mathfrak{h}^*, \quad Q^\vee(\lambda) := \sum_{j \in J} \mathbb{Z} h_{\phi_j} \subset \mathfrak{h}.$$

Then we know from [MP, Chs. 5.1 and 5.5] that there exist unique  $\mathbb{Z}$ -linear isomorphisms  $\Psi : Q^\lambda \rightarrow Q(\lambda)$  and  $\Psi^\vee : (Q^\lambda)^\vee \rightarrow Q^\vee(\lambda)$  such that  $\Psi(\beta_j) = \phi_j$  and  $\Psi^\vee(\beta_j^\vee) = h_{\phi_j}$  for each  $j \in J$ . Furthermore, we have

$$(5.1.1) \quad x(y) = \Psi(x)(\Psi^\vee(y)) \quad \text{for all } x \in Q^\lambda \text{ and } y \in (Q^\lambda)^\vee.$$

**Lemma 5.1.1.** *The GCM  $A(\lambda) = (2(\phi_j|\phi_i)/(\phi_i|\phi_i))_{i,j \in J}$  decomposes, after re-ordering the indices, into a direct sum of GCMs of finite type and those of affine type.*

*Proof.* Suppose that there exists a subset  $J'$  of  $J$  such that the submatrix  $A(\lambda)_{J'} := (2(\phi_j|\phi_i)/(\phi_i|\phi_i))_{i,j \in J'}$  of  $A(\lambda)$  is of indefinite type. Then, by the classification theorem of GCMs (see [Ka, Ch. 4]), there exists an element  $\beta = \sum_{j \in J'} k_j \beta_j$  such that  $k_j \in \mathbb{Z}_{\geq 1}$  and  $\beta(\beta_j^\vee) < 0$  for all  $j \in J'$ . We set  $\alpha := \Psi(\beta) \in \sum_{j \in J'} \mathbb{Z}_{\geq 1} \phi_j \subset Q_+$ . Then, by (5.1.1), we see that  $0 > \beta(\beta_j^\vee) = \alpha(h_{\phi_j}) = 2(\alpha|\phi_j)/(\phi_j|\phi_j)$  for all

$j \in J'$ . Thus we obtain  $(\alpha|\alpha) < 0$  for some  $\alpha \in \sum_{i \in I} \mathbb{R}\alpha_i$ . This is a contradiction since the bilinear form  $(\cdot|\cdot)$  restricted to  $\sum_{i \in I} \mathbb{R}\alpha_i$  is positive-semidefinite.  $\square$

By Lemma 5.1.1, the GCM  $A(\lambda) = (2(\phi_j|\phi_i)/(\phi_i|\phi_i))_{i,j \in J}$  satisfies the assumption on the GCM  $A = (a_{ij})_{i,j \in I}$  in §3.

*Remark 5.1.2.* We set  $D(\lambda) := \text{diag}(\varepsilon_j^\lambda)_{j \in J}$  with  $\varepsilon_j^\lambda := 2/(\phi_j|\phi_j)$  for  $j \in J$ , and  $B(\lambda) := (b_{ij}^\lambda)_{i,j \in J}$  with  $b_{ij}^\lambda := (\phi_i|\phi_j)$  for  $i, j \in J$ . Then we have  $A(\lambda) = D(\lambda)B(\lambda)$ , where the matrix  $B(\lambda)$  is a symmetric matrix with rational entries  $b_{ij}^\lambda = (\phi_i|\phi_j)$ .

**Lemma 5.1.3.** *Let  $\lambda \in \mathfrak{h}^*$ . Then we have*

- (1)  $\Delta(\omega^*(\lambda)) = \omega^*(\Delta(\lambda))$ .
- (2)  $\Pi(\omega^*(\lambda)) = \omega^*(\Pi(\lambda))$ .
- (3)  $\omega^*W(\lambda)(\omega^*)^{-1} = W(\omega^*(\lambda))$ .
- (4)  $\omega^*(W(\lambda) \circ \lambda) = W(\omega^*(\lambda)) \circ \omega^*(\lambda)$ .

*Proof.* Part (1) follows from Lemma 3.1.4 (4). Part (2) then follows from the definitions of  $\Pi(\lambda)$  and  $\Pi(\omega^*(\lambda))$ . Part (3) follows from part (1) together with Lemma 3.1.4 (4). Part (4) follows from part (3) since, for  $w \in W(\lambda)$ ,

$$\begin{aligned} \omega^*(w(\lambda + \rho) - \rho) &= \omega^*w(\lambda + \rho) - \rho = (\omega^*w(\omega^*)^{-1})\omega^*(\lambda + \rho) - \rho \\ &= (\omega^*w(\omega^*)^{-1})(\omega^*(\lambda) + \rho) - \rho. \end{aligned}$$

This proves the lemma.  $\square$

Now we define

$$\mathcal{C} := \{\lambda \in \mathfrak{h}^* \mid (\lambda + \rho|\delta) \neq 0\} = \{\lambda \in \mathfrak{h}^* \mid (\lambda + \rho)(c) \neq 0\},$$

which does not depend on the choice of the Weyl vector  $\rho$ . Since  $w(\delta) = \delta$  for all  $w \in W$ , the set  $\mathcal{C}$  is stable under the dot-action (3.2.5) of the Weyl group  $W$ .

Here we recall that the null root  $\delta = \sum_{i \in I} a_i \alpha_i \in Q_+$  spans the radical of the bilinear form  $(\cdot|\cdot)$  restricted to  $\sum_{i \in I} \mathbb{R}\alpha_i$ , and the positive integers  $a_i$  for  $i \in I$  are relatively prime. Therefore we deduce that  $\omega^*(\delta) = \delta$ , and hence  $\omega^*(\mathcal{C}) = \mathcal{C}$ .

For  $\lambda \in \mathcal{C}$ , we set

$$\mathcal{O}[\lambda] := \mathcal{O}\{W(\lambda) \circ \lambda\}.$$

**Proposition 5.1.4** ([Ku1, Th. 1.7]). *For each  $\lambda \in \mathcal{C}$ , the Verma module  $M(\lambda)$  is in the category  $\mathcal{O}[\lambda]$ .*

**5.2. Translation functors.** We define an equivalence relation on the set  $\mathcal{C}$  by

$$\lambda \sim \mu \iff \mu \in W(\lambda) \circ \lambda, \quad \lambda, \mu \in \mathcal{C}.$$

Let  $\tilde{\mathcal{C}}$  be the set of (representatives of) equivalence classes of  $\mathcal{C}$  under the equivalence relation  $\sim$ . Then we have the following decomposition result.

**Proposition 5.2.1** ([Ku1, Cor. 2.13 (a)]). *Let  $V$  be a  $\mathfrak{g}$ -module in the category  $\mathcal{O}\{\mathcal{C}\}$ . Then the module  $V$  decomposes as a direct sum*

$$V = \bigoplus_{\lambda \in \tilde{\mathcal{C}}} V[\lambda],$$

where the  $\mathfrak{g}$ -module  $V[\lambda]$  is in the category  $\mathcal{O}[\lambda]$  for  $\lambda \in \tilde{\mathcal{C}}$ . Moreover,  $V[\lambda]$  is the unique maximal submodule of  $V$  that lies in  $\mathcal{O}[\lambda]$ . In particular, this decomposition of  $V$  is unique.

For  $\lambda \in \mathcal{C}$ , we denote by  $P_\lambda$  the projection functor from  $\mathcal{O}\{\mathcal{C}\}$  to  $\mathcal{O}[\lambda]$  defined by

$$P_\lambda(V) := V[\lambda].$$

It follows from Proposition 5.2.1 that the functor  $P_\lambda$  is exact.

To define the translation functor, we need the following:

**Lemma 5.2.2** ([KT2, Lem. 3.4]). *Let  $\lambda, \mu \in \mathcal{C}$ ,  $\Lambda \in \mathfrak{h}^*$ , and  $x \in W$  be such that  $\mu - \lambda = x(\Lambda)$ . Then the  $\mathfrak{g}$ -module  $V \otimes_{\mathbb{C}} L(\Lambda)$  is in the category  $\mathcal{O}\{\mathcal{C}\}$  for every  $\mathfrak{g}$ -module  $V$  in the category  $\mathcal{O}[\lambda]$ .*

Let  $P := \{\lambda \in \mathfrak{h}^* \mid \lambda(h_i) \in \mathbb{Z} \text{ for all } i \in I\}$  be the set of integral weights. Note that for  $\lambda, \mu \in \mathfrak{h}^*$  with  $\mu - \lambda \in P$ , we have  $\Delta(\lambda) = \Delta(\mu)$  since  $h_\alpha \in Q^\vee$  for  $\alpha \in \Delta^{re}$ . The elements  $\lambda, \mu \in \mathcal{C}$  are said to satisfy condition (TR) if

$$(TR) \quad \omega^*(\lambda) = \lambda, \omega^*(\mu) = \mu, \quad x^{-1}(\mu - \lambda) \in P_+ \quad \text{for some } x \in \widetilde{W}.$$

In this case, we set  $\Lambda := x^{-1}(\mu - \lambda) \in P_+$ . Note that the  $W$ -orbit of  $\mu - \lambda$  intersects  $P_+$  in exactly one point by (the proof of) [Ka, Prop. 3.12 b)]. We further see that  $\omega^*(\Lambda) = \Lambda$  since

$$\begin{aligned} \omega^*(\Lambda) &= \omega^*(x^{-1}(\mu - \lambda)) = \omega^*x^{-1}(\mu - \lambda) = x^{-1}\omega^*(\mu - \lambda) \\ &= x^{-1}(\omega^*(\mu - \lambda)) = x^{-1}(\mu - \lambda) = \Lambda. \end{aligned}$$

Let  $\lambda, \mu \in \mathcal{C}$  be elements satisfying condition (TR) with  $\mu - \lambda = x(\Lambda)$  for  $x \in \widetilde{W}$  and  $\Lambda \in P_+$ . Then, by Lemma 5.2.2, the  $\mathfrak{g}$ -module  $V \otimes_{\mathbb{C}} L(\Lambda)$  is in  $\mathcal{O}\{\mathcal{C}\}$  for a  $\mathfrak{g}$ -module  $V$  in  $\mathcal{O}[\lambda]$ , and hence the module  $P_\mu(V \otimes_{\mathbb{C}} L(\Lambda))$  is well-defined and in  $\mathcal{O}[\mu]$  by Proposition 5.2.1. Thus we can define the translation functor following [J] and [DGK].

**Definition 5.2.3.** Let  $\lambda, \mu \in \mathcal{C}$  be elements satisfying condition (TR) with  $\mu - \lambda = x(\Lambda)$  for  $x \in \widetilde{W}$  and  $\Lambda \in P_+$ . Then the functor  $T_\mu^\lambda : \mathcal{O}[\lambda] \rightarrow \mathcal{O}[\mu]$  is defined by

$$T_\mu^\lambda(V) := P_\mu(V \otimes_{\mathbb{C}} L(\Lambda)),$$

for a  $\mathfrak{g}$ -module  $V$  in the category  $\mathcal{O}[\lambda]$ . This (exact) functor  $T_\mu^\lambda$  is called the (Jantzen) translation functor from  $\lambda$  to  $\mu$ .

Now let  $\lambda, \mu \in \mathcal{C}$  be elements satisfying condition (TR) with  $\mu - \lambda = x(\Lambda)$  for  $x \in \widetilde{W}$  and  $\Lambda \in P_+$ , and let  $\eta \in \mathcal{C}$  be a symmetric weight such that  $\eta \sim \lambda$ . Then we have  $\eta = w \circ \lambda$  for some  $w \in W(\lambda)$ . Since  $\omega^*(\eta) = \eta$ , there exists a unique linear automorphism

$$(5.2.1) \quad \tau_\omega : M(\eta) \rightarrow M(\eta),$$

such that

$$(5.2.2) \quad \tau_\omega(xv) = \omega^{-1}(x)\tau_\omega(v) \quad \text{for } x \in \mathfrak{g}, v \in M(\eta)$$

and  $\tau_\omega|_{M(\eta)_\eta} = \text{id}$ . Let  $\{0\} \subset Y_1 \subsetneq Y_2 \subset M(\eta)$  be a filtration of  $M(\eta)$  by  $\tau_\omega$ -stable  $\mathfrak{g}$ -submodules  $Y_1$  and  $Y_2$ , and let  $V := Y_2/Y_1$  be a quotient  $\mathfrak{g}$ -module. Then  $\tau_\omega : M(\eta) \rightarrow M(\eta)$  induces a linear automorphism  $\overline{\tau}_\omega : V \rightarrow V$  satisfying

$$(5.2.3) \quad \overline{\tau}_\omega(xv) = \omega^{-1}(x)\overline{\tau}_\omega(v) \quad \text{for } x \in \mathfrak{g}, v \in V.$$

Since the  $\Lambda$  above is a symmetric weight, there exists a unique linear automorphism

$$(5.2.4) \quad \tau_\omega : L(\Lambda) \rightarrow L(\Lambda)$$

satisfying  $\tau_\omega(xv) = \omega^{-1}(x)\tau_\omega(v)$  for  $x \in \mathfrak{g}$ ,  $v \in L(\Lambda)$  with  $\tau_\omega|_{L(\Lambda)_\Lambda} = \text{id}$ . (Here we have used the same symbol  $\tau_\omega$  as for  $M(\eta)$ , but no serious confusion will arise.) Thus, by tensoring this  $\tau_\omega$  with the map  $\bar{\tau}_\omega : V \rightarrow V$ , we obtain a linear automorphism  $\bar{\tau}_\omega \otimes \tau_\omega : V \otimes_{\mathbb{C}} L(\Lambda) \rightarrow V \otimes_{\mathbb{C}} L(\Lambda)$  satisfying

$$(5.2.5) \quad (\bar{\tau}_\omega \otimes \tau_\omega)(xv) = \omega^{-1}(x)(\bar{\tau}_\omega \otimes \tau_\omega)(v) \quad \text{for } x \in \mathfrak{g}, v \in V \otimes_{\mathbb{C}} L(\Lambda).$$

Since  $V$  is in  $\mathcal{O}[\lambda]$ , the  $\mathfrak{g}$ -module  $V \otimes_{\mathbb{C}} L(\Lambda)$  is in  $\mathcal{O}\{\mathcal{C}\}$  by Lemma 5.2.2. Hence, by Proposition 5.2.1, we have a decomposition

$$V \otimes_{\mathbb{C}} L(\Lambda) = \bigoplus_{\xi \in \tilde{\mathcal{C}}} P_\xi(V \otimes_{\mathbb{C}} L(\Lambda)).$$

**Proposition 5.2.4.** *With the notation above, we have*

$$(\bar{\tau}_\omega \otimes \tau_\omega)(P_\xi(V \otimes_{\mathbb{C}} L(\Lambda))) = P_{\omega^*(\xi)}(V \otimes_{\mathbb{C}} L(\Lambda))$$

for each  $\xi \in \tilde{\mathcal{C}}$ .

*Proof.* First, we note that for  $\xi \in \mathcal{C}$ , its equivalence class under  $\sim$  is  $W(\xi) \circ \xi$ , and the equivalence class of  $\omega^*(\xi)$  under  $\sim$  is  $W(\omega^*(\xi)) \circ \omega^*(\xi) = \omega^*(W(\xi) \circ \xi)$  by Lemma 5.1.3 (4). In other words, for  $\xi, \xi' \in \mathcal{C}$ , we have  $\xi \sim \xi'$  if and only if  $\omega^*(\xi) \sim \omega^*(\xi')$ .

Let us show that the  $\mathfrak{g}$ -module  $(\bar{\tau}_\omega \otimes \tau_\omega)(P_\xi(V \otimes_{\mathbb{C}} L(\Lambda)))$  is in  $\mathcal{O}[\omega^*(\xi)]$  for each  $\xi \in \mathcal{C}$ . To simplify the notation, we write  $f := \bar{\tau}_\omega \otimes \tau_\omega$  and  $U := P_\xi(V \otimes_{\mathbb{C}} L(\Lambda))$ . Then the map  $Y \mapsto f(Y)$  gives a bijection from the  $\mathfrak{g}$ -submodules of  $U$  to those of  $f(U)$ . Furthermore, for  $\mathfrak{g}$ -submodules  $Y_1 \subsetneq Y_2$  of  $U$  with  $Y_2/Y_1 \cong L(\varphi)$  for  $\varphi \in \mathcal{C}$ , we have an isomorphism of  $\mathfrak{g}$ -modules:  $f(Y_2)/f(Y_1) \cong f(Y_2/Y_1) \cong \tau_\omega(L(\varphi)) = L(\omega^*(\varphi))$ , where  $\tau_\omega : L(\varphi) \rightarrow L(\omega^*(\varphi))$  is defined as the map (4.1). Thus, for  $\varphi \in \mathcal{C}$ , the module  $L(\varphi)$  is isomorphic to an irreducible subquotient of  $U$  if and only if  $L(\omega^*(\varphi))$  is isomorphic to an irreducible subquotient of  $f(U)$ . Hence the  $\mathfrak{g}$ -module  $f(U)$  is in  $\mathcal{O}[\omega^*(\xi)]$  since  $U$  is in  $\mathcal{O}[\xi]$ .

Since the decomposition

$$V \otimes_{\mathbb{C}} L(\Lambda) = \bigoplus_{\xi \in \tilde{\mathcal{C}}} P_\xi(V \otimes_{\mathbb{C}} L(\Lambda))$$

is unique by Proposition 5.2.1, we obtain that

$$f(P_\xi(V \otimes_{\mathbb{C}} L(\Lambda))) = P_{\omega^*(\xi)}(V \otimes_{\mathbb{C}} L(\Lambda))$$

for  $\xi \in \tilde{\mathcal{C}}$ , as desired. □

From Proposition 5.2.4, we see that for the symmetric weight  $\mu \in \mathcal{C}$ ,

$$(\bar{\tau}_\omega \otimes \tau_\omega)(P_\mu(V \otimes_{\mathbb{C}} L(\Lambda))) = P_\mu(V \otimes_{\mathbb{C}} L(\Lambda)), \text{ i.e., } (\bar{\tau}_\omega \otimes \tau_\omega)(T_\mu^\lambda(V)) = T_\mu^\lambda(V).$$

Now we define the twining character  $\text{ch}^\omega(T_\mu^\lambda(V))$  of the module  $T_\mu^\lambda(V)$  by

$$\text{ch}^\omega(T_\mu^\lambda(V)) := \sum_{\chi \in (\mathfrak{h}^*)^0} \text{Tr}((\bar{\tau}_\omega \otimes \tau_\omega)|_{(T_\mu^\lambda(V))_\chi}) e(\chi).$$

We set

$$\mathcal{C}^+ := \{\xi \in \mathcal{C} \mid (\xi + \rho)(h_\alpha) \geq 0 \text{ for all } \alpha \in \Delta(\xi)_+\},$$

and for  $\xi \in \mathcal{C}$  we set

$$\Delta_0(\xi) := \{\alpha \in \Delta^{re} \mid (\xi + \rho)(h_\alpha) = 0\} \subset \Delta(\xi), \quad \Delta_0(\xi)_+ := \Delta_0(\xi) \cap \Delta_+.$$

Note that all of the above do not depend on the choice of the Weyl vector  $\rho$ . Then we have the following proposition, which will be crucial for the proof of our main result.

**Proposition 5.2.5.** *Let  $\lambda, \mu \in \mathcal{C}$  be elements satisfying condition (TR) with  $\mu - \lambda = x(\Lambda)$  for  $x \in \widetilde{W}$  and  $\Lambda \in P_+$ . Assume, in addition, that  $\lambda, \mu \in \mathcal{C}^+$  and  $\Delta_0(\lambda) \subset \Delta_0(\mu)$ . Then we have for each  $w \in W(\lambda) \cap \widetilde{W}$ ,*

$$\text{ch}^\omega(T_\mu^\lambda(M(w \circ \lambda))) = \text{ch}^\omega(M(w \circ \mu)).$$

*Proof.* Let  $w \in W(\lambda) \cap \widetilde{W}$ , and set  $\eta := w \circ \lambda \in \mathcal{C}$ . Since  $x, w \in \widetilde{W}$ , we have  $\omega^*(\Lambda) = \Lambda$  and  $\omega^*(\eta) = \eta$ . Hence we have a linear automorphism

$$\overline{\tau}_\omega \otimes \tau_\omega : M(\eta) \otimes_{\mathbb{C}} L(\Lambda) \rightarrow M(\eta) \otimes_{\mathbb{C}} L(\Lambda)$$

satisfying (5.2.5) with  $V = M(\eta)$ . We define for  $\chi = \Lambda - \sum_{i \in I} k_i \alpha_i \in P(L(\Lambda))$ ,  $\text{depth}_\Lambda(\chi) := \sum_{i \in I} k_i \in \mathbb{Z}_{\geq 0}$ . We write  $P(L(\Lambda)) = \{\chi_i\}_{i \in \mathbb{Z}_{\geq 1}}$  and arrange them so that

- (1)  $\chi_1 = \Lambda$ ;
- (2)  $\text{depth}_\Lambda(\chi_i) > \text{depth}_\Lambda(\chi_j)$  implies  $i > j$ .

Now, for each  $k \in \mathbb{Z}_{\geq 1}$ , we set  $P(\Lambda)_k := \{\chi \in P(L(\Lambda)) \mid \text{depth}_\Lambda(\chi) = k\}$ . Here we note that  $\text{depth}_\Lambda(\omega^*(\chi)) = \text{depth}_\Lambda(\chi)$  for  $\chi \in P(L(\Lambda))$ . Hence we can reindex the elements in  $P(\Lambda)_k$  so that the elements in the same  $\omega^*$ -orbit are indexed by consecutive numbers for all  $k \in \mathbb{Z}_{\geq 1}$ . Then we take a basis  $\{v_i\}_{i \in \mathbb{Z}_{\geq 1}}$  of  $L(\Lambda)$  consisting of weight vectors arranged in such a way that

- (1)  $L(\Lambda)_\Lambda = \mathbb{C}v_1$ ;
- (2) if  $v_i$  is of weight  $\chi_{t_i} \in P(L(\Lambda))$  with  $t_i \in \mathbb{Z}_{\geq 1}$  for  $i \geq 1$ , then  $i > j$  implies  $t_i \geq t_j$ .

We set for  $i \geq 1$ ,  $R_i := \sum_{1 \leq j \leq i} U(\mathfrak{g})(v_\eta \otimes v_j)$ , where  $v_\eta$  is the highest weight vector of  $M(\eta)$ . Then, by [DGK, Lem. 5.8] (see also the proof of [MP, Prop. 6.8.1]), the  $\mathfrak{g}$ -module  $R := M(\eta) \otimes_{\mathbb{C}} L(\Lambda)$  has a highest weight series  $\{0\} = R_0 \subset R_1 \subset R_2 \subset \dots \subset R$  such that

- (1)  $R = \bigcup_{i \geq 0} R_i$  with  $\text{ch } R = \sum_{i \geq 1} \text{ch}(R_i/R_{i-1})$ ;
- (2)  $R_i/R_{i-1} \cong M(\eta + \chi_{t_i})$  for  $i \geq 1$ .

By applying the exact functor  $P_\mu$ , we obtain the following filtration of  $P_\mu(R) = T_\mu^\lambda(M(\eta))$ :  $\{0\} = P_\mu(R_0) \subset P_\mu(R_1) \subset P_\mu(R_2) \subset \dots \subset P_\mu(R)$  such that  $P_\mu(R) = \bigcup_{i \geq 0} P_\mu(R_i)$ . We have for  $i \geq 1$ ,  $P_\mu(R_i)/P_\mu(R_{i-1}) \cong P_\mu(R_i/R_{i-1}) \cong P_\mu(M(\eta + \chi_{t_i}))$ . Here  $\eta + \chi_{t_i} = w \circ \lambda + \chi_{t_i} \in \mathcal{C}$  is equivalent to  $\mu$  under  $\sim$  if and only if  $w \circ \lambda + \chi_{t_i} = y \circ \mu$  for some  $y \in W(\mu) = W(\lambda)$ . In this case, we have  $w^{-1}y \circ \mu - \lambda = w^{-1}(\chi_{t_i}) \in P(L(\Lambda))$ . Then, by [KT2, Lem. 3.5], we see that  $w^{-1}y \circ \mu = \mu$ . Hence we have  $\chi_{t_i} = w(\mu - \lambda) = wx(\Lambda)$ , and  $w \circ \lambda + \chi_{t_i} = w \circ \mu$ . We know that  $\dim_{\mathbb{C}} L(\Lambda)_{\chi_{t_i}} = \dim_{\mathbb{C}} L(\Lambda)_{wx(\Lambda)} = \dim_{\mathbb{C}} L(\Lambda)_\Lambda = 1$ . Therefore, by Proposition 5.1.4, we conclude that there exists a unique integer  $i_0 \geq 1$  such that  $P_\mu(R_{i_0})/P_\mu(R_{i_0-1}) \neq \{0\}$ . Furthermore, we have  $w \circ \lambda + \chi_{t_{i_0}} = w \circ \mu$  with

$\chi_{t_{i_0}} = wx(\Lambda)$  and  $P_\mu(R_{i_0})/P_\mu(R_{i_0-1}) \cong M(w \circ \mu)$ . In particular, we have  $P_\mu(R_i) = P_\mu(R_{i-1})$  for all  $i \neq i_0$ . Hence we deduce that

$$\{0\} = P_\mu(R_0) = \cdots = P_\mu(R_{i_0-1}) \subsetneq P_\mu(R_{i_0}) = P_\mu(R_{i_0+1}) = \cdots \cong M(w \circ \mu),$$

and so

$$T_\mu^\lambda(M(w \circ \lambda)) = P_\mu(R) = \bigcup_{i \geq 0} P_\mu(R_i) = P_\mu(R_{i_0}) \cong M(w \circ \mu).$$

Thus, by Lemma 4.2 together with Proposition 5.2.4, we have the following commutative diagram for some  $c \in \mathbb{C} \setminus \{0\}$ :

$$\begin{array}{ccc} T_\mu^\lambda(M(w \circ \lambda)) & \xrightarrow{\cong} & M(w \circ \mu) \\ \overline{\tau_\omega} \otimes \tau_\omega \downarrow & & \downarrow c\tau_\omega \\ T_\mu^\lambda(M(w \circ \lambda)) & \xrightarrow{\cong} & M(w \circ \mu). \end{array}$$

We want to prove that  $c = 1$ . For each  $\xi \not\sim \mu \in \mathcal{C}$ , we have by Proposition 5.1.4,

$$P_\xi(R_{i_0})/P_\xi(R_{i_0-1}) \cong P_\xi(R_{i_0}/R_{i_0-1}) \cong P_\xi(M(w \circ \mu)) = \{0\},$$

and hence  $P_\xi(R_{i_0}) = P_\xi(R_{i_0-1})$ . On the other hand, we have by Proposition 5.2.1,

$$R_{i_0} = P_\mu(R_{i_0}) \oplus \left( \bigoplus_{\substack{\xi \in \tilde{\mathcal{C}} \\ \xi \not\sim \mu}} P_\xi(R_{i_0}) \right), \quad R_{i_0-1} = \bigoplus_{\substack{\xi \in \tilde{\mathcal{C}} \\ \xi \not\sim \mu}} P_\xi(R_{i_0-1}),$$

since  $P_\mu(R_{i_0-1}) = \{0\}$ . Thus we have  $R_{i_0} = R_{i_0-1} \oplus P_\mu(R_{i_0})$ , where  $P_\mu(R_{i_0}) = T_\mu^\lambda(M(w \circ \lambda))$ . We know from the proof of [MP, Prop. 6.8.1] that for  $i \geq 1$ ,  $R_i = \sum_{1 \leq j \leq i} (U(\mathfrak{n}_-)v_\eta) \otimes \mathbb{C}v_j$ . Recall that  $v_{i_0}$  is a weight vector of weight  $\chi_{t_{i_0}} = wx(\Lambda)$ , where  $\omega^*(\chi_{t_{i_0}}) = \chi_{t_{i_0}}$ . Hence we deduce from the indexing of  $\{v_i\}_{i \in \mathbb{Z}_{\geq 1}}$  that

$$(\overline{\tau_\omega} \otimes \tau_\omega)(R_{i_0}) = R_{i_0}, \quad (\overline{\tau_\omega} \otimes \tau_\omega)(R_{i_0-1}) = R_{i_0-1},$$

since  $\tau_\omega(v_j) \in L(\Lambda)_{\omega^*(\chi_{t_j})}$  for  $j \geq 1$ . Also,  $P_\mu(R_{i_0}) = T_\mu^\lambda(M(w \circ \lambda))$  is  $\overline{\tau_\omega} \otimes \tau_\omega$ -stable by Proposition 5.2.4. Thus we have the following commutative diagram:

$$\begin{array}{ccc} R_{i_0}/R_{i_0-1} & \xrightarrow{\cong} & P_\mu(R_{i_0}) \\ \overline{\tau_\omega} \otimes \tau_\omega \downarrow & & \downarrow \overline{\tau_\omega} \otimes \tau_\omega \\ R_{i_0}/R_{i_0-1} & \xrightarrow{\cong} & P_\mu(R_{i_0}), \end{array}$$

where the left vertical map  $\overline{\tau_\omega} \otimes \tau_\omega$  is induced from the map  $\overline{\tau_\omega} \otimes \tau_\omega : R_{i_0} \rightarrow R_{i_0}$ . We know from the proof of [MP, Prop. 6.8.1] that the quotient  $\mathfrak{g}$ -module  $R_{i_0}/R_{i_0-1}$  is a highest weight module (in fact, a Verma module) of highest weight  $\eta + \chi_{t_{i_0}}$  with highest weight vector  $v_\eta \otimes v_{i_0} + R_{i_0-1}$ . Hence, in order to prove that  $c = 1$ , it suffices to show that  $(\overline{\tau_\omega} \otimes \tau_\omega)(v_\eta \otimes v_{i_0}) = v_\eta \otimes v_{i_0}$ . Since we have  $(\overline{\tau_\omega} \otimes \tau_\omega)(v_\eta \otimes v_{i_0}) = \overline{\tau_\omega}(v_\eta) \otimes \tau_\omega(v_{i_0}) = v_\eta \otimes \tau_\omega(v_{i_0})$  and  $\chi_{t_{i_0}} = wx(\Lambda)$ , we need only show that  $\tau_\omega|_{L(\Lambda)_{wx(\Lambda)}} = \text{id}$ . Since  $wx \in \widetilde{W}$  and  $\Lambda \in P_+$ , we have by Theorem 4.5 that  $(wx)^{-1} \text{ch}^\omega(L(\Lambda)) = \text{ch}^\omega(L(\Lambda))$ . In particular, we have  $\text{Tr}(\tau_\omega|_{L(\Lambda)_{wx(\Lambda)}}) = \text{Tr}(\tau_\omega|_{L(\Lambda)_\Lambda}) = 1$ . Since  $\dim_{\mathbb{C}} L(\Lambda)_{wx(\Lambda)} = \dim_{\mathbb{C}} L(\Lambda)_\Lambda = 1$ , we deduce that  $\tau_\omega|_{L(\Lambda)_{wx(\Lambda)}} = \text{id}$ .  $\square$

**Corollary 5.2.6.** *Let  $\lambda, \mu \in \mathcal{C}$  be as in Proposition 5.2.5. Then, for  $w \in W(\lambda) \cap \widetilde{W}$ , we have*

$$\text{ch}^\omega(T_\mu^\lambda(L(w \circ \lambda))) = \begin{cases} \text{ch}^\omega(L(w \circ \mu)) & \text{if } w(\Delta_0(\mu)_+ \setminus \Delta_0(\lambda)_+) \subset -\Delta(\lambda)_+, \\ 0 & \text{otherwise.} \end{cases}$$

*Proof.* We set  $\eta := w \circ \lambda$ . Since  $\eta$  is a symmetric weight, we have an exact sequence of  $\overline{\tau}_\omega$ -stable  $\mathfrak{g}$ -modules:  $\{0\} \rightarrow J(\eta) \hookrightarrow M(\eta) \twoheadrightarrow L(\eta) \rightarrow \{0\}$ , where  $J(\eta)$  is the unique maximal proper submodule of  $M(\eta)$ . Note that the map  $\tau_\omega : L(\eta) \rightarrow L(\eta)$  in (4.2) is nothing but the map  $\overline{\tau}_\omega : M(\eta)/J(\eta) \rightarrow M(\eta)/J(\eta)$  induced from the map  $\tau_\omega : M(\eta) \rightarrow M(\eta)$  in (5.2.1). Since the functor  $(\cdot) \otimes_{\mathbb{C}} L(\Lambda)$  is exact, we obtain an exact sequence of  $\overline{\tau}_\omega \otimes \tau_\omega$ -stable  $\mathfrak{g}$ -modules

$$(5.2.6) \quad \{0\} \rightarrow J(\eta) \otimes_{\mathbb{C}} L(\Lambda) \hookrightarrow M(\eta) \otimes_{\mathbb{C}} L(\Lambda) \twoheadrightarrow L(\eta) \otimes_{\mathbb{C}} L(\Lambda) \rightarrow \{0\}.$$

Since  $\omega^*(\mu) = \mu$ , by Proposition 5.2.4 we have an exact sequence of  $\overline{\tau}_\omega \otimes \tau_\omega$ -stable  $\mathfrak{g}$ -modules

$$\{0\} \rightarrow T_\mu^\lambda(J(\eta)) \hookrightarrow T_\mu^\lambda(M(\eta)) \twoheadrightarrow T_\mu^\lambda(L(\eta)) \rightarrow \{0\}.$$

From (5.2.6) we have the commutative diagram

$$\begin{array}{ccc} (M(\eta) \otimes_{\mathbb{C}} L(\Lambda))/(J(\eta) \otimes_{\mathbb{C}} L(\Lambda)) & \xrightarrow{\cong} & (M(\eta)/J(\eta)) \otimes_{\mathbb{C}} L(\Lambda) \\ \overline{\overline{\tau}_\omega \otimes \tau_\omega} \downarrow & & \downarrow \overline{\tau}_\omega \otimes \tau_\omega \\ (M(\eta) \otimes_{\mathbb{C}} L(\Lambda))/(J(\eta) \otimes_{\mathbb{C}} L(\Lambda)) & \xrightarrow{\cong} & (M(\eta)/J(\eta)) \otimes_{\mathbb{C}} L(\Lambda), \end{array}$$

where the left vertical map  $\overline{\overline{\tau}_\omega \otimes \tau_\omega}$  is induced from the map  $\overline{\tau}_\omega \otimes \tau_\omega : M(\eta) \otimes_{\mathbb{C}} L(\Lambda) \rightarrow M(\eta) \otimes_{\mathbb{C}} L(\Lambda)$ . Since we have the following commutative diagram from the exactness of  $P_\mu$  together with Proposition 5.2.4,

$$\begin{array}{ccc} P_\mu(M(\eta) \otimes_{\mathbb{C}} L(\Lambda))/P_\mu(J(\eta) \otimes_{\mathbb{C}} L(\Lambda)) & \xrightarrow{\cong} & P_\mu((M(\eta) \otimes_{\mathbb{C}} L(\Lambda))/(J(\eta) \otimes_{\mathbb{C}} L(\Lambda))) \\ \overline{\overline{\tau}_\omega \otimes \tau_\omega} \downarrow & & \downarrow \overline{\overline{\tau}_\omega \otimes \tau_\omega} \\ P_\mu(M(\eta) \otimes_{\mathbb{C}} L(\Lambda))/P_\mu(J(\eta) \otimes_{\mathbb{C}} L(\Lambda)) & \xrightarrow{\cong} & P_\mu((M(\eta) \otimes_{\mathbb{C}} L(\Lambda))/(J(\eta) \otimes_{\mathbb{C}} L(\Lambda))), \end{array}$$

we obtain the commutative diagram

$$\begin{array}{ccc} T_\mu^\lambda(M(\eta))/T_\mu^\lambda(J(\eta)) & \xrightarrow{\cong} & T_\mu^\lambda(M(\eta)/J(\eta)) \\ \overline{\overline{\tau}_\omega \otimes \tau_\omega} \downarrow & & \downarrow \overline{\tau}_\omega \otimes \tau_\omega \\ T_\mu^\lambda(M(\eta))/T_\mu^\lambda(J(\eta)) & \xrightarrow{\cong} & T_\mu^\lambda(M(\eta)/J(\eta)), \end{array}$$

where the left vertical map  $\overline{\overline{\tau}_\omega \otimes \tau_\omega}$  is induced from the map  $\overline{\tau}_\omega \otimes \tau_\omega : T_\mu^\lambda(M(\eta)) \rightarrow T_\mu^\lambda(M(\eta))$ . By (the proof of) Proposition 5.2.5, we have that  $T_\mu^\lambda(M(\eta)) \cong M(w \circ \mu)$ . Furthermore, by [KT2, Prop. 3.8] we know that

$$T_\mu^\lambda(J(\eta)) = \begin{cases} J & \text{if } w(\Delta_0(\mu)_+ \setminus \Delta_0(\lambda)_+) \subset -\Delta(\lambda)_+, \\ T_\mu^\lambda(M(\eta)) & \text{otherwise,} \end{cases}$$

where  $J$  is the unique maximal proper submodule of  $T_\mu^\lambda(M(\eta)) \cong M(w \circ \mu)$ .

Now assume that  $w(\Delta_0(\mu)_+ \setminus \Delta_0(\lambda)_+) \subset -\Delta(\lambda)_+$ . (Otherwise, it is obvious that  $\text{ch}^\omega(T_\mu^\lambda(L(\eta))) = \text{ch}^\omega(T_\mu^\lambda(M(\eta)/J(\eta))) = \text{ch}^\omega(T_\mu^\lambda(M(\eta))/T_\mu^\lambda(J(\eta))) = 0$ .) Then,

by the proof of Proposition 5.2.5, we have the following commutative diagram:

$$\begin{CD} T_\mu^\lambda(M(\eta))/T_\mu^\lambda(J(\eta)) @>\simeq>> M(w \circ \mu)/J(w \circ \mu) \\ @V{\overline{\tau_\omega} \otimes \tau_\omega}VV @VV{\tau_\omega}V \\ T_\mu^\lambda(M(\eta))/T_\mu^\lambda(J(\eta)) @>\simeq>> M(w \circ \mu)/J(w \circ \mu), \end{CD}$$

where  $J(w \circ \mu)$  is the unique maximal proper submodule of  $M(w \circ \mu)$ . Consequently, we obtain  $\text{ch}^\omega(T_\mu^\lambda(L(\eta))) = \text{ch}^\omega(T_\mu^\lambda(M(\eta))/T_\mu^\lambda(J(\eta))) = \text{ch}^\omega(L(w \circ \mu))$ , as desired.  $\square$

6. TWINING CHARACTER FORMULA OF KAC-WAKIMOTO TYPE

We prove a formula of Kac-Wakimoto type for twining characters.

**6.1. Suto filtration.** We will use a partial ordering  $\leq$  on  $\mathfrak{h}^*$  defined by

$$\chi \leq \chi' \iff \chi' - \chi \in Q_+, \quad \chi, \chi' \in \mathfrak{h}^*.$$

Let  $\eta \in (\mathfrak{h}^*)^0$ . Then, as in §5.2, there exists a unique linear automorphism  $\tau_\omega : M(\eta) \rightarrow M(\eta)$  satisfying (5.2.2) with  $\tau_\omega|_{M(\eta)_\eta} = \text{id}$ . Let  $\{0\} \subset Y_1 \subsetneq Y_2 \subset M(\eta)$  be a filtration of  $M(\eta)$  by  $\tau_\omega$ -stable  $\mathfrak{g}$ -submodules  $Y_1$  and  $Y_2$ , and  $\bar{V} := Y_2/Y_1$  a quotient  $\mathfrak{g}$ -module. Then, again as in §5.2,  $\tau_\omega : M(\eta) \rightarrow M(\eta)$  induces a linear automorphism  $\overline{\tau_\omega} : \bar{V} \rightarrow \bar{V}$  satisfying (5.2.3).

I owe the proof of the next proposition to Professor Kiyokazu Suto, so the filtration given below is called a Suto filtration.

**Proposition 6.1.1.** *We keep the notation above. Let  $\varphi \in (\mathfrak{h}^*)^0$  be a symmetric weight. Then there exist a finite sequence  $V_0, V_1, \dots, V_t, t \geq 1$ , of  $\mathfrak{g}$ -submodules of  $V$  and a subset  $T \subset \{1, \dots, t\}$  with the following properties:*

- (1)  $V = V_t \supset V_{t-1} \supset \dots \supset V_1 \supset V_0 = \{0\}$ ;
- (2)  $\overline{\tau_\omega}(V_i) \subset V_i$  for all  $0 \leq i \leq t$ ;
- (3) if  $i \notin T$ , then the quotient  $\mathfrak{g}$ -module  $V_i/V_{i-1}$  has no weight  $\chi \geq \varphi$ ;
- (4) if  $i \in T$ , then there exist  $p_i \in \mathbb{Z}_{\geq 1}, \xi_i \geq \varphi$ , and  $\mathfrak{g}$ -submodules  $V_{i,0}, \dots, V_{i,p_i-1}$  of  $V_i$  such that

$$V_i = \sum_{k=0}^{p_i-1} V_{i,k}, \quad V_i/V_{i-1} = \bigoplus_{k=0}^{p_i-1} \pi_i(V_{i,k}),$$

$$\overline{\tau_\omega}(V_{i,k}) = V_{i,k+1}, \quad \pi_i(V_{i,k}) \cong L((\omega^*)^k(\xi_i)) \quad \text{for } 0 \leq k \leq p_i - 1,$$

where  $\pi_i : V_i \rightarrow V_i/V_{i-1}$  is the natural quotient map,  $p_i$  is the smallest positive integer  $p$  such that  $(\omega^*)^p(\xi_i) = \xi_i$ , and  $V_{i,p_i} := V_{i,0}$ .

*Proof.* We set

$$a(V, \varphi) := \sum_{\substack{\xi \in \mathfrak{h}^* \\ \xi \geq \varphi}} \dim_{\mathbb{C}} V_\xi.$$

We prove the proposition by induction on  $a(V, \varphi)$ . If  $a(V, \varphi) = 0$ , then  $V = V_1 \supset V_0 = \{0\}$  is the required filtration with  $T = \emptyset$ . Let  $a(V, \varphi) > 0$ . Choose a maximal element  $\xi \in P(V)$  with respect to the partial ordering  $\leq$  on  $\mathfrak{h}^*$  such that  $\xi \geq \varphi$ , and take the smallest positive integer  $p$  for which  $(\omega^*)^p(\xi) = \xi$ . Then we have  $(\overline{\tau_\omega})^p(V_\xi) = V_{(\omega^*)^p(\xi)} = V_\xi$ . Since the  $\xi$ -weight space  $V_\xi$  is a finite-dimensional

vector space over  $\mathbb{C}$ , there exist  $c_\xi \in \mathbb{C} \setminus \{0\}$  and  $0 \neq v \in V_\xi$  such that  $(\bar{\tau}_\omega)^p(v) = c_\xi v$ . Note that we have  $\mathfrak{n}_+ v = 0$  from the maximality of  $\xi \in P(V)$ . We set  $v_k := (\bar{\tau}_\omega)^k(v) \in V_{(\omega^*)^k(\xi)}$  for  $0 \leq k \leq p-1$  (so  $v_0 = v$ ). We note that for each  $0 \leq k \leq p-1$ , the weight  $(\omega^*)^k(\xi)$  is also a maximal element in  $P(V)$  since  $\chi \geq \chi'$  implies  $\omega^*(\chi) \geq \omega^*(\chi')$ . Hence we have  $\mathfrak{n}_+ v_k = 0$  for all  $0 \leq k \leq p-1$ . Set  $U_k := U(\mathfrak{g})v_k$  for  $0 \leq k \leq p-1$ . Then the  $\mathfrak{g}$ -module  $U_k$  is a highest weight module of highest weight  $(\omega^*)^k(\xi)$  with highest weight vector  $v_k$ . We also have  $\bar{\tau}_\omega(U_k) = U_{k+1}$  for  $0 \leq k \leq p-1$ , where  $U_p := U_0$  since  $\bar{\tau}_\omega(xv_k) = \omega^{-1}(x)\bar{\tau}_\omega(v_k)$  for  $x \in U(\mathfrak{g})$  and  $0 \leq k \leq p-1$ , and  $\bar{\tau}_\omega(v_k) = v_{k+1}$  for  $0 \leq k \leq p-2$  with  $\bar{\tau}_\omega(v_{p-1}) = (\bar{\tau}_\omega)^p(v_0) = c_\xi v_0$ . For each  $0 \leq k \leq p-1$ , let  $J_k$  be the unique maximal proper submodule of  $U_k$ , so that we have  $U_k/J_k \cong L((\omega^*)^k(\xi))$ . Note that we have  $\chi \not\geq (\omega^*)^k(\xi)$  for all  $\chi \in P(J_k)$ . We see that  $\bar{\tau}_\omega(J_k) = J_{k+1}$  for  $0 \leq k \leq p-1$ , where  $J_p := J_0$ .

We set

$$U := \sum_{k=0}^{p-1} U_k, \quad J := \sum_{k=0}^{p-1} J_k.$$

Obviously we have  $\bar{\tau}_\omega(U) = U$  and  $\bar{\tau}_\omega(J) = J$ . Let  $\pi : U \rightarrow U/J$  be the natural quotient map. Then we have for  $0 \leq k \leq p-1$ ,  $\pi(U_k) = (U_k + J)/J \cong U_k/(U_k \cap J)$ . Now let us show that  $U_k \cap J = J_k$ . Since the inclusion  $J_k \subset U_k \cap J$  is obvious, we will show that  $U_k \cap J \subset J_k$ . For this purpose, it suffices to show that  $U_k \cap J$  is a proper submodule of  $U_k$ . Suppose now that  $U_k \cap J = U_k$ , i.e.,  $U_k \subset J$ . Then the weight vector  $v_k$  of weight  $(\omega^*)^k(\xi)$  is an element of  $J = \sum_{k=0}^{p-1} J_k$ . Thus we have  $(\omega^*)^k(\xi) \in P(J) \subset \bigcup_{k=0}^{p-1} P(J_k)$ , and hence  $(\omega^*)^k(\xi) \not\geq (\omega^*)^{k'}(\xi) \in P(V)$  for some  $k'$  with  $0 \leq k' \neq k \leq p-1$ , which contradicts the maximality of  $(\omega^*)^k(\xi) \in P(V)$ . Thus we conclude that  $U_k \cap J = J_k$ , and hence  $\pi(U_k) \cong U_k/J_k \cong L((\omega^*)^k(\xi))$  for  $0 \leq k \leq p-1$ . Therefore we obtain

$$U/J = \pi(U) = \sum_{k=0}^{p-1} \pi(U_k) = \bigoplus_{k=0}^{p-1} \pi(U_k)$$

because  $\pi(U_k) \cong L((\omega^*)^k(\xi))$  for  $0 \leq k \leq p-1$  and  $(\omega^*)^k(\xi) \neq (\omega^*)^{k'}(\xi)$  for  $0 \leq k \neq k' \leq p-1$ . Thus we have a filtration  $V \supset U \supset J \supset \{0\}$ . Since we have  $a(J, \varphi) < a(V, \varphi)$  and  $a(V/U, \varphi) < a(V, \varphi)$ , we can use the induction hypothesis to get the required filtrations for  $J$  and for  $V/U$ . Combining them with the filtration  $V \supset U \supset J \supset \{0\}$ , we obtain the required filtration for  $V$ .  $\square$

Here we note that, for each  $1 \leq i \leq t$ ,  $\xi_i \in \mathfrak{h}^*$  is a symmetric weight if and only if  $p_i = 1$ . However, even if  $\xi_i$  is a symmetric weight, the restriction of  $\bar{\tau}_\omega$  to the highest weight space  $(V_i/V_{i-1})_{\xi_i}$  is not an identity operator  $\text{id}$ , but its nonzero scalar multiple  $c_{\xi_i} \text{id}$ . More precisely, by Lemma 4.2 we have the following commutative diagram:

$$(6.1.1) \quad \begin{array}{ccc} V_i/V_{i-1} & \xrightarrow{\cong} & L(\xi_i) \\ \bar{\tau}_\omega \downarrow & & \downarrow c_{\xi_i} \bar{\tau}_\omega \\ V_i/V_{i-1} & \xrightarrow{\cong} & L(\xi_i). \end{array}$$

Let us take  $\xi \in (\mathfrak{h}^*)^0$ , and fix  $\varphi \in (\mathfrak{h}^*)^0$  such that  $\xi \geq \varphi$ . Then we construct a (Suto) filtration given by Proposition 6.1.1. We define a number  $[V : L(\xi)]^\omega \in \mathbb{C}$

by

$$[V : L(\xi)]^\omega := \sum_{\substack{1 \leq i \leq t \\ \xi_i = \xi}} c_{\xi_i}.$$

Now we define the twining character  $\text{ch}^\omega(V)$  of  $V$  by

$$\text{ch}^\omega(V) := \sum_{\chi \in (\mathfrak{h}^*)^0} \text{Tr}(\bar{\tau}_\omega|_{V_\chi}) e(\chi),$$

and similarly we set

$$\text{ch}^\omega(V_j/V_{j-1}) := \sum_{\chi \in (\mathfrak{h}^*)^0} \text{Tr}(\bar{\tau}_\omega|_{(V_j/V_{j-1})_\chi}) e(\chi).$$

From Proposition 6.1.1, we can easily deduce that

$$(6.1.2) \quad \text{ch}^\omega(V) = \sum_{j=1}^t \text{ch}^\omega(V_j/V_{j-1}) \equiv_\varphi \sum_{\substack{\xi \in (\mathfrak{h}^*)^0 \\ [V:L(\xi)] \neq 0}} [V : L(\xi)]^\omega \text{ch}^\omega(L(\xi)),$$

where the symbol  $\equiv_\varphi$  means that the coefficients of  $e(\chi)$  on both sides of  $\equiv_\varphi$  are equal for each  $\chi \in (\mathfrak{h}^*)^0$  with  $\chi \geq \varphi$ . From this fact, we first see that the number  $[V : L(\xi)]^\omega$  is independent of the (Suto) filtration given by Proposition 6.1.1 and the choice of  $\varphi \in (\mathfrak{h}^*)^0$  with  $\xi \geq \varphi$ , since the coefficient of  $e(\xi)$  in  $\text{ch}^\omega(L(\xi))$  equals 1. We then see

$$(6.1.3) \quad \text{ch}^\omega(V) = \sum_{\substack{\xi \in (\mathfrak{h}^*)^0 \\ [V:L(\xi)] \neq 0}} [V : L(\xi)]^\omega \text{ch}^\omega(L(\xi)).$$

**6.2. Some preliminary lemmas.** Let  $\lambda \in \mathfrak{h}^*$ . Since  $\alpha \in \Pi(\lambda) \subset \Delta(\lambda)_+$  is a positive real root (say,  $\alpha = w(\alpha_i)$  for  $w \in W$  and  $i \in I$ ), we can write the dual real root  $h_\alpha = w(h_i) = 2\nu^{-1}(\alpha)/(\alpha|\alpha) \in Q^\vee$  as

$$h_\alpha = \sum_{i \in I} l_i^\alpha h_i \quad \text{with } l_i^\alpha \in \mathbb{Z}_{\geq 0}.$$

Denote by  $g_\alpha \in \mathbb{Z}_{\geq 1}$  the greatest common divisor of the integers  $\sum_{k=0}^{N_i-1} l_{\omega^k(i)}^\alpha \in \mathbb{Z}_{\geq 0}$  for  $i \in \hat{I}$ . We say that  $\lambda \in \mathfrak{h}^*$  satisfies condition (WI) for  $\alpha$  if

$$(WI) \quad \lambda(h_\alpha) \in \mathbb{Z} \text{ is a multiple of } g_\alpha \in \mathbb{Z}_{\geq 1}.$$

*Remark 6.2.1.* If  $\alpha$  is a simple root  $\alpha_i \in \Pi$ , then we have  $h_\alpha = h_i$  and  $g_\alpha = 1$ , and hence  $\lambda$  satisfies condition (WI) for this  $\alpha$ . Therefore, if  $\lambda$  is integral, i.e.,  $\lambda \in P$ , then  $\lambda$  satisfies condition (WI) for all  $\alpha \in \Pi(\lambda) = \Pi$ .

**Lemma 6.2.2.** *Let  $\lambda \in \mathfrak{h}^*$  be an element satisfying condition (WI) for an arbitrarily fixed  $\alpha \in \Pi(\lambda)$ . Then there exists an element  $\gamma_0 \in \mathfrak{h}^*$  such that  $\gamma_0 \in P$ ,  $\omega^*(\gamma_0) = \gamma_0$ , and  $\gamma_0(h_\alpha) = -\lambda(h_\alpha)$ .*

*Proof.* Recall from [FSS, §3.2] the definition of the diagram automorphism  $\omega : \mathfrak{g} \rightarrow \mathfrak{g}$ . Since  $\mathfrak{g}$  is an affine Lie algebra, we have  $\mathfrak{h} = (\bigoplus_{i \in I} \mathbb{C}h_i) \oplus \mathbb{C}D$  for some suitably

chosen element  $D \in \mathfrak{h}$  (not necessarily equal to the scaling element  $d$  in [Ka, Ch. 6]). We defined  $\omega$  by

$$\begin{cases} \omega(e_i) := e_{\omega(i)} & \text{for } i \in I, \\ \omega(f_i) := f_{\omega(i)} & \text{for } i \in I, \\ \omega(h_i) := h_{\omega(i)} & \text{for } i \in I, \\ \omega(D) := D. \end{cases}$$

Since the integer  $\lambda(h_\alpha)$  is a multiple of  $g_\alpha \in \mathbb{Z}_{\geq 1}$  by assumption, there exist integers  $z_i \in \mathbb{Z}$ ,  $i \in \widehat{I}$ , such that

$$\sum_{i \in \widehat{I}} z_i \left( \sum_{k=0}^{N_i-1} l_{\omega^k(i)}^\alpha \right) = -\lambda(h_\alpha).$$

We define an element  $\gamma_0 \in \mathfrak{h}^*$  by

$$\begin{cases} \gamma_0(h_{\omega^k(i)}) := z_i \in \mathbb{Z} & \text{for } i \in \widehat{I} \text{ and } 0 \leq k \leq N_i - 1, \\ \gamma_0(D) := 1. \end{cases}$$

Then we have  $\gamma_0 \in P$  and  $\omega^*(\gamma_0) = \gamma_0$ . Further, we have

$$\gamma_0(h_\alpha) = \sum_{i \in I} l_i^\alpha \gamma_0(h_i) = \sum_{i \in \widehat{I}} z_i \left( \sum_{k=0}^{N_i-1} l_{\omega^k(i)}^\alpha \right) = -\lambda(h_\alpha).$$

This proves the lemma. □

We assume that  $\lambda \in \mathfrak{h}^*$  is a symmetric weight, i.e.,  $\omega^*(\lambda) = \lambda$ . Then, by Lemma 5.1.3 (2), we have  $\omega^*(\Pi(\lambda)) = \Pi(\lambda)$ . Write  $\Pi(\lambda) = \{\phi_j\}_{j \in J}$  as in §5.1. Then  $\omega^*$  induces a bijection  $\omega^\lambda : J \rightarrow J$  by:  $\omega^*(\phi_j) = \phi_{(\omega^\lambda)^{-1}(j)}$  for  $j \in J$ . We note that

$$2(\phi_{\omega^\lambda(j)}|\phi_{\omega^\lambda(i)})/(\phi_{\omega^\lambda(i)}|\phi_{\omega^\lambda(i)}) = 2(\phi_j|\phi_i)/(\phi_i|\phi_i) \quad \text{for } i, j \in J.$$

In other words, the bijection  $\omega^\lambda : J \rightarrow J$  is a diagram automorphism for the GCM  $A(\lambda) = (2(\phi_j|\phi_i)/(\phi_i|\phi_i))_{i,j \in J}$ . Let  $N^\lambda$  be the order of  $\omega^\lambda : J \rightarrow J$ , and  $N_j^\lambda$  the number of elements in the  $\omega^\lambda$ -orbit of  $j \in J$  in  $J$ . We choose a set of representatives  $\widehat{J}$  of the  $\omega^\lambda$ -orbits in  $J$ , and set

$$\check{J} := \{j \in \widehat{J} \mid \sum_{k=0}^{N_j^\lambda-1} 2(\phi_{(\omega^\lambda)^k(j)}|\phi_j)/(\phi_j|\phi_j) > 0\}.$$

Set, for each  $j \in \widehat{J}$ ,

$$\psi_j := \sum_{k=0}^{N_j^\lambda-1} \phi_{(\omega^\lambda)^k(j)} \in (\mathfrak{h}^*)^0.$$

**Lemma 6.2.3.** *Let  $\lambda \in (\mathfrak{h}^*)^0$ . Fix an arbitrary  $j_0 \in \check{J}$ . Then there exists an element  $\theta \in \mathfrak{h}^*$  such that  $\theta \in P$ ,  $\omega^*(\theta) = \theta$ ,  $(\theta|\delta) > 0$ , and  $(\theta|\psi_{j_0}) = 0$ ,  $(\theta|\psi_j) > 0$  for  $j \neq j_0 \in \widehat{J}$ .*

*Proof.* First, we note that if  $\xi \in \mathfrak{h}^*$  is a symmetric weight, then for  $j \in \widehat{J}$ ,

$$(\xi|\psi_j) = (\xi|\sum_{k=0}^{N_j^\lambda-1} \phi_{(\omega^\lambda)^k(j)}) = \sum_{k=0}^{N_j^\lambda-1} (\xi|(\omega^*)^{-k}(\phi_j)) = N_j^\lambda \cdot (\xi|\phi_j).$$

We define an element  $\zeta \in \mathfrak{h}^*$  by

$$\begin{cases} \zeta(h_i) := 2 & \text{for all } i \in I, \\ \zeta(D) := 1. \end{cases}$$

It follows that  $\zeta \in P$ ,  $\omega^*(\zeta) = \zeta$ , and  $(\zeta|\delta) = \sum_{i \in I} a_i(\zeta|\alpha_i) = \sum_{i \in I} a_i \varepsilon_i^{-1} \zeta(h_i) > 0$ .

To simplify the notation, we write  $\alpha^\vee$  instead of  $h_\alpha \in Q^\vee$  for the dual real root of  $\alpha \in \Delta^{re}$ . We now set

$$\theta := \zeta - (\zeta(\phi_{j_0}^\vee)/\psi_{j_0}(\phi_{j_0}^\vee)) \psi_{j_0} \in \mathfrak{h}^*.$$

Here we note that since  $j_0 \in \check{J}$ , we have  $(\psi_{j_0}|\phi_{j_0}) > 0$ , i.e.,  $\psi_{j_0}(\phi_{j_0}^\vee) > 0$ . It follows that  $\omega^*(\theta) = \theta$ , and  $(\theta|\delta) = (\zeta|\delta) > 0$  since  $(\delta|Q) = 0$  implies  $(\delta|\phi_j) = 0$  for all  $j \in J$ . Note that  $\zeta(\phi_{j_0}^\vee) \in 2\mathbb{Z}$  since  $\phi_{j_0}^\vee \in Q^\vee$ , and that  $\psi_{j_0}(\phi_{j_0}^\vee) = 2(\psi_{j_0}|\phi_{j_0})/(\phi_{j_0}|\phi_{j_0}) > 0$  implies  $\psi_{j_0}(\phi_{j_0}^\vee) = 1, 2$  as we have indicated just above Proposition 3.2.2. Hence we have  $\theta \in P$  since  $\psi_{j_0} \in Q \subset P$ . It is obvious from the definition that  $\theta(\phi_{j_0}^\vee) = 0$ , i.e.,  $(\theta|\phi_{j_0}) = 0$ , and hence  $(\theta|\psi_{j_0}) = 0$ . For  $j \neq j_0 \in \widehat{J}$ , we have

$$(\theta|\phi_j) = (\zeta|\phi_j) - ((\zeta|\phi_{j_0})/(\psi_{j_0}|\phi_{j_0})) \cdot (\psi_{j_0}|\phi_j).$$

Here we have  $(\zeta|\phi_j) > 0$  for all  $j \in \widehat{J}$  since  $\phi_j \in \Pi(\lambda) \subset \Delta_+$ , and  $(\psi_{j_0}|\phi_j) = \sum_{k=0}^{N_{j_0}^\lambda-1} (\phi_{(\omega^\lambda)^k(j_0)}|\phi_j) \leq 0$  since  $j \neq (\omega^\lambda)^k(j_0)$  for any  $0 \leq k \leq N_{j_0}^\lambda - 1$ . Thus we conclude that  $(\theta|\phi_j) > 0$ , and hence  $(\theta|\psi_j) > 0$ .  $\square$

Let  $\lambda \in \mathcal{C}$  be a symmetric weight satisfying condition (WI) for all  $\alpha \in \Pi(\lambda)$ . Fix an arbitrary  $j_0 \in \check{J}$ . Then, by Lemma 6.2.2, there exists  $\gamma_0 \in \mathfrak{h}^*$  such that  $\gamma_0 \in P$ ,  $\omega^*(\gamma_0) = \gamma_0$ , and  $(\gamma_0|\phi_{j_0}) = -(\lambda|\phi_{j_0})$ . Since  $\omega^*(\gamma_0) = \gamma_0$  and  $\omega^*(\lambda) = \lambda$ , we have

$$(\gamma_0|\psi_{j_0}) = N_{j_0}^\lambda(\gamma_0|\phi_{j_0}) = -N_{j_0}^\lambda(\lambda|\phi_{j_0}) = -(\lambda|\psi_{j_0}).$$

Take  $\theta \in \mathfrak{h}^*$  as in Lemma 6.2.3, and set for each positive integer  $L \in \mathbb{Z}_{\geq 0}$ ,

$$(6.2.1) \quad \mu_0 := \lambda + \gamma_0 - \rho, \quad \mu := \mu_0 + L\theta.$$

Then we see that  $\omega^*(\mu) = \mu$ , and  $\mu - \lambda = \gamma_0 - \rho + L\theta \in P$  since  $\rho \in P$ . Also, since  $(\theta|\delta) > 0$ , we have  $(\mu + \rho|\delta) = (\lambda + \gamma_0|\delta) + L(\theta|\delta) \neq 0$ , and hence  $\mu \in \mathcal{C}$  for sufficiently large  $L$ . It follows that  $(\mu + \rho|\psi_{j_0}) = (\lambda + \gamma_0|\psi_{j_0}) + L(\theta|\psi_{j_0}) = 0$ . Furthermore, for each  $j \neq j_0 \in \widehat{J}$ , we have  $(\mu + \rho|\psi_j) = (\lambda + \gamma_0|\psi_j) + L(\theta|\psi_j) > 0$  for sufficiently large  $L$  (depending on  $j \in \widehat{J}$ ) because  $\phi_j \in \Pi(\lambda) \subset \Delta(\lambda)$  implies  $(\lambda|\phi_j) \in \mathbb{R}$  and hence  $(\lambda + \gamma_0|\psi_j) \in \mathbb{R}$ . Here we recall that  $J$  is a finite set from [KT2, Lem. 2.3]. Therefore, if  $L$  is sufficiently large, then we have  $(\mu + \rho|\psi_j) > 0$  for all  $j \neq j_0 \in \widehat{J}$ . In particular,  $\mu$  is an element of  $\mathcal{C}^+$  since  $\mu - \lambda \in P$  implies  $\Delta(\mu) = \Delta(\lambda)$ .

**Lemma 6.2.4.** *We keep the notation above. There exists some  $x \in \widetilde{W}$  such that  $x^{-1}(\mu - \lambda) \in P_+$ .*

*Proof.* Since  $(\gamma_0 - \rho|\delta) \in \mathbb{R}$ , we have  $(\mu - \lambda|\delta) = (\gamma_0 - \rho|\delta) + L(\theta|\delta) > 0$  for sufficiently large  $L$ . If the Dynkin diagram  $S(A)$  of the GCM  $A = (a_{ij})_{i,j \in I}$  is of type  $A_{n-1}^{(1)}$  and the diagram automorphism  $\omega : I \rightarrow I$  is a cyclic permutation of  $I$  of order  $n$ , then we have for each  $i_0 \in I$ ,

$$(\mu - \lambda|\delta) = (\mu - \lambda|\sum_{i \in I} a_i \alpha_i) = \sum_{i \in I} a_i(\mu - \lambda|\alpha_i) = n a_{i_0}(\mu - \lambda|\alpha_{i_0})$$

since  $\omega^*(\delta) = \delta$  and  $\omega^*(\mu - \lambda) = \mu - \lambda$ . Thus  $(\mu - \lambda|\delta) > 0$  implies  $(\mu - \lambda|\alpha_i) > 0$  for all  $i \in I$ , hence  $\mu - \lambda \in P_+$ . Therefore, by the comment just above Proposition 3.2.2, we may assume that the matrix  $\hat{A} = \check{A} = (\hat{a}_{ij})_{i,j \in \check{I}}$  is a GCM of affine type, where  $\hat{I} = \check{I}$ .

Let  $\hat{\delta} = \sum_{i \in \hat{I}} \hat{a}_i \hat{\alpha}_i \in \hat{\mathfrak{h}}^*$  be the null root of the orbit Lie algebra  $\hat{\mathfrak{g}}$  of affine type, where the  $\hat{a}_i$ ,  $i \in \hat{I}$ , are relatively prime positive integers. Then we have  $P_\omega^*(\hat{\delta}) \in Q_+$  since  $P_\omega^*(\hat{\alpha}_i) = s_i \beta_i$  by (3.2.4). In addition, we have for all  $i \in \hat{I}$ ,  $(P_\omega^*(\hat{\delta})|\beta_i) = s_i^{-1}(P_\omega^*(\hat{\delta})|P_\omega^*(\hat{\alpha}_i)) = s_i^{-1}(\hat{\delta}|\hat{\alpha}_i) = 0$ . Since  $P_\omega^*(\hat{\delta})$  is a symmetric weight, we obtain  $(P_\omega^*(\hat{\delta})|\alpha_i) = N_i^{-1}(P_\omega^*(\hat{\delta})|\beta_i) = 0$  for all  $i \in \hat{I}$ , and hence for all  $i \in I$ . In other words, the element  $P_\omega^*(\hat{\delta}) \in Q_+$  is in the radical of the bilinear form  $(\cdot|\cdot)$  restricted to  $\sum_{i \in I} \mathbb{R}\alpha_i$ . Therefore we deduce that  $P_\omega^*(\hat{\delta}) = k\delta$  for some positive integer  $k \in \mathbb{Z}_{\geq 1}$ .

Since  $\mu - \lambda \in \mathfrak{h}^*$  and  $\omega^*(\mu - \lambda) = \mu - \lambda$ , there exists  $\hat{\kappa} \in \hat{\mathfrak{h}}^*$  such that  $P_\omega^*(\hat{\kappa}) = \mu - \lambda$ . Note that  $(\hat{\kappa}|\hat{\alpha}_i) \in \mathbb{R}$  for all  $i \in \hat{I} = \check{I}$  since  $(\hat{\kappa}|\hat{\alpha}_i) = (P_\omega^*(\hat{\kappa})|P_\omega^*(\hat{\alpha}_i)) = (\mu - \lambda|s_i \beta_i) = s_i N_i (\mu - \lambda|\alpha_i)$ . Moreover, we have  $(\hat{\kappa}|\hat{\delta}) = (P_\omega^*(\hat{\kappa})|P_\omega^*(\hat{\delta})) = k(\mu - \lambda|\delta) > 0$ . Hence, by (the proof of) [Ka, Prop. 5.8 b)], there exists  $\hat{x} \in \hat{W}$  such that  $(\hat{x}(\hat{\kappa})|\hat{\alpha}_i) \geq 0$  for all  $i \in \hat{I}$ . Then, by Proposition 3.3.1, we have  $P_\omega^*(\hat{x}(\hat{\kappa})) = \Theta(\hat{x})(P_\omega^*(\hat{\kappa})) = \Theta(\hat{x})(\mu - \lambda)$ . Set  $x := \Theta(\hat{x}) \in \widetilde{W} \subset W$  and  $\Lambda := P_\omega^*(\hat{x}(\hat{\kappa})) \in (\mathfrak{h}^*)^0$ . Then we have  $\Lambda = x(\mu - \lambda) \in P$  since  $\mu - \lambda \in P$ . Furthermore, we have for each  $i \in \hat{I}$ ,

$$\begin{aligned} 0 &\leq (\hat{x}(\hat{\kappa})|\hat{\alpha}_i) = (P_\omega^*(\hat{x}(\hat{\kappa}))|P_\omega^*(\hat{\alpha}_i)) = (\Lambda|s_i \beta_i) \\ &= s_i (\Lambda| \sum_{k=0}^{N_i-1} \alpha_{\omega^k(i)}) = s_i N_i (\Lambda|\alpha_i) \end{aligned}$$

since  $\omega^*(\Lambda) = \Lambda$ . Thus we obtain  $\Lambda \in P_+$ . □

In particular, the elements  $\lambda, \mu \in \mathcal{C}$  satisfy condition (TR) with  $\mu - \lambda = x(\Lambda)$  for  $x \in \widetilde{W}$  and  $\Lambda \in P_+$ .

**6.3. About Weyl groups.** We use the notation of §5.1. Let  $\lambda \in (\mathfrak{h}^*)^0$ . Then, as in §6.2,  $\omega^*$  induces a bijection of  $\Pi(\lambda) = \{\phi_j\}_{j \in J}$ , and so a diagram automorphism  $\omega^\lambda : J \rightarrow J$  by  $\omega^*(\phi_j) = \phi_{(\omega^\lambda)^{-1}(j)}$  for  $j \in J$ . By Lemma 5.1.1, the GCM  $A(\lambda) = (2(\phi_j|\phi_i)/(\phi_i|\phi_i))_{i,j \in J}$  decomposes, after reordering the indices, into a direct sum of GCMs of finite type and those of affine type. Note that in the decomposition  $A(\lambda) = D(\lambda)B(\lambda)$  given in Remark 5.1.2, we have  $\varepsilon_{\omega^\lambda(j)}^\lambda = \varepsilon_j^\lambda$  for all  $j \in J$  since  $(\omega^*(\phi_j)|\omega^*(\phi_j)) = (\phi_j|\phi_j)$ . Hence we can apply the setting in §3 to the Kac-Moody algebra  $\mathfrak{g}^\lambda = \mathfrak{g}(A(\lambda))$  with Cartan subalgebra  $\mathfrak{h}^\lambda$  and simple roots  $\Pi^\lambda = \{\beta_j\}_{j \in J} \subset (\mathfrak{h}^\lambda)^*$ . Let  $(\cdot|\cdot)^\lambda$  be the nondegenerate, symmetric, invariant bilinear form on  $\mathfrak{g}^\lambda$  corresponding to the decomposition  $A(\lambda) = D(\lambda)B(\lambda)$  above. We also denote by  $\omega^\lambda : \mathfrak{g}^\lambda \rightarrow \mathfrak{g}^\lambda$  the diagram automorphism of the Lie algebra  $\mathfrak{g}^\lambda$  induced from the bijection  $\omega^\lambda : J \rightarrow J$ , and by  $(\omega^\lambda)^* : (\mathfrak{h}^\lambda)^* \rightarrow (\mathfrak{h}^\lambda)^*$  the dual map of the restriction of  $\omega^\lambda$  to  $\mathfrak{h}^\lambda$ . Note that we have the following commutative diagram from the definitions:

$$\begin{array}{ccc} Q^\lambda & \xrightarrow{\Psi} & Q(\lambda) \\ (\omega^\lambda)^* \downarrow & & \downarrow \omega^* \\ Q^\lambda & \xrightarrow{\Psi} & Q(\lambda), \end{array}$$

where  $\Psi : Q^\lambda \rightarrow Q(\lambda)$  is the  $\mathbb{Z}$ -linear isomorphism in §5.1. Furthermore, we know from [MP, §5.1 and §5.5] that there exists a unique group isomorphism  $\Xi : W^\lambda \rightarrow W(\lambda)$  satisfying  $\Xi(r_j^\lambda) = r_{\phi_j}$  for each  $j \in J$ . In addition, we have the following commutative diagram for each  $w \in W^\lambda$ :

$$\begin{array}{ccc} Q^\lambda & \xrightarrow{\Psi} & Q(\lambda) \\ w \downarrow & & \downarrow \Xi(w) \\ Q^\lambda & \xrightarrow{\Psi} & Q(\lambda). \end{array}$$

Set

$$\widetilde{W}^\lambda := \{w \in W^\lambda \mid (\omega^\lambda)^* w = w(\omega^\lambda)^*\}.$$

**Lemma 6.3.1.** *We have  $\Xi(\widetilde{W}^\lambda) = W(\lambda) \cap \widetilde{W}$ .*

*Proof.* Let us show that for each  $w \in W^\lambda$ ,

$$\Xi(((\omega^\lambda)^*)^{-1} w (\omega^\lambda)^*) = (\omega^*)^{-1} \Xi(w) \omega^*,$$

from which the lemma immediately follows, since  $\Xi : W^\lambda \rightarrow W(\lambda)$  is a group isomorphism. To show the equality above, we may assume that  $w = r_j^\lambda$  for  $j \in J$ . Then, by Lemma 3.1.4 (2) and (4), we have

$$\begin{aligned} \Xi(((\omega^\lambda)^*)^{-1} r_j^\lambda (\omega^\lambda)^*) &= \Xi(r_{\omega^\lambda(j)}^\lambda) = r_{\phi_{\omega^\lambda(j)}} = r_{(\omega^*)^{-1}(\phi_j)} \\ &= (\omega^*)^{-1} r_{\phi_j} \omega^* = (\omega^*)^{-1} \Xi(r_j^\lambda) \omega^*, \end{aligned}$$

as desired. □

Now recall from §3.3 that the group  $\widetilde{W}^\lambda$  is a Coxeter group with the canonical generator system  $\{w_j^\lambda \mid j \in \check{J}\}$ , where  $w_j^\lambda \in \widetilde{W}^\lambda$  for  $j \in \check{J}$  is defined by using  $r_{(\omega^\lambda)^k(j)}^\lambda$  with  $0 \leq k \leq N_j^\lambda - 1$  in exactly the same way as  $w_i \in \widetilde{W}$  for  $i \in \check{I}$  is defined by using  $r_{\omega^k(i)}$  with  $0 \leq k \leq N_i - 1$  in (3.3.1). Therefore, by Lemma 6.3.1, we deduce that the group  $W(\lambda) \cap \widetilde{W}$  is a Coxeter group with the canonical generator system  $\{s_j := \Xi(w_j^\lambda) \mid j \in \check{J}\}$ . We denote the length function of  $W(\lambda) \cap \widetilde{W} \cong \widetilde{W}^\lambda$  by  $\widehat{\ell}_\lambda : W(\lambda) \cap \widetilde{W} \rightarrow \mathbb{Z}_{\geq 0}$ .

*Remark 6.3.2.* From the argument above, we see that  $W(\lambda) \cap \widetilde{W} = \{1\}$  if and only if  $\check{J} = \emptyset$ .

**6.4. Proof of the main theorem.** Let  $\lambda \in \mathfrak{h}^*$  be a symmetric weight such that  $\lambda \in \mathcal{C}^+$  and  $\Delta_0(\lambda) = \emptyset$ . Then, by arguments similar to those in the proof of [Ka, Prop. 3.12], we see that for each  $w \in W(\lambda)$ ,  $w \circ \lambda \in \lambda - \sum_{\alpha \in \Pi(\lambda)} \mathbb{Z}_{\geq 0} \alpha$ , and hence  $w \circ \lambda \leq \lambda$  with equality if and only if  $w = 1$ .

Let  $w \in W(\lambda) \cap \widetilde{W}$ . Then the module  $M(w \circ \lambda)$  is in the category  $\mathcal{O}[\lambda]$ . Hence  $[M(w \circ \lambda) : L(\xi)] > 0$  for  $\xi \in \mathfrak{h}^*$  implies  $\xi = y \circ \lambda$  for a unique  $y \in W(\lambda)$ . If

$\omega^*(\xi) = \xi$ , then we have

$$\begin{aligned} \omega^*y(\lambda + \rho) - \rho &= \omega^*y(\lambda + \rho) - \omega^*(\rho) = \omega^*(y \circ \lambda) \\ &= y \circ \lambda = y(\lambda + \rho) - \rho = y\omega^*(\lambda + \rho) - \rho, \end{aligned}$$

from which we have  $((\omega^*)^{-1}y\omega^*) \circ \lambda = y \circ \lambda$ . Since  $(\omega^*)^{-1}y\omega^* \in W(\lambda)$ , we have  $(\omega^*)^{-1}y\omega^* = y$ , i.e.,  $y \in \widetilde{W}$ . Therefore, by (6.1.3), we see that

$$\text{ch}^\omega(M(w \circ \lambda)) = \sum_{y \in W(\lambda) \cap \widetilde{W}} [M(w \circ \lambda) : L(y \circ \lambda)]^\omega \text{ch}^\omega(L(y \circ \lambda)).$$

Since the unique maximal proper submodule  $J(w \circ \lambda)$  of  $M(w \circ \lambda)$  is  $\tau_\omega$ -stable and  $M(w \circ \lambda)/J(w \circ \lambda) = L(w \circ \lambda)$ , we obtain that  $[M(w \circ \lambda) : L(w \circ \lambda)]^\omega = 1$ .

*Remark 6.4.1.* From the argument above, we see that if  $W(\lambda) \cap \widetilde{W} = \{1\}$ , i.e.,  $\check{J} = \emptyset$ , then we have  $\text{ch}^\omega(L(\lambda)) = \text{ch}^\omega(M(\lambda))$ .

Now let us take an arbitrary  $w \in W(\lambda) \cap \widetilde{W}$ , and set  $\eta := w \circ \lambda \in (\mathfrak{h}^*)^0$ . Fix  $\varphi \in (\mathfrak{h}^*)^0$ , and take a (Suto) filtration of the Verma module  $M(\eta)$  given by Proposition 6.1.1

$$(6.4.1) \quad M(\eta) = V_t \supset V_{t-1} \supset \cdots \supset V_1 \supset V_0 = \{0\}.$$

By (6.1.2), we have

$$\text{ch}^\omega(M(\eta)) = \sum_{i=1}^t \text{ch}^\omega(V_i/V_{i-1}) \equiv_\varphi \sum_{y \in W(\lambda) \cap \widetilde{W}} [M(\eta) : L(y \circ \lambda)]^\omega \text{ch}^\omega(L(y \circ \lambda)).$$

Assume, in addition, that there exists a symmetric weight  $\mu \in \mathcal{C}^+$  such that  $\mu - \lambda = x(\Lambda)$  for  $x \in \widetilde{W}$  and  $\Lambda \in P_+$ . From (6.4.1), by tensoring with the module  $L(\Lambda)$ , we get a filtration of  $M(\eta) \otimes_{\mathbb{C}} L(\Lambda)$  by  $\overline{\tau}_\omega \otimes \tau_\omega$ -stable  $\mathfrak{g}$ -submodules

$$(6.4.2) \quad \begin{aligned} M(\eta) \otimes_{\mathbb{C}} L(\Lambda) &= V_t \otimes_{\mathbb{C}} L(\Lambda) \supset V_{t-1} \otimes_{\mathbb{C}} L(\Lambda) \supset \cdots \\ &\cdots \supset V_1 \otimes_{\mathbb{C}} L(\Lambda) \supset V_0 \otimes_{\mathbb{C}} L(\Lambda) = \{0\}, \end{aligned}$$

where for  $i \notin T$ , the quotient module  $(V_i \otimes_{\mathbb{C}} L(\Lambda))/(V_{i-1} \otimes_{\mathbb{C}} L(\Lambda))$  has no weight  $\chi \geq \varphi + \Lambda$ . By applying the exact functor  $P_\mu$  to this filtration and using Proposition 5.2.4, we obtain a filtration of  $T_\mu^\lambda(M(\eta))$  by  $\overline{\tau} \otimes \tau_\omega$ -stable  $\mathfrak{g}$ -submodules

$$T_\mu^\lambda(M(\eta)) = T_\mu^\lambda(V_t) \supset T_\mu^\lambda(V_{t-1}) \supset \cdots \supset T_\mu^\lambda(V_1) \supset T_\mu^\lambda(V_0) = \{0\}.$$

**Lemma 6.4.2.** *With the notation above, we have*

$$\text{ch}^\omega(T_\mu^\lambda(M(w \circ \lambda))) \equiv_{\varphi+\Lambda} \sum_{y \in W(\lambda) \cap \widetilde{W}} [M(\eta) : L(y \circ \lambda)]^\omega \text{ch}^\omega(T_\mu^\lambda(L(y \circ \lambda))).$$

*Proof.* As in the proof of Corollary 5.2.6, we obtain the following commutative diagram for  $1 \leq i \leq t$  from the filtration (6.4.2):

$$\begin{array}{ccc} T_\mu^\lambda(V_i)/T_\mu^\lambda(V_{i-1}) & \xrightarrow{\cong} & T_\mu^\lambda(V_i/V_{i-1}) \\ \overline{\tau}_\omega \otimes \tau_\omega \downarrow & & \downarrow \overline{\tau}_\omega \otimes \tau_\omega \\ T_\mu^\lambda(V_i)/T_\mu^\lambda(V_{i-1}) & \xrightarrow{\cong} & T_\mu^\lambda(V_i/V_{i-1}), \end{array}$$

where the left vertical map  $\overline{\tau_\omega \otimes \tau_\omega}$  is induced from the map  $\tau_\omega \otimes \tau_\omega : V_i \otimes_{\mathbb{C}} L(\Lambda) \rightarrow V_i \otimes_{\mathbb{C}} L(\Lambda)$ . Thus we have

$$\mathrm{ch}^\omega(T_\mu^\lambda(M(\eta))) = \sum_{i=1}^t \mathrm{ch}^\omega(T_\mu^\lambda(V_i)/T_\mu^\lambda(V_{i-1})) = \sum_{i=1}^t \mathrm{ch}^\omega(T_\mu^\lambda(V_i/V_{i-1})).$$

Now recall that for each  $1 \leq i \leq t$ , the diagram (6.1.1) commutes for some  $c_{\xi_i} \in \mathbb{C} \setminus \{0\}$ . By tensoring with  $L(\Lambda)$  and then applying the exact functor  $P_\mu$  (using Proposition 5.2.4), we obtain the commutative diagram

$$\begin{array}{ccc} T_\mu^\lambda(V_i/V_{i-1}) & \xrightarrow{\cong} & T_\mu^\lambda(L(\xi_i)) \\ \tau_\omega \otimes \tau_\omega \downarrow & & \downarrow c_{\xi_i} \tau_\omega \otimes \tau_\omega \\ T_\mu^\lambda(V_i/V_{i-1}) & \xrightarrow{\cong} & T_\mu^\lambda(L(\xi_i)). \end{array}$$

Hence we have for  $1 \leq i \leq t$ ,  $\mathrm{ch}^\omega(T_\mu^\lambda(V_i/V_{i-1})) = c_{\xi_i} \mathrm{ch}^\omega(T_\mu^\lambda(L(\xi_i)))$ . Therefore we have

$$\sum_{i=1}^t \mathrm{ch}^\omega(T_\mu^\lambda(V_i/V_{i-1})) \equiv_{\varphi+\Lambda} \sum_{y \in W(\lambda) \cap \widetilde{W}} [M(\eta) : L(y \circ \lambda)]^\omega \mathrm{ch}^\omega(T_\mu^\lambda(L(y \circ \lambda))).$$

This completes the proof.  $\square$

Since  $\varphi \in (\mathfrak{h}^*)^0$  is arbitrary in Lemma 6.4.2, we obtain the following:

**Proposition 6.4.3.** *Let  $\lambda, \mu \in \mathcal{C}^+$  be elements satisfying condition (TR) with  $\mu - \lambda = x(\Lambda)$  for  $x \in \widetilde{W}$  and  $\Lambda \in P_+$ . Assume, in addition, that  $\Delta_0(\lambda) = \emptyset$ . Then, for each  $w \in W(\lambda) \cap \widetilde{W}$ , we have*

$$\mathrm{ch}^\omega(M(w \circ \lambda)) = \sum_{y \in W(\lambda) \cap \widetilde{W}} [M(w \circ \lambda) : L(y \circ \lambda)]^\omega \mathrm{ch}^\omega(L(y \circ \lambda)),$$

$$\mathrm{ch}^\omega(T_\mu^\lambda(M(w \circ \lambda))) = \sum_{y \in W(\lambda) \cap \widetilde{W}} [M(w \circ \lambda) : L(y \circ \lambda)]^\omega \mathrm{ch}^\omega(T_\mu^\lambda(L(y \circ \lambda))).$$

We are now in a position to state our main result.

**Theorem 6.4.4.** *Let  $\mathfrak{g} = \mathfrak{g}(A)$  be the Kac-Moody algebra associated to a GCM  $A = (a_{ij})_{i,j \in I}$  of affine type, and  $\omega : I \rightarrow I$  a diagram automorphism. Let  $\lambda \in \mathfrak{h}^*$  be a symmetric weight such that  $\lambda \in \mathcal{C}^+$  and  $\Delta_0(\lambda) = \emptyset$ .*

(a) *When  $W(\lambda) \cap \widetilde{W} = \{1\}$  (i.e.,  $\check{J} = \emptyset$ ), we have*

$$\begin{aligned} \mathrm{ch}^\omega(L(\lambda)) &= \mathrm{ch}^\omega(M(\lambda)) \\ &= e(\lambda) \cdot \left( \sum_{w \in \widetilde{W}} (-1)^{\widehat{\ell}(w)} e(w(\rho) - \rho) \right)^{-1}, \end{aligned}$$

where  $\widehat{\ell} : \widetilde{W} \rightarrow \mathbb{Z}$  denotes the length function of the Coxeter group  $\widetilde{W}$ .

(b) When  $W(\lambda) \cap \widetilde{W} \neq \{1\}$  (i.e.,  $\check{J} \neq \emptyset$ ), we further assume that  $\lambda$  satisfies condition (WI) for all  $\alpha \in \Pi(\lambda)$ . In this case, we have

$$\begin{aligned} \text{ch}^\omega(L(\lambda)) &= \sum_{w \in W(\lambda) \cap \widetilde{W}} (-1)^{\widehat{\ell}_\lambda(w)} \text{ch}^\omega(M(w \circ \lambda)) \\ &= \frac{\sum_{w \in W(\lambda) \cap \widetilde{W}} (-1)^{\widehat{\ell}_\lambda(w)} e(w(\lambda + \rho))}{\sum_{w \in \widetilde{W}} (-1)^{\widehat{\ell}(w)} e(w(\rho))}, \end{aligned}$$

where  $\widehat{\ell}_\lambda : W(\lambda) \cap \widetilde{W} \rightarrow \mathbb{Z}$  denotes the length function of the Coxeter group  $W(\lambda) \cap \widetilde{W}$ .

*Proof.* Part (a) is already proved by Remark 6.4.1 combined with Theorem 4.4. We will prove part (b). So we assume that  $\check{J} \neq \emptyset$ . Recall that for  $w \in W(\lambda) \cap \widetilde{W}$ , we have  $w \circ \lambda \leq \lambda$ , and  $w \circ \lambda = \lambda$  implies  $w = 1$ . Hence we can write  $(W(\lambda) \cap \widetilde{W}) \circ \lambda = \{\lambda_i \mid i \in \mathbb{Z}_{\geq 1}\}$  and reindex them in such a way that

- (1)  $\lambda_1 = \lambda$ ;
- (2)  $\lambda_i = y_i \circ \lambda$  for a unique  $y_i \in W(\lambda) \cap \widetilde{W}$ ,  $i \in \mathbb{Z}_{\geq 1}$ ;
- (3)  $\text{depth}_\lambda(\lambda_i) < \text{depth}_\lambda(\lambda_j)$  implies  $i < j$ .

By Proposition 6.4.3, we have for  $i \geq 1$ ,

$$(6.4.3) \quad \text{ch}^\omega(M(\lambda_i)) = \sum_{j \geq i} [M(y_i \circ \lambda) : L(y_j \circ \lambda)]^\omega \text{ch}^\omega(L(\lambda_j)),$$

where  $[M(y_i \circ \lambda) : L(y_i \circ \lambda)]^\omega = 1$ . We may view (6.4.3) above as a system of linear equations whose matrix is lower triangular with all the diagonal entries equal to 1. Thus we may invert this system to obtain for  $i \geq 1$ ,

$$\text{ch}^\omega(L(\lambda_i)) = \sum_{j \geq i} c(y_i, y_j) \text{ch}^\omega(M(\lambda_j)) \quad \text{with } c(y_i, y_j) \in \mathbb{C},$$

where  $c(y_i, y_i) = 1$ . In particular, for  $\lambda_1 = \lambda$ , we have

$$\text{ch}^\omega(L(\lambda)) = \sum_{j \geq 1} c(1, y_j) \text{ch}^\omega(M(\lambda_j)) \quad \text{with } c(1, 1) = 1.$$

We set  $c(y) := c(1, y)$  for  $y \in W(\lambda) \cap \widetilde{W}$ . Then we can write

$$(6.4.4) \quad \text{ch}^\omega(L(\lambda)) = \sum_{y \in W(\lambda) \cap \widetilde{W}} c(y) \text{ch}^\omega(M(y \circ \lambda)) \quad \text{with } c(1) = 1.$$

We want to determine the numbers  $c(y) \in \mathbb{C}$  for  $y \in W(\lambda) \cap \widetilde{W}$ . Let  $\mu \in \mathcal{C}^+$  be a symmetric weight such that  $\mu - \lambda = x(\Lambda)$  for  $x \in \widetilde{W}$  and  $\Lambda \in P_+$ . By Proposition 6.4.3, we have for  $i \geq 1$ ,

$$\text{ch}^\omega(T_\mu^\lambda(M(\lambda_i))) = \sum_{j \geq i} [M(y_i \circ \lambda) : L(y_j \circ \lambda)]^\omega \text{ch}^\omega(T_\mu^\lambda(L(\lambda_j))).$$

Hence, in the same way as for  $\text{ch}^\omega(L(\lambda))$  above, we obtain

$$\text{ch}^\omega(T_\mu^\lambda(L(\lambda))) = \sum_{y \in W(\lambda) \cap \widetilde{W}} c(y) \text{ch}^\omega(T_\mu^\lambda(M(y \circ \lambda))),$$

where the  $c(y)$ 's for  $y \in W(\lambda) \cap \widetilde{W}$  are exactly the same numbers as those for  $\text{ch}^\omega(L(\lambda))$  in (6.4.4). Let us fix an arbitrary  $j_0 \in \widetilde{J}$ , and take  $\mu \in (\mathfrak{h}^*)^0$  as in (6.2.1). Then we have  $\mu \in \mathcal{C}^+$ ,  $\mu - \lambda = x(\Lambda)$  for  $x \in \widetilde{W}$ ,  $\Lambda \in P_+$ , and  $(\mu + \rho|\psi_{j_0}) = 0$ ,  $(\mu + \rho|\psi_j) > 0$  for all  $j \neq j_0 \in \widetilde{J}$ . Then, by Proposition 5.2.5 and Corollary 5.2.6, we have

$$\text{ch}^\omega(T_\mu^\lambda(M(y \circ \lambda))) = \text{ch}^\omega(M(y \circ \mu)) \quad \text{for each } y \in W(\lambda) \cap \widetilde{W},$$

and  $\text{ch}^\omega(T_\mu^\lambda(L(\lambda))) = 0$ . Consequently, we obtain

$$0 = \sum_{y \in W(\lambda) \cap \widetilde{W}} c(y) \text{ch}^\omega(M(y \circ \mu)).$$

Here we note that the twining characters  $\text{ch}^\omega(M(\xi))$ ,  $\xi \in (\mathfrak{h}^*)^0$ , are linearly independent over  $\mathbb{C}$ . Thus we get, for each  $y \in W(\lambda) \cap \widetilde{W}$ ,

$$0 = \sum_{\substack{w \in W(\lambda) \cap \widetilde{W} \\ w \circ \mu = y \circ \mu}} c(w).$$

Now let  $w \in W(\lambda) \cap \widetilde{W}$  be such that  $w \circ \mu = \mu$ . Since  $(\mu + \rho|\psi_j) \geq 0$  for all  $j \in \widetilde{J}$ , we have  $(\mu + \rho|\phi_j) \geq 0$  for all  $j \in J$  since  $\omega^*(\mu + \rho) = \mu + \rho$ . Hence, by [MP, Prop. 5.6.3], we see that  $w(\mu + \rho) = \mu + \rho$  for  $w \in W(\lambda)$  implies

$$w \in \langle r_{\phi_{(\omega^\lambda)^k(j_0)}} \mid 0 \leq k \leq N_{j_0}^\lambda - 1 \rangle \subset W(\lambda).$$

Recall from §6.3 that there exists a group isomorphism  $\Xi : W^\lambda \rightarrow W(\lambda)$  satisfying  $\Xi(r_j^\lambda) = r_{\phi_j}$  for each  $j \in J$ . Thus we have

$$w' := \Xi^{-1}(w) \in \langle r_{(\omega^\lambda)^k(j_0)}^\lambda \mid 0 \leq k \leq N_{j_0}^\lambda - 1 \rangle \subset W^\lambda.$$

Furthermore, by Lemma 6.3.1, we have that  $w' \in \widetilde{W}^\lambda$ . Let  $\zeta \in (\mathfrak{h}^\lambda)^*$  be such that  $(\omega^\lambda)^*(\zeta) = \zeta$  and  $(\zeta|\beta_j)^\lambda > 0$  for all  $j \in J$ , where  $(\cdot|\cdot)^\lambda$  is the (induced) bilinear form on  $(\mathfrak{h}^\lambda)^*$ . Set  $\zeta' := w'(\zeta)$ . It follows that  $(\zeta'|\beta_{j_0})^\lambda > 0$  or  $(\zeta'|\beta_{j_0})^\lambda < 0$  since  $(\zeta'|\beta_{j_0})^\lambda = (w'(\zeta)|\beta_{j_0})^\lambda = (\zeta|(w')^{-1}(\beta_{j_0}))^\lambda$  and

$$(w')^{-1}(\beta_{j_0}) \in \left( \sum_{0 \leq k \leq N_{j_0}^\lambda - 1} \mathbb{Z}_{\geq 0} \beta_{(\omega^\lambda)^k(j_0)} \right) \cup \left( \sum_{0 \leq k \leq N_{j_0}^\lambda - 1} \mathbb{Z}_{\leq 0} \beta_{(\omega^\lambda)^k(j_0)} \right).$$

Hence we have either  $(\zeta'|\beta_{(\omega^\lambda)^k(j_0)})^\lambda > 0$  for all  $0 \leq k \leq N_{j_0}^\lambda - 1$ , or  $(\zeta'|\beta_{(\omega^\lambda)^k(j_0)})^\lambda < 0$  for all  $0 \leq k \leq N_{j_0}^\lambda - 1$ , since  $(\omega^\lambda)^*(\zeta') = \zeta'$ . It is known from [FSS, §5.1] that  $w_{j_0}^\lambda(\beta_{j_0}) \in \sum_{0 \leq k \leq N_{j_0}^\lambda - 1} \mathbb{Z}_{\leq 0} \beta_{(\omega^\lambda)^k(j_0)}$ . So we have either  $(\zeta'|\beta_{j_0})^\lambda > 0$  or  $(w_{j_0}^\lambda(\zeta')|\beta_{j_0})^\lambda > 0$ , and hence either  $(\zeta'|\beta_{(\omega^\lambda)^k(j_0)})^\lambda > 0$  for all  $0 \leq k \leq N_{j_0}^\lambda - 1$ , or  $(w_{j_0}^\lambda(\zeta')|\beta_{(\omega^\lambda)^k(j_0)})^\lambda > 0$  for all  $0 \leq k \leq N_{j_0}^\lambda - 1$ , since  $w_{j_0}^\lambda \in \widetilde{W}^\lambda$  implies  $(\omega^\lambda)^*(w_{j_0}^\lambda(\zeta')) = w_{j_0}^\lambda(\zeta')$ . Thus, by (the proof of) [Ka, Prop. 3.12 a) and b)], we deduce that  $w'(\zeta) = \zeta$  or  $w_{j_0}^\lambda w'(\zeta) = \zeta$ , and hence  $w' = 1$  or  $w' = w_{j_0}^\lambda$ , since  $w_{j_0}^\lambda \in \widetilde{W}^\lambda$  is generated by  $r_{(\omega^\lambda)^k(j_0)}^\lambda$  for  $0 \leq k \leq N_{j_0}^\lambda - 1$ . Therefore, we obtain that  $w = 1$  or  $w = s_{j_0}$ , where  $s_{j_0} = \Xi(w_{j_0}^\lambda)$ .

Summarizing the arguments above, we have  $c(y) + c(ys_{j_0}) = 0$  for  $y \in W(\lambda) \cap \widetilde{W}$ . Since  $j_0 \in \check{J}$  is arbitrary, we obtain

$$(6.4.5) \quad c(y) + c(ys_j) = 0 \quad \text{for } y \in W(\lambda) \cap \widetilde{W} \text{ and } j \in \check{J},$$

where  $c(1) = 1$ . Here we note that  $W(\lambda) \cap \widetilde{W}$  is a Coxeter group with the canonical generator system  $\{s_j \mid j \in \check{J}\}$ . Thus, we see by induction on the length  $\widehat{\ell}_\lambda(y)$  of  $y \in W(\lambda) \cap \widetilde{W}$  that the numbers  $c(y)$  for  $y \in W(\lambda) \cap \widetilde{W}$  are determined uniquely by the relations (6.4.5) together with  $c(1) = 1$ . Furthermore, the numbers  $(-1)^{\widehat{\ell}_\lambda(y)}$  satisfy the relations (6.4.5) together with  $(-1)^{\widehat{\ell}_\lambda(1)} = (-1)^0 = 1$ . Therefore we deduce that

$$(6.4.6) \quad c(y) = (-1)^{\widehat{\ell}_\lambda(y)} \quad \text{for } y \in W(\lambda) \cap \widetilde{W}.$$

Combining (6.4.6) with Theorem 4.4, we complete the proof of the theorem.  $\square$

*Remark 6.4.5.* We have assumed an additional condition on  $\lambda$  in part (b) of Theorem 6.4.4 in order to make the clever choice of the translation functor  $T_\mu^\lambda$ . We expect that the same formula will still hold without this condition.

*Remark 6.4.6.* If the symmetric weight  $\lambda \in (\mathfrak{h}^*)^0$  is dominant integral, i.e.,  $\lambda \in P_+$ , then  $\lambda$  satisfies the assumption of Theorem 6.4.4 (b). In this case, we have  $W(\lambda) = W$  and hence  $W(\lambda) \cap \widetilde{W} = \widetilde{W}$ . Then, Theorem 6.4.4 is nothing but Theorem 4.5, which is the main result of [FSS]. Recall from the comment just above Proposition 3.2.2 that  $\widetilde{W} = \{1\}$  if and only if the GCM  $A = (a_{ij})_{i,j \in I}$  is of type  $A_{n-1}^{(1)}$  and the diagram automorphism  $\omega : I \rightarrow I$  is a cyclic permutation of  $I$  of order  $n$ .

*Remark 6.4.7.* If the diagram automorphism  $\omega : I \rightarrow I$  is the identity map, then any element  $\lambda \in \mathcal{C}^+$  such that  $\Delta_0(\lambda) = \emptyset$  satisfies the assumption of Theorem 6.4.4 (b). This is because when we write the dual real root  $h_\alpha$  of  $\alpha \in \Delta^{re}$  as  $h_\alpha = \sum_{i \in I} l_i^\alpha h_i \in Q^\vee$ , the integers  $l_i^\alpha$  for  $i \in I$  are relatively prime. In this case, we have  $\widetilde{W} = W$ , and hence  $W(\lambda) \cap \widetilde{W} = W(\lambda)$ . Then, Theorem 6.4.4 is just the well-known character formula by Kac-Wakimoto ([KW]) for the ordinary character  $\text{ch} L(\lambda)$ .

*Remark 6.4.8.* Theorem 6.4.4 holds also for a finite-dimensional simple Lie algebra  $\mathfrak{g}$  when  $\mathcal{C}$  is replaced by  $\mathfrak{h}^*$  and so  $\mathcal{C}^+$  by the set

$$\{\xi \in \mathfrak{h}^* \mid (\xi + \rho)(h_\alpha) \geq 0 \quad \text{for all } \alpha \in \Delta(\xi)_+\}.$$

Since the proof in the finite-dimensional case is similar to and simpler than that in the affine case, we omit it.

## REFERENCES

- [B] R. E. Borcherds, *Generalized Kac-Moody algebras*, J. Algebra **115** (1988), 501–512. MR **89g**:17004
- [DGK] V. V. Deodhar, O. Gabber, and V. Kac, *Structure of some categories of representations of infinite-dimensional Lie algebras*, Adv. Math. **45** (1982), 92–116. MR **83i**:17012
- [FRS] J. Fuchs, U. Ray, and C. Schweigert, *Some automorphisms of generalized Kac-Moody algebras*, J. Algebra **191** (1997), 518–540. MR **98i**:17025
- [FSS] J. Fuchs, B. Schellekens, and C. Schweigert, *From Dynkin diagram symmetries to fixed point structures*, Comm. Math. Phys. **180** (1996), 39–97. MR **98i**:17021
- [J] J. C. Jantzen, *Moduln mit einem Höchsten Gewicht*, Lecture Notes in Math. Vol. 750, Springer-Verlag, Berlin, 1979. MR **81m**:17011

- [Ka] V. G. Kac, *Infinite Dimensional Lie Algebras*, (3rd edition), Cambridge Univ. Press, Cambridge, 1990. MR **92k**:17038
- [KW] V. G. Kac and M. Wakimoto, *Modular invariant representations of infinite-dimensional Lie algebras and superalgebras*, Proc. Natl. Acad. Sci. USA **85** (1988), 4956–4960. MR **89j**:17019
- [KK] S.-J. Kang and J.-H. Kwon, *Graded Lie superalgebras, supertrace formula, and orbit Lie superalgebras*, Proc. London Math. Soc. **81** (2000), 675–724. MR **2001f**:17057
- [KT1] M. Kashiwara and T. Tanisaki, *Kazhdan-Lusztig conjecture for symmetrizable Kac-Moody Lie algebras. III—Positive rational case—*, Asian J. Math. **2** (1998), 779–832. MR **2001f**:17050
- [KT2] M. Kashiwara and T. Tanisaki, *Characters of irreducible modules with non-critical highest weights over affine Lie algebras*, Representations and Quantizations (Shanghai, 1998), China High. Educ. Press, Beijing, 2000, pp. 275–296. MR **2001k**:17037
- [Ku1] S. Kumar, *Extension of the category  $\mathcal{O}^g$  and a vanishing theorem for the Ext functor for Kac-Moody algebras*, J. Algebra **108** (1987), 472–491. MR **89h**:17038
- [Ku2] S. Kumar, *Toward proof of Lusztig’s conjecture concerning negative level representations of affine Lie algebras*, J. Algebra **164** (1994), 515–527. MR **95h**:17028
- [MP] R. V. Moody and A. Pianzola, *Lie Algebras With Triangular Decompositions*, Canadian Mathematical Society series of monographs and advanced texts, A Wiley-Interscience Publication, John Wiley & Sons, New York, 1995. MR **96d**:17025
- [N1] S. Naito, *Embedding into Kac-Moody algebras and construction of folding subalgebras for generalized Kac-Moody algebras*, Japan. J. Math. (New Series) **18** (1992), 155–171. MR **93h**:17062
- [N2] S. Naito, *Character formula of Kac-Wakimoto type for generalized Kac-Moody algebras*, J. Pure Appl. Algebra **166** (2002), 105–123.

INSTITUTE OF MATHEMATICS, UNIVERSITY OF TSUKUBA, TSUKUBA, IBARAKI 305-8571, JAPAN  
E-mail address: [naito@math.tsukuba.ac.jp](mailto:naito@math.tsukuba.ac.jp)