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PRINCIPAL NILPOTENT ORBITS AND REDUCIBLE PRINCIPAL SERIES

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Abstract. Let G be a split reductive p-adic group. In this paper, we establish an explicit link between principal nilpotent orbits of G and the irreducible constituents of principal series of G. A geometric characterization of certain irreducible constituents is also provided.

1. Introduction

Let G be a split reductive p-adic group. That is, $G = \mathbf{G}(F)$, where $\mathbf{G}$ is a split reductive algebraic group defined over a p-adic local field F of characteristic zero. Let $\mathfrak{g}$ be its Lie algebra. Then G acts on $\mathfrak{g}$ by the adjoint action which breaks $\mathfrak{g}$ up into G-orbits. The set of nilpotent elements, being G-stable, breaks up into *nilpotent orbits*. Let $\mathcal{N}$ be the set of nilpotent orbits. Similarly, the set of all regular nilpotent elements breaks up into G-orbits, and are called *principal nilpotent orbits*.

Let B be a Borel subgroup of G. B can be written as $B = TN$, where T is a maximal split torus and N is a maximal nilpotent subgroup. Let λ be a unitary character of T. We can extend λ to B by letting it act trivially on N, and then viewing λ as a character of B. We denote the induced representation $\mathrm{Ind}_B^G\lambda$ by π_λ. A representation π_λ arising in this way is called a *unitary principal series representation*. Because the character λ is unitary, so is π_λ. Since a unitary representation is semisimple, we can decompose π_λ into the sum of its irreducible constituents: $\pi_\lambda = \bigoplus_{\xi\in\Sigma_\lambda} m_\xi \xi$, where Σ_λ is the set of all irreducible constituents of π_λ and m_ξ is the multiplicity of ξ in π_λ.

Studying the irreducible constituents of unitary principal series representations is a very interesting problem in representation theory for the following reason. The Langlands correspondence predicts that the set of irreducible admissible representations of G breaks up into finite sets (called L-packets) indexed by the *Langlands parameters*. The Langlands parameters are homomorphisms from the Weil-Deligne group W_F' into the L-group ${}^L\mathbf{G}$ of G. By the principle of functoriality (a consequence of the Langlands correspondence), the irreducible constituents of a single unitary principal representation π_λ should be in a single L-packet. In other words, the various irreducible constituents in Σ_λ are *L-indistinguishable* in the sense of Langlands [3]. In general, understanding the structure of an L-packet is a very delicate question. Therefore, understanding irreducible constituents of unitary principal series representations offers some new insights into this general problem.

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The purpose of this paper is to find links between the set $\mathcal{P}$ of principal nilpotent orbits and the set Σ_λ of irreducible constituents of π_λ. At first glance, it seems that there is no obvious relation between them. Yet there is indeed a link, not in literature but known to experts, in the case of real groups. For a split reductive real Lie group G, the group R^{uni}, consisting of the central elements of G of order two, acts on $\mathcal{P}$ simply transitively. Given a unitary principal series representation π_λ, Knapp [13], [14] and Knapp and Stein [15] constructed a subgroup R_λ of the Weyl group W of G, called the R-group. For any $w \in R_\lambda$, we can define a normalized intertwining operator $\iota_\lambda(w)$ intertwining π_λ with itself. The normalization is not canonical but depends on the choice of an irreducible constituent of π_λ. It is proved that the map ι_λ induces an algebra isomorphism from the group algebra $\mathbb{C}[R_\lambda]$ to the commuting algebra $\mathcal{C}(\pi_\lambda)$ of π_λ. Furthermore, R_λ is isomorphic to a direct product of copies of $\mathbb{Z}/2\mathbb{Z}$; in particular, it is abelian. Therefore, the set $R_\lambda^\wedge$ of the characters of irreducible representations of R_λ has an abelian group structure. By the virtue of the isomorphism ι_λ, $R_\lambda^\wedge$ can be identified with Σ_λ, but not canonically.

Through representation theory, one can construct a surjective set map ρ from $\mathcal{P}$ to Σ_λ. If we choose the basepoint of $\mathcal{P}$ and an irreducible constituent of π_λ simultaneously, we can identify $\mathcal{P}$ and Σ_λ with R^{uni} and $R_\lambda^\wedge$ respectively. By these identifications, ρ induces a surjective set map $\overline{\rho}$ from R^{uni} to $R_\lambda^\wedge$. Remarkably, $\overline{\rho}$, being only a map between a priori sets, is actually is a group homomorphism, and is canonical; i.e., it is independent of the choices for basepoints. In other words, if Q_λ denotes the kernel of $\overline{\rho}$, then two principal nilpotent orbits have the same image under ρ if and only if they are in the same Q_λ-orbit.

Furthermore, we have a geometric characterization of irreducible constituents associated via $\overline{\rho}$ with the subsets of the principal nilpotent orbits for the real groups. Barbasch and Vogan [1] showed that representations of G have the asymptotic expansion of the distribution characters; especially the Fourier transform of the first terms of the expansion is a linear combination of the G-invariant measures on certain nilpotent orbits. Kostant [16] and Vogan [25] proved that a representation admits a Whittaker model if and only if the nilpotent orbits which appear in the first terms of the expansion are principal nilpotent orbits. In the case of principal series representations, since they admit Whittaker models, the first terms are principal nilpotent orbits. By theorems of Matumoto [17] and the multiplicity one theorem in [16], all leading coefficients are 1. Given a principal nilpotent orbit $\mathcal{O}$, the leading terms of the character expansion of the irreducible constituent $\rho(\mathcal{O})$ are exactly the orbits $Q_\lambda \cdot \mathcal{O}$ in $\mathcal{P}$.

The results above are proved as follows. Let $\overline{\mathbf{G}}$ be the adjoint group of $\mathbf{G}$ and p the projection map from $\mathbf{G}$ to $\overline{\mathbf{G}}$. The group $\overline{\mathbf{G}}(\mathbb{R})$ has a unique principal nilpotent orbit, and furthermore, its unitary principal series representations are always irreducible. The group $p(\mathbf{G}(\mathbb{R}))$ is a subgroup of $\overline{\mathbf{G}}(\mathbb{R})$, and it can be viewed as a subgroup of $\overline{\mathbf{G}}(\mathbb{C})$ via the inclusion from $\overline{\mathbf{G}}(\mathbb{R})$ to $\overline{\mathbf{G}}(\mathbb{C})$. Since the projection map p can be defined from $\overline{\mathbf{G}}(\mathbb{C})$ to $\overline{\mathbf{G}}(\mathbb{C})$, we define the group $\widetilde{\mathbf{G}}(\mathbb{R})$ as the subgroup $p^{-1}\big(p(\mathbf{G}(\mathbb{R}))\big)$ of $\mathbf{G}(\mathbb{C})$, where we view $p(\mathbf{G}(\mathbb{R}))$ as a subgroup of $\overline{\mathbf{G}}(\mathbb{C})$. Then $\widetilde{\mathbf{G}}(\mathbb{R})$ has also only one principal nilpotent orbit and we can extend the principal series representation π_λ of $\mathbf{G}(\mathbb{R})$ to the representation $\widetilde{\pi}_\lambda$ of $\widetilde{\mathbf{G}}(\mathbb{R})$, which is also irreducible. π_λ and $\widetilde{\pi}_\lambda$ share the same representation space. From this, it is easy to deduce that the R-group R_λ of π_λ is $\widetilde{\mathbf{G}}(\mathbb{R})/\mathbf{G}(\mathbb{R})$. The group $\widetilde{\mathbf{G}}(\mathbb{R})/\mathbf{G}(\mathbb{R})$ is isomorphic to $\overline{\mathbf{G}}(\mathbb{R})/p(\mathbf{G}(\mathbb{R}))$. One can show that the group $\overline{\mathbf{G}}(\mathbb{R})/p(\mathbf{G}(\mathbb{R}))$ can be identified with a subgroup of the dual group $R^\wedge$ of R^{uni} and is isomorphic to R_λ. Then the injective map from R_λ to $R^\wedge$, through $\overline{\mathbf{G}}(\mathbb{R})/\mathbf{G}(\mathbb{R})$, is the dual map of $\overline{\rho}$. The geometric statement can be proved by the theorems in [17].

It is Harish-Chandra's belief that the results in the archimedean cases should have nonarchimedean analogues. He called this phenomenon the *Lefschetz Principle*. Guided by this philosophy, it is natural to ask whether a similar link between $\mathcal{P}$ and Σ_λ exists in the nonarchimedean case.

However, development on the nonarchimedean side has been rather slow. For SL_2, Gel'fand, Graev, and Pyatetskii-Shapiro (see [9]) have given a complete description of their irreducible constituents. The representation space of a unitary principal series representation π_λ can be realized on the space of functions on the maximal nilpotent subgroup N, which is isomorphic to $\mathbb{G}_a$. The dual Lie algebra $\mathfrak{n}^*$ of N can be partitioned into two parts according to λ. After we take the Fourier transform, the irreducible constituents can be characterized as functions supported on one of two parts of $\mathfrak{n}^*$. Gelbart and Knapp [7] introduced a new technique to treat SL_n. They provide a description of irreducible constituents through a series of Fourier transforms on certain vector spaces. In another paper [8], they also found relations with the Langlands program.

The above works deal only with SL_n and do not involve principal nilpotent orbits. In this paper, we will prove the analogous statement for split reductive p-adic groups. More precisely, we will give an explicit relation between principal nilpotent orbits and *generic* irreducible constituents of principal series representation, where generic constituents are those which admit Whittaker models. The major difference between the archimedean case and the nonarchimedean one is that R-groups are not abelian in general. Hence, it is impossible for the map ρ to be surjective in general. The adjustment needed for the nonarchimedean case is to replace irreducible constituents with generic ones.

On the geometric side, let Γ be the Galois group $\mathrm{Gal}(\overline{F}/F)$, where $\overline{F}$ is the algebraic closure of F. Let $\underline{G} = \mathbf{G}(\overline{F})$, the $\overline{F}$ points of $\mathbf{G}$, and $Z(\underline{G})$ be the center of $\underline{G}$. Then the first Galois cohomology group $H^1(\Gamma, Z(\underline{G}))$ acts simply transitively on the set $\mathcal{P}$ of principal nilpotent orbits. On the representation side, a unitary principal series representation is not irreducible in general, and its decomposition is determined by its commuting algebra $\mathcal{C}(\lambda) = \mathrm{End}(\pi_\lambda)$ of intertwining operators. In particular, there is a bijection $\xi \mapsto r_\xi$ between Σ_λ, the set of irreducible constituents of π_λ, and $R_\lambda^\wedge$, the set of irreducible characters of an explicit finite subgroup R_λ of Weyl group $W(T, G)$.

We introduce two more constructions into the picture:

– for each $\mathcal{O} \in \mathcal{P}$, we construct a generic irreducible constituent $\rho(\mathcal{O})$ of π_λ;
– we construct a canonical pairing $\langle , \rangle \colon R_\lambda \times H^1(\Gamma, Z(\underline{G})) \to \mathbb{C}^*$.

The main result is:

Theorem. (i) *The construction* $\rho\colon \mathcal{P} \to \Sigma_\lambda$ *induces a bijection* $Q_\lambda \backslash \mathcal{P} \xrightarrow{\sim} \Sigma_\lambda^{\mathrm{gen}}$, *where* Q_λ *is the right kernel of* $\langle , \rangle$, *and* $\Sigma_\lambda^{\mathrm{gen}}$ *is the subset of generic representations in* Σ_λ.

(ii) *The composite* $\mathcal{P} \xrightarrow{\rho} \Sigma_\lambda \xrightarrow{r} \mathrm{Hom}(R_\lambda, \mathbb{C}^*)$ *is the same as the composite* $\mathcal{P} \simeq H^1(\Gamma, Z(\underline{G})) \xrightarrow{\langle , - \rangle_\lambda^{-1}} \mathrm{Hom}(R_\lambda, \mathbb{C}^*)$.

The construction $r\colon \Sigma_\lambda \to \mathrm{Hom}(R_\lambda, \mathbb{C}^*)$ depends on the choice of a generic representation, which determines an identification $\mathcal{P} \simeq H^1(\Gamma, Z(\underline{G}))$.

The construction $\mathcal{O} \mapsto \rho(\mathcal{O})$ can also be characterized in terms of Harish-Chandra's local character expansion. The following theorem gives a geometric description of generic irreducible constituents.

Theorem. *The set of principal nilpotent orbits appearing in the local character expansion of $\rho(\mathcal{O})$ is precisely $Q_\lambda \cdot \mathcal{O}$.*

Our proof of the above results relies on four ingredients, namely, intertwining operators, Whittaker functionals, R-groups and local character expansions. The relations among these four can be described in the following way.

We start with a unitary character λ of T. For all w in $W(T, G)$, we can define a formal intertwining operator $A(\lambda, w)$ from π_λ to $\pi_{w\lambda}$; $A(\lambda, w)$ can be proved to be well-defined in the sense of analytic continuation. Consider the subgroup W_λ of $W(T, G)$ consisting of all elements which fix λ. Then for all w in W_λ, $A(\lambda, w)$ is an endomorphism of π_λ. However, the map from W_λ to the commuting algebra $\mathcal{C}(\pi_\lambda)$ by $A(\lambda, w)$ is not a homomorphism.

Now given a generic character χ of N, one can prove that π_λ has a unique Whittaker functional $\delta_{\lambda,\chi}$ associated with χ. Using the uniqueness of the Whittaker functional, one can define the local coefficient $C(\lambda, \chi, w)$. Let $\mathfrak{a}(\lambda, \chi, w)$ be the product of $C(\lambda, \chi, w)$ and $A(\lambda, w)$. Then the new map $\iota_{\lambda,\chi}$ from W_λ to $\mathcal{C}(\pi_\lambda)$ by $\mathfrak{a}(\lambda, \chi, w)$ is a homomorphism. It can be proved that there is a subgroup R_λ of W_λ, called the R-group, such that the homomorphism from $\mathbb{C}[R_\lambda]$ to $\mathcal{C}(\pi_\lambda)$ induced by $\iota_{\lambda,\chi}$ is an isomorphism. Therefore, $\iota_{\lambda,\chi}$ induces a bijection between Σ_λ and the set of the irreducible representations of R_λ.

By a theorem of Harish-Chandra, any representation has a local character expansion, which is a sum over all nilpotent orbits. In the case of π_λ, the leading terms are principal nilpotent orbits and the coefficients of leading terms are all equal to 1. Fix a principal nilpotent orbit $\mathcal{O}$. We vary w in R_λ and look at the coefficient $c_{\mathcal{O}}(w)$ of $\mathcal{O}$ in the local character expansion of $\mathfrak{a}(\lambda, \chi, w)\pi_\lambda$. We can prove that the map $c_{\mathcal{O}}(w)$ is a character of R_λ; therefore $c_{\mathcal{O}}(w)$ is an irreducible representation of R_λ. Now we can define a map ρ from $\mathcal{P}$ to Σ_λ as follows: given an $\mathcal{O}$ in $\mathcal{P}$, the character $c_{\mathcal{O}}(w)$ is an irreducible representation of R_λ, and through the bijection described in the previous paragraph, we can associate it with the element $\rho(\mathcal{O})$ of Σ_λ.

To get an explicit description of ρ, the key is to get a criterion for when two different principal nilpotent orbits map to the same character of R_λ. The answer lies on the normalization of formal intertwining operators; i.e., the local coefficient $C(\lambda, \chi, w)$. Given a principal nilpotent orbit $\mathcal{O}$ and an element Y in the intersection of $\mathcal{O}$ and $\bar{\mathfrak{n}}$, we can construct a generic character χ_Y of N. Different choices of Y do not change the associated Whittaker models; therefore, it makes sense that we use the notation $C(\lambda, \mathcal{O}, w)$. The crucial point is that for any two principal nilpotent orbits $\mathcal{O}$ and $\mathcal{O}'$, $\rho(\mathcal{O})$ is equal to $\rho(\mathcal{O}')$ if and only if for all w in R_λ, $C(\lambda, \mathcal{O}, w)$ is equal to $C(\lambda, \mathcal{O}', w)$. It remains to compute the local coefficient $C(\lambda, \mathcal{O}', w)$. We write down concrete Whittaker vectors and use the factorization of local coefficients to reduce to the SL_2 case. After expressing local coefficients explicitly, we get our theorem.

This paper is organized as follows. In §2, we define the intertwining operators between unitary principal series representations and we move on to the multiplicity one theorem, Whittaker functionals and local coefficients. Based on local coefficients, we define the normalized intertwining operators and state Keys' reducibility theorem. In §3, we will summarize the results of Mœglin and Waldspurger in [18]. Their results are about the coefficients of local character expansion, which provide the foundation of the geometric side of our theorems. In §4, we formulate theorems linking principal nilpotent orbits and the irreducible constituents of unitary principal series representations; mainly we use the Whittaker models and their relation to the coefficients of local character expansion. In the final section, we prove our theorems. The main technique is to compute the local coefficients for different generic characters.

1.1. **Preliminary Notations.**

1.1.1. Let F be a nonarchimedean local field of characteristic 0; i.e., a finite extension of $\mathbb{Q}_p$ for some prime number p. Furthermore, we require $p \neq 2$. Let $\overline{F}$ be an algebraic closure of F and Γ the Galois group of $\overline{F}$ over F. Let $\mathfrak{D}$ be the ring of integers of F. Fix a uniformizing element ϖ of F. Let q be the cardinality of $\mathfrak{D}/\varpi\mathfrak{D}$, $|\cdot|_F$ the normalized absolute value on F and v_F the normalized valuation on F; i.e., $|x|_F = q^{-v_F(x)}$, for all $x \in F$. We write ψ for a chosen additive character of F which is trivial on $\mathfrak{D}$ and nontrivial on $\varpi^{-1}\mathfrak{D}$.

1.1.2. Let $\mathbf{G}$ be a connected split reductive group defined over F. Fix a Borel subgroup $\mathbf{B}$ of $\mathbf{G}$ and write $\mathbf{B} = \mathbf{TN}$, where $\mathbf{T}$ and $\mathbf{N}$ represent a split maximal torus and the unipotent radical of $\mathbf{B}$ respectively. Write $\mathbf{N}^-$ as the unipotent radical of the Borel subgroup opposite to $\mathbf{B}$. Let Δ be the set of roots and $\Delta^\vee$ the set of coroots. The Borel subgroup $\mathbf{B}$ determines the set Δ^+ of positive roots. Let Π (resp. Π^-) be the set of simple roots contained in Δ^+ (resp. Δ^-). A standard parabolic $\mathbf{P}$ of $\mathbf{G}$ is a parabolic subgroup $\mathbf{P}$ containing $\mathbf{B}$. There is a one-to-one correspondence between the set of standard parabolic subgroups $\mathbf{P}$ of $\mathbf{G}$ and the set of subsets $\Pi_{\mathbf{P}}$ of Π. Write $\mathbf{P} = \mathbf{M_P N_P}$, where $\mathbf{M_P} \supset \mathbf{T}$ is a Levi factor, and $\mathbf{N_P} \subset \mathbf{N}$ is the unipotent radical. Let $\mathbf{A_P}$ be the split component in the center of $\mathbf{M_P}$. Let $W(\mathbf{A_P})$ be the Weyl group of $\mathbf{A_P}$ in $\mathbf{G}$. Sometimes we use $W(\mathbf{A_P})$ or just W for the Weyl group if there is no confusion. Define $\mathbf{N_P^-}$ in the same way as $\mathbf{N}^-$. We will use G to denote $\mathbf{G}(F)$. Similarly, for K, B, T, N, P, M_P, N_P, N_P^-, and A_P. By abuse of notation, we use $\underline{G}$ to denote $\mathbf{G}(\overline{F})$ if there is no confusion. For a subset θ of Π, we denote by $\mathbf{P}_\theta$ the corresponding standard parabolic subgroup, and by $\mathbf{M}_\theta$ the Levi factor of $\mathbf{P}_\theta$. Similar notation holds for other subgroups.

1.1.3. For each root α, there is a homomorphism ζ_α from $SL(2,F)$ into a subgroup of G. Define x_α, y_α, and h_α as follows:

$$\zeta_\alpha \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} = x_\alpha(t), \quad \zeta_\alpha \begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix} = y_\alpha(t), \text{ and } \zeta_\alpha \begin{pmatrix} s & 0 \\ 0 & s^{-1} \end{pmatrix} = h_\alpha(s),$$

where $t \in F$ and $s \in F^*$. Assume that $x_{-\alpha} = y_\alpha$ and h_α is the coroot $\alpha^\vee$ associated with α. Let $X(T)$ be the set of the rational characters of T and $X_*(T)$ the set of 1-parameter subgroups. Then the set $\Delta^\vee$ of coroots is a subset of $X_*(T)$. We write $N_{\alpha,n} = \{x_\alpha(t) | v_F(t) \geq n\}$.

Let $W = N(T)/T$ be the Weyl group of G. For each root α, let s_α be the reflection of W associated to α. Then the action of s_α on T is defined by $t \mapsto t(\alpha^\vee \circ \alpha(t))^{-1}$. Define $\tilde{s}_\alpha$ to be

$$\tilde{s}_\alpha = x_\alpha(1)x_{-\alpha}(-1)x_\alpha(1) = \zeta_\alpha \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

It is a representative of s_α in G. Throughout this paper, we fix the choices of ζ_α. Then they determine the representatives $\tilde{s}_\alpha$ of s_α.

1.1.4. Let the German letter $\mathfrak{g}$ denote the Lie algebra of the group G. Let κ be a symmetric nondegenerate G-invariant bilinear form on $\mathfrak{g}$ with values in F. G acts on $\mathfrak{g}$ by the adjoint representation. We know $\mathfrak{g}$ can be decomposed as the direct sum of one-dimensional eigenspaces $\mathfrak{n}_\alpha$ of T with eigencharacters $\alpha \in \Delta$ and the Lie algebra $\mathfrak{t}$ of T. Also, there is a nondegenerate, invariant bilinear form κ on $\mathfrak{g}$. For all $\alpha \in \Delta^+$, let X_α be the element of $\mathfrak{n}_\alpha$ such that $x_\alpha(t) = \exp(tX_\alpha)$. Similarly, for all $\beta \in \Delta^-$, define $Y_\beta \in \mathfrak{n}_\beta$ so that $x_\beta(t) = \exp(tY_\beta)$.

2. Intertwining Operators and Reducible Principal Series Representations

In this section, we will describe the relations among intertwining operators, commuting algebras of principal series, and Whittaker functionals. These are the foundations of our theorems. In §2.1, we define the intertwining operators and their factorization and analytic continuations; mostly we follow Shahidi [22]. Next, in §2.2, we state Rodier's multiplicity one theorem of Whittaker functionals from Casselman and Shalika [5]. Then we use the language of Whittaker functionals to define the local coefficients and their factorization, which are also taken from Shahidi [22]. In §2.3, we state Keys' structure theorem of the commuting algebras of principal series representations.

2.1. **Intertwining operators.** In this subsection, we define a memormorphic family of intertwining operators. These are first given by integrals, which only converge on suitable domains. Harish-Chandra showed that they could be extended by analytic continuation. Determining the pole of intertwining operators is a very hard problem. It is equivalent to finding the zeros of Plancherel measure. To study its properties, we have a factorization theorem of intertwining operators, which says that the intertwining operators have a factorization into the rank one intertwining operators. This theorem reduces the problem to the case of rank one. However, this reduction also requires the knowledge of the the zeros of rank one operators, in order to understand the cancellations between poles and zeros. So far the zeros and poles for rank one operators are still unknown in general.

2.1.1. Let $\mathbf{P}$ be a standard parabolic subgroup of $\mathbf{G}$. Let $X(\mathbf{M_P})_F$ be the group of F-rational characters of $\mathbf{M_P}$. We define

$$\mathfrak{a}_P = \mathrm{Hom}(X(\mathbf{M_P})_F, \mathbb{R}),$$

the real Lie algebra of $\mathbf{A_P}$. Then

$$\mathfrak{a}_P^* = X(\mathbf{M_P})_F \otimes_{\mathbb{Z}} \mathbb{R} = X(\mathbf{A_P})_F \otimes_{\mathbb{Z}} \mathbb{R}, \text{ and } (\mathfrak{a}_P^*)_{\mathbb{C}} = \mathfrak{a}_P^* \otimes_{\mathbb{R}} \mathbb{C}.$$

Set

$$\rho_P = \frac{1}{2} \sum_{\alpha \in \Delta^+ - \Delta^+_{\mathbf{P}}} \alpha,$$

where $\Delta^+_{\mathbf{P}}$ is the subset of positive roots in the linear span of $\Pi_{\mathbf{P}}$. In the case when $\mathbf{P}$ is the Borel subgroup $\mathbf{B}$, we will drop the subscript $\mathbf{P}$ for the corresponding notation.

Let H_P be the homomorphism from $\mathbf{M_P}$ to $\mathfrak{a}_P$ defined by

$$q^{\langle \eta, H_P(m)\rangle} = |\eta(m)|_F,$$

for all $\eta \in X(\mathbf{M_P})_F$ and $m \in M$.

2.1.2. Fix an irreducible unitary representation (σ, V_σ) of M. Let $(V_\sigma)_K$ be the subspace of K-finite vectors. That is,

$$(V_\sigma)_K = \{v \in V_\sigma | \dim(\langle k \cdot v\rangle_{k\in K_0}) < \infty, \text{ some open compact subgroup } K_0 \text{ in } G\}.$$

For all $\nu \in (\mathfrak{a}_P^*)_{\mathbb{C}}$, let $\pi_{\nu,\sigma}$ be the unitarily induced representation

$$\pi_{\nu,\sigma} = \mathrm{Ind}_{MN}^G \sigma \otimes q^{\langle \nu, H_p(\ )\rangle} \otimes 1.$$

More precisely, the representation space $V_{\nu,\sigma}$ of $\pi_{\nu,\sigma}$ consists of all the smooth functions f from G into $(V_\sigma)_K$ which satisfy

$$f(gnm) = \sigma(m^{-1}) q^{\langle -\nu-\rho_P, H_P(m)\rangle} f(g),$$

where $m \in M_P$ and $n \in N_p$. The representation $\pi_{\nu,\sigma}$ acts by left inverse translations. For $\nu = 0$, we write π_σ for $\pi_{0,\sigma}$ and V_σ for $V_{0,\sigma}$.

2.1.3. Fix a $w \in W(\mathbf{A})$ (the Weyl group of $\mathbf{G}$) such that $w(\Pi_{\mathbf{P}}) \subset \Pi$. We choose a reduced expression $w = \Pi s_{\alpha_i}$ and it determines a representative $\tilde{w} = \Pi \tilde{s}_{\alpha_i}$ of w in G. This representative does not depend on the choices of reduced expressions (for Chevalley group, see Part b of Lemma 83 of [24]; for the connected reductive p-adic group, see page 112 of [4]). Let $(N_P)_w = N_P \cap \tilde{w} N_P^- \tilde{w}^{-1}$. Given a function $f \in V_{\nu,\sigma}$, we define

$$A(\nu,\sigma,w)f(g) = \int_{(N_P)_w} f(gn\tilde{w})dn.$$

The integral converges absolutely if

$$(*) \qquad \Re\langle \nu, \alpha^\vee\rangle \gg 0, \quad \forall \alpha \in \Pi - \Pi_{\mathbf{P}}.$$

Moreover, it extends to a meromorphic function of ν on all of $(\mathfrak{a}_P^*)_{\mathbb{C}}$ (*cf.* [22]). In addition, away from its poles, it defines an intertwining operator between $\pi_{\nu,\sigma}$ and $\pi_{w\nu,w\sigma}$, where $w\sigma(m') = \sigma(w^{-1}m'w)$ with $m' \in M'_P = wM_Pw^{-1}$. We write $A(\sigma, w)$ for $A(0, \sigma, w)$.

Fix a proper subset θ of Π and $\alpha \in \Pi - \theta$. Let $\Omega = \theta \cup \{\alpha\}$. Define $\overline{\theta}$ to be $w_{l,\Omega} w_{l,\theta}\theta \subset \Omega$, where $w_{l,\Omega}$ and $w_{l,\theta}$ denote the longest elements in the Weyl groups of $\mathbf{M}_\Omega$ and $\mathbf{M}_\theta$ respectively. We call $\overline{\theta}$ the *conjugate* of θ in Ω. For two subsets θ, θ' of Π, let

$$W(\theta,\theta') = \{w \in W | w(\theta) = \theta'\}.$$

If $W(\theta,\theta')$ is nonempty, we say θ and θ' are *associated*.

The following lemma is the basis for the factorization of intertwining operators $A(\nu,\sigma,w)$.

2.1.4. Lemma (Shahidi [22], Lemma 2.1.2)**.** *Suppose θ and θ' are associated. Let $w \in W(\theta,\theta')$. There exists a family of subsets $\theta_1, \theta_2, \ldots, \theta_{n+1} \subset \Pi$ such that*

(i) *$\theta_1 = \theta$ and $\theta_{n+1} = \theta'$;*
(ii) *fix $1 \le i \le n$; then there is a root $\alpha_i \in \Pi - \theta_i$ such that θ_{i+1} is the conjugate of θ_i in $\Omega_i = \theta_i \cup \{\alpha_i\}$;*
(iii) *set $w_i = w_{l,\Omega_i} w_{l,\theta_i} \in W(\theta_i, \theta_{i+1})$ for $1 \le i \le n$, then $w = w_n w_{n-1} \cdots w_1$.*

2.1.5. Now we are ready to state a factorization theorem of intertwining operators $A(\nu, \sigma, w)$.

Theorem (Shahidi [22], Theorem 2.1.1)**.** *Fix $\theta, \theta' \in \Pi$ and choose $w \in W(\theta, \theta')$. Let $\theta_1, \theta_2, \ldots, \theta_{n+1} \subset \Pi$, and $w_i \in W(\theta_i, \theta_{i+1})$ be as in Lemma 2.1.4. Take $\nu \in (\mathfrak{a}_P^*)_{\mathbb{C}}$ which satisfies $(*)$. Then each ν_i also satisfies $(*)$ with respect to $\mathbf{A}_{\mathbf{P}_{\theta_i}}$ and*

$$(**) \qquad A(\nu, \sigma, w) = A(\nu_n, \sigma_n, w_n) \cdots A(\nu_1, \sigma_1, w_1),$$

where $\nu_i = w_{i-1}(\nu_{i-1})$, $\sigma_i = w_{i-1}(\sigma_{i-1})$, $2 \le i \le n$, $\nu_1 = \nu$, and $\sigma_1 = \sigma$. Moreover, $(**)$ holds for all $\nu \in (\mathfrak{a}_P^*)_{\mathbb{C}}$ away from their poles in the sense of analytic continuation.

Remark. In particular, if θ and θ' are empty sets, w can be expressed as a product of simple reflections and $A(\nu, \sigma, w)$ is the composition of rank one intertwining operators.

2.2. **Whittaker functionals and local coefficients I: definitions.** In this subsection, we state the multiplicity one result for principal series representations. Then we can define Whittaker functionals and local coefficients. We will also state some basic properties of local coefficients. The multiplicity one result is due to Rodier for the p-adic cases. Using the multiplicity one result and Whittaker functionals, Shahidi defined local coefficients, which are closely related to intertwining operators. Similarly, there is a factorization theorem for local coefficients. It reduces the computation of local coefficients to the case of rank one.

2.2.1. Let χ be a smooth complex character of N, and $\mathbb{C}_\chi$ the corresponding one-dimensional N-module.

Let (π, V) be any smooth representation of N. We define $V_{(\chi)}$ to be the Jacquet space of the twisted representation $\pi \otimes \chi^{-1}$; i.e., if $V_\chi(N)$ is the subspace of V spanned by $\{\pi(n)v - \chi(n)v | n \in N, v \in V\}$, then $V_{(\chi)} = V/V_\chi(N)$.

Proposition (Casselman and Shalika [5], Propositions 1.1 and 1.2)**.** *If V' is a space on which N acts by χ, then the functor $V \mapsto V_{(\chi)}$ induces an isomorphism*

$$\mathrm{Hom}_N(V, V') \cong \mathrm{Hom}_{\mathbb{C}}(V_{(\chi)}, V').$$

Moreover, the functor $V \mapsto V_{(\chi)}$ is exact.

2.2.2. Since $N/[N,N]$ is generated by N_α, $\alpha \in \Pi$, the character χ of N is determined uniquely by its restriction χ_α on each N_α. We fix a parameterization x_α of N_α (*cf.* §1.1.3), and identify N_α with $\mathbb{G}_a(F) \cong F$. In this setting, χ_α is an additive character of F. We say χ is *generic* if no χ_α is trivial. For the rest of this section, we assume χ is generic.

We define $\mathrm{Ind}_N^G \mathbb{C}_\chi$ to be the space of all $f \colon G \to \mathbb{C}$ such that

(1) $f(gn) = \chi^{-1}(n) f(g)$;
(2) there exists an open subgroup $K \subset G$ such that $f(kg) = f(g)$ for all $g \in G, k \in K$.

G acts on it by the left inverse translations. If (π, V) is an admissible representation of G, we call a G-embedding of V into $\mathrm{Ind}_N^G \mathbb{C}_\chi$ a *Whittaker model* for V. There is a close relationship between the space $V_{(\chi)}$ and Whittaker models. Combining Frobenius reciprocity and Proposition 2.2.1, we have the following proposition.

Proposition (Casselman and Shalika [5], Proposition 1.3)**.** *Let χ be a generic character of N, and V a smooth representation of G. Then there is a natural isomorphism:*

$$\mathrm{Hom}_G(V, \mathrm{Ind}_N^G \mathbb{C}_\chi) \cong \mathrm{Hom}_{\mathbb{C}}(V_{(\chi)}, \mathbb{C}).$$

We say a smooth admissible representation (π, V) *admits a Whittaker model* or is *generic* if there is a generic character χ of N such that $\mathrm{Hom}_G(V, \mathrm{Ind}_N^G \mathbb{C}_\chi)$ is nontrivial; in this case we call (π, V) χ*-generic* if we wish to specify the generic character χ. From Proposition 2.2.2, we know the multiplicity of (π, V) occurring in $\mathrm{Ind}_N^G \mathbb{C}_\chi$ is the dimension of $V_{(\chi)}$. If $\dim V_{(\chi)} \leq 1$, we say (π, V) satisfies the multiplicity one.

Given a smooth admissible representation (π, V) which is χ-generic, we call a linear functional δ on V a *Whittaker functional* if for all $n \in N, v \in V$,

$$\delta(\pi(n)v) = \chi(n)\delta(v).$$

By definition, we can view a Whittaker functional as an element of $\mathrm{Hom}_{\mathbb{C}}(V_{(\chi)}, \mathbb{C})$.

2.2.3. Fix a subset θ of Π. Let (σ, V) be an admissible representation of M_P, and (π_σ, V_σ) the induced representation of G. Because of the Bruhat decomposition, we have

$$G = \bigsqcup_{w \in W(A_\theta)'} PwP_\theta,$$

where $W(A_\theta)' = \{w \in W | w^{-1}(\theta) \subset \Delta^+\}$. Therefore V_σ is filtered by P-stable subspaces

$$V_n = \{f \in V_\sigma | \mathrm{Supp}(f) \subset \bigsqcup_{w \in W(A_\theta)'_n} PwP_\theta\},$$

where $W(A_\theta)'_n = \{w \in W(A_\theta)' | \dim(PwP_\theta/P_\theta) \geq n\}$. Let $w_{l,\theta}$ and w_l be the longest elements of $W(A_\theta)$ and W respectively. It follows that $w_\theta = w_{l,\theta} w_l$ is the longest element of $W(A_\theta)'$ and $Pw_\theta P_\theta/P_\theta$ is the unique open double coset in G/P_θ. Let d_θ be its dimension. Then we have a natural injection $i\colon V_{d_\theta} \hookrightarrow V_\sigma$.

Theorem (Rodier, *cf.* [5], Theorem 1.4)**.** *If χ is a generic character of N, then the inclusion $i\colon V_{d_\theta} \hookrightarrow V_\sigma$ induces an isomorphism of $(V_{d_\theta})_{(\chi)}$ with $(V_\sigma)_{(\chi)}$.*

If θ is the empty set, then $P_\theta = B$, $M_\theta = T$, and $N_\theta = N$. Let σ be a one-dimensional representation of T; i.e., a quasi-character of T. We have the following multiplicity one theorem and the construction of the Whittaker functional.

Proposition. (Casselman and Shalika [5], Corollary 1.8, and Shahidi [22], Proposition 3.1) *Assume P_θ is the Borel subgroup B of G, σ one-dimensional, χ a generic character of N, and $\tilde{w}_l \in N_G(T)$ representing the longest element of W (cf. §2.1.3). For all $f \in V_{d_\theta} \subset V_\sigma$, the space of the induced representation π_σ, the functional*

$$(\flat) \qquad \delta_{\sigma,\chi}(f) = \int_N f(n\tilde{w}_l)\chi^{-1}(n)dn,$$

extends uniquely to a basis element of the one-dimensional space $\mathrm{Hom}_N(V_\sigma, \mathbb{C}_\chi)$*. As a consequence,* $\dim\big((V_\sigma)_{(\chi)}\big)$ *is equal to 1. Furthermore, let λ be a character of T, and $(\pi_{\nu,\lambda}, V_{\nu,\lambda})$ the induced representation defined in §2.1.2. Then the functional $\delta_{\nu,\lambda,\chi}$ is an entire function of ν and, furthermore, for every ν and λ, $\delta_{\nu,\lambda,\chi}$ is nontrivial and is a basis element of* $\mathrm{Hom}_N(V_{\nu,\lambda}, \mathbb{C}_\chi)$.

2.2.4. From now on, we only consider the case $P = B$ and λ a character of T. By Proposition 2.2.3, for *any* generic character χ of N, the induced representation $(\pi_{\nu,\sigma}, V_{\nu,\sigma})$ has the multiplicity one theorem and a canonical choice of the Whittaker functional $\delta_{\nu,\lambda,\chi}$. Recall that we define an intertwining operator $A(\nu, \lambda, w)$ in §2.1.3. We have the following theorem for the change of Whittaker functionals for the intertwined representations.

Theorem (Shahidi [22], Theorem 3.1)**.** *For all $w \in W$, there is a complex number $C(\nu, \lambda, \chi, w)$ such that*

$$\delta_{\nu,\lambda,\chi} = C(\nu, \lambda, \chi, w)\delta_{w\nu,w\lambda,\chi}A(\nu, \lambda, w), \tag{$\flat\flat$}$$

Furthermore, as a function of ν, it is meromorphic on $\mathfrak{a}^*_{\mathbb{C}}$.

We call the number $C(\nu, \lambda, \chi, w)$, the *local coefficient* attached to ν, λ, χ, and w. We write $C(\lambda, \chi, w)$ for $C(0, \lambda, \chi, w)$.

2.2.5. By definition of the local coefficients and the factorization of intertwining operators (*cf.* Theorem 2.1.5), we obtain the factorization of local coefficients.

Proposition. *Let $w \in W$, which is of length n. Denote w_i to be simple reflections as in Lemma 2.1.4. Then*

$$C(\nu, \lambda, \chi, w) = \prod_{i=1}^{i=n} C(\nu_i, \lambda_i, \chi, w_i), \tag{$\flat\flat\flat$}$$

where $\nu_i = w_{i-1}(\nu_{i-1})$, $\lambda_i = w_{i-1}(\lambda_{i-1})$, $2 \le i \le n$, $\nu_1 = \nu$, and $\lambda_1 = \lambda$.

2.3. **Normalized intertwining operators and the reducibility of principal series representations.** In this subsection, we will define the normalized intertwining operators and state the theorem of the reducibility of principal series representations on split reductive groups. The main tool here is the normalized intertwining operators. In [10] and [11], Keys proved that the normalized intertwining operators satisfy cocycle conditions and then form a group. Furthermore, he determined the structure of commuting algebras of principal series representations. The result is similar to Knapp and Stein in the archimedean cases.

2.3.1. Let λ be a unitary character of the maximal split torus T. Let $(\pi_{\nu,\lambda}, V_{\nu,\lambda})$ be the induced representations defined in §2.1.2. We write (π_λ, V_λ) for $(\pi_{0,\lambda}, V_{0,\lambda})$. The representation (π_λ, V_λ) is called the *unitary principal series* representation.

Let (π, V) be a representation of G. Define the *commuting algebra* to be the subalgebra $\{A \colon V \to V | A\pi(g) = \pi(g)A, \forall g \in G\}$ of $\mathrm{End}_{\mathbb{C}}(V)$. The algebra structure of $\mathcal{C}(\pi)$ is defined naturally, and is defined over $\mathbb{C}$.

2.3.2. In §2.1.3, for $\nu \in \mathfrak{a}^*_{\mathbb{C}}$ and $w \in W$, we define intertwining operators $A(\nu, \lambda, w)$ from $V_{\nu,\lambda}$ to $V_{w\nu,w\lambda}$, which intertwine $(\pi_{\nu,\lambda}, V_{\nu,\lambda})$ and $(\pi_{w\nu,w\lambda}, V_{w\nu,w\lambda})$. Recall that $A(\nu, \lambda, w)$ is well-defined only when ν satisfies $(*)$ in §2.1.3, and we can extend it to the whole $\mathfrak{a}^*_{\mathbb{C}}$ by analytic continuation. In §2.2.4, we define the local coefficients $C(\nu, \lambda, \chi, w)$, where χ is a generic character. Define the *normalized intertwining operator* $\mathfrak{a}(\nu, \lambda, \chi, w)$ by

$$\mathfrak{a}(\nu, \lambda, \chi, w) = C(\nu, \lambda, \chi, w)A(\nu, \lambda, w).$$

As usual, we write $\mathfrak{a}(\lambda, \chi, w)$ for $\mathfrak{a}(0, \lambda, \chi, w)$.

2.3.3. The reason for the normalization of $A(\nu, \lambda, w)$ is that we need the following property for our intertwining operators.

Proposition. (Chapter 1 and the proof of Theorem 2 of [10], and Proposition 3.1.4 of [22]) *The normalized intertwining operator $\mathfrak{a}(\nu, \lambda, \chi, w)$ satisfies the cocycle condition; i.e, for any $w', w'' \in W$, the equation*

$$\mathfrak{a}(\nu, \lambda, \chi, w'w'') = \mathfrak{a}(w''\nu, w''\lambda, \chi, w')\mathfrak{a}(\nu, \lambda, \chi, w'')$$

holds away from the poles. Notice that there is no condition imposed on the length of w', w''. If λ is unitary and $\Re\nu = 0$, $\mathfrak{a}(\nu, \lambda, \chi, w)$ is holomorphic; therefore, the cocycle condition always holds for λ unitary and $\Re\nu = 0$,.

2.3.4. Given a character λ of T, define $W_\lambda = \{w \in W | w\lambda = \lambda\}$. Then by Proposition 2.3.3 the map from W_λ to $\mathcal{C}(\pi_\lambda)$, denoted by $\iota_{\lambda,\chi}$, defined by $w \mapsto \mathfrak{a}(\lambda, \chi, w)$ is a representation of W_λ.

Let $\lambda_\alpha = \lambda \circ \alpha^\vee$, where $\alpha \in \Pi$. Define $\Delta_\lambda^- = \{\alpha \in \Delta^- | \lambda_\alpha \equiv 1\}$ and $W'_\lambda = \langle s_\beta \rangle_{\beta \in \Delta_\lambda^-}$. Indeed, it is easy to show that $W'_\lambda \subset W_\lambda$. We only need to prove that for all $\beta \in \Delta_\lambda^-$, $s_\beta\lambda = \lambda$. By definition, $s_\beta(t) = t(\beta^\vee \circ \beta(t))^{-1}$. Since $\lambda_\beta \equiv 1$,

$$s_\beta\lambda(t) = \lambda(s_\beta(t)) = \lambda(t(\beta^\vee \circ \beta(t))^{-1}) = \lambda(t)\lambda(\beta^\vee \circ \beta(t))^{-1} = \lambda(t).$$

Define a subgroup R_λ of W_λ, called the *R-group* of π_λ,

$$\begin{aligned} R_\lambda &= \{w \in W_\lambda | \alpha > 0 \quad \text{and} \quad \lambda_\alpha \equiv 1 \Rightarrow w\alpha \in \Delta^-\} \\ &= \{w \in W_\lambda | w(\Delta_\lambda^+) = \Delta_\lambda^+\}. \end{aligned}$$

2.3.5. The reducibility of a representation (π, V) is determined by its commuting algebra $\mathcal{C}(\pi)$. In general, it is not easy to compute commuting algebras. However, in the case of principal series representations of split reductive groups, it can be computed explicitly. The following theorem is due to Keys.

Theorem (Keys [10], Chapter 1, Section 3, Theorem 1, and [11])**.** *Keep the notations above. We can write W_λ as a semi-direct product*

$$W_\lambda = W'_\lambda \rtimes R_\lambda.$$

Furthermore, given a generic character χ of N, W'_λ is the group consisting of all w in W_λ such that $\mathfrak{a}(\lambda, \chi, w)$ are scalars. The operators in $\{\mathfrak{a}(\lambda, \chi, w) | w \in R_\lambda\}$ are linearly independent and they form a basis of $\mathcal{C}(\pi_\lambda)$. It implies that the map from the group algebra $\mathbb{C}[R_\lambda]$ to the commuting algebra $\mathcal{C}(\pi_\lambda)$ induced by $\iota_{\lambda,\chi}$ is an algebra isomorphism.

3. The Degenerate Whittaker Models for p-adic Groups

In this section, we state some results of Mœglin and Waldspurger [18] related to our program. We also provide the sketch of the proof since part of it will be used later. The foundation of this section is built on Harish-Chandra's result for the local expansions of characters. He proved the existence of the local expansions of characters for the admissible distributions. In [18], Mœglin and Waldspurger provided the interpretation of certain coefficients in terms of degenerate Whittaker models. Their theorem gives us the link between Whittaker Models and the coefficients in the local expansions. We will use it in the later sections to construct the geometric side of our theorem. Since their theorem requires that the characteristic of the residue field of F is not equal to 2, we need this assumption for the rest of this paper.

3.1. **Harish-Chandra's theorem for the local expansions of characters.** In this subsection, we will state Harish-Chandra's theorem for the local expansions of characters, following Harish-Chandra's lecture in [6]. He proved that any admissible distribution has a local character expansion supported on the nilpotent cone.

3.1.1. Let $\mathcal{N}$ be the set of all nilpotent G-orbits of $\mathfrak{g}$. Recall from §1.1.1 that the character ψ of F is trivial on the ring of integers $\mathfrak{D}$ and is nontrivial on $\varpi^{-1}\mathfrak{D}$. Fix a symmetric nondegenerate, G-invariant bilinear form κ on $\mathfrak{g}$ with values in F. Define the *Fourier transform* $\hat{}$ by

$$\hat{f}(Y) = \int_{\mathfrak{g}} \psi(\kappa(Y, X)) f(X) dX,$$

where $f \in C_c^\infty(\mathfrak{g})$, dX is a Haar measure on the additive group of $\mathfrak{g}$, and $Y \in \mathfrak{g}$. Then $f \mapsto \hat{f}$ is a linear bijection of $C_c^\infty(\mathfrak{g})$ onto itself. Furthermore, $\hat{\hat{f}}(Y) = f(-Y)$ if we normalize the Haar measure dX properly.

For any distribution T on $\mathfrak{g}$, we define the *Fourier transform* $\hat{T}$ of the distribution T by

$$\hat{T}(f) = T(\hat{f}).$$

3.1.2. Fix a G-invariant distribution Θ on an open G-invariant subset U of G. Let K_0 be an open compact subgroup of G, and γ an element in U. We say Θ is (G, K_0)-*admissible* at γ if:

1. $\gamma K_0 \subset U$.
2. For any open subgroup K of K_0 and $\underline{d} \in K^\wedge$, on γK_0,

   $$\Theta * \chi_{\underline{d}} = 0, \text{ unless } G \text{ intertwines } 1_{K_0} \text{ with } \underline{d}.$$

   Here $K^\wedge$ is the set of all irreducible representations of K, $\chi_{\underline{d}}$ is the character of $\underline{d}$, $*$ means the convolution, and 1_{K_0} is the class of trivial representation of K_0.

We say that Θ is G-*admissible* at γ if it is (G, K_0)-admissible at γ for some open compact subgroup K_0 of G. An *admissible distribution* on G means a distribution which is G-admissible at every point.

Consider the following example of admissible distributions on G. Let (π, V) be an admissible irreducible representation of G. For $f \in C_c^\infty(G)$, define

$$\pi(f) = \int_G f(g)\pi(g) dg,$$

where dg is the Haar measure on G. Then $\pi(f)$ is an operator of finite rank. Define the *character* Θ_π of π by

$$\Theta_\pi(f) = \operatorname{tr} \pi(f), \quad f \in C_c^\infty(G),$$

where tr is the trace of the operator $\pi(f)$ on V. Then Θ_π is a G-invariant distribution on G. By admissibility, we can choose an open compact subgroup K of G which is sufficiently small such that the K-invariant subspace of V is nontrivial. Then it is easy to check that Θ_π is (G, K)-admissible at every point.

Theorem (Harish-Chandra [6], Theorems 19 and 20). *Let Θ be a G-invariant distribution on an open G-invariant subset U of G. Let γ be a semisimple element in U. Then if Θ is G-admissible at γ, it coincides with a locally summable function around γ. Let M and $\mathfrak{m}$ be the centralizers of γ in G and $\mathfrak{g}$ respectively. Then for* *each $\mathcal{O} \in \mathcal{N}_M$, the set of all nilpotent M-orbits in $\mathfrak{m}$, we can choose unique complex numbers $c_{\mathcal{O}}(\Theta)$ such that*

$$\Theta(\gamma \exp Y) = \sum_{\mathcal{O}\in\mathcal{N}_M} c_{\mathcal{O}}(\Theta)\hat{\mu}_{\mathcal{O}}(Y), \text{ for } Y \text{ sufficiently near zero in } \mathfrak{m},$$

where $\mu_{\mathcal{O}}$ is the M-invariant measure on $\mathfrak{m}$ corresponding to $\mathcal{O}$ and $\hat{\mu}_{\mathcal{O}}$ is the Fourier transform of $\mu_{\mathcal{O}}$ on $\mathfrak{m}$.

3.1.3. We apply this theorem in the case when γ is the identity element of G and Θ is the character Θ_π of a smooth irreducible representation (π, V) of G. In this case, $M = G$, $\mathfrak{m} = \mathfrak{g}$, and $\mathcal{N}_M = \mathcal{N}$. For any $Y \in \mathcal{O}, \mathcal{O} \in \mathcal{N}$, we define an alternating form κ_Y with values in F:

$$(Z, X) \in \mathfrak{g} \times \mathfrak{g} \mapsto \kappa_Y(Z, X) \colon = \kappa(Y, [X, Z]),$$

where κ is the symmetric nondegenerate G-invariant bilinear form on $\mathfrak{g}$ which we chose before. Therefore, we can restate Harish-Chandra's theorem as follows:

Let (π, V) be a smooth irreducible representation of G. There is a neighborhood $\mathcal{V}$ of the identity element, isomorphic via the logarithm map to a neighborhood of 0 of $\mathfrak{g}$, such that we have the equality

$$(\star) \qquad \forall f \in C_c^\infty(\mathcal{V}), \quad \Theta_\pi(f) = \sum_{\mathcal{O}\in\mathcal{N}} c_{\mathcal{O}} \int_{\mathcal{O}} \widehat{f \circ \exp}(X) d_{\mu_{\mathcal{O}}} X,$$

where $\mathcal{N}$ is the set of all nilpotent orbits, $c_{\mathcal{O}}$ are constants, $\hat{}$ is the Fourier transform of the invariant measure $d_{\mu_{\mathcal{O}}}$. The Haar measures can be normalized suitably in the following way. Using the bilinear forms of ψ and κ, we can define a self-dual measure on F and $\mathfrak{g}$, respectively. The unique self-dual measure κ_Y on the tangent space of $\mathcal{O}$ at Y induces a G-invariant measure on $\mathcal{O}$. We also fix a Haar measure on G so that the Jacobian of the exponential map is the identity map of the neighborhood of 0.

3.2. **The theorem of Mœglin and Waldspurger.** In this subsection, we will state Mœglin and Waldspurger's theorem about the leading coefficients of the local expansions of characters in Harish-Chandra's theorem. They introduced a notation called the *degenerate Whittaker model* and proved that the leading coefficients are equal to the dimension of degenerate Whittaker models. Thus, in particular, they are all integers.

3.2.1. The main result of [18] tells us how to compute the coefficients $c_{\mathcal{O}}$ for certain nilpotent orbits. We need some notation to formulate it. Let us fix a nilpotent orbit $\mathcal{O}$ and choose an element Y of $\mathcal{O}$. We also choose a one-dimensional split torus $\varphi \colon F^* \to G$ satisfying

$$(\star\star) \qquad \forall s \in F^*, \quad \varphi(s) Y \varphi(s)^{-1} = s^{-2} Y.$$

The existence of such a φ is ensured by the theory of the SL_2-triplets ([3], §11).

We can break up $\mathfrak{g}$ by the adjoint action of φ, i.e.,

$$\mathfrak{g} = \sum_{i\in\mathbb{Z}} \mathfrak{g}_i, \text{ where } \mathfrak{g}_i = \{X \in \mathfrak{g} | \forall s \in F^*, \ \varphi(s) X \varphi(s)^{-1} = s^i X\}.$$

Set $\mathfrak{g}^i = \sum_{j\geq i} \mathfrak{g}_j$. Let $Y^\sharp$ be the centralizer of Y in $\mathfrak{g}$. We can choose a complement $\mathfrak{g}_Y$ of $Y^\sharp$ in $\mathfrak{g}$ stabilized by $\varphi(F^*)$. We decompose $\mathfrak{g}_Y$:

$$\mathfrak{g}_Y = \sum_{i\in\mathbb{Z}} \mathfrak{g}_{Yi}, \quad \text{where } \mathfrak{g}_{Yi} := \mathfrak{g}_Y \cap \mathfrak{g}_i.$$

Set $\mathfrak{g}_Y^i = \mathfrak{g}_Y \cap \mathfrak{g}^i$.

Let U (resp. U') be the unipotent subgroup of G whose Lie algebra is $\mathfrak{g}^1$ (resp. $\mathfrak{g}^2$) and U'' the subgroup of U generated by U' and the stabilizer in U of Y. One can prove that the function

$$\chi_Y : \gamma \mapsto \psi(\kappa(Y, \log\gamma))$$

is a character of U'' (*cf.* [18]).

Define

$$V_{U'',Y} = V/\{(\pi(u) - \chi_Y(u))v\}_{u\in U'',v\in V}.$$

We say that V *admits a degenerate Whittaker model* relative to (Y, φ) if $V_{U'',Y} \neq 0$. If $U'' \neq U$, $U/U'' \cap \mathrm{Ker}\chi_Y$ is a Heisenberg group with center $U''/U'' \cap \mathrm{Ker}\chi_Y$. Denote by $\mathcal{J}$ the irreducible representation of this group with central character χ_Y.

We set

$$V_{\varphi,Y} = \begin{cases} V_{U'',Y} & \text{if } U = U'', \\ \mathrm{Hom}_U(\mathcal{J}, V_{U'',Y}) & \text{if } U \neq U''. \end{cases}$$

We call $V_{\varphi,Y}$ the *space of the degenerate Whittaker forms* on V relative to (Y, φ). In general, this space depends on the choice of φ (for example if $Y = 0$, we can take φ so that either $U'' = 1$ or U'' is a maximal unipotent subgroup of G; in the first case $V_{\varphi,Y} = V$ and in the second case $V_{\varphi,Y}$ is the usual Jacquet module relative to U'').

There is an alternative way to describe $V_{\varphi,Y}$. We choose a basis $\{Z_i\}$ of $\mathfrak{g}_{Y1}$ and set $\mathfrak{m}_1 = \sum \mathfrak{D} Z_i$. Define $L_{Y1} = \exp \mathfrak{m}_1$. Then

$$V_{\varphi,Y} = (V_{U'',Y})^{L_{Y1}},$$

where the exponent indicates that we take the space of invariants of L_{Y1}.

3.2.2. We denote $\mathcal{N}_{\mathrm{tr}}(\pi)$ to be the set of nilpotent orbits for which $c_{\mathcal{O}} \neq 0$ (*cf.* $(\star)$). Also, $\mathcal{N}_{Wh}(\pi)$ represents the set of nilpotent orbits containing an element Y for which there is a 1-parameter subgroup φ satisfying $(\star\star)$ and for which $V_{\varphi,Y} \neq 0$. We define a partial ordering on $\mathcal{N}$ by $\mathcal{O} \leq \mathcal{O}' \Leftrightarrow \overline{\mathcal{O}} \subset \overline{\mathcal{O}'}$, where $^-$ is the closure operator for the usual topology. We denote $\mathrm{Max}\mathcal{N}_{\mathrm{tr}}(\pi)$ (resp. $\mathrm{Max}\mathcal{N}_{Wh}(\pi)$) to be the set of maximal elements of $\mathcal{N}_{\mathrm{tr}}(\pi)$ (resp. $\mathcal{N}_{Wh}(\pi)$). Now we can state the main theorem in [18].

Theorem (Mœglin and Waldspurger [18], Theorem I.16 and Corollary I.17)**.** *We keep all notation in this section. Let (π, V) be a smooth irreducible representation of G. Then the set of maximal elements of $\mathcal{N}_{\mathrm{tr}}(\pi)$ coincides with the set of maximal elements of $\mathcal{N}_{Wh}(\pi)$; i.e.,* $\mathrm{Max}\mathcal{N}_{\mathrm{tr}}(\pi) = \mathrm{Max}\mathcal{N}_{Wh}(\pi)$. *Furthermore, let $\mathcal{O}$ be a maximal element of $\mathcal{N}_{\mathrm{tr}}(\pi)$. Then the coefficient $c_{\mathcal{O}}$ is equal to the dimension* $\dim V_{\varphi,Y}$ *of the space of degenerate Whittaker forms related to an arbitrary Y of $\mathcal{O}$ and an arbitrary 1-parameter subgroup φ, satisfying $(\star\star)$. In particular, $c_{\mathcal{O}}$ is an integer.*

Sketch of the proof. The main idea of the proof is that in certain cases, the space of co-invariants $V_{U'',Y}$ can be expressed as the limit of an inductive system of semi-invariants for a family of compact subgroups of G, which is relative to certain characters of these groups.

3.2.3. We adopt all notation in this subsection. Given a nilpotent orbit $\mathcal{O}$, choose an element Y in $\mathcal{O}$. We can find a free $\mathfrak{D}$-submodule L of $\mathfrak{g}$ which satisfies $\mathfrak{g} = L \otimes_{\mathfrak{D}} F$, $[L, L] \subset L$, and $Y \in L$. Set

$$\mathfrak{g}_Y^- := \sum_{i \leq 0} \mathfrak{g}_{Y i} \text{ and } \mathfrak{g}_Y^2 = \mathfrak{g}_Y \cap \mathfrak{g}^2.$$

We can choose a base $Y_1, ..., Y_s$ of $\mathfrak{g}_Y^-$, $X_1, ..., X_s$ of $\mathfrak{g}_Y^2$, and $Z_1, ..., Z_{2r}$ of $\mathfrak{g}_{Y1}$ such that $\kappa_Y(Y_i, X_j) = \delta_{ij}$ and $\kappa_Y(Z_i, Z_j) = \delta_{i,2r-j+1}$, where $\delta_{ij} = 1$ if $i = j$ and 0 otherwise. Set

$$\mathfrak{g}_Y' := \sum_{1 \leq i \leq s} (\mathfrak{D} Y_i + \mathfrak{D} X_i) + \sum_{1 \leq i \leq r} (\mathfrak{D} Z_i + \mathfrak{D} Z_{2r-i+1}),$$

and

$$L' := \mathfrak{g}_Y' + \sum_i L \cap Y^\sharp \cap \mathfrak{g}_i.$$

Let $t := \varphi(\varpi)$. Define

$$G_n := \exp \varpi^n L', \quad G_n' :\ = t^{-n} G_n t^n.$$

3.2.4. Let $\chi_{Y,n}$ be a map from G_n to $\mathbb{C}^*$ defined by $\gamma \in G_n \mapsto \psi \circ \kappa(\varpi^{-2n} Y, \log \gamma)$. Indeed, $\chi_{Y,n}$ is a character of G_n for n large enough (*cf.* [18], Lemma I.6). For such n, set $\chi'_{Y,n}$ to be the character of G_n' which maps $\gamma \mapsto \chi_{Y,n}(t^n \gamma t^{-n})$.

3.2.5. Define

$$V_n := \{v \in V | \forall \gamma \in G_n, \pi(\gamma) v = \chi_{Y,n}(\gamma) v\},$$

$$V_n' := \{v \in V | \forall \gamma \in G_n', \pi(\gamma) v = \chi'_{Y,n}(\gamma) v\},$$

and the maps

$$\begin{aligned} I_{n,m} \colon V_n \to V_m, &\quad v \mapsto \int_{G_m} \chi_{Y,m}(\gamma^{-1}) \pi(\gamma t^{m-n}) v d\gamma, \\ I'_{n,m} \colon V_n' \to V_m', &\quad v \mapsto \int_{G_m'} \chi'_{Y,m}(\gamma^{-1}) \pi(\gamma) v d\gamma, \\ I_n \colon V \to V_n, &\quad v \mapsto \int_{G_n} \chi_{Y,n}(\gamma^{-1}) \pi(\gamma) v d\gamma. \end{aligned}$$

3.2.6. Let $U'' = \exp \mathfrak{g}_Y^1 \exp \mathfrak{g}^2$ as above. We define a function χ_Y on U'' by

$$\chi_Y \colon \gamma \in U'' \mapsto \psi \circ \kappa(Y, \log \gamma).$$

In fact, χ_Y is a character of U'' (*cf.* [18], I.7).

3.2.7. Let (π, V) be a smooth irreducible representation of G. We denote $U''_{\chi_Y} V$ to be the U''-submodule of V generated by $\langle(\pi(u) - \chi_Y(u))v\rangle_{v\in V, u\in U''}$. Let $V_{U'',Y} = V/U''_{\chi_Y} V$ and $V_{\varphi,Y} = (V_{U'',Y})^{L_{Y1}}$ as we defined above. Then there is a natural map j

$$(\star\star\star) \qquad j\colon V'_n/V'_{n,\chi_Y} \to (V_{U'',Y})^{L_{Y1}} = V_{\varphi,Y},$$

where $V'_{n,\chi_Y} = \bigcup_{m>n} \mathrm{Ker} I'_{n,m}$ (*cf.* [18], I.9). Furthermore, if $V_{\varphi,Y} \neq 0$, then V_n and V'_n are nontrivial for n large enough (*cf.* [18], Lemma I.10).

Let $\varphi_n(\gamma) = \chi_{Y,n}(\gamma^{-1})$ if $\gamma \in G_m$ and equal to 0 otherwise. The key of the entire proof is that we use φ_n as testing functions in $(\star)$, and, in fact that we can really compute their Fourier transforms explicitly. Therefore we can get the information about coefficients $c_{\mathcal{O}}$.

Remark. Later on we will need the above result. For the case that interests us, $U'' = N$, χ_Y is generic, and $V_{\varphi,Y} = V_{\chi_Y}$. When n is large enough, j induces an isomorphism from V'_n to V_{χ_Y}. In particular, if V admits a Whittaker model (*cf.* §2.2.2), i.e., $V_{\chi_Y} \neq \{0\}$, then for all nontrivial Whittaker functional δ on V, δ is nonzero on V'_n.

3.2.8. The next proposition gives us the geometric property of the degenerate Whittaker forms.

Proposition ([18], Proposition I.11). *If $V_{\varphi,Y} \neq 0$, then there exists a nilpotent orbit $\mathcal{O}$ of $\mathfrak{g}$ appearing in the expression of Θ_π on a neighborhood of the identity element with a nonzero coefficient such that $Y \in \overline{\mathcal{O}}$.*

3.2.9. The next lemma tells us the explicit information about the coefficients $c_{\mathcal{O}}$ for which $\mathcal{O}$ are maximal around all nilpotent orbits $\mathcal{O}'$ with nonzero coefficients $c'_{\mathcal{O}}$.

Lemma ([18], Lemma I.12). *We denote $\mathcal{O} = G\cdot Y$ and suppose that $c_{\mathcal{O}}$ is nonzero in $(\star)$ and the Y does not belong to the closure of another orbit $\mathcal{O}'$ for which $c'_{\mathcal{O}} \neq 0$, i.e., $\mathcal{O}$ is a maximal element of $N_{\mathrm{tr}}(\pi)$. Then for all large n, $\dim V_n$ is finite and independent of n. Also it is equal to $c_{\mathcal{O}}$.*

3.2.10. It can be proved that for n, m large enough, the map j in $(\star\star\star)$ is injective and its image is exactly $V_{\varphi,Y}$ (*cf.* [18], Corollary I.14). Also, for n large enough, the map $I'_{n,m}$ is injective (*cf.* [18], Lemma I.15). Combining these results, we get

$$c_{\mathcal{O}} = \dim V_n = \dim(\Im j) = \dim V_{\varphi,Y},$$

where $\mathcal{O} \in \mathrm{Max}\, \mathcal{N}_{\mathrm{tr}}(\pi)$.

3.2.11. To prove $\mathrm{Max}\, \mathcal{N}_{Wh}(\pi) = \mathrm{Max}\, \mathcal{N}_{\mathrm{tr}}(\pi)$, we first pick a maximal element $\mathcal{O}$ of $N_{\mathrm{tr}}(\pi)$. Fix Y in $\mathcal{O}$ and we choose an arbitrary one-parameter subgroup satisfying $(\star\star)$ relative to Y. To show that V admits a degenerate Whittaker model relative to Y and this one-parameter subgroup, it is enough to show that V'_{n,χ_Y} is trivial for n large. This is true since $I'_{n,m}$ is injective for large n, m.

Now let $\mathcal{O}$ be a maximal element of $N_{Wh}(\pi)$; according to Proposition 3.2.8, there exists an orbit $\mathcal{O}'$ of $N_{\mathrm{tr}}(\pi)$ so that $\overline{\mathcal{O}'} \supset \overline{\mathcal{O}}$. By the definition of the order relation over the set of nilpotent orbits of $\mathfrak{g}$, we know that there exists a maximal element $\mathcal{O}''$ of $N_{\mathrm{tr}}(\pi)$, possibly equal to $\mathcal{O}'$, so that $\overline{\mathcal{O}''} \supset \overline{\mathcal{O}}$. The first part of the proof shows that we have $\mathcal{O}'' \in N_{Wh}(\pi)$, so $\mathcal{O}'' = \mathcal{O}$. This finishes the proof. □

4. Principal Series Representations and Principal Nilpotent Orbits

In this section, we discuss some possible ways to link principal series representations and principal nilpotent orbits and formulate the relations between them. Those relations will be proved in the next section. The main tools that we use are the multiplicity one theorem of Whittaker models and the theorem of Mœglin and Waldspurger. Furthermore, using Whittaker models, we can describe the nature of our maps. At the same time, we can get the geometric picture of our maps by the theorem of Mœglin and Waldspurger. More precisely, our maps can determine the lead coefficients of the local expansions of characters.

4.1. **The construction of the maps ρ, $\overline{\rho}$, and ϱ.** In this subsection, we will construct maps linking irreducible constituents of principal series and principal nilpotent orbits from representation theory. At first, we give a parametrization of principal nilpotent orbits. Then we use the multiplicity one theorem and the theorem of Mœglin and Waldspurger to construct the maps from principal nilpotent orbits to the generic irreducible constituents of principal series representations.

4.1.1. Define a subset of regular nilpotent elements

$$\mathsf{P} = \{ \sum_{\alpha\in\Pi^-} Z_\alpha | Z_\alpha \in \mathfrak{n}_{-\alpha}\backslash\{0\}\},$$

The set P has a useful structure. Define an abelian group

$$\tilde{T} = \{(t_\alpha)_{\alpha\in\Pi^-} | t_\alpha \in F^*\} = \prod_{\alpha\in\Pi^-} F^*_\alpha.$$

Then $\tilde{T}$ acts on P simply transitively by

$$\tilde{t}(\sum_{\alpha\in\Pi^-} Z_\alpha) = \sum_{\alpha\in\Pi^-} t_\alpha Z_\alpha,\ \tilde{t}\in\tilde{T}.$$

The action of a maximal split torus T on P is defined by adjoint and it can be expressed as follows

$$\mathrm{Ad}(t)(\sum_{\alpha\in\Pi^-} Z_\alpha) = \sum_{\alpha\in\Pi^-} \mathrm{Ad}(t)Z_\alpha = \sum_{\alpha\in\Pi^-} \alpha(t)Z_\alpha.$$

There are two important facts.

(1) For all $\mathcal{O}\in\mathcal{P}$, $\mathcal{O}\cap\mathsf{P}$ is nonempty.
(2) Two elements in P are conjugated to each other by G if and only if they are conjugated by T.

Therefore, instead of working on $\mathcal{P}$, we can just work on P. The following is a well-known result for the parametrization of principal nilpotent orbits for split groups.

Proposition. *The abelian group $H^1(\Gamma, Z(\underline{G}))$ acts on the principal nilpotent orbits simply transitively.*

Proof. Let ϕ be the map from T to $\tilde{T}$ defined by $t \to \prod_{\alpha\in\Pi^-}\alpha(t)$. From the discussion above, we know $\tilde{T}/\phi(T)$ acts on $\mathcal{P}$ simply transitively. The short exact sequence

$$1 \to Z(\underline{G}) \to \underline{T} \to \prod_{\alpha\in\Pi^-} \overline{F}^*_\alpha \to 1$$

gives rise to a long exact sequence

$$1 \to Z(\underline{G})^\Gamma \to T \to \tilde{T} \to H^1(\Gamma, Z(\underline{G})) \to H^1(\Gamma, \underline{T}) \to \cdots$$

$H^1(\Gamma, \underline{T})$ is trivial by Hilbert's Theorem 90. Therefore, $H^1(\Gamma, Z(\underline{G}))$ is isomorphic to $\tilde{T}/\phi(T)$ as desired. □

4.1.2. There is a fact about the generic character of N and P.

Lemma. *Given $Y \in \mathsf{P}$, define a map $\chi_{_Y}$ from N to $\mathbb{C}$ by*

$$\chi_{_Y} : n \mapsto \psi(\kappa(Y, \log \gamma)).$$

Then $\chi_{_Y}$ is a generic character and all generic characters can be constructed this way.

Proof. In §3.2.6, we define the character $\chi_{_Y}$ on U'' in the same way. The difference between U'' and N is the weight 1 space. However, there is no weight 1 space for a regular nilpotent element. Therefore, U'' is equal to N and $\chi_{_Y}$ is a character and defined on N.

Now given a generic character χ, we know χ is uniquely determined by χ_α, for all $\alpha \in \Pi$ (*cf.* §2.2.2). χ_α is an additive character of F. There is an element $b_\alpha \in F^*$ such that $\chi_\alpha(f) = \psi(b_\alpha f)$ for all $f \in F$. We can choose $Z_{-\alpha} \in \mathfrak{n}_{-\alpha}$ such that $\kappa(Z_{-\alpha}, X_\alpha) = b_\alpha$. Let Y be $\sum_{\beta\in\Pi} Z_{-\beta}$. For all $\alpha \in \Pi$, we have

$$\begin{aligned}(\chi_{_Y})_\alpha(f) = \chi_{_Y} \circ x_\alpha(f) &= \psi(\kappa(\sum_{\beta\in\Pi} Z_{-\beta}, f X_\alpha)) \\ &= \psi(f \cdot \kappa(Z_{-\alpha}, X_\alpha)) = \psi(f \cdot b_\alpha) = \chi_\alpha(f).\end{aligned}$$

Therefore, $\chi_{_Y} = \chi$. □

4.1.3. Let χ be a generic character of N and (π, V) be an admissible representation of G. Recall that we define a functor $*_{(\chi)}$ from the representations of G to vector spaces as follows:

$$V_{(\chi)} = V / \langle (\pi(n) - \chi(n))v|_{n\in N, v\in V} \rangle,$$

and this functor $*_{(\chi)}$ is exact (*cf.* Proposition 2.2.1).

Proposition. *Given a generic character χ of N, and a character λ of T, there is a unique irreducible constituent $V_{\lambda,\chi}$ of the principal series representation (π_λ, V_λ) such that $\dim(V_{\lambda,\chi})_{(\chi)}$ is equal to 1. Furthermore, the multiplicity of $V_{\lambda,\chi}$ in V_λ is 1.*

Proof. From Proposition 2.2.3, we know that

$$\dim(\mathrm{Hom}_G(V_\lambda, \mathrm{Ind}_N^G \chi)) = \dim(V_\lambda)_\chi = 1.$$

Let $V_\lambda = \bigoplus_{\xi\in\Sigma_\lambda} m_\xi V_\xi$. Since the functor $*_{(\chi)}$ is exact, we have

$$1 = \dim(V_\lambda)_\chi = \sum_{\xi\in\Sigma_\lambda} m_\xi \dim(V_\xi)_\chi.$$

The right-hand side is a sum of nonnegative integers. Hence, only one will be nonzero and equal to one. We conclude that both dimension and multiplicity are one. The proposition is proved. □

4.1.4. From Proposition 4.1.3, we can derive more information about the coefficients $c_{\mathcal{O}}, \mathcal{O} \in \mathcal{P}$, in equation ($\star$) in §3.1.3.

Corollary. *Let λ be a character of T and (π_λ, V_λ) the principal series representation. According to Theorem 3.1.2, there is an equality which is valid for any sufficiently small neighborhood $\mathcal{V}$ of the identity element:*

$$\forall f \in C_c^\infty(\mathcal{V}), \quad \Theta_{\pi_\lambda}(f) = \sum_{\mathcal{O}\in\mathcal{N}} c_{\mathcal{O}} \int_{\mathcal{O}} \widehat{f \circ \exp}(X) d_{\mu_{\mathcal{O}}} X,$$

where Θ_{π_λ} is the character of π_λ, $\mathcal{N}$ is the set of all nilpotent orbits, $c_{\mathcal{O}}$ are constants. Let $\mathcal{P} \subset \mathcal{N}$ be the set of principal nilpotent orbits. Then for any $\mathcal{O} \in \mathcal{P}$, $c_{\mathcal{O}}$ are equal to 1.

Proof. From Theorem 3.2.2, we know that

$$c_{\mathcal{O}} = \dim(V/\langle(\pi_\lambda(u) - \chi_Y(u))v|_{u\in U'',\ v\in V}\rangle)^{L_{Y1}}.$$

Here $\mathcal{O}$ is a maximal nilpotent orbit with nonzero $c_{\mathcal{O}}$; i.e., $\mathcal{O}$ belongs to the set $\mathrm{Max}\mathcal{N}_{\mathrm{tr}}(\pi_\lambda)$ of the maximal elements of $\mathcal{N}_{\mathrm{tr}}(\pi_\lambda)$, where $\mathcal{N}_{\mathrm{tr}}(\pi_\lambda)$ is the set of nilpotent orbits $\mathcal{O}$ with nonzero $c_{\mathcal{O}}$ in ($\star$) (*cf.* §3.1.3).

Fix a Y in $\mathcal{O} \cap \mathsf{P}$. Let U be the subgroup of N whose weights are greater than 1, with respect to an $\mathfrak{sl}_2-$triple containing Y. Let U' be the centralizer of Y in N. U'' is the subgroup of N generated by U and U'. Let χ_Y be a character of U'' defined by $\gamma \mapsto \psi(\kappa(Y, \log\gamma))$. Choose the complement $\mathfrak{g}_Y$ of the centralizer of Y in $\mathfrak{g}$. Choose a basis $\{Z_i\}$ of the weight 1 subspace of $\mathfrak{g}_Y$. Then L_{Y1} is the subgroup $\exp\mathfrak{m}_1$, where $\mathfrak{m}_1 = \sum_i \mathfrak{D} Z_i$.

Let $\mathcal{O}$ be a principal nilpotent orbit and $Y \in \mathcal{O} \cap \mathsf{P}$ be a regular nilpotent element. There is no weight 1 space in the $\mathfrak{sl}_2$ decomposition. Therefore, L_{Y1} is trivial and U is equal to N. Hence U'' is just N. Furthermore, χ_Y is generic since Y is regular.

Recall the definition $(V_\lambda)_{U'',Y}$ of the degenerate Whittaker model on V_λ:

$$(V_\lambda)_{U'',Y} = (V_\lambda/\langle(\pi_\lambda(u) - \chi_Y(u))v|_{u\in U'',\ v\in V_\lambda}\rangle),$$

and the definition $(V_\lambda)_{\varphi,Y}$ of the degenerate Whittaker form on V_λ

$$(V_\lambda)_{\varphi,Y} = (V_\lambda/\langle(\pi_\lambda(u) - \chi_Y(u))v|_{u\in U'',\ v\in V_\lambda}\rangle)^{L_{Y1}},$$

where φ is a homomorphism from F^* to G satisfying ($\star\star$) in §3.2.1. We have

$$\begin{aligned} (V_\lambda)_{\varphi,Y} &= (V_\lambda)_{U'',Y} && \text{by } L_{Y1} = 0 \\ &= (V_\lambda)_{(\chi_Y)} && \text{by } U'' = N \\ &\neq 0. && \text{by Propostion 4.1.3.} \end{aligned}$$

By Theorem 3.2.2, we get that $c_{\mathcal{O}}$ is nonzero since the degenerate Whittaker form is nonzero and $\mathcal{O}$ has the maximal dimension. We apply Theorem 3.2.2 to get

$$c_{\mathcal{O}} = \dim(V_\lambda)_{\varphi,Y} = \dim(V_\lambda)_{(\chi_Y)} = 1.$$

The proof is completed. ∎

4.1.5. Recall that in §2.3.2, for all $w \in W$, we define the normalized intertwining operators $\mathfrak{a}(\lambda, \chi, w)$ between (π_λ, V_λ) and $(\pi_{w\lambda}, V_{w\lambda})$. Let $W_\lambda = \{w \in W | w\lambda = \lambda\}$. Then for all $w \in W_\lambda$, the intertwining operator $\mathfrak{a}(\lambda, \chi, w)$ intertwines the principal series representation (π_λ, V_λ) with itself. Given a root α, we define a character λ_α of F^* to be $\lambda \circ \alpha^\vee$. We denote $\Delta_\lambda^+ = \{\alpha \in \Delta^+ | \lambda_\alpha \equiv 1\}$. Set W'_λ to be the subgroup of W_λ generated by s_α, $\alpha \in \Delta_\lambda^+$. The R-group R_λ is the subgroup of W_λ which preserves Δ_λ^+. Theorem 2.3.5 tells us that the map $\iota_{\lambda,\chi}$ from W_λ to $\mathcal{C}(\pi_\lambda)$ defined by $w \in W_\lambda \mapsto \mathfrak{a}(\lambda, \chi, w) \in \mathcal{C}(\pi_\lambda)$ is a group representation. Furthermore, the kernel of $\iota_{\lambda,\chi}$ is W'_λ and the image of R_λ forms a basis of $\mathcal{C}(\pi_\lambda)$; i.e, $\mathbb{C}[R_\lambda] \cong \mathcal{C}(\pi_\lambda)$.

Proposition. *Let $R_\lambda^\wedge$ be the set of irreducible representations of R_λ and Σ_λ the set of the irreducible constituents of π_λ. There is a one-to-one correspondence between Σ_λ and $R_\lambda^\wedge$. This correspondence depends on the choice of the generic character χ of N.*

Proof. We decompose the principal series representation (π_λ, V_λ) into $\sum_{\xi \in \Sigma_\lambda} m_\xi \xi$, where m_ξ is the multiplicity of ξ which occurs in π_λ. From the decomposition of $\pi_\lambda = \sum_{\xi \in \Sigma_\lambda} m_\xi \xi$, we get $\mathcal{C}(\pi_\lambda)$ is isomorphic to $\sum_{\xi \in \Sigma_\lambda} M(m_\xi, \mathbb{C})$, where $M(m_\xi, \mathbb{C})$ is the matrix algebra over an m_ξ-dimensional $\mathbb{C}$ vector space. On the other hand, look at the R-group side; the group algebra $\mathbb{C}[\pi_\lambda]$ of R_λ can be expressed as $\sum_{\tau \in R_\lambda^\wedge} M(\dim \tau, \mathbb{C})$, where $\dim \tau$ is the dimension of the representation space of τ. Therefore there is a one-to-one correspondence between Σ_λ and $R_\lambda^\wedge$. Notice that this correspondence depends on the choice of $\iota_{\lambda,\chi}$; i.e., the choice of the generic character χ of N. □

4.1.6. Define $\Sigma_\lambda^{\mathrm{gen}}$ to be a subset of Σ_λ consisting of all generic irreducible constituents of π_λ:

$$\Sigma_\lambda^{\mathrm{gen}} = \{\xi \in \Sigma_\lambda | \xi \text{ is generic}\}.$$

We can partition Σ_λ into the generic part $\Sigma_\lambda^{\mathrm{gen}}$ and the nongeneric part Σ'_λ. According to Proposition 4.1.3, we know the multiplicity of a generic irreducible representation which occurs in principal series must be equal to 1. Therefore, we have

$$\pi_\lambda = \sum_{\tau' \in \Sigma_\lambda^{\mathrm{gen}}} 1 \cdot \tau' + \sum_{\tau'' \in \Sigma'_\lambda} m_{\tau''} \cdot \tau''.$$

Notice that the multiplicity $m_{\tau''}$ of τ'' may or may not be equal to 1. Let Σ_λ^1 be the subset of Σ_λ consisting of those ξ with $m_\xi = 1$. Hence Σ^{gen} is a subset of Σ_λ^1. Then the bijection in Proposition 4.1.5 induces a bijection Σ_λ^1 and $\mathrm{Hom}(R_\lambda, \mathbb{C}^*)$.

Proposition. *Let ρ^{-1} be a map from Σ^{gen} to the power set $2^{\mathcal{P}}$ of $\mathcal{P}$ defined by*

$$\rho^{-1}(\xi) = \mathrm{Max}\,\mathcal{N}_{\mathrm{tr}}(\xi) \cap \mathcal{P} = \mathcal{P}_\xi, \ \xi \in \Sigma^{gen},$$

where $\mathrm{Max}\,\mathcal{N}_{\mathrm{tr}}$ *is defined in §3.2.2. Then:*

(i) $\bigcup_{\xi \in \Sigma^{gen}} \mathcal{P}_\xi = \mathcal{P}$.
(ii) *If $\xi, \xi' \in \Sigma^{gen}$ and $\xi \neq \xi'$, then $\mathcal{P}_\xi \cap \mathcal{P}_{\xi'} = \emptyset$.*

Proof. First, we show that $\mathcal{P}_\xi$ is nonempty for all $\xi \in \Sigma^{\mathrm{gen}}$. Since (ξ, W) is generic, $\dim W_\chi \neq 0$ for some generic χ. By Lemma 4.1.2, there is a $Y \in \mathsf{P}$ such that $\chi_Y = \chi$. Let $\mathcal{O}_Y \in \mathcal{P}$ be the principal nilpotent orbit containing Y. Following the same argument as Corollary 4.1.4, we know $\mathcal{O}_Y \in \mathrm{Max}\,\mathcal{N}_{\mathrm{tr}}(\xi)$. Therefore, $\mathcal{O}_Y \in \mathrm{Max}\,\mathcal{N}_{\mathrm{tr}}(\xi) \cap \mathcal{P} = \mathcal{P}_\xi$. It concludes that $\mathcal{P}_\xi$ is nonempty.

For (i), given $\mathcal{O} \in \mathcal{P}$, we can pick up a $Y \in \mathcal{O} \cap \mathsf{P}$. By Proposition 4.1.3, we know there is a unique irreducible constituent ξ_{χ_Y} which is χ_Y-generic. As the same argument above, we know $\mathcal{O} = \mathcal{O}_Y \in \operatorname{Max} \mathcal{N}_{\mathrm{tr}}(\xi_{\chi_Y})$. Hence, $\mathcal{O} \in \mathcal{P}_{\xi_{\chi_Y}}$. This proves (i).

For (ii), if $\mathcal{O} \in \mathcal{P}_\xi \cap \mathcal{P}_{\xi'}$ and $Y \in \mathcal{O} \cap \mathsf{P}$, then by additivity of characters, $\dim V_{\lambda(\chi_Y)} \geq c_{\mathcal{O}}(\Theta_\xi) + c_{\mathcal{O}}(\Theta_{\xi'}) \geq 2$. This contradicts Corollary 4.1.4. □

4.1.7. By the proposition above, we have a map ρ from $\mathcal{P} \to \Sigma_\lambda^{\mathrm{gen}} \subset \Sigma_\lambda$, which is the inverse map of ρ^{-1}:

$$\begin{aligned} \rho \colon &[\text{the set } \mathcal{P} \text{ of principal nilpotent orbits}] \\ &\quad \to [\text{the set } \Sigma_\lambda \text{ of irreducible constituents}]. \end{aligned} \tag{1}$$

By abuse of notation, we use $R_\lambda^\wedge$ to denote the set of characters of irreducible representations of R_λ. Composing ρ with the correspondence between Σ_λ^1 and $\operatorname{Hom}(R_\lambda, \mathbb{C}^*)$, we define $\overline{\rho}$:

$$\overline{\rho} \colon [\text{the set } \mathcal{P} \text{ of principal nilpotent orbits}] \to \operatorname{Hom}(R_\lambda, \mathbb{C}^*) \hookrightarrow R_\lambda^\wedge.$$

Note that $\overline{\rho}$ depends on the choice of a generic character of N.

By Proposition 4.1.1, $H^1(\Gamma, Z(\underline{G}))$ parametrizes $\mathcal{P}$. We can identify $\mathcal{P}$ with $H^1(\Gamma, Z(\underline{G}))$, if we choose a principial nilpotent orbit as a basepoint of $\mathcal{P}$. Using the identification above and the correspondence between Σ_λ and $\operatorname{Hom}(R_\lambda, \mathbb{C}^*)$, we can modify (1) as

$$\varrho \colon H^1(\Gamma, Z(\underline{G})) \to \operatorname{Hom}(R_\lambda, \mathbb{C}^*). \tag{1'}$$

Our main goal is to write down precise formulas for those maps. We provide a way to approach it from group representations by Whittaker models.

Remark. Consider the map

$$\overline{\rho} \colon \mathcal{P} \to \operatorname{Hom}(R_\lambda, \mathbb{C}^*).$$

This map does not seem to be natural; we have a natural choice of basepoint in $\operatorname{Hom}(R_\lambda, \mathbb{C}^*)$, the trivial representation, but it is not clear what a natural choice of basepoint of principal nilpotent orbits is. It can be clarified in the following way.

The correspondence between Σ_λ and $\operatorname{Hom}(R_\lambda, \mathbb{C}^*)$ replies on $\iota_{\lambda,\chi}$, which does depend on the choices of the generic character χ of N. A regular nilpotent Y can produce a generic character χ_Y of N by §3.2.6. Hence, the principal nilpotent $\mathcal{O}_Y$ which contains Y can be the choice of the basepoint of $\mathcal{P}$.

Furthermore, after we pick up $\mathcal{O}_Y$ as the basepoint of $\mathcal{P}$, the map

$$\varrho \colon H^1(\Gamma, Z(\underline{G})) \to \operatorname{Hom}(R_\lambda, \mathbb{C}^*)$$

is well-defined. Although ϱ is a map between two groups, there is no reason a priori that it should be a group homomorphism. However, remarkably, our Main Theorem can show it is indeed a homomorphism.

4.2. **The Whittaker version of ρ.** In this subsection, we use the Whittaker models to describe the nature of our maps. The main idea is to see the change of the leading coefficients of the local expansions under the intertwining operators. Then we can produce a natural map from the principal nilpotent orbits to the characters of R_λ. Finally, we are able to formulate our main theorem and geometric theorem.

4.2.1. Let I be an intertwining operator of a representation (π, V). Recall the character Θ_π of (π, V) is a distribution on G defined by

$$\Theta_\pi(f) = \operatorname{tr}\pi(f),\ \forall f \in C_c^\infty(G),$$

where $\pi(f) = \int_G f(g)\pi(g)dg$ is of finite rank and tr is the trace function. Note that tr is well-defined since the rank of $\pi(f)$ is finite. We define a new distribution $\Theta_{I\circ\pi}$ on G by

$$\Theta_{I\circ\pi}(f) = \operatorname{tr}(I \circ \pi(f)),\ \forall f \in C_c^\infty(G).$$

Note that the trace $\operatorname{tr}(I \circ \pi(f))$ is well-defined also because $\pi(f)$ is of finite rank. $\Theta_{I\circ\pi}$ is G-invariant since I intertwines π with itself.

We have the following lemma:

Lemma. *Let I be an intertwining operator of a unitary representation (π, V) of finite length. Define a distribution $\Theta_{I\circ\pi}$ as above. Then $\Theta_{I\circ\pi}$ is G-admissible (cf. Theorem 3.1.2).*

Proof. Since π is unitary and of finite length, we can reduce to the case of $\pi = m\xi$, where ξ is irreducible and m is its multiplicity. We can identify the intertwining operator I as an $m\times m$-matrix M and it is easy to see that $\operatorname{tr}(I\circ\pi(f)) = \operatorname{tr}(M)\operatorname{tr}(\xi(f))$. Therefore, we have $\Theta_{I\circ\pi} = \operatorname{tr}(M)\Theta_\xi$. Θ_ξ is admissible, so is its multiple. Therefore $\Theta_{I\circ\pi}$ is admissible. □

4.2.2. From the previous lemma, $\Theta_{I\circ\pi}$ satisfies the assumption of Theorem 3.1.2. Therefore, $\Theta_{I\circ\pi}$ has a local expansion around a neighborhood $\mathcal{V}$ of the identity element of G:

$$\forall f \in C_c^\infty(\mathcal{V}), \quad \Theta_{I\circ\pi}(f) = \sum_{\mathcal{O}\in\mathcal{N}} c_{\mathcal{O}}(I) \int_{\mathcal{O}} \widehat{f \circ \exp}(X) d_{\mu_{\mathcal{O}}} X, \tag{2}$$

where $\mathcal{N}$ is the set of all nilpotent orbits, $c_{\mathcal{O}}(I)$ are constants, and $\widehat{\ }$ is the Fourier transform of the invariant measure $d_{\mu_{\mathcal{O}}}$. Unlike Theorem 3.2.2, we cannot expect that all $c_{\mathcal{O}}(I)$ are integers. For example, if I is a scalar multiple a of the identity, then $\operatorname{tr}(I \circ \pi) = a \operatorname{tr}(\pi)$.

We now return to principal series representations. Given a generic character χ of N, we have defined the intertwining operators $\mathfrak{a}(\lambda, \chi, w)$, for all $\lambda \in W_\lambda$. The map $w \in R_\lambda \mapsto \mathfrak{a}(\lambda, \chi, w)$ is a group isomorphism. According to the previous discussion, for all $w \in R_\lambda$, we have an equality of distributions

$$\Theta_{\mathfrak{a}(\lambda,\chi,w)\circ\pi_\lambda} = \sum_{\mathcal{O}\in\mathcal{N}} c_{\mathcal{O}}(\mathfrak{a}(\lambda, \chi, w))\hat{\mu}_{\mathcal{O}},$$

where $\hat{\mu}_{\mathcal{O}}$ is the Fourier transform of Haar measure on $\mathcal{O}$. To ease the notation, we denote $c_{\mathcal{O}}(\mathfrak{a}(\lambda, \chi, w)) = c_{\mathcal{O}}(w)$.

Consider the map $w \in R_\lambda \mapsto c_{\mathcal{O}}(w)$. By abuse of notation, we still use $c_{\mathcal{O}}$ to denote this map. It is easy to see that $c_{\mathcal{O}}$ is a character. From the proof of Lemma 4.2.1, we know the coefficients of $c_{\mathcal{O}}(w)$ are the trace of the representations of R_λ which are induced by its restriction on the irreducible constituents $\rho(\mathcal{O})$. Since the multiplicity of $\rho(\mathcal{O})$ is 1, we get that the trace is the same as the representation, which is of one-dimension. Therefore they are a character of R_λ.

Philosophically, we can not distinguish the elements in the same $\mathcal{P}_\xi$ using the characters of R_λ. Thus, we expect the following lemma:

Lemma. *Given any two principal nilpotent orbits $\mathcal{O}$, $\mathcal{O}'$, if $\mathcal{O}$, $\mathcal{O}'$ belong to the same subset $\mathcal{P}_\xi$ of $\mathcal{P}$ for some $\xi \in \Sigma_\lambda^{\text{gen}}$, then two characters $c_{\mathcal{O}}$, $c_{\mathcal{O}'}$ of R_λ are equal.*

Proof. Let ξ be a generic constituent of π_λ and $\mathcal{O}$, $\mathcal{O}'$ be two principal nilpotent orbits associated to ξ. Since the multiplicity of ξ is 1, any intertwining operator acts as a scalar by its restriction on ξ. Note that nonzero coefficients $c_{\mathcal{O}}$ and $c_{\mathcal{O}'}$ appear in the local expansion of Θ_ξ. Let $c_\xi(w)$ be the constant such that $\mathfrak{a}(\lambda,\chi,w)|_\xi = c_\xi(w)\text{id}$. Then $\Theta_{(\mathfrak{a}(\lambda,\chi,w)|_\xi)\circ\xi} = c_\xi(w)\Theta_\xi$. Therefore $c_{\mathcal{O}}(w) = c_\xi(w)$ and $c'_{\mathcal{O}}(w) = c_\xi(w)$. □

4.2.3. By the previous lemma, for all $\xi \in \Sigma_\lambda^{\text{gen}}$, we can define a degree-one character c_ξ of R_λ so that $c_\xi = c_{\mathcal{O}}$ for all $\mathcal{O} \in \mathcal{P}_\xi$. Therefore, the map from $\mathcal{P}$ to $R_\lambda^\wedge$ defined by $\mathcal{O} \mapsto c_\xi$ is exactly the map $\overline{\rho}$. In conclusion, we have the following proposition:

Proposition. *Fix a generic character χ of N and then it determines the unique monomorphism from R_λ into the commuting algebra of π_λ. Then the map from $\mathcal{P}$ to $R_\lambda^\wedge$ defined by $\mathcal{O} \mapsto c_\xi$ is the same as the map $\overline{\rho}$.*

4.2.4. Now we can formulate our main theorem for the maps ρ, $\overline{\rho}$, and ϱ. We will prove it in a later section.

Main Theorem. *Let $\mathcal{P}$ be the set of principal nilpotent orbits on $\mathfrak{g}$. The abelian group $H^1(\Gamma, Z(\underline{G}))$ acts on $\mathcal{P}$ simply transitively. Give a character λ of T. Let R_λ be the R-group of π_λ, Σ_λ the set of irreducible constituents of π_λ, and $\Sigma_\lambda^{\text{gen}}$ the set of generic irreducible constituents of π_λ. Write $R_\lambda^\wedge$ for the set of characters of irreducible representations of R_λ. Fix a principal nilpotent orbit $\mathcal{O}$ and an element $Y \in \mathcal{O} \cap \mathsf{P}$. Using $\mathcal{O}$ and χ_Y, we can uniquely determine the identifications $\mathcal{P} \simeq H^1(\Gamma, Z(\underline{G}))$ and $\Sigma_\lambda \xrightarrow{\sim} R_\lambda^\wedge$. Then there is a canonical pairing $\langle , \rangle_\lambda$ of $R_\lambda \times H^1(\Gamma, Z(\underline{G}))$, which will be defined in §5.2, such that*

(i) *the map $\rho\colon \mathcal{P} \to \Sigma_\lambda$ induces a bijection $Q_\lambda \backslash \mathcal{P} \xrightarrow{\sim} \Sigma_\lambda^{\text{gen}}$, where Q_λ is the right kernel of $\langle , \rangle_\lambda$;*

(ii) *the composite $\mathcal{P} \xrightarrow{\rho} \Sigma_\lambda \xrightarrow{\sim} R_\lambda^\wedge$ is the same as the composite*
$$\mathcal{P} \simeq H^1(\Gamma, Z(\underline{G})) \xrightarrow{\langle , - \rangle_\lambda^{-1}} R_\lambda^\wedge;$$
both are equal to $\overline{\rho}$;

(iii) *the map ϱ is equal to $H^1(\Gamma, Z(\underline{G})) \xrightarrow{\langle , - \rangle_\lambda^{-1}} \text{Hom}(R_\lambda, \mathbb{C}^*) \hookrightarrow R_\lambda^\wedge$.*

Remark. The R-groups are not abelian in general. Therefore, the map ϱ is not onto $R_\lambda^\wedge$. Even in the cases of abelian R-groups, ϱ is not onto $\text{Hom}(R_\lambda, \mathbb{C}^*)$ in general, either. Just look at the case of G_2, the algebraic group of Dynkin diagram type G_2. The R-groups can be $\mathbb{Z}/2\mathbb{Z}$ but there is only one principal nilpotent orbit since G_2 is adjoint.

4.2.5. From the previous discussion and Main Theorem, we have the following geometric characterization of generic irreducible constituents:

Theorem. *The set of principal nilpotent orbits appearing in the local character expansion of the generic irreducible constituent $\rho(\mathcal{O})$ is precisely $Q_\lambda \cdot \mathcal{O}$; i.e.,*
$$\Theta_{\rho(\mathcal{O})} = \sum_{\mathcal{S} \in Q_\lambda \cdot \mathcal{O}} c_{\mathcal{S}}(\rho(\mathcal{O}))\hat{\mu}_{\mathcal{S}} + \textit{ lower dimensional terms.}$$

Proof. Let ξ be an irreducible admissible representation of G. By the proof of Corollary 4.1.4, for $\mathcal{S} \in \mathcal{P}$ and $Y \in \mathcal{S} \cap \mathsf{P}$, $c_{\mathcal{S}}$ is equal to $\dim\bigl(\mathrm{Hom}_G(\xi, \mathrm{Ind}_N^G \chi_Y)\bigr)$. It means that for any $Y \in \mathcal{S} \cap \mathsf{P}$, $c_{\mathcal{S}}$ appears in the local expansion if and only if ξ is χ_Y-generic. Given a generic irreducible constituent σ, by the construction of ρ, $\rho(\mathcal{S}) = \sigma$ if and only if σ is χ_Y-generic. Therefore the set of principal nilpotent orbits appearing in the local character expansion of σ is $\rho^{-1}(\sigma)$. Applying the argument above to $\mathcal{O} \in \mathcal{P}$, from the Main Theorem, we know $\rho^{-1}(\rho(\mathcal{O}))$ is $Q_\lambda \cdot \mathcal{O}$. This ends the proof. □

5. The proof of the Main Theorem

In this section, we will give the proof of the Main Theorem. The idea of the proof is based on an observation about local coefficients. The way to determine which principal nilpotent orbit goes with which irreducible constituent is to look at the change of local coefficients for different generic characters of N. We calculate the change of local coefficients by the the method of Keys and the proof of the theorem of Mœglin and Waldspurger. Then, we are able to prove our theorem using the information obtained from local coefficients.

5.1. **Whittaker functionals and local coefficients II: the rule of changes.** In this subsection, we study how the local coefficients behave when we vary generic characters. There is indeed a nice formula for them. A similar result has been obtained by Shahidi [23]. However, it might be possible to extend our method to the nongeneric case. The idea here is inspired by Keys' computation in [11]. We construct the canonical vectors, called the *standard functions*, in the representation spaces of principal series presentations. Using the multiplicity one theorem and the proof of the theorem of Mœglin and Waldspurger, we can show that the standard functions are unique up to scalars. Then we calculate their changes under the intertwining operators. It can be proved that in our cases the ratios of the standard functions under the intertwining operators is closely related to local coefficients.

5.1.1. Let $\overline{\mathbf{G}}$ be the adjoint group of $\mathbf{G}$, p the projection from $\mathbf{G}$ to $\overline{\mathbf{G}}$, $\overline{\mathbf{T}}$ the maximal split torus of $\overline{\mathbf{G}}$, and $\overline{T}$ the F-rational points of $\overline{\mathbf{T}}$. For all roots $\alpha \in \Delta$, $\alpha = \overline{\alpha} \circ p$. Write $\mathbf{1}$ for the identity element of $\overline{T}$. Let w_l be the longest element of the Weyl group and we choose a representative $\tilde{w}_l$ as we did in §2.1.3. Define a $\overline{T}$-action on P as follows:

$$\bar{t}\Bigl(\sum_{\alpha \in \Pi^-} Z_\alpha\Bigr) = \sum_{\alpha \in \Pi^-} \sigma^{-1}(\overline{\alpha})(\bar{t}) Z_\alpha,$$

where $\bar{t} \in \overline{T}$, $\overline{\alpha}$ are the root of $\overline{\mathbf{G}}$ associated to the root α of $\mathbf{G}$, and σ is a permutation of Π^- defined by

$$\sigma(\alpha) = -w_l(\alpha),\ \alpha \in \Pi^-.$$

Since $\overline{\mathbf{G}}$ is adjoint, the map $\overline{T}$ to $\tilde{T} = \prod_{\alpha \in \Pi^-} F_\alpha^*$, defined by $\bar{t} \mapsto \prod_{\alpha \in \Pi^-} \sigma^{-1}(\overline{\alpha})(\bar{t})$, is an isomorphism. Hence, by Proposition 4.1.1, the $\overline{T}$-action on P is simply transitive.

There are two T-actions on P.

(i) In §4.1.1), we define a T-action on P by the adjoint action.

(ii) The group $p(T)$ is a subgroup of $\overline{T}$. We can define a T-action on P via the $\overline{T}$-action on P; i.e.,

$$t \cdot (\sum_{\alpha\in\Pi^-} Z_\alpha) = \sum_{\alpha\in\Pi^-} \sigma^{-1}(\overline{\alpha})\big(p(t)\big)Z_\alpha = \sum_{\alpha\in\Pi^-} \sigma^{-1}\alpha(t)Z_\alpha,$$

where $t \in T$ and σ is defined as above.

The reason why we discuss the T-action on P is because G-orbits on P is the same as T-orbits on it; however, the T-action here is the action in (1) above.

The two actions are different but they just differ up to an automorphism $t \mapsto (\tilde{w}_l t \tilde{w}_l^{-1})^{-1}$. Hence, for our purpose, since the orbits are the same, it does not hurt to use the latter T-action; i.e., the T-action via $p(T)$ and $\overline{T}$-action. Using the latter action can simplify our notation. Hence, we will use it for the rest of this paper. By the proof of Proposition 4.1.1, $\overline{T}/p(T)$ is isomorphic to $H^1(\Gamma, Z(\underline{G}))$ and acts simply transitively on $\mathcal{P}$.

5.1.2. Given $\overline{t} \in \overline{T}$, let $Y_{\overline{t}}$ be a regular nilpotent element in P defined by

$$Y_{\overline{t}} = \sum_{\alpha\in\Pi^-} c_\alpha \sigma^{-1}(\overline{\alpha})(\overline{t})Y_\alpha,$$

where Y_α is defined in §1.1.4, and c_α is the constants such that

$$\kappa(c_\alpha Y_\alpha, \tilde{w}_l Y_{\sigma^{-1}(\alpha)} \tilde{w}_l^{-1}) = 1.$$

We define $Y'_\alpha = c_\alpha Y_\alpha$ and then rewrite the definition of $Y_{\overline{t}}$ as $\sum_{\alpha\in\Pi} \sigma^{-1}(\overline{\alpha})(\overline{t})Y'_\alpha$ and we have

$$\kappa(Y'_\alpha, \tilde{w}_l Y_{\sigma^{-1}(\alpha)} \tilde{w}_l^{-1}) = 1.$$

Clearly, for all $\overline{t}, \overline{t}' \in \overline{T}$, we have $\overline{t}' Y_{\overline{t}} = Y_{\overline{t}'\overline{t}}$. Choose a homomorphism $\varphi_{\overline{t}}$ for $Y_{\overline{t}}$ from F^* to G satisfying $(\star\star)$ in §3.2.1. Let $t_{\overline{t}} = \varphi_{\overline{t}}(\varpi)$.

Recall some definitions from §3.2. In §3.2.3, for all $\overline{t} \in \overline{T}$, we defined the groups $G_{n,\overline{t}}$, $G'_{n,\overline{t}}$ which are only dependent on the nilpotent element $Y_{\overline{t}}$. $G_{n,\overline{t}}$ can be written as the product $B^-_{n,\overline{t}}N_{n,\overline{t}}$, where $B^-_{n,\overline{t}} = G_{n,\overline{t}} \cap B^-$ and $N_{n,\overline{t}} = G_{n,\overline{t}} \cap N$ (*cf.* [18] I.3(2)). Furthermore, for all $Y_{\overline{t}}$, by adjusting Y_i in §3.2.3, we can choose our X_i in the definition of $G_{n,\overline{t}}$ to be always the same as X_α in §1.1.4. Notice that there is no Z_i for regular nilpotent elements. Therefore, we can rewrite the product as

$$G_{n,\overline{t}} = B^-_{n,\overline{t}}N_n, \text{ where } N_n = \prod_{\alpha\in\Delta^+} N_{\alpha,n}.$$

Let $\chi_{\overline{t},n}$ be a character of $G_{n,\overline{t}}$ defined by $\gamma \in G_{n,\overline{t}} \mapsto \psi(\kappa(\varpi^{-2n}Y_{\overline{t}}, \log\gamma))$ and $\chi'_{\overline{t},n}$ a character of of $G'_{n,\overline{t}}$ which maps $\gamma \in G'_{n,\overline{t}} \mapsto \chi_{\overline{t},n}(t^n_{\overline{t}}\gamma t^{-n}_{\overline{t}})$.

Now we start to define some new notation. Let $\tilde{G}_{n,\overline{t}} = \tilde{w}_l G_{n,\overline{t}} \tilde{w}_l^{-1}$. As above, $\tilde{G}_{n,\overline{t}}$ is a product of N_n^- and $B_{n,\overline{t}}$, where $N_n^- = N^- \cap \tilde{G}_{n,\overline{t}}$ and $B_{n,\overline{t}} = B \cap \tilde{G}_{n,\overline{t}}$. Since $\forall\beta \in \Delta$, $\tilde{w}_l x_\beta(t) \tilde{w}_l^{-1} = x_{w_l\beta}(\pm t)$ (*cf.* Part b of Lemma 20 of [24]), we have

$$N_n^- = \prod_{\alpha\in\Delta^-} N_{\alpha,n},$$

where $N_{\alpha,n}$ is defined in §1.1.3. Define a character $\tilde{\chi}_{\overline{t},n}$ of $\tilde{G}_{n,\overline{t}}$ by

$$\tilde{\chi}_{\overline{t},n}(\gamma) = \chi_{\overline{t},n}(\tilde{w}_l^{-1}\gamma\tilde{w}_l), \ \gamma \in G_{n,\overline{t}}.$$

Since $\chi_{\overline{t},n}$ is trivial on $B^-_{n,\overline{t}}$ (*cf.* [18], I.9), $\tilde{\chi}_{\overline{t},n}$ is trivial on $B_{n,\overline{t}}$. For all $\alpha \in \Pi^-$, define an additive character $\tilde{\chi}_{\overline{t},n,\alpha}$ of F by

$$\tilde{\chi}_{\overline{t},n,\alpha} = \tilde{\chi}_{\overline{t},n} \circ x_\alpha.$$

Lemma. *For $a \in F^*$, let m_a be the endomorphism on F defined by multiplying a and ψ_n the composition $\psi \circ m_{\varpi^{-2n}}$. Then $\tilde{\chi}_{\overline{t},n,\alpha} = \psi_n \circ m_{\overline{\alpha}(\overline{t})}$.*

Proof. From the definition of $\tilde{\chi}_{\overline{t},n,\alpha}$, for all $s \in F$, we have

$$\begin{aligned}\tilde{\chi}_{\overline{t},n,\alpha}(s) =& \psi(\varpi^{-2n}\kappa(\sum_{\beta\in\Pi^-} \sigma^{-1}(\overline{\beta})(\overline{t})Y'_\beta, s\tilde{w}_l Y_\alpha \tilde{w}_l^{-1}))\\ =& \psi(\varpi^{-2n}\overline{\alpha}(\overline{t})\kappa(Y'_{\sigma(\alpha)}, \tilde{w}_l Y_\alpha \tilde{w}_l^{-1})s) = \psi(\varpi^{-2n}\overline{\alpha}(\overline{t})s).\end{aligned}$$

The third equality is due to the orthogonality of the Killing form. The lemma follows. □

5.1.3. Consider a subspace of $\tilde{V}_{\nu,\lambda,n}$ defined by

$$\tilde{V}_{\nu,\lambda,n,\overline{t}} = \{v \in V_{\nu,\lambda} | \pi_\lambda(\gamma)v = \tilde{\chi}_{\overline{t},n}(\gamma)v, \forall\gamma \in \tilde{G}_{n,\overline{t}}\}.$$

Recall that we define $V_{\nu,\lambda,n,\overline{t}}$ and $V'_{\nu,\lambda,n,\overline{t}}$ in §3.2.5 as follows:

$$\begin{aligned}V_{\nu,\lambda,n,\overline{t}} &= \{v \in V_{\nu,\lambda} | \pi_\lambda(\gamma)v = \chi_{\overline{t},n}(\gamma)v, \forall\gamma \in G_{n,\overline{t}}\},\\ V'_{\nu,\lambda,n,\overline{t}} &= \{v \in V_{\nu,\lambda} | \pi_\lambda(\gamma)v = \chi'_{\overline{t},n}(\gamma)v, \forall\gamma \in G'_{n,\overline{t}}\}.\end{aligned}$$

We have

$$\tilde{V}_{\nu,\lambda,n,\overline{t}} = \pi_{\nu,\lambda}(\tilde{w}_l^{-1})V_{\nu,\lambda,n,\overline{t}},\ V'_{\nu,\lambda,n,\overline{t}} = \pi_{\nu,\lambda}(t^n_{\overline{t}})V_{\nu,\lambda,n,\overline{t}},$$

and

$$V'_{\nu,\lambda,n,\overline{t}} = \pi_{\nu,\lambda}(t^n_{\overline{t}}\tilde{w}_l)\tilde{V}_{\nu,\lambda,n,\overline{t}}.$$

Write $\tilde{V}_{\lambda,n,\overline{t}}$ to be $\tilde{V}_{0,\lambda,n,\overline{t}}$, the same for $V_{\lambda,n,\overline{t}}$ and $V'_{\lambda,n,\overline{t}}$.

Lemma. *For all $\overline{t} \in \overline{T}$ and n large enough, any vector v in $V_{\nu,\lambda}$, which satisfies $\pi_{\nu,\lambda}(\gamma)v = \tilde{\chi}_{\overline{t},n}(\gamma)v, \forall\gamma \in \tilde{G}_{n,\overline{t}}$, is unique up to a scalar.*

Proof. For all $\overline{t} \in \overline{T}$ and n large enough, from §3.2.10, we know the dimension of $V_{\nu,\lambda,n,\overline{t}}$ is equal to the dimension of $(V_{\nu,\lambda})_{(\chi_{\overline{t}})}$. Furthermore, the dimension of $(V_{\nu,\lambda})_{(\chi_{\overline{t}})}$ is 1 by the multiplicity one theorem (*cf.* Proposition 2.2.3). According to the equalities above, the dimension of $\tilde{V}_{\nu,\lambda,n,\overline{t}}$ is equal to that of $V_{\nu,\lambda,n,\overline{t}}$. Therefore, it is also equal to 1. This completes the proof. □

5.1.4. Define a *standard function*, on N^-B, which is open and dense in G, by

$$f_{\nu,\lambda,n,\overline{t}}(vmn) = \begin{cases} 0 & \text{if } v \notin N_n^-;\\ \tilde{\chi}^{-1}_{\overline{t},n}(v)\lambda(m)^{-1}q^{\langle -\nu-\rho, H(m)\rangle} & \text{if } v \in N_n^-,\end{cases}$$

where $m \in T$, $n \in N$, $\nu \in \mathfrak{a}^*_{\mathbb{C}}$, ρ is the half sum of positive roots, H is a map defined in §2.1.2, and we take n large enough such that $\lambda^{-1}q^{\langle -\nu-\rho, H(-)\rangle}$ is trivial on $T \cap B_{n,\overline{t}}$. From the definition, $f_{\nu,\lambda,n,\overline{t}}$ is in $V_{\nu,\lambda}$.

Proposition. *$f_{\nu,\lambda,n,\overline{t}}$ belongs to $\tilde{V}_{\nu,\lambda,n,\overline{t}}$.*

Proof. Let $\gamma \in \tilde{G}_{n,\bar{t}}$ and $\gamma = vmn$; i.e., $\gamma^{-1} = n^{-1}m^{-1}v^{-1}$, where $v \in N_n^-$, $m \in T \cap B_{n,\bar{t}}$ and $n \in N$. Since $\tilde{\chi}_{\bar{t},n}$ is a character, therefore $\tilde{\chi}_{\bar{t},n}^{-1}(\gamma^{-1}) = \tilde{\chi}_{\bar{t},n}(\gamma) = \tilde{\chi}_{\bar{t},n}(v)$. On the other hand, $\pi_{\nu,\lambda}(\gamma)$ acts on $f_{\nu,\lambda,n,\bar{t}}(g)$ as $f_{\nu,\lambda,n,\bar{t}}(\gamma^{-1}g)$. Assume $g = g_{n^-}g_m g_n,\ g_{n^-} \in N_n^-, g_m \in T, g_n \in N$. $\gamma^{-1}g_{n^-} = n^{-1}m^{-1}v^{-1}g_{n^-} = v'm'n'$, where $v' \in N_n^-$, $m' \in T \cap B_{n,\bar{t}}$ and $n' \in N$. Since $\tilde{\chi}_{\bar{t},n}$ is a character, $\tilde{\chi}_{\bar{t},n}^{-1}(v') = \tilde{\chi}_{\bar{t},n}(v)\tilde{\chi}_{\bar{t},n}^{-1}(g_{n^-})$. From the assumption that $\lambda^{-1}q^{\langle -\nu-\rho, H(-)\rangle}$ is trivial on $T \cap B_{n,\bar{t}}$, we get

$$\begin{aligned} f_{\nu,\lambda,n,\bar{t}}(\gamma^{-1}g) =& f_{\nu,\lambda,n,\bar{t}}(v'm'n'g_m g_n) = f_{\nu,\lambda,n,\bar{t}}\big(v'm'g_m(g_m^{-1}n'g_m)g_n\big) \\ &= f_{\nu,\lambda,n,\bar{t}}(v'm'g_m) = \tilde{\chi}_{\bar{t},n}^{-1}(v')f_{\nu,\lambda,n,\bar{t}}(g_m) \\ &= \tilde{\chi}_{\bar{t},n}(v)\tilde{\chi}_{\bar{t},n}^{-1}(g_{n^-})f_{\nu,\lambda,n,\bar{t}}(g_m) = \tilde{\chi}_{\bar{t},n}(\gamma)f_{\nu,\lambda,n,\bar{t}}(g). \end{aligned}$$

This completes the proof. □

5.1.5. For all $w \in W$, consider the action of $A(\nu,\lambda,w)$ on $f_{\nu,\lambda,n,\bar{t}}$. If ν satisfies $(*)$ in §2.1.3, then $A(\nu,\lambda,w)f_{\nu,\lambda,n,\bar{t}}$ converges and $A(\nu,\lambda,w)f_{\nu,\lambda,n,\bar{t}} \in V_{w\nu,w\lambda}$. Since $A(\nu,\lambda,w)$ intertwines $V_{\nu,\lambda}$ and $V_{w\nu,w\lambda}$, we have that for any $g \in \tilde{G}_{n,\bar{t}}$,

$$\pi_{w\lambda}(g)A(\nu,\lambda,w)f_{\nu,\lambda,n,\bar{t}} = A(\nu,\lambda,w)\pi_\lambda(g)f_{\nu,\lambda,n,\bar{t}} = \tilde{\chi}_{\bar{t},n}A(\nu,\lambda,w)f_{\nu,\lambda,n,\bar{t}}.$$

Hence, the function $A(\nu,\lambda,w)f_{\nu,\lambda,n,\bar{t}}$ belongs to $\tilde{V}_{w\nu,w\lambda,n,\bar{t}}$; it must be a scalar multiple of $f_{w\nu,w\lambda,n,\bar{t}}$. Define

$$(\#) \qquad A(\nu,\lambda,w)f_{\nu,\lambda,n,\bar{t}}(g) = \gamma(\nu,\lambda,n,\bar{t},w)f_{w\nu,w\lambda,n,\bar{t}}(g), \quad \forall g \in G.$$

We write $\gamma(\lambda,n,\bar{t},w)$ to be $\gamma(0,\lambda,n,\bar{t},w)$. We will prove later that $\gamma(\nu,\lambda,n,\bar{t},w)$ is defined for all ν by analytic continuation. For the moment, we assume that this is true. Then we have the following important equality.

Theorem. *If $w \in W$ satisfies $w\lambda = \lambda$, then the local coefficients $C(\lambda,\chi_{\bar{t}},w)$ are equal to $\gamma(\lambda,n,\bar{t},w)^{-1}$.*

Proof. From the remark in §3.2.7, we know the Whittaker functional $\delta_{\lambda,\chi_{\bar{t}}}$ acts nontrivially on $V'_{\lambda,n,\bar{t}}$. Let $k = t_{\bar{t}}^n\tilde{w}_l$ (*cf.* §5.1.3). Then $\pi_\lambda(k)f_{\lambda,n,\bar{t}}$ belongs to $V'_{\lambda,n,\bar{t}}$. Therefore $\delta_{\lambda,\chi_{\bar{t}}}\big((\pi_\lambda(k)f_{\lambda,n,\bar{t}})\big)$ is nonzero. We apply $\delta_{\lambda,\chi_{\bar{t}}} \circ \pi_{\nu,\lambda}k$ on the definition of $\gamma(\lambda,n,\bar{t},w)$. We obtain

$$\begin{aligned} & \delta_{\lambda,\chi_{\bar{t}}}\big(\pi_\lambda(k)A(\lambda,w)f_{\lambda,n,\bar{t}}\big) = \delta_{\lambda,\chi_{\bar{t}}}\big(\pi_\lambda(k)\gamma(\lambda,n,\bar{t},w)f_{w\lambda,n,\bar{t}}\big), \\ \Longrightarrow\quad & \delta_{\lambda,\chi_{\bar{t}}}\big(A(\lambda,w)\pi_\lambda(k)f_{\lambda,n,\bar{t}}\big) = \gamma(\lambda,n,\bar{t},w)\delta_{\lambda,\chi_{\bar{t}}}\big(\pi_\lambda(k)f_{\lambda,n,\bar{t}}\big), \\ \Longrightarrow\quad & C(\lambda,\chi_{\bar{t}},w)^{-1}\delta_{\lambda,\chi_{\bar{t}}}\big(\pi_\lambda(k)f_{\lambda,n,\bar{t}}\big) = \gamma(\lambda,n,\bar{t},w)\delta_{\lambda,\chi_{\bar{t}}}\big(\pi_\lambda(k)f_{\lambda,n,\bar{t}}\big), \\ \Longrightarrow\quad & C(\lambda,\chi_{\bar{t}},w)^{-1} = \gamma(\lambda,n,\bar{t},w). \end{aligned}$$

This is what we need. □

5.1.6. Now we would like to compute the $\gamma(\nu,\lambda,n,\bar{t},w)$. We prove that the resulting function has an analytic continuation. As a by-product, we also get the formula when we vary $\bar{t}$ in $\overline{T}$.

For any $w \in W$, we consider the product decomposition of $A(\nu,\lambda,w)$ into rank-one operators corresponding to the reduced expression. Then $\gamma(\nu,\lambda,n,\bar{t},w)$ can be given by a product formula in terms of rank-one operators. Therefore, we only need to compute the cases for simple reflection w.

For all $\alpha \in \Pi^-$, set $\lambda_{\nu,\alpha} = \lambda_\nu \circ \alpha^\vee = \lambda_\nu \circ h_\alpha$. As usual, $\lambda_\alpha = \lambda_{0,\alpha}$. We write $\gamma(\nu,\lambda,n,\mathbf{1},w)$ as $\gamma(\nu,\lambda,n,w)$. The following lemma is crucial for our computation.

Lemma. *Let w be the simple reflection $s_\alpha \in W$. Then for n large enough, we have*

$$\gamma(\nu,\lambda,n,\overline{t},w) = \lambda_{\nu,\alpha}(\overline{\alpha}(\overline{t}))^{-1}\gamma(\nu,\lambda,n,w),$$

and

$$\gamma(\nu,\lambda,n,w) = \lambda_{\nu,\alpha}(\varpi)^{2n}\Gamma(\lambda_{\nu,\alpha}),$$

where $\Gamma(\lambda_{\nu,\alpha})$ is the gamma function of [20]*. Obviously, $\lambda_{\nu,\alpha}(\varpi)^{2n}\Gamma(\lambda_{\nu,\alpha})$ can be defined for all ν by analytic continuation. As a consequence, $\gamma(\nu,\lambda,n,\overline{t},w)$ can be defined for all $\nu \in \mathfrak{a}_{\mathbb{C}}^*$ by analytic continuation.*

Proof. We evaluate (#) in §5.1.5 at $g = 1$. Then the right-hand side is exact by $\gamma(\nu,\lambda,n,\overline{t},w)$. The left-hand side is $A(\nu,\lambda,w)f_{\nu,\lambda,n,\overline{t}}(1)$. Then for suitable ν,

$$\begin{aligned}
\gamma(\nu,\lambda,n,\overline{t},w) &= \int_{N_\alpha} f_{\nu,\lambda,n,\overline{t}}(n\tilde{s}_\alpha)dn \\
&= \int_F f_{\nu,\lambda,n,\overline{t}}\big(\zeta_\alpha(\begin{pmatrix}1 & t\\ 0 & 1\end{pmatrix}\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix})\big)dt \\
&= \int_{F\backslash 0} f_{\nu,\lambda,n,\overline{t}}\big(\zeta_\alpha(\begin{pmatrix}1 & 0\\ 1/t & 1\end{pmatrix}\begin{pmatrix}-t & 1\\ 0 & -1/t\end{pmatrix})\big)dt \\
&= \int_{F\backslash 0} f_{\nu,\lambda,n,\overline{t}}\big(x_{-\alpha}(1/t)\alpha^\vee(-t)x_\alpha(1)\big)dt \\
&= \int_{v_F(t)\leq -n} \tilde{\chi}_{\overline{t},n}^{-1}\circ x_{-\alpha}(1/t)(\lambda_\nu\circ\alpha^\vee(-t))^{-1}|t^{\langle\rho,\alpha^\vee\rangle}|_F^{-1}dt \\
&= \int_{v_F(t)\leq -n} \tilde{\chi}_{\overline{t},n,\alpha}^{-1}(1/t)\lambda_{\nu,\alpha}^{-1}(-t)|t|_F^{-1}dt \\
&= \int_{v_F(t)\leq -n} [(\psi_n\circ m_{\overline{\alpha}(\overline{t})})^{-1}(1/t)]\lambda_{\nu,\alpha}^{-1}(-t)|t|_F^{-1}dt \\
&= \int_{v_F(s)\geq n} \psi^{-1}(-\varpi^{-2n}\overline{\alpha}(\overline{t})s)\lambda_{\nu,\alpha}(s)|s|_F^{-1}ds \\
&= \lambda_{\nu,\alpha}(\varpi^{2n}\overline{\alpha}(\overline{t})^{-1})\int_{v_F(s)\geq -n+v_F(\overline{\alpha}(\overline{t}))} \psi(s)\lambda_{\nu,\alpha}(s)|s|_F^{-1}ds \\
&= \lambda_{\nu,\alpha}^{-1}(\overline{\alpha}(\overline{t}))\lambda_{\nu,\alpha}(\varpi^{2n})\Gamma(\lambda_{\nu,\alpha}).
\end{aligned}$$

The last equality holds for n large enough (*cf.* Chapter 2, §2, §6, (4) of [9]). Now the proof follows. □

5.1.7. Combining Theorem 5.1.5 with Lemma 5.1.6, we have the following corollary:

Corollary. *Let* **G** *be a split reductive group defined over F and* **B** *be its Borel subgroup.* **B** *can be written as* **TN***, where* **T** *is the maximal split torus and* **N** *is the maximal nilpotent subgroup. Let λ be a nontrivial character of T, the F-rational points of* **T***. Let σ be a permutation of Π^- defined by $\sigma(\alpha) = -w_l(\alpha)$, where w_l is the longest element of the Weyl group W. Given an $\overline{t} \in \overline{T} = \prod_{\alpha\in\Pi^-} F_\alpha^*$, we define a regular nilpotent element $Y_{\overline{t}} = \sum_{\alpha\in\Pi^-}\sigma^{-1}(\overline{\alpha})(\overline{t})Y'_\alpha$ as in §5.1.2. Set* **1** *to be the identity element of $\overline{T}$. For any $w \in W$, we can factorize $w = s_{\alpha_n}s_{\alpha_{n-1}}\cdots s_{\alpha_1}$ (cf. §2.1.4), where $\alpha_i \in \Pi^-$. For $i \geq 2$ (resp. $i = 1$), we define $w'_i = s_{\alpha_{i-1}}\cdots s_{\alpha_1}$ (resp. w'_1 is the identity). Let $\lambda_i = w'_i\lambda$. and $(\lambda_i)_{\alpha_i} = \lambda_i\circ\alpha_i^\vee$. If $w\lambda = \lambda$, then we* *have the following equality for local coefficients:*

$$C(\lambda, \chi_{\overline{t}}, w) = \Big(\prod_{i=1}^{i=n} (\lambda_i)_{\alpha_i}(\overline{\alpha}_i(\overline{t}))\Big) C(\lambda, \chi_{Y_{\mathbf{1}}}, w).$$

5.2. **The proof of the Main Theorem.** In this subsection, we prove the Main Theorem. The job here is just to interpret the change of local coefficients into our picture and the theorem follows.

5.2.1. We recall the product terms in Corollary 5.1.7. Define a pairing $\langle , \rangle_\lambda$ on $R_\lambda \times \overline{T}$ as

$$\langle w, \overline{t} \rangle_\lambda : = \prod_{i=1}^{i=n} (\lambda_i)_{\alpha_i}(\overline{\alpha}_i(\overline{t})),$$

where $w \in R_\lambda$ and $\overline{t} \in \overline{T}$. For all $w \in R_\lambda$, we define

$$\lambda_w(\overline{t}) = \langle w, \overline{t} \rangle_\lambda,$$

where $\overline{t} \in \overline{T}$.

Apparently λ_w is a character of $\overline{T}$.

5.2.2. Lemma. *Fix an element $\overline{t}$ of $\overline{T}$. Let $Y_{\overline{t}}$ be the regular nilpotent element associated with $\overline{t}$ and $\chi_{\overline{t}}$ the generic character of N associated with $Y_{\overline{t}}$. Then there is the unique generic irreducible constituent $\pi_{\lambda,\overline{t}}$ of π_λ such that the restriction of $\iota_{\lambda,\chi_{\overline{t}}}(R_\lambda)$ on $\pi_{\lambda,\overline{t}}$ is a trivial character of R_λ.*

Proof. Let $\overline{t}$ be an element of $\overline{T}$ and $\chi_{\overline{t}}$ be the generic character of N associated to the regular nilpotent element $Y_{\overline{t}}$. There is a unique irreducible constituent $(\pi_{\lambda,\overline{t}}, V_\lambda(\chi_{\overline{t}}))$ such that

$$\dim\big(V_\lambda(\chi_{\overline{t}})_{(\chi_{\overline{t}})}\big) = 1.$$

Therefore, the Whittaker functional $\delta_{\lambda,\chi_{\overline{t}}}$ is nontrivial on this unique irreducible constituent and is trivial on the others. Let $\mathfrak{a}(\lambda, \chi_{\overline{t}}, w)$ be the normalized intertwining operators. From the definition of $\mathfrak{a}(\lambda, \chi_{\overline{t}}, w)$, we have

$$\delta_{\lambda,\chi_{\overline{t}}} = \delta_{\lambda,\chi_{\overline{t}}} \mathfrak{a}(\lambda, \chi_{\overline{t}}, w).$$

Since the multiplicity of $\pi_{\lambda,\overline{t}}$ is equal to 1. The restriction of $\mathfrak{a}(\lambda, \chi_{\overline{t}}, w)$ on $V_\lambda(\chi_{\overline{t}})$ must be a scalar multiple $c(w, \overline{t})$ of identity. Choose a vector $v \in V_\lambda(\chi_{\overline{t}})$ such that $\delta_{\lambda,\chi_{\overline{t}}}(v) \neq 0$. Apply this vector on the both sides of the equality above:

$$\delta_{\lambda,\chi_{\overline{t}}}(v) = \delta_{\lambda,\chi_{\overline{t}}} \mathfrak{a}(\lambda, \chi_{\overline{t}}, w)(v) = c(w, \overline{t}) \delta_{\lambda,\chi_{\overline{t}}}(v).$$

Therefore, $c(w, \overline{t}) = 1$, for all $w \in R_\lambda$. In other words, the generic irreducible constituent $\pi_{\lambda,\overline{t}}$ corresponds to the trivial character of R_λ (*cf.* Remark 4.1.7). Since $\mathbb{C}[R_\lambda]$ is isomorphic to the commuting algebra $\mathcal{C}(\pi_\lambda)$ and $\iota_{\lambda,\chi_{\overline{t}}}(R_\lambda)$ forms a basis of $\mathcal{C}(\pi_\lambda)$, $\pi_{\lambda,\overline{t}}$ is the unique irreducible constituent of π_λ such that the restriction of $\iota_{\lambda,\chi_{\overline{t}}}(R_\lambda)$ on $\pi_{\lambda,\overline{t}}$ is a trivial character of R_λ. The proof is completed. □

5.2.3. To simplify notation, we write $V_\lambda(\chi_{\overline{t}})$ (resp. $\pi_{\lambda,\overline{t}}$, $\delta_{\lambda,\chi_{\overline{t}}}$, $C(\lambda,\chi_{\overline{t}},w)$, and $\mathfrak{a}(\lambda,\chi_{\overline{t}},w)$) as $V_{\overline{t}}$ (resp. $\pi_{\overline{t}}$, $\delta_{\overline{t}}$, $C(\lambda,\overline{t},w)$, and $\mathfrak{a}(\lambda,\overline{t},w)$).

Now we determine which principal nilpotent orbits are associated with a given generic irreducible constituent. Instead of dealing with principal nilpotent orbits, we consider the subset P of regular nilpotent elements defined in §4.1.1 . We know that that P is parametrized by $\overline{T}$. We can define a map $\tilde{\rho}$, an extension of ρ in §4.1.7, from $\overline{T}$ to Σ_λ defined by $\overline{t}\mapsto(\pi_{\overline{t}},V_{\overline{t}})$. Our main goal is to find a criterion to determine when $(\pi_{\overline{t}},V_{\overline{t}})$ and $(\pi_{\overline{t}'},V_{\overline{t}'})$ are the same. This is achieved in the next lemma.

Lemma. *Let $\overline{t}$ and $\overline{t}'$ be two elements of $\overline{T}$. Then a necessary and sufficient condition for $(\pi_{\overline{t}},V_{\overline{t}})=(\pi_{\overline{t}'},V_{\overline{t}'})$ is that for all $w\in R_\lambda$, $\lambda_w(\overline{t}^{-1}\overline{t}')=1$. In other words, $V_{\overline{t}}=V_{\overline{t}'}$ if and only if $\overline{t}$ and $\overline{t}'$ are in the same orbit of Q_λ, where Q_λ is the intersection of all kernels of λ_w.*

Proof. Fix an element $\overline{t}$ of $\overline{T}$. According to Lemma 5.2.2, $(\pi_{\overline{t}},V_{\overline{t}})$ can be uniquely characterized as follows:

For all $w\in R_\lambda$, the restriction of $\mathfrak{a}(\lambda,\overline{t},w)$ on $V_{\overline{t}}$ is the identity.

Let $\overline{t}'$ be another element of $\overline{T}$. We know that $(\pi_{\overline{t}'},V_{\overline{t}'})$ is the unique irreducible constituent which is characterized as above. For all $w\in R_\lambda$, $\mathfrak{a}(\lambda,\overline{t},w)$ and $\mathfrak{a}(\lambda,\overline{t}',w)$ only differ up to a scalar. Therefore, the necessary and sufficient condition for $V'_{\overline{t}}=V_{\overline{t}}$ is that for all $w\in R_\lambda$, $\mathfrak{a}(\lambda,\overline{t},w)=\mathfrak{a}(\lambda,\overline{t}',w)$.

Recall the definition of $\mathfrak{a}(\lambda,\overline{t},w)$ and $\mathfrak{a}(\lambda,\overline{t}',w)$:

$$\mathfrak{a}(\lambda,\overline{t},w)=C(\lambda,\overline{t},w)A(\lambda,w),\ \text{ and }\mathfrak{a}(\lambda,\overline{t}',w)=C(\lambda,\overline{t}',w)A(\lambda,w).$$

Then $\mathfrak{a}(\lambda,\overline{t},w)=\mathfrak{a}(\lambda,\overline{t}',w)$ if and only if $C(\lambda,\overline{t},w)=C(\lambda,\overline{t}',w)$. Applying Corollary 5.1.7, the equality holds if and only if $\lambda_w(\overline{t})=\lambda_w(\overline{t}')$. Since λ_w is a character of $\overline{T}$. The conditions are equivalent to $\lambda_w(\overline{t}^{-1}\overline{t}')=1$. This completes the proof. ☐

5.2.4. Now we obtain the criterion for when two elements in P have the same image of the extended map $\tilde{\rho}$. Without a doubt, $\tilde{\rho}$ must factor through ρ; i.e., if two elements are in the same orbit, then their images are the same. We know that any two elements are in the same orbits if and only if they differ by an element of $p(T)$. Hence, $\tilde{\rho}$ factors through ρ if and only if for all $w\in R_\lambda$, $\lambda_w(p(T))=1$. The following lemma tells us that this is true as predicted.

Lemma. *For all $w\in W$, we have the following equality:*

$$\lambda_w(p(t))=\lambda(t(w^{-1}t)^{-1}),\quad\forall t\in T.$$

Remark. If $w\lambda=\lambda$, then $\lambda(t(wt)^{-1})=1$, $\forall t\in T$. As a consequence, $\lambda_w(p(T))=1$.

Proof. Recall the definition of λ_w:

$$\lambda_w(\overline{t})=\prod_{i=1}^{i=n}(\lambda_i)_{\alpha_i}(\overline{\alpha}_i(\overline{t})),$$

where $\overline{t}\in\overline{T}$. If $\overline{t}=p(t),\ t\in T$, then λ_w becomes

$$\lambda_w(p(t))=\prod_{i=1}^{i=n}(\lambda_i)_{\alpha_i}(\overline{\alpha}_i(p(t)))=\prod_{i=1}^{i=n}(\lambda_i)_{\alpha_i}(\alpha_i(t)).$$

The last equality holds since $\overline{\alpha}_i \circ p = \alpha_i$. Now we prove our lemma by induction on the length $l(w)$ of w. If $l(w) = 1$, then $w = s_\alpha$ for some $\alpha \in \Pi$. Compute directly:

$$\begin{aligned}\lambda_w(p(t)) &= \lambda_\alpha(\alpha(t)) = \lambda(\alpha^\vee \circ \alpha(t)) \\ &= \lambda\big(t(t(\alpha^\vee \circ \alpha(t))^{-1})^{-1}\big) = \lambda\big(ts_\alpha(t)^{-1}\big) = \lambda\big(ts_\alpha^{-1}(t)^{-1}\big).\end{aligned}$$

Therefore the length 1 is true.

Assume $l(w) = n-1$ is true; i.e, for any $t \in T$ and $w' = s_{\alpha_{n-1}} s_{\alpha_{n-2}} \cdots s_{\alpha_1}$, we have

$$\lambda_{w'}(p(t)) = \lambda(t({w'}^{-1}t)^{-1}).$$

Let $l(w) = n$ and $w = s_{\alpha_n} s_{\alpha_{n-1}} s_{\alpha_{n-2}} \cdots s_{\alpha_1} = s_{\alpha_n} w'$, where $w' = s_{\alpha_{n-1}} s_{\alpha_{n-2}} \cdots s_{\alpha_1}$. Then

$$\begin{aligned}\lambda(t(w^{-1}t)^{-1}) =& \lambda(t({w'}^{-1} s_{\alpha_n} t)^{-1}) \\ =& \lambda\big(t({w'}^{-1}(s_{\alpha_n} t))^{-1}\big) \\ =& \lambda\big(t(s_{\alpha_n} t)^{-1}(s_{\alpha_n} t)({w'}^{-1}(s_{\alpha_n} t))^{-1}\big) \\ =& \lambda\big(t(s_{\alpha_n} t)^{-1}\big)\lambda\big((s_{\alpha_n} t)({w'}^{-1}(s_{\alpha_n} t))^{-1}\big) \\ =& \lambda\big(t(s_{\alpha_n} t)^{-1}\big)\lambda_{w'}\big(p(s_{\alpha_n} t)\big) \\ =& \lambda\big(t(s_{\alpha_n} t)^{-1}\big)\lambda_{w'}\big(p(t(\alpha_n^\vee \circ \alpha_n(t))^{-1})\big) \\ =& \lambda\big(t(s_{\alpha_n} t)^{-1}\big)\lambda_{w'}\big(p((\alpha_n^\vee \circ \alpha_n(t))^{-1})\big)\lambda_{w'}\big(p(t)\big) \\ =& \lambda\big(t(s_{\alpha_n} t)^{-1}\big)\lambda\big((\alpha_n^\vee \circ \alpha_n(t))^{-1}({w'}^{-1}((\alpha_n^\vee \circ \alpha_n(t))^{-1}))^{-1}\big)\lambda_{w'}\big(p(t)\big) \\ =& \lambda\big(t(s_{\alpha_n} t)^{-1}\big)\lambda\big((\alpha_n^\vee \circ \alpha_n(t))^{-1}\big)\lambda\big({w'}^{-1}((\alpha_n^\vee \circ \alpha_n(t))\big)\lambda_{w'}\big(p(t)\big) \\ =& \lambda\big((\alpha_n^\vee \circ \alpha_n(t))\big)\lambda\big((\alpha_n^\vee \circ \alpha_n(t))^{-1}\big)\lambda_n(\alpha_n^\vee \circ \alpha_n(t))\lambda_{w'}\big(p(t)\big) \\ =& (\lambda_n)_{\alpha_n}(\alpha_n(t)) \prod_{i=1}^{i=n-1} (\lambda_i)_{\alpha_i}(\alpha_i(t)) \\ =& \prod_{i=1}^{i=n} (\lambda_i)_{\alpha_i}(\alpha_i(t)) \\ =& \lambda_w(p(t)).\end{aligned}$$

This completes the induction step. □

5.2.5. *Proof.* Now we start to prove the Main Theorem. For part (i), according to Lemma 5.2.4 above, for all $w \in R_\lambda \subset W_\lambda$, λ_w are trivial on $p(T)$. Therefore, λ_w can be thought as a character of $\overline{T}/p(T) \simeq H^1(\Gamma, Z(\underline{G}))$. Define Q_λ to be the intersection of all kernels of λ_w. Clearly, $Q_\lambda = \mathsf{Q}_\lambda/p(T)$, where Q_λ is defined in §5.2.3. According to Lemma 5.2.3, for any two $Y, Y' \in \mathsf{P}$, $\tilde{\rho}(Y) = \tilde{\rho}(Y')$ if and only if they are in the same orbits of Q_λ. Then we can translate our arguments in terms of principal nilpotent orbits by replacing Q_λ to Q_λ. Therefore part (i) holds.

For part (ii), without loss of generality, we choose the generic character $\chi_{\mathbf{1}}$ of N, where $\mathbf{1}$ is the identity element of $\overline{T}$. According to Proposition 4.2.3, we need to determine the character c_i of R_λ, $i \in \Sigma_\lambda^{\mathrm{gen}}$. Recall that c_i is defined by

$$\mathfrak{a}(\lambda, \chi_{\mathbf{1}}, w)|_{\pi_i} = c_i(w)\mathrm{id},$$

where $w \in R_\lambda$. Let $\bar{t} \in \overline{T}$ and $\pi_{\bar{t}} \in \Sigma_\lambda^{\mathrm{gen}}$ be the unique $\chi_{\bar{t}}$- generic irreducible constituent of π_λ. We know the restriction of $\mathfrak{a}(\lambda, \chi_{\bar{t}}, w)$ on $\pi_{\bar{t}}$ is the identity operator. By Corollary 5.1.7 and the definition of $\mathfrak{a}(\lambda, \chi_{\bar{t}}, w)$ and $\mathfrak{a}(\lambda, \chi_{\mathbf{1}}, w)$, we have

$$\mathfrak{a}(\lambda, \chi_{\mathbf{1}}, w)|_{\pi_{\bar{t}}} = \langle w, \bar{t}\rangle_\lambda^{-1}\mathfrak{a}(\lambda, \chi_{\bar{t}}, w)|_{\pi_{\bar{t}}} = \langle w, \bar{t}\rangle_\lambda^{-1}\mathrm{id}.$$

Therefore, $c_{\pi_{\overline{t}}}(w) = \langle w, \overline{t}\rangle_{\lambda}^{-1}$. This is exactly the statement of (ii).

Part (iii) is just a reformulation of part (ii). This completes the proof. □

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