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SOME CLOSED FORMULAS FOR CANONICAL BASES OF FOCK SPACES

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Abstract. We give some closed formulas for certain vectors of the canonical bases of the Fock space representation of $U_v(\widehat{\mathfrak{sl}}_n)$. As a result, a combinatorial description of certain parabolic Kazhdan-Lusztig polynomials for affine type A is obtained.

1. Introduction

Let $\mathcal{F}_v$ be the Fock space representation of $U_v(\widehat{\mathfrak{sl}}_n)$ introduced by Hayashi [H] and further studied by Misra and Miwa [MM], Stern [St], and Kashiwara, Miwa and Stern [KMS]. It has a standard basis $\Sigma = \{s(\lambda) \mid \lambda \in \mathcal{P}\}$ indexed by the set $\mathcal{P}$ of all integer partitions. In [LT1] two canonical bases $B = \{G(\lambda) \mid \lambda \in \mathcal{P}\}$ and $B^- = \{G^-(\lambda) \mid \lambda \in \mathcal{P}\}$ of $\mathcal{F}_v$ have been constructed. The subset of B consisting of the $G(\lambda)$'s for which λ is n-regular coincides with Kashiwara's lower global basis (or Lusztig's canonical basis) of the irreducible sub-representation of $\mathcal{F}_v$ generated by the highest weight vector $s(\emptyset)$.

The main motivation for introducing the bases B and B^- was their conjectural relation with the decomposition matrices of the q-Schur algebras $\mathcal{S}_m(q)$ defined by Dipper and James [DJ] in connection with the modular representation theory of the finite groups $GL_m(\mathbb{F}_q)$ in non-describing characteristic. Conjecture 5.2 of [LT1] was proved by Varagnolo and Vasserot [VV], who established that the coefficients of the expansion of $G^-(\lambda)$ on the basis Σ are equal to the Kazhdan-Lusztig polynomials appearing in Lusztig's character formula for $U_\zeta(\mathfrak{sl}_r)$, where ζ is a complex primitive nth root of 1.

Let ℓ be a prime number coprime to q and such that the multiplicative order of q in $\overline{\mathbb{F}}_\ell^*$ is equal to n. To $w \in \mathbb{N}^*$ one associates a "large n-core partition" $\rho = \rho(w)$ (see below, Definition 1). Set $m = nw + |\rho|$, and let $B_{w,\rho}$ be the unipotent block of $\overline{\mathbb{F}}_\ell GL_m(\mathbb{F}_q)$ containing the Specht-type modules $S(\lambda)$ labelled by partitions λ with n-core ρ and n-weight w (see [DJ]). Assume that $w < \ell$, and let $S(\lambda) = \mathrm{rad}^0(S(\lambda)) \supset \mathrm{rad}^1(S(\lambda)) \supset \mathrm{rad}^2(S(\lambda)) \supset \dots$ denote the radical series of $S(\lambda)$. In [Mi] the graded composition multiplicities

$$\mathrm{rad}_{\lambda,\mu}(v) = \sum_{i \geqslant 0} [\mathrm{rad}^i(S(\lambda))/\mathrm{rad}^{i+1}(S(\lambda)) : D(\mu)]\, v^i$$

of all $S(\lambda)$ in $B_{w,\rho}$ were computed explicitly in terms of the Littlewood-Richardson coefficients, using a Morita equivalence between $B_{w,\rho}$ and the principal block of

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$\overline{\mathbb{F}}_\ell GL_n(\mathbb{F}_q)\wr\mathfrak{S}_w$ established in [HM]. This Morita equivalence is similar to a Morita equivalence for blocks of symmetric groups conjectured long ago by Rouquier (see [R]) and recently proved by Chuang and Kessar [CK]. It was also conjectured in [Mi] that

$$\mathrm{rad}_{\lambda,\mu}(v) = d_{\lambda,\mu}(v)\,,$$

where $d_{\lambda,\mu}(v)$ denotes the v-decomposition number of [LT1], that is, the coefficient of $s(\lambda)$ in the expansion of $G(\mu)$. The main result of this paper is a proof of this conjecture. Simultaneously, we also determine a similar expression for the coefficient $e_{\lambda,\mu}(-v^{-1})$ of $s(\mu)$ in the expansion of $G^-(\lambda)$. As a consequence we obtain a closed and combinatorial expression for two families of parabolic affine Kazhdan-Lusztig polynomials. It is remarkable that in fact all these polynomials are just monomials. Another consequence is that we obtain a representation-theoretical interpretation of these particular v-decomposition numbers: the exponent of v in $d_{\lambda,\mu}(v)$ indicates to which layer of the radical filtration of $S(\lambda)$ the copies of the simple module $D(\mu)$ belong.

In terms of the Fock space, the block $B_{w,\rho}$ corresponds to a distinguished weight space. More precisely, there is a natural bijection between the set of n-core partitions and the set of extremal weights $\sigma(\Lambda_0)$ of $\mathcal{F}_v$. Here Λ_0 is the highest weight of $\mathcal{F}_v$ and σ belongs to the affine Weyl group $W = \widetilde{\mathfrak{S}}_n$ of $\widehat{\mathfrak{sl}}_n$. Let $\sigma_\rho(\Lambda_0)$ be the extremal weight corresponding to ρ in this bijection. Then our results give some closed formulas in terms of the Littlewood-Richardson coefficients for the Σ-expansions of all the $G(\mu)$'s and $G^-(\lambda)$'s of weight $\Lambda_{w,\rho} := \sigma_\rho(\Lambda_0) - w\delta$, where δ is the imaginary root.

Let $P(\mathcal{F}_v)$ denote the set of weights of $\mathcal{F}_v$. We have the decomposition $P(\mathcal{F}_v) = \bigsqcup_{w\in\mathbb{N}} \mathcal{O}_w$, where $\mathcal{O}_w = \{\Lambda = \sigma\Lambda_0 - w\delta \mid \sigma \in W\}$ is the W-orbit of $\Lambda_0 - w\delta$. For $\Lambda \in P(\mathcal{F}_v)$, let $T(\Lambda)$ (resp. $T^-(\Lambda)$) denote the transition matrix from Σ to B (resp. from Σ to B^-) in the weight space $\mathcal{F}_v(\Lambda)$. Having computed the matrices $T(\Lambda_{w,\rho})$ and $T^-(\Lambda_{w,\rho})$, it is easy to determine $T(\Lambda)$ and $T^-(\Lambda)$ for many other weights Λ. Indeed, suppose that $\Lambda \in P(\mathcal{F}_v)$ and α_i is a simple root of $\widehat{\mathfrak{sl}}_n$ such that $\Lambda + \alpha_i \notin P(\mathcal{F}_v)$. Let σ_i be the simple reflection of W associated with α_i. Then $T(\Lambda) = T(\sigma_i\Lambda)$ and $T^-(\Lambda) = T^-(\sigma_i\Lambda)$. It follows that, for each w, the orbit $\mathcal{O}_w$ can be divided into a finite number of classes on which the matrices $T(\Lambda)$ and $T^-(\Lambda)$ remain constant. The class of $\Lambda_{w,\rho}$ for which we have found closed formulas is always infinite; hence, for each w our computations give the transition matrices for an infinite number of weights of $\mathcal{O}_w$. Moreover, in the case of $\widehat{\mathfrak{sl}}_2$ this class is the only infinite one and therefore our formulas calculate in this case the matrices for all but a finite number of weights in each $\mathcal{O}_w$. These facts, which are easily deduced from the theory of crystal bases, are the Fock space counterpart of Scopes' results [Sc] about Morita equivalence for blocks of the symmetric groups, and of the analogues of these results for blocks of Hecke algebras and unipotent blocks of $GL_n(\mathbb{F}_q)$ [Jo1, Jo2].

The paper is structured as follows. In Section 2 we review the correspondence between n-core partitions and the W-orbit of the fundamental weight Λ_0 of $\widehat{\mathfrak{sl}}_n$. Section 3 and 4 recall the main facts about the Fock representations of $\widehat{\mathfrak{sl}}_n$ and $U_v(\widehat{\mathfrak{sl}}_n)$, and introduce the canonical bases. In Section 5 we consider the space $\mathcal{S}$ of symmetric functions in n independent sets of variables $A_0, \dots, A_{n-1}$ with coefficients in $\mathbb{C}(v)$. The standard basis of $\mathcal{S}$ is given by the products of Schur functions $s_{\underline{\lambda}} = s_{\lambda^0}(A_0)\cdots s_{\lambda^{n-1}}(A_{n-1})$. We introduce two new bases $\{\eta_{\underline{\lambda}}\}$ and

$\{\psi_{\underline{\lambda}}\}$ which are canonical with respect to a certain bar-involution sending v to v^{-1} and two crystal lattices $\mathcal{L}$ and $\mathcal{L}^-$, and we calculate their expansions on the standard basis in terms of the Littlewood-Richardson coefficients. Heuristically, we regard $\mathcal{S}$ (or rather its specialization $\mathcal{S}_1$ at $v = 1$) as the carrier space of the "canonical commutation relations" representation of the homogeneous Heisenberg subalgebra of $\widehat{\mathfrak{gl}}_n$. In fact, the sum of the homogeneous components of $\mathcal{S}_1$ of degree less than ℓ can be identified to the sum of the complexified Grothendieck groups

$$\bigoplus_{w<\ell} G(B(\overline{\mathbb{F}}_\ell GL_n(\mathbb{F}_q) \wr \mathfrak{S}_w))\,,$$

where B means the principal block. In this identification the vectors $s_{\underline{\lambda}}$ are mapped to the classes of the unipotent ordinary irreducible modules, and the vectors $\psi_{\underline{\lambda}}$ to the classes of the simple modules. In Section 6 we state our main result (Theorem 8), namely that the canonical bases of certain weight spaces of $\mathcal{F}_v$ coincide with the canonical bases of the corresponding homogeneous components of $\mathcal{S}$ under a natural vector space isomorphism. At $v = 1$ this vector space isomorphism is essentially the natural intertwining operator between the principal and the homogeneous realization of the basic representation of $\widehat{\mathfrak{sl}}_n$ (see [LL, Lei]). This is the counterpart in this setting of the Morita equivalence of [HM], and it gives immediately the above-mentioned formulas for $d_{\lambda,\mu}(v)$ and $e_{\lambda,\mu}(v)$ when λ and μ have n-core ρ and n-weight $\leqslant w$ (Corollary 10). Section 7 is devoted to the proof of Theorem 8. Finally, Section 8 develops in the context of the Fock space the combinatorics underlying Scopes' isometries between blocks of symmetric groups.

After this paper was completed, we received a preprint by Chuang and Tan which had some overlap with ours and contained the formula for $d_{\lambda,\mu}(v)$ given in Corollary 10 below, but only for partitions μ which are n-regular. This preprint is now published [CT].

2. Combinatorics of partitions and the affine Weyl group

2.1. A partition is a finite nonincreasing sequence of positive integers. We shall denote by $\mathcal{P}$ the set of all partitions. By convention, $\mathcal{P}$ contains the empty partition $\emptyset$. To a partition

$$\lambda = (\lambda_1, \ldots, \lambda_k)$$

one associates the infinite decreasing sequence of β-numbers

$$\beta(\lambda) = (\lambda_1, \lambda_2 - 1, \lambda_3 - 2, \ldots, \lambda_i - i + 1, \ldots)\,,$$

where λ_i is assumed to be 0 for $i > k$.

Fix an integer $n \geqslant 2$, and form an infinite abacus with n runners labelled $1, \ldots, n$ from left to right. The positions on the ith runner are labelled by the integers having residue i modulo n. By placing a bead on each β-number of λ one gets the abacus representation of λ. As is well known, sliding all the beads in the n-abacus representation of λ as high as they will go produces the n-core $\lambda_{(n)}$ of λ. This is illustrated in Figure 1 for $\lambda = (6, 4, 3, 1, 1, 1)$ and $n = 3$. In that case $\beta(\lambda) = (6, 3, 1, -2, -3, -4, -6, -7, -8, \ldots)$.

Note that in the abacus representation of a partition, the number of occupied positive positions is always equal to the number of vacant nonpositive positions. In particular, we see that the n-core partitions are in one-to-one correspondence with the n-tuples of integers $(a_1, \ldots, a_n)$ such that $\sum_i a_i = 0$. Thus, as shown in Figure 1, the 3-core $(3, 1)$ is associated to the 3-tuple $(0, -1, 1)$.

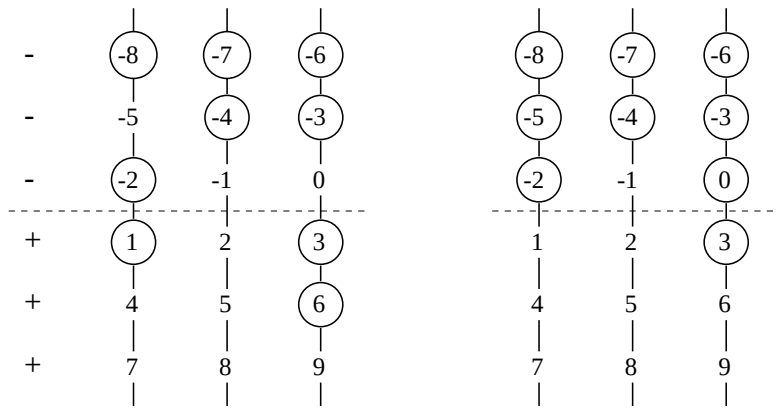

FIGURE 1. The 3-abacus representation of $\lambda = (6, 4, 3, 1, 1, 1)$ and of its 3-core $\lambda_{(3)} = (3, 1)$.

The next definition introduces certain "large n-core partitions". They were first considered by Rouquier as labels of certain good blocks of the symmetric groups.

Definition 1. Let $w \in \mathbb{N}^*$. The partition $\rho = \rho(w)$ associated with w is the n-core corresponding to the n-tuple

$$(a_1, \dots, a_n) = \begin{cases} \left(\frac{(1-n)(w-1)}{2}, \frac{(3-n)(w-1)}{2}, \dots, \frac{(n-3)(w-1)}{2}, \frac{(n-1)(w-1)}{2}\right) & \text{if } n \text{ or } w \text{ is odd,} \\ \left(\frac{w}{2}, \frac{3w-2}{2}, \dots, \frac{(n-1)w-n+2}{2}, \frac{(1-n)w+n-2}{2}, \dots, \frac{2-3w}{2}, -\frac{w}{2}\right) & \text{otherwise.} \end{cases}$$

Example 2. For $n = 4$ and $w = 3$, we have

$$(a_1, a_2, a_3, a_4) = (-3, -1, 1, 3) \quad \text{and} \quad \rho = (12, 9, 6^2, 4^2, 2^3, 1^3).$$

For $n = 4$ and $w = 4$, we have

$$(a_1, a_2, a_3, a_4) = (2, 5, -5, -2) \quad \text{and} \quad \rho = (18, 15, 12, 9^2, 7^2, 5^2, 3^3, 2^3, 1^3).$$

2.2. Let $\mathfrak{g} = \widehat{\mathfrak{sl}}_n$ be the affine Lie algebra of type $A^{(1)}_{n-1}$ [Ka]. Denote by $\Lambda_0, \dots, \Lambda_{n-1}$ the fundamental weights of $\mathfrak{g}$, by $\alpha_0, \dots, \alpha_{n-1}$ its simple roots, and by δ the imaginary root. They are related by

$$\begin{cases} \alpha_0 & = & 2\Lambda_0 - \Lambda_1 - \Lambda_{n-1} + \delta, \\ \alpha_i & = & -\Lambda_{i-1} + 2\Lambda_i - \Lambda_{i+1} \qquad (i = 1, \dots, n-2), \\ \alpha_{n-1} & = & -\Lambda_0 - \Lambda_{n-2} + 2\Lambda_{n-1}, \end{cases}$$

if $n \geqslant 3$, and

$$\begin{cases} \alpha_0 & = & 2\Lambda_0 - 2\Lambda_1 + \delta, \\ \alpha_1 & = & -2\Lambda_0 + 2\Lambda_1, \end{cases}$$

if $n = 2$. The free $\mathbb{Z}$-modules $P = \left(\bigoplus_{i=0}^{n-1} \mathbb{Z}\Lambda_i\right) \oplus \mathbb{Z}\delta$ and $Q = \bigoplus_{i=0}^{n-1} \mathbb{Z}\alpha_i$ are called respectively the weight lattice and the root lattice of $\mathfrak{g}$. Let a_{ij} be the coefficient of Λ_j in α_i. The $n \times n$ matrix $A = (a_{ij})$ is called the Cartan matrix of $\mathfrak{g}$. One defines

a nondegenerate symmetric bilinear form on P by

$$\begin{cases} (\alpha_i, \alpha_j) &= a_{ij} \quad (0 \leqslant i, j \leqslant n-1), \\ (\Lambda_0, \Lambda_0) &= 0, \\ (\Lambda_0, \alpha_0) &= 1, \\ (\Lambda_0, \alpha_i) &= 0 \qquad (1 \leqslant i \leqslant n-1). \end{cases}$$

The Weyl group W of $\mathfrak{g}$ is the subgroup of $GL(P)$ generated by the simple reflections σ_i defined by

$$\sigma_i(\Lambda) = \Lambda - (\alpha_i, \Lambda)\alpha_i \qquad (\Lambda \in P,\ 0 \leqslant i \leqslant n-1).$$

This is a Coxeter group of type $A^{(1)}_{n-1}$, and it is isomorphic to the semi-direct product of the symmetric group $\mathfrak{S}_n$ by the root lattice $Q_0 = \bigoplus_{i=1}^{n-1} \mathbb{Z}\alpha_i$ of the finite-dimensional Lie algebra $\mathfrak{g}_0 = \mathfrak{sl}_n$.

2.3. It will be convenient to realize P and Q as lattices in a $\mathbb{Q}$-vector space X of dimension $n+2$. Namely, we set

$$X = \left(\bigoplus_{i=1}^{n} \mathbb{Q}\varepsilon_i \right) \oplus \mathbb{Q}\Lambda_0 \oplus \mathbb{Q}\delta$$

and we identify the simple roots and fundamental weights to the following elements of X:

$$\alpha_i = \varepsilon_i - \varepsilon_{i+1} \quad (i = 1, \ldots, n-1), \qquad \alpha_0 = \varepsilon_n - \varepsilon_1 + \delta,$$
$$\Lambda_i = \varepsilon_1 + \cdots + \varepsilon_i - \frac{i}{n}(\varepsilon_1 + \cdots + \varepsilon_n) + \Lambda_0 \qquad (i = 1, \ldots n-1).$$

Note that the root lattice Q_0 of $\mathfrak{g}_0$ gets identified in this realization to

$$Q_0 = \left\{ \sum_{i=1}^{n} a_i \varepsilon_i \in X \ ;\ a_i \in \mathbb{Z},\ \sum_{i=1}^{n} a_i = 0 \right\}.$$

It is easy to check that the above bilinear form is the restriction to P of the bilinear form on X given by

$$\begin{cases} (\varepsilon_i, \varepsilon_j) &= \delta_{ij} \quad (1 \leqslant i, j \leqslant n), \\ (\Lambda_0, \Lambda_0) &= 0, \\ (\Lambda_0, \varepsilon_i) &= 0 \qquad (1 \leqslant i \leqslant n), \\ (\delta, \delta) &= 0, \\ (\delta, \varepsilon_i) &= 0 \qquad (1 \leqslant i \leqslant n), \\ (\Lambda_0, \delta) &= 1. \end{cases}$$

This implies that the action of W on P can be lifted to X by setting

$$\begin{cases} \sigma_0(\Lambda_0) = \Lambda_0 - \alpha_0, \\ \sigma_0(\delta) = \delta, \\ \sigma_0(\varepsilon_1) = \varepsilon_n + \delta, \\ \sigma_0(\varepsilon_n) = \varepsilon_1 - \delta, \\ \sigma_0(\varepsilon_j) = \varepsilon_j, \qquad (j \neq 1, n) \end{cases} \qquad \begin{cases} \sigma_i(\Lambda_0) = \Lambda_0, \\ \sigma_i(\delta) = \delta, \\ \sigma_i(\varepsilon_i) = \varepsilon_{i+1}, \\ \sigma_i(\varepsilon_{i+1}) = \varepsilon_i, \\ \sigma_i(\varepsilon_j) = \varepsilon_j, \qquad (j \neq i, i+1) \end{cases} \qquad (i \neq 0).$$

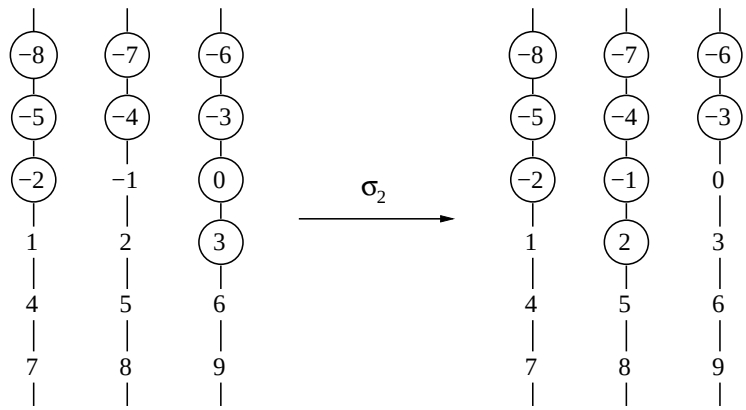

FIGURE 2. The 3-core partition $(3,1)$ is mapped under σ_2 to the 3-core partition (2).

2.4. The stabilizer W_0 of Λ_0 is the subgroup of W generated by $\sigma_1, \dots, \sigma_{n-1}$. Therefore, it is isomorphic to the symmetric group $\mathfrak{S}_n$. Hence, the orbit $W\Lambda_0 \simeq W/W_0$ is in a natural one-to-one correspondence with the root lattice Q_0 of $\mathfrak{g}_0$. Namely, for $\sigma \in W$ write

$$\sigma(\Lambda_0) = \Lambda_0 + d\delta + \sum_{i=1}^{n} a_i \varepsilon_i \,.$$

By the formulas of 2.3 for the action of W on X, we see that the integer coordinates a_i satisfy $\sum_i a_i = 0$, and that the projection $\sigma(\Lambda_0) \mapsto \sum_{i=1}^{n} a_i \varepsilon_i$ induces a bijection from $W\Lambda_0$ to Q_0.

Comparing with 2.1, we obtain that the orbit $W\Lambda_0$ can be identified with the set $\mathcal{C}_n$ of n-core partitions. In other words, there is a natural action of the affine Weyl group W on $\mathcal{C}_n$, which can be easily described using the abacus representation of the elements of $\mathcal{C}_n$. Namely, since W_0 acts on Q_0 by permuting the vectors ε_i, we see that for $i = 1, \dots, n-1$, the action of σ_i on an n-core partition is obtained by switching the beads of the ith and $(i+1)$th runners of its abacus representation. This is illustrated in Figure 2. On the other hand, we have

$$\begin{aligned} &\sigma_0\left(\Lambda_0 + d\delta + \sum_{i=1}^{n} a_i \varepsilon_i\right) \\ &\qquad = \Lambda_0 + (d-1)\delta + (a_n+1)\varepsilon_1 + a_2\varepsilon_2 + \cdots + a_{n-1}\varepsilon_{n-1} + (a_1-1)\varepsilon_n \,, \end{aligned}$$

so that the action of σ_0 on an n-core partition is obtained by first switching the beads of the first and last runners of its abacus representation, and then adding one more bead to the first runner and removing one bead from the last runner.

3. The Fock space representation of $\mathfrak{g}$

3.1. Let $V(\Lambda_0)$ denote the irreducible $\mathfrak{g}$-module with highest weight Λ_0 ([Ka], 9). It is the simplest infinite-dimensional $\mathfrak{g}$-module, and as such, it is often called the basic representation of $\mathfrak{g}$. The easiest way to construct it is to embed it into a larger representation of $\mathfrak{g}$ called the Fock space representation [JiMi]. The Fock space $\mathcal{F}$

has a distinguished $\mathbb{C}$-basis $\Sigma = \{s(\lambda)\ ;\ \lambda \in \mathcal{P}\}$, on which the Chevalley generators e_i, f_i $(0 \leqslant i \leqslant n-1)$ of $\mathfrak{g}$ act by

$$e_i s(\lambda) = \sum_{\mu} s(\mu), \qquad f_i s(\lambda) = \sum_{\nu} s(\nu), \tag{1}$$

where the first sum is over all partitions μ obtained by removing from the Young diagram of λ a node with n-residue i, and the second sum is over all ν's obtained from λ by adding an i-node. (Recall that the n-residue of the node on row j and column k is $k-j$ mod n.) Denoting by $\emptyset$ the empty partition, we see immediately that $s(\emptyset)$ is a highest weight vector of $\mathcal{F}$ with weight Λ_0. The cyclic submodule $U(\mathfrak{g})\, s(\emptyset)$ is isomorphic to $V(\Lambda_0)$.

These infinite-dimensional $\mathbb{C}$-vector spaces are graded by putting deg $s(\lambda) = m$ if λ is a partition of m. The dimension of the degree m component of $V(\Lambda_0)$ is known to be equal to the number of n-regular partitions of m.

For a $\mathfrak{g}$-module M and a weight $\Lambda \in P$, we denote by $M(\Lambda)$ the Λ-weight space of M. Let $P(M)$ denote the set of weights of M, that is, the set of $\Lambda \in P$ such that $M(\Lambda)$ is nonzero. It is known (see [Ka], 12.6) that

$$P(V(\Lambda_0)) = P(\mathcal{F}) = \{\sigma(\Lambda_0) - w\delta;\ \sigma \in W,\ w \in \mathbb{N}\}.$$

Recall from 2.4 that there is a natural bijection, say γ, from the orbit $W\Lambda_0$ to the set $\mathcal{C}_n$ of n-cores. Then the weight-space $\mathcal{F}(\Lambda)$ associated with the weight $\Lambda = \sigma(\Lambda_0) - w\delta$ consists precisely of the span of the vectors $s(\lambda)$ where λ has n-core $\gamma(\sigma(\Lambda_0))$ and n-weight w.

3.2. The Fock space is an integrable $\mathfrak{g}$-module, that is, the operators e_i and f_i are locally nilpotent on $\mathcal{F}$. Define the following linear operators on $\mathcal{F}$:

$$r_i = \exp(e_i)\ \exp(-f_i)\ \exp(e_i) \qquad (0 \leqslant i \leqslant n-1),$$

(see [Ka] 3.8). They satisfy $r_i e_i = -f_i r_i$ and $r_i f_i = -e_i r_i$; hence,

$$r_i^{-1} = \exp(f_i)\ \exp(-e_i)\ \exp(f_i).$$

Moreover, $r_i(\mathcal{F}(\Lambda)) = \mathcal{F}(\sigma_i(\Lambda))$ for all $\Lambda \in P(\mathcal{F})$.

It is easy to see that the divided powers $e_i^k/k!$ and $f_i^k/k!$ preserve the $\mathbb{Z}$-lattice $\mathcal{F}_{\mathbb{Z}}$ spanned by the basis $\{s(\lambda)\ ;\ \lambda \in \mathcal{P}\}$. It follows that r_i also preserves $\mathcal{F}_{\mathbb{Z}}$ for all i.

3.3. $\mathcal{F}$ is endowed with a canonical scalar product defined by

$$\langle s(\lambda)\,,\, s(\mu)\rangle = \delta_{\lambda,\mu} \qquad (\lambda, \mu \in \mathcal{P}).$$

This form is compatible with the actions of $\mathfrak{g}$ and of the r_i, in the sense that

$$\langle e_i x\,,\, y\rangle = \langle x\,,\, f_i y\rangle, \quad \langle r_i x\,,\, y\rangle = \langle x\,,\, r_i^{-1} y\rangle \qquad (x, y \in \mathcal{F},\ 0 \leqslant i \leqslant n-1).$$

Hence we see that r_i is an orthogonal transformation of $\mathcal{F}$.

3.4. For $k \in \mathbb{N}^*$, define an endomorphism d_k of $\mathcal{F}$ by

$$d_k\, s(\lambda) = \sum_{\mu} (-1)^{\mathrm{spin}(\mu/\lambda)}\, s(\mu)$$

where the sum is over all μ such that μ/λ is a horizontal n-ribbon strip of weight k, and spin(μ/λ) is the spin of the strip (see [LT2]). It is known that the d_k pairwise commute, and also commute with the action of $\mathfrak{g}' = [\mathfrak{g}, \mathfrak{g}]$ on $\mathcal{F}$. Moreover, $\mathcal{F}$

becomes a cyclic module under the action of $\mathcal{A} := U(\mathfrak{g}') \otimes \mathbb{C}[d_k\,;\, k \geqslant 1]$, that is, $\mathcal{F} \simeq \mathcal{A}\, s(\emptyset)$.

4. The Fock space representation of U_v

4.1. Let $U_v = U_v(\widehat{\mathfrak{sl}}_n)$ be the quantum affine algebra of type $A^{(1)}_{n-1}$ over the field $K := \mathbb{C}(v)$. Let $\mathcal{F}_v$ be the v-deformation of the Fock space introduced by Hayashi [H] and further studied by Misra and Miwa [MM]. As a vector space $\mathcal{F}_v = K \otimes_{\mathbb{C}} \mathcal{F}$, and $\Sigma = \{s(\lambda) \,|\, \lambda \in \mathcal{P}\}$ is a K-basis of $\mathcal{F}_v$. The Chevalley generators E_i, F_i $(0 \leqslant i \leqslant n-1)$ of U_v act on $\mathcal{F}_v$ via some simple v-analogues of the formulas (1), as follows.

Let λ and μ be two Young diagrams such that μ is obtained from λ by adding an i-node γ. Such a node is called a removable i-node of μ, or an indent i-node of λ. Let $I^r_i(\lambda,\mu)$ (resp. $R^r_i(\lambda,\mu)$) be the number of indent i-nodes of λ (resp. of removable i-nodes of λ) situated to the right of γ (γ not included). Set $N^r_i(\lambda,\mu) = I^r_i(\lambda,\mu) - R^r_i(\lambda,\mu)$. Then

$$F_i s(\lambda) = \sum_{\mu} v^{N^r_i(\lambda,\mu)} s(\mu), \tag{2}$$

where the sum is over all partitions μ such that μ/λ is an i-node. Similarly,

$$E_i s(\mu) = \sum_{\lambda} v^{-N^l_i(\lambda,\mu)} s(\lambda), \tag{3}$$

where the sum is over all partitions λ such that μ/λ is an i-node, and $N^l_i(\lambda,\mu)$ is defined as $N^r_i(\lambda,\mu)$ but replacing right by left.

4.2. The cyclic submodule $U_v\, s(\emptyset)$ is isomorphic to the basic representation $V_v(\Lambda_0)$ of U_v. As in 3.4, define operators D_k $(k \geqslant 1)$ acting on $\mathcal{F}$ by

$$D_k\, s(\lambda) = \sum_{\mu} (-v)^{-\mathrm{spin}(\mu/\lambda)}\, s(\mu)\,,$$

the sum being over all μ such that μ/λ is a horizontal n-ribbon strip of weight k. They pairwise commute and they also commute with the action of the subalgebra U'_v of U_v obtained by omitting the degree generator of the Cartan part. The Fock space $\mathcal{F}_v$ becomes a cyclic module under the action of $\mathcal{A}_v := U'_v \otimes K[D_k\,;\, k \geqslant 1]$, generated by $s(\emptyset)$.

4.3. By Kashiwara's theory of crystal bases [K1, K2, K3], $V_v(\Lambda_0)$ has two canonical bases dual to each other called the lower global base and the upper global base.

In [LT1] (see also [LT2]) the canonical bases $B = \{G(\lambda)\}$ and $B^- = \{G^-(\lambda)\}$ of $\mathcal{F}_v$ were introduced. We shall review their definition.

Let $x \mapsto \overline{x}$ be the bar involution of $\mathcal{F}_v$ defined in [LT1]. It is the unique semi-linear map satisfying $\overline{s(\emptyset)} = s(\emptyset)$ and

$$\overline{F_i\, x} = F_i \overline{x}\,, \qquad \overline{D_k\, x} = D_k \overline{x}\,, \qquad (0 \leqslant i \leqslant n-1,\ k \geqslant 1,\ x \in \mathcal{F}_v).$$

Let L (resp. L^-) be the free $\mathbb{Z}[v]$-module (resp. $\mathbb{Z}[v^{-1}]$-module) with basis $s(\lambda)$. The bases B and B^- are characterized by the properties

$$\overline{G(\lambda)} = G(\lambda), \quad \overline{G^-(\lambda)} = G^-(\lambda),$$
$$G(\lambda) \equiv s(\lambda) \bmod vL, \quad G^-(\lambda) \equiv s(\lambda) \bmod v^{-1}L^-,$$

for $\lambda \in \mathcal{P}$. The subset of B consisting of the $G(\lambda)$ for which λ is n-regular coincides with the lower global base of $V_v(\Lambda_0)$.

4.4. Define polynomials $d_{\lambda,\mu}(v)$ and $e_{\lambda,\mu}(v)$ by

$$G(\mu) = \sum_{\lambda} d_{\lambda,\mu}(v) s(\lambda)$$

and

$$G^-(\lambda) = \sum_{\mu} e_{\lambda,\mu}(-v^{-1}) s(\mu).$$

The polynomials $d_{\lambda,\mu}(v)$ and $e_{\lambda,\mu}(v)$ are known to be parabolic Kazhdan-Lusztig polynomials and they belong to $\mathbb{N}[v]$ (see [VV, LT2, KT]).

5. Symmetric functions and Heisenberg algebras

5.1. Let Sym denote the K-algebra of symmetric functions in a countable set X of indeterminates (see [Mcd]). It is known that Sym is a polynomial algebra in the power-sums symmetric functions

$$p_k = \sum_{x \in X} x^k\,, \qquad (k \geqslant 1).$$

Other systems of generators are the elementary symmetric functions e_k and the complete symmetric functions h_k. The Schur functions s_λ form a linear basis of Sym, and we denote by $\langle \cdot\,, \cdot \rangle$ the scalar product for which this basis is orthonormal. We may identify Sym to the Fock space representation $\mathcal{F}_v$ of $U_v(\widehat{\mathfrak{sl}}_n)$ by identifying the basis $\{s_\lambda\}$ with the standard basis Σ of $\mathcal{F}_v$.

For $f \in \mathrm{Sym}$, denote by $\hat{f}$ the operator of multiplication by f and by D_f the adjoint operator, that is,

$$\langle D_f(g)\,,\, h \rangle = \langle g\,,\, fh \rangle \qquad (f, g, h \in \mathrm{Sym}).$$

It is known that $D_{p_k} = k\,(\partial/\partial p_k)$ (see [Mcd], I.5 Ex.3). It follows that the operators $\hat{f}$, D_f ($f \in \mathrm{Sym}$) generate the enveloping algebra of the Heisenberg Lie algebra $\mathfrak{H}$ spanned by the operators $\hat{p}_k$, $\partial/\partial p_k$ ($k \in \mathbb{N}^*$) and the identity. Thus, Sym $\simeq \mathcal{F}_v$ can be regarded as the "canonical commutation relations" representation of $\mathfrak{H}$.

5.2. We introduce the generating function

$$H(t, X) = \sum_{k \geq 0} h_k t^k = \prod_{x \in X} (1 - xt)^{-1}$$

and we put $H(X) := H(1, X)$. Then, if we denote by Y a second countable set of indeterminates, and write $XY = \{xy \mid x \in X, y \in Y\}$, we have the Cauchy identity (see [Mcd], I.4.3)

$$\sum_{\lambda \in \mathcal{P}} s_\lambda(X) s_\lambda(Y) = H(XY) = \prod_{x \in X, y \in Y} (1 - xy)^{-1}. \tag{4}$$

More generally, if $U = \{u_\lambda\,,\, \lambda \in \mathcal{P}\}$, $V = \{v_\lambda\,,\, \lambda \in \mathcal{P}\}$ are two homogeneous bases of Sym with $\deg u_\lambda = \deg v_\lambda = |\lambda|$, then U and V are adjoint to each other with respect to $\langle \cdot\,, \cdot \rangle$ if and only if

$$\sum_{\lambda \in \mathcal{P}} u_\lambda(X) v_\lambda(Y) = H(XY)$$

(see [Mcd], I.4.6).

5.3. Let $A_0, \dots, A_{n-1}$ be n countable sets of indeterminates. We denote by

$$\mathcal{S} = \mathrm{Sym}(A_0, \dots, A_{n-1})$$

the algebra over K of functions symmetric in each set $A_0, \dots, A_{n-1}$ separately. This is the polynomial algebra in the variables $p_k(A_i)$ $(k \in \mathbb{N}^*,\ 0 \leqslant i \leqslant n-1)$. A linear basis of $\mathcal{S}$ is given by the products

$$s_{\underline{\lambda}} := s_{\lambda^0}(A_0) \cdots s_{\lambda^{n-1}}(A_{n-1}), \qquad \underline{\lambda} = (\lambda^0, \cdots, \lambda^{n-1}) \in \mathcal{P}^n.$$

The space $\mathcal{S}$ carries a scalar product defined by

$$\langle f_0(A_0) \cdots f_{n-1}(A_{n-1})\,,\, g_0(A_0) \cdots g_{n-1}(A_{n-1}) \rangle = \langle f_0, g_0 \rangle \cdots \langle f_{n-1}, g_{n-1} \rangle$$

for $f_0, \dots, f_{n-1}, g_0, \dots, g_{n-1} \in \mathrm{Sym}$. Clearly, the basis $\{ s_{\underline{\lambda}} \mid \underline{\lambda} \in \mathcal{P}^n \}$ is orthonormal for this scalar product.

As in 5.1, we regard $\mathcal{S}$ as the "canonical commutation relations" representation of the Heisenberg algebra $\mathfrak{H}_n$ generated by the operators

$$\hat{p}_k(A_i), \quad \frac{\partial}{\partial p_k(A_i)}, \qquad (k \in \mathbb{N}^*,\ 0 \leqslant i \leqslant n-1).$$

In the rest of this section we are going to introduce two canonical bases of $\mathcal{S}$.

5.4. Given $q_1(v), \dots, q_r(v) \in \mathbb{Z}[v, v^{-1}]$ and $i_1, \dots, i_r \in \{0, \dots, n-1\}$, we define symmetric functions of the formal set of variables $q_1(v)A_{i_1} + \cdots + q_r(v)A_{i_r}$ as follows. For $f \in \mathrm{Sym}$ we let $f(q_1(v)A_{i_1} + \cdots + q_r(v)A_{i_r})$ denote the image of f by the algebra homomorphism from Sym to $\mathcal{S}$ which maps p_k to

$$p_k(q_1(v)A_{i_1} + \cdots + q_r(v)A_{i_r}) := q_1(v^k)p_k(A_{i_1}) + \cdots + q_r(v^k)p_k(A_{i_r})\,.$$

(For the reader familiar with the language of λ-rings [Kn], we consider $A_0, \dots, A_{n-1}$, v as elements of a λ-ring, v being invertible of rank 1.) For example, for $j \in \{0, \dots, n-2\}$ we define

$$\begin{aligned} &p_k(A_j + vA_{j+1} + \cdots + v^{n-1-j}A_{n-1}) \\ &\qquad := p_k(A_j) + v^k p_k(A_{j+1}) + \cdots + v^{(n-j-1)k} p_k(A_{n-1})\,. \end{aligned}$$

Then we have (see [Mcd], I.5.9)

$$\begin{aligned} &s_\lambda(A_j + vA_{j+1} + \cdots + v^{n-1-j}A_{n-1}) \\ &\quad = \sum_{\alpha^j, \dots, \alpha^{n-1} \in \mathcal{P}} v^{\sum_{j \leqslant i \leqslant n-1}(i-j)|\alpha^i|}\, c^\lambda_{\alpha^j, \dots, \alpha^{n-1}}\, s_{\alpha^j}(A_j) \cdots s_{\alpha^{n-1}}(A_{n-1})\,, \end{aligned}$$

where the $c^\lambda_{\alpha^j, \dots, \alpha^{n-1}}$ are the Littlewood-Richardson coefficients, that is,

$$c^\lambda_{\alpha^j, \dots, \alpha^{n-1}} := \langle s_{\alpha^j} \cdots s_{\alpha^{n-1}}\,,\, s_\lambda \rangle\,. \tag{5}$$

Similarly, for $j = 1, \dots, n-1$, we set

$$p_k(A_j - vA_{j-1}) := p_k(A_j) - v^k p_k(A_{j-1})\,,$$

and we have (see [Mcd], I.3.10)

$$\begin{aligned} s_\lambda(A_j - vA_{j-1}) &= \sum_{\beta \in \mathcal{P}} s_{\lambda/\beta}(A_j) s_\beta(-vA_{j-1}) \\ &= \sum_{\alpha, \beta \in \mathcal{P}} (-v)^{|\beta|}\, c^\lambda_{\alpha,\beta}\, s_\alpha(A_j) s_{\beta'}(A_{j-1})\,, \end{aligned} \tag{6}$$

where β' denotes the partition conjugate to β.

5.5. For $\underline{\lambda} = (\lambda^0, \dots, \lambda^{n-1}) \in \mathcal{P}^n$ we write $|\underline{\lambda}| = \sum_{0 \leqslant i \leqslant n-1} |\lambda^i|$. Consider two bases of $\mathcal{S}$ $U = \{u_{\underline{\lambda}}\,,\, \underline{\lambda} \in \mathcal{P}^n\}$ and $V = \{v_{\underline{\lambda}}\,,\, \underline{\lambda} \in \mathcal{P}^n\}$ consisting of homogeneous elements of degree $\deg u_{\underline{\lambda}} = \deg v_{\underline{\lambda}} = |\underline{\lambda}|$. Then 5.2 may clearly be generalized, as follows.

Lemma 3. *The bases U and V are adjoint to each other with respect to $\langle \cdot\,, \cdot \rangle$ if and only if*

$$\sum_{\underline{\lambda} \in \mathcal{P}^n} u_{\underline{\lambda}}(A_0, \dots, A_{n-1})\, v_{\underline{\lambda}}(B_0, \dots, B_{n-1}) = H(A_0 B_0) \cdots H(A_{n-1} B_{n-1}).$$

5.6. For $\underline{\lambda} = (\lambda^0, \dots, \lambda^{n-1}) \in \mathcal{P}^n$, we define the following elements of $\mathcal{S}$:

$$\begin{aligned}
\eta_{\underline{\lambda}}(v) &= s_{\lambda^0}(A_0)\, s_{\lambda^1}(A_1 - vA_0)\, s_{\lambda^2}(A_2 - vA_1) \;\cdots\; s_{\lambda^{n-1}}(A_{n-1} - vA_{n-2})\,, \\
\varphi_{\underline{\lambda}}(v) &= s_{\lambda^0}(A_0 + vA_1 + \cdots + v^{n-1}A_{n-1})\, s_{\lambda^1}(A_1 + vA_2 + \cdots + v^{n-2}A_{n-1}) \\
&\qquad \cdots\; s_{\lambda^{n-3}}(A_{n-3} + vA_{n-2} + v^2 A_{n-1})\, s_{\lambda^{n-2}}(A_{n-2} + vA_{n-1})\, s_{\lambda^{n-1}}(A_{n-1}).
\end{aligned}$$

It is easy to see that these are two bases of $\mathcal{S}$. By 5.3, their expansions on the orthonormal basis $\{s_{\underline{\lambda}}\}$ are readily computed in terms of the Littlewood-Richardson coefficients. In particular,

Lemma 4. *For $\underline{\lambda}, \underline{\mu} \in \mathcal{P}^n$ we have*

$$\langle s_{\underline{\lambda}}\,,\, \eta_{\underline{\mu}}(v) \rangle = (-v)^{\delta(\underline{\lambda}, \underline{\mu})} \sum_{\substack{\alpha^0, \dots, \alpha^n \\ \beta^0, \dots, \beta^{n-1}}} \prod_{0 \leqslant j \leqslant n-1} c^{\mu^j}_{\alpha^j \beta^j} c^{\lambda^j}_{\beta^j (\alpha^{j+1})'}\,,$$

where $\alpha^0, \dots, \alpha^n, \beta^0, \dots, \beta^{n-1}$ run through $\mathcal{P}$ subject to the conditions

$$|\alpha^i| = \sum_{0 \leqslant j \leqslant i-1} |\lambda^j| - |\mu^j|\,, \qquad |\beta^i| = |\mu^i| + \sum_{0 \leqslant j \leqslant i-1} |\mu^j| - |\lambda^j|\,,$$

and

$$\delta(\underline{\lambda}, \underline{\mu}) := \sum_{1 \leqslant j \leqslant n-1} j(|\mu^j| - |\lambda^j|)\,.$$

Here we have used the convention that an empty sum is equal to 0. Thus $|\alpha^0| = 0$ and $|\beta^0| = |\mu^0|$, so that $c^{\mu^0}_{\alpha^0 \beta^0} c^{\lambda^0}_{\beta^0 (\alpha^1)'}$ is in fact equal to $c^{\lambda^0}_{\mu^0 (\alpha^1)'}$. Similarly, $|\alpha^n| = 0$ and $c^{\mu^{n-1}}_{\alpha^{n-1} \beta^{n-1}} c^{\lambda^{n-1}}_{\beta^{n-1} (\alpha^n)'}$ reduces to $c^{\mu^{n-1}}_{\alpha^{n-1} \lambda^{n-1}}$.

Proof. Using (6) we obtain

$$\eta_{\underline{\mu}}(v) = \sum_{\alpha^1, \dots, \alpha^{n-1}} (-v)^{\sum_i |\alpha^i|} s_{\mu^0}(A_0) \prod_{1 \leqslant j \leqslant n-1} s_{\mu^j/\alpha^j}(A_j) s_{(\alpha^j)'}(A_{j-1})\,.$$

Therefore,

$$\begin{aligned}
\langle s_{\underline{\lambda}}\,,\, \eta_{\underline{\mu}}(v) \rangle \;&=\; \sum_{\alpha^1, \dots, \alpha^{n-1}} (-v)^{\sum_i |\alpha^i|} \langle s_{\mu^0} s_{(\alpha^1)'}\,,\, s_{\lambda^0} \rangle \langle s_{\mu^{n-1}/\alpha^{n-1}}\,,\, s_{\lambda^{n-1}} \rangle \\
&\qquad \times \prod_{1 \leqslant j \leqslant n-2} \langle s_{\mu^j/\alpha^j} s_{(\alpha^{j+1})'}\,,\, s_{\lambda^j} \rangle\,,
\end{aligned}$$

and the result follows from the identity $\langle s_{\alpha/\beta}\, s_\gamma\,,\, s_\delta \rangle = \sum_\varepsilon c^\alpha_{\beta\varepsilon} c^\delta_{\varepsilon\gamma}$ and from the fact that $c^\gamma_{\alpha\beta}$ is nonzero only if $|\alpha| + |\beta| = |\gamma|$. $\square$

5.7. For $\underline{\lambda} = (\lambda^0, \dots, \lambda^{n-1}) \in \mathcal{P}^n$ we set $\underline{\lambda}' = ((\lambda^{n-1})', \dots, (\lambda^0)')$. Let $\prec$ denote any total ordering of $\mathcal{P}$ such that $|\lambda| < |\mu|$ implies $\lambda \prec \mu$. We also denote by $\prec$ the corresponding reverse lexicographic ordering of $\mathcal{P}^n$, that is, $\underline{\lambda} \prec \underline{\mu}$ if and only if there exists k such that $\lambda^k \prec \mu^k$ and $\lambda^j = \mu^j$ for $j > k$.

The functions $\eta_{\underline{\lambda}}$ and $\varphi_{\underline{\lambda}}$ enjoy the following properties.

Proposition 5. (i) *For $\underline{\lambda}, \underline{\mu} \in \mathcal{P}^n$ there holds $\langle s_{\underline{\lambda}}\,,\, \eta_{\underline{\mu}}(v)\rangle = \langle s_{\underline{\mu}'}\,,\, \eta_{\underline{\lambda}'}(v)\rangle$.*

(ii) *For $\underline{\lambda}, \underline{\mu} \in \mathcal{P}^n$ there holds $\langle \varphi_{\underline{\lambda}}(v)\,,\, \eta_{\underline{\mu}}(v)\rangle = \delta_{\underline{\lambda}\,\underline{\mu}}$.*

(iii) *The matrices*

$$\Big[\langle s_{\underline{\lambda}}\,,\, \eta_{\underline{\mu}}(v)\rangle\Big]_{\underline{\lambda},\underline{\mu}\in\mathcal{P}^n}\,, \qquad \Big[\langle \varphi_{\underline{\lambda}}(v)\,,\, s_{\underline{\mu}}\rangle\Big]_{\underline{\lambda},\underline{\mu}\in\mathcal{P}^n}$$

are lower unitriangular if their rows and columns are arranged according to the total ordering $\succ$.

Proof. (i) follows from Lemma 4 and the symmetry property $c^{\gamma}_{\alpha\beta} = c^{\gamma'}_{\alpha'\beta'}$ of the Littlewood-Richardson coefficients.

To prove (ii) we use Lemma 3. We have to show that

$$\sum_{\underline{\lambda}\in\mathcal{P}} \eta_{\underline{\lambda}}(A_0, \dots, A_{n-1}, v)\, \varphi_{\underline{\lambda}}(B_0, \dots, B_{n-1}, v) = H(A_0B_0)\cdots H(A_{n-1}B_{n-1})\,.$$

The left-hand side is equal to

$$\begin{aligned}
&\sum_{\lambda^0} s_{\lambda^0}(A_0)s_{\lambda^0}(B_0 + \cdots + v^{n-1}B_{n-1}) \sum_{\lambda^1} s_{\lambda^1}(A_1 - vA_0)s_{\lambda^1}(B_1 + \cdots + v^{n-2}B_{n-1}) \\
&\qquad \times \cdots \times \sum_{\lambda^{n-1}} s_{\lambda^{n-1}}(A_{n-1} - vA_{n-2})s_{\lambda^{n-1}}(B_{n-1}) \\
&\quad = H(A_0(B_0 + \cdots + v^{n-1}B_{n-1}))H((A_1 - vA_0)(B_1 + \cdots + v^{n-2}B_{n-1})) \\
&\qquad \times \cdots \times H((A_{n-1} - vA_{n-2})B_{n-1}) \\
&\quad = H(A_0B_0 + \cdots + A_{n-1}B_{n-1})\,.
\end{aligned}$$

Here we have used (4) and the basic identities

$$H(X)H(Y) = H(X+Y), \quad H(X)H(-X) = 1$$

valid for any formal sets of variables X and Y. Hence (ii) is proved.

Suppose that $\langle s_{\underline{\lambda}}\,,\, \eta_{\underline{\mu}}(v)\rangle \neq 0$. Then, by Lemma 4 we see that there exists $\alpha^{n-1} \in \mathcal{P}$ such that

$$c^{\mu^{n-1}}_{\alpha^{n-1}\lambda^{n-1}} \neq 0,$$

hence either $\lambda^{n-1} = \mu^{n-1}$ or $|\lambda^{n-1}| < |\mu^{n-1}|$. In the second case $\underline{\lambda} \prec \underline{\mu}$, and in the first one we have $\alpha^{n-1} = 0$, therefore

$$c^{\mu^{n-2}}_{\alpha^{n-2}\beta^{n-2}} c^{\lambda^{n-2}}_{\beta^{n-2}(\alpha^{n-1})'} = c^{\mu^{n-2}}_{\alpha^{n-2}\lambda^{n-2}} \neq 0.$$

It follows that either $\lambda^{n-2} = \mu^{n-2}$ or $|\lambda^{n-2}| < |\mu^{n-2}|$. In the second case, we obtain that $\underline{\lambda} \prec \underline{\mu}$, and in the first one we have $\alpha^{n-2} = 0$. Thus, by induction we see that $\langle s_{\underline{\lambda}}\,,\, \eta_{\underline{\mu}}(v)\rangle \neq 0$. implies that $\underline{\lambda} \preceq \underline{\mu}$. Moreover, clearly $\langle s_{\underline{\mu}}\,,\, \eta_{\underline{\mu}}(v)\rangle = 1$. Next, by (ii) we have

$$s_{\underline{\lambda}} = \sum_{\underline{\mu}} \langle s_{\underline{\lambda}}\,,\, \eta_{\underline{\mu}}(v)\rangle\, \varphi_{\underline{\mu}}(v).$$

Hence the expansion of $s_{\underline{\lambda}}$ on $\{\varphi_{\underline{\mu}}(v)\}$ involves only multi-partitions $\underline{\mu} \preceq \underline{\lambda}$, and by solving this unitriangular system we obtain that $\langle s_{\underline{\lambda}}\,,\, \varphi_{\underline{\mu}}(v)\rangle \neq 0$ implies that $\underline{\lambda} \succeq \underline{\mu}$ and $\langle \varphi_{\underline{\mu}}(v)\,,\, s_{\underline{\mu}}\rangle = 1$. $\square$

5.8. Define

$$S = \Big[\langle s_{\underline{\lambda}}\,,\, \eta_{\underline{\mu}}(v)\rangle\Big]_{\underline{\lambda},\underline{\mu}\in\mathcal{P}^n}\,, \qquad A = S\,\overline{S^{-1}} = \Big[a_{\underline{\lambda}\,\underline{\mu}}(v)\Big]_{\underline{\lambda},\underline{\mu}\in\mathcal{P}^n}\,,$$

where $\overline{S^{-1}}$ denotes the matrix obtained from S^{-1} by changing v into v^{-1}. We introduce a semi-linear involution $f \mapsto \overline{f}$ on $\mathcal{S}$ by requiring that

$$\overline{s_{\underline{\mu}}} = \sum_{\underline{\lambda}} a_{\underline{\lambda}\,\underline{\mu}}(v)\, s_{\underline{\lambda}}\,, \qquad \overline{q(v)\,x} = q(v^{-1})\,\overline{x} \qquad (\underline{\mu}\in\mathcal{P}^n,\ x\in\mathcal{S},\ q(v)\in K)\,.$$

The identity $A\overline{S} = S$ shows that

$$\overline{\eta_{\underline{\mu}}(v)} = \eta_{\underline{\mu}}(v). \tag{7}$$

On the other hand, let $\mathcal{L}$ be the $\mathbb{Z}[v]$-lattice in $\mathcal{S}$ spanned by the vectors $s_{\underline{\lambda}}$. It follows easily from Lemma 4 that

$$\eta_{\underline{\lambda}}(v) \equiv s_{\underline{\lambda}} \mod v\mathcal{L}\,. \tag{8}$$

By Proposition 5 (iii) S is unitriangular, hence A is also unitriangular. So by a classical argument (see [Lu], 7.10), the basis $\{\eta_{\underline{\lambda}}(v)\}$ is uniquely determined by (7) and (8). This is the canonical basis of $\mathcal{S}$ associated with the involution $f \mapsto \overline{f}$ and the lattice $\mathcal{L}$.

5.9. Define

$$\psi_{\underline{\lambda}}(v^{-1}) = s_{\lambda^0}(A_0)s_{\lambda^1}(v^{-1}A_0 + A_1)\cdots s_{\lambda^{n-1}}(v^{-n+1}A_0 + \cdots + v^{-1}A_{n-2} + A_{n-1})\,.$$

Obviously, $\{\psi_{\underline{\lambda}}(v^{-1})\}$ is another basis of $\mathcal{S}$. Let $\mathcal{L}^-$ be the $\mathbb{Z}[v^{-1}]$-lattice in $\mathcal{S}$ spanned by the vectors $s_{\underline{\lambda}}$. One can check easily that

$$\psi_{\underline{\lambda}}(v^{-1}) \equiv s_{\underline{\lambda}} \mod v^{-1}\mathcal{L}^-\,. \tag{9}$$

On the other hand, for $k = 1, \ldots, n$ we have the formal identities

$$v^{-k+1}A_0 + \cdots + v^{-1}A_{k-2} + A_{k-1} = [k]A_0 + \sum_{1\leqslant j\leqslant k-1} [k-j](A_j - vA_{j-1})\,,$$

where the coefficients $[j] := (v^j - v^{-j})/(v - v^{-1})$ are bar-invariant. Therefore the expansion of $\psi_{\underline{\lambda}}(v^{-1})$ on the basis $\{\eta_{\underline{\mu}}\}$ will have only bar-invariant coefficients. Hence

$$\overline{\psi_{\underline{\lambda}}(v^{-1})} = \psi_{\underline{\lambda}}(v^{-1}), \tag{10}$$

and $\{\psi_{\underline{\lambda}}(v^{-1})\}$ is the canonical basis of $\mathcal{S}$ associated with $f \mapsto \overline{f}$ and the lattice $\mathcal{L}^-$.

5.10. The coefficients of the expansion of $\psi_{\underline{\lambda}}(v^{-1})$ on $\{s_{\underline{\lambda}}\}$ are given by the next lemma, which is proved in the same way as Lemma 4.

Lemma 6. *For $\underline{\lambda}, \underline{\mu} \in \mathcal{P}^n$ we have*

$$\langle \psi_{\underline{\lambda}}(v^{-1})\,,\, s_{\underline{\mu}} \rangle = v^{-\delta(\underline{\mu},\underline{\lambda})} \sum_{\alpha} \prod_{0 \leqslant j \leqslant n-1} c^{\lambda^j}_{\alpha^j_0,\alpha^j_1,\ldots,\alpha^j_j}\,, c^{\mu^j}_{\alpha^j_j,\alpha^{j+1}_j,\ldots,\alpha^{n-1}_j}$$

where the sum is over all families of partitions $\alpha = (\alpha^j_i \in \mathcal{P},\ 0 \leqslant i \leqslant j \leqslant n-1)$ subject to the conditions

$$\sum_j |\alpha^j_i| = |\mu^i|\,, \qquad \sum_i |\alpha^j_i| = |\lambda^j|.$$

Let $x \mapsto x'$ denote the semi-linear involution of $\mathcal{S}$ defined by $(s_{\underline{\lambda}})' = s_{\underline{\lambda}'}$. The next proposition is easily checked by expansion on $\{s_{\underline{\lambda}}\}$.

Proposition 7. *The adjoint basis $\{\varphi_{\underline{\lambda}}(v)\}$ of the canonical basis $\{\eta_{\underline{\lambda}}(v)\}$ is related to the basis $\{\psi_{\underline{\lambda}}(v^{-1})\}$ by*

$$(\varphi_{\underline{\lambda}}(v))' = \psi_{\underline{\lambda}'}(v^{-1})\,.$$

6. Comparison of the canonical bases of $\mathcal{F}_v$ and $\mathcal{S}$

6.1. We fix $w \geqslant 1$. Let $\rho = (\rho_1, \ldots, \rho_l)$ be the large n-core associated with w (see Definition 1). Let $\mathcal{P}(\rho)$ be the set of partitions with n-core ρ and let $\mathcal{P}(\rho, w)$ be the subset of partitions with n-weight $\leqslant w$. Note that ρ is equal to its conjugate partition ρ'. It follows that the map $\lambda \mapsto \lambda'$ induces a bijection of $\mathcal{P}(\rho, w)$.

To $\lambda \in \mathcal{P}(\rho)$ we associate its n-quotient $\underline{\lambda} = (\lambda^0, \ldots, \lambda^{n-1}) \in \mathcal{P}^n$ (see [JK]). The n-quotient of a partition is well-defined only up to circular permutations. We remove this ambiguity by requiring that the partition $\rho + (n) = (\rho_1 + n, \rho_2, \ldots, \rho_l)$ have n-quotient $(\emptyset, \ldots, \emptyset, (1))$. It is well-known that $\lambda \mapsto \underline{\lambda}$ is a bijection from $\mathcal{P}(\rho)$ onto $\mathcal{P}^n$. We denote by $|\lambda|_n$ the n-weight of λ, that is, $|\lambda|_n = \sum_i |\lambda^i|$.

We shall also use the n-sign of λ defined as follows (see, e.g., [LLT2, LT2]). Write $k = |\lambda|_n$. One can obtain the n-core ρ of λ by removing successively k n-ribbons $R_1, \ldots, R_k$ from the diagram of λ. Let $h_1, \ldots, h_k$ denote the respective heights of these ribbons. Then $(-1)^{h_1 + \cdots + h_k - k}$ does not depend on the choice of the sequence $(R_1, \ldots, R_k)$ and is denoted by $\varepsilon_n(\lambda)$.

Comparing Lemma 4 with [Mi] we can see that for $\lambda, \mu \in \mathcal{P}(\rho, w)$

$$\mathrm{rad}_{\lambda,\mu}(v) = \varepsilon_n(\lambda)\varepsilon_n(\mu)\, \langle \eta_{\underline{\mu}}(v)\,, s_{\underline{\lambda}} \rangle \tag{11}$$

where $\mathrm{rad}_{\lambda,\mu}(v)$ has been defined in Section 1. This motivates the next definition.

6.2. Let $\mathcal{F}_v(\rho)$ denote the subspace of $\mathcal{F}_v$ spanned by all $s(\lambda)$ with $\lambda \in \mathcal{P}(\rho)$. We define a linear map Φ from $\mathcal{F}_v(\rho)$ to $\mathcal{S}$ by setting

$$\Phi(s(\lambda)) = \varepsilon_n(\lambda)\, s_{\underline{\lambda}}\,.$$

This is an isomorphism of vector spaces, which has already been considered in [LL] for $n = 2$ and in [Lei] in general. Our main result is

Theorem 8. *For $\lambda \in \mathcal{P}(\rho, w)$ one has*

$$G(\lambda) = \varepsilon_n(\lambda)\, \Phi^{-1}(\eta_{\underline{\lambda}}(v))\,, \tag{12}$$

$$G^-(\lambda) = \varepsilon_n(\lambda)\, \Phi^{-1}(\psi_{\underline{\lambda}}(v^{-1}))\,. \tag{13}$$

Theorem 8 will be proved in Section 7.

Example 9. Let $n=3$, $w=3$, $\rho=(6,4,2,2,1,1)$ and $\lambda=(12,4,4,3,1,1)$. Then

$$\begin{aligned} G(\lambda) &= s_{(12,4^2,3,1^2)} + v\, s_{(12,4,2^2,1^5)} + v\, s_{(9,6,5,3,1^2)} + v\, s_{(9,4^2,3^2,2)} \\ &\quad + v^2\, s_{(9,4^2,3,1^5)} + v^2\, s_{(6^2,5,3^2,2)} + v^2\, s_{(6,4^2,3^2,2^2,1)} + v^3\, s_{(6,4^2,3^2,2,1^3)}. \end{aligned}$$

Therefore,

$$\begin{aligned} \Phi(G(\lambda)) &= -s_{(1)}(A_1)s_{(2)}(A_2) + v\, s_{(1)}(A_0)s_{(2)}(A_2) + v\, s_{(2)}(A_1)s_{(1)}(A_2) \\ &\quad + v\, s_{(1^2)}(A_1)s_{(1)}(A_2) - v^2\, s_{(1)}(A_0)s_{(1)}(A_1)s_{(1)}(A_2) \\ &\quad - v^2\, s_{(2,1)}(A_1) - v^2\, s_{(1^3)}(A_1) + v^3\, s_{(1)}(A_0)s_{(1^2)}(A_1). \end{aligned}$$

On the other hand, $(\lambda^0,\lambda^1,\lambda^2)=(\emptyset,(1),(2))$, $\varepsilon_3(\lambda)=-1$ and

$$\begin{aligned} \eta_{\underline{\lambda}}(v) &= s_{(1)}(A_1 - vA_0)\, s_{(2)}(A_2 - vA_1) \\ &= \big(s_{(1)}(A_1) - v\, s_{(1)}(A_0)\big)\big(s_{(2)}(A_2) - v\, s_{(1)}(A_1)s_{(1)}(A_2) + v^2\, s_{(1^2)}(A_1)\big) \\ &= -\Phi(G(\lambda))\,. \end{aligned}$$

Theorem 8, Lemma 4 and Lemma 6 readily imply

Corollary 10. *Conjecture 5.5 of* [Mi] *is true and we have for* $\lambda,\mu\in\mathcal{P}(\rho,w)$,

$$d_{\lambda,\mu}(v) = v^{\delta(\underline{\lambda},\underline{\mu})} \sum_{\substack{\alpha^0,\dots,\alpha^n \\ \beta^0,\dots,\beta^{n-1}}} \prod_{0\leqslant j\leqslant n-1} c^{\mu^j}_{\alpha^j\beta^j} c^{\lambda^j}_{\beta^j(\alpha^{j+1})'},$$

where $\alpha^0,\dots,\alpha^n,\beta^0,\dots,\beta^{n-1}$ *run through* $\mathcal{P}$ *subject to the conditions*

$$|\alpha^i| = \sum_{0\leqslant j\leqslant i-1} |\lambda^j| - |\mu^j|\,, \qquad |\beta^i| = |\mu^i| + \sum_{0\leqslant j\leqslant i-1} |\mu^j| - |\lambda^j|\,,$$

and

$$\delta(\underline{\lambda},\underline{\mu}) := \sum_{1\leqslant j\leqslant n-1} j(|\mu^j| - |\lambda^j|)\,.$$

Moroever, we also have

$$e_{\lambda,\mu}(v) = v^{\delta(\underline{\mu},\underline{\lambda})} \sum_{\alpha} \prod_{0\leqslant k\leqslant n-1} c^{\mu^k}_{\alpha^k_k,\alpha^{k+1}_k,\dots,\alpha^{n-1}_k} c^{\lambda^k}_{\alpha^k_0,\alpha^k_1,\dots,\alpha^k_k}\,,$$

where the sum is over all families of partitions $\alpha=(\alpha^j_i\in\mathcal{P},\ 0\leqslant i\leqslant j\leqslant n-1)$ *subject to the conditions*

$$\sum_j |\alpha^j_i| = |\mu^i|\,, \qquad \sum_i |\alpha^j_i| = |\lambda^j|.$$

This can be regarded as a combinatorial description of the parabolic Kazhdan-Lusztig polynomials $d_{\lambda,\mu}(v)$ and $e_{\lambda,\mu}(v)$ in this case. In particular, we note that these polynomials are just monomials when λ and μ belong to $\mathcal{P}(\rho,w)$.

When $n=2$ and μ is a strict partition, $d_{\lambda,\mu}(1)$ is a decomposition number for the Hecke algebra over a field of characteristic 0 at $q=-1$ [LLT1, Ar]. In this case, the decomposition numbers for partitions with a large core have been first calculated by James and Mathas [JM].

Finally, we deduce from Proposition 5 (i) and Proposition 7 the following interesting symmetries

Corollary 11. *For* $\lambda,\mu\in\mathcal{P}(\rho,w)$ *there holds*

$$d_{\lambda,\mu}(v) = d_{\mu',\lambda'}(v)\,, \qquad e_{\lambda,\mu}(v) = e_{\mu',\lambda'}(v)\,.$$

7. Proof of Theorem 8

7.1. We want to construct certain elements of $U_v(\widehat{\mathfrak{sl}}_n)$ whose action on $s(\rho) \in \mathcal{F}_v(\rho)$ corresponds via Φ to the multiplication by $e_k(A_j - vA_{j-1})$ in $\mathcal{S}$ for $k \leqslant w$. Let $r \in \{0, \dots, n-1\}$ be the residue of ρ_1 modulo n. Thus $F_i s(\rho) = 0$ if $i \neq r$. For convenience we allow the indices i of the Chevalley generators F_i to belong to $\mathbb{Z}$ by setting $F_i = F_{i \bmod n}$.

Definition 12. For $j \in \{0, \dots, n-2\}$ and $k \geqslant 1$ we set

$$H_{j,k} = F^{(k)}_{j+1+r} \cdots F^{(k)}_{n-2+r} F^{(k)}_{n-1+r} F^{(k)}_{j+r} \cdots F^{(k)}_{r+1} F^{(k)}_{r} \,.$$

Example 13. Let $n = 4$, $w = 3$, and $\rho = (12, 9, 6^2, 4^2, 2^3, 1^3)$. Then $r = 0$ and

$$\begin{aligned} H_{0,k} &= F^{(k)}_1 F^{(k)}_2 F^{(k)}_3 F^{(k)}_0 \,, \\ H_{1,k} &= F^{(k)}_2 F^{(k)}_3 F^{(k)}_1 F^{(k)}_0 \,, \\ H_{2,k} &= F^{(k)}_3 F^{(k)}_2 F^{(k)}_1 F^{(k)}_0 \,. \end{aligned}$$

For $w = 2$ we have $\rho = (6, 3^2, 1^3)$, $r = 2$ and

$$\begin{aligned} H_{0,k} &= F^{(k)}_3 F^{(k)}_0 F^{(k)}_1 F^{(k)}_2 \,, \\ H_{1,k} &= F^{(k)}_0 F^{(k)}_1 F^{(k)}_3 F^{(k)}_2 \,, \\ H_{2,k} &= F^{(k)}_1 F^{(k)}_0 F^{(k)}_3 F^{(k)}_2 \,. \end{aligned}$$

Let $x \mapsto \overline{x}$ denote the bar involution of $U_v(\widehat{\mathfrak{sl}}_n)$. By definition $\overline{F^{(k)}_i} = F^{(k)}_i$ for all i, k, hence

$$\overline{H_{j,k}} = H_{j,k} \,, \qquad (0 \leqslant j \leqslant n-2,\ k \geqslant 1). \tag{14}$$

Proposition 14. *Let $\lambda \in \mathcal{P}(\rho)$ with n-weight $u < w$. Then, for $k \leqslant w - u$,*

$$\Phi(H_{j,k} s(\lambda)) = (-1)^{k(n-j-2)} \varepsilon_n(\lambda)\, e_k(A_{j+1} - vA_j) s_{\underline{\lambda}} \,.$$

The proof of Proposition 14 relies on the following combinatorial properties of the n-core partition ρ established in [CK].

Lemma 15. *Let $\lambda, \mu \in \mathcal{P}(\rho, w)$ with $|\mu|_n = |\lambda|_n - 1$. If $\mu \subset \lambda$ (i.e., the Young diagram of μ is contained in that of λ), then there exists $i \in \{0, \dots, n-1\}$ such that $\mu^j = \lambda^j$ for $j \neq i$ and $\mu^i \subset \lambda^i$. Moreover, λ/μ is the Young diagram of the hook partition $(i+1, 1^{n-i-1})$.*

Thus, if $\lambda \in \mathcal{P}(\rho, w)$ and if we consider an n-ribbon tiling of λ/ρ, we see that all ribbons contributing to λ^i have the same shape $(i+1, 1^{n-i-1})$ and the same spin $n - i - 1$. Let us say that such ribbons have color i. In fact, for $i < j$ any ribbon of colour i is situated to the left of any ribbon of colour j, as shown in Figure 3 with $n = 4$. Moreover, there is a unique n-ribbon tiling of λ/μ, and since the ribbons with different colors do not mix, for any ribbon tableau T of shape λ/μ the corresponding n-tuple of tableaux $(t_0, \dots, t_{n-1})$ under the Stanton-White correspondence [SW, LLT2] is obtained in a trivial way, as illustrated in Figure 4.

Proof of Proposition 14. Since $\mathrm{wt}(H_{j,k}) = -k\delta$, all the $s(\mu)$ occurring in the expansion of $H_{j,k}(s(\lambda))$ have weight $\mathrm{wt}(s(\lambda)) - k\delta$, hence the corresponding partitions μ are obtained from λ by adding k n-ribbons. Since by definition $H_{j,k} =$

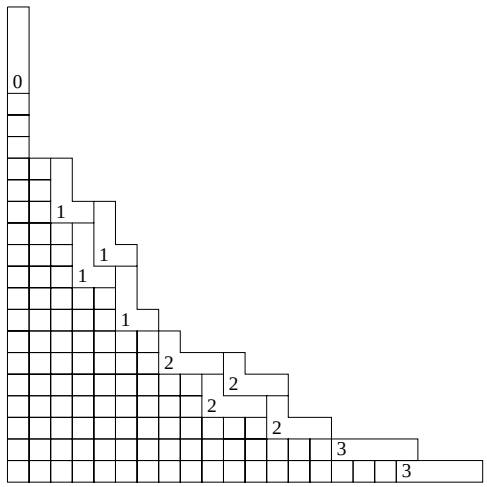

FIGURE 3. The ribbon tiling of λ/ρ for some $\lambda \in \mathcal{P}(\rho)$. Each ribbon is labelled with its color.

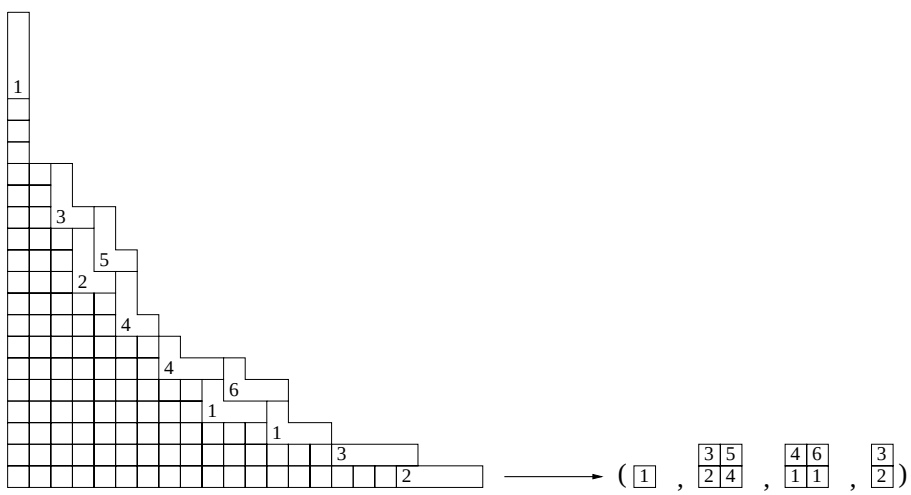

FIGURE 4. The Stanton-White bijection for a ribbon tableau of shape $\lambda \in \mathcal{P}(\rho)$.

$F^{(k)}_{j+1+r} \cdots F^{(k)}_{r}$, all these ribbons must be such that one of their ends is a removable cell of μ with content equal to $j+1+r \bmod n$. With our convention for the definition of the n-quotient, this means that all these ribbons have either colour j or $j+1$. Moreover, the ribbons of colour $j+1$ must form a vertical ribbon strip and those of colour j must form a horizontal ribbon strip. (For the definition of a horizontal ribbon strip see [LLT2, LT2]. A vertical ribbon strip is the transpose of a horizontal ribbon strip.)

Conversely, let μ be a partition obtained from λ by adding a horizontal ribbon strip of s ribbons of colour j and a vertical ribbon strip of $k-s$ ribbons of colour $j+1$. Then there is a unique way of obtaining μ from λ by adding in that order :

k cells with content r mod n,
k cells with content $r+1$ mod n,
...
k cells with content $r+j$ mod n,
k cells with content $n-1+r$ mod n,
...
k cells with content $r+j+1$ mod n.

Hence $s(\mu)$ occurs in $H_{j,k}(s(\lambda))$ and its coefficient is a single power of v. Then an elementary but tedious calculation based on (2) shows that this power is precisely v^s. It follows that

$$\begin{aligned}\Phi(H_{j,k}s(\lambda)) &= \varepsilon_n(\lambda)\sum_{0\leqslant s\leqslant k}(-1)^{s(n-1-j)+(k-s)(n-2-j)}v^s\,h_s(A_j)e_{k-s}(A_{j+1})\,s_{\underline{\lambda}}\\ &= (-1)^{k(n-2-j)}\varepsilon_n(\lambda)\,e_k(A_{j+1}-vA_j)\,s_{\underline{\lambda}}\,,\end{aligned}$$

by [Mcd], (5.16), (5.17), and equation (6). □

7.2. Recall the operators D_k ($k\geqslant 1$) of 4.2.

Proposition 16. *Let $\lambda\in\mathcal{P}(\rho)$ with n-weight $u<w$. Then, for $k\leqslant w-u$,*

$$\Phi(D_k s(\lambda)) = \varepsilon_n(\lambda)\,h_k(v^{-n+1}A_0+\cdots+v^{-1}A_{n-2}+A_{n-1})\,s_{\underline{\lambda}}\,.$$

Proof. Let μ be such that μ/λ is a horizontal ribbon strip containing k ribbons. Let k_i be the number of these ribbons which have colour i. Then, by Lemma 15, the spin of μ/λ is equal to

$$\mathrm{spin}(\mu/\lambda) = \sum_{0\leqslant i\leqslant n-1} k_i(n-1-i)\,,$$

and the n-quotient $\underline{\mu}=(\mu^0,\dots,\mu^{n-1})$ of μ is obtained from $\underline{\lambda}=(\lambda^0,\dots,\lambda^{n-1})$ by adding to each λ^i a horizontal strip of size k_i. Using [Mcd], (5.16), it follows that

$$\begin{aligned}\Phi(D_k s(\lambda)) &= \left(\varepsilon_n(\lambda)\sum_{k_0+\cdots+k_{n-1}=k}(-v)^{\sum_i -k_i(n-1-i)}\,h_{k_0}(A_0)\cdots h_{k_{n-1}}(A_{n-1})\right)s_{\underline{\lambda}}\\ &= \varepsilon_n(\lambda)\;h_k(v^{-n+1}A_0+\cdots+v^{-1}A_{n-2}+A_{n-1})\,s_{\underline{\lambda}}\,.\end{aligned}$$

□

7.3. Let $\lambda\in\mathcal{P}(\rho,w)$ and recall that

$$\eta_{\underline{\lambda}}(v) = s_{\lambda^0}(A_0)s_{\lambda^1}(A_1-vA_0)\cdots s_{\lambda^{n-1}}(A_{n-1}-vA_{n-2})\,.$$

Each Schur function $s_{\lambda^i}(A_i-vA_{i-1})$ is a polynomial in the elementary symmetric functions $e_k(A_i-vA_{i-1})$ with coefficients in $\mathbb{Z}$. On the other hand, using the formal identity

$$[n]A_0 = (v^{-n+1}A_0+\cdots+v^{-1}A_{n-2}+A_{n-1}) - \sum_{1\leqslant j\leqslant n-1}[n-j](A_j-vA_{j-1})$$

where $[j]:=(v^j-v^{-j})/(v-v^{-1})$, we see that $s_{\lambda^0}(A_0)$ can be expressed as a polynomial in the variables $e_k(A_j-vA_{j-1})$ and $h_k(v^{-n+1}A_0+\cdots+v^{-1}A_{n-2}+A_{n-1})$ with coefficients in $\mathbb{C}(v)$ invariant under $v\mapsto v^{-1}$. Therefore by Proposition 14 and Proposition 16, since $|\underline{\lambda}|\leqslant w$ the vector $\Phi^{-1}(\eta_{\underline{\lambda}}(v))$ can be obtained by applying to $s(\rho)$ a polynomial in the operators $H_{j,k}$ and D_k with bar-invariant coefficients.

Since by (14) and Section 4.3 the operators $H_{j,k}$ and D_k are bar-invariant, as well as $s(\rho)$, it follows that $\Phi^{-1}(\eta_{\underline{\lambda}}(v))$ is bar-invariant. Moreover, we obviously have

$$\varepsilon_n(\lambda)\,\Phi^{-1}(\eta_{\underline{\lambda}}(v)) \equiv s(\lambda) \bmod vL\,,$$

hence (12) is proved.

Finally, (13) follows easily from (12). Indeed, (12) implies that if $y \in \mathcal{S}$ is bar-invariant of degree $\leqslant w$, then $\Phi^{-1}(y) \in \mathcal{F}(\rho)$ is bar-invariant, because y can be expressed as a linear combination of the $\eta_{\underline{\lambda}}(v)$ with bar-invariant coefficients. Therefore for $\lambda \in \mathcal{P}(\rho, w)$, the vector $\varepsilon_n(\lambda)\,\Phi^{-1}(\psi_{\underline{\lambda}}(v^{-1}))$ is bar-invariant, and since it obviously coincides modulo $v^{-1}L^-$ with $s(\lambda)$, it has to be equal to $G^-(\lambda)$.

Thus, Theorem 8 is proved.

8. The Scopes isometries

8.1. In [Sc], Scopes has introduced certain bijections between sets of partitions with given n-cores. More precisely, let τ be an n-core partition, and write $\alpha = \sum_{i=1}^n a_i\varepsilon_i$ for the corresponding element of Q_0, as in 2.4. Fix $i \in \{0, \dots, n-1\}$ and let $\sigma_i(\tau)$ denote the element of $\mathcal{C}_n$ obtained from τ via the action of W described in 2.4. Let $k_i = a_{i+1} - a_i$ if $i \neq 0$ and $k_i = a_1 - a_n - 1$ if $i = 0$. Thus, k_i is equal to the number of beads transfered from the $(i+1)$th runner to the ith runner when one computes $\sigma_i(\tau)$ by means of the abacus representation when $i \neq 0$, and from the first to the last runner when $i = 0$. (If $k_i < 0$, we understand that the beads are transfered from the ith runner to the $(i+1)$th runner.)

Now let $\mathcal{P}_0(\tau, w)$ and $\mathcal{P}_0(\sigma_i(\tau), w)$ be the sets of partitions with n-weight w, and n-core τ and $\sigma_i(\tau)$ respectively. For $\lambda \in \mathcal{P}_0(\tau, w)$ we deduce from 3.2 and 3.3 that, r_i being an orthogonal transformation of $\mathcal{F}_{\mathbb{Z}}$,

$$r_i(s(\lambda)) = \pm s(\mu)$$

for some $\mu \in \mathcal{P}_0(\sigma_i(\tau), w)$. Hence r_i induces a signed bijection $\lambda \mapsto \pm\mu$ from $\mathcal{P}_0(\tau, w)$ to $\mathcal{P}_0(\sigma_i(\tau), w)$.

8.2. In the case when $w \leqslant k_i$, it is easy to show that no negative sign occurs in the previous bijection, as we shall now see. Let $\sigma(\Lambda_0) \in W\Lambda_0$ be the extremal weight corresponding to the n-core partition τ, and let $\Lambda = \sigma(\Lambda_0) - w\delta$.

Lemma 17. *Scopes' condition that $w \leqslant k_i$ is equivalent to the fact that $\Lambda - \alpha_i$ does not belong to $P(\mathcal{F})$.*

Proof. Let us write $\sigma(\Lambda_0) = \Lambda_0 + d\delta + \sum_{i=1}^n a_i\varepsilon_i$. Then $(\sigma(\Lambda_0)\,,\,\alpha_i) = a_i - a_{i+1}$ if $i \neq 0$, and $(\sigma(\Lambda_0)\,,\,\alpha_0) = 1 + a_n - a_1$. Hence, in all cases we have $k_i = -(\sigma(\Lambda_0)\,,\,\alpha_i) = -(\Lambda\,,\,\alpha_i)$. Now

$$\begin{aligned}
(\Lambda\,,\,\Lambda) &= (\sigma(\Lambda_0) - w\delta\,,\,\sigma(\Lambda_0) - w\delta) \\
&= (\sigma(\Lambda_0)\,,\,\sigma(\Lambda_0)) + w^2(\delta\,,\,\delta) - 2w(\sigma(\Lambda_0)\,,\,\delta) \\
&= (\Lambda_0\,,\,\Lambda_0) - 2w(\Lambda_0\,,\,\delta) \\
&= -2w,
\end{aligned}$$

so that $w = -(\Lambda\,,\,\Lambda)/2$. Therefore,

$$w \leqslant k_i \quad\Longleftrightarrow\quad (\Lambda\,,\,\Lambda) - 2(\Lambda\,,\,\alpha_i) \geqslant 0 \quad\Longleftrightarrow\quad (\Lambda - \alpha_i\,,\,\Lambda - \alpha_i) \geqslant 1\,.$$

It is known that a weight $\gamma \in P$ occurs in the weight system $P(\mathcal{F})$ of the Fock space if and only if $\gamma \in \Lambda_0 + Q$ and $(\gamma\,,\,\gamma) \leqslant 0$. Hence $w \leqslant k_i$ is indeed equivalent to $\Lambda - \alpha_i \notin P(\mathcal{F})$. □

Lemma 18. *If $w \leqslant k_i$, we have*

$$r_i\, x = (e_i^{k_i}/k_i!)\, x\,, \qquad (x \in \mathcal{F}(\Lambda)).$$

Proof. If $w \leqslant k_i$ by Lemma 17 we have $f_i\mathcal{F}(\Lambda) = \{0\}$. Hence, if $x \in \mathcal{F}(\Lambda)$, then

$$r_i\, x = \exp(-f_i)\, \exp(e_i)\, \exp(-f_i)\, x = \exp(-f_i)\, \exp(e_i)\, x\,.$$

Since $P(\mathcal{F})$ is W-invariant, Scopes' condition also implies that $\sigma_i(\Lambda) + \alpha_i$ does not belong to $P(\mathcal{F})$. Therefore we have $e_i\mathcal{F}(\sigma_i(\Lambda)) = \{0\}$, hence $e_i^m\, x = 0$ if $m > k_i$. Since $r_i\, x \in \mathcal{F}(\sigma_i(\Lambda))$, we see that the contributions of the monomials e_i^m with $m < k_i$ must also cancel, and we simply obtain $r_i\, x = (e_i^{k_i}/k_i!)\, x$. □

It follows that if $w \leqslant k_i$, the bijection from $\mathcal{P}_0(\tau, w)$ to $\mathcal{P}_0(\sigma_i(\tau), w)$ induced by r_i consists in associating to λ the partition μ obtained by removing from λ the k_i removable nodes with n-residue equal to i. (The previous Lemmas show that this is always possible and in a unique way.) So we recover Scopes' description [Sc] of the bijection. We shall denote this bijection by π_i.

8.3. Let A be the subring of K consisting of the functions without pole at $v = 0$, and let

$$L_A = A \otimes_{\mathbb{Z}[v]} L\,.$$

By [MM] the A-module L_A is a lower crystal lattice at $v = 0$ in the sense of Kashiwara [K1, K2]. Let $\overline{A}$ be the subring of K consisting of the functions without pole at $v = \infty$. It follows that the $\overline{A}$-module $\overline{L}_A$ is a lower crystal lattice at $v = \infty$. Finally, let $\mathcal{F}_v^{\mathrm{int}}$ be the $\mathbb{C}[v, v^{-1}]$-module with basis $\{G(\lambda)\}$. Then we have

$$\mathcal{F}_v \simeq K \otimes_A L_A \simeq K \otimes_{\overline{A}} \overline{L}_A \simeq K \otimes_{\mathbb{C}[v,v^{-1}]} \mathcal{F}_v^{\mathrm{int}}\,.$$

Moreover, if we set $E = \mathcal{F}_v^{\mathrm{int}} \cap L_A \cap \overline{L}_A$, then clearly E is a $\mathbb{C}$-vector space with basis $\{G(\lambda)\}$, and the map $G(\lambda) \mapsto s(\lambda) \mathrm{mod} v L_A$ is an isomorphism of $\mathbb{C}$-vector spaces from E to L_A/vL_A. Hence, $(\mathcal{F}_v^{\mathrm{int}}, L_A, \overline{L}_A)$ is a balanced triple in the sense of [K3], and $\{G(\lambda)\}$ is a lower global base of $\mathcal{F}_v$.

In the sequel, in order to simplify notation, we write λ instead of $s(\lambda)$ mod vL_A. The Kashiwara operators $\tilde{E}_i$ and $\tilde{F}_i$ act on L_A/vL_A. Since $\{\lambda\}$ is a crystal basis of L_A/vL_A [MM], for each partition λ, $\tilde{E}_i\lambda$ (resp. $\tilde{F}_i\lambda$) is a single partition μ or 0. The combinatorial description of $\tilde{E}_i\lambda$ (resp. $\tilde{F}_i\lambda$) was given in [MM] (see also [LLT1]).

8.4. Let $\langle\cdot\,,\,\cdot\rangle_v$ be the scalar product on $\mathcal{F}_v$ defined by

$$\langle s(\lambda)\,,\, s(\mu)\rangle_v = v^{-|\lambda|_n}\, \delta_{\lambda,\mu} \qquad (\lambda, \mu \in \mathcal{P}),$$

where $|\lambda|_n$ denotes the n-weight of λ. Let $B^* = \{G^*(\lambda)\}$ be the adjoint basis of B with respect to $\langle\cdot\,,\,\cdot\rangle_v$. The basis B^* is a renormalization of the basis $\{G^\dagger(\lambda)\}$ of [LT1], namely

$$G^*(\lambda) = v^{|\lambda|_n}\, G^\dagger(\lambda).$$

Then we have

$$v^{|\lambda|_n} s(\lambda) = \sum_\mu d_{\lambda,\mu}(v) G^*(\mu)\,, \qquad G^*(\lambda) = v^{|\lambda|_n} \sum_\mu e_{\lambda',\mu'}(-v) s(\mu). \tag{15}$$

8.5. The scalar product $\langle \cdot\,,\,\cdot \rangle_v$ satisfies

$$\langle E_i x\,,\,y\rangle_v = \langle x\,,\,F_i y\rangle_v, \qquad (x, y \in \mathcal{F}_v,\ 0 \leqslant i \leqslant n-1),$$

(see [LLT1], 8.1). Therefore B^* is an upper global base. Hence, using Lemma 5.1.1 of [K3], we get

Lemma 19. *Let $\lambda \in \mathcal{P}$ and $i \in \{0, \dots, n-1\}$. Let k be the maximal integer such that $\tilde{E}_i^k \lambda \neq 0$. Then $E_i^{(k)} G^*(\lambda) = G^*(\tilde{E}_i^k \lambda)$.*

8.6. Let us return to the setting of 8.1 and 8.2. We fix an n-core τ, an integer $i \in \{0, \dots, n-1\}$, and we take $w \leqslant k_i$. Then, for all $\lambda \in \mathcal{P}_0(\tau, w)$, the maximal integer k such that $\tilde{E}_i^k \lambda \neq 0$ is $k = k_i$, and it follows easily from the combinatorial descriptions of $\tilde{E}_i \lambda$ and $E_i s(\lambda)$ that

$$\tilde{E}_i^{k_i} \lambda = \pi_i(\lambda), \quad E_i^{(k_i)} s(\lambda) = s(\pi_i(\lambda)), \qquad \lambda \in \mathcal{P}_0(\tau, w), \tag{16}$$

where $\pi_i : \mathcal{P}_0(\tau, w) \longrightarrow \mathcal{P}_0(\sigma_i(\tau), w)$ is Scopes' bijection. Using Lemma 19 we obtain

$$E_i^{(k_i)} G^*(\lambda) = G^*(\pi_i(\lambda)), \qquad \lambda \in \mathcal{P}_0(\tau, w). \tag{17}$$

Then, combining equations (16), (17), (15), we obtain

Theorem 20. *For all $\lambda, \mu \in \mathcal{P}_0(\tau, w)$, there holds*

$$d_{\lambda,\mu}(v) = d_{\pi_i(\lambda), \pi_i(\mu)}(v)\,, \qquad e_{\lambda,\mu}(v) = e_{\pi_i(\lambda), \pi_i(\mu)}(v)\,,$$

where π_i denotes Scopes' bijection.

8.7. Let $\rho = \rho(w)$ and $\sigma_\rho(\Lambda_0)$ be the large n-core associated with w and the corresponding extremal weight, respectively. Assume that $\Lambda \in \mathcal{O}_w$ can be reached from $\Lambda_{w,\rho} = \sigma_\rho(\Lambda_0) - w\delta$ by a sequence of reflections

$$\Lambda_{w,\rho} \xrightarrow{\sigma_{i_1}} \Lambda^1 \xrightarrow{\sigma_{i_2}} \Lambda^2 \xrightarrow{\sigma_{i_3}} \cdots \xrightarrow{\sigma_{i_s}} \Lambda^s = \Lambda$$

such that $\Lambda^j - \alpha_{i_j} \notin P(\mathcal{F})$ for all j. It follows immediately from Theorem 20 that the transition matrices $T(\Lambda)$ and $T^-(\Lambda)$ are equal to $T(\Lambda_{w,\rho})$ and $T^-(\Lambda_{w,\rho})$ respectively and are given by Corollary 10.

In the case of $\widehat{\mathfrak{sl}}_2$, the orbit $\mathcal{O}_w$ consists of the weights

$$\Lambda^{2k} = (\sigma_1\sigma_0)^k(\Lambda_0 - w\delta), \qquad \Lambda^{2k+1} = \sigma_0(\sigma_1\sigma_0)^k(\Lambda_0 - w\delta), \qquad (k \in \mathbb{N})\,,$$

and it easy to see that our formulas calculate the canonical bases for all Λ^j with $j \geqslant w-1$.

In the general case, the class of weights of $\mathcal{O}_w$ to which the formulas for $\Lambda_{w,\rho}$ can be transferred is also infinite, but there is still an infinite number of weights of $\mathcal{O}_w$ for which the formulas do not apply.

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