

VIRTUAL CRYSTALS AND FERMIONIC FORMULAS OF TYPE $D_{n+1}^{(2)}$, $A_{2n}^{(2)}$, AND $C_n^{(1)}$

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ABSTRACT. We introduce “virtual” crystals of the affine types $\mathfrak{g} = D_{n+1}^{(2)}$, $A_{2n}^{(2)}$ and $C_n^{(1)}$ by naturally extending embeddings of crystals of types B_n and C_n into crystals of type A_{2n-1} . Conjecturally, these virtual crystals are the crystal bases of finite dimensional $U'_q(\mathfrak{g})$ -modules associated with multiples of fundamental weights. We provide evidence and in some cases proofs of this conjecture. Recently, fermionic formulas for the one-dimensional configuration sums associated with tensor products of the finite dimensional $U'_q(\mathfrak{g})$ -modules were conjectured by Hatayama et al. We provide proofs of these conjectures in specific cases by exploiting duality properties of crystals and rigged configuration techniques. For type $A_{2n}^{(2)}$ we also conjecture a new fermionic formula coming from a different labeling of the Dynkin diagram.

1. INTRODUCTION

The quantized universal enveloping algebra $U_q(\mathfrak{g})$ associated with a symmetrizable Kac–Moody Lie algebra \mathfrak{g} was introduced independently by Drinfeld [5] and Jimbo [9] in their study of two-dimensional solvable lattice models in statistical mechanics. The parameter q corresponds to the temperature of the underlying model. Kashiwara [15] showed that at zero temperature or $q = 0$ the representations of $U_q(\mathfrak{g})$ have bases, which he coined crystal bases, with a beautiful combinatorial structure and favorable properties such as uniqueness and stability under tensor products.

1.1. Finite affine crystals. The underlying algebras \mathfrak{g} of affine crystals are affine Kac–Moody algebras. There are two main categories of affine crystals: (1) crystal bases of infinite dimensional integrable highest weight $U_q(\mathfrak{g})$ -modules, and (2) crystal bases of finite dimensional $U'_q(\mathfrak{g})$ -modules, where $U'_q(\mathfrak{g})$ is a subalgebra of $U_q(\mathfrak{g})$ without the degree operator. The former category is well understood. For instance, it is known [16] that an irreducible integrable $U_q(\mathfrak{g})$ -module has a unique crystal basis. On the other hand, the latter is still far from well understood. It is not even known which finite dimensional $U'_q(\mathfrak{g})$ -modules have a crystal basis.

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In [8, 7] it was conjectured that there exists a family of finite dimensional $U'_q(\mathfrak{g})$ -modules $\{W_s^{(r)}\}$ having crystal bases $\{B^{r,s}\}$ ($1 \leq r \leq n, s \geq 1$), where $n+1$ is the number of vertices in the Dynkin diagram of \mathfrak{g} . If such crystals indeed exist, one can define one-dimensional configuration sums, which play an important role in the study of phase transitions in two-dimensional exactly solvable lattice models. If \mathfrak{g} is of type $A_n^{(1)}$, the existence of the crystal $B^{r,s}$ is settled in [14], and the one-dimensional configuration sums contain the Kostka polynomials [33, 38, 41, 44], which turn up in the theory of symmetric functions [36], combinatorics [31], the study of subgroups of finite abelian groups [3], and Kazhdan–Lusztig theory [21, 34]. In certain limits they are branching functions of integrable highest weight modules [6, 38]. For all affine types and $s = 1$ it is shown in [19] that $B^{r,1}$ exists (but not that it has the desired decomposition as a classical crystal in all cases). For type $A_{2n}^{(2)}$, the structure of $B^{r,1}$ has been described explicitly in [10]. The existence and structure of $B^{r,s}$ has been proven for various other subcases [12, 14, 28, 29, 47].

1.2. Virtual crystals. Classical finite dimensional crystals of type C_n and B_n can be embedded into crystals of type A_{2n-1} [2]. This embedding has been extended to certain perfect affine crystals [18]. Following this idea we consider “virtual” crystals $V^{r,s}$ for types $D_{n+1}^{(2)}$, $A_{2n}^{(2)}$, $A_{2n}^{(2)\dagger}$ (the Dynkin diagram $A_{2n}^{(2)}$ with different distinguished vertex 0), and $C_n^{(1)}$ by naturally extending these embeddings to finite affine crystals. We conjecture that these virtual crystals $V^{r,s}$ are exactly the conjectured crystals $B^{r,s}$ of [8, 7]. This is proven for $s = 1$ and types $A_{2n}^{(2)}$, $A_{2n}^{(2)\dagger}$, and $C_n^{(1)}$. For type $A_{2n}^{(2)\dagger}$ the work was already done by combining [2] and [10]. As further evidence we show that the virtual crystals have the expected decomposition into classical components for all $s \geq 1$ in type $D_{n+1}^{(2)}$. In all cases the virtual crystal $V^{r,s}$ is shown to have at least the desired classical components. We give an explicit characterization of elements in $V^{r,1}$, which is used to establish the equality between the one-dimensional sums (defined in terms of the virtual crystals $V^{r,1}$) and the fermionic formulas mentioned below.

1.3. Fermionic formulas. Crystal theory provides a general setting for the definition of one-dimensional configuration sums, denoted by X . Explicit formulas M , called fermionic formulas, for X have recently been conjectured in [8, 7]. Fermionic formulas are expressions for certain polynomials in q (or q -series) which are sums of products of q -binomial coefficients. These expressions play a preponderant role in Bethe Ansatz studies of spin chain models. At $q = 1$ they give the number of solutions to Bethe equations of the underlying model. Kirillov and Reshetikhin [24] obtained fermionic expressions for the Kostka polynomials from the Bethe Ansatz study of the XXX model. In a separate development Kedem and McCoy [22] discovered fermionic expressions for the branching functions of the coset pair $(A_3^{(1)})_1 \oplus (A_3^{(1)})_1 / (A_3^{(1)})_2$ based on the Bethe Ansatz study of the 3-state Potts chain. This led to a flurry of works on fermionic expressions for characters and branching functions of conformal field theories (see [7] for references). At present many of these fermionic expressions are understood as certain limits of M or its level truncation.

Branching functions and one-dimensional configuration sums are, however, not the only places where fermionic formulas have appeared. Kirillov and Reshetikhin

[25] conjectured that for \mathfrak{g} of nontwisted affine type, the coefficients of the decomposition of the representations of $U'_q(\mathfrak{g})$ naturally associated with multiples of the fundamental weights into direct sums of irreducible representations of $U_q(\overline{\mathfrak{g}})$ are given by the fermionic formulas at $q = 1$. Here $\overline{\mathfrak{g}}$ is the simple Lie algebra associated to the Kac–Moody algebra \mathfrak{g} . Chari [4] proved a number of cases of this conjecture. Recently, it was conjectured by Lusztig [35] and Nakajima [37] that quiver varieties are also related to fermionic formulas.

The conjecture $X = M$ was proven in [26] for type $A_n^{(1)}$. The main technique used in the proof is a statistic preserving bijection between crystals and rigged configurations. Rigged configurations are combinatorial objects which label all terms in the fermionic formula and also originate from the Bethe Ansatz [23, 24]. Here we prove the conjecture $X = M$ of [8, 7] for types $D_{n+1}^{(2)}$, $A_{2n}^{(2)}$ and $C_n^{(1)}$ for tensor products of crystals of the form $B^{r,1}$, assuming that $V^{r,1} = B^{r,1}$ for type $D_{n+1}^{(2)}$. We also conjecture a new fermionic formula of type $A_{2n}^{(2)\dagger}$.

1.4. Duality on rigged configurations. The proof of $X = M$ is achieved by an explicit characterization of the rigged configurations in the image of the embedding of the above crystals into crystals of type $A_{2n-1}^{(1)}$ combined with the bijection between type $A_{2n-1}^{(1)}$ crystals and rigged configurations [26]. An important tool that is used in the proofs and the characterizations is the contragredient duality on crystals and rigged configurations.

1.5. Content of the paper. Section 2 gives a review of crystal theory, in particular, the definition of crystal graphs, dual crystals, simple crystals, the combinatorial R -matrix, the (intrinsic) energy function and the definition of the one-dimensional configuration sums. In section 3 the combinatorial structure of crystals of type $A_n^{(1)}$ is discussed in detail, and all results and definitions of section 2 are stated in terms of operations on Young tableaux. Irreducible components of tensor products of crystals of type $A_n^{(1)}$ can be parametrized in terms of Littlewood–Richardson tableaux. These are introduced in section 4. They are of importance in section 5 where we define rigged configurations and recall the bijection between Littlewood–Richardson tableaux and rigged configurations of [26]. The relation between the duality and this bijection is given in Theorem 5.7. Virtual crystals $V^{r,s}$ for types $D_{n+1}^{(2)}$, $A_{2n}^{(2)}$, $A_{2n}^{(2)\dagger}$, and $C_n^{(1)}$ are introduced in section 6. Their explicit characterization for $s = 1$ is given in section 6.12 in terms of classical virtual crystals $V(\lambda)$. Finally, in section 7 the fermionic formulas for $s = 1$ and types $D_{n+1}^{(2)}$, $A_{2n}^{(2)}$ and $C_n^{(1)}$ are proved. The proof relies on the main Theorem 7.1 which gives a characterization of the rigged configurations in the image of the virtual crystals under the bijection of section 5. For type $A_{2n}^{(2)\dagger}$ a similar characterization is given in Conjecture 7.3 which yields a new fermionic formula given in section 7.6. Appendix A is reserved for the proof of Theorem 7.1.

Finally, it worth mentioning that the techniques of this paper have been applied in [39] to obtain fermionic formulas for level-restricted one-dimensional configuration sums of type $C_n^{(1)}$.

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2. CRYSTAL THEORY

In this section some results from affine crystal theory are reviewed. In addition, we show that the tensor category of simple crystal bases of irreducible finite-dimensional modules over quantized affine algebras, can be enhanced by including a grading function which we call the intrinsic energy. This result (Proposition 2.13) is central to the definition of the energy function of a virtual crystal, notions that are introduced in section 6. A prerequisite for the explicit formula for the intrinsic energy of a tensor product, is the discussion in section 2.12 of isomorphisms and energy functions indexed by reduced words.

2.1. Crystals. Let \mathfrak{g} be a symmetrizable Kac-Moody algebra, P the weight lattice, I the index set for the vertices of the Dynkin diagram of \mathfrak{g} , $\{\alpha_i \in P \mid i \in I\}$ the simple roots, and $\{h_i \in P^* = \text{Hom}_{\mathbb{Z}}(P, \mathbb{Z}) \mid i \in I\}$ the simple coroots. Let $U_q(\mathfrak{g})$ be the quantized universal enveloping algebra of \mathfrak{g} [17]. A $U_q(\mathfrak{g})$ -crystal is a nonempty set B equipped with maps $\text{wt} : B \rightarrow P$ and $\tilde{e}_i, \tilde{f}_i : B \rightarrow B \cup \{\emptyset\}$ for all $i \in I$, satisfying

$$(2.1) \quad \tilde{f}_i(b) = b' \Leftrightarrow \tilde{e}_i(b') = b \text{ if } b, b' \in B,$$

$$(2.2) \quad \text{wt}(\tilde{f}_i(b)) = \text{wt}(b) - \alpha_i \text{ if } \tilde{f}_i(b) \in B,$$

$$(2.3) \quad \langle h_i, \text{wt}(b) \rangle = \varphi_i(b) - \epsilon_i(b).$$

Here for $b \in B$

$$(2.4) \quad \begin{aligned} \epsilon_i(b) &= \max\{n \geq 0 \mid \tilde{e}_i^n(b) \neq \emptyset\}, \\ \varphi_i(b) &= \max\{n \geq 0 \mid \tilde{f}_i^n(b) \neq \emptyset\}. \end{aligned}$$

(It is assumed that $\varphi_i(b), \epsilon_i(b) < \infty$ for all $i \in I$ and $b \in B$.) A $U_q(\mathfrak{g})$ -crystal B can be viewed as a directed edge-colored graph (the crystal graph) whose vertices are the elements of B , with a directed edge from b to b' labeled $i \in I$, if and only if $\tilde{f}_i(b) = b'$.

2.2. Morphisms of crystals. A morphism $\psi : B_1 \rightarrow B_2$ of $U_q(\mathfrak{g})$ -crystals is a map $\psi : B_1 \cup \{\emptyset\} \rightarrow B_2 \cup \{\emptyset\}$ such that

$$(2.5) \quad \psi(\emptyset) = \emptyset.$$

$$(2.6) \quad \text{If } \psi(b) \neq \emptyset \text{ for } b \in B_1, \text{ then}$$

$$\begin{aligned} \text{wt}(\psi(b)) &= \text{wt}(b), \\ \epsilon_i(\psi(b)) &= \epsilon_i(b), \text{ and} \\ \varphi_i(\psi(b)) &= \varphi_i(b). \end{aligned}$$

$$(2.7) \quad \begin{aligned} \text{For } b \in B_1 \text{ such that } \psi(b) \neq \emptyset \text{ and } \psi(\tilde{e}_i(b)) \neq \emptyset, \\ \psi(\tilde{e}_i(b)) &= \tilde{e}_i(\psi(b)). \end{aligned}$$

$$(2.8) \quad \begin{aligned} \text{For } b \in B_1 \text{ such that } \psi(b) \neq \emptyset \text{ and } \psi(\tilde{f}_i(b)) \neq \emptyset, \\ \psi(\tilde{f}_i(b)) &= \tilde{f}_i(\psi(b)). \end{aligned}$$

2.3. Tensor products of crystals. Let B_1, B_2, \dots, B_L be $U_q(\mathfrak{g})$ -crystals. The Cartesian product $B_L \times \dots \times B_2 \times B_1$ has the structure of a $U_q(\mathfrak{g})$ -crystal using the so-called signature rule [30]. The resulting crystal is denoted $B = B_L \otimes \dots \otimes B_2 \otimes B_1$ and its elements (b_L, \dots, b_1) are written $b_L \otimes \dots \otimes b_1$ where $b_j \in B_j$. The reader

is warned that our convention is opposite to that of Kashiwara [17]. Fix $i \in I$ and $b = b_L \otimes \cdots \otimes b_1 \in B$. The i -signature of b is the word consisting of the symbols $+$ and $-$ given by

$$\underbrace{\cdots}_{\varphi_i(b_L) \text{ times}} \underbrace{\cdots}_{\epsilon_i(b_L) \text{ times}} \cdots \underbrace{\cdots}_{\varphi_i(b_1) \text{ times}} \underbrace{\cdots}_{\epsilon_i(b_1) \text{ times}} .$$

The reduced i -signature of b is the subword of the i -signature of b , given by the repeated removal of adjacent symbols $+ -$ (in that order); it has the form

$$\underbrace{\cdots}_{\varphi \text{ times}} \underbrace{\cdots}_{\epsilon \text{ times}} .$$

If $\varphi = 0$, then $\tilde{f}_i(b) = \emptyset$; otherwise,

$$\tilde{f}_i(b_L \otimes \cdots \otimes b_1) = b_L \otimes \cdots \otimes b_{j+1} \otimes \tilde{f}_i(b_j) \otimes \cdots \otimes b_1$$

where the rightmost symbol $-$ in the reduced i -signature of b comes from b_j . Similarly, if $\epsilon = 0$, then $\tilde{e}_i(b) = \emptyset$; otherwise,

$$\tilde{e}_i(b_L \otimes \cdots \otimes b_1) = b_L \otimes \cdots \otimes b_{j+1} \otimes \tilde{e}_i(b_j) \otimes \cdots \otimes b_1$$

where the leftmost symbol $+$ in the reduced i -signature of b comes from b_j . It is not hard to verify that this well-defines the structure of a $U_q(\mathfrak{g})$ -crystal with $\varphi_i(b) = \varphi$ and $\epsilon_i(b) = \epsilon$ in the above notation, with weight function

$$(2.9) \quad \text{wt}(b_L \otimes \cdots \otimes b_1) = \sum_{j=1}^L \text{wt}(b_j).$$

This tensor construction is easily seen to be associative. The case of two tensor factors is given explicitly by

$$(2.10) \quad \tilde{f}_i(b_2 \otimes b_1) = \begin{cases} \tilde{f}_i(b_2) \otimes b_1 & \text{if } \epsilon_i(b_2) \geq \varphi_i(b_1), \\ b_2 \otimes \tilde{f}_i(b_1) & \text{if } \epsilon_i(b_2) < \varphi_i(b_1) \end{cases}$$

and

$$(2.11) \quad \tilde{e}_i(b_2 \otimes b_1) = \begin{cases} \tilde{e}_i(b_2) \otimes b_1 & \text{if } \epsilon_i(b_2) > \varphi_i(b_1), \\ b_2 \otimes \tilde{e}_i(b_1) & \text{if } \epsilon_i(b_2) \leq \varphi_i(b_1). \end{cases}$$

2.4. Crystals for subalgebras. For a subset $J \subset I$ define the J -components of a $U_q(\mathfrak{g})$ -crystal B to be the connected components of the crystal graph using only edges labeled with elements of J . Say that $b \in B$ is a J -highest weight vector if $\epsilon_i(b) = 0$ for all $j \in J$. A highest weight vector is an I -highest weight vector.

2.5. Highest weight crystals. Let $P^+ = \{\lambda \in P \mid \langle h_i, \lambda \rangle \geq 0 \text{ for all } i \in I\}$. For $\lambda \in P^+$ let $B(\lambda)$ be the crystal basis of the irreducible integrable $U_q(\mathfrak{g})$ -module of highest weight λ . By [17] $B(\lambda)$ has the structure of a $U_q(\mathfrak{g})$ -crystal in the sense of section 2.1 where \tilde{e}_i and \tilde{f}_i are the modified Chevalley generators. $B(\lambda)$ is connected and has a unique highest weight vector, denoted u_λ ; it is the unique element in $B(\lambda)$ of weight λ .

2.6. Dual crystals. The notion of a dual crystal is given in [17, Section 7.4]. Let B be a $U_q(\mathfrak{g})$ -crystal. Then there is a $U_q(\mathfrak{g})$ -crystal denoted B^\vee obtained from B by reversing arrows. That is, $B^\vee = \{b^\vee \mid b \in B\}$ with

$$\begin{aligned}
 \text{wt}(b^\vee) &= -\text{wt}(b) \\
 \epsilon_i(b^\vee) &= \varphi_i(b) \\
 \varphi_i(b^\vee) &= \epsilon_i(b) \\
 \tilde{e}_i(b^\vee) &= (\tilde{f}_i(b))^\vee \\
 \tilde{f}_i(b^\vee) &= (\tilde{e}_i(b))^\vee.
 \end{aligned}
 \tag{2.12}$$

Proposition 2.1. [17] *There is an isomorphism $(B_2 \otimes B_1)^\vee \cong B_1^\vee \otimes B_2^\vee$ given by $(b_2 \otimes b_1)^\vee \mapsto b_1^\vee \otimes b_2^\vee$.*

2.7. Extremal vectors. Let W be the Weyl group of \mathfrak{g} , $\{s_i \mid i \in I\}$ the simple reflections in W . Let B be the crystal graph of an integrable $U_q(\mathfrak{g})$ -module. Say that $b \in B$ is an extremal vector of weight $\lambda \in P$ provided that $\text{wt}(b) = \lambda$ and there exists a family of elements $\{b_w \mid w \in W\} \subset B$ such that

1. $b_w = b$ for $w = e$.
2. If $\langle h_i, w\lambda \rangle \geq 0$, then $\tilde{e}_i(b_w) = \emptyset$ and $\tilde{f}_i^{\langle h_i, w\lambda \rangle}(b_w) = b_{s_i w}$.
3. If $\langle h_i, w\lambda \rangle \leq 0$, then $\tilde{f}_i(b_w) = \emptyset$ and $\tilde{e}_i^{\langle h_i, w\lambda \rangle}(b_w) = b_{s_i w}$.

2.8. Affine case. We now fix notation for the affine case, which, for the most part, follows [11]. Let \mathfrak{g} be an affine Kac-Moody algebra over \mathbb{C} . Let I be the index set for the Dynkin diagram, P the weight lattice, $P^* = \text{Hom}_{\mathbb{Z}}(P, \mathbb{Z})$, $\langle \cdot, \cdot \rangle$ the natural pairing $P^* \otimes P \rightarrow \mathbb{Z}$, $\{\alpha_i \mid i \in I\} \subset P$ the simple roots, $\{h_i \mid i \in I\} \subset P^*$ the simple coroots, $\{\Lambda_i \mid i \in I\} \subset P$ the fundamental weights, c the canonical central element, and $\delta \in P$ the null root. One has $P = \mathbb{Z}\delta \oplus \bigoplus_{i \in I} \mathbb{Z}\Lambda_i$.

Let the distinguished vertex in the Dynkin diagram be labeled $0 \in I$ as specified in [11]. Let $\bar{\mathfrak{g}} \subset \mathfrak{g}$ be the simple Lie algebra whose Dynkin diagram is $J = I - \{0\}$ and let $\mathfrak{g}' \subset \mathfrak{g}$ be the derived subalgebra. Denote the weight lattices of $\bar{\mathfrak{g}} \subset \mathfrak{g}' \subset \mathfrak{g}$ by \bar{P} , P_{cl} , and P respectively. There are natural projections

$$P \xrightarrow{\text{cl}} P_{\text{cl}} \twoheadrightarrow \bar{P}.$$

$P_{\text{cl}} \cong P/\mathbb{Z}\delta$ and $\bar{P} \cong P/(\mathbb{Z}\delta \oplus \mathbb{Z}\Lambda_0)$. We shall identify P_{cl} and \bar{P} with the sublattices of P given by $\bigoplus_{i \in I} \mathbb{Z}\Lambda_i$ and $\bigoplus_{i \in J} \mathbb{Z}\bar{\Lambda}_i$ where $\bar{\Lambda}_i = \Lambda_i - \langle c, \Lambda_i \rangle \Lambda_0$. Let $\bar{P}^+ = \bigoplus_{i \in J} \mathbb{N}\bar{\Lambda}_i$.

Denote by $U_q(\bar{\mathfrak{g}})$, $U'_q(\mathfrak{g})$, and $U_q(\mathfrak{g})$ the quantized universal enveloping algebras of $\bar{\mathfrak{g}}$, \mathfrak{g}' , and \mathfrak{g} respectively [17]. Let W and \bar{W} be the Weyl groups of \mathfrak{g} and $\bar{\mathfrak{g}}$; they are generated by the simple reflections $\{s_i \mid i \in I\}$ and $\{s_i \mid i \in J\}$ respectively.

2.9. Simple crystals. Following [1], say that a $U'_q(\mathfrak{g})$ -crystal B is *simple* if

1. B is the crystal basis of a finite dimensional integrable $U'_q(\mathfrak{g})$ -module.
2. There is a weight $\lambda \in \bar{P}^+$ such that B has a unique vector (denoted $u(B)$) of weight λ , and the weight of any extremal vector of B is contained in $\bar{W}\lambda$.

In the definition of simple crystal in [1], condition 1 is not present. However, we always want to assume both conditions, so it is convenient to include condition 1 in the definition above.

Theorem 2.2 ([1]).

1. *Simple crystals are connected.*
2. *The tensor product of simple crystals is simple.*

Remark 2.3. Suppose $\Psi : B \rightarrow B'$ is an isomorphism of simple $U'_q(\mathfrak{g})$ -crystals. Then Ψ is the only such isomorphism. Since Ψ must preserve weight and send extremal vectors to extremal vectors, it follows that $\Psi(u(B)) = u(B')$. Since B is connected, the rest of the map Ψ is determined by the requirement that Ψ be a $U'_q(\mathfrak{g})$ -crystal morphism.

Suppose B_j are simple $U'_q(\mathfrak{g})$ -crystals. By Theorem 2.2, $B = B_L \otimes \cdots \otimes B_1$ is simple, with

$$(2.13) \quad u(B) = u(B_L) \otimes \cdots \otimes u(B_1).$$

2.10. Level and perfectness. For the $U'_q(\mathfrak{g})$ -crystal B , define $\epsilon, \varphi : B \rightarrow P_{cl}$ by

$$(2.14) \quad \begin{aligned} \epsilon(b) &= \sum_{i \in I} \epsilon_i(b) \Lambda_i \\ \varphi(b) &= \sum_{i \in I} \varphi_i(b) \Lambda_i. \end{aligned}$$

Define the level of B by

$$(2.15) \quad \text{level}(B) = \min\{\langle c, \epsilon(b) \rangle \mid b \in B\}.$$

Say that $b \in B$ is minimal if $\langle c, \epsilon(b) \rangle = \text{level}(B)$.

Let B be simple. Say that B is perfect of level ℓ if $\ell = \text{level}(B)$ and ϵ and φ are bijections from the set of minimal elements of B to $(P_{cl}^+)_{\ell} = \{\lambda \in P_{cl} \mid \langle c, \lambda \rangle = \ell\}$.

2.11. Combinatorial R -matrix. The following two results essentially appear in [13] in the case that $B_1 = B_2 = B_3$. They are a consequence of the existence of a combinatorial analogue of the R -matrix on the affinizations of finite $U'_q(\mathfrak{g})$ -crystals and the fact that it satisfies the Yang-Baxter equation.

Theorem 2.4 ([13, Section 4]). *Suppose B_1 and B_2 are simple $U'_q(\mathfrak{g})$ -crystals. Then*

1. *There is a unique isomorphism of $U'_q(\mathfrak{g})$ -crystals $\sigma = \sigma_{B_2, B_1} : B_2 \otimes B_1 \rightarrow B_1 \otimes B_2$.*
2. *There is a function $H = H_{B_2, B_1} : B_2 \otimes B_1 \rightarrow \mathbb{Z}$, unique up to global additive constant, such that H is constant on J -components and, for all $b_2 \in B_2$ and $b_1 \in B_1$, if $\sigma(b_2 \otimes b_1) = b'_1 \otimes b'_2$, then*

$$(2.16) \quad H(\tilde{\epsilon}_0(b_2 \otimes b_1)) = H(b_2 \otimes b_1) + \begin{cases} -1 & \text{if } \epsilon_0(b_2) > \varphi_0(b_1) \text{ and } \epsilon_0(b'_1) > \varphi_0(b'_2), \\ 1 & \text{if } \epsilon_0(b_2) \leq \varphi_0(b_1) \text{ and } \epsilon_0(b'_1) \leq \varphi_0(b'_2), \\ 0 & \text{otherwise.} \end{cases}$$

We shall call the maps σ and H the local isomorphism and local energy function on $B_2 \otimes B_1$. The pair (σ, H) is called the combinatorial R -matrix.

By Remark 2.3 and (2.13),

$$(2.17) \quad \sigma(u(B_2) \otimes u(B_1)) = u(B_1) \otimes u(B_2).$$

It is convenient to normalize the local energy function H by requiring that

$$(2.18) \quad H(u(B_2) \otimes u(B_1)) = 0.$$

With this convention it follows by definition that

$$(2.19) \quad H_{B_1, B_2} \circ \sigma_{B_2, B_1} = H_{B_2, B_1}$$

as operators on $B_2 \otimes B_1$.

The following observation, which follows from the uniqueness of the local isomorphism and energy function, is obvious but useful.

Proposition 2.5. *Suppose B_1, B_2 , and B'_1 are simple $U'_q(\mathfrak{g})$ -crystals and there is a $U'_q(\mathfrak{g})$ -crystal isomorphism $\Psi : B_1 \rightarrow B'_1$. Then*

$$(2.20) \quad \sigma_{B_2, B_1} = (\Psi^{-1} \otimes 1_{B_2}) \circ \sigma_{B_2, B'_1} \circ (1_{B_2} \otimes \Psi),$$

$$(2.21) \quad H_{B_2, B_1} = H_{B_2, B'_1} \circ (1_{B_2} \otimes \Psi).$$

Suppose B_j is a simple $U'_q(\mathfrak{g})$ -crystal for $1 \leq j \leq L$. By abuse of notation let σ_k and H_k denote the local isomorphism and local energy function acting in the k -th and $(k + 1)$ -st tensor positions (from the right).

Theorem 2.6 ([13]). *There is a unique $U'_q(\mathfrak{g})$ -crystal isomorphism $B_3 \otimes B_2 \otimes B_1 \rightarrow B_1 \otimes B_2 \otimes B_3$ given by either side of the Yang-Baxter equation*

$$(2.22) \quad \sigma_1 \circ \sigma_2 \circ \sigma_1 = \sigma_2 \circ \sigma_1 \circ \sigma_2.$$

Moreover,

$$(2.23) \quad H_1 + H_2\sigma_1 = H_1\sigma_2 + H_2\sigma_1\sigma_2,$$

$$(2.24) \quad H_2 + H_1\sigma_2 = H_2\sigma_1 + H_1\sigma_2\sigma_1.$$

Observe that (2.23) and (2.24) are equivalent; the latter is obtained from the former by composing on the right by $\sigma_1\sigma_2$, using (2.22), and the identities $H_1\sigma_1 = H_1$ and $H_2\sigma_2 = H_2$ which hold by (2.19).

2.12. Isomorphisms and energy functions indexed by reduced words. Let B_j be a simple $U'_q(\mathfrak{g})$ -module for $1 \leq j \leq L$ and let $B = B_L \otimes \cdots \otimes B_1$. Let $a = (a_1, a_2, \dots, a_p)$ be a sequence of indices in the set $\{1, 2, \dots, L - 1\}$. Let s_j denote the permutation that exchanges the j -th and $(j + 1)$ -st positions. Define $\text{perm}(a)$, the permutation of the set $\{1, 2, \dots, L\}$ associated with the sequence a , by

$$(2.25) \quad \text{perm}(a) = s_{a_1} s_{a_2} \cdots s_{a_p}.$$

Denote by $\sigma_a : B \rightarrow \text{perm}(a)B$ the $U'_q(\mathfrak{g})$ -crystal isomorphism given by

$$(2.26) \quad \sigma_a = \sigma_{a_1} \sigma_{a_2} \cdots \sigma_{a_p}$$

and the energy function $E_a : B \rightarrow \mathbb{Z}$ by

$$(2.27) \quad E_a = \sum_{j=1}^p H_{a_j} \sigma_{a_{j+1}} \sigma_{a_{j+2}} \cdots \sigma_{a_p}.$$

Proposition 2.7. *With the above notation, let B' be a tensor product obtained from B by reordering the factors. Then there is a unique $U'_q(\mathfrak{g})$ -crystal isomorphism $\sigma : B \rightarrow B'$, and $\sigma = \sigma_a$ for any sequence a such that $B' = \text{perm}(a)B$.*

Proof. By Theorem 2.2, B and B' are simple since they are tensor products of simple crystals. Let a be any sequence such that $B' = \text{perm}(a)B$. Then $\sigma_a : B \rightarrow B'$ is an isomorphism of simple $U'_q(\mathfrak{g})$ -crystals. But there is a unique $U'_q(\mathfrak{g})$ -crystal isomorphism $B \rightarrow B'$ by Remark 2.3. \square

Remark 2.8. If a' is obtained from a by a sequence of exchanges of the form $(\dots, i, j, \dots) \rightarrow (\dots, j, i, \dots)$ where $|i - j| > 1$, then it is easy to see that $E_a = E_{a'}$.

Remark 2.9. In general, $E_{(1,2,1)} \neq E_{(2,1,2)}$. Take $B_1 = B_2 = B_3 = B^{1,1}$ in $A_1^{(1)}$ with $b = 1 \otimes 2 \otimes 1$ (see sections 3.2 and 3.4). Then $E_{(1,2,1)}(b) = 1$ and $E_{(2,1,2)} = 2$.

For later use we define a few specific reduced words. Let

$$(2.28) \quad \begin{aligned} a^{(1)} &= \emptyset, \\ a^{(k)} &= (1, 2, \dots, k - 1, a^{(k-1)}) \quad \text{for } k > 1. \end{aligned}$$

Note that $\text{perm}(a^{(L)})$ is the permutation that reverses the numbers $\{1, 2, \dots, L\}$. For ℓ and m positive integers summing to L , define

$$(2.29) \quad \begin{aligned} a^{(\ell,1)} &= (1, 2, \dots, \ell), \\ a^{(\ell,m)} &= (m, m + 1, \dots, \ell + m - 1, a^{(\ell,m-1)}) \quad \text{for } m > 1. \end{aligned}$$

Observe that $\text{perm}(a^{(\ell,m)})$ is the shortest permutation that exchanges the numbers $\{1, 2, \dots, \ell\}$ past the numbers $\{\ell + 1, \ell + 2, \dots, \ell + m\}$.

2.13. Tensoring tensor products. Let ℓ and m be positive integers. Let B_j be a simple $U'_q(\mathfrak{g})$ -crystal for $1 \leq j \leq \ell + m$. Then $B'_2 = B_{\ell+m} \otimes \dots \otimes B_{\ell+2} \otimes B_{\ell+1}$ and $B'_1 = B_\ell \otimes \dots \otimes B_1$ are simple crystals by Theorem 2.2. We compute the local isomorphism $\sigma_{B'_2, B'_1}$ and energy function $H_{B'_2, B'_1}$ explicitly.

Proposition 2.10. *With the above notation,*

$$(2.30) \quad \sigma_{B'_2, B'_1} = \sigma_{a^{(\ell,m)}},$$

$$(2.31) \quad H_{B'_2, B'_1} = E_{a^{(\ell,m)}}.$$

Proposition 2.10 is proved by induction. The first nontrivial cases are for three tensor factors.

Proposition 2.11.

$$(2.32) \quad \sigma_{B_3, B_2 \otimes B_1} = \sigma_1 \sigma_2,$$

$$(2.33) \quad H_{B_3, (B_2 \otimes B_1)} = H_2 + H_1 \sigma_2,$$

$$(2.34) \quad \sigma_{(B_3 \otimes B_2), B_1} = \sigma_2 \sigma_1,$$

$$(2.35) \quad H_{(B_3 \otimes B_2), B_1} = H_1 + H_2 \sigma_1.$$

Proof. We will prove the case $\ell = 2$ and $m = 1$ as the case $\ell = 1$ and $m = 2$ is similar. We have $\sigma = \sigma_1 \sigma_2$. It must be shown that $H_{B_3, B_2 \otimes B_1} = H_2 + H_1 \sigma_2$. Let us fix $b \in B_3 \otimes B_2 \otimes B_1$. For any isomorphism Ψ that reorders $B_3 \otimes B_2 \otimes B_1$ by a composition of local isomorphisms, define $\text{pos}(\Psi) \in \{1, 2, 3\}$ to be the position in the threefold tensor $\Psi(b)$ where \tilde{e}_0 acts (see section 2.3). Let $\Delta H_2 = H_2(b) - H_2(\tilde{e}_0 b)$, $\Delta H_1 \sigma_2 = H_1(\sigma_2 b) - H_1(\sigma_2 \tilde{e}_0 b)$, $\Delta = \Delta H_2 + \Delta H_1 \sigma_2$. Write $\sigma = \sigma_{B_3, (B_2 \otimes B_1)}$ and let $\text{pos}'(id)$ (resp. $\text{pos}'(\sigma)$) have value L (left) and R (right) on the two-fold tensor product $B_3 \otimes (B_2 \otimes B_1)$ (resp. $(B_2 \otimes B_1) \otimes B_3$). Then the rows of the following

table give all the possibilities:

| pos(<i>id</i>) | pos(σ_2) | pos($\sigma_1\sigma_2$) | pos'(<i>id</i>) | pos'(σ) | ΔH_2 | $\Delta H_1\sigma_2$ | Δ |
|------------------|-------------------|---------------------------|-------------------|------------------|--------------|----------------------|----------|
| 3 | 3 | 3 | <i>L</i> | <i>L</i> | 1 | 0 | 1 |
| 3 | 2 | 2 | <i>L</i> | <i>L</i> | 0 | 1 | 1 |
| 3 | 2 | 1 | <i>L</i> | <i>R</i> | 0 | 0 | 0 |
| 2 | 3 | 3 | <i>R</i> | <i>L</i> | 0 | 0 | 0 |
| 2 | 2 | 2 | <i>R</i> | <i>L</i> | 0 | 0 | 0 |
| 2 | 2 | 1 | <i>R</i> | <i>R</i> | -1 | 0 | -1 |
| 1 | 1 | 2 | <i>R</i> | <i>L</i> | 0 | 0 | 0 |
| 1 | 1 | 1 | <i>R</i> | <i>R</i> | 0 | -1 | -1 |

The columns pos'(*id*), pos'(σ), and Δ agree with the defining conditions for *H* in Theorem 2.4. □

The next case is $m = 1$.

Proposition 2.12.

$$(2.36) \quad \sigma_{B_{\ell+1},(B_\ell \otimes \dots \otimes B_1)} = \sigma_1 \sigma_2 \dots \sigma_\ell,$$

$$(2.37) \quad H_{B_{\ell+1},(B_\ell \otimes \dots \otimes B_1)} = \sum_{j=1}^{\ell} H_j \sigma_{j+1} \sigma_{j+2} \dots \sigma_\ell.$$

Proof. Formula (2.36) holds by Remark 2.3. Formula (2.37) is proved by induction on ℓ , by applying (2.32) and (2.33) for $B'_3 \otimes (B'_2 \otimes B'_1)$ where $B'_3 = B_{\ell+1}$, $B'_2 = B_\ell$, and $B'_1 = B_{\ell-1} \otimes \dots \otimes B_1$. □

Proof of Proposition 2.10. Again (2.30) follows by Remark 2.3. The proof of (2.31) follows by induction on m . The base of the induction ($m = 1$) is given by Proposition 2.12. For the induction step, (2.34) and (2.35) are applied for $(B'_3 \otimes B'_2) \otimes B'_1$ where $B'_3 = B_{\ell+m}$, $B'_2 = B_{\ell+m-1} \otimes \dots \otimes B_{\ell+1}$ and $B'_1 = B_\ell \otimes \dots \otimes B_1$. □

2.14. Graded simple crystals. Let B be a simple $U'_q(\mathfrak{g})$ -crystal, equipped with a function $D = D_B : B \rightarrow \mathbb{Z}$ called its intrinsic energy, which is required to be constant on J -components and defined up to a global additive constant. Call the pair (B, D) a graded simple $U'_q(\mathfrak{g})$ -crystal. We normalize the intrinsic energy function by the requirement that

$$(2.38) \quad D_B(u(B)) = 0.$$

Let (B_j, D_j) be graded simple $U'_q(\mathfrak{g})$ -crystals for $j = 1, 2$. Write π for the projection onto the rightmost tensor factor. Define $D_{B_2 \otimes B_1} : B_2 \otimes B_1 \rightarrow \mathbb{Z}$ by

$$(2.39) \quad D_{B_2 \otimes B_1} = H_{B_2, B_1} + D_1 \pi + D_2 \pi \sigma_{B_2, B_1}.$$

From now on we suppress the operator π , observing that the operator D_j always acts on the rightmost tensor factor, which will be in B_j . The subscripts of H and σ indicate positions.

Observe that (2.38) holds for $B = B_2 \otimes B_1$ if it is assumed that it holds for B_1 and B_2 and that (2.18) holds. This can be seen using (2.13) and (2.17).

Proposition 2.13. *Graded simple crystals form a tensor category.*

Proof. It must be shown that the tensor product construction for graded simple crystals is associative. This is true if the grading is ignored. It suffices to show that

$$D_{B_3 \otimes (B_2 \otimes B_1)} = D_{(B_3 \otimes B_2) \otimes B_1}.$$

By Proposition 2.11 and (2.39) we have

$$(2.40) \quad \begin{aligned} D_{B_3 \otimes (B_2 \otimes B_1)} &= H_{B_3, B_2 \otimes B_1} + D_{B_2 \otimes B_1} + D_3 \sigma_{B_3, B_2 \otimes B_1} \\ &= H_2 + H_1 \sigma_2 + H_1 + D_1 + D_2 \sigma_1 + D_3 \sigma_1 \sigma_2 \end{aligned}$$

and

$$(2.41) \quad \begin{aligned} D_{(B_3 \otimes B_2) \otimes B_1} &= H_{(B_3 \otimes B_2), B_1} + D_1 + D_{B_3 \otimes B_2} \sigma_{B_3 \otimes B_2, B_1} \\ &= H_1 + H_2 \sigma_1 + D_1 + (H_1 + D_2 + D_3 \sigma_1) \sigma_2 \sigma_1 \\ &= H_1 + H_2 \sigma_1 + H_1 \sigma_2 \sigma_1 + D_1 + D_2 \sigma_2 \sigma_1 + D_3 \sigma_2 \sigma_1 \sigma_2 \\ &= H_1 + H_2 \sigma_1 + H_1 \sigma_2 \sigma_1 + D_1 + D_2 \sigma_1 + D_3 \sigma_1 \sigma_2. \end{aligned}$$

In the second computation, (2.22) is used. Also used is $D_j \sigma_2 = D_j$, which holds because D_j acts on the rightmost tensor factor, which is not changed by σ_2 . One sees that (2.40) and (2.41) are equal by (2.24). \square

Let (B_j, D_j) be graded simple $U'_q(\mathfrak{g})$ -crystals for $1 \leq j \leq L$ and let $u_j = u(B_j)$. Let $B = B_L \otimes \cdots \otimes B_1$. Following [38] define the energy function $E_B : B \rightarrow \mathbb{Z}$ by

$$(2.42) \quad E_B = \sum_{1 \leq i < j \leq L} H_i \sigma_{i+1} \sigma_{i+2} \cdots \sigma_{j-1}.$$

By the normalization assumption (2.18) it follows that

$$(2.43) \quad E_B(u(B)) = 0.$$

The following formula was motivated by the definition of the D energy function in [7, Section 3.3].

Proposition 2.14. *The intrinsic energy D_B for the L -fold tensor product $B = B_L \otimes \cdots \otimes B_1$ is given by*

$$(2.44) \quad D_B = E_B + \sum_{j=1}^L D_j \sigma_1 \sigma_2 \cdots \sigma_{j-1}.$$

Proof. The proof proceeds by induction on L . For $L \leq 2$ (2.44) holds by definition. Suppose it holds for $L - 1$ tensor factors. Let $B'_2 = B_L$ and $B'_1 = B_{L-1} \otimes \cdots \otimes B_1$. Applying the definition (2.39) for $B'_2 \otimes B'_1$, induction, and Proposition 2.10, we have

$$\begin{aligned} D_B &= H_{B'_2, B'_1} + D_{B'_1} + D_{B'_2} \sigma_{B'_2, B'_1} \\ &= \sum_{i=1}^{L-1} H_i \sigma_{i+1} \sigma_{i+2} \cdots \sigma_{L-1} + E_{B'_1} \\ &\quad + \sum_{j=1}^{L-1} D_{B_j} \sigma_1 \sigma_2 \cdots \sigma_{j-1} + D_{B_L} \sigma_1 \cdots \sigma_{L-1} \end{aligned}$$

which is evidently the right-hand side of (2.44). \square

The energy functions are insensitive to reordering of tensor factors.

Proposition 2.15. *Let B' be a reordering of $B = B_L \otimes \cdots \otimes B_1$ and $\sigma : B \rightarrow B'$ any composition of local isomorphisms. Then $D_{B'}\sigma = D_B$ and $E_{B'}\sigma = E_B$.*

Proof. The proof immediately reduces to the case $\sigma = \sigma_j$. Let us prove the first assertion, as the second follows from it. Write $B'_3 = B_L \otimes \cdots \otimes B_{j+2}$, $B'_1 = B_{j-1} \otimes \cdots \otimes B_1$, $B'_2 = B_{j+1} \otimes B_j$ and $B''_2 = B_j \otimes B_{j+1}$. By (2.19) it follows that $D_{B'_2} \circ \sigma' = D_{B''_2}$ where $\sigma' : B_{j+1} \otimes B_j \rightarrow B_j \otimes B_{j+1}$. Therefore, $B'_2 \cong B''_2$ as graded simple crystals. Tensoring both on the left by B'_3 and on the right by B'_1 , the result is a pair of isomorphic graded simple crystals $B \cong B'$ via the map $\sigma = 1_{B'_3} \otimes \sigma' \otimes 1_{B'_1}$. In particular, $D_B = D_{B'} \circ \sigma$. This argument implicitly uses Proposition 2.13 and Remark 2.3. \square

2.15. The crystals $B^{r,s}$. We recall conjectures regarding the crystals $B^{r,s}$. Recall Chevalley’s partial order on \overline{P} , defined by $\mu \succeq \lambda$ if and only if $\mu - \lambda \in \bigoplus_{i \in J} \mathbb{N}\alpha_i$.

Conjecture 2.16 ([7]). *For each $r \in J$ and $s \geq 1$, there exists an irreducible finite dimensional integrable $U'_q(\mathfrak{g})$ -module $W_s^{(r)}$ with simple crystal basis $B^{r,s}$ having a unique extremal vector $u(B^{r,s})$ of weight $s\overline{\Lambda}_r$, and a prescribed $U_q(\tilde{\mathfrak{g}})$ -crystal decomposition of the form $B^{r,s} \cong B(s\overline{\Lambda}_r) \oplus B$, where B is a direct sum of $U_q(\tilde{\mathfrak{g}})$ -crystals of the form $B(\lambda)$ where $\lambda \in \overline{P}^+$ and $s\overline{\Lambda}_r \triangleright \lambda$. Moreover, there is a prescribed intrinsic energy function $D = D_{B^{r,s}} : B^{r,s} \rightarrow \mathbb{Z}$, that is constant on J -components, such that $0 = D(u(B^{r,s})) > D(b)$ and b is any element not in the J -component of $u(B^{r,s})$.*

Conjecture 2.17 ([19]). *$W_s^{(r)}$ is the universal $U'_q(\mathfrak{g})$ -module generated by an extremal vector of weight $s\overline{\Lambda}_r$.*

This is known to hold for $s = 1$ [19].

Conjecture 2.18 ([7]). *There is a unique $b^\natural \in B^{r,s}$ such that $\varphi(b^\natural) = \text{level}(B^{r,s})\Lambda_0$. Moreover,*

$$(2.45) \quad D_{B^{r,s}}(b) = H(b \otimes b^\natural) - H(u(B^{r,s}) \otimes b^\natural)$$

where $H = H_{B^{r,s}, B^{r,s}}$ is the local energy function.

Remark 2.19. Observe that the function $B^{r,s} \rightarrow \mathbb{Z}$ given by $b \mapsto H(b \otimes b^\natural)$ is constant on J -components. By definition $\varphi_i(b^\natural) = 0$ for $i \in J$. By (2.10), $\tilde{f}_i(b \otimes b^\natural) = \tilde{f}_i(b) \otimes b^\natural$ for all $b \in B^{r,s}$. The assertion follows from the fact that H is constant on J -components of $B^{r,s} \otimes B^{r,s}$, by Theorem 2.4.

2.16. One-dimensional sums. Let $\lambda \in \overline{P}$ and $B = B_L \otimes \cdots \otimes B_1$ where $B_j = B^{r_j, s_j}$ for $1 \leq j \leq L$. The set $\mathcal{P}(B, \lambda)$ of (classically restricted) paths in B of weight λ is the set of J -highest weight vectors in B of weight λ . Define the classically restricted one-dimensional sum

$$(2.46) \quad X(B, \lambda; q) = \sum_{b \in \mathcal{P}(B, \lambda)} q^{D_B(b)}.$$

3. CRYSTALS OF TYPE $A_n^{(1)}$

In this section we discuss the combinatorial properties of type $A_n^{(1)}$ in detail.

3.1. The root system A_{n-1} . In this section let $\tilde{\mathfrak{g}}$ be the complex simple Lie algebra of type A_{n-1} . Let $J = \{1, 2, \dots, n-1\}$ be the index set for the vertices of its Dynkin diagram. Let us realize the weight lattice \overline{P} explicitly inside the real span of the hyperplane in \mathbb{R}^n orthogonal to the vector $e = (1, 1, \dots, 1) \in \mathbb{R}^n$. Let $\{\varepsilon_i \mid 1 \leq i \leq n\}$ be the standard basis of \mathbb{R}^n . Then one may take $\alpha_i = \varepsilon_i - \varepsilon_{i+1}$ for $i \in J$. Identifying the weight lattice of $\mathfrak{gl}(n)$ with \mathbb{Z}^n , there is a projection $\text{wt}_{\text{sl}} : \mathbb{Z}^n \rightarrow \overline{P}$ defined by $v \mapsto v - \frac{(e,v)}{n}e$ where (\cdot, \cdot) is the standard bilinear form on \mathbb{R}^n . In coordinates it is given by $\text{wt}_{\text{sl}}(a_1, a_2, \dots, a_n) = \sum_{i=1}^{n-1} (a_i - a_{i+1})\overline{\Lambda}_i$. In particular, $\overline{\Lambda}_i = \text{wt}_{\text{sl}}(1^i, 0^{n-i})$ for $i \in J$. The highest root of $\tilde{\mathfrak{g}}$ is $\theta = (1, 0^{n-2}, -1)$. If λ is a partition $(\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n) \in \mathbb{N}^n$, then $\text{wt}_{\text{sl}}(\lambda) \in \overline{P}^+$. By abuse of notation an element of \mathbb{Z}^n is sometimes identified with its image under wt_{sl} .

3.2. Crystal graphs of type A_{n-1} . Recall that for $\lambda \in \overline{P}^+$, $B(\lambda)$ is the crystal of the irreducible integrable $U_q(\tilde{\mathfrak{g}})$ -module of highest weight λ . We now review the explicit realization of $B(\lambda)$ given in [20]. The vertices of $B(\lambda)$ are given by the set of (semistandard) tableaux of shape λ in the alphabet $[n] = \{1, 2, \dots, n\}$. The $U_q(\tilde{\mathfrak{g}})$ -crystal structure shall be defined by embedding tableaux into the set of words.

The crystal graph of $B(\overline{\Lambda}_1)$ (where we recall that $\overline{\Lambda}_1$ is the image under wt_{sl} of the partition $(1, 0^{n-1})$) is given by the set $[n]$ with $\tilde{f}_i(i) = i + 1$ for all $i \in J$ and all other elements sent to \emptyset . One defines $\text{wt}(i) = \overline{\Lambda}_i - \overline{\Lambda}_{i-1}$ for all $i \in [n]$.

The set of words in the alphabet $[n]$ is the crystal graph of the tensor algebra of $B(\overline{\Lambda}_1)$; its explicit structure is given by the signature rule (see section 2.3). The weight function wt on this crystal graph is given as follows. Let the content of a word u be the element $\text{cont}(u) = (m_1(u), m_2(u), \dots, m_n(u)) \in \mathbb{N}^n$ where $m_j(u)$ is the number of occurrences of the letter j in u . Then $\text{wt}(u) = \text{wt}_{\text{sl}}(\text{cont}(u)) \in \overline{P}$.

We identify the tableau $t \in B(\lambda)$ with its column-reading word, which is by definition $t = c_1 c_2 \dots$ where c_j is the strictly decreasing word given by the j -th column of t . This defines an embedding of $B(\lambda)$ into the set of words in the alphabet $[n]$. It is easy to see that \tilde{e}_i and \tilde{f}_i stabilize the image of this embedding. The $U_q(\tilde{\mathfrak{g}})$ -crystal structure on $B(\lambda)$ is defined by declaring this embedding to be a morphism of $U_q(\tilde{\mathfrak{g}})$ -crystals. The highest weight vector u_λ of $B(\lambda)$ is given explicitly by the Yamanouchi tableau, the tableau of shape λ whose i -th row is filled with the letter i for all i .

3.3. Schensted's insertion. Schensted [40] gave an insertion algorithm that associates to a word u a tableau $\mathbb{P}(u)$ of partition shape. Given an alphabet (totally ordered set) A , let \sim be the equivalence relation on words in the alphabet A defined by $u \sim v$ if and only if $\mathbb{P}(u) = \mathbb{P}(v)$. Let $\text{Plac}(A)$ denote the set of \sim -classes. It follows from [40] that $\mathbb{P}(\mathbb{P}(u)\mathbb{P}(v)) = \mathbb{P}(uv)$, so that the multiplication on words given by juxtaposition, descends to a multiplication in $\text{Plac}(A)$. $\text{Plac}(A)$ is called the plactic monoid. Knuth [27] showed that the set of relations in $\text{Plac}(A)$ is generated by relations of the following form, where $x, y, z \in A$ and u, v are words:

$$(3.1) \quad \begin{aligned} uxzyv &\sim uzxyv && \text{for } x \leq y < z, \\ uyzxv &\sim uyxzv && \text{for } x < y \leq z. \end{aligned}$$

A column word is one that is strictly decreasing.

Let $B \subseteq A$ be a subinterval. For a word u in the alphabet A , denote by $u|_B$ the subword of u obtained by erasing all of the letters not in B . It follows immediately from the relations (3.1) that if $u \sim v$, then $u|_B \sim v|_B$.

It is a well-known and remarkable fact that Schensted’s algorithm gives a morphism of type A_{n-1} crystal graphs.

Theorem 3.1. *For all words u in the alphabet $[n]$ and all $i \in J$, $\mathbb{P}(\tilde{e}_i(u)) = \tilde{e}_i\mathbb{P}(u)$.*

Proof. It suffices to show that if $u \sim v$ is one of the relations (3.1), then $e_i(u) \sim e_i(v)$ is also. The result is entirely straightforward unless $\{x, y, z\} = \{i, i + 1\}$ and \tilde{e}_i changes one of x, y, z . In this case (ignoring identical left and right factors) one has $(i + 1)(i + 1)i \sim (i + 1)i(i + 1)$. Applying \tilde{e}_i to both sides one obtains $i(i + 1)i \sim (i + 1)ii$. \square

3.4. The crystals $B^{r,s}$ of type $A_{n-1}^{(1)}$. Let \mathfrak{g} be the affine algebra of type $A_{n-1}^{(1)}$. Write $I = J \cup \{0\} = \{0, 1, \dots, n - 1\}$.

Theorem 3.2 ([14]). *For every $r \in J$ and $s \geq 1$ there is a finite dimensional integrable $U_q(A_{n-1}^{(1)})$ -module $W_s^{(r)}$ with crystal basis $B^{r,s}$ such that*

1. $B^{r,s} \cong B(s\bar{\Lambda}_r)$ as $U_q(A_{n-1})$ -crystals.
2. $D_{B^{r,s}} = 0$.
3. $B^{r,s}$ is perfect of level s .

Moreover, let ψ be the rotation of the Dynkin diagram of type $A_{n-1}^{(1)}$ that sends the simple root α_i to α_{i+1} where subscripts are taken modulo n . Then there is a unique bijection $\psi : B^{r,s} \rightarrow B^{r,s}$ such that

$$(3.2) \quad \psi^n = 1,$$

$$(3.3) \quad \text{wt} \circ \psi = \psi \circ \text{wt},$$

$$(3.4) \quad \tilde{f}_i = \psi^{-1} \circ \tilde{f}_{i+1} \circ \psi,$$

$$(3.5) \quad \tilde{e}_i = \psi^{-1} \circ \tilde{e}_{i+1} \circ \psi$$

where $\psi : \bar{P} \rightarrow \bar{P}$ is induced by the map $\mathbb{Z}^n \rightarrow \mathbb{Z}^n$ given by $(a_1, a_2, \dots, a_n) \mapsto (a_2, a_3, \dots, a_n, a_1)$.

The operators \tilde{e}_0 and \tilde{f}_0 are computed explicitly in [44]. It suffices to define ψ^{-1} . For $t \in B^{r,s}$ define $\psi^{-1}(t) \in B^{r,s}$ by the property

$$(3.6) \quad \psi^{-1}(t)|_{[n-1]} = \mathbb{P}(t|_{[2,n]}) - 1$$

where $t + r$ is the tableau obtained by adding the integer r to each letter. This definition determines the positions in $\psi^{-1}(t)$ of the letters in the subinterval $[n - 1]$. But it also determines the positions of the letters n as they must fill up the rest of the rectangular shape.

Example 3.3. Let $r = 3, s = 4, n = 5$. Then a tableau $t \in B^{r,s}$ and $\psi^{-1}(t)$ are given below:

$$\begin{aligned}
 t &= \begin{array}{|c|c|c|c|} \hline 1 & 1 & 2 & 3 \\ \hline 2 & 2 & 3 & 4 \\ \hline 3 & 4 & 5 & 5 \\ \hline \end{array} & t|_{[2,n]} &= \begin{array}{|c|c|c|c|} \hline & & 2 & 3 \\ \hline 2 & 2 & 3 & 4 \\ \hline 3 & 4 & 5 & 5 \\ \hline \end{array} & \mathbb{P}(t|_{[2,n]}) &= \begin{array}{|c|c|c|c|} \hline 2 & 2 & 2 & 3 \\ \hline 3 & 3 & 4 & 5 \\ \hline 4 & 5 & & \\ \hline \end{array} \\
 \\
 \mathbb{P}(t|_{[2,n]}) - 1 &= \begin{array}{|c|c|c|c|} \hline 1 & 1 & 1 & 2 \\ \hline 2 & 2 & 3 & 4 \\ \hline 3 & 4 & & \\ \hline \end{array} & \psi^{-1}(t) &= \begin{array}{|c|c|c|c|} \hline 1 & 1 & 1 & 2 \\ \hline 2 & 2 & 3 & 4 \\ \hline 3 & 4 & 5 & 5 \\ \hline \end{array}
 \end{aligned}$$

Since $B^{r,s}$ is perfect of level s , there is a unique $b^{\natural} \in B^{r,s}$ such that $\varphi(b^{\natural}) = s\Lambda_0$; it is the tableau in $B^{r,s}$ whose i -th row from the bottom consists of s copies of the value $n + 1 - i$. Since $B^{r,s}$ consists of a single J -component, Conjecture 2.18 holds by Remark 2.19.

3.5. Inhomogeneous paths. Let $R = (R_1, R_2, \dots, R_L)$ be a sequence of rectangular partitions. Say that R_j has r_j rows and s_j columns where $s_j \geq 1$ and $1 \leq r_j \leq n - 1$ for all $1 \leq j \leq L$. Define the $U'_q(\mathfrak{g})$ -crystal

$$(3.7) \quad B_R = B^{r_L, s_L} \otimes \dots \otimes B^{r_2, s_2} \otimes B^{r_1, s_1}.$$

The elements of B_R are called inhomogeneous paths. Of course one could compute \tilde{e}_0 and \tilde{f}_0 on B_R using the signature rule and the above rule for \tilde{e}_0 and \tilde{f}_0 on single tensor factors, but it is simpler to compute them as follows. Let $\psi : B_R \rightarrow B_R$ be the bijection given by $\psi(b_L \otimes \dots \otimes b_1) = \psi(b_L) \otimes \dots \otimes \psi(b_1)$. Then \tilde{e}_0 and \tilde{f}_0 as operators on B_R may be defined by the equations (3.5) and (3.4).

3.6. Simplicity of tensor products of $B^{r,s}$. Consider the finite dimensional integrable $U'_q(\mathfrak{g})$ -module $W_s^{(r)}$ with crystal basis $B^{r,s}$ (see Theorem 3.2). We observe that the crystal $B^{r,s}$ is simple (see section 2.9). Let $\lambda = s\bar{\Lambda}_r$, with corresponding partition $(s^r, 0^{n-r}) \in \mathbb{Z}^n$. It is not hard to check that the extremal vectors of $B^{r,s}$ are in bijection with the orbit $\overline{W}\lambda$. In the notation of section 2.7, for every $w \in \overline{W}$, let b_w be the unique tableau in $B^{r,s}$ of content $w(s^r, 0^{n-r}) \in \mathbb{Z}^n$. This shows that $B^{r,s}$ is simple. The crystal B_R of the previous section is a tensor product of simple crystals. So B_R is simple and hence connected.

3.7. Dual crystals. Let $N \geq \lambda_1$ and $\lambda^\vee = (N - \lambda_n, \dots, N - \lambda_2, N - \lambda_1)$. For a column word c in the alphabet $[n]$, let c^\vee be the column word that uses precisely those letters that are not in c . For a sequence of column words $u = c_1 c_2 \dots c_N$, define

$$(3.8) \quad u^\vee = c_N^\vee \dots c_1^\vee.$$

Proposition 3.4. *There is a unique $U_q(\overline{\mathfrak{g}})$ -crystal isomorphism $B(\lambda)^\vee \rightarrow B(\lambda^\vee)$.*

Proof. It is easy to see that the map $B(\lambda) \rightarrow B(\lambda^\vee)$ given by $t \mapsto t^\vee$ (where t^\vee is defined in (3.8)) is a well-defined bijection that satisfies (2.12). For uniqueness, by definition the map \vee takes highest weight vectors to lowest weight vectors. But $B(\lambda)$ has a unique highest weight vector and $B(\lambda^\vee)$ has a unique lowest weight vector. \square

Proposition 3.5. *Let $u = c_1 \dots c_N$ where each c_j is a column word. Then $\mathbb{P}(u^\vee) = \mathbb{P}(u)^\vee$.*

Proof. In the statement, $\mathbb{P}(u)^\vee$ is computed with respect to a factorization into N column words, the last several of which may be empty. It suffices to show that if u, u', v, v' are column words such that $uv \sim u'v'$ with u' one letter longer than u , then $v^\vee u^\vee \sim v'^\vee u'^\vee$. This reduction, together with the proof of this special case, can both be seen by considering a jeu de taquin on skew tableaux having at most two columns. \square

Example 3.6. Let $n = 6$. One has $uv = (42)(6521) \sim (642)(521) \sim (6421)(52) = \mathbb{P}(uv)$ and dually $(uv)^\vee = (43)(6531) \sim (643)(531) \sim (6431)(53) = \mathbb{P}(uv)^\vee$.

Proposition 3.7. *The $U'_q(\mathfrak{g})$ -crystal $B^{r,s}$ has dual crystal $B^{n-r,s}$.*

Proof. Observe that as $U_q(\overline{\mathfrak{g}})$ -crystals one has $(B^{r,s})^\vee \cong B^{n-r,s}$ using $N = s$ columns. By the uniqueness in Proposition 3.4 it follows that the crystals are also dual as $U'_q(\mathfrak{g})$ -crystals by the same map. \square

3.8. Reversing the Dynkin diagram.

Theorem 3.8. *Let B be the crystal basis of a finite dimensional $U_q(\overline{\mathfrak{g}})$ -module (resp. $U'_q(\mathfrak{g})$ -module). Then there is an involution $B \rightarrow B$ denoted $b \mapsto b^*$, such that*

$$(3.9) \quad \begin{aligned} \tilde{f}_i(b^*) &= \tilde{e}_{n-i}(b)^*, \\ \tilde{e}_i(b^*) &= \tilde{f}_{n-i}(b)^*, \\ \text{wt}(b^*) &= w_0 \text{wt}(b) \end{aligned}$$

for all $i \in J$ (resp. $i \in I$ with subscripts taken modulo n) where w_0 is the longest element of \overline{W} .

Proof. Consider the orientation-reversing automorphism of the Dynkin diagram of type $A_{n-1}^{(1)}$ that sends α_i to α_{n-i} for all i with subscripts taken modulo n . The existence of this automorphism implies the existence of a map $*$: $B \rightarrow B$ satisfying (3.9). Since the Dynkin diagram automorphism is an involution, so is the induced map $*$. \square

The following result is easily verified.

Proposition 3.9. *Suppose there are crystal graphs B_j and bijections $*$: $B_j \rightarrow B_j$ satisfying (3.9). Let $B = B_L \otimes \cdots \otimes B_1$ and $B^* = B_1 \otimes \cdots \otimes B_L$. Then the map $*$: $B \rightarrow B^*$ given by*

$$b_L \otimes \cdots \otimes b_1 \mapsto b_1^* \otimes \cdots \otimes b_L^*$$

satisfies (3.9).

Given a word u , let u^* be the word obtained by replacing each letter i by $n+1-i$, and reversing the resulting word. Clearly, if u is a column word, then so is u^* . If $u = c_1 c_2 \dots c_N$ where c_j is a column word for all j , then by definition $u^* = c_N^* \dots c_1^*$, which is a sequence of column words. In particular, let $t \in B(\lambda)$. It is easy to check that t^* is a skew tableau whose shape is given by the 180 degree rotation of λ . Define $t^{\text{ev}} = \mathbb{P}(t^*)$. The tableau t^{ev} is called the evacuation [32] of the tableau t . Observe that if λ is a rectangle, then $t^{\text{ev}} = t^*$.

Proposition 3.10.

1. *The map $u \mapsto u^*$ is an involution satisfying (3.9).*
2. *For any word u , $\mathbb{P}(u^*) = \mathbb{P}(u)^{\text{ev}}$.*

3. The map $B(\lambda) \rightarrow B(\lambda)$ given by $t \mapsto t^{\text{ev}}$ is the unique map satisfying (3.9) (in which $*$ is replaced by ev).

Proof. Part 1 is straightforward. For part 2 it suffices to show that if $u \sim v$ is a relation of the form (3.1), then so is $u^* \sim v^*$, but this is also straightforward. Part 3 follows from part 2, the connectedness of $B(\lambda)$, and the fact that \mathbb{P} is a morphism of crystal graphs. \square

We remark that the uniqueness in part 3, together with Theorem 3.8, imply that $t \mapsto t^{\text{ev}}$ is an involution.

3.9. Combining duality and Dynkin reversal.

Proposition 3.11. *There is a bijection $B(\lambda) \rightarrow B(\lambda^\vee)$ given by $t \mapsto t^{\text{ev}\vee} = t^{\vee\text{ev}}$ such that*

$$(3.10) \quad \begin{aligned} \tilde{f}_i(t^{\text{ev}\vee}) &= \tilde{f}_{n-i}(t)^{\text{ev}\vee}, \\ \tilde{e}_i(t^{\text{ev}\vee}) &= \tilde{e}_{n-i}(t)^{\text{ev}\vee}, \\ \text{wt}(t^{\text{ev}\vee}) &= -w_0(\text{wt}(t)) \end{aligned}$$

for all $t \in B(\lambda)$ and $i \in J$. Moreover, if λ is a rectangle, then (3.10) also holds for $i = 0$.

Proof. \vee and $*$ obviously commute on sequences of column words. Proposition 3.5 implies that $t^{\text{ev}\vee} = t^{\vee\text{ev}}$ for all $t \in B(\lambda)$. Equation (3.10) follows from (2.12) and Theorem 3.8. \square

A similar statement holds for tensor products.

3.10. Local isomorphism. Let $B_j = B^{r_j, s_j}$ for $j = 1, 2$. The existence and uniqueness of the local isomorphism $\sigma : B_2 \otimes B_1 \rightarrow B_1 \otimes B_2$ (see section 2.11) is guaranteed by the simplicity of $B^{r, s}$ (see section 3.6). Observe that as a $U_q(\bar{\mathfrak{g}})$ -crystal $B_2 \otimes B_1$ is multiplicity-free, since $B_j \cong B(s_j \bar{\Lambda}_j)$ is indexed by a rectangular partition for $j = 1, 2$ [45]. Therefore, the isomorphism $\sigma : B_2 \otimes B_1 \rightarrow B_1 \otimes B_2$ is uniquely specified by the property that $\sigma(b_2 \otimes b_1)$ is the unique element $b'_1 \otimes b'_2 \in B_1 \otimes B_2$ such that $\mathbb{P}(b_2 b_1) = \mathbb{P}(b'_1 b'_2)$. The bijection σ is described quite explicitly in [42].

Example 3.12. Let

$$b_2 = \begin{array}{|c|c|} \hline 1 & 2 \\ \hline 3 & 3 \\ \hline \end{array} \quad \text{and} \quad b_1 = \begin{array}{|c|c|} \hline 1 & 1 \\ \hline \end{array}$$

Then

$$\mathbb{P}(b_2 b_1) = \begin{array}{|c|c|c|} \hline 1 & 1 & 1 \\ \hline 2 & 3 & \\ \hline 3 & & \\ \hline \end{array}$$

Hence

$$b'_1 = \begin{array}{|c|c|} \hline 1 & 3 \\ \hline \end{array} \quad \text{and} \quad b'_2 = \begin{array}{|c|c|} \hline 1 & 1 \\ \hline 2 & 3 \\ \hline \end{array}$$

Proposition 3.13. *The following diagram commutes:*

$$(3.11) \quad \begin{array}{ccc} B_2 \otimes B_1 & \xrightarrow{\sigma} & B_1 \otimes B_2 \\ \downarrow \vee & & \downarrow \vee \\ B_1^\vee \otimes B_2^\vee & \xrightarrow{\sigma} & B_2^\vee \otimes B_1^\vee \end{array}$$

Proof. Let $b_j \in B_j$ for $j = 1, 2$ and $\sigma(b_2 \otimes b_1) = b'_1 \otimes b'_2$. Then

$$\mathbb{P}((b_2 b_1)^\vee) = \mathbb{P}(b_2 b_1)^\vee = \mathbb{P}(b'_1 b'_2)^\vee = \mathbb{P}((b'_1 b'_2)^\vee)$$

by the definition of σ and Proposition 3.5. This suffices by the definition of σ . \square

Proposition 3.14. *The following diagram commutes:*

$$(3.12) \quad \begin{array}{ccc} B_2 \otimes B_1 & \xrightarrow{\sigma} & B_1 \otimes B_2 \\ * \downarrow & & \downarrow * \\ B_1 \otimes B_2 & \xrightarrow{\sigma} & B_2 \otimes B_1 \end{array}$$

Proof. Let b_j, b'_j be as before for $j = 1, 2$.

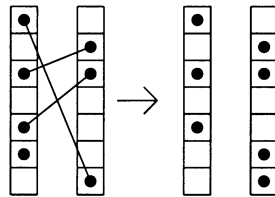
$$\mathbb{P}((b_2 b_1)^*) = \mathbb{P}(b_2 b_1)^{\text{ev}} = \mathbb{P}(b'_1 b'_2)^{\text{ev}} = \mathbb{P}((b'_1 b'_2)^*)$$

by the definition of σ and Proposition 3.10. This suffices by the definition of σ . \square

3.11. Local isomorphism for single columns. In the case that $B_j = B^{r_j,1}$ for $j = 1, 2$ there is an explicit construction for the local isomorphism σ which will be useful later. For $\bar{\mathfrak{g}} = A_{n-1}$, represent $b \in B^{r,1}$ by a column of height n with a dot at height i if and only if the letter i occurs in b . Now $\sigma : B^{r_2,1} \otimes B^{r_1,1} \rightarrow B^{r_1,1} \otimes B^{r_2,1}$ with $r_2 \geq r_1$ is obtained as follows [38]. Let $b_2 \otimes b_1 \in B_2 \otimes B_1$ and $b'_1 \otimes b'_2 = \sigma(b_2 \otimes b_1)$.

1. Pick the highest dot in b_1 . Call it \bullet_a . Connect \bullet_a with the highest dot in b_2 not higher than \bullet_a (assuming periodic boundary conditions if necessary).
2. Repeat this with all unconnected dots in b_1 .
3. Slide all $(r_2 - r_1)$ unpaired dots from b_2 to b_1 . The result is $b'_1 \otimes b'_2$.

Example 3.15. Let $n = 7$, $b_2 = 7532$ and $b_1 = 651$. Then



so that $b'_1 = 753$ and $b'_2 = 6521$.

3.12. Energy function. Explicit formulas may be given for the energy functions in type A . Let $B_j = B^{r_j, s_j}$ with J -highest weight vector u_j for $j = 1, 2$. Given the normalization (2.18) one has the explicit formula [44, 41]

$$(3.13) \quad H(b_2 \otimes b_1) = -\min(r_1, r_2) \min(s_1, s_2) + d_{\max\{s_1, s_2\}}(\text{shape}(\mathbb{P}(b_2 b_1)))$$

where $d_s(\lambda)$ is the number of cells of λ that lie in columns strictly to the right of the s -th column. It is not hard to see that the values of H are nonpositive integers.

Remark 3.16. In [44, 41] a different normalization was used, namely, only the d function was present.

Let $R = (R_1, \dots, R_L)$ and $b = b_L \otimes \dots \otimes b_1 \in B_R$ where B_R is defined in (3.7). Write D_R and E_R for D_{B_R} and E_{B_R} in (2.44) and (2.42). Since $D_{B^{r,s}} = 0$,

$$(3.14) \quad D_R = E_R.$$

Proposition 3.17. *For all $b \in B_R$,*

$$(3.15) \quad E_{R^{\vee*}}(b^{\vee*}) = E_R(b).$$

Proof. Let $B = B_R$, B' be any reordering of the tensor factors of B_R , and $\sigma : B \rightarrow B'$ any composition of local isomorphisms. By Propositions 3.13 and 3.14,

$$(3.16) \quad \sigma(b^{\vee*}) = \sigma(b)^{\vee*}.$$

Using this one may reduce to the case $R = (R_1, R_2)$. The crystal B_R is connected (see section 3.6). It then suffices to check (3.15) for the single element $u_2 \otimes u_1$ where u_j is the J -highest weight vector of B_j , by (3.16) and Proposition 3.11. Again by Proposition 3.11 $(u_2 \otimes u_1)^{\vee*}$ is a J -highest weight vector of the same weight (namely $-w_0(\text{wt}(u_2 \otimes u_1))$) as $v_2 \otimes v_1$ where v_j is the unique J -highest weight vector in B_j^\vee . But there is only one such J -highest weight vector in $B_{R^{\vee*}}$ of this weight, so the two vectors must agree. It follows that both sides of (3.15) evaluate to zero. \square

3.13. Kostka polynomials. Let λ be a partition (or an element of \overline{P}^+ via wt_{sl} as in section 3.1), $R = (R_1, \dots, R_L)$ a sequence of rectangles and B_R as in (3.7). Define

$$(3.17) \quad K_{\lambda,R}(q) = q^{\|R\|} X(B_R, \lambda; q),$$

$$(3.18) \quad \tilde{K}_{\lambda,R}(q) = X(B_R, \lambda; q^{-1})$$

with $X(B_R, \lambda; q)$ as in (2.46) and

$$\|R\| = \sum_{1 \leq i < j \leq L} \min(r_i, r_j) \min(s_i, s_j).$$

It was shown in [38] (see also [44, 41]) that $K_{\lambda,R}(q)$ is the Kostka polynomial $K_{\lambda\mu}(q)$ if R is a sequence of single rowed partitions where $r_i = 1$ and $s_i = \mu_i$. Hence we call $K_{\lambda,R}(q)$ the generalized *Kostka polynomial*. Both $K_{\lambda,R}(q)$ and $\tilde{K}_{\lambda,R}(q)$ have nonnegative integer coefficients.

4. LITTLEWOOD–RICHARDSON TABLEAUX

4.1. Definition of Littlewood–Richardson tableaux. The J -components of B_R are parametrized by the combinatorial objects called Littlewood–Richardson (LR) tableaux. These objects, being in the multiplicity space, should be considered a completely different kind of object than the elements of B_R . It is a beautiful coincidence in type A that operators that act on the representation space such as crystal operators and duality, also act in the multiplicity space in certain situations.

We recall the notion of an R -LR word [42], which can be formulated as follows. Let $(\eta_1, \eta_2, \dots, \eta_L) \in \mathbb{N}^L$ be a sequence of positive integers summing to N , $A = \{1 < 2 < \dots < N\}$, A_1 the subinterval of A given by the first η_1 numbers, A_2 the next η_2 numbers, etc. Say that a word u in the alphabet A is η -balanced if $\tilde{e}_i(u) = \tilde{f}_i(u) = 0$ for all $i \in A$ that are not maximum in one of the subintervals A_j .

Let $R = (R_1, \dots, R_L)$ be a sequence of rectangular partitions such that R_j has η_j rows and μ_j columns. Let $\gamma(R) = (\mu_1^{\eta_1}, \dots, \mu_L^{\eta_L}) \in \mathbb{N}^N$ be obtained by juxtaposing the parts of the R_j . Define the set $W(R)$ of R -LR words by $u \in W(R)$ if and only if u has content $\gamma(R)$ and u is η -balanced. This definition is equivalent to the one in [42].

Conversely, for an η -balanced word u , let $\gamma \in \mathbb{N}^N$ be the content of u . Then $\gamma_i = \gamma_{i'}$ for all i, i' in the same subalphabet A_j . Let μ_j be this common value for the subalphabet A_j and let R_j be the rectangular partition having η_j rows and μ_j columns. Then u is an R -LR word.

Let $\text{LR}(R)$ be the subset of $W(R)$ consisting of tableaux of partition shape (via the identification of a tableau with its column-reading word) and $\text{LR}(\lambda, R)$ the subset of $\text{LR}(R)$ consisting of tableaux of partition shape λ . The set $\text{LR}(\lambda, R)$ is empty unless λ has at most N parts.

4.2. Recording tableaux. The way in which the LR tableaux parametrize the multiplicity space of B_R , is by suitable recording tableaux for Schensted's insertion.

Consider a given factorization $u = c_N \cdots c_2 c_1$ of u where c_j is a column word in the alphabet $[n]$. Let $Q = \mathbb{Q}(c_N \cdots c_2 c_1)$ be the unique filling of the Ferrers diagram of the shape of $\mathbb{P}(c_N \cdots c_2 c_1)$ such that the shape of $Q|_{[j]}$ is the shape of $\mathbb{P}(c_j \cdots c_1)$ for all j . Since each c_j is a column word it follows that the transpose Q^t of Q is a semistandard tableau. In more traditional language, Q is the recording tableau for the column insertion of the word $c_N \cdots c_2 c_1$ such that the insertion of the letters in c_j are recorded by the letter j .

Fix $R = (R_1, \dots, R_L)$ and let $R^t = (R_1^t, \dots, R_L^t)$ where t denotes transpose. Let n be a positive integer such that $n \geq \mu_j$ for all j . One may regard elements of B_{R^t} as tableaux of a fixed skew shape in the alphabet $[n]$. The following result, up to labeling of the recording tableaux, is a special case of the main theorem in [46].

Proposition 4.1. *There is a bijection $B_{R^t} \rightarrow \bigcup_{\lambda} B(\lambda^t) \times \text{LR}(\lambda, R)$ given by $b \mapsto (\mathbb{P}(b), \mathbb{Q}(b)^t)$ where λ runs over partitions with $\lambda_1 \leq n$.*

Recall that the operation $^\vee$ was defined for tableaux t in the alphabet $[n]$ with at most N columns and for partitions with at most n rows and N columns. We define an operation $^\wedge$ which is a similar kind of operation but with n and N exchanged.

Let λ^\wedge be the partition obtained by skewing the rectangle (n^N) by λ and rotating 180 degrees, or equivalently $\lambda_i^\wedge = n - \lambda_{N+1-i}$. Let $R^\wedge = (R_1^\wedge, \dots, R_L^\wedge)$ where $R_j^\wedge = ((n - \mu_j)^{\eta_j})$ for each j . This complements the widths of the rectangles in R .

For a sequence u of n column words in the alphabet $[N]$, we write u^\wedge for the dual sequence of column words instead of u^\vee .

Proposition 4.2.

$$\mathbb{Q}((c_N \cdots c_2 c_1)^{\vee*})^t = \mathbb{Q}(c_N \cdots c_2 c_1)^{t^\wedge}.$$

Proof. Let Q_1 and Q_2 be the tableaux on the left- and right-hand sides. Let v be obtained from u by replacing c_N by \emptyset and let Q'_1 and Q'_2 be the corresponding tableaux for v . By induction on the number of nonempty factors, $Q'_1 = Q'_2$. By definition the first $N - 1$ column words of $u^{\vee*}$ and $v^{\vee*}$ agree. To conclude $Q_1 = Q_2$ it suffices to show:

1. $Q'_i = Q_i|_{[N-1]}$ for $i = 1, 2$.
2. Q_1 and Q_2 have the same shape.

Part 1 holds by the definition of \mathbb{Q} . For part 2 one has

$$\begin{aligned} \text{shape}(\mathbb{Q}(u^{\vee*})^t) &= \text{shape}(\mathbb{Q}(u^{\vee*}))^t = \text{shape}(\mathbb{P}(u^{\vee*}))^t \\ &= \text{shape}(\mathbb{P}(u^\vee)^{\text{ev}})^t = \text{shape}(\mathbb{P}(u^\vee))^t = \text{shape}(\mathbb{P}(u)^\vee)^t \\ &= \text{shape}(\mathbb{P}(u))^{\vee t} = \text{shape}(\mathbb{P}(u))^{t^\wedge} = \text{shape}(\mathbb{Q}(u))^{t^\wedge} \\ &= \text{shape}(\mathbb{Q}(u)^{t^\wedge}) \end{aligned}$$

by the fact that the \mathbb{P} and \mathbb{Q} tableaux of a given word have the same shape, Propositions 3.10 and 3.5, the fact that ev is shape-preserving, and the rule for how \vee and \wedge change shapes. \square

Proposition 4.3. *There is a bijection $\text{LR}(\lambda, R) \rightarrow \text{LR}(\lambda^\wedge, R^\wedge)$ given by $t \mapsto t^\wedge$.*

Proof. Let $t \in \text{LR}(\lambda, R)$. Then t^\wedge is a tableau of shape λ^\wedge by Proposition 3.4. It has the correct content to be R^\wedge -LR by definition since the corresponding rectangles in R and R^\wedge have the same heights. Since t is η -balanced, t^\wedge is also by (2.12). Thus the desired map is well defined. Since \wedge is an involution the map is bijective. \square

Example 4.4. Let $\mu = (3, 2, 1)$ and $\eta = (2, 3, 1)$ and $n = 5$. Then $N = 6$ and R and R^\wedge (whose rectangles have their rows are filled with the letters of their corresponding subintervals) are given by

| | | | | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|--|---|---|--|---|---|--|---|--|--|
| R | | | | | | | | | | | | | | | | | | |
| <table border="1" style="border-collapse: collapse; margin: auto;"> <tr><td style="padding: 2px 5px;">1</td><td style="padding: 2px 5px;">1</td><td style="padding: 2px 5px;">1</td></tr> <tr><td style="padding: 2px 5px;">2</td><td style="padding: 2px 5px;">2</td><td style="padding: 2px 5px;">2</td></tr> <tr><td style="padding: 2px 5px;">3</td><td style="padding: 2px 5px;">3</td><td></td></tr> <tr><td style="padding: 2px 5px;">4</td><td style="padding: 2px 5px;">4</td><td></td></tr> <tr><td style="padding: 2px 5px;">5</td><td style="padding: 2px 5px;">5</td><td></td></tr> <tr><td style="padding: 2px 5px;">6</td><td></td><td></td></tr> </table> | 1 | 1 | 1 | 2 | 2 | 2 | 3 | 3 | | 4 | 4 | | 5 | 5 | | 6 | | |
| 1 | 1 | 1 | | | | | | | | | | | | | | | | |
| 2 | 2 | 2 | | | | | | | | | | | | | | | | |
| 3 | 3 | | | | | | | | | | | | | | | | | |
| 4 | 4 | | | | | | | | | | | | | | | | | |
| 5 | 5 | | | | | | | | | | | | | | | | | |
| 6 | | | | | | | | | | | | | | | | | | |

| | | | | | | | | | | | | | | | | | | | |
|--|---|---|---|---|---|--|---|---|---|---|---|---|---|---|---|---|---|---|---|
| R^\wedge | | | | | | | | | | | | | | | | | | | |
| <table border="1" style="border-collapse: collapse; margin: auto;"> <tr><td style="padding: 2px 5px;">1</td><td style="padding: 2px 5px;">1</td><td></td></tr> <tr><td style="padding: 2px 5px;">2</td><td style="padding: 2px 5px;">2</td><td></td></tr> <tr><td style="padding: 2px 5px;">3</td><td style="padding: 2px 5px;">3</td><td style="padding: 2px 5px;">3</td></tr> <tr><td style="padding: 2px 5px;">4</td><td style="padding: 2px 5px;">4</td><td style="padding: 2px 5px;">4</td></tr> <tr><td style="padding: 2px 5px;">5</td><td style="padding: 2px 5px;">5</td><td style="padding: 2px 5px;">5</td></tr> <tr><td style="padding: 2px 5px;">6</td><td style="padding: 2px 5px;">6</td><td style="padding: 2px 5px;">6</td><td style="padding: 2px 5px;">6</td></tr> </table> | 1 | 1 | | 2 | 2 | | 3 | 3 | 3 | 4 | 4 | 4 | 5 | 5 | 5 | 6 | 6 | 6 | 6 |
| 1 | 1 | | | | | | | | | | | | | | | | | | |
| 2 | 2 | | | | | | | | | | | | | | | | | | |
| 3 | 3 | 3 | | | | | | | | | | | | | | | | | |
| 4 | 4 | 4 | | | | | | | | | | | | | | | | | |
| 5 | 5 | 5 | | | | | | | | | | | | | | | | | |
| 6 | 6 | 6 | 6 | | | | | | | | | | | | | | | | |

Let $\lambda = (5, 4, 3, 1, 0, 0)$. Then $\lambda^\wedge = (5, 5, 4, 2, 1, 0)$. Below is a tableau $t \in \text{LR}(\lambda, R)$ and its dual $t^\wedge \in \text{LR}(\lambda^\wedge, R^\wedge)$.

| | | | | | | | | | | | | | | | | | | | | |
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4.3. Generalized automorphisms of conjugation. Given $1 \leq p \leq L - 1$, let $\tau_p R$ be obtained from R by exchanging the p -th and $(p + 1)$ -st rectangles. There is a bijection [42] given by the generalized automorphisms of conjugation $\tau_p : \text{LR}(\lambda, R) \rightarrow \text{LR}(\lambda, \tau_p R)$. It extends uniquely to a bijection $\tau_p : W(R) \rightarrow W(\tau_p R)$ by $\mathbb{P}(\tau_p u) = \tau_p \mathbb{P}(u)$ and $\mathbb{Q}(\tau_p u) = \mathbb{Q}(u)$.

The bijections τ_p and the combinatorial R -matrices σ_p are related as follows (which may also be used as the definition of τ_p).

Proposition 4.5. *For any $b \in B_{R^t}$ and $1 \leq p \leq L - 1$, $\mathbb{Q}(\sigma_p(b))^t = \tau_p \mathbb{Q}(b)^t$.*

Proposition 4.6. *The following diagram commutes:*

$$(4.1) \quad \begin{array}{ccc} \mathrm{LR}(\lambda, R) & \xrightarrow{\tau_p} & \mathrm{LR}(\lambda, \tau_p R) \\ \wedge \downarrow & & \downarrow \wedge \\ \mathrm{LR}(\lambda^\wedge, R^\wedge) & \xrightarrow{\tau_p} & \mathrm{LR}(\lambda^\wedge, \tau_p R^\wedge) \end{array}$$

Proof. Let $Q \in \mathrm{LR}(R)$. By Proposition 4.1 there is a $b \in B_{R^t}$ such that $\mathbb{Q}(b)^t = Q$. Then

$$\begin{aligned} \tau_p(Q^\wedge) &= \tau_p(\mathbb{Q}(b)^{t^\wedge}) = \tau_p(\mathbb{Q}(b^{\vee*})^t) = \mathbb{Q}(\sigma_p(b^{\vee*}))^t \\ &= \mathbb{Q}(\sigma_p(b)^{\vee*})^t = \mathbb{Q}(\sigma_p(b))^{t^\wedge} = \tau_p(\mathbb{Q}(b)^t)^\wedge = \tau_p(Q)^\wedge \end{aligned}$$

by Propositions 4.2, 4.5, 3.13 and 3.14. \square

4.4. Embeddings. There are embeddings $\theta_R : \mathrm{LR}(\lambda, R) \rightarrow \mathrm{LR}(\lambda, \mathrm{rows}(R))$ where $\mathrm{rows}(R)$ is the sequence of single-rowed shapes of sizes given by $\gamma(R)$; they are defined as compositions of two kinds of elementary embeddings [43, 41]. The second kind of elementary embedding is given by the τ_p . The first kind is given as follows. Suppose that $R_1 = (k^a)$ and $R_2 = (k^b)$ where $a - 1 \geq b + 1$. Write $R' = ((k^{a-1}), (k^{b+1}), R_3, \dots, R_L)$. Then there is an embedding $\iota_{k,a,b} : \mathrm{LR}(\lambda, R) \hookrightarrow \mathrm{LR}(\lambda, R')$. This extends to an embedding $W(R) \rightarrow W(R')$ by $u \mapsto u'$ where $\mathbb{P}(u') = \iota_{k,a,b}(\mathbb{P}(u))$ and $\mathbb{Q}(u') = \mathbb{Q}(u)$.

Again fix n . We have $R^\wedge = (((n-k)^a), ((n-k)^b), R_3^\wedge, \dots, R_L^\wedge)$ and $R'^\wedge = (((n-k)^{a-1}), ((n-k)^{b+1}), R_3^\wedge, \dots, R_L^\wedge)$.

Proposition 4.7. *The following diagram commutes:*

$$\begin{array}{ccc} \mathrm{LR}(\lambda, R) & \xrightarrow{\iota_{k,a,b}} & \mathrm{LR}(\lambda, R') \\ \wedge \downarrow & & \downarrow \wedge \\ \mathrm{LR}(\lambda^\wedge, R^\wedge) & \xrightarrow{\iota_{n-k,a,b}} & \mathrm{LR}(\lambda^\wedge, R'^\wedge) \end{array}$$

Proof. One reduces to the two rectangle case by restriction to the subinterval $A_1 \cup A_2$. Recalling that the tensor product of two rectangles is multiplicity-free and using the duality symmetry of tensor product multiplicities, one has

$$|\mathrm{LR}(\lambda, R)| = |\mathrm{LR}(\lambda^\wedge, R^\wedge)| \leq |\mathrm{LR}(\lambda, R')| = |\mathrm{LR}(\lambda^\wedge, R'^\wedge)| \leq 1$$

from which the result follows. \square

Corollary 4.8. *The following diagram commutes:*

$$\begin{array}{ccc} \mathrm{LR}(\lambda, R) & \xrightarrow{\theta_R} & \mathrm{LR}(\lambda, \mathrm{rows}(R)) \\ \wedge \downarrow & & \downarrow \wedge \\ \mathrm{LR}(\lambda^\wedge, R^\wedge) & \xrightarrow{\theta_{R^\wedge}} & \mathrm{LR}(\lambda^\wedge, \mathrm{rows}(R)^\wedge) \end{array}$$

Proof. This holds by Propositions 4.6 and 4.7. \square

4.5. Generalized charge and duality. Let $R = (R_1, \dots, R_L)$ such that R_j has η_j rows and μ_j columns. The generalized charge is a function $\text{LR}(\lambda, R) \rightarrow \mathbb{N}$ defined as follows. For $L = 1$ charge_R is the zero function. For $L = 2$ and $u \in W(R_1, R_2)$, let $d_{R_2, R_1}(u)$ be the number of cells in the shape of $\mathbb{P}(u)$ to the right of the $\max\{\mu_1, \mu_2\}$ -th column (see (3.13)). For general L and $t \in \text{LR}(\lambda, R)$, define [42, 41]

$$\text{charge}_R(t) = \frac{1}{L!} \sum_{\tau \in \mathfrak{S}_L} \sum_{j=1}^{L-1} (L-j)d_j(\tau t)$$

where $d_j(\tau t)$ refers to the above function d applied to the shape of the \mathbb{P} tableau of the restriction of τt to the j -th and $(j + 1)$ -st subalphabets for τR .

Proposition 4.9 ([44]). *For all $b \in B_{R^t}$, $-E_{R^t}(b) = \text{charge}_R(\mathbb{Q}(b)^t)$.*

Corollary 4.10. *For all $t \in \text{LR}(R)$, $\text{charge}_R(t) = \text{charge}_{R^\wedge}(t^\wedge)$.*

Proof. This follows from Propositions 4.1, 4.9, and 3.17. □

5. RIGGED CONFIGURATIONS

Rigged configurations are combinatorial objects which were first introduced in the context of Bethe Ansatz studies of spin models in statistical mechanics [23, 24]. Here we recall their definition and a bijection from Littlewood–Richardson tableaux (or equivalently classically restricted paths of type $A_n^{(1)}$) to rigged configurations. The relation between duality and this bijection is given in Theorem 5.7. We will need these results later in section 7 in the proof of fermionic formulas of type $D_{n+1}^{(2)}$, $A_{2n}^{(2)}$ and $C_n^{(1)}$. In section 7 we will also introduce rigged configurations associated to $D_{n+1}^{(2)}$, $A_{2n}^{(2)}$ and $C_n^{(1)}$. The rigged configurations of this section correspond to the algebra $A_n^{(1)}$.

5.1. Definition of rigged configurations. Let λ be a partition and let $R = (R_1, \dots, R_L)$ be a sequence of rectangles with R_j having r_j rows and s_j columns.

The set of admissible configurations $C(\lambda, R)$ is the set of all sequences of partitions $\nu = (\nu^{(1)}, \nu^{(2)}, \dots)$ subject to the constraints

$$|\nu^{(k)}| = -\sum_{j=1}^k \lambda_j + \sum_{a=1}^L s_a \min\{r_a, k\},$$

$$P_i^{(k)}(\nu) \geq 0$$

for $k \geq 1$ and $i \geq 0$. Here $P_i^{(k)}(\nu)$ is the vacancy number of the parts (strings) of length i in $\nu^{(k)}$ defined as

$$P_i^{(k)}(\nu) = Q_i(\nu^{(k-1)}) - 2Q_i(\nu^{(k)}) + Q_i(\nu^{(k+1)}) + Q_i(\xi^{(k)}(R)),$$

where $\nu^{(0)}$ is the empty partition, $Q_i(\rho)$ is the size of the first i columns of the partition ρ , and $\xi^{(k)}(R)$ is the partition whose parts are the widths of the rectangles in R of height k .

The set $\text{RC}(\lambda, R)$ of rigged configurations is defined as follows. An element of $(\nu, J) \in \text{RC}(\lambda, R)$ consists of a configuration $\nu \in C(\lambda, R)$ together with a rigging J . The rigging J is a double sequence of partitions

$$J = \{J^{(k,i)}\}_{i,k \geq 1}$$

such that $J^{(k,i)}$ is a partition in a box of width $P_i^{(k)}(\nu)$ and height $m_i(\nu^{(k)})$ where $m_i(\nu^{(k)})$ is the number of parts of $\nu^{(k)}$ of size i .

The cocharge of a rigged configuration $(\nu, J) \in \text{RC}(\lambda, R)$ is defined by

$$cc(\nu, J) = cc(\nu) + \sum_{i,k \geq 1} |J^{(k,i)}|$$

with

$$cc(\nu) = \sum_{k,i \geq 1} \alpha_i^{(k)} (\alpha_i^{(k)} - \alpha_i^{(k+1)}),$$

where $\alpha_i^{(k)}$ is the size of the i -th column of $\nu^{(k)}$.

A rigged configuration (ν, J) can also be viewed as follows. Each part of $J^{(k,i)}$ labels a part of length i in $\nu^{(k)}$. The pair (i, x) of a part of $\nu^{(k)}$ together with its label is called a string. Then (ν, J) is the multiset of strings. A string (i, x) is called singular if $x = P_i^{(k)}(\nu)$.

5.2. Duality and rigged configurations. Let λ be a partition and R a sequence of rectangles R_a with r_a rows and s_a columns. Fix n such that $n \geq \lambda_1$ and $n \geq s_a$ for all $1 \leq a \leq L$, and let $N = \sum_{a=1}^L r_a$.

Proposition 5.1.

1. There is a bijection $\wedge : \text{C}(\lambda^t, R^t) \rightarrow \text{C}(\lambda^{\wedge t}, R^{\wedge t})$ defined by $\nu^{\wedge(k)} = \nu^{(n-k)}$ for all $1 \leq k \leq n - 1$.
2. $P_i^{(k)}(\nu^{\wedge}) = P_i^{(n-k)}(\nu)$ for all $i \geq 1$ and $1 \leq k \leq n - 1$.
3. There is an induced bijection $\wedge : \text{RC}(\lambda^t, R^t) \rightarrow \text{RC}(\lambda^{\wedge t}, R^{\wedge t})$ defined by $\nu^{\wedge(k)} = \nu^{(n-k)}$ and $J^{\wedge(k,i)} = J^{(n-k,i)}$ for all $1 \leq k \leq n - 1$ and $i \geq 1$.
4. $cc(\nu, J^{\wedge}) = cc(\nu, J)$.

Proof. Note that $|\nu^{\wedge(k)}| = |\nu^{(n-k)}| = -\sum_{j=1}^{n-k} \lambda_j^t + \sum_{a=1}^L r_a \min\{s_a, n - k\}$. Since $\sum_j \lambda_j^t = \sum_a r_a s_a$, $\sum_{j > n-k} \lambda_j^t = Nk - \sum_{j > n-k} \lambda_{n+1-j}^t$ and $N = \sum_a r_a$, it follows that $|\nu^{\wedge(k)}| = -\sum_{j=1}^k \lambda_j^t + \sum_{a=1}^L r_a \min\{n - s_a, k\}$, so that indeed $\nu^{\wedge(k)} \in \text{C}(\lambda^{\wedge t}, R^{\wedge t})$. The other assertions follow easily from the definitions. \square

5.3. Bijection between LR tableaux and rigged configurations. In [26] the existence of several bijections between LR tableaux and rigged configurations was established. Let $\bar{\phi}$ and $\tilde{\phi}$ be the columnwise quantum and coquantum number bijections, respectively.

Let us describe the algorithm for the bijection $\bar{\phi}_R : \text{LR}(\lambda, R) \rightarrow \text{RC}(\lambda^t, R^t)$ explicitly.

Let $t \in \text{LR}(\lambda, R)$ be an LR tableau for a partition λ and a sequence of rectangles $R = (R_1, \dots, R_L)$ where R_a has r_a rows and s_a columns. Set $\tilde{A}_j = [|R_1| + \dots + |R_{j-1}| + 1, |R_1| + \dots + |R_j|]$. Relabel the letters in t such that the i -th occurrence (from the left) of the a -th largest letter in \tilde{A}_j is mapped to $|R_1| + \dots + |R_{j-1}| + (i - 1)r_j + a$. Call the relabelled tableau T . Set $|\lambda| = M$. Let Z_j be the tableau of shape R_j such that the columns of Z_j are successively labelled by the letters in \tilde{A}_j .

The rigged configuration $(\nu, J) = \bar{\phi}_R(t)$ is obtained recursively by constructing a rigged configuration $(\nu, J)_{(x)}$ for each letter $1 \leq x \leq M$ occurring in T . Set $(\nu, J)_{(0)} = \emptyset$. Suppose that $x \in \tilde{A}_j$, and denote the column index of x in T by c and the column index of x in Z_j by c' . Define the numbers $\ell^{(k)}$ for $c' \leq k < c$

TABLE 1. Example for the bijection algorithm (see Example 5.2)

| x | $(\nu, J)_{(x)}^{(1)}$ | $(\nu, J)_{(x)}^{(2)}$ | $(\nu, J)_{(x)}^{(3)}$ |
|-----|------------------------|--|------------------------|
| 6 | 1 \square 1 | 0 \square 0 | |
| 7 | 1 $\square \square$ 1 | 0 \square 0 | |
| 9 | 1 $\square \square$ 1 | 0 \square 0 | |
| 10 | 2 $\square \square$ 1 | 0 $\begin{smallmatrix} \square \\ \square \end{smallmatrix}$ 0 | 0 \square 0 |
| 11 | 3 $\square \square$ 1 | 0 $\begin{smallmatrix} \square & \square \\ \square & 0 \end{smallmatrix}$ 0 | 0 \square 0 |
| 13 | 1 $\square \square$ 1 | 0 $\begin{smallmatrix} \square & \square \\ \square & 0 \end{smallmatrix}$ 0 | 0 \square 0 |

as follows. Let $\ell^{(c-1)}$ be the length of the longest singular string in $(\nu, J)_{(x-1)}^{(c-1)}$. Now select inductively a singular string in $(\nu, J)_{(x-1)}^{(k)}$ for $k = c - 2, c - 3, \dots, c'$ whose length $\ell^{(k)}$ is maximal such that $\ell^{(k)} \leq \ell^{(k+1)}$; if no such string exists, set $\ell^{(k)} = 0$. Then $(\nu, J)_{(x)}$ is obtained from $(\nu, J)_{(x-1)}$ by adding one box to the selected strings with labels such that they remain singular, leaving all other strings unchanged. Then the image of t under $\overline{\phi}_R$ is given by $(\nu, J) = (\nu, J)_{(M)}$.

For the above algorithm it is necessary to be able to compute the vacancy numbers of an intermediate configuration $\nu_{(x)}$. Suppose x occurs in Z_j in column c' . In general, $R_{(x)} = (R_1, \dots, R_{j-1}, \text{shape}(Z_j|_{[1,x]}))$ is not a sequence of rectangles. If $\text{shape}(Z_j|_{[1,x]})$ is not a rectangle, one splits it into two rectangles, one of width c' and one of width $c' - 1$. The vacancy numbers are calculated with respect to this new sequence of rectangles.

By definition set $\ell^{(k)} = \infty$ for $k \geq c$ and $\ell^{(k)} = 0$ if $k < c'$.

Example 5.2. Let $\lambda = (4, 3, 2, 2, 1, 1)$, $R = ((1, 1, 1), (2), (2, 2, 2, 2))$. Consider $t \in \text{LR}(\lambda, R)$ given by

$$t = \begin{array}{cccc} 1 & 4 & 5 & 5 \\ 2 & 6 & 6 & \\ 3 & 7 & & \\ 4 & 8 & & \\ 7 & & & \\ 8 & & & \end{array}$$

Then

$$T = \begin{array}{cccc} 1 & 5 & 6 & 10 \\ 2 & 7 & 11 & \\ 3 & 12 & & \\ 4 & 13 & & \\ 8 & & & \\ 9 & & & \end{array}$$

The nontrivial steps of the above algorithm for $\overline{\phi}_R$ are given in Table 1. A rigged partition is represented by its Ferrers diagram where to the right of each part

the corresponding rigging is indicated. The vacancy numbers are given to the left of each part. For example $R_{(10)} = ((1, 1, 1), (2), (2, 1, 1, 1))$ so that the vacancy numbers of $(\nu, J)_{(10)}$ are calculated with respect to the sequence of rectangles $((1, 1, 1), (2), (2), (1, 1, 1))$.

Let comp be the involution on rigged configurations that complements the quantum numbers. More precisely $\text{comp}(\nu, J) = (\nu, \tilde{J})$ where \tilde{J} is obtained from J by complementing each partition $J^{(k,i)}$ within the box of width $P_i^{(k)}(\nu)$ and height $m_i(\nu^{(k)})$. By definition, the coquantum number bijection is $\tilde{\phi}_R = \text{comp} \circ \bar{\phi}_R$.

It was shown in [26] that the bijection $\tilde{\phi}_R$ preserves the statistics.

Theorem 5.3 ([26, Theorem 8.3]). *For $t \in \text{LR}(\lambda, R)$ we have*

$$\text{charge}_R(t) = cc(\tilde{\phi}_R(t)).$$

5.4. Bijection from paths to rigged configurations. Proposition 4.1 yields a bijection $B_R \rightarrow \bigcup_{\lambda} B(\lambda) \times \text{LR}(\lambda^t, R^t)$. An element $b \in \mathcal{P}(B_R, \lambda)$ will be mapped to (y, t) where y is the unique Yamanouchi tableau of shape and content λ (see the end of section 3.2). Hence the above bijection restricts to a bijection between paths and LR tableaux $\mathcal{P}(B_R, \lambda) \rightarrow \text{LR}(\lambda^t, R^t)$. This in turn induces a bijection between paths and rigged configurations. It will be useful for later to state this bijection explicitly. By abuse of notation we denote the bijection induced by $\bar{\phi}$ also by $\bar{\phi} : \mathcal{P}(B_R, \lambda) \rightarrow \text{RC}(\lambda, R)$, and similarly for $\tilde{\phi}$.

Let $b = b_L \otimes \cdots \otimes b_1 \in \mathcal{P}(B_R, \lambda)$. Recall that R_i in $R = (R_1, \dots, R_L)$ has r_i rows and s_i columns. Factor each step b_i into columns $b_i = u_i^{s_i} \cdots u_i^1$. Under the bijection $\mathcal{P}(B_R, \lambda) \rightarrow \text{LR}(\lambda^t, R^t)$, which maps $b \in \mathcal{P}(B_R, \lambda)$ to $t \in \text{LR}(\lambda^t, R^t)$, a letter $c \in u_i^j$ is mapped to the letter $\sum_{k=1}^{i-1} s_k + j$ in column c in t . As before the rigged configuration (ν, J) corresponding to b is constructed recursively, this time on $1 \leq i \leq L, 1 \leq j \leq s_i$ and increasing $c \in u_i^j$ in this order. Suppose the letter c is at height c' in u_i^j . Then select singular strings of length $\ell^{(c-1)}, \ell^{(c-2)}, \dots, \ell^{(c')}$ in the $(c-1)$ -th, $(c-2)$ -th, \dots , c' -th rigged partition such that the $\ell^{(k)}$ are maximal subject to the condition $\ell^{(c-1)} \geq \ell^{(c-2)} \geq \dots \geq \ell^{(c')}$. As before add a box to each selected string, keeping it singular and leaving all other strings unchanged. If $c \in u_i^j$ and $j < s_i$, the vacancy number is calculated with respect to the sequence of rectangles $(R_1, \dots, R_{i-1}, (j^{c'}), ((s_i - j)^{c'-1}))$ where c' is the height of c in b_i .

Example 5.4. Let $\lambda = (6, 4, 2, 1), R = ((3), (1, 1), (4, 4))$ and

$$b = \begin{array}{|c|c|c|c|} \hline 1 & 1 & 2 & 3 \\ \hline 2 & 2 & 3 & 4 \\ \hline \end{array} \otimes \begin{array}{|c|} \hline 1 \\ \hline 2 \\ \hline \end{array} \otimes \begin{array}{|c|c|c|} \hline 1 & 1 & 1 \\ \hline \end{array}$$

in $\mathcal{P}(B_R, \lambda)$. Then the corresponding LR tableau in $\text{LR}(\lambda^t, R^t)$ is the t of Example 5.2. For example the letter 4 in $u_3^1 = \begin{array}{|c|} \hline 3 \\ \hline 4 \\ \hline \end{array}$ corresponds to the letter 5 in the fourth column of t and will select singular string lengths $\ell^{(3)} \geq \ell^{(2)}$ in the second and third rigged partition. Similarly, the letter 3 in $u_3^2 = \begin{array}{|c|} \hline 2 \\ \hline 3 \\ \hline \end{array}$ corresponds to the letter 6 in the third column of t and will select a singular string of length $\ell^{(2)}$ in the second rigged partition. For example, the rigged configuration in Table 1 for $x = 10$ corresponds

to the intermediary paths

$$\begin{array}{|c|c|c|c|} \hline 1 & 1 & 2 & 3 \\ \hline & & & 4 \\ \hline \end{array} \otimes \begin{array}{|c|} \hline 1 \\ \hline 2 \\ \hline \end{array} \otimes \begin{array}{|c|c|c|} \hline 1 & 1 & 1 \\ \hline \end{array}.$$

5.5. Duality under the bijection.

Lemma 5.5. *Let $R = (R_1, \dots, R_L)$ where R_L is a single row with at least two boxes. Fix $t \in \text{LR}(\lambda, R)$ and let t' (resp. t'') be obtained from t by removing the rightmost (resp. two rightmost) letters L from t . Denote by $\ell^{(k)}$ (resp. $\bar{\ell}^{(k)}$) the lengths of the selected strings under the bijection algorithm for $\bar{\phi}$ going from t' to t (resp. t'' to t'). Then*

$$\ell^{(k)} \leq \bar{\ell}^{(k-1)}.$$

Proof. Let $(\nu, J) = \bar{\phi}(t)$, $(\nu', J') = \bar{\phi}(t')$, and $(\nu'', J'') = \bar{\phi}(t'')$. The vacancy numbers change by

$$(5.1) \quad P_i^{(k)}(\nu') - P_i^{(k)}(\nu'') = \chi(\bar{\ell}^{(k-1)} < i \leq \bar{\ell}^{(k)}) - \chi(\bar{\ell}^{(k)} < i \leq \bar{\ell}^{(k+1)}).$$

Let c (resp. \bar{c}) be the column index of the box t/t' (resp. t'/t''). Then $c > \bar{c}$. Since $\bar{\ell}^{(\bar{c})} = \infty$, it follows from (5.1) that there are no singular strings of length i with $\bar{\ell}^{(\bar{c}-1)} < i$ in $(\nu', J')^{(\bar{c})}$. Hence $\ell^{(\bar{c})} \leq \bar{\ell}^{(\bar{c}-1)}$. By induction on k it follows that $\ell^{(k)} \leq \bar{\ell}^{(k+1)} \leq \bar{\ell}^{(k)}$ for $k < \bar{c}$. Since by (5.1) the strings in the range $\bar{\ell}^{(k-1)} < i \leq \bar{\ell}^{(k)}$ are nonsingular, we must have $\ell^{(k)} \leq \bar{\ell}^{(k-1)}$. □

Theorem 5.6 ([26, Theorem 8.3]). *The following diagram commutes:*

$$\begin{array}{ccc} \text{LR}(\lambda, R) & \xrightarrow{\tilde{\phi}_R} & \text{RC}(\lambda^t, R^t) \\ \theta_R \downarrow & & \downarrow \text{inclusion} \\ \text{LR}(\lambda, \text{rows}(R)) & \xrightarrow{\tilde{\phi}_{\text{rows}(R)}} & \text{RC}(\lambda^t, \text{rows}(R)^t) \end{array}$$

Theorem 5.7. *The following diagram commutes:*

$$\begin{array}{ccc} \text{LR}(\lambda, R) & \xrightarrow{\tilde{\phi}_R} & \text{RC}(\lambda^t, R^t) \\ \wedge \downarrow & & \downarrow \wedge \\ \text{LR}(\lambda^\wedge, R^\wedge) & \xrightarrow{\tilde{\phi}_{R^\wedge}} & \text{RC}(\lambda^{\wedge t}, R^{\wedge t}) \end{array}$$

The diagram also commutes with $\bar{\phi}$ replacing $\tilde{\phi}$.

Proof. By Corollary 4.8 and Theorem 5.6, one may reduce to the case that $R = (R_1, \dots, R_L)$ is a sequence of single-rowed shapes. Recall that $\bar{\phi}_R = \text{comp} \circ \tilde{\phi}_R$. By Proposition 5.1 it is obvious that comp commutes with the duality map on rigged configurations. Thus it suffices to prove the theorem where $\bar{\phi}$ replaces $\tilde{\phi}$.

We proceed by induction on L . The theorem is true for $L = 0$. Let $t \in \text{LR}(\lambda, R)$ and $c_1 < \dots < c_\ell$ be the column indices of the letters L in t . Then in t^\wedge the column indices of the letters L are in the alphabet $\{1 < \dots < n\}$ with letters $n + 1 - c_\ell, \dots, n + 1 - c_1$ omitted. Call them c_i^\wedge for $1 \leq i \leq n - \ell$. Let t' be the tableau obtained from t by removing all letters L and $R' = (R_1, \dots, R_{L-1})$. By induction $\overline{\phi}_{R'^\wedge}(t'^\wedge) = \overline{\phi}_{R'}(t')^\wedge$. Hence it suffices to show that the addition of the letters L to t' under $\overline{\phi}_R$ is (up to reversal of the order of all partitions) equal to the addition of the letters L to t'^\wedge under $\overline{\phi}_{R^\wedge}$.

Set $(\nu_0, J_0) := \overline{\phi}_{R'}(t')$ and let (ν_{i-1}, J_{i-1}) be the rigged configuration corresponding to t' with letters L added in columns c_1, \dots, c_{i-1} . Adding the letter L in column c_i has the effect on the rigged configurations of selecting singular strings in $(\nu, J)^{(k)}$ for $k = c_i - 1, c_i - 2, \dots, i$ of length $\ell_i^{(k)}$ maximal such that $\ell_i^{(c_i-1)} \geq \ell_i^{(c_i-2)} \geq \dots \geq \ell_i^{(i)} \geq \ell_i^{(i-1)} = \dots = \ell_i^{(0)} = 0$, adding a box to the selected strings and making them singular again. By Lemma 5.5, $\ell_i^{(k)} \leq \ell_{i-1}^{(k-1)}$. It follows that $\ell_i^{(k)}$ with $1 \leq i \leq \ell$ and $i \leq k < c_i$ is uniquely defined to be the length of the maximal singular string in $(\nu_0, J_0)^{(k)}$ such that

$$(5.2) \quad \ell_i^{(k)} \leq \min\{\ell_i^{(k+1)}, \ell_{i-1}^{(k-1)}\}$$

where $\ell_i^{(k+1)} = \infty$ if $k \geq c_i - 1$ and $\ell_{i-1}^{(k-1)} = \infty$ if $i = 1$ or $k > c_{i-1}$. Define a matrix M with ℓ rows and $c_\ell - \ell$ columns with entries $M_{ij} = \ell_i^{(j+i-1)}$. The entries are weakly increasing along a row and weakly decreasing along a column. Some of the entries may be ∞ .

Now do the analogous construction for t^\wedge . Call the selected strings under $\overline{\phi}$, $s_i^{(k)}$ for $1 \leq i \leq n - \ell$ and $i \leq k < c_i^\wedge$. By the same arguments as before they are uniquely defined as the lengths of the maximal singular string in $(\nu_0^\wedge, J_0^\wedge)^{(k)} = \overline{\phi}_{R'^\wedge}(t'^\wedge)$ such that $s_i^{(k)} \leq \min\{s_i^{(k+1)}, s_{i-1}^{(k-1)}\}$ where $s_i^{(k+1)} = \infty$ if $k \geq c_i^\wedge - 1$ and $s_{i-1}^{(k-1)} = \infty$ if $i = 1$ or $k > c_{i-1}^\wedge$. This yields a matrix M^\wedge with entries $M_{ij}^\wedge = s_i^{(j+i-1)}$.

It is clear from (5.2) that the entries in M can either be defined inductively row by row, top to bottom, right to left, or column by column, right to left, top to bottom. The same is true for M^\wedge . Since by induction $(\nu_0, J_0)^{(k)} = (\nu_0^\wedge, J_0^\wedge)^{(n-k)}$ for $1 \leq k < n$, it follows that $M_{i,j} = M_{c_\ell - \ell + 1 - j, \ell + 1 - i}^\wedge$ or equivalently $\ell_i^{(k)} = s_{c_\ell - \ell - k + i}^{(c_\ell - k)}$. This implies $(\nu_\ell, J_\ell)^{(k)} = (\nu_\ell^\wedge, J_\ell^\wedge)^{(n-k)}$ as desired. \square

Example 5.8. Let $R = ((1), (3), (2), (4), (3), (2), (4))$ and

$$t = \begin{array}{|c|c|c|c|c|c|} \hline 1 & 2 & 2 & 3 & 4 & 5 \\ \hline 2 & 4 & 4 & 5 & 7 & 7 \\ \hline 3 & 5 & 6 & 7 & & \\ \hline 4 & 7 & & & & \\ \hline 6 & & & & & \\ \hline \end{array} \quad \text{so that} \quad t' = \begin{array}{|c|c|c|c|c|c|} \hline 1 & 2 & 2 & 3 & 4 & 5 \\ \hline 2 & 4 & 4 & 5 & & \\ \hline 3 & 5 & 6 & & & \\ \hline 4 & & & & & \\ \hline 6 & & & & & \\ \hline \end{array}$$

Then $c_1 = 2, c_2 = 4, c_3 = 5, c_4 = 6$ and

$$\begin{aligned}
 (\nu_0, J_0) &= 1 \begin{array}{|c|} \hline \square \\ \hline \end{array} 1 & 0 \begin{array}{|c|c|} \hline \square & \square \\ \hline \square & 0 \\ \hline \end{array} & 0 \begin{array}{|c|} \hline \square \\ \hline \square \\ \hline \square \\ \hline \square \\ \hline \square \\ \hline \end{array} 0 & 1 \begin{array}{|c|} \hline \square \\ \hline \square \\ \hline \end{array} 1 & 0 \begin{array}{|c|} \hline \square \\ \hline \end{array} 0 \\
 (\nu_1, J_1) &= 1 \begin{array}{|c|c|} \hline \square & \square \\ \hline \end{array} 1 & 1 \begin{array}{|c|c|} \hline \square & \square \\ \hline \square & 0 \\ \hline \end{array} & 0 \begin{array}{|c|} \hline \square \\ \hline \square \\ \hline \square \\ \hline \square \\ \hline \square \\ \hline \end{array} 0 & 1 \begin{array}{|c|} \hline \square \\ \hline \square \\ \hline \end{array} 1 & 0 \begin{array}{|c|} \hline \square \\ \hline \end{array} 0 \\
 (\nu_2, J_2) &= 1 \begin{array}{|c|c|} \hline \square & \square \\ \hline \end{array} 1 & 1 \begin{array}{|c|c|} \hline \square & \square \\ \hline \square & 1 \\ \hline \square & 0 \\ \hline \end{array} & 0 \begin{array}{|c|c|} \hline \square & \square \\ \hline \square & 0 \\ \hline \square & 0 \\ \hline \square & 0 \\ \hline \square & 0 \\ \hline \end{array} & 1 \begin{array}{|c|} \hline \square \\ \hline \square \\ \hline \end{array} 1 & 0 \begin{array}{|c|} \hline \square \\ \hline \end{array} 0 \\
 (\nu_3, J_3) &= 1 \begin{array}{|c|c|} \hline \square & \square \\ \hline \end{array} 1 & 1 \begin{array}{|c|c|} \hline \square & \square \\ \hline \square & 1 \\ \hline \square & 0 \\ \hline \end{array} & 0 \begin{array}{|c|c|} \hline \square & \square \\ \hline \square & 0 \\ \hline \square & 0 \\ \hline \square & 0 \\ \hline \square & 0 \\ \hline \end{array} & 1 \begin{array}{|c|c|} \hline \square & \square \\ \hline \square & 0 \\ \hline \end{array} 1 & 0 \begin{array}{|c|} \hline \square \\ \hline \end{array} 0 \\
 (\nu_4, J_4) &= 1 \begin{array}{|c|c|} \hline \square & \square \\ \hline \end{array} 1 & 1 \begin{array}{|c|c|} \hline \square & \square \\ \hline \square & 1 \\ \hline \square & 0 \\ \hline \end{array} & 0 \begin{array}{|c|c|} \hline \square & \square \\ \hline \square & 0 \\ \hline \square & 0 \\ \hline \square & 0 \\ \hline \square & 0 \\ \hline \end{array} & 1 \begin{array}{|c|c|} \hline \square & \square \\ \hline \square & 1 \\ \hline \square & 0 \\ \hline \end{array} 1 & 0 \begin{array}{|c|c|} \hline \square & \square \\ \hline \end{array} 0
 \end{aligned}$$

so that

$$M = \begin{pmatrix} \ell_1^{(1)} & \ell_1^{(2)} \\ \ell_2^{(2)} & \ell_2^{(3)} \\ \ell_3^{(3)} & \ell_3^{(4)} \\ \ell_4^{(4)} & \ell_4^{(5)} \end{pmatrix} = \begin{pmatrix} 1 & \infty \\ 0 & 1 \\ 0 & 1 \\ 0 & 1 \end{pmatrix}.$$

On the other hand, $c_1^\wedge = 4, c_2^\wedge = 6$ so that

$$\begin{aligned}
 (\nu_0^\wedge, J_0^\wedge) &= 0 \begin{array}{|c|} \hline \square \\ \hline \end{array} 0 & 1 \begin{array}{|c|} \hline \square \\ \hline \square \\ \hline \end{array} 1 & 0 \begin{array}{|c|} \hline \square \\ \hline \square \\ \hline \square \\ \hline \square \\ \hline \square \\ \hline \end{array} 0 & 0 \begin{array}{|c|c|} \hline \square & \square \\ \hline \square & 0 \\ \hline \end{array} 0 & 1 \begin{array}{|c|} \hline \square \\ \hline \end{array} 1 \\
 (\nu_1^\wedge, J_1^\wedge) &= 0 \begin{array}{|c|c|} \hline \square & \square \\ \hline \end{array} 0 & 1 \begin{array}{|c|c|} \hline \square & \square \\ \hline \square & 0 \\ \hline \end{array} 1 & 0 \begin{array}{|c|} \hline \square \\ \hline \square \\ \hline \square \\ \hline \square \\ \hline \square \\ \hline \end{array} 0 & 1 \begin{array}{|c|c|} \hline \square & \square \\ \hline \square & 0 \\ \hline \end{array} 0 & 1 \begin{array}{|c|} \hline \square \\ \hline \end{array} 1 \\
 (\nu_2^\wedge, J_2^\wedge) &= 0 \begin{array}{|c|c|} \hline \square & \square \\ \hline \end{array} 0 & 1 \begin{array}{|c|c|} \hline \square & \square \\ \hline \square & 1 \\ \hline \square & 0 \\ \hline \end{array} 1 & 0 \begin{array}{|c|} \hline \square \\ \hline \square \\ \hline \square \\ \hline \square \\ \hline \square \\ \hline \end{array} 0 & 1 \begin{array}{|c|c|} \hline \square & \square \\ \hline \square & 1 \\ \hline \square & 0 \\ \hline \end{array} 0 & 1 \begin{array}{|c|c|} \hline \square & \square \\ \hline \end{array} 1
 \end{aligned}$$

and

$$M^\wedge = \begin{pmatrix} s_1^{(1)} & s_1^{(2)} & s_1^{(3)} & s_1^{(4)} \\ s_2^{(2)} & s_2^{(3)} & s_2^{(4)} & s_2^{(5)} \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & \infty \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

So indeed $\ell_i^{(k)} = s_{2-k+i}^{(6-k)}$.

5.6. An embedding. For $r < n$ there is a unique embedding of multiplicity-free $U_q(A_{2n-1})$ -crystals

$$i_{r,s} : B^{2n-r,s} \otimes B^{r,s} \rightarrow B^{2n-r-1,s} \otimes B^{r+1,s}.$$

Explicitly it is given by $i_{r,s}(u \otimes v) = v' \otimes u'$ where $\mathbb{P}(uv) = \mathbb{P}(v'u')$. By (3.13),

$$H_{B^{2n-r,s}, B^{r,s}}(u \otimes v) = H_{B^{2n-r-1,s}, B^{r+1,s}}(v' \otimes u') + s.$$

This map also preserves \tilde{e}_0 and \tilde{f}_0 when such operators act on the left tensor factor. An analogous map can be defined on rigged configurations. Let $R = (R_1, \dots, R_L)$ be a sequence of rectangles such that $R_{L-1} = (s^r)$ and $R_L = (s^{2n-r})$ and R^+ the same sequence of rectangles with R_{L-1} replaced by (s^{r+1}) and R_L replaced by (s^{2n-r-1}) . Define $j_{r,s}$ by the following commutative diagram:

$$\begin{array}{ccc} \text{RC}(\lambda^t, R^t) & \xrightarrow{\text{inclusion}} & \text{RC}(\lambda^t, R^{+t}) \\ \text{RCtr} \downarrow & & \downarrow \text{RCtr} \\ \text{RC}(\lambda, R) & \xrightarrow{j_{r,s}} & \text{RC}(\lambda, R^+). \end{array}$$

Here $\text{RCtr} : \text{RC}(\lambda^t, R^t) \rightarrow \text{RC}(\lambda, R)$ is the RC-transpose map as defined in [26, Section 7]. It can be checked by direct computation that for $(\nu, J) \in \text{RC}(\lambda, R)$ the rigged configuration $j_{r,s}(\nu, J)$ is obtained by adding a singular string of length s to the k -th rigged partition for all $r + 1 \leq k \leq 2n - r - 1$.

Theorem 5.9. *Let B be any tensor product of type $A_{2n-1}^{(1)}$ crystals of the form $B^{r',s'}$ and let R (resp. R^+) be the sequence of rectangles associated with $B_R = B^{2n-r,s} \otimes B^{r,s} \otimes B$ (resp. $B_{R^+} = B^{2n-r-1,s} \otimes B^{r+1,s} \otimes B$). The following diagram commutes:*

$$\begin{array}{ccc} \mathcal{P}(B_R, \lambda) & \xrightarrow{i_{r,s} \otimes \text{id}_B} & \mathcal{P}(B_{R^+}, \lambda) \\ \bar{\phi} \downarrow & & \downarrow \bar{\phi} \\ \text{RC}(\lambda, R) & \xrightarrow{j_{r,s}} & \text{RC}(\lambda, R^+). \end{array}$$

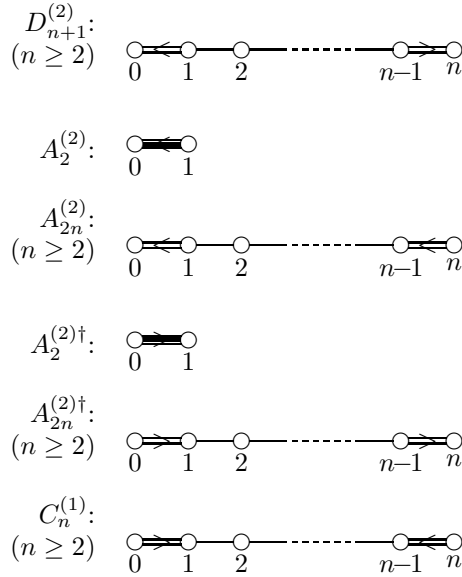
Proof. The proof follows from the Evacuation Theorem [26, Theorem 5.6], the Transpose Theorem [26, Theorem 7.1] and $\mathbb{P}(u^*v^*) = \mathbb{P}(v'^*u'^*)$ for $i_{r,s}(u \otimes v) = v' \otimes u'$ which is true by Proposition 3.10. \square

6. CRYSTALS OF TYPE $D_{n+1}^{(2)}, A_{2n}^{(2)}$ AND $C_n^{(1)}$

Let \mathfrak{g} be an affine Lie algebra of type $D_{n+1}^{(2)}, A_{2n}^{(2)}, A_{2n}^{(2)\dagger}$, or $C_n^{(1)}$, defined by the Dynkin diagrams with distinguished vertex 0 given in Table 2.

There is an embedding $P \rightarrow P^A$ of the weight lattice of \mathfrak{g} into that of $A_{2n-1}^{(1)}$ which preserves distance up to a fixed scalar factor. This suggests that $U'_q(\mathfrak{g})$ -crystals can be embedded into $U'_q(A_{2n-1}^{(1)})$ -crystals. We introduce the notion of a virtual $U'_q(\mathfrak{g})$ -crystal, which is by definition a suitable subset of a $U'_q(A_{2n-1}^{(1)})$ -crystal. We define the virtual $U'_q(\mathfrak{g})$ -crystal $V^{r,s}$ and give some evidence that it agrees with the crystal $B^{r,s}$ of Conjecture 2.16 (whose existence was conjectured in [7, Conj. 2.1]). These conjectures are proved for crystals $B^{r,1}$ of types $\mathfrak{g} = C_n^{(1)}, A_{2n}^{(2)}, A_{2n}^{(2)\dagger}$.

TABLE 2.



6.1. **Affine Lie algebras $D_{n+1}^{(2)}$, $A_{2n}^{(2)}$ and $C_n^{(1)}$.** We use the notation of section 2.8. The null root δ is given explicitly by

$$\delta = \begin{cases} \alpha_0 + \alpha_1 + \cdots + \alpha_{n-1} + \alpha_n & \text{for } \mathfrak{g} = D_{n+1}^{(2)}, \\ 2\alpha_0 + 2\alpha_1 + \cdots + 2\alpha_{n-1} + \alpha_n & \text{for } \mathfrak{g} = A_{2n}^{(2)}, \\ \alpha_0 + 2\alpha_1 + \cdots + 2\alpha_{n-1} + 2\alpha_n & \text{for } \mathfrak{g} = A_{2n}^{(2)\dagger}, \\ \alpha_0 + 2\alpha_1 + \cdots + 2\alpha_{n-1} + \alpha_n & \text{for } \mathfrak{g} = C_n^{(1)}. \end{cases}$$

To each \mathfrak{g} we associate two simple Lie algebras $\bar{\mathfrak{g}}$ and $\tilde{\mathfrak{g}}$,

| | | | | |
|------------------------|-----------------|----------------|-----------------------|-------------|
| \mathfrak{g} | $D_{n+1}^{(2)}$ | $A_{2n}^{(2)}$ | $A_{2n}^{(2)\dagger}$ | $C_n^{(1)}$ |
| $\bar{\mathfrak{g}}$ | B_n | C_n | B_n | C_n |
| $\tilde{\mathfrak{g}}$ | B_n | B_n | B_n | C_n |

Note that $\bar{\mathfrak{g}} = \tilde{\mathfrak{g}}$ except for $A_{2n}^{(2)}$. The algebra $\tilde{\mathfrak{g}}$ will be used in section 7.

Let $(\cdot|\cdot)$ be the standard bilinear form on P normalized by

$$(6.1) \quad (\delta|\lambda) = \langle c, \lambda \rangle \quad \text{for any } \lambda \in P.$$

6.2. **Embedding of weight lattices.** Let

$$(6.2) \quad \gamma = \begin{cases} 1 & \text{for } \mathfrak{g} = D_{n+1}^{(2)}, A_{2n}^{(2)\dagger}, \\ 2 & \text{for } \mathfrak{g} = A_{2n}^{(2)}, C_n^{(1)}, \end{cases}$$

$$\gamma' = \begin{cases} 1 & \text{for } \mathfrak{g} = D_{n+1}^{(2)}, A_{2n}^{(2)}, \\ 2 & \text{for } \mathfrak{g} = A_{2n}^{(2)\dagger}, C_n^{(1)}. \end{cases}$$

We use the superscript and subscript A to denote the root system $A_{2n-1}^{(1)}$ (or A_{2n-1}). There is an embedding of weight lattices $\Psi : P \rightarrow P^A$ defined by

$$\begin{aligned}
 \Psi(\Lambda_0) &= \gamma' \Lambda_0^A, \\
 \Psi(\Lambda_i) &= \Lambda_i^A + \Lambda_{2n-i}^A \quad \text{for } 0 < i < n, \\
 \Psi(\Lambda_n) &= \gamma \Lambda_n^A, \\
 \Psi(\delta) &= \gamma \delta^A.
 \end{aligned}
 \tag{6.3}$$

This induces the embedding $\Psi : \overline{P} \rightarrow \overline{P}^A$ given by

$$\begin{aligned}
 \Psi(\overline{\Lambda}_i) &= \overline{\Lambda}_i^A + \overline{\Lambda}_{2n-i}^A \quad \text{for } 1 \leq i < n, \\
 \Psi(\overline{\Lambda}_n) &= \gamma \overline{\Lambda}_n^A.
 \end{aligned}
 \tag{6.4}$$

One may verify that

$$\begin{aligned}
 \Psi(\alpha_0) &= \gamma' \alpha_0^A, \\
 \Psi(\alpha_i) &= \alpha_i^A + \alpha_{2n-i}^A \quad \text{for } 0 < i < n, \\
 \Psi(\alpha_n) &= \gamma \alpha_n^A.
 \end{aligned}
 \tag{6.5}$$

Observe that

$$(\Psi(u)|\Psi(v))_A = \gamma \gamma' (u|v) \quad \text{for } u, v \in P
 \tag{6.6}$$

under the normalization (6.1). Note that if $\lambda \in \overline{P}^A$, then $\lambda \in \text{Im}(\Psi)$ if and only if the following conditions hold:

$$\begin{aligned}
 \langle h_i, \lambda \rangle &= \langle h_{2n-i}, \lambda \rangle \quad \text{for } 1 \leq i \leq n-1, \\
 \langle h_n, \lambda \rangle &\text{ is divisible by } \gamma, \\
 \langle h_0, \lambda \rangle &\text{ is divisible by } \gamma'.
 \end{aligned}
 \tag{6.7}$$

The third condition is a consequence of the other two. Let w_0 be the longest element in the Weyl group of type A_{2n-1} . Then for $\lambda \in \text{Im}(\Psi) \cap \overline{P}^A$,

$$-w_0(\lambda) = \lambda
 \tag{6.8}$$

since this holds for $\Psi(\overline{\Lambda}_i)$ for $1 \leq i \leq n$.

6.3. Virtual $U'_q(\mathfrak{g})$ -crystals. We wish to define a category of $U'_q(\mathfrak{g})$ -crystals which are realized as subsets of $U'_q(A_{2n-1}^{(1)})$ -crystals. Let \widehat{V} be a finite $U'_q(A_{2n-1}^{(1)})$ -crystal. In view of (6.5) we define the virtual $U'_q(\mathfrak{g})$ raising and lowering operators on \widehat{V} by

$$\begin{aligned}
 \tilde{e}_0 &= (\tilde{e}_0^A)^{\gamma'} & \tilde{f}_0 &= (\tilde{f}_0^A)^{\gamma'}, \\
 \tilde{e}_i &= \tilde{e}_{2n-i}^A \tilde{e}_i^A & \tilde{f}_i &= \tilde{f}_{2n-i}^A \tilde{f}_i^A \quad \text{for } 0 < i < n, \\
 \tilde{e}_n &= (\tilde{e}_n^A)^\gamma & \tilde{f}_n &= (\tilde{f}_n^A)^\gamma.
 \end{aligned}
 \tag{6.9}$$

Let $V \subset \widehat{V}$ be a nonempty subset such that

$$\text{Im}(\text{wt}_A|_V) \subset \text{Im}(\Psi).
 \tag{6.10}$$

Then V has a virtual weight function $\text{wt} : V \rightarrow P$ defined by

$$\Psi(\text{wt}(b)) = \text{wt}_A(b).
 \tag{6.11}$$

This is well defined since Ψ is injective.

A virtual $U'_q(\mathfrak{g})$ -crystal is a pair (V, \widehat{V}) such that V satisfies (6.10) and is closed under the virtual raising and lowering operators \tilde{e}_i and \tilde{f}_i for $0 \leq i \leq n$. Sometimes the larger crystal \widehat{V} (called the ambient crystal) is omitted in the notation.

Proposition 6.1. *A virtual $U'_q(\mathfrak{g})$ -crystal (V, \widehat{V}) is a $U'_q(\mathfrak{g})$ -crystal in the sense of section 2.1 (see also 2.8).*

Proof. All the properties are immediate except for (2.3). Since \tilde{e}_i and \tilde{e}_{2n-i} (resp. \tilde{f}_i and \tilde{f}_{2n-i}) commute for $1 \leq i \leq n-1$, by the definitions we have for $b \in V$,

$$(6.12) \quad \epsilon_0(b) = \lfloor \epsilon_0^A(b)/\gamma' \rfloor \qquad \varphi_0(b) = \lfloor \varphi_0^A(b)/\gamma' \rfloor,$$

$$(6.13) \quad \epsilon_i(b) = \min\{\epsilon_i^A(b), \epsilon_{2n-i}^A(b)\} \qquad \text{for } 1 \leq i \leq n-1,$$

$$(6.14) \quad \varphi_i(b) = \min\{\varphi_i^A(b), \varphi_{2n-i}^A(b)\} \qquad \text{for } 1 \leq i \leq n-1,$$

$$(6.15) \quad \epsilon_n(b) = \lfloor \epsilon_n^A(b)/\gamma \rfloor \qquad \varphi_n(b) = \lfloor \varphi_n^A(b)/\gamma \rfloor.$$

Here $\lfloor x \rfloor$ denotes the largest integer smaller or equal to x . First let us verify (2.3) for $i = 0$. We have that

$$\varphi_0^A(b) - \epsilon_0^A(b) = \langle h_0^A, \text{wt}_A(b) \rangle$$

is a multiple of γ' by (6.7) applied to $\text{wt}_A(b)$ and (2.3) for b viewed as an element of \widehat{V} . For $b \in V$ we have

$$\varphi_0(b) - \epsilon_0(b) = \frac{\varphi_0^A(b) - \epsilon_0^A(b)}{\gamma'} = \frac{1}{\gamma'} \langle h_0^A, \text{wt}_A(b) \rangle = \langle h_0, \text{wt}(b) \rangle$$

by (6.12), (6.11), and (6.3). A similar calculation establishes (2.3) for $i = n$. Let $1 \leq i \leq n-1$. Observe that

$$(6.16) \quad \varphi_i^A(b) - \epsilon_i^A(b) = \langle h_i^A, \text{wt}_A(b) \rangle = \langle h_{2n-i}^A, \text{wt}_A(b) \rangle = \varphi_{2n-i}^A(b) - \epsilon_{2n-i}^A(b)$$

using (2.3) for $b \in \widehat{V}$ and (6.7). Suppose first that $\varphi_i^A(b) \leq \varphi_{2n-i}^A(b)$. Then $\epsilon_i^A(b) \leq \epsilon_{2n-i}^A(b)$ and one has

$$\varphi_i(b) - \epsilon_i(b) = \varphi_i^A(b) - \epsilon_i^A(b).$$

If $\varphi_i^A(b) > \varphi_{2n-i}^A(b)$, then $\epsilon_i^A(b) > \epsilon_{2n-i}^A(b)$ and

$$\varphi_i(b) - \epsilon_i(b) = \varphi_{2n-i}^A(b) - \epsilon_{2n-i}^A(b).$$

In either case (2.3) follows by (6.16) and (6.3). □

A morphism of virtual $U'_q(\mathfrak{g})$ -crystals $(V_1, \widehat{V}_1) \rightarrow (V_2, \widehat{V}_2)$ is a morphism $\widehat{V}_1 \rightarrow \widehat{V}_2$ of the ambient $U'_q(A_{2n-1}^{(1)})$ -crystals which restricts to a morphism $V_1 \rightarrow V_2$ of $U'_q(\mathfrak{g})$ -crystals.

6.4. Tensor products of virtual crystals. Let (V_1, \widehat{V}_1) and (V_2, \widehat{V}_2) be virtual $U'_q(\mathfrak{g})$ -crystals. We wish to define their tensor product. Define the virtual crystal operators \tilde{e}_i and \tilde{f}_i on $\widehat{V}_2 \otimes \widehat{V}_1$ by (6.9). Define $\text{wt}_{V_2 \otimes V_1} : V_2 \otimes V_1 \rightarrow P$ by (2.9). Unfortunately, the subset $V_2 \otimes V_1 \subset \widehat{V}_2 \otimes \widehat{V}_1$ need not be closed under \tilde{e}_i and \tilde{f}_i .

Example 6.2. Let $n = 2$, $\mathfrak{g} = C_2^{(1)}$, and let $\widehat{V}_2 = \widehat{V}_1 = B_A^{2,2}$ be $U'_q(A_{2n-1}^{(1)})$ -crystals. Let (V_2, \widehat{V}_2) and (V_1, \widehat{V}_1) be the virtual crystals generated by the elements b_2 and b_1 where

$$b_2 = \begin{array}{|c|c|} \hline 1 & 1 \\ \hline 2 & 3 \\ \hline \end{array} \qquad b_1 = \begin{array}{|c|c|} \hline 1 & 1 \\ \hline 2 & 2 \\ \hline \end{array}.$$

Then $\tilde{f}_2(b_2 \otimes b_1) = \tilde{f}_2^A(b_2) \otimes \tilde{f}_2^A(b_1) \notin V_2 \otimes V_1$. This disagrees with the usual tensor product structure on $U'_q(\mathfrak{g})$ -crystals defined by (2.10).

Let $b \in \widehat{V}$. Consider the following conditions:

$$(6.17) \quad \begin{aligned} &\varphi_i^A(b) = \varphi_{2n-i}^A(b) \text{ and } \epsilon_i^A(b) = \epsilon_{2n-i}^A(b) \text{ for } 1 \leq i \leq n-1, \\ &\varphi_0^A(b) \text{ and } \epsilon_0^A(b) \text{ are divisible by } \gamma', \\ &\varphi_n^A(b) \text{ and } \epsilon_n^A(b) \text{ are divisible by } \gamma. \end{aligned}$$

Say that b is i -aligned for $1 \leq i \leq n-1$ if the first condition holds, 0-aligned if the second holds, and n -aligned if the third holds. Say that (V, \widehat{V}) is an aligned virtual $U'_q(\mathfrak{g})$ -crystal if, for every $b \in V$, b is i -aligned for all $0 \leq i \leq n$.

Let $1 \leq i \leq n-1$. If b is i -aligned, then

$$(6.18) \quad \epsilon_i(b) = \epsilon_i^A(b) = \epsilon_{2n-i}^A(b) \quad \text{and} \quad \varphi_i(b) = \varphi_i^A(b) = \varphi_{2n-i}^A(b).$$

If b is n -aligned, then

$$(6.19) \quad \epsilon_n(b) = \frac{1}{\gamma} \epsilon_n^A(b) \quad \text{and} \quad \varphi_n(b) = \frac{1}{\gamma} \varphi_n^A(b).$$

If b is 0-aligned, then

$$(6.20) \quad \epsilon_0(b) = \frac{1}{\gamma'} \epsilon_0^A(b) \quad \text{and} \quad \varphi_0(b) = \frac{1}{\gamma'} \varphi_0^A(b).$$

Remark 6.3. Let (V, \widehat{V}) be a virtual $U_q(\mathfrak{g})$ -crystal and $b \in \widehat{V}$. If $\gamma = 1$, then b is n -aligned and if $\gamma' = 1$, then b is 0-aligned.

Proposition 6.4. *Aligned virtual $U'_q(\mathfrak{g})$ -crystals form a tensor category.*

Proof. Let (V_j, \widehat{V}_j) be aligned virtual $U'_q(\mathfrak{g})$ -crystals for $j = 1, 2$. Since they are aligned, it follows that for all $b_2 \otimes b_1 \in V_2 \otimes V_1$ and all $i \in I$, \tilde{e}_i and \tilde{f}_i either act entirely on the left factor or on the right factor of $b_2 \otimes b_1$. Therefore, the operators \tilde{e}_i and \tilde{f}_i coincide with those given by the $U'_q(\mathfrak{g})$ -crystal structure on the tensor product $V_2 \otimes V_1$ defined by (2.11) and (2.10), which define a tensor category. Finally, it is easy to verify that $(V_2 \otimes V_1, \widehat{V}_2 \otimes \widehat{V}_1)$ is aligned. \square

If (V_2, \widehat{V}_2) and (V_1, \widehat{V}_1) are aligned virtual $U'_q(\mathfrak{g})$ -crystals then define their tensor product to be the aligned virtual $U'_q(\mathfrak{g})$ -crystal $(V_2 \otimes V_1, \widehat{V}_2 \otimes \widehat{V}_1)$.

6.5. Virtual combinatorial R -matrix and energy function. Say that the virtual $U'_q(\mathfrak{g})$ -crystal (V, \widehat{V}) is simple if

1. \widehat{V} is a simple $U'_q(A_{2n-1}^{(1)})$ -crystal.
2. V , with its virtual $U'_q(\mathfrak{g})$ -crystal structure, is isomorphic to a simple $U'_q(\mathfrak{g})$ -crystal.
3. $u(V) = u(\widehat{V})$.

The following result shows that local isomorphisms and energy functions exist for aligned simple virtual crystals.

Proposition 6.5. *Let (V_1, \widehat{V}_1) and (V_2, \widehat{V}_2) be aligned simple virtual $U'_q(\mathfrak{g})$ -crystals. Write*

$$\begin{aligned} \widehat{\sigma} &= \sigma_{\widehat{V}_2, \widehat{V}_1} : \widehat{V}_2 \otimes \widehat{V}_1 \rightarrow \widehat{V}_1 \otimes \widehat{V}_2, \\ \widehat{H} &= H_{\widehat{V}_2, \widehat{V}_1} : \widehat{V}_2 \otimes \widehat{V}_1 \rightarrow \mathbb{Z}, \\ \sigma &= \sigma_{V_2, V_1} : V_2 \otimes V_1 \rightarrow V_1 \otimes V_2, \\ H &= H_{V_2, V_1} : V_2 \otimes V_1 \rightarrow \mathbb{Z} \end{aligned}$$

for the local isomorphisms and energy functions for the pair of simple $U'_q(A_{2n-1}^{(1)})$ -crystals \widehat{V}_1 and \widehat{V}_2 , and the pair of simple $U'_q(\mathfrak{g})$ -crystals V_1 and V_2 . Then

$$(6.21) \quad \sigma = \widehat{\sigma}|_{V_2 \otimes V_1},$$

$$(6.22) \quad H = \frac{1}{\gamma'} \widehat{H}|_{V_2 \otimes V_1}.$$

Proof. $(V_2 \otimes V_1, \widehat{V}_2 \otimes \widehat{V}_1)$ is an aligned simple virtual $U'_q(\mathfrak{g})$ -crystal, by Proposition 6.4 and Theorem 2.2, generated from the extremal vector

$$u := u(V_2 \otimes V_1) = u(V_2) \otimes u(V_1) = u(\widehat{V}_2) \otimes u(\widehat{V}_1) = u(\widehat{V}_2 \otimes \widehat{V}_1),$$

which holds by (2.13) and property 3 of the definition of the simple virtual crystal (V_j, \widehat{V}_j) . Similarly, $(V_1 \otimes V_2, \widehat{V}_1 \otimes \widehat{V}_2)$ is an aligned simple virtual $U'_q(\mathfrak{g})$ -crystal with generator $u' := u(V_1 \otimes V_2) = u(\widehat{V}_1 \otimes \widehat{V}_2)$. Now $\widehat{\sigma}$ is a $U'_q(A_{2n-1}^{(1)})$ -crystal isomorphism such that $\widehat{\sigma}(u) = u'$. As such it intertwines with the virtual operators \widetilde{e}_i and \widetilde{f}_i for $0 \leq i \leq n$. Therefore, $\widehat{\sigma}|_{V_2 \otimes V_1}$ is a $U'_q(\mathfrak{g})$ -crystal isomorphism from the $U'_q(\mathfrak{g})$ -component of u to that of u' , that is, from $V_2 \otimes V_1$ to $V_1 \otimes V_2$. But there is a unique such map, namely, σ . This proves (6.21).

By abuse of notation, let $H : V_2 \otimes V_1 \rightarrow \frac{1}{\gamma'}\mathbb{Z}$ be defined by (6.22). It suffices to show that H satisfies the defining properties of the energy function given in Theorem 2.4. This is entirely straightforward except for (2.16). Let $b_j \in V_j$ for $j = 1, 2$ and write $\sigma(b_2 \otimes b_1) = b'_1 \otimes b'_2$.

Suppose first that $\epsilon_0(b_2) > \varphi_0(b_1)$ and $\epsilon_0(b'_1) > \varphi_0(b'_2)$. Since (V_j, \widehat{V}_j) is aligned for $j = 1, 2$, this means

$$\epsilon_0^A(b_2) = \gamma' \epsilon_0(b_2) > \gamma' \varphi_0(b_1) = \varphi_0^A(b_1)$$

and

$$\epsilon_0^A(b'_1) = \gamma' \epsilon_0(b'_1) > \gamma' \varphi_0(b'_2) = \varphi_0^A(b'_2)$$

by (6.20). Applying \widetilde{e}_0^A to $b_2 \otimes b_1$ γ' times and using (2.16), one has

$$\widehat{H}(\widetilde{e}_0^A(b_2 \otimes b_1)) = \widehat{H}((\widetilde{e}_0^A)^{\gamma'}(b_2 \otimes b_1)) = \widehat{H}(b_2 \otimes b_1) - \gamma'.$$

Dividing by γ' one obtains (2.16) for H in the first case. The other cases are similar. \square

6.6. Virtual intrinsic energy. Let (V, \widehat{V}) be an aligned simple virtual $U'_q(\mathfrak{g})$ -crystal. Suppose the ambient simple crystal \widehat{V} is graded by the intrinsic energy function $D_{\widehat{V}}$. Then V has an inherited intrinsic energy function $D_V : V \rightarrow \frac{1}{\gamma'}\mathbb{Z}$, defined by

$$(6.23) \quad D_V(b) = \frac{1}{\gamma'} D_{\widehat{V}}(b).$$

Call $(V, \widehat{V}; D_V, D_{\widehat{V}})$ an aligned graded simple virtual crystal.

Let $(V_j, \widehat{V}_j; D_{V_j}, D_{\widehat{V}_j})$ be an aligned graded simple virtual $U'_q(\mathfrak{g})$ -crystal for $j = 1, 2, \dots, L$. Consider the simple $U'_q(\mathfrak{g})$ -crystal $V = V_L \otimes \dots \otimes V_1$ and the simple $U'_q(A_{2n-1}^{(1)})$ -crystal $\widehat{V} = \widehat{V}_L \otimes \dots \otimes \widehat{V}_1$. There are two ways to form an intrinsic energy function for V . One way is to use the fact that it is a tensor product of graded simple $U'_q(\mathfrak{g})$ -crystals (V_j, D_{V_j}) using (2.44). The other way is to use the inherited intrinsic energy function (6.23) coming from the fact that (V, \widehat{V}) is an aligned virtual crystal and $(\widehat{V}, D_{\widehat{V}})$ is a graded simple crystal (with $D_{\widehat{V}}$ defined by (2.44) in terms of $D_{\widehat{V}_j}$). It follows from (6.22) that the two definitions agree.

6.7. Virtual crystal $V^{r,s}$. For \mathfrak{g} of type $D_{n+1}^{(2)}$, $A_{2n}^{(2)}$, $A_{2n}^{(2)\dagger}$, or $C_n^{(1)}$, we wish to define a virtual $U'_q(\mathfrak{g})$ -crystal $V^{r,s}$ ($1 \leq r \leq n, s \geq 1$). Let $\widehat{V}^{r,s}$ be the $U'_q(A_{2n-1}^{(1)})$ -crystal

$$(6.24) \quad \widehat{V}^{r,s} = \begin{cases} B_A^{2n-r,s} \otimes B_A^{r,s} & \text{if } r < n, \\ B_A^{n,s} & \text{if } r = n \text{ and } \mathfrak{g} = D_{n+1}^{(2)}, \\ (B_A^{n,s})^{\otimes 2} & \text{if } r = n \text{ and } \mathfrak{g} = A_{2n}^{(2)}, A_{2n}^{(2)\dagger}, \\ B_A^{n,2s} & \text{if } r = n \text{ and } \mathfrak{g} = C_n^{(1)}. \end{cases}$$

The virtual crystal $(V^{r,s}, \widehat{V}^{r,s})$ is defined as the subgraph of $\widehat{V}^{r,s}$ generated from the extremal vector $u(\widehat{V}^{r,s})$ by applying the virtual operators \tilde{e}_i and \tilde{f}_i ($0 \leq i \leq n$).

Conjecture 6.6. $V^{r,s}$ is aligned.

For $1 \leq i \leq n$, define

$$(6.25) \quad \mu_i = \begin{cases} 2 & \text{if } \mathfrak{g} = A_{2n}^{(2)\dagger} \text{ and } i = n, \\ 1 & \text{otherwise.} \end{cases}$$

The crystals $B^{r,s}$ of Conjecture 2.16 come equipped with the following prescribed decomposition as $U_q(\overline{\mathfrak{g}})$ -crystals:

$$(6.26) \quad B^{r,s} \cong \begin{cases} B(s\overline{\Lambda}_n) & \text{if } r = n \text{ and } \mathfrak{g} = D_{n+1}^{(2)}, C_n^{(1)}, \\ \bigoplus_{\lambda} B(\lambda) & \text{otherwise} \end{cases}$$

where $\lambda \in \overline{P}^+$ has the form $\lambda = \sum_{i=1}^r m_i \mu_i \overline{\Lambda}_i$ such that $\sum_{i=1}^r m_i \leq s$, $s - m_r \in \gamma' \mathbb{Z}$, and $m_i \in \gamma' \mathbb{Z}$ for $1 \leq i \leq r - 1$. Moreover, $B^{r,s}$ comes equipped with the following intrinsic energy function:

$$(6.27) \quad D_{B^{r,s}}(b) = \begin{cases} 0 & \text{if } r = n \text{ and } \mathfrak{g} = D_{n+1}^{(2)}, C_n^{(1)}, \\ (\overline{\Lambda}_n | \lambda - s\mu_r \overline{\Lambda}_r) & \text{if } b \in B(\lambda) \subseteq B^{r,s}. \end{cases}$$

Conjecture 6.7. The virtual crystal $V^{r,s}$ is the simple crystal $B^{r,s}$ of the $U'_q(\mathfrak{g})$ -module $W_s^{(r)}$, with extremal vector $u(V^{r,s}) = u(\widehat{V}^{r,s})$ of weight $s\mu_r \overline{\Lambda}_r$. In particular, $V^{r,s}$ has the $U_q(\overline{\mathfrak{g}})$ -crystal decomposition given by (6.26) and $D_{V^{r,s}} = D_{B^{r,s}}$.

It will be shown in Proposition 6.18 that $V^{r,s}$ has at least the $U_q(\overline{\mathfrak{g}})$ -components specified by (6.26) and that the virtual intrinsic energy on these components agrees with (6.27). If either $\mathfrak{g} = D_{n+1}^{(2)}$ or $s = 1$, it will be shown in Theorems 6.21, 6.23, 6.25, and 6.27 that $V^{r,s}$ is aligned and has exactly the decomposition (6.26).

6.8. Self-duality. We now consider the problem of giving explicit conditions for membership in the virtual crystal $V^{r,s}$ as a subset of $\widehat{V}^{r,s}$. We shall see that a necessary condition for membership in $V^{r,s}$ is self-duality up to local isomorphism.

Let $\widehat{V}_{\text{op}}^{r,s}$ be the $U'_q(A_{2n-1}^{(1)})$ -crystal obtained by reversing the order of the tensor factors in $\widehat{V}^{r,s}$. Let $V_{\text{op}}^{r,s}$ be defined as $V^{r,s}$ is, except with $\widehat{V}^{r,s}$ replaced by $\widehat{V}_{\text{op}}^{r,s}$. There is a $U'_q(A_{2n-1}^{(1)})$ -crystal isomorphism $\sigma : \widehat{V}^{r,s} \cong \widehat{V}_{\text{op}}^{r,s}$ where σ is the identity or the local isomorphism according to whether $\widehat{V}^{r,s}$ has one or two tensor factors. The map σ induces a $U'_q(\mathfrak{g})$ -crystal isomorphism $V^{r,s} \rightarrow V_{\text{op}}^{r,s}$.

Proposition 6.8. *For every $b \in V^{r,s}$,*

$$(6.28) \quad b^{\vee*} = \sigma(b).$$

Proof. First let $b = u(\widehat{V}^{r,s})$. Since $\widehat{V}_{\text{op}}^{r,s}$ is multiplicity-free as a $U_q(A_{2n-1})$ -crystal, it suffices to show that both $b^{\vee*}$ and $\sigma(b)$ are A_{2n-1} -highest weight vectors of the same weight. Since σ is a $U'_q(A_{2n-1}^{(1)})$ -crystal isomorphism, $\sigma(b)$ is a $U_q(A_{2n-1})$ -highest weight vector of weight $\text{wt}_A(b)$. $b^{\vee*} \in \widehat{V}_{\text{op}}^{r,s}$ is an A_{2n-1} -highest weight vector of weight $\text{wt}_A(b^{\vee*}) = -w_0(\text{wt}_A(b))$ by (3.10). Since $\text{wt}_A(b) = \Psi(s\mu_r\bar{\Lambda}_r)$, it follows that $-w_0(\text{wt}_A(b)) = \text{wt}_A(b)$ by (6.8). So (6.28) holds.

Now suppose $b \in \widehat{V}^{r,s}$ satisfies (6.28) and $\tilde{f}_i(b) \neq \emptyset$ for some $0 \leq i \leq n$. It will be shown that $\tilde{f}_i(b)$ also satisfies (6.28). This, together with a similar proof with \tilde{e}_i replacing \tilde{f}_i , suffices. Let $0 \leq j \leq 2n$. We have

$$(6.29) \quad \tilde{f}_j^A(b)^{\vee*} = \tilde{e}_j^A(b^{\vee})^* = \tilde{f}_{2n-j}^A(b^{\vee*}) = \tilde{f}_{2n-j}^A(\sigma(b)) = \sigma(\tilde{f}_{2n-j}^A(b))$$

by (2.12), Theorem 3.8, (6.28), and the fact that σ is a $U'_q(A_{2n-1}^{(1)})$ -crystal isomorphism. (6.28) holds for $\tilde{f}_0(b)$ and $\tilde{f}_n(b)$ by applying (6.29) with $j = 0$ and $j = n$ respectively. (6.28) holds for $\tilde{f}_i(b)$ for $1 \leq i \leq n-1$ by observing that \tilde{f}_j^A and \tilde{f}_{2n-j}^A commute, and applying (6.29) for $\tilde{f}_j^A(b)$ and then for $\tilde{f}_{2n-j}^A\tilde{f}_j^A(b)$. \square

Proposition 6.9. *For all $b \in \widehat{V}^{r,s}$ such that (6.28) holds and for all $1 \leq i \leq n-1$, b is i -aligned.*

Proof. Let $1 \leq i \leq n-1$. Since σ is an isomorphism of $U'_q(A_{2n-1}^{(1)})$ -crystals we have

$$\epsilon_i^A(b) = \epsilon_i^A(\sigma(b)) = \epsilon_i^A(b^{\vee*}) = \varphi_{2n-i}^A(b^{\vee}) = \epsilon_{2n-i}^A(b)$$

by Theorem 3.8 and (2.12). The proof for φ is similar. \square

6.9. Virtual $U_q(\bar{\mathfrak{g}})$ -crystals. Recall that for the affine algebras \mathfrak{g} under consideration $\bar{\mathfrak{g}}$ is either of type B_n or C_n . A pair (V, \widehat{V}) is a virtual $U_q(\bar{\mathfrak{g}})$ -crystal provided that \widehat{V} is a finite $U_q(A_{2n-1})$ -crystal, $V \subset \widehat{V}$ is a subset that satisfies (6.10) and is closed under the virtual operators \tilde{e}_i and \tilde{f}_i for $1 \leq i \leq n$ defined by (6.9). An aligned virtual $U_q(\bar{\mathfrak{g}})$ -crystal is one that consists of elements that are i -aligned for $1 \leq i \leq n$.

Let $\lambda \in \bar{P}^+$ and $\lambda^A = \Psi(\lambda) \in (\bar{P}^A)^+$. Since the virtual operators \tilde{e}_i and \tilde{f}_i (for $1 \leq i \leq n$) are comprised of type $U_q(A_{2n-1})$ operators and the latter preserve the property of being a tableau of a given shape, it follows that $B_A(\lambda^A)$ is closed under these virtual operators. Therefore we may define $V(\lambda)$ to be the virtual

$U_q(\bar{\mathfrak{g}})$ -crystal given by the subset of $B_A(\lambda^A)$ generated from u_{λ^A} using the virtual operators \tilde{e}_i and \tilde{f}_i for $1 \leq i \leq n$.

The following crucial theorem is due to T. Baker [2]. Our approach, which emphasizes the self-duality property, could be used to give an elegant proof of this result.

Theorem 6.10. *$V(\lambda)$ is aligned and is isomorphic to $B(\lambda)$ as a $U_q(\bar{\mathfrak{g}})$ -crystal.*

Proposition 6.11. *Let (V, \hat{V}) be an aligned $U_q(\bar{\mathfrak{g}})$ -crystal. Then every $b \in V$ is in the connected component of a virtual $U_q(\bar{\mathfrak{g}})$ -highest weight vector u , and u is a $U_q(A_{2n-1})$ -highest weight vector when viewed as an element of \hat{V} .*

Proof. Let $b \in V$. Suppose $\epsilon_j^A(b) > 0$ for some $1 \leq j \leq 2n - 1$. Since (V, \hat{V}) is an aligned virtual $U_q(\bar{\mathfrak{g}})$ -crystal, it follows that $\tilde{e}_i(b) \in V$ where $i = j$ for $1 \leq j \leq n$ and $i = 2n - j$ for $n < j \leq 2n - 1$. Replacing b with $\tilde{e}_i(b)$ and continuing in this manner, eventually one has $\epsilon_j^A(b) = 0$ for $1 \leq j \leq 2n - 1$. \square

Proposition 6.12. *Let $\lambda \in \bar{P}^+$ and $b \in B_A(\lambda^A)$. If $b \in V(\lambda)$, then*

$$(6.30) \quad \mathbb{P}(b^{\vee*}) = b.$$

Moreover, if $\bar{\mathfrak{g}} = B_n$ and b satisfies (6.30), then $b \in V(\lambda)$.

Proof. The proof that elements of $V(\lambda)$ satisfy (6.30) is similar to the proof of Proposition 6.8 and is omitted. Suppose $\bar{\mathfrak{g}} = B_n$. Let $b \in B_A(\lambda^A)$ be such that (6.30) holds. Let $V' \subset B_A(\lambda^A)$ be the virtual $U_q(B_n)$ -crystal generated by b . It suffices to show that $u_{\lambda^A} \in V'$. By the proof of Proposition 6.8, (6.30) holds for any element in V' . But (6.30) implies that V' is i -aligned for $1 \leq i \leq n - 1$, in a manner similar to the proof of Proposition 6.9. By Remark 6.3 V' is n -aligned. Therefore, $(V', B_A(\lambda^A))$ is an aligned $U_q(B_n)$ -crystal. By Proposition 6.11 $u_{\lambda^A} \in V'$. \square

6.10. $U_q(C_n)$ -crystals. For this section let $\bar{\mathfrak{g}} = C_n$. The crystal graphs of the irreducible integrable $U_q(C_n)$ -modules $B(\lambda)$ for $\lambda \in \bar{P}^+$, were constructed explicitly by Kashiwara and Nakashima [20] in terms of certain tableaux.

The crystal graph $B(\bar{\Lambda}_1)$ is given by

$$(6.31) \quad \boxed{1} \xrightarrow{1} \boxed{2} \xrightarrow{2} \dots \xrightarrow{n-1} \boxed{n} \xrightarrow{n} \boxed{\bar{n}} \xrightarrow{n-1} \dots \xrightarrow{2} \boxed{\bar{2}} \xrightarrow{1} \boxed{\bar{1}}$$

with $\text{wt}(i) = \bar{\Lambda}_i - \bar{\Lambda}_{i-1} = -\text{wt}(\bar{i})$ for $1 \leq i \leq n$.

Let \mathbb{A} be the set of vertices of $B(\bar{\Lambda}_1)$, totally ordered so that the elements appear from smallest to largest going from left to right in (6.31). Let $|u|$ denote the length of the word u . A subset $A \subset [n]$ is often identified with the column word having precisely the letters in A . Write $A^c = [n] - A$, $\bar{A} = \{\bar{i} \mid i \in A\}$, and $\bar{A}^c = [\bar{n}] - \bar{A} = \bar{A}^c$. In particular, if u and v are column words in the alphabet $[n]$, then $\bar{v}u$ is a column word in the alphabet \mathbb{A} and $(\bar{v}u)^{\vee*} = \bar{v}^c v^c$. If w is a column word in the alphabet \mathbb{A} , then it can be written uniquely in the form $w = \bar{v}u$ where u and v are column words in the alphabet $[n]$. In this case write $u = w_+$ and $v = w_-$.

The crystal $B(\bar{\Lambda}_r)$ may be defined as the connected component in the tensor product $B(\bar{\Lambda}_1)^{\otimes r}$ of the column word $r(r-1)\cdots 21$ where the tensor symbols are omitted. Explicitly, $B(\bar{\Lambda}_r)$ is the set of column words P of length r in the alphabet $[n] \cup [\bar{n}]$ that satisfy the one-column condition [20]:

$$(6.32) \quad \text{If } i \text{ and } \bar{i} \text{ are both in } P, \text{ then } |P_+|_{[i]} + |P_-|_{[\bar{i}]} \leq i.$$

Let $\lambda \in \overline{P}^+$. Write $\lambda = \sum_{j=1}^p \overline{\Lambda}_{m_j}$ where $m_1 \geq m_2 \geq \dots \geq m_p$. Then $B(\lambda)$ may be defined as the connected component of $u_{\overline{\Lambda}_{m_1}} \otimes \dots \otimes u_{\overline{\Lambda}_{m_p}}$ in $B(\overline{\Lambda}_{m_1}) \otimes \dots \otimes B(\overline{\Lambda}_{m_p})$.

Lemma 6.13. *P satisfies (6.32) if and only if $P_-^c P_+$ is a tableau.*

Proof. The following are equivalent:

1. P does not satisfy (6.32).
2. There is an index $1 \leq i \leq n$ such that $i \in P_+, i \notin P_-^c$, and $|P_+|_{[i]} > |P_-^c|_{[i]}$.
3. There is an index $1 \leq i \leq n$ such that $|P_+|_{[i]} > |P_-^c|_{[i]}$.
4. $P_-^c P_+$ is not a tableau.

Each of the above conditions is obviously equivalent to the next except for 3 implies 2. For that case just take i to be minimal. If $i \notin P_+$ or $i \in P_-^c$, then the index $i - 1$ satisfies 3, contradicting minimality. \square

We now recall the explicit isomorphism $B(\lambda) \rightarrow V(\lambda)$ given in [2]. The first case is $\lambda = \overline{\Lambda}_r$. For $P \in B(\overline{\Lambda}_r)$, let

$$(6.33) \quad \begin{aligned} K &= P_+ \cap P_-, \\ J &= \max\{A \subset (P_+ \cup P_-)^c \mid |A| = |K| \text{ and } A < K\} \end{aligned}$$

where the maximum is computed with respect to the partial order on column words that says $u \leq v$ if and only if uv is a tableau. This given, let

$$(6.34) \quad Q_{\pm} = (P_{\pm} - K) \cup J.$$

Define the map $\overline{\Psi} : B(\overline{\Lambda}_r) \rightarrow B_A(\overline{\Lambda}_{2n-r} + \overline{\Lambda}_r)$ by

$$(6.35) \quad P \mapsto \overline{Q}_+^c P_-^c \otimes \overline{Q}_- P_+$$

where $B_A(\overline{\Lambda}_{2n-r} + \overline{\Lambda}_r)$ is regarded as the $U_q(A_{2n-1})$ -subcrystal of $B_A(\overline{\Lambda}_{2n-r}) \otimes B_A(\overline{\Lambda}_r)$ by taking the column-reading word. Lemma 6.13 ensures that the column-reading word indeed corresponds to a tableau.

Let $\lambda = \overline{\Lambda}_{m_1} + \overline{\Lambda}_{m_2} + \dots + \overline{\Lambda}_{m_p}$ for $1 \leq m_j \leq n$, $\lambda^A = \Psi(\lambda)$, and $b = P^{(1)} \otimes \dots \otimes P^{(p)} \in B(\lambda)$ where $P^{(j)} \in B(\overline{\Lambda}_{m_j})$. Define $\overline{\Psi} : B(\lambda) \rightarrow B_A(\lambda^A)$ by

$$(6.36) \quad \overline{\Psi}(b) = \mathbb{P}(\overline{\Psi}(P^{(1)})\overline{\Psi}(P^{(2)}) \dots \overline{\Psi}(P^{(p)})).$$

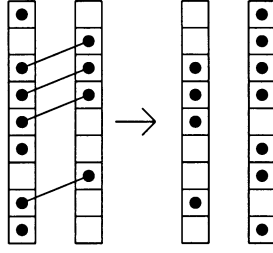
We observe that Q_+ and Q_- may be computed from P by local isomorphisms of type A for tensor products of columns.

Lemma 6.14.

$$(6.37) \quad \sigma(P_-^c \otimes P_+) = Q_+ \otimes Q_-^c.$$

Proof. The construction in the definition of Q_+ and Q_- is equivalent to the algorithm of section 3.11 used to compute $\sigma(P_-^c \otimes P_+)$. \square

Example 6.15. Let $n = 9$ and $P = \overline{388763}$. Then $P_- = 83$, $P_+ = 8763$, $K = 83$, $J = 52$, $Q_- = 52$, and $Q_+ = 7652$. On the other hand, $\sigma(P_-^c \otimes P_+)$ may be computed using the algorithm described in section 3.11. Note that $P_-^c = 9765421$ so that



K is the set of all heights which has a dot on the right, but no dot on the left. $(P_+ \cup P_-)^c$ is the set of heights with a dot on the left, but no dot on the right. Hence J is obtained by selecting from top to bottom for each element $i \in K$ the maximal $h < i$ with a dot on the left and no dot on the right at height h . The elements in J are all selected by the algorithm for the computation of σ as described in section 3.11, and hence both computations of J are equivalent.

For the following Lemma, we identify \mathbb{A} with $[2n]$ via the bijection $i \mapsto i$ and $\bar{i} \mapsto 2n + 1 - i$ for $i \in [n]$.

Lemma 6.16. *Let u and v be column words in the alphabet $[n]$. Then there is a unique pair of column words u' and v' in the alphabet $[n]$ such that $|u| = |u'|$, $|v| = |v'|$, and (defining $s = \overline{u'}^c v$ and $t = \overline{v'}^c u$)*

$$(6.38) \quad \sigma(s \otimes t) = s^{\vee*} \otimes t^{\vee*}$$

where $\sigma : B_A^{2n-k,1} \otimes B_A^{k,1} \rightarrow B_A^{k,1} \otimes B_A^{2n-k,1}$ is the local isomorphism of type $A_{2n-1}^{(1)}$ with $k = n + |u| - |v|$.

Proof. For existence, define u' and v' by

$$(6.39) \quad \sigma(v \otimes u) = u' \otimes v'.$$

By definition u' and v' are column words of the correct length. Applying the map \vee^* to (6.39), by Propositions 3.13 and 3.14 and the fact that σ is an involution, one has

$$(6.40) \quad \sigma(\overline{u'}^c \otimes \overline{v'}^c) = \overline{v}^c \otimes \overline{u}^c.$$

By definition $s = \overline{u'}^c v$ and $t = \overline{v'}^c u$, so that s and $t^{\vee*}$ are column words of the same length and t and $s^{\vee*}$ are column words of the same length. Thus to prove (6.38) it is enough to show that $\mathbb{P}(st) = \mathbb{P}((st)^{\vee*})$. Since the shapes of both $\mathbb{P}(st)$ and $\mathbb{P}((st)^{\vee*})$ have two columns, it suffices to show that

1. $\mathbb{P}(st|_{\mathbb{B}}) = \mathbb{P}((st)^{\vee*}|_{\mathbb{B}})$ for $\mathbb{B} = [n]$ and for $\mathbb{B} = \overline{[n]}$.
2. $\mathbb{P}(st)$ and $\mathbb{P}((st)^{\vee*})$ have the same shape.

Now $s^{\vee*} = \overline{v}^c u'$ and $t^{\vee*} = \overline{u}^c v'$. We have

$$\mathbb{P}((st)^{\vee*}|_{[n]}) = \mathbb{P}(u'v') = \mathbb{P}(vu) = \mathbb{P}(st|_{[n]})$$

by (6.39) and

$$\mathbb{P}((st)^{\vee*}|_{\overline{[n]}}) = \mathbb{P}(\overline{v}^c \overline{u}^c) = \mathbb{P}(\overline{u'}^c \overline{v'}^c) = \mathbb{P}(st|_{\overline{[n]}})$$

by (6.40). This proves 1. Condition 2 is equivalent to $Q_1 = \mathbb{Q}(st)^t$ and $Q_2 = \mathbb{Q}((st)^{\vee*})^t$ have the same shape. But $Q_2 = \mathbb{Q}(st)^{t^\wedge} = Q_1^\wedge$ by Proposition 4.2 with $N = 2$. So it is enough to show that $\text{shape}(Q_1)^\wedge = \text{shape}(Q_1)$. But this holds since

shape(Q_1) has at most two rows and exactly $2n$ cells (since \wedge is taken within the $2 \times 2n$ rectangle).

For uniqueness, observe that $\mathbb{Q}(vu)^t$ is a tableau with $|u|$ ones and $|v|$ twos, and $\mathbb{Q}(u'v')^t$ is a tableau of the same shape with $|v|$ ones and $|u|$ twos. The second tableau is uniquely specified by this property. \square

Proposition 6.17. *Let $b = u \otimes v \in B_A(\overline{\Lambda}_{2n-r} + \overline{\Lambda}_r)$. Write $u_1 = u|_{[n]}$ and $u_2 = u|_{[n+1, 2n]}$ and similarly for v . The following are equivalent:*

1. $b \in V(\overline{\Lambda}_r)$.
2. (6.30) holds, u_1v_1 is a tableau and $|u_1| - |v_1| = n - r$.
3. (6.30) holds, u_2v_2 is a tableau and $|u_2| - |v_2| = n - r$.

Proof. 1 implies 2: Let $P \mapsto b$ under the $U_q(C_n)$ -crystal isomorphism $B(\overline{\Lambda}_r) \rightarrow V(\overline{\Lambda}_r)$, given by (6.35). Then $u_2 = \overline{Q}_+^c$, $u_1 = P_-^c$, $v_2 = \overline{Q}_-$, and $v_1 = P_+$ with Q_\pm defined by Lemma 6.14. By Lemma 6.13 u_1v_1 is a tableau, with $|u_1| - |v_1| = n - |P_-| - |P_+| = n - |P| = n - r$. 2 implies 1: Let P be such that $P_-^c = u|_{[n]}$ and $P_+ = v|_{[n]}$. By Lemma 6.13, $P \in B(\overline{\Lambda}_r)$. By Lemma 6.16 it follows that $P \mapsto b$ under the isomorphism $B(\overline{\Lambda}_r) \rightarrow V(\overline{\Lambda}_r)$.

For the equivalence of 2 and 3, suppose that P and Q are such that (6.37) holds. Then the following are equivalent:

1. $P_-^c P_+$ is a tableau.
2. $Q_+ Q_-^c$ is the column-reading word of a skew two-column tableau with unique southeast corner.
3. $\overline{Q}_-^c \overline{Q}_+$ is a tableau.
4. $\overline{Q}_+^c \overline{Q}_-$ is a tableau.

Moreover, it is easily seen that $|P_-^c| - |P_+| = |\overline{Q}_+^c| - |\overline{Q}_-|$. Now suppose 2 holds.

Write b in the form $\overline{Q}_+^c P_-^c \otimes \overline{Q}_- P_+$. By the proof of Lemma 6.16 it follows that (6.37) holds. Thus 3 follows. The proof that 3 implies 2 is entirely similar. \square

6.11. $V^{r,s}$ contains the prescribed $U_q(\mathfrak{g})$ -components.

Proposition 6.18. *Let $\lambda \in \overline{P}^+$ be such that $B(\lambda)$ is a summand specified by (6.26). Then there is a unique $U_q(\mathfrak{g})$ -crystal embedding $\iota_\lambda : V(\lambda) \rightarrow V^{r,s}$ defined by $b' \mapsto b$ where $b' = \mathbb{P}(b)$, and $D_{V^{r,s}} = D_{B^{r,s}}$ on $\text{Im}(\iota_\lambda)$.*

Proof. Let $\lambda = \sum_{j=1}^r m_j \overline{\Lambda}_j \in \overline{P}^+$ be any weight such that $\Psi(\lambda)$ occurs in the $U_q(A_{2n-1})$ -crystal decomposition of $\widehat{V}^{r,s}$. $B_A(\Psi(\lambda))$ occurs in $\widehat{V}^{r,s}$ with multiplicity one, and on this component $D_{\widehat{V}^{r,s}}$ has value 0 if $\widehat{V}^{r,s}$ has one tensor factor and value $-rs + \sum_{j=1}^r jm_j$ otherwise [42, 41]. Let $v_\lambda \in \widehat{V}^{r,s}$ be the unique $U_q(A_{2n-1})$ -highest weight vector in $\widehat{V}^{r,s}$ with $\text{wt}_A(v_\lambda) = \Psi(\lambda)$. There is an embedding of virtual $U_q(\mathfrak{g})$ -crystals $V(\lambda) \rightarrow V^{r,s}$ given by $u \mapsto v$ where $u = \mathbb{P}(v)$. The map is a morphism of virtual $U_q(\mathfrak{g})$ -crystals since \mathbb{P} is a morphism of $U_q(A_{2n-1})$ -crystals.

Now let $\lambda \in \overline{P}^+$ be a weight appearing in (6.26). Then $B_A(\Psi(\lambda))$ occurs in $\widehat{V}^{r,s}$. By the argument in the previous paragraph it suffices to show that $v_\lambda \in V^{r,s}$. It can be shown that there is an explicit sequence of virtual operators going from $u(\widehat{V}^{r,s})$ to v_λ . Moreover, one has $D_{V^{r,s}}(v_\lambda) = 0$ if $\widehat{V}^{r,s}$ has a single tensor factor,

and otherwise

$$D_{V^{r,s}}(v_\lambda) = \frac{1}{\gamma'} D_{\widehat{V}^{r,s}}(v_\lambda) = \frac{1}{\gamma'} (-rs + \sum_{j=1}^r jm_j) = (\lambda - s\mu_r \overline{\Lambda}_r | \overline{\Lambda}_n) = D_{B^{r,s}}(v_\lambda).$$

□

Instead of writing this sequence of operators in full generality, we give the following illustrative example.

Example 6.19. Let $\mathfrak{g} = C_n^{(1)}$, $n = 4$, $r = 3$, $s = 5$, and $\lambda = 2\overline{\Lambda}_2 + \overline{\Lambda}_3$. We work in $V_{\text{op}}^{r,s}$ and $\widehat{V}_{\text{op}}^{r,s}$ for convenience. Using the explicit rules for computing the operators of type $A_{2n-1}^{(1)}$,

$$u(\widehat{V}_{\text{op}}^{3,5}) \xrightarrow{\tilde{e}_2^4 \tilde{e}_1^4 \tilde{e}_0^2} v_{4\overline{\Lambda}_2 + \overline{\Lambda}_3} \xrightarrow{\tilde{e}_1^2 \tilde{e}_0} v_{2\overline{\Lambda}_1 + 2\overline{\Lambda}_2 + \overline{\Lambda}_3} \xrightarrow{\tilde{e}_0} v_{2\overline{\Lambda}_2 + \overline{\Lambda}_3}.$$

Each of the above vectors has the form $b \otimes u(B_A^{5,5})$, where b is given respectively by

| | | | | |
|---|---|---|---|---|
| 1 | 1 | 1 | 1 | 1 |
| 2 | 2 | 2 | 2 | 2 |
| 3 | 3 | 3 | 3 | 3 |

| | | | | |
|---|---|---|---|---|
| 1 | 1 | 1 | 1 | 1 |
| 2 | 2 | 2 | 2 | 2 |
| 3 | 6 | 6 | 6 | 6 |

| | | | | |
|---|---|---|---|---|
| 1 | 1 | 1 | 1 | 1 |
| 2 | 2 | 2 | 6 | 6 |
| 3 | 6 | 6 | 7 | 7 |

| | | | | |
|---|---|---|---|---|
| 1 | 1 | 1 | 6 | 6 |
| 2 | 2 | 2 | 7 | 7 |
| 3 | 6 | 6 | 8 | 8 |

This shows that $v_\lambda \in V^{r,s}$.

6.12. Characterization of $V^{r,s}$. We characterize the elements of $V^{r,s}$ inside $\widehat{V}^{r,s}$ using the self-duality condition (6.28).

Proposition 6.20. *Let $\mathfrak{g} = D_{n+1}^{(2)}$ and $b \in \widehat{V}^{r,s}$. Then $b \in V^{r,s}$ if and only if (6.28) holds.*

Proof. The forward direction holds by Proposition 6.8. For the reverse direction, suppose $b \in \widehat{V}^{r,s}$ satisfies (6.28). Let $V \subseteq \widehat{V}^{r,s}$ be the virtual $U'_q(\mathfrak{g})$ -crystal generated by b . It follows from the proof of Proposition 6.8 that every element of V satisfies (6.28). But V is automatically aligned by Remark 6.3 and Proposition 6.9. By Proposition 6.11 there is a $U_q(\overline{\mathfrak{g}})$ -highest weight vector $u \in V$ that is also a $U_q(A_{2n-1})$ -highest weight vector. Since u satisfies (6.28), $\text{wt}_A(u) \in \text{Im}(\Psi) \cap (\overline{P}^A)^+$. Since $u \in \widehat{V}^{r,s}$, it follows that $\text{wt}_A(u) = \Psi(\lambda)$ for some λ appearing in (6.26). By Proposition 6.18 $u \in V^{r,s}$. It follows that $V = V^{r,s}$. □

Theorem 6.21. *Let $\mathfrak{g} = D_{n+1}^{(2)}$. Then the $U'_q(\mathfrak{g})$ -crystal $V^{r,s}$ is aligned and has the $U_q(\overline{\mathfrak{g}})$ -crystal decomposition given in (6.26).*

Proof. This follows from the proof of Proposition 6.20. □

Lemma 6.22. *Let b be an element of the $U_q(A_{2n-1})$ -crystal $B_A(\overline{\Lambda}_{2n-r}) \otimes B_A(\overline{\Lambda}_r)$ such that (6.30) holds for $\mathbb{P}(b)$. Let c_j be the number of occurrences of j in b for $1 \leq j \leq 2n$. Then $c_j + c_{2n+1-j} = 2$ for all $1 \leq j \leq n$.*

Proof. The lemma follows directly from $\text{wt}_A(b) = -w_0 \text{wt}_A(b)$, which holds by (6.30) for $\mathbb{P}(b)$, (2.12) and (3.9). □

Theorem 6.23. *Let $\mathfrak{g} = C_n^{(1)}$. Then $V^{r,1}$ is an aligned $U'_q(C_n^{(1)})$ -crystal and $V^{r,1} \cong V(\overline{\Lambda}_r)$ as a $U_q(C_n)$ -crystal.*

Recall that the characterization of the $U_q(C_n)$ -crystal $V(\overline{\Lambda}_r)$ was given in Proposition 6.17.

Proof. By Proposition 6.18 $V(\overline{\Lambda}_r) \subseteq V^{r,1}$ and the embedding is inclusion. To show equality it suffices to show that $V(\overline{\Lambda}_r)$ is closed under \tilde{f}_0 and \tilde{e}_0 . To check this we use the explicit computation of the 0-string in $U'_q(A_{2n-1}^{(1)})$ -crystals given in section 3.4. Let $b = u \otimes v \in V(\overline{\Lambda}_r) \subset B_A(\overline{\Lambda}_{2n-r} + \overline{\Lambda}_r)$ such that $\tilde{f}_0(b) \neq \emptyset$. Write $\tilde{f}_0(b) = w \otimes x \in \widehat{V}^{r,1}$. Then u and v both contain $2n$ and do not contain 1. Computing $\tilde{f}_0 = (\tilde{f}_0^A)^2$ on b , in all cases w and x are obtained from u and v respectively by removing $2n$ from the bottom and adding 1 at the top. Thus wx is a tableau. $\tilde{f}_0(b) \in V^{r,1}$ satisfies (6.28) by Proposition 6.8. Let w_j and x_j be defined as u_j for $j = 1, 2$. Then w_2x_2 is a tableau, being obtained from u_2v_2 by removing $2n$ from the bottom of each column. From this, one also sees that $|w_2| - |x_2| = n - r$. By Proposition 6.17, $\tilde{f}_0(b) \in V(\overline{\Lambda}_r)$. In an entirely similar manner, one may show that $V(\overline{\Lambda}_r)$ is closed under \tilde{e}_0 .

By Theorem 6.10 it remains to show that the elements of $V(\overline{\Lambda}_r) \subseteq \widehat{V}^{r,1}$ are 0-aligned. By Lemma 6.22, we need only check the case that 1 and $2n$ occur in b once each. Since uv is a tableau, $1 \in u$. If $2n \in u$, then $\epsilon_0^A(b) = \varphi_0^A(b) = 0$ and b is 0-aligned. So assume $2n \in v$. If $r < n$, then $\widehat{V}^{r,1}$ has two tensor factors, and by (2.11) and (2.10) b is 0-aligned. So let $r = n$. Write $u = \widehat{u}1$ and $v = (2n)\widehat{v}$. We will show that $\epsilon_0^A(b) = 0$, as the proof of $\varphi_0^A(b) = 0$ is similar. $\tilde{e}_0^A = \psi \circ \tilde{e}_{2n-1}^A \circ \psi^{-1}$ by (3.5). Thus it is enough to show that $\epsilon_{2n-1}(\psi^{-1}(b)) = 0$. Since $\psi^{-1}(b)$ has a single $2n - 1$ and a single $2n$, this holds if $2n - 1$ is in the right-hand column of $\psi^{-1}(b)$. But

$$\psi^{-1}(b)|_{[2n-1]} = \mathbb{P}(b|_{[2,2n]}) = \mathbb{P}(u\widehat{v}) = \sigma(u \otimes \widehat{v})$$

by (3.6). So it suffices to show that $2n - 1$ remains in the right-hand column in passing from $u \otimes \widehat{v}$ to $\sigma(u \otimes \widehat{v})$. But there is only one letter that moves from the right hand column to the left under σ , and it must be in v_1 (and therefore is not $2n - 1$) since $u_1v_1 = (uv)|_{[n]}$ is a tableau with columns of equal size. \square

Proposition 6.24. *Let $\mathfrak{g} = A_{2n}^{(2)}$, $V = \bigoplus_{k=0}^r \text{Im}(\iota_{\overline{\Lambda}_k}) \subseteq \widehat{V}^{r,1}$, $b = u \otimes v \in \widehat{V}^{r,1}$, and u_j and v_j defined as in Proposition 6.17 for $j = 1, 2$. Then $b \in V$ if and only if*

1. b satisfies (6.28).
2. u_2v_2 is a tableau.
3. $|u_2| - |v_2| \geq n - r$.

Proof. The following are equivalent:

- b satisfies (6.28).
- $\mathbb{P}(b^{V^*}) = \mathbb{P}(\sigma(b))$.
- $\mathbb{P}(b)$ satisfies (6.30).

The equivalence of the first two items follows from the fact that $\widehat{V}^{r,1}$ is multiplicity-free as a $U_q(A_{2n-1})$ -crystal and \mathbb{P} is a $U_q(A_{2n-1})$ -crystal morphism. The equivalence of the second and third items is a consequence of Propositions 3.5 and 3.10, and the definition of σ .

Write $\mathbb{P}(b) = u' \otimes v'$ where u' and v' are column words. Since u and v are column words of lengths $2n - r$ and r respectively, it follows that u' and v' have lengths $2n - k$ and k for some $0 \leq k \leq r$. Moreover, one can pass between b and $\mathbb{P}(b)$ using a two column jeu de taquin. Let u'_j and v'_j be defined in a manner similar to u_j for $j = 1, 2$.

Suppose first that $b \in V$. Then $\mathbb{P}(b) \in V(\overline{\Lambda}_k)$ for some $0 \leq k \leq r$. By Proposition 6.17, $\mathbb{P}(b)$ satisfies (6.30). By the above argument b satisfies (6.28). By Proposition 6.17 $u'_2 v'_2$ is a tableau and $|u'_2| - |v'_2| = n - k \geq n - r$. To finish the forward direction it is enough to show that $u'_2 = u_2$ and $v'_2 = v_2$. In passing from $u' \otimes v'$ to $u \otimes v$, some letters go from the left column to the right. Since all letters in v'_2 are blocking letters of u'_2 , the only letters that can block u'_1 from moving to the right are those in v'_1 . But by Proposition 6.17, $|u'_1| - |v'_1| = n - k$, so there are $n - k$ letters in u'_1 available to move to the right. The smallest $r - k$ of these, actually do move. In particular, $u'_2 = u_2$ and $v'_2 = v_2$.

For the reverse direction, suppose $b \in \widehat{V}^{r,1}$ satisfies the three properties. Since $\mathbb{P}(b)$ is a tableau, it suffices to show that $\mathbb{P}(b)$ satisfies the properties in Proposition 6.17 for k . Since b satisfies (6.28), $\mathbb{P}(b)$ satisfies (6.30), arguing as above. This time, passing from $u \otimes v$ to $u' \otimes v'$, we assume that $u_2 v_2$ is a tableau. That means that all the letters of v_2 cannot move to the left, and so stay in their column. The left column only gets larger, and can only get larger by letters in the interval $[n]$. Therefore, $u'_2 = u_2$ and $v'_2 = v_2$ again. \square

Theorem 6.25. *Let $\mathfrak{g} = A_{2n}^{(2)}$. Then the $U_q(A_{2n}^{(2)})$ -crystal $V^{r,1}$ is aligned, and as a $U_q(C_n)$ -crystal,*

$$V^{r,1} \cong \bigoplus_{k=0}^r V(\overline{\Lambda}_k).$$

Moreover, the $U_q(C_n)$ -crystal embedding $\iota_{\overline{\Lambda}_k} : V(\overline{\Lambda}_k) \rightarrow V^{r,1}$ can be computed by $\iota_{\overline{\Lambda}_k} = i_{r-1,1} \circ i_{r-2,1} \circ \dots \circ i_{k,1}$ with $i_{r,s}$ as defined in section 5.6.

Proof. Observe that alignedness follows from the above $U_q(C_n)$ -crystal decomposition, since $\gamma' = 1$ for $\mathfrak{g} = A_{2n}^{(2)}$ and the virtual $U_q(C_n)$ -crystals $V(\overline{\Lambda}_k)$ are i -aligned for $1 \leq i \leq n$. The map $i_{r-1,1} \circ \dots \circ i_{k,1}$ coincides with $\iota_{\overline{\Lambda}_k}$ by Proposition 6.18, since both are $U_q(C_n)$ -crystal embeddings $V(\overline{\Lambda}_k) \rightarrow \widehat{V}^{r,1}$. Let V be as in Proposition 6.18, which asserts that $V \subseteq V^{r,1}$. To show equality it suffices to show that V is closed under \tilde{e}_0 and \tilde{f}_0 .

Recall that for $\mathfrak{g} = A_{2n}^{(2)}$, $\widehat{V}^{r,1} = B_A^{2n-r,1} \otimes B_A^{r,1}$ and $\tilde{f}_0 = \tilde{f}_0^A$. Let $b = u \otimes v$ be such that $\tilde{f}_0(b) \neq \emptyset$. Write $\tilde{f}_0(b) = w \otimes x$. Let w_j and x_j be defined as u_j is for $j = 1, 2$. It will be shown that $\tilde{f}_0(b)$ satisfies the three properties in Proposition 6.24. Since $b \in V \subseteq V^{r,1}$, $\tilde{f}_0(b)$ satisfies (6.28) by Proposition 6.8.

Suppose first that $\tilde{f}_0(u \otimes v) = u \otimes \tilde{f}_0(v)$. Then $u = w$ and $v_2 = (2n)x_2$. It is easily seen that properties 2 and 3 of Proposition 6.24 hold for $\tilde{f}_0(b)$, so that $\tilde{f}_0(b) \in V$.

Otherwise, suppose $\tilde{f}_0(u \otimes v) = \tilde{f}_0(u) \otimes v$. Then $x = v$, $2n \in u$, $1 \notin u$, and w is obtained from u by removing $2n$ from the bottom and putting 1 at the top. In particular, $u_2 = (2n)w_2$ and $v_2 = x_2$. The only way that $w_2 x_2$ fails to be a tableau is if $n = r$, when the first column is shorter than the second. Since $u_2 v_2$ is a tableau with columns of equal length, $2n \in v_2$. By Lemma 6.22 there are no ones present. But then $\tilde{f}_0(u \otimes v) = u \otimes \tilde{f}_0(v)$, contrary to assumption. Therefore, $w_2 x_2$ is a tableau.

To check property 3 of Proposition 6.24 it is enough to show $|u_2| - |v_2| > n - r$. Suppose not, that is, equality holds. Following the proof of Proposition 6.24, write $u' \otimes v' = \mathbb{P}(u \otimes v)$ and define u'_j and v'_j . Then $u'_1 \otimes v'_1 = \mathbb{P}(u_1 \otimes v_1)$. As in the

mentioned proof it can be shown that $u' \otimes v' \in V(\overline{\Lambda}_k)$ for some $0 \leq k \leq r$ and that $u'_2 = u_2$ and $v'_2 = v_2$. By our assumption it follows that $k = r$ and hence that $u' = u$ and $v' = v$. In particular, uv is a tableau. By Lemma 6.22 $1 \in v$ or $2n \in v$ but not both. But $1 \in v$ is not possible since uv is a tableau and $1 \notin u$. And if $2n \in v$, then $\tilde{f}_0^A(u \otimes v) = u \otimes \tilde{f}_0^A(v)$, contrary to our assumption.

The proof that V is closed under \tilde{e}_0 is similar. □

Proposition 6.26. *Let $\mathfrak{g} = A_{2n}^{(2)\dagger}$ and $1 \leq r \leq n$. Recall that $V(\mu_r \overline{\Lambda}_r) \subset B_A(\overline{\Lambda}_{2n-r} + \overline{\Lambda}_r) \subset \widehat{V}^{r,1}$. Then $b \in V(\mu_r \overline{\Lambda}_r)$ if and only if $b \in \widehat{V}^{r,1}$, b is a tableau, and b satisfies (6.28).*

Proof. This follows from Proposition 6.12. □

Theorem 6.27. *Let $\mathfrak{g} = A_{2n}^{(2)\dagger}$. Then the virtual $U'_q(A_{2n}^{(2)\dagger})$ -crystal $V^{r,1}$ is aligned and is isomorphic to $V(\mu_r \overline{\Lambda}_r)$ as a $U_q(B_n)$ -crystal.*

Proof. It suffices to show that $(V(\mu_r \overline{\Lambda}_r, \widehat{V}^{r,1}))$ is 0-aligned and closed under $\tilde{e}_0 = (\tilde{e}_0^A)^2$ and $\tilde{f}_0 = (\tilde{f}_0^A)^2$. Let $b \in V(\mu_r \overline{\Lambda}_r)$. By Proposition 6.26 and Lemma 6.22, b is a tableau and $c_1 + c_{2n} = 2$ where c_j is the number of occurrences of j in b for $1 \leq j \leq 2n$. Write $b = u \otimes v$ where u and v are column words of type A_{2n-1} . Suppose first that $c_1 = 2$. Then $\epsilon_0^A(b) = 2$ and $\varphi_0^A(b) = 0$, so that b is 0-aligned. Writing $\tilde{e}_0(b) = w \otimes x$ the column words w and x are obtained from u and v by removing 1 from the top and putting $2n$ at the bottom. Clearly, $\tilde{e}_0(b)$ is a tableau. Since $\tilde{e}_0(b) \in V^{r,1}$, it satisfies (6.28) by Proposition 6.8 and is therefore in $V(\mu_r \overline{\Lambda}_r)$ by Proposition 6.26. An entirely similar argument applies for the case $c_1 = 0$. Finally, suppose $c_1 = 1$. Since b is a tableau, $1 \in u$. Regardless of where the single symbol $2n$ appears in b , one has $\epsilon_0^A(b) = \varphi_0^A(b) = 0$, so that b is again 0-aligned. □

6.13. $B^{r,1}$ for $\mathfrak{g} = C_n^{(1)}$. The goal of this section is to establish Conjecture 6.7 for $V^{r,1}$ of type $C_n^{(1)}$. The $U'_q(C_n^{(1)})$ -module $W_1^{(r)}$ exists and has a simple crystal basis $B^{r,1}$ [1, 19]. We use the symmetry of the Dynkin diagram $C_n^{(1)}$ to derive properties which uniquely define the structure of the affine crystal $B^{r,1}$. Then we show that the virtual crystal $V^{r,1}$ satisfies these properties.

We state some generalities which must be satisfied by $B^{r,s}$ assuming it exists. Consider the automorphism of the Dynkin diagram $C_n^{(1)}$ given by $i \mapsto n - i$ for $0 \leq i \leq n$; it has order two. It induces an involution on crystal bases of $U'_q(C_n^{(1)})$ -modules. Let $P \rightarrow P$ be the linear involution given by $\Lambda_i \mapsto \Lambda_{n-i}$ for $i \in I$. There is an induced linear involution $\psi_C : \overline{P} \rightarrow \overline{P}$ given by $\overline{\Lambda}_i \mapsto \overline{\Lambda}_{n-i} - \overline{\Lambda}_n$ for all $i \in J$. Identifying $\overline{P} \cong \mathbb{Z}^n$ one has

$$(6.41) \quad \psi_C(a_1, a_2, \dots, a_n) = (-a_n, \dots, -a_2, -a_1).$$

The existence of the Dynkin diagram automorphism implies the following result.

Proposition 6.28. *Let B be the crystal basis of an irreducible integrable $U'_q(C_n^{(1)})$ -module. Then there is an automorphism ψ_C of B with the following properties:*

$$(6.42) \quad \psi_C^2 = 1,$$

$$(6.43) \quad \text{wt} \circ \psi_C = \psi_C \circ \text{wt},$$

$$(6.44) \quad \psi_C \circ \tilde{f}_i = \tilde{f}_{n-i} \circ \psi_C \quad \text{for all } i \in I,$$

$$(6.45) \quad \psi_C \circ \tilde{e}_i = \tilde{e}_{n-i} \circ \psi_C \quad \text{for all } i \in I.$$

In particular,

$$(6.46) \quad \tilde{f}_0 = \psi_C \circ \tilde{f}_n \circ \psi_C \quad \text{and} \quad \tilde{e}_0 = \psi_C \circ \tilde{e}_n \circ \psi_C.$$

Since the $U_q(C_n)$ -crystal structure of $B^{r,s}$ is prescribed by (6.26), it only remains to determine the automorphism ψ_C .

Remark 6.29. Let $J' = \{1, 2, \dots, n - 1\}$. By Proposition 6.28 ψ_C acts on the set of J' -highest weight vectors in $B^{r,s}$ and is uniquely determined by this action.

Define the map $\psi_C : \widehat{V}^{r,s} \rightarrow \widehat{V}^{r,s}$ by

$$(6.47) \quad \psi_C = \begin{cases} \psi^n \otimes \psi^n & \text{if } r < n, \\ \psi^n & \text{otherwise,} \end{cases}$$

where ψ is the order $2n$ automorphism of type $A_{2n-1}^{(1)}$.

Proposition 6.30. *The map ψ_C defined by (6.47) stabilizes the virtual crystal $V^{r,s}$ and satisfies the conditions of Proposition 6.28 for $V^{r,s}$.*

Proof. ψ_C satisfies the conditions of Proposition 6.28 for all elements of $\widehat{V}^{r,s}$; this follows immediately from the known properties of ψ recorded in Theorem 3.2. To see that ψ_C stabilizes $V^{r,s}$, by (6.44) and (6.45) it is enough to show that $\psi_C(u(\widehat{V}^{r,s})) \in V^{r,s}$. But this element is of extremal weight and is easily seen to be in the $U_q(C_n)$ -component of $u(\widehat{V}^{r,s})$ by direct computation. \square

Proposition 6.31. *$V(\overline{\Lambda}_r)$ is stable under ψ_C . In particular, $V^{r,1} = V(\overline{\Lambda}_r)$ as sets.*

Proof. Let $b = u \otimes v \in V^{r,1}$. Write $u = u_2 u_1$ and $v = v_2 v_1$ where $u_1 = u|_{[n]}$ and $u_2 = u|_{[n+1,2n]}$ and similarly for v . By Proposition 6.17 $u_2 v_2$ is a tableau and $|u_2| - |v_2| = n - r$.

Suppose first that $r < n$. By direct computation $\psi_C(b) = (u_1 + n)(u_2 - n) \otimes (v_1 + n)(v_2 - n)$ where $t \pm n$ means to add or subtract n from each entry in the tableau t . Applying Proposition 6.17 again, it follows that $\psi_C(b) \in V(\overline{\Lambda}_r)$ since $(u_2 v_2) - n$ has the same above properties that $u_2 v_2$ does.

Otherwise, let $r = n$. Then $|u_j| = |v_j|$ for $j = 1, 2$. Now

$$\psi^n(uv)|_{[n]} = \mathbb{P}(uv|_{[n+1,2n]}) - n = \mathbb{P}(u_2 v_2) - n = (u_2 - n)(v_2 - n)$$

which has two columns of equal length. This means that $\psi^n(uv)|_{[n+1,2n]}$ has two columns of equal length, so that it is equal to its \mathbb{P} tableau. But $\mathbb{P}(\psi^n(uv)|_{[n+1,2n]}) = \mathbb{P}(uv|_{[n]}) + n = (u_1 + n)(v_1 + n)$. Therefore, the columns of $\psi_C(uv)$ are given by $(u_1 + n)(u_2 - n)$ and $(v_1 + n)(v_2 - n)$, and the previous argument goes through. \square

Theorem 6.32. *The virtual crystal $V^{r,1}$ is the crystal graph $B^{r,1}$ of the $U'_q(C_n^{(1)})$ -module $W_1^{(r)}$.*

Proof. By [1] it is known that there is an irreducible integrable $U'_q(C_n^{(1)})$ -module $W_1^{(r)}$ with crystal basis which is isomorphic to $B(\overline{\Lambda}_r)$ as a $U_q(C_n)$ -crystal. Since the J' -highest weight vectors in $B(\overline{\Lambda}_r)$ have distinct weights, it follows by Remark 6.29 that the map ψ_C of Proposition 6.28 for the crystal basis of the module $W_1^{(r)}$ is uniquely determined by (6.42), (6.43), (6.44), and (6.45) where only $i \in J'$ are used. By Proposition 6.31, the involution ψ_C of $\widehat{V}^{r,1}$ restricts to an involution on $V^{r,1}$. As such it satisfies the properties of Proposition 6.28, by Proposition 6.30. \square

6.14. $B^{r,1}$ for $\mathfrak{g} = A_{2n}^{(2)\dagger}$. The following result is obtained by combining results in [10] and [2].

Theorem 6.33. *For $\mathfrak{g} = A_{2n}^{(2)\dagger}$, the virtual crystal $V^{r,1}$ is isomorphic to the crystal graph $B^{r,1}$ of the $U'_q(A_{2n}^{(2)\dagger})$ -module $W_1^{(r)}$.*

Proof. In [10] a $U'_q(A_{2n}^{(2)\dagger})$ -module $W_1^{(r)}$ was constructed. Its crystal basis $B^{r,1}$ was shown to be isomorphic to $B(\mu_r \bar{\Lambda}_r)$ as a $U_q(B_n)$ -crystal. This agrees with the decomposition specified by (6.26). Combining Theorem 6.10 of [2] and the explicit action of \tilde{e}_0 and \tilde{f}_0 computed in [10], it follows that $B^{r,1} \cong V^{r,1}$ as $U'_q(A_{2n}^{(2)\dagger})$ -crystals. \square

6.15. $B^{r,1}$ for $\mathfrak{g} = A_{2n}^{(2)}$. The structure of $B^{r,1}$ of type $A_{2n}^{(2)}$ can be deduced from that of the opposite Dynkin labeling $A_{2n}^{(2)\dagger}$. To distinguish various objects defined for $A_{2n}^{(2)\dagger}$ we shall use the symbol \dagger . There is an isomorphism of Dynkin diagrams $A_{2n}^{(2)\dagger} \rightarrow A_{2n}^{(2)}$ given by $i \mapsto n - i$ for $0 \leq i \leq n$. This induces an isomorphism of weight lattices $\psi_{A^{(2)}} : P^\dagger \rightarrow P$ given by $\psi_{A^{(2)}}(\Lambda_i^\dagger) = \Lambda_{n-i}$ for $0 \leq i \leq n$.

Theorem 6.34. *For $\mathfrak{g} = A_{2n}^{(2)}$, the virtual crystal $V^{r,1}$ is isomorphic to the crystal graph $B^{r,1}$ of the $U'_q(A_{2n}^{(2)})$ -module $W_1^{(r)}$.*

Proof. The $U'_q(A_{2n}^{(2)\dagger})$ -module $W_{1\dagger}^{(r)}$ is also the $U'_q(A_{2n}^{(2)})$ -module $W_1^{(r)}$. Therefore, their crystal bases $B_{\dagger}^{r,1}$ and $B^{r,1}$ are equal as sets (but not as weighted crystals). By abuse of notation we write $B^{r,1}$ for either crystal, using \dagger to distinguish the two structures on this set. The isomorphism of Dynkin diagrams implies that $\tilde{f}_i = \tilde{f}_{n-i}^\dagger$ on $B^{r,1}$. Therefore, there is an automorphism $\psi_{A^{(2)}} : B^{r,1} \rightarrow B^{r,1}$ such that

$$(6.48) \quad \begin{aligned} \psi_{A^{(2)}}(\tilde{f}_i(b)) &= \tilde{f}_{n-i}(\psi_{A^{(2)}}(b)) \quad \text{for } 0 \leq i \leq n, \\ \text{wt}(\psi_{A^{(2)}}(b)) &= \psi_{A^{(2)}}(\text{wt}_\dagger(b)) \end{aligned}$$

for all $b \in B^{r,1}$. We claim that $B^{r,1} \cong V^{r,1}$ as $U'_q(A_{2n}^{(2)})$ -crystals. In light of the above facts and the $U'_q(A_{2n}^{(2)\dagger})$ -crystal isomorphism $B_{\dagger}^{r,1} \rightarrow V_{\dagger}^{r,1}$, it is enough to show that there is a unique bijection $\psi_{A^{(2)}} : V_{\dagger}^{r,1} \rightarrow V^{r,1}$ such that (6.48) holds for all $b \in V_{\dagger}^{r,1}$. But the map $\psi_{A^{(2)}}$ is unique since $V_{\dagger}^{r,1}$ is connected as a virtual $U_q(B_n)$ -crystal and there is no choice for $\psi_{A^{(2)}}(u(V_{\dagger}^{r,1}))$ by weight considerations. Recall that $\widehat{V}^{r,1} = \widehat{V}_{\dagger}^{r,1} = B_A^{2n-r,1} \otimes B_A^{r,1}$. It is easily verified that the automorphism $\psi^n \otimes \psi^n$ of the $U'_q(A_{2n-1}^{(1)})$ -crystal $\widehat{V}^{r,1}$ satisfies (6.48). \square

7. FERMIONIC FORMULAS

In this section we prove the fermionic formulas associated with crystals of type $D_{n+1}^{(2)}$, $A_{2n}^{(2)}$ and $C_n^{(1)}$ of the form $V^{rL,1} \otimes \dots \otimes V^{r1,1}$ as conjectured in [7, 8] and conjecture a new fermionic formula of type $A_{2n}^{(2)\dagger}$. We use the virtual crystals discussed in section 6 and results [26] on bijections between paths of type $A_n^{(1)}$ and rigged configurations.

7.1. Main theorem. Let $V^{r,s}$ be a virtual crystal of type $D_{n+1}^{(2)}$, $A_{2n}^{(2)}$ or $C_n^{(1)}$ as defined in section 6.7. Denote by $\Psi^{r,s} : V^{r,s} \rightarrow \widehat{V}^{r,s}$ the inclusion of the virtual crystal into the corresponding ambient crystal. The image of $\Psi^{r,1}$ is given in Propositions 6.17, 6.20 and 6.24 for type $C_n^{(1)}$, $D_{n+1}^{(2)}$ and $A_{2n}^{(2)}$, respectively. Under the bijection $\overline{\phi}$ from paths of type $A_{2n-1}^{(1)}$ to rigged configurations as described in section 5.4 this image can also be explicitly characterized in terms of rigged configurations as stated in the theorem below. This theorem will be essential in the proof of the fermionic formulas.

Let $R = (R_1, \dots, R_L)$ be a sequence of rectangles where R_i has r_i rows and s_i columns. Define $V_R = V^{r_L, s_L} \otimes \dots \otimes V^{r_1, s_1}$, $\widehat{V}_R = \widehat{V}^{r_L, s_L} \otimes \dots \otimes \widehat{V}^{r_1, s_1}$, and let $\Psi_R : V_R \rightarrow \widehat{V}_R$ denote the inclusion $\Psi_R = \Psi^{r_L, s_L} \otimes \dots \otimes \Psi^{r_1, s_1}$. Let \tilde{R} be the sequence of rectangles corresponding to the underlying tensor product of crystals $B_A^{r,s}$ (see (6.24)).

Theorem 7.1. *Let $R = (R_1, \dots, R_L)$ be a sequence of single columns where R_i has height r_i and V_R the corresponding virtual crystal of type $C_n^{(1)}$, $A_{2n}^{(2)}$ or $D_{n+1}^{(2)}$. The image $\text{Im}(\overline{\phi} \circ \Psi_R)$ of $\overline{\phi} \circ \Psi_R : \mathcal{P}(V_R, \cdot) \rightarrow \text{RC}(\cdot, \tilde{R})$ is characterized by the set of rigged configurations (ν, J) satisfying the properties:*

- 1, 2, 3 for type $C_n^{(1)}$,
- 1, 3 for type $A_{2n}^{(2)}$,
- 1 for type $D_{n+1}^{(2)}$

where

1. $(\nu, J)^{(k)} = (\nu, J)^{(2n-k)}$, i.e., they are contragredient self-dual.
2. All parts of $\nu^{(n)}$ are even.
3. All riggings in $(\nu, J)^{(n)}$ are even.

The proof of Theorem 7.1 is given in Appendix A.

Conjecture 7.2. *Theorem 7.1 holds for any sequence of rectangles R .*

A similar characterization of $\text{Im}(\overline{\phi} \circ \Psi_R)$ seems to exist for type $A_{2n}^{(2)\dagger}$ also. The image of $\Psi^{r,1}$ is described explicitly in Proposition 6.26.

Conjecture 7.3. *Let V_R be a virtual crystal of type $A_{2n}^{(2)\dagger}$. The image $\text{Im}(\overline{\phi} \circ \Psi_R)$ of $\overline{\phi} \circ \Psi_R : \mathcal{P}(V_R, \cdot) \rightarrow \text{RC}(\cdot, \tilde{R})$ is characterized by the set of rigged configurations (ν, J) satisfying the properties:*

1. $(\nu, J)^{(k)} = (\nu, J)^{(2n-k)}$, i.e., they are contragredient self-dual.
2. All parts in $J^{(n,i)}$ are congruent to i modulo 2.

We believe that a proof of this conjecture for tensor products of single columns can be given in a similar fashion to the proof of Theorem 7.1 as given in Appendix A.

7.2. Rigged configurations of type $C_n^{(1)}$. Given a sequence of rectangles $R = (R_1, \dots, R_L)$ and a partition λ , let $\nu = (\nu^{(1)}, \dots, \nu^{(n)})$ be a sequence of partitions with the properties

1. $|\nu^{(a)}| = -\sum_{j=1}^a \lambda_j + \sum_{i=1}^L s_i \min(r_i, a)$ for $1 \leq a \leq n$,
2. $\nu^{(n)}$ has only even parts.

Define vacancy numbers as follows:

$$(7.1) \quad \begin{aligned} P_i^{(a)}(\nu) &= Q_i(\nu^{(a-1)}) - 2Q_i(\nu^{(a)}) + Q_i(\nu^{(a+1)}) + Q_i(\xi^{(a)}(R)) \quad \text{for } 1 \leq a < n, \\ P_i^{(n)}(\nu) &= Q_i(\nu^{(n-1)}) - Q_i(\nu^{(n)}) + \frac{1}{2}Q_i(2\xi^{(n)}(R)). \end{aligned}$$

Denote by $C_{\mathfrak{g}}(\lambda, R)$ where $\mathfrak{g} = C_n^{(1)}$ the set of $(\nu^{(1)}, \dots, \nu^{(n)})$ satisfying the two properties listed above and $P_i^{(a)}(\nu) \geq 0$ for all $1 \leq a \leq n$ and $i \geq 0$.

The set of rigged configurations of type $\mathfrak{g} = C_n^{(1)}$, denoted by $RC_{\mathfrak{g}}(\lambda, R)$, is given by (ν, J) such that $\nu \in C_{\mathfrak{g}}(\lambda, R)$ and J is a double sequence of partitions $J = \{J^{(a,i)}\}_{\substack{1 \leq a \leq n \\ i \geq 1}}$ such that the partition $J^{(a,i)}$ lies in a box of width $P_i^{(a)}(\nu)$ and height $m_i(\nu^{(a)})$.

The cocharge for $(\nu, J) \in RC_{\mathfrak{g}}(\lambda, R)$ is defined by

$$cc_{\mathfrak{g}}(\nu, J) = cc_{\mathfrak{g}}(\nu) + \sum_{a=1}^n \sum_{i \geq 1} |J^{(a,i)}|$$

where

$$cc_{\mathfrak{g}}(\nu) = \sum_{i \geq 1} \left(\sum_{a=1}^{n-1} \alpha_i^{(a)} (\alpha_i^{(a)} - \alpha_i^{(a+1)}) + \frac{1}{2} \alpha_i^{(n) 2} \right).$$

7.3. Rigged configurations of type $A_{2n}^{(2)}$. Given a sequence of rectangles $R = (R_1, \dots, R_L)$ and a partition λ , let $\nu = (\nu^{(1)}, \dots, \nu^{(n)})$ be a sequence of partitions such that

$$|\nu^{(a)}| = - \sum_{j=1}^a \lambda_j + \sum_{i=1}^L s_i \min(r_i, a) \quad \text{for } 1 \leq a \leq n.$$

Define vacancy numbers as follows:

$$(7.2) \quad \begin{aligned} P_i^{(a)}(\nu) &= Q_i(\nu^{(a-1)}) - 2Q_i(\nu^{(a)}) + Q_i(\nu^{(a+1)}) + Q_i(\xi^{(a)}(R)) \quad \text{for } 1 \leq a < n, \\ P_i^{(n)}(\nu) &= Q_i(\nu^{(n-1)}) - Q_i(\nu^{(n)}) + Q_i(\xi^{(n)}(R)). \end{aligned}$$

Denote by $C_{\mathfrak{g}}(\lambda, R)$ where $\mathfrak{g} = A_{2n}^{(2)}$ the set of $(\nu^{(1)}, \dots, \nu^{(n)})$ satisfying the constraint above and $P_i^{(a)}(\nu) \geq 0$ for all $1 \leq a \leq n$ and $i \geq 0$.

The set of rigged configurations of type $\mathfrak{g} = A_{2n}^{(2)}$, denoted by $RC_{\mathfrak{g}}(\lambda, R)$, is given by (ν, J) such that $\nu \in C_{\mathfrak{g}}(\lambda, R)$ and J is a double sequence of partitions $J = \{J^{(a,i)}\}_{\substack{1 \leq a \leq n \\ i \geq 1}}$ such that the partition $J^{(a,i)}$ lies in a box of width $P_i^{(a)}(\nu)$ and height $m_i(\nu^{(a)})$.

The cocharge for $(\nu, J) \in RC_{\mathfrak{g}}(\lambda, R)$ is defined by

$$cc_{\mathfrak{g}}(\nu, J) = cc_{\mathfrak{g}}(\nu) + 2 \sum_{a=1}^n \sum_{i \geq 1} |J^{(a,i)}|$$

where

$$cc_{\mathfrak{g}}(\nu) = \sum_{i \geq 1} \left(\sum_{a=1}^{n-1} 2\alpha_i^{(a)} (\alpha_i^{(a)} - \alpha_i^{(a+1)}) + \alpha_i^{(n)2} \right).$$

7.4. Rigged configurations of type $D_{n+1}^{(2)}$. Let $\lambda = (\lambda_1, \dots, \lambda_n)$ be a composition satisfying $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n \geq 0$ and either $\lambda_i \in \mathbb{Z}$ for all i or $\lambda_i \in \mathbb{Z} + \frac{1}{2}$ for all i , and $R = (R_1, \dots, R_L)$ a sequence of rectangles. Then let $\nu = (\nu^{(1)}, \dots, \nu^{(n)})$ be a sequence of partitions with the properties

$$|\nu^{(a)}| = -\sum_{j=1}^a \lambda_j + \sum_{\substack{i=1 \\ r_i < n}}^L s_i \min(r_i, a) + \frac{1}{2} \sum_{\substack{i=1 \\ r_i = n}}^L s_i a \quad \text{for } 1 \leq a \leq n.$$

Define vacancy numbers as follows:

$$(7.3) \quad \begin{aligned} P_i^{(a)}(\nu) &= Q_i(\nu^{(a-1)}) - 2Q_i(\nu^{(a)}) + Q_i(\nu^{(a+1)}) + Q_i(\xi^{(a)}(R)) \quad \text{for } 1 \leq a < n, \\ P_i^{(n)}(\nu) &= 2Q_i(\nu^{(n-1)}) - 2Q_i(\nu^{(n)}) + Q_i(\xi^{(n)}(R)). \end{aligned}$$

Denote by $C_{\mathfrak{g}}(\lambda, R)$ where $\mathfrak{g} = D_{n+1}^{(2)}$ the set of $(\nu^{(1)}, \dots, \nu^{(n)})$ satisfying the constraint above and $P_i^{(a)}(\nu) \geq 0$ for all $1 \leq a \leq n$ and $i \geq 0$.

The set of rigged configurations of type $\mathfrak{g} = D_{n+1}^{(2)}$, denoted by $RC_{\mathfrak{g}}(\lambda, R)$, is given by (ν, J) such that $\nu \in C_{\mathfrak{g}}(\lambda, R)$ and J is a double sequence of partitions $J = \{J^{(a,i)}\}_{\substack{1 \leq a \leq n \\ i \geq 1}}$ such that the partition $J^{(a,i)}$ lies in a box of width $P_i^{(a)}(\nu)$ and height $m_i(\nu^{(a)})$.

The cocharge for $(\nu, J) \in RC_{\mathfrak{g}}(\lambda, R)$ is defined by

$$cc_{\mathfrak{g}}(\nu, J) = cc_{\mathfrak{g}}(\nu) + \sum_{i \geq 1} \left(\sum_{a=1}^{n-1} 2|J^{(a,i)}| + |J^{(n,i)}| \right)$$

where

$$cc_{\mathfrak{g}}(\nu) = \sum_{i \geq 1} \left(\sum_{a=1}^{n-1} 2\alpha_i^{(a)} (\alpha_i^{(a)} - \alpha_i^{(a+1)}) + \alpha_i^{(n)2} \right).$$

7.5. Fermionic formulas. It follows from Theorem 5.3 and Proposition 4.9 that

$$(7.4) \quad -E_R(b) = cc(\tilde{\phi}(b)) \quad \text{for } b \in \mathcal{P}(B_R^A, \Lambda).$$

By Theorem 7.1 (and Conjecture 7.2), (7.4), (6.23) and the definitions of rigged configurations in sections 7.2, 7.3 and 7.4, we can express the one-dimensional sum of type $\mathfrak{g} = C_n^{(1)}, A_{2n}^{(2)}, D_{n+1}^{(2)}$ in terms of rigged configurations

$$(7.5) \quad X(B_R, \Lambda; q^{-1}) = \sum_{(\nu, J) \in RC_{\mathfrak{g}}(\lambda, R)} q^{cc_{\mathfrak{g}}(\nu, J)}$$

where λ is the ‘‘partition’’ corresponding to the weight Λ . That is, if $\Lambda = \overline{\Lambda}_{k_1} + \dots + \overline{\Lambda}_{k_\ell}$, then λ is the ‘‘partition’’ with columns of height k_1, \dots, k_ℓ . In the case of $\mathfrak{g} = D_{n+1}^{(2)}$, $\overline{\Lambda}_n$ yields a column of height n and width $\frac{1}{2}$ (explaining the quotation marks around partition). The contribution of $\{J^{(a,i)}\}$ to $cc_{\mathfrak{g}}(\nu, J)$ can be evaluated

explicitly by noting that the generating function of partitions in a box of dimension $m \times p$ is the q -binomial coefficient

$$\begin{bmatrix} m+p \\ m \end{bmatrix}_q = \frac{(q)_{m+p}}{(q)_m(q)_p}$$

for $p, m \in \mathbb{N}$ and zero otherwise, where $(q)_m = (1-q)(1-q^2) \cdots (1-q^m)$. We also introduce t_a, t_a^\vee ($1 \leq a \leq n$) by

$$(t_a)_{1 \leq a \leq n} = \begin{cases} (2, 2, \dots, 2, 1) & \text{for } C_n^{(1)}, \\ (1, 1, \dots, 1, 1) & \text{otherwise,} \end{cases}$$

$$(t_a^\vee)_{1 \leq a \leq n} = \begin{cases} (2, 2, \dots, 2, 1) & \text{for } D_{n+1}^{(2)}, \\ (2, 2, \dots, 2, 2) & \text{for } A_{2n}^{(2)}, \\ (1, 1, \dots, 1, 1) & \text{for } C_n^{(1)}. \end{cases}$$

Then (7.5) may be recast into the form

$$(7.6) \quad X(B_R, \Lambda; q^{-1}) = \sum_{\nu \in C_{\mathfrak{g}}(\lambda, R)} q^{cc_{\mathfrak{g}}(\nu)} \prod_{a=1}^n \prod_{i \geq 1} \begin{bmatrix} m_i(\nu^{(a)}) + P_i^{(a)}(\nu) \\ m_i(\nu^{(a)}) \end{bmatrix}_{q_a}$$

where $q_a = q^{t_a^\vee}$. This formula is known as the fermionic formula.

To compare this with the fermionic formula stated in [7, Eq. (4.5)] several definitions are necessary. Set

$$p_i^{(a)} = \begin{cases} P_i^{(a)}(\nu) & \text{for } 1 \leq a < n, \\ P_{\gamma' i}^{(n)}(\nu) & \text{for } a = n, \end{cases}$$

$$m_i^{(a)} = \begin{cases} m_i(\nu^{(a)}) & \text{for } 1 \leq a < n, \\ m_{\gamma' i}(\nu^{(n)}) & \text{for } a = n \end{cases}$$

with γ' as in (6.2). Let $L_i^{(a)}$ be the number of rectangles in R with a rows and i columns. Let $\tilde{\alpha}_a, \tilde{\Lambda}_a$ ($a = 1, \dots, n$) be the simple roots and fundamental weights of $\tilde{\mathfrak{g}}$ and let \tilde{P} be its weight lattice. To relate \tilde{P} with \bar{P} we introduce a \mathbb{Z} -linear map $\iota : \bar{P} \rightarrow \tilde{P}$ by

$$(7.7) \quad \iota(\bar{\Lambda}_a) = \varepsilon_a \tilde{\Lambda}_a \quad \text{for } 1 \leq a \leq n$$

where ε_a is specified as

$$\varepsilon_a = \begin{cases} 2 & \text{if } \mathfrak{g} = A_{2n}^{(2)} \text{ and } a = n, \\ 1 & \text{otherwise.} \end{cases}$$

Note that (7.7) induces $\iota(\alpha_a) = \varepsilon_a \tilde{\alpha}_a$. With these definitions we obtain

$$(7.8) \quad p_i^{(a)} = \sum_{j \geq 1} L_j^{(a)} \min(i, j) - \frac{1}{t_a^\vee} \sum_{b=1}^n (\tilde{\alpha}_a | \tilde{\alpha}_b) \sum_{k \geq 1} \min(t_b i, t_a k) m_k^{(b)}$$

for all $i \geq 1$ and $1 \leq a \leq n$,

$$(7.9) \quad \sum_{a=1}^n \sum_{i \geq 1} i m_i^{(a)} \tilde{\alpha}_a = \iota \left(\sum_{a=1}^n \sum_{i \geq 1} i L_i^{(a)} \bar{\Lambda}_a - \Lambda \right)$$

and

$$(7.10) \quad cc_{\mathfrak{g}}(\nu) = cc(\{m\}) := \frac{1}{2} \sum_{a,b=1}^n (\tilde{\alpha}_a | \tilde{\alpha}_b) \sum_{j,k \geq 1} \min(t_b j, t_a k) m_j^{(a)} m_k^{(b)}.$$

Then (7.6) is equivalent to

$$(7.11) \quad X(B_R, \Lambda; q^{-1}) = \sum_{\{m\}} q^{cc(\{m\})} \prod_{a=1}^n \prod_{i \geq 1} \left[\begin{matrix} m_i^{(a)} + p_i^{(a)} \\ m_i^{(a)} \end{matrix} \right]_{q_a}$$

where the sum is over all $\{m_i^{(a)} \in \mathbb{N} \mid 1 \leq a \leq n, i \geq 1\}$ subject to the constraint (7.9) and $p_i^{(a)}$ is given by (7.8). This is [7, Eq. (4.5)] with $q \rightarrow 1/q$ and $\ell \rightarrow \infty$.

There are also level-restricted versions of the one-dimensional configuration sums which are q -analogues of fusion coefficients. In [8, Eq. (4.8)] and [7, Conj. 3.1 and Eq. (4.5)] fermionic formulas for the level-restricted configuration sums were conjectured for $\lambda = 0$. For type $C_n^{(1)}$ these and generalizations thereof for general λ are proven in [39] using the virtual crystal techniques developed in this paper.

7.6. New conjectured fermionic formula for $A_{2n}^{(2)\dagger}$. Similar to the derivation of (7.11) from Theorem 7.1 and Conjecture 7.2, a fermionic formula of type $A_{2n}^{(2)\dagger}$ can be derived from Conjecture 7.3 which to our knowledge is new. Namely, we conjecture that for B_R a crystal of type $A_{2n}^{(2)\dagger}$ and Λ a dominant integral weight

$$(7.12) \quad X(B_R, \Lambda; q^{-1}) = \sum_{\{m\}} q^{cc(\{m\})} \prod_{a=1}^{n-1} \prod_{i \geq 1} \left[\begin{matrix} m_i^{(a)} + p_i^{(a)} \\ m_i^{(a)} \end{matrix} \right] \\ \prod_{\substack{i \geq 1 \\ i \text{ even}}} \left[\begin{matrix} m_i^{(n)} + p_i^{(n)} \\ m_i^{(n)} \end{matrix} \right] \prod_{\substack{i \geq 1 \\ i \text{ odd}}} q^{m_i^{(n)}/2} \left[\begin{matrix} m_i^{(n)} + p_i^{(n)} - 1 \\ m_i^{(n)} \end{matrix} \right].$$

Here $cc(\{m\})$ is as in (7.10) and $p_i^{(a)}$ as in (7.8) with $(t_a)_{1 \leq a \leq n} = (t_a^\vee)_{1 \leq a \leq n} = (1, 1, \dots, 1, 1)$. The sum is over all $\{m_i^{(a)} \in \mathbb{N} \mid 1 \leq a \leq n, i \geq 1\}$ subject to constraint (7.9) with $\varepsilon_n = 2$ and $\varepsilon_a = 1$ for $1 \leq a < n$. Note that a summand in (7.12) is only nonzero if $p_i^{(a)} \geq 0$ for all $1 \leq a \leq n, i \geq 1$ and $p_i^{(n)} \geq 1$ if $m_i^{(n)} > 0$ and i is odd.

APPENDIX A. PROOF OF THEOREM 7.1

First we show that if $(\nu, J) \in \text{Im}(\bar{\phi} \circ \Psi_R)$, then (ν, J) has the properties of the theorem. Later we will show that the converse holds, that is, if (ν, J) satisfies the the properties, then $(\nu, J) \in \text{Im}(\bar{\phi} \circ \Psi_R)$.

By Propositions 6.17, 6.20 and 6.24 the image of Ψ_R yields paths that are contra-redient self-dual up to application of the combinatorial R -matrix for all three types $D_{n+1}^{(2)}$, $A_{2n}^{(2)}$ and $C_n^{(1)}$. It was shown in [26] that under the bijection $\bar{\phi}$ the combinatorial R -matrix on LR tableaux yields the identity map on rigged configurations. Hence point 1 follows from Theorem 5.7 in all three cases.

To prove point 3 for type $A_{2n}^{(2)}$ and points 2 and 3 for type $C_n^{(1)}$ we proceed by induction on L . Let $p = p_L \otimes \dots \otimes p_1 \in \mathcal{P}(V_R, \cdot)$. By induction $(\nu, J) = \bar{\phi} \circ \Psi_{R'}(p_{L-1} \otimes \dots \otimes p_1)$ satisfies point 3 for $A_{2n}^{(2)}$ and points 2 and 3 for $C_n^{(1)}$ where $R' = (R_1, \dots, R_{L-1})$. Let r' be the height of R_L . For type $C_n^{(1)}$, $p_L =$

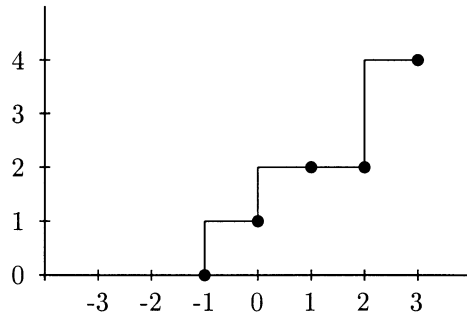


FIGURE 1. Example of an ℓ -slide for $n = 4$ and with $\ell_3^{(4-1)} = 0$, $\ell_3^{(4+0)} = 1$, $\ell_3^{(4+1)} = 2$, $\ell_3^{(4+2)} = 2$, $\ell_3^{(4+3)} = 4$

$u \otimes v \in V_C(\overline{\Lambda}_r)$ for some one column tableaux u and v of height $2n - r$ and r respectively where $r = r'$. For type $A_{2n}^{(2)}$, $p_L = i_{r'-1,1} \circ i_{r'-2,1} \circ \dots \circ i_{r,1}(u \otimes v)$ for some $u \otimes v \in V_C(\overline{\Lambda}_r)$ by Theorem 6.25. Let $\ell_i^{(k)}$ for $1 \leq i \leq r$ (resp. $s_i^{(k)}$ for $1 \leq i \leq 2n - r$) be the lengths of the selected strings corresponding to the i -th letter in v (resp. u) under $\overline{\phi}$ as described in section 5.4. By definition and by Lemma 5.5 the following inequalities hold

$$(A.1) \quad \ell_i^{(k)} \leq \ell_i^{(k+1)}, \quad s_i^{(k)} \leq s_i^{(k+1)},$$

$$(A.2) \quad \ell_i^{(k)} \leq \ell_{i-1}^{(k-1)}, \quad s_i^{(k)} \leq s_{i-1}^{(k-1)}.$$

Since $r \leq n$, the number of boxes added to $\nu^{(n)}$ by v is $d := r - |v|_{[n]}$. Similarly, the boxes added to the central partition by u is $n - |u|_{[n]}$ which is equal to d by Proposition 6.17. Let a (resp. b) be such that $u_a, u_{a+1}, \dots, u_{a+d-1}$ (resp. $v_b, v_{b+1}, \dots, v_{b+d-1}$) add boxes to $\nu^{(n)}$. Since by (A.1) and (A.2) $\ell_{i+1}^{(k)} \leq \ell_i^{(k)}$ and $s_{i+1}^{(k)} \leq s_i^{(k)}$ it suffices to show that

$$(A.3) \quad s_{a+i}^{(n)} = \ell_{b+i}^{(n)} + 1 \quad \text{for } 0 \leq i < d.$$

Hence, in this case, point 2 follows directly by induction for type $C_n^{(1)}$. Since the vacancy numbers $P_m^{(n)}$ are all even by the symmetry of point 1 and the fact that under the embedding all rectangles in \tilde{R} of height n have even widths, point 3 is also satisfied. For type $A_{2n}^{(2)}$ equation (A.3) also ensures that after the addition of $u \otimes v$ all riggings are even since the vacancy numbers are even. The step p_L is obtained from $u \otimes v$ by the application of $i_{r'-1,1} \circ \dots \circ i_{r,1}$. By Theorem 5.9 this induces $j_{r'-1,1} \circ \dots \circ j_{r,1}$ on rigged configurations which adds singular strings of length 1. Again, since the $j_{k,1}$ do not change the vacancy numbers and all vacancy numbers are even, point 3 is satisfied for type $A_{2n}^{(2)}$. It remains to prove (A.3).

To prove (A.3) we need to study the properties of $\ell_i^{(k)}$ and $s_i^{(k)}$ more closely. To this end it is useful to think of $\ell_i^{(k)}$ and $s_i^{(k)}$ in geometrical terms. Draw dots, called bolts, at $(j, \ell_i^{(n+j)})$ in the xy -plane whenever $\ell_i^{(n+j)}$ is defined and not infinity. Connect two bolts $(j, \ell_i^{(n+j)})$ and $(j - 1, \ell_i^{(n+j-1)})$ by a horizontal line from $(j - 1, \ell_i^{(n+j)})$ to $(j, \ell_i^{(n+j)})$ and a vertical line from $(j - 1, \ell_i^{(n+j-1)})$ to $(j - 1, \ell_i^{(n+j)})$. For fixed i call this the i -th ℓ -slide. In the same fashion define the i -th s -slide by replacing $\ell_i^{(k)}$ by $s_i^{(k)}$ everywhere. An example is given in Figure 1.

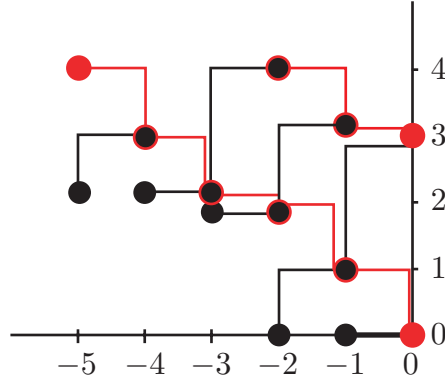


FIGURE 2. Example of folded ℓ -slides for $n = 6$ and v the single row with entries 3, 5, 6, 9, 12. The positive parts are red whereas the negative parts are black. Nut-bolt pairs are denoted by red circles with black interior.

Lemma A.1. *ℓ -slides (resp. s -slides) do not cross.*

Proof. This follows directly from the conditions (A.2). □

Call the set of all points (x, y) of a slide S with $x > 0$ (resp. $x < 0$) the positive (resp. negative) part of S . Let us now define a folding operation on slides by mapping the point (x, y) of a slide S to $(-|x|, y)$. A bolt that changes under folding transforms to a “nut”.

Let us now define nut-bolt pairs as follows. By the symmetry of the rigged configuration (ν, J) and the property of v that there is no v_k with $v_k = 2n - v_b + 1$ (otherwise the duality property of Proposition 6.17 cannot hold) the b -th folded ℓ -slide must cross each k -th folded ℓ -slide with $k < b$ and $v_k > 2n - v_b$ in at least one bolt. If $\ell_k^{(n-j)} = \ell_b^{(n+j)}$ with $j > 0$, then call $[\ell_k^{(n-j)}, \ell_b^{(n+j)}]$ a nut-bolt pair (note that here we also allow $k = b$). By induction on $i \geq 0$ the $(b + i)$ -th folded ℓ -slide must cross each k -th folded ℓ -slide with $k < b + i$ and $v_k > 2n - v_{b+i}$ in at least one unclaimed bolt. Call $[\ell_k^{(n-j)}, \ell_{b+i}^{(n+j)}]$ a nut-bolt pair if $\ell_k^{(n-j)} = \ell_{b+i}^{(n+j)}$ and $[\ell_k^{(n-j)}, \ell_{b+i-1}^{(n+j)}]$ is not a nut-bolt pair. An example is given in Figure 2.

We will use the following two lemmas to prove (A.3).

Lemma A.2. *For $i \geq 0$ and $j > 0$ the following must be nut-bolt pairs:*

$$(A.4) \quad [\ell_k^{(n-j)}, \ell_{b+i}^{(n+j)}] \text{ for some } k \quad \text{or} \quad [s_{a+i-j}^{(n-j)}, \ell_{b+i}^{(n+j)}];$$

$$(A.5) \quad [s_k^{(n-j)}, s_{a+i}^{(n+j)}] \text{ for some } k \quad \text{or} \quad [\ell_{b+i-j}^{(n-j)}, s_{a+i}^{(n+j)}].$$

If $[s_{a+i-j}^{(n-j)}, \ell_{b+i}^{(n+j)}]$ is not a nut-bolt pair, then

$$(A.6) \quad s_{a+i-j}^{(n-j)} > \ell_{b+i}^{(n+j)},$$

and similarly if $[\ell_{b+i-j}^{(n-j)}, s_{a+i}^{(n+j)}]$ is not a nut-bolt pair, then

$$(A.7) \quad s_{a+i}^{(n+j)} > \ell_{b+i-j}^{(n-j)}.$$

Lemma A.3. *Let $j > 1$. If $[\ell_k^{(n-j)}, \ell_{b+i}^{(n+j)}]$ is a nut-bolt pair and $[\ell_k^{(n-j+1)}, \ell_{b+i-1}^{(n+j-1)}]$ is not a nut-bolt pair, then $\ell_{b+i}^{(n+j)} < s_{a+i-j}^{(n-j)} \leq \max\{\ell_k^{(n-j+1)}, \ell_k^{(n-j)} + 1\}$.*

The proof of these lemmas is given by an intertwined induction on i and j . We will give the proofs separately, but the inductive step of each will rely on the validity of both lemmas for either the same i and greater j , or smaller i .

Proof of Lemma A.2. We prove (A.4) by induction on increasing i and decreasing j . Suppose that for given i and j , $[\ell_k^{(n-j)}, \ell_{b+i}^{(n+j)}]$ is not a nut-bolt pair for any k . We want to show that then $[s_{a+i-j}^{(n-j)}, \ell_{b+i}^{(n+j)}]$ is a nut-bolt pair.

First assume that j is maximal, that is $n + j = v_{b+i} - 1$. Note that from the duality and the tableau condition of $u \otimes v$ as stated in Proposition 6.17 it follows that $u_{a+i-j} > n - j$ and $u_{a+i-j-1} \leq n - j$ for $n + j = v_{b+i} - 1$. Since we know already that the final rigged configuration has to be symmetric, i.e., satisfies condition 1 of Theorem 7.1, and since $s_k^{(n-j)} \leq s_{a+i-j}^{(n-j)}$ for all $k > a + i - j$ by (A.2), we therefore must have $s_{a+i-j}^{(n-j)} = \ell_{b+i}^{(n+j)}$ so that $[s_{a+i-j}^{(n-j)}, \ell_{b+i}^{(n+j)}]$ forms a nut-bolt pair.

Now assume that $n + j < v_{b+i} - 1$. Then by induction (A.4) must hold for $j + 1$.

Assume that $[s_{a+i-j-1}^{(n-j-1)}, \ell_{b+i}^{(n+j+1)}]$ is a nut-bolt pair. If $[s_{a+i-j}^{(n-j+1)}, \ell_{b+i-1}^{(n+j-1)}]$ is also a nut-bolt pair, then by symmetry $s_{a+i-j}^{(n-j)} = \ell_{b+i}^{(n+j)}$ so that $[s_{a+i-j}^{(n-j)}, \ell_{b+i}^{(n+j)}]$ indeed forms a nut-bolt pair. If $[\ell_k^{(n-j+1)}, \ell_{b+i-1}^{(n+j-1)}]$ is a nut-bolt pair for some k or $\ell_{b+i-1}^{(n+j-1)} = \infty$, then $\ell_k^{(n-j+1)} > \ell_{b+i}^{(n+j+1)}$ since otherwise $[\ell_k^{(n-j)}, \ell_{b+i}^{(n+j)}]$ would be a nut-bolt pair which contradicts our assumptions. Then again by symmetry $s_{a+i-j}^{(n-j)} = \ell_{b+i}^{(n+j)}$ and $[s_{a+i-j}^{(n-j)}, \ell_{b+i}^{(n+j)}]$ forms a nut-bolt pair.

Otherwise, assume that $[\ell_k^{(n-j-1)}, \ell_{b+i}^{(n+j+1)}]$ is a nut-bolt pair for some k . If $[\ell_k^{(n-j)}, \ell_{b+i-1}^{(n+j)}]$ is also a nut-bolt pair, then $[\ell_{k'}^{(n-j+1)}, \ell_{b+i-1}^{(n+j-1)}]$ cannot be a nut-bolt pair for any k' since otherwise $[\ell_{k'}^{(n-j)}, \ell_{b+i}^{(n+j)}]$ would be a nut-bolt pair which contradicts our assumptions. Hence $[s_{a+i-j}^{(n-j+1)}, \ell_{b+i-1}^{(n+j-1)}]$ must be a nut-bolt pair. By the change of vacancy numbers there are no singular strings of length $\ell_k^{(n-j-1)} < h \leq \ell_k^{(n-j)}$ in the $(n - j)$ -th rigged partition after the addition of all letters up to u_{a+i-1} . Since $s_{a+i-j}^{(n-j)} \leq s_{a+i-j}^{(n-j+1)} = \ell_{b+i-1}^{(n+j-1)} \leq \ell_k^{(n-j)}$, this shows by symmetry that $s_{a+i-j}^{(n-j)} = \ell_{b+i}^{(n+j)}$ and hence $[s_{a+i-j}^{(n-j)}, \ell_{b+i}^{(n+j)}]$ forms a nut-bolt pair. If $[\ell_k^{(n-j)}, \ell_{b+i-1}^{(n+j)}]$ is not a nut-bolt pair, then $\ell_k^{(n-j-1)} < s_{a+i-j-1}^{(n-j-1)} \leq \ell_k^{(n-j)}$ by lemma A.3 (note that $\ell_k^{(n-j)} > \ell_k^{(n-j-1)}$ since otherwise $[\ell_k^{(n-j)}, \ell_{b+i}^{(n+j)}]$ would be a nut-bolt pair which contradicts our assumptions). By the same arguments as before there are no singular strings of length $\ell_k^{(n-j-1)} < h \leq \ell_k^{(n-j)}$ in the $(n - j)$ -th rigged partition after the addition of all letters up to u_{a+i-1} . Using $s_{a+i-j}^{(n-j)} \leq s_{a+i-j-1}^{(n-j-1)}$ (A.4) follows again.

The statements (A.5), (A.6) and (A.7) are proven in a similar fashion. □

Proof of Lemma A.3. The lower bound follows directly from (A.6).

The upper bound is proven by increasing induction on i and decreasing induction on j .

Fix i . Let j be maximal, that is $n + j = v_{b+i} - 1$. Assume that $[\ell_{k'}^{(n-j+1)}, \ell_{b+i-1}^{(n+j-1)}]$ is a nut-bolt pair for some $k' \leq k$. The conditions of the lemma require that

$k' < k$. By definition $\ell_{b+i}^{(n+j)}$ is the length of the largest singular string in $(\nu, J)^{(n+j)}$ subject to the condition $\ell_{b+i}^{(n+j)} \leq \ell_{b+i-1}^{(n+j-1)}$. There is a singular string of length $\ell_{k'}^{(n-j)} \geq \ell_k^{(n-j)}$ in $(\nu, J)^{(n-j)}$. Hence by symmetry $[\ell_{k'}^{(n-j)}, \ell_{b+i}^{(n+j)}]$ must form a nut-bolt pair so that $k' = k$. This contradicts $k' < k$. Therefore, by (A.4), either $\ell_{b+i-1}^{(n+j-1)} = \infty$ or $[s_{a+i-j}^{(n-j+1)}, \ell_{b+i-1}^{(n+j-1)}]$ is a nut-bolt pair. By the definition of $\ell_{b+i}^{(n+j)}$ there is no singular string of length $\ell_{b+i}^{(n+j)} < h \leq \ell_{b+i-1}^{(n+j-1)}$ in $(\nu, J)^{(n+j)}$ and hence by symmetry in $(\nu, J)^{(n-j)}$. Using that $s_{a+i-j}^{(n-j)} \leq s_{a+i-j}^{(n-j+1)}$ and taking into account the change in vacancy numbers this implies that $s_{a+i-j}^{(n-j)} \leq \max\{\ell_k^{(n-j+1)}, \ell_k^{(n-j)} + 1\}$.

If $n + j < v_{b+i} - 1$, then by (A.4) and (A.5) and the crossing conditions on ℓ -slides exactly one of the following three must be a nut-bolt pair for $\ell_{b+i}^{(n+j+1)}$: (1) $[s_{a+i-j-1}^{(n-j-1)}, \ell_{b+i}^{(n+j+1)}]$; (2) $[\ell_k^{(n-j-1)}, \ell_{b+i}^{(n+j+1)}]$; (3) $[\ell_{k-1}^{(n-j-1)}, \ell_{b+i}^{(n+j+1)}]$. We are going to treat each case separately.

Assume that case (1) holds. If $\ell_{b+i-1}^{(n+j-1)} \geq \ell_{b+i}^{(n+j+1)}$, then by the definition of $\ell_{b+i}^{(n+j)}$ there are no singular strings of length $\ell_{b+i}^{(n+j)} < h \leq \ell_{b+i}^{(n+j+1)}$ in $(\nu, J)^{(n+j)}$. By symmetry there are no singular strings of length $\ell_k^{(n-j)} < h \leq \ell_{b+i}^{(n+j+1)}$ in $(\nu, J)^{(n-j)}$. Taking into account the change of vacancy number, there are still no singular strings of length $\max\{\ell_k^{(n-j+1)}, \ell_k^{(n-j)} + 1\} < h \leq \ell_{b+i}^{(n+j+1)}$ in the $(n - j)$ -th rigged partition after the addition of all letters up to u_{a+i-1} . Since $s_{a+i-j}^{(n-j)} \leq s_{a+i-j-1}^{(n-j-1)} = \ell_{b+i}^{(n+j+1)}$ by (A.2), it follows that $s_{a+i-j}^{(n-j)} \leq \max\{\ell_k^{(n-j+1)}, \ell_k^{(n-j)} + 1\}$. If $\ell_{b+i-1}^{(n+j-1)} < \ell_{b+i}^{(n+j+1)}$, then $[s_{a+i-j}^{(n-j+1)}, \ell_{b+i-1}^{(n+j-1)}]$ must be a nut-bolt pair. By the same arguments as above there are no singular strings of length $\max\{\ell_k^{(n-j+1)}, \ell_k^{(n-j)} + 1\} < h \leq \ell_{b+i-1}^{(n+j-1)}$ in the $(n - j)$ -th rigged partition after the addition of all letters u_{a+i-1} . Since $s_{a+i-j}^{(n-j)} \leq s_{a+i-j}^{(n-j+1)} = \ell_{b+i-1}^{(n+j-1)}$ by (A.1) this implies $s_{a+i-j}^{(n-j)} \leq \max\{\ell_k^{(n-j+1)}, \ell_k^{(n-j)} + 1\}$.

Assume that case (2) holds. Then the conditions of Lemma A.3 are satisfied for j replaced by $j + 1$ and by induction $s_{a+i-j-1}^{(n-j-1)} = \ell_k^{(n-j-1)} + 1$. Note that $\ell_k^{(n-j)} = \ell_k^{(n-j-1)}$ in case (2). Since $\ell_{b+i-1}^{(n+j-1)} > \ell_k^{(n-j+1)}$ it follows that $s_{a+i-j}^{(n-j+1)} > \ell_k^{(n-j+1)} \geq \ell_k^{(n-j)}$. Hence by (A.2) it follows that $s_{a+i-j}^{(n-j)} = \ell_k^{(n-j)} + 1$.

Assume that case (3) holds. Let us first show that $s_{a+i-j}^{(n-j)} \leq \ell_{k-1}^{(n-j)} + 1$. If $[\ell_{k-1}^{(n-j)}, \ell_{b+i-1}^{(n+j)}]$ is not a nut-bolt pair, then by induction $s_{a+i-j-1}^{(n-j-1)} \leq \ell_{k-1}^{(n-j)}$ (note that $\ell_{k-1}^{(n-j)} > \ell_{k-1}^{(n-j-1)}$ since otherwise $[\ell_{k-1}^{(n-j)}, \ell_{b+i}^{(n+j)}]$ would be a nut-bolt pair which contradicts our assumptions). Since $s_{a+i-j}^{(n-j)} \leq s_{a+i-j-1}^{(n-j-1)}$ by (A.2) it follows that $s_{a+i-j}^{(n-j)} \leq \ell_{k-1}^{(n-j)}$. If $[\ell_{k-1}^{(n-j)}, \ell_{b+i-1}^{(n+j)}]$ is a nut-bolt pair, then (a) $[s_{a+i-j}^{(n-j+1)}, \ell_{b+i-1}^{(n+j-1)}]$ or (b) $[\ell_{k-1}^{(n-j+1)}, \ell_{b+i-1}^{(n+j-1)}]$ is a nut-bolt pair. In case (a) $s_{a+i-j}^{(n-j)} \leq s_{a+i-j}^{(n-j+1)} = \ell_{b+i-1}^{(n+j-1)} \leq \ell_{k-1}^{(n-j)}$ so that the assertion holds. In case (b) the assumptions of the lemma hold for i replaced by $i - 1$ so that by induction $s_{a+i-j-1}^{(n-j)} = \ell_{k-1}^{(n-j)} + 1$. Since $s_{a+i-j}^{(n-j)} \leq s_{a+i-j}^{(n-j+1)} \leq s_{a+i-j-1}^{(n-j)}$ by (A.1) and (A.2) the assertion holds.

There are no singular strings of length $\max\{\ell_{k-1}^{(n-j-1)}, \ell_k^{(n-j+1)}\} < h \leq \ell_{k-1}^{(n-j)}$ in the $(n - j)$ -th rigged partition after the addition of all letters up to u_{a+i-1} because of the change of the vacancy numbers. Furthermore, there are no singular strings

of length $\ell_{b+i}^{(n+j)} < h \leq \min\{\ell_{b+i}^{(n+j+1)}, \ell_{b+i-1}^{(n+j-1)}\}$ in $(\nu, J)^{(n+j)}$ by the definition of $\ell_{b+i}^{(n+j)}$. Hence by symmetry and change in vacancy numbers, there are no singular strings of length $\max\{\ell_k^{(n-j+1)}, \ell_k^{(n-j)} + 1\} < h \leq \min\{\ell_{b+i}^{(n+j+1)}, \ell_{b+i-1}^{(n+j-1)}\}$ in the $(n-j)$ -th rigged partition after the addition of all letters up to u_{a+i-1} . Now assume that $s_{a+i-j}^{(n-j)} \leq \ell_{k-1}^{(n-j)}$. Then these conditions and the fact that $[s_{a+i-j}^{(n-j+1)}, \ell_{b+i-1}^{(n+j-1)}]$ is a nut-bolt pair if $\ell_{b+i-1}^{(n+j-1)} \leq \ell_{b+i}^{(n+j+1)}$ imply that $s_{a+i-j}^{(n-j)} \leq \max\{\ell_k^{(n-j+1)}, \ell_k^{(n-j)} + 1\}$.

Hence the only case left to consider is $s_{a+i-j}^{(n-j)} = \ell_{k-1}^{(n-j)} + 1$. By the above arguments this case can only occur when $[\ell_{k-1}^{(n-j)}, \ell_{b+i-1}^{(n+j)}]$ and $[\ell_{k-1}^{(n-j+1)}, \ell_{b+i-1}^{(n+j-1)}]$ are nut-bolt pairs with $s_{a+i-j-1}^{(n-j)} = \ell_{k-1}^{(n-j)} + 1$. We will show that $s_{a+i-j}^{(n-j)} = \ell_{k-1}^{(n-j)} + 1$ cannot occur unless $\ell_k^{(n-j)} = \ell_{k-1}^{(n-j)}$ in which case Lemma A.3 obviously holds. Assume that $\ell_k^{(n-j)} < \ell_{k-1}^{(n-j)}$. By the definition of $\ell_{b+i-1}^{(n+j)}$ and symmetry there is no singular string of length $\ell_{k-1}^{(n-j)} + 1$ available for $s_{a+i-j}^{(n-j)}$ unless (i) $\ell_{b+i-1}^{(n+j+1)} = \ell_{b+i-1}^{(n+j)}$ or (ii) $\ell_{b+i-2}^{(n+j-1)} = \ell_{b+i-1}^{(n+j)}$. Assume case (i) holds. By (A.4) either $[s_{a+i-j-2}^{(n-j-1)}, \ell_{b+i-1}^{(n+j-1)}]$ or $[\ell_{k-2}^{(n-j-1)}, \ell_{b+i-1}^{(n+j+1)}]$ has to be a nut-bolt pair. However, the first case yields a contradiction since $s_{a+i-j-2}^{(n-j-1)} \geq s_{a+i-j-1}^{(n-j)}$ so that $[\ell_{k-2}^{(n-j-1)}, \ell_{b+i-1}^{(n+j+1)}]$ must be a nut-bolt pair. Assume that case (ii) holds. Then $[\ell_{k-2}^{(n-j+1)}, \ell_{b+i-2}^{(n+j-1)}]$ must be a nut-bolt pair. By (A.2) this implies that $\ell_{k-2}^{(n-j-1)} = \ell_{k-2}^{(n-j)} = \ell_{k-1}^{(n-j)}$ and $[\ell_{k-2}^{(n-j-1)}, \ell_{b+i-1}^{(n+j+1)}]$ must be a nut-bolt pair. Hence in both case (i) and case (ii) $[\ell_{k-2}^{(n-j-1)}, \ell_{b+i-1}^{(n+j+1)}]$ must be a nut-bolt pair and $\ell_{k-2}^{(n-j-1)} = \ell_{k-1}^{(n-j)}$. Now, there is no singular string of length $\ell_{k-1}^{(n-j)} + 1$ for $s_{a+i-j}^{(n-j)}$ unless $\ell_{k-2}^{(n-j-1)} = \ell_{k-2}^{(n-j)}$. But then $[\ell_{k-2}^{(n-j)}, \ell_{b+i-2}^{(n+j)}]$ must be a nut-bolt pair since otherwise $[\ell_{k-2}^{(n-j)}, \ell_{b+i-1}^{(n+j)}]$ is a nut-bolt pair which contradicts our assumptions. This in turn implies that $\ell_{b+i-2}^{(n+j-1)} = \ell_{k-1}^{(n-j)}$. By (A.4) either $[s_{a+i-j-1}^{(n-j+1)}, \ell_{b+i-2}^{(n+j-1)}]$ or $[\ell_{k-2}^{(n-j+1)}, \ell_{b+i-2}^{(n+j-1)}]$ must be a nut-bolt pair. However, the first case contradicts $s_{a+i-j}^{(n-j)} = \ell_{k-1}^{(n-j)} + 1$ since $s_{a+i-j}^{(n-j)} \leq s_{a+i-j-1}^{(n-j+1)}$. Hence $[\ell_{k-2}^{(n-j+1)}, \ell_{b+i-2}^{(n+j-1)}]$ must be a nut-bolt pair. By induction $s_{a+i-j-2}^{(n-j)} = \ell_{k-1}^{(n-j)} + 1$. In summary, both $[\ell_{k-2}^{(n-j)}, \ell_{b+i-2}^{(n+j)}]$ and $[\ell_{k-2}^{(n-j+1)}, \ell_{b+i-2}^{(n+j-1)}]$ are nut-bolt pairs with $s_{a+i-j-2}^{(n-j)} = \ell_{k-1}^{(n-j)} + 1$. Now repeat the entire argument which shows that both $[\ell_{k-3}^{(n-j)}, \ell_{b+i-3}^{(n+j)}]$ and $[\ell_{k-3}^{(n-j+1)}, \ell_{b+i-3}^{(n+j-1)}]$ are nut-bolt pairs with $s_{a+i-j-3}^{(n-j)} = \ell_{k-1}^{(n-j)} + 1$, etc. Since there are only finitely many ℓ -slides, these conditions must eventually break down which shows that $s_{a+i-j}^{(n-j)} = \ell_{k-1}^{(n-j)} + 1$ cannot occur when $\ell_k^{(n-j)} < \ell_{k-1}^{(n-j)}$. This concludes the proof of Lemma A.3. \square

Proof of (A.3). We prove (A.3) by induction on i .

Induction beginning: $i = 0$. If $v_b = n + 1$, then by Proposition 6.17 also $u_a = n + 1$ and $u_{a-1}, v_{b-1} < n$. This implies that $\ell_b^{(n)}$ is the largest singular string in $(\nu, J)^{(n)}$ and $s_a^{(n)}$ is the largest singular string in the central rigged partition after the addition of v . Since $\ell_b^{(n)} + 1$ is singular it follows that $s_a^{(n)} \geq \ell_b^{(n)} + 1$. After the addition of v the vacancy numbers of the strings of length $h > \ell_b^{(n)}$ in the central rigged partition decrease by one. However, since all labels are even no string of length $h > \ell_b^{(n)} + 1$ becomes singular. Hence $s_a^{(n)} = \ell_b^{(n)} + 1$.

Now assume that $v_b > n + 1$.

If $[s_{a-1}^{(n-1)}, \ell_b^{(n+1)}]$ is a nut-bolt pair, then by (A.2) $s_a^{(n)} \leq s_{a-1}^{(n-1)} = \ell_b^{(n+1)}$. Furthermore, $\ell_{b-1}^{(n-1)} > \ell_b^{(n+1)}$ since otherwise the folding of the b -th ℓ -slide would not cross the $(b-1)$ -th ℓ -slide in a bolt. This implies that $s_a^{(n+1)} > \ell_b^{(n+1)}$. Also $s_{a-1}^{(n-1)} > \ell_b^{(n)}$ since otherwise $[s_{a-1}^{(n-1)}, \ell_b^{(n+1)}]$ was not a nut-bolt pair. Since there is a singular string of length $\ell_b^{(n)} + 1$, this implies that $s_a^{(n)} \geq \ell_b^{(n)} + 1$. After the addition of v , the vacancy numbers corresponding to the strings of length $\ell_b^{(n)} < h \leq \ell_b^{(n+1)}$ in the central rigged partition have decreased by 1. Again, since there are no singular strings of length $\ell_b^{(n)} < h \leq \ell_b^{(n+1)}$ in $(\nu, J)^{(n)}$ by the definition of $\ell_b^{(n)}$ and all labels are even, it follows that there are no singular strings of length $\ell_b^{(n)} + 1 < h \leq \ell_b^{(n+1)}$ after the addition of v . Hence $s_a^{(n)} = \ell_b^{(n)} + 1$.

If $[\ell_{b-1}^{(n-1)}, \ell_b^{(n+1)}]$ is a nut-bolt pair, then by (A.6) and (A.7) we have $s_{a-1}^{(n-1)} > \ell_{b-1}^{(n-1)}$ and $s_a^{(n+1)} > \ell_b^{(n+1)}$. This implies $s_a^{(n)} \geq \ell_b^{(n)} + 1$. After the addition of v the vacancy number in the central rigged partition corresponding to the strings of length $\ell_b^{(n)} < h \leq \ell_b^{(n+1)}$ decreases by one and of length $\ell_b^{(n+1)} < h$ increases by one. Hence there are no singular strings of length $h > \ell_b^{(n+1)}$ and by the now familiar arguments also not for $\ell_b^{(n)} + 1 < h \leq \ell_b^{(n+1)}$ since all labels are even. This proves $s_a^{(n)} = \ell_b^{(n)} + 1$.

Finally, let $[\ell_b^{(n-1)}, \ell_b^{(n+1)}]$ be a nut-bolt pair. Note that this case can only happen if $u_{a-1} = n$. We will show that $s_{a-1}^{(n-1)} = \ell_b^{(n)} + 1$. Since by (A.7) $s_a^{(n+1)} > \ell_b^{(n)}$, it then follows that $s_a^{(n)} = \ell_b^{(n)} + 1$. If $v_b = n + 2$, then $v_{b-1} < n$ for $[\ell_b^{(n-1)}, \ell_b^{(n+1)}]$ holds. The length of the longest singular string in $(\nu, J)^{(n+1)}$ is $\ell_b^{(n)}$ and hence by symmetry also in $(\nu, J)^{(n-1)}$. This forces $s_{a-1}^{(n-1)} = \ell_b^{(n)} + 1$. Hence assume $v_b > n + 2$. If $[\ell_b^{(n-2)}, \ell_b^{(n+2)}]$ is a nut-bolt pair, then $s_{a-1}^{(n-1)} = \ell_b^{(n)} + 1$ holds by Lemma A.3. If $[\ell_{b-1}^{(n-2)}, \ell_b^{(n+2)}]$ is a nut-bolt pair, then the vacancy numbers in the $(n-1)$ -st rigged partition corresponding to strings of length $\ell_{b-1}^{(n-2)} < h \leq \ell_{b-1}^{(n-1)}$ are increased by one after the addition of v so that there are no singular strings of this length. By Lemma A.3 $\ell_{b-1}^{(n-2)} < s_{a-2}^{(n-2)} \leq \ell_{b-1}^{(n-1)}$ (note that the case $\ell_{b-1}^{(n-1)} = \infty$ is included here). Hence using $s_{a-1}^{(n-1)} \leq s_{a-2}^{(n-2)}$ and (A.6) this implies $\ell_b^{(n-1)} < s_{a-1}^{(n-1)} \leq \ell_{b-1}^{(n-2)}$. By the definition of $\ell_b^{(n+1)}$ there are no singular strings in $(\nu, J)^{(n+1)}$ of length $\ell_b^{(n)} < h \leq \ell_{b-1}^{(n-2)}$ and hence by symmetry also not in $(\nu, J)^{(n-1)}$. After the addition of v there is a singular string of length $\ell_b^{(n-1)} + 1$ so that $s_{a-1}^{(n-1)} = \ell_b^{(n)} + 1$. To conclude, assume that $[s_{a-2}^{(n-2)}, \ell_b^{(n+2)}]$ is a nut-bolt pair. Then by (A.2) and (A.6) $\ell_b^{(n-1)} < s_{a-1}^{(n-1)} \leq s_{a-2}^{(n-2)}$. By the same arguments as in the previous case there are no singular strings of length $\ell_{b-1}^{(n-1)} + 1 < h \leq s_{a-2}^{(n-2)}$ in the $(n-1)$ -st rigged partition after the addition of v so that $s_{a-1}^{(n-1)} = \ell_b^{(n)} + 1$ as asserted.

Induction step: $i-1 \rightarrow i$. Assume that $[s_{a+i-1}^{(n-1)}, \ell_{b+i}^{(n+1)}]$ is a nut-bolt pair. Then $\ell_{b+i}^{(n+1)} > \ell_{b+i}^{(n)}$, $s_{a+i}^{(n+1)} \geq \ell_{b+i}^{(n+1)}$ and $\ell_{b+i-1}^{(n-1)} > \ell_{b+i}^{(n+1)}$. Since there is a singular string of length $\ell_{b+i}^{(n)}$ after the addition of v and the letters u_1, \dots, u_{a+i-1} , it follows that $\ell_{b+i}^{(n)} < s_{a+i}^{(n)} \leq s_{a+i-1}^{(n-1)}$ by (A.2). By the definition of $\ell_{b+i}^{(n)}$ there are no singular

strings of length $\ell_{b+i}^{(n)} < h \leq \ell_{b+i}^{(n+1)}$ in $(\nu, J)^{(n)}$. The vacancy numbers of the central rigged partition corresponding to the strings of length $\ell_{b+i}^{(n)} < h \leq \ell_{b+i}^{(n+1)}$ decrease by one. Hence due to the even labels there are no singular strings of length $\ell_{b+i}^{(n)} + 1 < h \leq \ell_{b+i}^{(n+1)}$ in the central rigged partition after the addition of all letters up to u_{a+i-1} . This proves (A.3).

Now assume that $[\ell_{b+i-1}^{(n-1)}, \ell_{b+i}^{(n+1)}]$ is a nut-bolt pair. By induction $s_{a+i-1}^{(n)} = \ell_{b+i-1}^{(n)} + 1$. The vacancy numbers of the central rigged partition after the addition of all letters up to u_{a+i-1} for the strings of length $\ell_{b+i-1}^{(n-1)} < h \leq \ell_{b+i-1}^{(n)}$ increase by one and for the strings of length $\ell_{b+i}^{(n)} < h \leq \ell_{b+i-1}^{(n-1)}$ decrease by one. Since there are no singular strings of length $\ell_{b+i}^{(n)} < h \leq \ell_{b+i}^{(n+1)} = \ell_{b+i-1}^{(n-1)}$ in $(\nu, J)^{(n)}$ by the definition of $\ell_{b+i}^{(n)}$ there are no singular strings of length $\ell_{b+i}^{(n)} + 1 < h \leq \ell_{b+i-1}^{(n)}$ in the central rigged partition after the addition of all letters up to u_{a+i-1} due to the even labels. Also $s_{a+i}^{(n)} \leq s_{a+i-1}^{(n)} = \ell_{b+i-1}^{(n)} + 1$. Hence (A.3) holds unless $s_{a+i}^{(n)} = s_{a+i-1}^{(n)}$ and $\ell_{b+i}^{(n)} < \ell_{b+i-1}^{(n)}$. We will show that in the present case this cannot occur. Assume that $s_{a+i}^{(n)} = s_{a+i-1}^{(n)}$ and $\ell_{b+i}^{(n)} < \ell_{b+i-1}^{(n)}$. This requires $s_{a+i-1}^{(n-1)} = s_{a+i-1}^{(n)}$ and $s_{a+i}^{(n+1)} = s_{a+i-1}^{(n)}$ by (A.2). By (A.4) either (1) $[s_{a+i-2}^{(n-1)}, \ell_{b+i-1}^{(n+1)}]$ or (2) $[\ell_k^{(n-1)}, \ell_{b+i-1}^{(n+1)}]$ forms a nut-bolt pair. Since $[\ell_{b+i-1}^{(n-1)}, \ell_{b+i}^{(n+1)}]$ is a nut-bolt pair by assumption, k in case (2) must be $k = b + i - 2$. Hence if case (2) holds, there is no singular string of length $s_{a+i-1}^{(n)}$ in the central rigged partition after the addition of all letters up to u_{a+i-1} by the change in vacancy number. Hence $s_{a+i}^{(n)} = s_{a+i-1}^{(n)}$ cannot hold. By the same arguments there is no singular string of length $s_{a+i-1}^{(n)}$ if $\ell_{b+i-2}^{(n-1)}, \ell_{b+i-1}^{(n+1)} > \ell_{b+i-1}^{(n)}$. Hence either (a) $\ell_{b+i-2}^{(n-1)} = \ell_{b+i-1}^{(n)}$ or (b) $\ell_{b+i-1}^{(n+1)} = \ell_{b+i-1}^{(n)}$. Since $s_{a+i-2}^{(n-1)} \geq s_{a+i-1}^{(n-1)} > \ell_{b+i-1}^{(n)}$, case (b) requires case (2) which we already argued yields a contradiction. Hence assume case (a) and case (1) holds. But then by (A.4) and (A.5) $[\ell_{b+i-2}^{(n-1)}, s_{a+i-1}^{(n+1)}]$ must be a nut-bolt pair which contradicts $s_{a+i-1}^{(n+1)} = \ell_{b+i-1}^{(n)} + 1$. This concludes the proof of (A.3) when $[\ell_{b+i-1}^{(n-1)}, \ell_{b+i}^{(n+1)}]$ is a nut-bolt pair.

Finally, assume that $[\ell_{b+i}^{(n-1)}, \ell_{b+i}^{(n+1)}]$ is a nut-bolt pair. If $\ell_{b+i}^{(n+2)} = \infty$, then $i = 1$ and $v_b = u_a = n + 1$, $v_{b+1} = u_{a+1} = n + 2$ and $u_{a-1}, v_{b-1} < n - 1$. By symmetry $[\ell_b^{(n-1)}, \ell_{b+1}^{(n+1)}]$ forms a nut-bolt pair which contradicts our assumption. Hence we may assume from now on that $\ell_{b+i}^{(n+2)} < \infty$. We will show that $s_{a+i-1}^{(n-1)} = \ell_{b+i}^{(n)} + 1$. Then by (A.6) and (A.2) it follows that $s_{a+i}^{(n)} = \ell_{b+i}^{(n)} + 1$. If $[s_{a+i-2}^{(n-2)}, \ell_{b+i}^{(n+2)}]$ is a nut-bolt pair, then $\ell_{b+i-1}^{(n-2)} \geq \ell_{b+i}^{(n+2)}$ since otherwise the $(b + i - 1)$ -st and the $(b + i)$ -th ℓ -slide would not cross in a bolt. Hence by the definition of $\ell_{b+i}^{(n+1)}$ there are no singular strings of length $\ell_{b+i}^{(n+1)} < h \leq \ell_{b+i}^{(n+2)}$ in $(\nu, J)^{(n+1)}$. By symmetry there are no singular strings of length $\ell_{b+i}^{(n+1)} < h \leq \ell_{b+i}^{(n+2)}$ in $(\nu, J)^{(n-1)}$. Since $s_{a+i-1}^{(n-1)} \leq s_{a+i-2}^{(n-2)}$ by (A.2) it follows that $s_{a+i-1}^{(n-1)} = \ell_{b+i}^{(n)} + 1$. If $[\ell_{b+i}^{(n-2)}, \ell_{b+i}^{(n+2)}]$ is a nut-bolt pair, then $s_{a+i-1}^{(n-1)} = \ell_{b+i}^{(n)} + 1$ follows from Lemma A.3. The last case to consider is that $[\ell_{b+i-1}^{(n-2)}, \ell_{b+i}^{(n+2)}]$ is a nut-bolt pair. Note that due to the change of vacancy numbers there are no singular strings of length $\ell_{b+i-1}^{(n-2)} < h \leq \ell_{b+i-1}^{(n-1)}$ in

the $(n - 1)$ -st rigged partition after the addition of all letters up to u_{a+i-1} . Also by the definition of $\ell_{b+i}^{(n+1)}$ and symmetry there are no singular strings of length $\ell_{b+i}^{(n-1)} + 1 < h \leq \ell_{b+i-1}^{(n-2)}$ in the same rigged partition. Hence, if $s_{a+i-1}^{(n-1)} \leq \ell_{b+i-1}^{(n-1)}$, then $s_{a+i-1}^{(n-1)} = \ell_{b+i}^{(n-1)} + 1$ as asserted. We may therefore restrict our attention to the case $s_{a+i-1}^{(n-1)} = \ell_{b+i-1}^{(n-1)} + 1$ with $\ell_{b+i}^{(n-1)} < \ell_{b+i-1}^{(n-1)}$ (note that $\ell_{b+i-1}^{(n-1)} = \ell_{b+i}^{(n-1)}$ would imply the assertion). We will show that these conditions cannot hold simultaneously. By Lemma A.3 $s_{a+i-1}^{(n-1)} = \ell_{b+i-1}^{(n-1)} + 1$ can only happen if (1) $[\ell_{b+i-1}^{(n-1)}, \ell_{b+i}^{(n+1)}]$ is a nut-bolt pair or (2) $\ell_{b+i-1}^{(n-2)} = \ell_{b+i-1}^{(n-1)}$. Note that case (2) implies case (1) since otherwise $[\ell_{b+i-1}^{(n-1)}, \ell_{b+i}^{(n+1)}]$ is a nut-bolt pair which contradicts our assumptions. Case (1) implies by induction that $s_{a+i-2}^{(n-1)} = \ell_{b+i-1}^{(n-1)} + 1$. In summary $[\ell_{b+i-1}^{(n-1)}, \ell_{b+i-1}^{(n+1)}]$ is a nut-bolt pair with $s_{a+i-2}^{(n-1)} = \ell_{b+i-1}^{(n-1)} + 1$. There are no singular strings of length $\ell_{b+i-1}^{(n-1)} + 1$ available in the $(n - 1)$ -st rigged partition for $s_{a+i-1}^{(n-1)}$ unless (a) $\ell_{b+i-1}^{(n+2)} = \ell_{b+i-1}^{(n-1)}$ or (b) $\ell_{b+i-2}^{(n)} = \ell_{b+i-1}^{(n-1)}$. In case (a) by (A.4) either $[s_{a+i-3}^{(n-2)}, \ell_{b+i-1}^{(n+2)}]$ or $[\ell_{b+i-2}^{(n-2)}, \ell_{b+i-1}^{(n+2)}]$ forms a nut-bolt pair. The first case contradicts $s_{a+i-3}^{(n-2)} \geq s_{a+i-2}^{(n-1)}$. In case (b) $\ell_{b+i-2}^{(n-2)} = \ell_{b+i-1}^{(n-1)}$ so that $[\ell_{b+i-2}^{(n-2)}, \ell_{b+i-1}^{(n+2)}]$ must form a nut-bolt pair. Hence in cases (a) and (b), $[\ell_{b+i-2}^{(n-2)}, \ell_{b+i-1}^{(n+2)}]$ forms a nut-bolt pair and $\ell_{b+i-2}^{(n-2)} = \ell_{b+i-1}^{(n-1)}$. By the change in vacancy numbers there is no singular string of length $\ell_{b+i-1}^{(n-1)} + 1$ available for $s_{a+i-1}^{(n-1)}$ unless $\ell_{b+i-2}^{(n-1)} = \ell_{b+i-1}^{(n-1)}$. Then $[\ell_{b+i-2}^{(n-1)}, \ell_{b+i-2}^{(n+1)}]$ must be a nut-bolt pair since otherwise $[\ell_{b+i-2}^{(n-1)}, \ell_{b+i-1}^{(n+1)}]$ forms a nut-bolt pair which contradicts our assumptions. By induction $s_{a+i-3}^{(n-1)} = \ell_{b+i-1}^{(n-1)} + 1$. Repeating the same arguments shows that $[\ell_{b+i-3}^{(n-1)}, \ell_{b+i-3}^{(n+1)}]$ forms a nut-bolt pair with $s_{a+i-4}^{(n-1)} = \ell_{b+i-1}^{(n-1)} + 1$, etc. Since there are only finitely many ℓ -slides, these conditions must be eventually broken which shows that $s_{a+i-1}^{(n-1)} = \ell_{b+i-1}^{(n-1)} + 1$ is not possible unless $\ell_{b+i}^{(n-1)} = \ell_{b+i-1}^{(n-1)}$. This concludes the proof of (A.3). \square

So far we have shown that the conditions of Theorem 7.1 are satisfied if $(\nu, J) \in \text{Im}(\overline{\phi} \circ \Psi_R)$. It remains to show the converse, that is, if (ν, J) satisfies the conditions of Theorem 7.1, then $(\nu, J) \in \text{Im}(\overline{\phi} \circ \Psi_R)$.

Let $p = p_L \otimes \cdots \otimes p_1 = \overline{\phi}^{-1}(\nu, J)$ where $p_i = b_i \tilde{b}_i \in B_A^{n,2}$ if R_i is a single column of height $r_i = n$ and p_i is of type $C_n^{(1)}$, $p_i = b_i \in B_A^{n,1}$ if R_i is a single column of height $r_i = n$ and p_i of type $D_{n+1}^{(2)}$, or $p_i = b_i \otimes \tilde{b}_i \in B_A^{2n-r_i,1} \otimes B_A^{r_i,1}$ otherwise. By Theorem 5.7 and the fact that the combinatorial R -matrix on paths yields the identity map on rigged configurations, condition 1 of Theorem 7.1 ensures that p is contragredient self-dual, that is $\sigma(b_i \otimes \tilde{b}_i) = b_i^{\vee*} \otimes \tilde{b}_i^{\vee*}$. For type $D_{n+1}^{(2)}$ this already shows that $(\nu, J) \in \text{Im}(\overline{\phi} \circ \Psi_R)$ by Proposition 6.20.

For type $A_{2n}^{(2)}$ act on (ν, J) with $j_{r,1}^{-1} \circ j_{r+1,1}^{-1} \circ \cdots \circ j_{r_L-1,1}^{-1}$ with r as small as possible. That is, first remove singular strings of length 1 from the k -th rigged partition for $r_L \leq k \leq 2n - r_L$ if possible. Repeat this for $r_L - 1, r_L - 2, \dots$ until no more such singular strings can be removed for $r - 1$.

By induction we may assume that $(\overline{\nu}, \overline{J}) = \overline{\phi}(p_{L-1} \otimes \cdots \otimes p_1)$ is in $\text{Im}(\overline{\phi} \circ \Psi_{R'})$ if it satisfies the properties of Theorem 7.1 where $R' = (R_1, \dots, R_{L-1})$. Set $u \otimes v = b_L \otimes \tilde{b}_L$ and $r = r_L$ for type $C_n^{(1)}$ and $u \otimes v = i_{r,1}^{-1} \circ i_{r+1,1}^{-1} \circ \cdots \circ i_{r_L-1,1}^{-1}(b_L \otimes \tilde{b}_L)$

for type $A_{2n}^{(2)}$. Then by Proposition 6.17 and Theorem 6.25 it suffices to prove that $(uv)|_{[n]}$ is a tableau, $|u|_{[n]} - |v|_{[n]} = n - r$ and $(\overline{\nu}, \overline{J})$ satisfies point 3 of Theorem 7.1 for type $A_{2n}^{(2)}$ and points 2 and 3 for type $C_n^{(1)}$.

The rigged configuration $(\overline{\nu}, \overline{J})$ is obtained from (ν, J) by the inverse algorithm to the one described in section 5.4. Let $(\nu, J)_{(u,i)}$ for $0 \leq i \leq 2n - r$ and $(\nu, J)_{(v,i)}$ for $0 \leq i \leq r$ be rigged configurations where $(\nu, J)_{(u,2n-r)} = (\nu, J)$, $(\nu, J)_{(u,0)} = (\nu, J)_{(v,r)}$ and $(\nu, J)_{(v,0)} = (\overline{\nu}, \overline{J})$. The rigged configuration $(\nu, J)_{(u,i-1)}$ is obtained from $(\nu, J)_{(u,i)}$ in the following way: Select singular strings of length $\tilde{s}_i^{(i)}, \tilde{s}_i^{(i+1)}, \dots$ in the i -th, $(i + 1)$ -th, \dots rigged partition in $(\nu, J)_{(u,i)}$ minimal such that $\tilde{s}_i^{(i)} \leq \tilde{s}_i^{(i+1)} \leq \dots$. If no such singular string exists in the k -th rigged partition set $\tilde{s}_i^{(k)} = \infty$. Let u_i be minimal such that $\tilde{s}_i^{(u_i)} = \infty$. Then remove a box from each of the selected strings in the i -th to $(u_i - 1)$ -th rigged partition in $(\nu, J)_{(u,i)}$ keeping them singular and leaving all other strings unchanged. The result is $(\nu, J)_{(u,i-1)}$. Define $\tilde{\ell}_i^{(k)}, v_i$ and $(\nu, J)_{(v,i-1)}$ in the analogous fashion starting with $(\nu, J)_{(v,i)}$. Then let u be the column tableau of height $2n - r$ with entries u_1, \dots, u_{2n-r} and v the column tableau of height r with entries v_1, \dots, v_r .

Similarly to (A.3) the following holds. Let a and d_u be defined such that $\tilde{s}_a^{(n)}, \tilde{s}_{a+1}^{(n)}, \dots, \tilde{s}_{a+d_u-1}^{(n)}$ are precisely all finite $\tilde{s}_i^{(n)}$. Define b and d_v similarly in terms of $\tilde{\ell}_i^{(n)}$. Then similar to the proof of (A.3) it can be shown that $d := d_u = d_v$ and

$$\tilde{\ell}_{b+i}^{(n)} = \tilde{s}_{a+i}^{(n)} - 1 \quad \text{for } 0 \leq i < d.$$

(For type $C_n^{(1)}$, $\tilde{s}_{a+i}^{(n)} \neq 1$ since by property 2 all parts in $\nu^{(n)}$ are even. For type $A_{2n}^{(2)}$ observe that $\tilde{s}_{a+i}^{(n)} \neq 1$ also due to the application of $j_{r,1}^{-1} \circ j_{r+1,1}^{-1} \circ \dots \circ j_{r_L-1,1}^{-1}$ to (ν, J) .) This proves, in particular, that $(\overline{\nu}, \overline{J})$ satisfies point 3 for type $A_{2n}^{(2)}$ and points 2 and 3 of Theorem 7.1 for type $C_n^{(1)}$. It also shows that $|u|_{[n]} - |v|_{[n]} = (n - d) - (r - d) = n - r$ as desired.

Finally, Lemma A.2 holds with $s_i^{(k)}$ and $\ell_i^{(k)}$ replaced by $\tilde{s}_i^{(k)}$ and $\tilde{\ell}_i^{(k)}$, respectively. The proof is similar to the proof of Lemma A.2. From this it follows that $\tilde{\ell}_{b+i}^{(n+j)} \leq \tilde{s}_{a+i-j}^{(n-j)}$. In particular, this implies that $\tilde{s}_{a+i-j}^{(n-j)} = \infty$ if $\tilde{\ell}_{b+i}^{(n+j)} = \infty$. Let c be minimal such that $\tilde{\ell}_{b+i}^{(n+c)} = \infty$, that is, $n + c = v_{b+i}$. Then the above shows that

$$(A.8) \quad u_{a+i-c} \leq n - c.$$

By the contragredient self-duality $(uv)|_{[n]}$ is a tableau if and only if $(uv)|_{[n+1,2n]}$ is a tableau. Now assume that $(uv)|_{[n+1,2n]}$ is not a tableau, that is, there exists an index i such that $u_{a+i'} \leq v_{b+i'}$ for all $0 \leq i' < i$ and $u_{a+i} > v_{b+i}$. Let $v_{b+i} = n + c$. Then it follows from duality that $u_{a+i-c} > n - c$. This contradicts (A.8). Hence $(uv)|_{[n]}$ must be a tableau as asserted.

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