REPRESENTATION THEORY
An Electronic Journal of the American Mathematical Society
Volume 7, Pages 181–195 (May 7, 2003)
S 1088-4165(03)00178-X

COUNTEREXAMPLES TO THE 0-1 CONJECTURE

TIMOTHY J. MCLARNAN AND GREGORY S. WARRINGTON

ABSTRACT. For permutations x and w, let $\mu(x, w)$ be the coefficient of highest possible degree in the Kazhdan-Lusztig polynomial $P_{x,w}$. It is well-known that the $\mu(x, w)$ arise as the edge labels of certain graphs encoding the representations of S_n. The 0-1 Conjecture states that the $\mu(x, w) \in \{0, 1\}$. We present two counterexamples to this conjecture, the first in S_{16}, for which x and w are in the same left cell, and the second in S_{10}. The proof of the counterexample in S_{16} relies on computer calculations.

1. Introduction

In studying the representations of Hecke algebras, Kazhdan and Lusztig [7] defined a class of polynomials now known as Kazhdan-Lusztig (K-L) polynomials $P_{x,w}$ that are indexed by pairs of elements in a Coxeter group. These polynomials carry important representation-theoretic and geometric information. Certain coefficients $\mu(x, w)$ are particularly important representation-theoretically in addition to controlling the recursive structure of the polynomials. While these coefficients $\mu(x, w)$ are easily seen to take varied (nonnegative) values in most Weyl groups, empirical evidence has suggested the following

0-1 Conjecture. For $x, w \in S_n$, $\mu(x, w) \in \{0, 1\}$.

If this conjecture were true, Kazhdan and Lusztig's construction of the irreducible representations of S_n (see [7]) would be embodied simply by graphs rather than *edge-labeled* (by the $\mu(x, w)$) graphs. However, as the following theorem shows, this is not the case.

Theorem 1. *Identify elements of S_{16} (resp., S_{10}) with permutations of the set $\{0, 1, \ldots, 9, a, b, c, d, e, f\}$ (resp., $\{0, 1, \ldots, 9\}$). We have the following two equalities:*

1. $\mu(54109832\mathrm{dc}76\mathrm{bafe}, \mathrm{c}810\mathrm{d}942\mathrm{fa}53\mathrm{b}6\mathrm{e}7) = 5$.
2. $\mu(4321098765, 9467182350) = 4$.

The first case offers the smallest counterexample to the 0-1 Conjecture with both permutations lying in the same left cell. The existence of such an example implies that the graphs describing the irreducible representations do, in fact, need to be edge-labeled. Exhaustive computer calculations by du Cloux [3] and both authors independently have shown that there are no counterexamples in S_9 or below. Hence, the counterexample given in part 2 of Theorem 1 occurs in the smallest possible group. The following corollary is immediate from either part of Theorem 1:

Received by the editors October 1, 2002 and, in revised form, March 24, 2003.
2000 *Mathematics Subject Classification.* Primary 05E15; Secondary 20F55.

Corollary 2. *The 0-1 Conjecture is false.*

So, in this sense, the combinatorics of the symmetric group is not simpler than that of other Weyl groups.

The possibility that $\mu(x, w) \in \{0, 1\}$ for any $x, w \in S_n$ was noticed by Lascoux and Schützenberger (see [5]), presumably noticed by Kazhdan and Lusztig, and certainly noticed independently by many others. In fact, Lascoux and Schützenberger [10] showed the 0-1 Conjecture to be true for Grassmannian permutations w. However, given the difficulty of examining S_n for $n \geq 9$ empirically, there has not appeared to be a consensus as to the truth of the conjecture.

Based on the work of Lascoux and Schützenberger, Garsia and McLarnan [5] list three progressively weaker conjectures related to the 0-1 Conjecture. The Lascoux-Schützenberger (L-S) graph has as vertices all members of a left cell. All pairs $\{x, w\}$ in which w covers x in the left weak Bruhat-Chevalley order are edges of this graph, as are the pairs produced by all possible applications of the L_i of Definition 10. Each such edge $\{x, w\}$ is easily checked to have $\mu(x, w) = 1$. There are three natural questions motivated by this construction:

1. Is this L-S graph identical to the "K-L graph" described by Kazhdan and Lusztig in [7]?
2. If one starts with the L-S graph and follows the recipe of Kazhdan and Lusztig for using the graph to associate a transition matrix to each permutation, does one obtain an irreducible representation for S_n corresponding to that left cell?
3. If not, then does one at least get some representation of S_n?

In Section 5, we elaborate on the computer calculations of the first author from the late 1980s that answered each of these questions in the negative. (Ochiai and Kako [12], in response to hearing about these original computations, confirmed these answers in the pursuit of computing the K-L graphs up through S_{15}.) Section 2 presents preliminary notation and definitions. Section 3 describes the algorithm used by the first author in 1989 to prove Theorem 1.1. In Section 4, we give the second author's combinatorial proof of Theorem 1.2.

2. Preliminaries

We will consider elements of S_n as permutations on the set $\{0, \ldots, n-1\}$. As a generating set $\mathcal{S}$, we will take the adjacent transpositions $s_i = (i, i+1)$ for $0 \leq i \leq n-2$. A one-line notation for a permutation w is afforded by writing the image of the n-tuple $[0, \ldots, n-1]$ under the action of w: $[w(0), w(1), \ldots, w(n-1)]$ (we often omit the commas and brackets). The *length function* for S_n is given by

$$l(w) = |\{0 \leq i < j < n : w(i) > w(j)\}|.$$

In Definition 4, we define the Bruhat-Chevalley partial order on S_n. (The definition we give is equivalent to more common descriptions such as the tableau criterion; see [1, 4, 6] and the references cited therein.)

Definition 3. Let $x, w \in S_n$, $p, q \in \mathbb{Z}$. Define the *rank function* $r_w(p, q) := |\{i \leq p : w(i) \geq q\}|$ and the *difference function* $d_{x,w}(p, q) := r_w(p, q) - r_x(p, q)$.

Definition 4. We define the *Bruhat-Chevalley partial order* "$\leq$" on S_n by setting $x \leq w$ if and only if $d_{x,w}(p, q) \geq 0$ for all p, q.

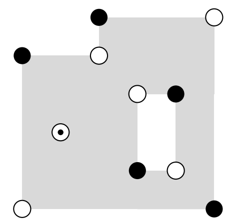

FIGURE 1. Bruhat picture for $x = [2, 0, 4, 1, 3, 5]$, $w = [5, 2, 3, 1, 4, 0]$.

Let $[x, w] = \{z : x \leq z \leq w\}$ in the Bruhat-Chevalley ordering. Billey and Warrington [2] prove the following result.

Lemma 5. *If $x(i) = w(i)$, $d_{x,w}(i, x(i)) = 0$ and $z \in [x, w]$, then $z(i) = x(i)$.*

We can view the Bruhat-Chevalley order graphically using "Bruhat pictures" determined by the function $d_{x,w}$. A typical picture is shown in Figure 1. Let $\mathrm{mat}(w)$ refer to the permutation matrix for w. Entries of $\mathrm{mat}(x)$ (resp., $\mathrm{mat}(w)$) are denoted by black disks (resp., open circles). Shading denotes regions in which $d_{x,w} \geq 1$. Successively darker shading denotes successively higher values of $d_{x,w}$. Positions corresponding to 1's of both $\mathrm{mat}(x)$ and $\mathrm{mat}(w)$ (termed "capitols") are denoted by a black disk and a larger concentric circle.

While not strictly necessary, these pictures help motivate results such as Lemma 5 and can be very helpful in computing with K-L polynomials. In fact, a number of the arguments in Section 3 were arrived at with the aid of these pictures. The reader of that section may benefit from constructing the appropriate Bruhat pictures.

Using the Bruhat-Chevalley order, there are several sets we can associate to any permutation w. We define the right and left descent sets of w and the set of flush elements of w to be

$$\mathrm{D_R}(w) = \{s \in \mathcal{S} : ws < w\}, \tag{1}$$

$$\mathrm{D_L}(w) = \{s \in \mathcal{S} : sw < w\} \text{ and} \tag{2}$$

$$\mathrm{Flush}(w) = \{x \leq w : \mathrm{D_R}(x) \supseteq \mathrm{D_R}(w) \text{ and } \mathrm{D_L}(x) \supseteq \mathrm{D_L}(w)\}. \tag{3}$$

We now give a combinatorial definition of the Kazhdan-Lusztig (K-L) polynomials applicable to any Coxeter group. For motivation and a more natural definition, we refer the reader to [6, 7]. In order to give the definition succinctly, we set

$$\mu(x, w) = \text{ coefficient of } q^{(l(w)-l(x)-1)/2} \text{ in } P_{x,w}, \tag{4}$$

and define $c_s(x) = 1$ if $xs < x$; $c_s(x) = 0$ if $xs > x$.

Theorem 6 ([7]). *There is a unique set of polynomials $\{P_{x,w}\}_{x,w \in S_n}$ such that, for all $x, w \in S_n$:*

1. $P_{w,w} = 1$,
2. $P_{x,w} = 0$ *when* $x \not\leq w$,

3. *If* $s \in \mathrm{D_R}(w)$, *then*

$$P_{x,w} = q^{c_s(x)} P_{x,ws} + q^{1-c_s(x)} P_{xs,ws} - \sum_{\substack{z \leq ws \\ zs < z}} \mu(z,ws) q^{\frac{l(w)-l(z)}{2}} P_{x,z}. \tag{5}$$

The analogous recursion with s *acting on the left holds when* $s \in \mathrm{D_L}(w)$. *Further, these polynomials satisfy the degree restriction*

$$\deg(P_{x,w}) \leq (l(w) - l(x) - 1)/2 \text{ when } x < w. \tag{6}$$

Note that $\mu(x,w)$ is the coefficient of the highest possible power of q in $P_{x,w}$ and that $\mu(x,w) = 0$ if $l(w) - l(x)$ is even.

The complexity of the K-L polynomials arises from the sum subtracted off in (5). We now introduce some notation to let us deal with these sums concisely. For $x, w \in S_n$ and $s \in \mathrm{D_R}(w)$, let

$$\begin{aligned}
\omega_{(\cdot s)}[x,ws] &= \{z : x \leq z < ws,\ zs < z,\ l(z) < l(ws) - 1, z \in \mathrm{Flush}(ws)\}, \\
\delta_{(\cdot s)}[x,ws] &= \{z : x \leq z < ws,\ zs < z,\ l(z) = l(ws) - 1\}, \\
\theta_{(\cdot s)}[x,ws] &= \delta_{(\cdot s)}[x,ws] \cup \omega_{(\cdot s)}[x,ws], \\
\Theta_{(\cdot s)}[x,ws] &= \sum_{z \in \theta_{(\cdot s)}[x,ws]} \mu(z,ws) q^{\frac{l(w)-l(z)}{2}} P_{x,z}.
\end{aligned}$$

Proposition 9.4 will imply that $\Theta_{(\cdot s)}[x,ws]$ is the sum appearing in (5). Let $z \in [x,ws]$. We say that z is *right s-flush* for this interval if $z \in \omega_{(\cdot s)}[x,ws]$. It is *right s-coatomic* for this interval if $z \in \delta_{(\cdot s)}[x,ws]$. The "left" versions are defined analogously (with "$(s\cdot)$" substituted for "$(\cdot s)$"). We will omit "left" and "right," as they will be clear from context.

We will need several additional properties of K-L polynomials that are not immediately apparent from the definition; we require the following notation:

Definition 7. For $w \in S_n$ and $0 \leq i_1 < \cdots < i_k \leq n-1$ for $k \leq n$, let $\mathrm{fl}[w(i_1), w(i_2), \ldots, w(i_k)]$ be the unique *flattened* permutation $[v(1), \ldots, v(k)] \in S_k$ such that $v(j) < v(k)$ precisely when $w(i_j) < w(i_k)$.

For example, if $w = 7461098253$, then

$$\mathrm{fl}[w(0), w(2), w(3), w(5), w(8), w(9)] = \mathrm{fl}[7,6,1,9,5,3] = 430521.$$

Definition 8. Let

$$\Delta(x,w) = \{i : x(i) \neq w(i) \text{ or } d_{x,w}(i, x(i)) \neq 0\}.$$

If $\Delta(x,w) = \{d_1, d_2, \ldots, d_k\}$ with $d_i < d_j$ for $i < j$, we get two *reduced* permutations by flattening x and w with respect to $\Delta(x,w)$:

$$\begin{aligned}
\widetilde{x} &= \mathrm{fl}([x(d_1), x(d_2), \ldots, x(d_k)]) \quad \text{and} \\
\widetilde{w} &= \mathrm{fl}([w(d_1), w(d_2), \ldots, w(d_k)]).
\end{aligned}$$

Note that $\widetilde{x}$ and $\widetilde{w}$ are permutations in S_k. For instance, if

$$x = 6491082753, \tag{7}$$

$$w = 9461782350, \tag{8}$$

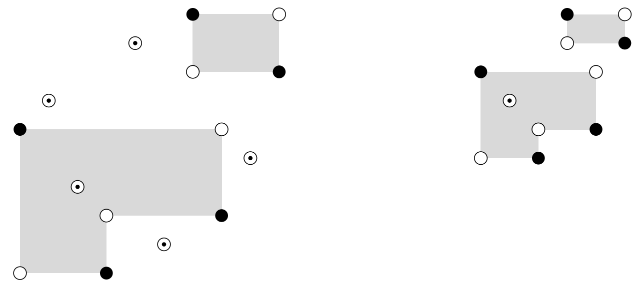

FIGURE 2. Sample Bruhat pictures for (7) and (9).

then $\Delta(x, w) = \{0, 2, 4, 6, 7, 9\}$, and

$$\widetilde{x} = 350142, \tag{9}$$

$$\widetilde{w} = 534120. \tag{10}$$

To obtain the Bruhat picture of the pair $\widetilde{x}$,$\widetilde{w}$ from that for x,w, one simply removes the capitols not surrounded by a shaded region (see Figure 2).

Proposition 9. *The K-L polynomials satisfy the following properties:*

1. *If* $s \in \mathrm{D_R}(w)$*, then* $P_{x,w} = P_{xs,w}$*. If* $s \in \mathrm{D_L}(w)$*, then* $P_{x,w} = P_{sx,w}$*.*
2. $P_{x,w} = P_{x^{-1},w^{-1}}$*.*
3. $P_{x,w} = P_{\widetilde{x},\widetilde{w}}$*.*
4. *If* $x \notin \mathrm{Flush}(w)$ *and* $l(x) < l(w) - 1$*, then* $\mu(x, w) = 0$*.*

The first two properties are standard and can be found in [7]. Proof of the third can be found in [2]; the fourth follows from the first property along with (6).

The $\mu(x, w)$ also satisfy an identity which we will be integral to the proof in Section 3. To state it, we make the following definitions:

Definition 10. Let $\mathcal{L}_k$ be the set of permutations w for which $s_k w < w$ or $s_{k+1} w < w$, but not both. Define an operator L_k acting on $\mathcal{L}_k$ by setting

$$(L_k w)^{-1}(j) = \begin{cases} w^{-1}(k+2), & \text{if } j = k, \\ w^{-1}(k), & \text{if } j = k+2, \\ w^{-1}(j), & \text{otherwise.} \end{cases}$$

In other words, $\mathcal{L}_k$ consists of all permutations in which k, $k + 1$, $k + 2$ do not appear either in increasing or decreasing order; and $L_k w$ is obtained from w by interchanging k, $k + 2$. For instance, $L_2[3, 1, 4, 0, 2] = [3, 1, 2, 0, 4]$. The operator L_k is called an elementary Knuth transformation. It is intimately connected to the Robinson-Schensted correspondence discussed below; for details, see [4, 8, 9].

Definition 11. For x and w comparable under the Bruhat-Chevalley order, set

$$\mu[x, w] = \begin{cases} \mu(x, w), & \text{if } x \leq w, \\ \mu(w, x), & \text{otherwise.} \end{cases}$$

Theorem 12 ([7])**.** *If* $x, w \in \mathcal{L}_k$*, then* $\mu[x, w] = \mu[L_k x, L_k w]$*.*

3. Computer Proof of Theorem 1.1

The example of Theorem 1.1 was found via a computer search for a counterexample to the 0-1 Conjecture. Since looking at every pair of permutations even in S_{10} is prohibitively expensive, we will cut down our search space by searching for a counterexample that is minimal in some sense. In particular, we will search for a counterexample $\{x, w\}$ in S_n with x and w in the same left cell which minimizes in order the following parameters.

1. n: i.e., $\mu(u, v) \leq 1$ for all $u, v \in S_{n-1}$ in the same left cell.
2. $l(w) - l(x)$: i.e., $u, v \in S_n$ in the same left cell with $l(v) - l(u) < l(w) - l(x)$ implies $\mu(u, v) \leq 1$.
3. $l(w)$: i.e., $v \in S_n$ with $l(v) < l(w)$ implies that there does not exist a $u \in S_n$ in the same left cell as v having $\mu(u, v) > 1$.

The key to searching efficiently turns out to be the Robinson-Schensted correspondence, which we now recall. The material in this section will be presented briefly—a more detailed exposition can be found in [5] (also see [4, 9]).

Let λ be a partition of n (denoted $\lambda \vdash n$) with $\lambda_1 \geq \lambda_2 \geq \cdots \lambda_k \geq 0$. We associate a Ferrers diagram consisting of left-justified rows of boxes with λ_i boxes in the i-th row from the bottom. A *standard tableau of shape* λ is an injective filling of these boxes with $0, 1, 2, \ldots, n-1$ such that the entries increase from left to right on rows and from bottom to top on columns. The *column word*, $\mathrm{cwd}(T)$, of a tableau T is obtained by reading the columns of T from top to bottom starting with the leftmost column. The *row word*, $\mathrm{rwd}(T)$, of T is obtained by reading the rows of T from left to right starting with the top row. Any word that can be obtained in this way from some tableau is called a *tableau word*. The *descent set* $D(T)$ of a tableau T is the set of indices i for which $i+1$ is strictly to the north and weakly to the west of i. In other words,

$$D(T) = D_L(\mathrm{rwd}(T)) = D_L(\mathrm{cwd}(T)).$$

The Robinson-Schensted correspondence gives a bijection between the elements $w \in S_n$ and the pairs of tableaux of the same shape $\lambda \vdash n$. For the specifics of the bijection, see, e.g., [4]. Via the Bruhat-Chevalley order on permutations, this correspondence induces a partial order on pairs of tableaux of the same shape which we will also denote by "$\leq$". The *left cell* indexed by the tableau Q consists of all pairs (P_w, Q) where P_w has the same shape as Q. Below we illustrate these definitions:

$$x = 4265013 \leftrightarrow (P_x, Q_x) = \left(\begin{array}{|c|c|c} \hline 4 & 6 & \\ \hline 2 & 5 & \\ \hline 0 & 1 & \multicolumn{1}{c|}{3} \\ \hline \end{array} , \begin{array}{|c|c|c} \hline 4 & 5 & \\ \hline 1 & 3 & \\ \hline 0 & 2 & \multicolumn{1}{c|}{6} \\ \hline \end{array} \right), \tag{11}$$

$$\mathrm{rwd}(P_x) = 4625013, \tag{12}$$

$$\mathrm{cwd}(P_x) = 4206513. \tag{13}$$

In order to describe the edge-labeled graphs defined by Kazhdan and Lusztig corresponding to *irreducible* representations of S_n, we need only consider pairs x, w lying in the same left cell. Thus, a counterexample among such pairs shows that the edge-labeling is necessary.

Fortunately, there is complete redundancy amongst the left cells with respect to the values of the $\mu[x, w]$:

Theorem 13 ([7]). *Let P_x, P_w, Q and Q' be tableaux of the same shape. Let x, w, x' and w' correspond, under the Robinson-Schensted correspondence, to the following pairs of tableaux:*

$$x \leftrightarrow (P_x, Q), \qquad w \leftrightarrow (P_w, Q),$$
$$x' \leftrightarrow (P_x, Q'), \qquad w' \leftrightarrow (P_w, Q').$$

Then $\mu[x, w] = \mu[x', w']$.

With Theorem 13, to search for a minimal counterexample, we effectively need only search over pairs $(P_x, Q), (P_w, Q)$ of tableaux of the same shape without regard to Q. One can get a sense of the savings by noting that $|S_{16}| = 20,922,789,888,000$, while the number of standard Young tableau of size 16 is a mere $46,206,736$. The task of considering all *pairs* of permutations is clearly infeasible. While one still needs to look at pairs of tableaux (of the same shape), even naively we need only consider $|S_{16}|$ pairs. And, with the proper filters, we can do much better.

We are now ready to present the main facts upon which the algorithm rests.

Lemma 14. *Let $x < w \in S_n$ with*

$$x \leftrightarrow (P_x, Q) \quad \text{and} \quad w \leftrightarrow (P_w, Q)$$

under the Robinson-Schensted correspondence. If the pair $\{x, w\}$ is a counterexample to the 0-1 Conjecture satisfying the above three minimality properties, then the following eight conditions must hold:

1. $D(P_w) \subseteq D(P_x)$.
2. $(P_w, Q') > (P_x, Q')$ *for all tableaux* Q'.
3. *The largest number, $n - 1$, sits strictly higher in P_w than in P_x.*
4. *If* $\mathrm{rwd}(P_w)^{-1}(k+2) < \mathrm{rwd}(P_w)^{-1}(k)$, *then*
$$\mathrm{rwd}(P_x)^{-1}(k+2) < \mathrm{rwd}(P_x)^{-1}(k+1) < \mathrm{rwd}(P_x)^{-1}(k).$$
5. *There do not exist $L_{i_1}, L_{i_2}, \ldots, L_{i_k}$ such that*

$$l(L_{i_k} \cdots L_{i_2} L_{i_1} P_w) - l(L_{i_k} \cdots L_{i_2} L_{i_1} P_x) < l(P_w) - l(P_x). \tag{14}$$

6. *There do not exist $L_{i_1}, L_{i_2}, \ldots, L_{i_k}$ satisfying both* (14) *and*

$$l(L_{i_j} L_{i_{j-1}} \cdots L_{i_1} P_w) - l(L_{i_j} L_{i_{j-1}} \cdots L_{i_1} P_x) = l(P_w) - l(P_x), \text{ all } j < k. \tag{15}$$

7. $l(w) - l(x)$ *is odd.*
8. *For no i are $0, 1, \ldots, i-1$ in identical positions in w and x and are $\widetilde{w}$ and $\widetilde{x}$ tableau words of the same shape.*

Proof. We give a brief justification for each condition:

1. This follows from Proposition 9.4 and the fact that $D(P_w) = D_L(\mathrm{rwd}(P_w))$.
2. Incomparability for some Q' would imply a contradiction by Theorem 13. So consider the case where $(P_w, Q') < (P_x, Q')$ for some Q'. As detailed in [5, Section 5], this implies that there exists some Q'' for which $w'' \leftrightarrow (P_w, Q'')$ and $x'' \leftrightarrow (P_w, Q'')$ are related in the *weak* Bruhat-Chevalley order and satisfy $l(w'') - l(x'') = 1$. But then, by definition, $\mu(x'', w'') = 1$. A contradiction then results by applying Theorem 13.
3. By the previous property, we must have $\mathrm{cwd}(P_w) \geq \mathrm{cwd}(P_x)$. This implies that $n - 1$ must be at least as high in P_w. If it is the same height, then by Proposition 9.3, you could delete it and get a counterexample in S_{n-1}.

4. Given minimality property 3, this is equivalent to the first property along with Theorem 13.
5. Knuth transformations preserve left cells; so by Theorem 13, existence would contradict minimality property 2.
6. This is a special case of the previous property.
7. If $l(w) - l(x)$ is even, then $\mu(x, w) = 0$.
8. Otherwise, $\widetilde{x}, \widetilde{w}$ lie in the same left cell and afford a smaller counterexample by Proposition 9.3. (Note that $l(w) - l(x) = l(\widetilde{w}) - l(\widetilde{x})$.)

□

We were able to write code to check quickly whether a pair (x, w) satisfies Properties 1, 3, 4, 7 and 8. It's more time-consuming to check 6. We found it slowest to check Properties 5 and 2.

Since we are working with such large groups, considerable care must be taken to check each of the above eight properties as efficiently as possible. For instance, it is impossibly slow to test property 2 by computing all Q' tableaux, doing inverse Robinson-Schensted, and checking the Bruhat-Chevalley relations. It is much faster to generate the pairs by doing Knuth transformations and to check for the Bruhat-Chevalley relation by seeing whether the Knuth transformation has destroyed the Bruhat-Chevalley relation which applied before the transformation.

The algorithm used to find a counterexample is as follows:

Step 1. Build up pairs of tableau P_x and P_w that satisfy properties 1, 3, 4 and 8 one letter at a time.

Step 2. Successively filter out those pairs not satisfying properties 7, 6, 5 and 2.

Step 3. For all remaining pairs P_x and P_w, choose Q' to minimize the length difference between $x \leftrightarrow (P_x, Q')$ and $w \leftrightarrow (P_w, Q')$.

Step 4. Compute $\mu(x, w)$. Filter out those pairs for which $\mu(x, w) \leq 1$.

No pairs in S_{13} or below make it through Step 2. In S_{14} and S_{15}, none make it through Step 4. But in S_{16}, the following pair of permutations passes all steps:

$$w = \text{c810d942fa53b6e7}, \quad l(w) = 53,$$
$$x = \text{54109832dc76bafe}, \quad l(x) = 32.$$

The difference in lengths is 21, and the leading coefficient (the coefficient of degree 10) of $P_{x,w}(q)$ is $\mu(x, w) = 5$. The K-L polynomial in its entirety is

$$P_{x,w}(q) = 5q^{10} + 72q^9 + 387q^8 + 1039q^7 + 1610q^6 + 1536q^5 + 931q^4 + 365q^3 + 92q^2 + 14q + 1.$$

This completes the proof of Theorem 1.1.

Exactly one other pair of permutations in S_{16} passes all the steps of this algorithm, affording a second counterexample to the 0-1 Conjecture:

$$w = \text{ca610fb732d84e95}, \quad l(w) = 60,$$
$$x = \text{76310cb542a98fed}, \quad l(x) = 39,$$
$$P_{x,w}(q) = 5q^{10} + 56q^9 + 231q^8 + 533q^7 + 776q^6 + 755q^5 + 501q^4 + 226q^3 + 67q^2 + 12q + 1.$$

To give an indication of the computational issues here, in S_{16} there are $16! = 20,922,689,888,000$ pairs of tableaux of the same shape, obviously too many to

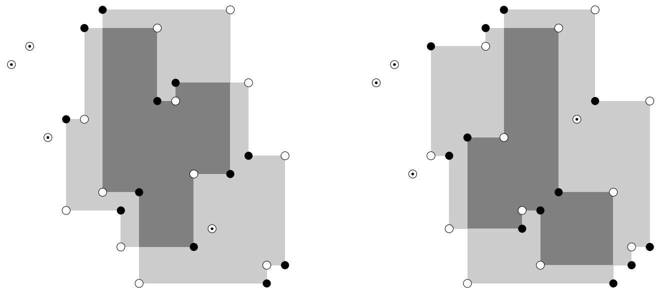

FIGURE 3. The Bruhat pictures for the two minimal counterexamples to the 0-1 Conjecture lying in S_{16}.

consider. By contrast, Step 1 of our algorithm produces 499,788 pairs. Of these, 246,399 satisfy property 7. Only 672 of these satisfy property 6, and only 288 also satisfy properties 5 and 2. Thus, in searching for a minimal counterexample to the 0-1 Conjecture in S_{16}, we need to compute $\mu(x, w)$ at only 288 pairs. The 2 pairs listed above are the only ones having $\mu(x, w) > 1$.

Although we have not been able completely to verify the results in this section without the use of computers, we are extremely confident of the truth of Theorem 1.1. The two authors began collaborating after we had independently written programs to compute K-L polynomials, and these programs agree on the values of the polynomials computed above. Only the first author has carried out the process of generating and filtering pairs to produce these counterexamples, but errors in that code would only affect the minimality of our examples. It seems extraordinarily unlikely that our completely independent computations of the K-L polynomials could be incorrect and yet agree. The computer code used in the proof of Theorem 1.1 (along with Java code for computing K-L polynomials) can be found at [11].

4. COMBINATORIAL PROOF OF THEOREM 1.2

The previous section describes a counterexample showing that for large enough n, labeled graphs are required for Kazhdan and Lusztig's description of the irreducible representations of S_n. In this section, we remove the condition that the two permutations lie in the same left cell. Such a counterexample is less interesting representation-theoretically, but it can be carried out entirely by hand, and it should lend insight into how and when $\mu(x, w)$ can be greater than 1. The counterexample we present was arrived at by close examination (using the "Bruhat pictures" of [2]) of the counterexample presented in the previous section.

We begin our proof of Theorem 1.2 by first calculating several intermediate K-L polynomials. The main tools in the proof are the defining recurrence relation (5) and parts 1 and 3 of Proposition 9. For each application of (5), there are usually several choices of the generator s. While our choices for s may seem *ad hoc*, they are actually carefully made, both to maximize the number of applications of Proposition 9.3 we can make and to simplify the calculation of the resulting Θ's.

Lemma 15. *The following equalities hold:*

1. $P_{1032,3120} = 1 + q$.
2. $P_{0213,2301} = 1 + q$.
3. $P_{315042,534120} = 1 + 3q + q^2$.
4. $P_{3106542,6345120} = 1 + 4q + 4q^2 + q^3$.

In the rest of this paper, for layout reasons, we sometimes write $P_{\substack{x\\w}}$ for $P_{x,w}$.

Proof. The first two equalities can be shown immediately using (5) or they can be found in [2]. For the third equality, we begin by expanding using (5) with $s = s_4$:

$$P_{\substack{315042\\534120}} = qP_{\substack{315042\\534102}} + P_{\substack{315024\\534102}} - \Theta_{(\cdot s_4)}[315042, 534102]. \tag{16}$$

Consider $\Theta_{(\cdot s_4)}[315042, 534102]$. It is clear that there are no s_4-coatomic elements. By Lemma 5, for any $z \in [315042, 534102]$, we need $z(5) = 2$. And for z to be s_4-flush, we need $\mathrm{D_R}(z) \supseteq \{s_0, s_2, s_3, s_4\}$. But these conditions cannot simultaneously be satisfied along with $z(0) \geq 3$ (necessary for $z \geq 315042$). So $\theta_{(\cdot s_4)}[315024, 534102] = \emptyset$ and $\Theta_{(\cdot s_4)}[315024, 534102] = 0$. Therefore,

$$\begin{aligned}
P_{\substack{315042\\534120}} &= qP_{\substack{21403\\42310}} + P_{\substack{315204\\534102}} && \text{(Proposition 9, parts 1 \& 3)}\\
&= qP_{\substack{21430\\42310}} + P_{\substack{20413\\42301}} && \text{(Proposition 9, parts 1 \& 3)}\\
&= qP_{\substack{1032\\3120}} + \Big(qP_{\substack{20413\\24301}} + P_{\substack{02413\\24301}} - \Theta_{(\cdot s_0)}[20413, 24301]\Big).
\end{aligned}$$

(The last equality follows from Proposition 9.3 and (5) with $s = s_0$.)

It is clear that there are no s_0-coatomic elements. If $z \in [20413, 24301]$, then it follows from Lemma 5 that $z(0) = 2$. But for z to be s_0-flush we need $\mathrm{D_R}(z) \supseteq \{s_0, s_1, s_2\}$. These two conditions cannot simultaneously be satisfied. Therefore, $\theta_{(\cdot s_0)}[20413, 24301] = \emptyset$ and $\Theta_{(\cdot s_0)}[20413, 24301] = 0$. Therefore,

$$\begin{aligned}
P_{\substack{315042\\534120}} &= q(1+q) + (qP_{\substack{0312\\3201}} + P_{\substack{04213\\24301}}) && \text{(Lemma 15.1; Proposition 9, parts 1 \& 3)}\\
&= q(1+q) + (q \cdot 1 + P_{\substack{0213\\2301}}) && \text{(Theorem 6.1; Proposition 9, parts 1 \& 3)}\\
&= q(1+q) + (q \cdot 1 + (1+q)) && \text{(Lemma 15.2)}\\
&= 1 + 3q + q^2.
\end{aligned}$$

For the fourth equality, we set $s = s_3$ in (5), and utilize Proposition 9.1:

$$P_{\substack{3106542\\6345120}} = (1+q)P_{\substack{3106542\\6341520}} - \Theta_{(\cdot s_3)}[3106542, 6341520].$$

There are no s_3-coatomic elements in $[3106542, 6341520]$. If z is in this interval, then by Lemma 5, $z(4) = 5$. For z to be s_3-flush, it must satisfy $\mathrm{D_R}(z) \supseteq \{s_0, s_2, s_3, s_4, s_5\}$. As these conditions cannot be simultaneously satisfied, we conclude that $\theta_{(\cdot s_3)}[3106542, 6341520] = \emptyset$; hence $\Theta_{(\cdot s_3)}[3106542, 6341520] = 0$. Therefore,

$$\begin{aligned}
P_{\substack{3106542\\6345120}} &= (1+q)P_{\substack{3106542\\6341520}}\\
&= (1+q)P_{\substack{315042\\534120}} && \text{(Proposition 9, parts 1 \& 3)}\\
&= (1+q)(1 + 3q + q^2) && \text{(Lemma 15.3)}\\
&= 1 + 4q + 4q^2 + q^3.
\end{aligned}$$

□

In addition to the above KL polynomials, we also need to compute several Θ's.

Lemma 16. *The following equalities hold:*

1. $\Theta_{(s_2\cdot)}[32170654, 72561340] = q^4$.
2. $\Theta_{(\cdot s_4)}[321087654, 835617240] = 0$.
3. $\Theta_{(\cdot s_3)}[4321098765, 9461782350] = q^4(1+q)$.

Proof. It is easily checked that there are no coatomic elements for any of the three cases above. Hence, in the following, we will assume that z is s-flush.

1. Let $x = 32170654$ and $v = s_2 w = 72561340$. To find the elements of $\omega_{(s_2\cdot)}[x,v]$ is straightforward but tedious. If $z \in \omega_{(s_2\cdot)}[x,v]$, then helpful facts about z include
   (a) $\{s_0, s_1, s_2, s_4, s_6\} \subseteq \mathrm{D_L}(z)$.
   (b) $\{s_0, s_3, s_6\} \subseteq \mathrm{D_R}(z)$.
   (c) $z(1) = 2$, which combined with 1a implies that $z(0) = 3$.
   (d) $z(2) \in \{1, 5, 7\}$.
   (e) To have $z \in [x, v]$ requires that $z^{-1}(1) \in \{2,3,4\}$, that $z^{-1}(7) \in \{2,3\}$, and that $z^{-1}(6) \in \{3,4,5\}$.

   Armed with these facts, it is not hard to find the nine elements of $\omega_{(s_2\cdot)}[x,v]$. Of these nine, only three have an odd length difference with respect to v (an even length difference with respect to w); only these three, which are shown in Table 1, can contribute to $\Theta_{(s_2\cdot)}[32170654, 72561340]$.

	z	$\widetilde{z}$	$\widetilde{v}$
z_1	32170654	2160543	6451230
z_2	32175640	10423	42301
z_3	32751640	0312	3120

TABLE 1. Cases for Lemma 16.1.

   We know from Lemma 15.4 and Proposition 9, parts 2 and 3 that $P_{z_1,v} = 1 + 4q + 4q^2 + q^3$. As $l(v) - l(z_1) = 7$, $\mu(z_1, v) = 1$. Finally, since $P_{z_1,z_1} = 1$, the only nonzero term of $\Theta_{(s_2\cdot)}[x,v]$ is $1 \cdot q^4 \cdot 1 = q^4$.
2. If $z \in [321087654, 835617240]$ then by Lemma 5, $z(5) = 7$. And if $z \in \omega_{(\cdot s_4)}[321087654, 835617240]$, then $\mathrm{D_R}(z) \supseteq \{s_0, s_3, s_4, s_5, s_7\}$. These two conditions cannot be satisfied simultaneously.
3. Let $x = 4321098765$ and $v = ws_3 = 9461782350$. To find the elements of $\omega_{(\cdot s_3)}[x,v]$ is again straightforward but still more tedious. If $z \in \omega_{(\cdot s_3)}[x,v]$, then:
   (a) $\mathrm{D_R}(z) \supseteq \{s_0, s_2, s_3, s_5, s_8\}$.
   (b) $\mathrm{D_L}(z) \supseteq \{s_0, s_3, s_5, s_8\}$.
   (c) $z(3) = 1$; $z(4) = 0$.
   (d) $z^{-1}(8) \leq 6$; $z^{-1}(7) \leq 7$.
   (e) $z^{-1}(9) \in \{0, 2, 5\}$.
   (f) $z^{-1}(8) \in \{5, 6\}$.
   (g) $z(9) \in \{2, 3, 5\}$.

These observations let us generate the 34 elements of $\omega_{(\cdot s_3)}[x,v]$. Since $l(v) = 30$, we can only have $\mu(z,v) \neq 0$ if $l(z)$ is odd. In Table 2, we list the seventeen of these z with an odd length difference with respect to v along with the corresponding $\widetilde{z}$ and $\widetilde{v}$.

	z	$\widetilde{z}$	$\widetilde{v}$
z_1	4371098265	326087154	8356712410
z_2	4371098652	32507641	73456120
z_3	4391087265	32706154	73561240
z_4	4391087652	3260541	6345120
z_5	6421098753	3106542	6345120
z_6	6471098352	230541	523410
z_7	6491082753	350142	534120
z_8	6491087352	24031	42310
z_9	7431098265	52076143	74561230
z_{10}	7431098652	4206531	6345120
z_{11}	7461098253	305412	534120
z_{12}	9421083765	102543	451230
z_{13}	9421086753	10342	34120
z_{14}	9431087265	205143	451230
z_{15}	9431087652	20431	34120
z_{16}	9461083752	0231	3120
z_{17}	9461087253	0312	3120

TABLE 2. Cases for Lemma 16.3.

By Proposition 9.4, we ascertain that the only z in the above table for which we might have $\mu(z,v) \neq 0$ is $z_5 = 6421098753$. By Proposition 9.3 and Lemma 15.4, $P_{z_5,v} = 1 + 4q + 4q^2 + q^3$. As $l(v) - l(z_5) = 7$, $\mu(z_5, v) = 1$. Proposition 9.3, along with Lemma 15.1, shows that $P_{x,z_5} = 1 + q$. The only nonzero contribution to the sum in (5) is therefore $1 \cdot q^4 \cdot (1+q) = q^4(1+q)$, as desired. $\square$

Proof of Theorem 1.2. By (5) with $s = s_3$ and Proposition 9.1,

$$P_{\substack{4321098765\\9467182350}} = (1+q)P_{\substack{4321098765\\9461782350}} - \Theta_{(\cdot s_3)}[4321098765, 9461782350]. \tag{17}$$

By Lemma 16.3, $\Theta_{(\cdot s_3)}[4321098765, 9461782350] = q^4(1+q)$. Using Proposition 9.3, we can therefore rewrite (17) as

$$P_{\substack{4321098765\\9467182350}} = (1+q)P_{\substack{321087654\\835671240}} - q^4(1+q). \tag{18}$$

Expanding using (5) with $s = s_4$ and applying Proposition 9.1 and Lemma 16.2, we get

$$P_{\substack{4321098765\\9467182350}} = (1+q)\left((1+q)P_{\substack{321087654\\835617240}} - 0\right) - q^4(1+q). \tag{19}$$

By Proposition 9, parts 1 and 3, this can be rewritten

$$P_{\substack{4321098765\\9467182350}} = (1+q)(1+q)P_{\substack{32170654\\73561240}} - q^4(1+q) \tag{20}$$

$$= (1+q)^2\left((1+q)P_{\substack{32170654\\72561340}} - q^4\right) - q^4(1+q). \tag{21}$$

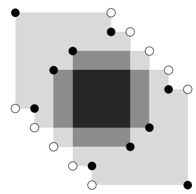

FIGURE 4. The Bruhat picture corresponding to the counterexample to the 0-1 Conjecture lying in S_{10}.

The second follows from the first by the left-hand version of (5) with $s = s_2$ and Lemma 16.1. Simplifying according to Proposition 9.3, we get

$$P_{\substack{4321098765\\9467182350}} = (1+q)^2\left((1+q)P_{\substack{2160543\\6451230}} - q^4\right) - q^4(1+q) \tag{22}$$

$$= (1+q)^2\left((1+q)(1+4q+4q^2+q^3) - q^4\right) - q^4(1+q) \tag{23}$$

$$= 1 + 7q + 19q^2 + 26q^3 + 17q^4 + 4q^5. \tag{24}$$

Going from (22) to (23) we use Lemma 15.4 and Proposition 9.2. Since $l(9467182350) - l(4321098765) = 11$, this completes the proof of the theorem. □

Remark 17. $\mu(x, w)$ can, in fact, be 2 or 3, though we have not yet found any examples in groups smaller than S_{14}:

$$w = 789\text{ab}0\text{cd}123456, \quad l(w) = 47,$$
$$x = 0759321\text{cba}486\text{d}, \quad l(x) = 32,$$
$$P_{x,w}(q) = 2q^7 + 111q^6 + 693q^5 + 1292q^4 + 908q^3 + 257q^2 + 29q + 1.$$

Also,

$$w = 789\text{ab}0\text{cd}123456, \quad l(w) = 47,$$
$$x = 0784321\text{cba}956\text{d}, \quad l(x) = 32,$$
$$P_{x,w}(q) = 3q^7 + 124q^6 + 716q^5 + 1346q^4 + 960q^3 + 263q^2 + 29q + 1.$$

5. Computations on Remaining Conjectures

In Section 1, we mention the conjectures that the L-S graph obtained by taking the trivial edges in a left cell and adding all the edges obtained from these by Knuth relations might be the same as the K-L graph, or at least that this graph might give rise to a representation of S_n. That the first of these conjectures is false follows at once from the counterexample in Theorem 1.1, since every edge of the L-S graph has weight 1. Thus, this conjecture must be false starting in S_{16}. In fact, a computer search for counterexamples to this conjecture, carried out in a manner analogous to that in Section 3, shows that the first counterexamples to this conjecture appear in S_{14}. We obtained these counterexamples before the counterexamples to the 0-1 Conjecture, and their existence inspired us to continue searching for edges of weight greater than 1.

A typical edge in S_{14} that cannot be obtained from any trivial edge by a sequence of Knuth relations is that joining $w = \text{db}630\text{c}7418295\text{a}$ and $x = 76530\text{db}4192\text{c}8\text{a}$.

Not only is the L-S graph not the same as the K-L graph, but the L-S graph fails to give rise to a representation of S_n, again starting at $n = 14$. This has been shown by the first author via computer calculations. In particular, take the tableaux whose column words are the permutations in the counterexample above,

$$P_w = \begin{array}{|c|c|c|c|c|}\cline{1-1} d \\ \cline{1-2} b & c \\ \cline{1-2} 6 & 7 \\ \cline{1-4} 3 & 4 & 8 & 9 \\ \hline 0 & 1 & 2 & 5 & a \\ \hline \end{array} \quad \text{and} \quad P_x = \begin{array}{|c|c|c|c|c|}\cline{1-1} 7 \\ \cline{1-2} 6 & d \\ \cline{1-2} 5 & b \\ \cline{1-4} 3 & 4 & 9 & c \\ \hline 0 & 1 & 2 & 8 & a \\ \hline \end{array}.$$

Choose any tableau Q of the same shape as P_x and P_w and let

$$x \leftrightarrow (P_x, Q) \text{ and } w \leftrightarrow (P_w, Q)$$

under the Robinson-Schensted correspondence. For an adjacent transposition s, let $A(s)$ be the matrix obtained by following the recipe of Kazhdan and Lusztig starting with the L-S graph, and let $B(s)$ be the matrix obtained by following the recipe of Kazhdan and Lusztig starting with the K-L graph. Since the $B(s)$ generate a representation, we know that for every commuting s and t we have the equality of the matrix entries

$$(B(s)B(t))_{w,tx} = (B(t)B(s))_{w,tx}. \tag{25}$$

If the $A(s)$ also generate a representation, then we should also have

$$(A(s)A(t))_{w,tx} = (A(t)A(s))_{w,tx} \tag{26}$$

for all choices of commuting s and t.

If $s = s_4$ and $t = s_b$, then the edge between x and w, present in the K-L graph and absent in the L-S graph, makes a contribution to the left-hand side of (25) via the term

$$B(s)_{w,x}B(t)_{x,tx} = 1 \cdot 1 = 1.$$

The corresponding term on the left-hand side of (26) is

$$A(s)_{w,x}A(t)_{x,tx} = 0 \cdot 1 = 0.$$

It is not obvious that this edge forces different contributions to the right-hand sides of (25) and (26). This suggests that for these values of x and w and s and t, (26) might well be false.

The representation here has dimension 68,640, and finding each matrix entry involves computing a K-L polynomial; but the matrices are sufficiently sparse such that nearly every term in the matrix products in (26) is obviously 0. It is therefore not difficult to show by computer that for this x, w, s, t we have

$$(A(s)A(t))_{w,tx} = 0 \neq 1 = (A(t)A(s))_{w,tx}.$$

Thus, this conjecture, like all those mentioned here, is also false.

Acknowledgements

The first author would like to thank his wife Ann, who shared a bedroom with many of the computer processor cycles used in this work, and to thank Adriano Garsia, who started him thinking about K-L representations. The second author would like to thank his wife Jill and also Sara Billey with whom several of the techniques used in this paper were developed.

References

1. S. Billey and V. Lakshmibai, *Singular Loci of Schubert Varieties*, Progress in Mathematics, no. 182, Birkhäuser Boston, 2000. MR **2001j:**14065
2. S. Billey and G. Warrington, *Maximal singular loci of Schubert varieties in* $SL(n)/B$, Trans. Amer. Math. Soc. (to appear).
3. F. du Cloux, Personal communication, 2002.
4. W. Fulton, *Young tableaux; with applications to representation theory and geometry*, London Mathematical Society Student Texts, vol. 35, Cambridge University Press, New York, 1997. MR **99f:**05119
5. A. M. Garsia and T. J. McLarnan, *Relations between Young's natural and the Kazhdan-Lusztig representations of* S_n, Adv. in Math. **69** (1988), no. 1, 32–92. MR **89f:**20016
6. J. E. Humphreys, *Reflection groups and Coxeter groups*, Cambridge University Press, 1990. MR **92h:**20002
7. D. Kazhdan and G. Lusztig, *Representations of Coxeter groups and Hecke algebras*, Invent. Math. **53** (1979), 165–184. MR **81j:**20066
8. Donald E. Knuth, *Permutations, matrices, and generalized Young tableaux*, Pacific J. Math. **34** (1970), 709–727. MR **42:**7535
9. ———, *The art of computer programming. Volume 3*, Addison-Wesley, 1973. MR **56:**4281
10. A. Lascoux and M.-P. Schützenberger, *Polynômes de Kazhdan & Lusztig pour les grassmanniennes*, Young tableaux and Schur functors in algebra and geometry (Toruń, 1980), Soc. Math. France, Paris, 1981, pp. 249–266. MR **83i:**14045
11. T. J. McLarnan and G. Warrington, *Counterexamples to the 0-1 conjecture*, arXiv:math.CO/0209221 (2002).
12. M. Ochiai and F. Kako, *Computational construction of* W*-graphs of Hecke algebras* $H(q,n)$ *for* n *up to* 15, Experiment. Math. **4** (1995), no. 1, 61–67. MR **96k:**20019

Department of Mathematics, Earlham College, Richmond, Indiana 47374
E-mail address: timm@earlham.edu

Department of Mathematics and Statistics, University of Massachusetts, Amherst, Massachusetts 01003
E-mail address: warrington@math.umass.edu