

PRESENTING GENERALIZED q -SCHUR ALGEBRAS

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ABSTRACT. We obtain a presentation by generators and relations for generalized Schur algebras and their quantizations. This extends earlier results obtained in the type A case. The presentation is compatible with Lusztig's modified form $\dot{\mathbf{U}}$ of a quantized enveloping algebra. We show that generalized Schur algebras inherit a canonical basis from $\dot{\mathbf{U}}$, that this gives them a cellular structure, and thus they are quasihereditary over a field.

INTRODUCTION

In [Do1] Donkin defined the notion of a generalized Schur algebra for an algebraic group, depending on the group and a finite saturated subset π of dominant weights. He also showed how to construct the generalized Schur algebra from the enveloping algebra of the complex Lie algebra of the same type as the given algebraic group, by an appropriate modification of the construction of Chevalley groups.

The purpose of this paper is to give a presentation by generators and relations for generalized Schur algebras and their quantizations. The presentation has the same form as [DG, Theorems 1.4, 2.4], for Schur algebras and q -Schur algebras in type A .

We approach this problem the other way around. First we define (in §1) an algebra $\mathbf{S}(\pi)$ (over $\mathbb{Q}(v)$, v an indeterminate) by generators and relations. It depends only on a Cartan matrix (of finite type) and a given saturated set π of dominant weights. We prove that this algebra is a finite-dimensional semisimple quotient of the quantized enveloping algebra \mathbf{U} determined by the Cartan matrix, and that it is a q -analogue of a generalized Schur algebra in Donkin's sense. We show that $\mathbf{S}(\pi)$ is isomorphic with the algebra $\dot{\mathbf{U}}/\dot{\mathbf{U}}[P]$ constructed by Lusztig in [Lu], where P is the complement of π in the set of dominant weights and where $\dot{\mathbf{U}}$ is his "modified form" of \mathbf{U} , using his "refined Peter-Weyl theorem," which for convenience we recall in §2. It follows that $\mathbf{S}(\pi)$ inherits a canonical basis and a cell datum (in the sense of Graham and Lehrer [GL]) from $\dot{\mathbf{U}}$. This provides $\mathbf{S}(\pi)$ with an extremely rigid structure, which essentially determines its representation theory in all possible specializations. In particular, ${}_R\mathbf{S}(\pi)$ is quasihereditary over any field R . These results are contained within §§3–5.

In §6 we consider the classical case. First we define an algebra $S(\pi)$ (over \mathbb{Q}) by generators and relations. The defining presentation of $S(\pi)$ is obtained from the

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defining presentation of $\mathbf{S}(\pi)$ by simply setting $v = 1$. Then we show that $S(\pi)$ is a generalized Schur algebra. It has essentially the same cell datum as does $\mathbf{S}(\pi)$, and it is quasihereditary when specialized to any field. (This also follows from results in [Do1].) The arguments are quite similar to those used in the quantum case, so many of them are omitted or sketched.

In §7 we consider some natural examples, and in §8 we obtain generators and relations for integral forms of generalized Schur algebras.

We mostly follow notational conventions of [Lu].

1. THE ALGEBRA $\mathbf{S}(\pi)$

1.1. Let $(a_{ij})_{1 \leq i, j \leq n}$ be a Cartan matrix of finite type. We do not assume it is indecomposable. The entries a_{ij} are integers and there is a vector (d_1, \dots, d_n) with entries $d_i \in \{1, 2, 3\}$ such that the matrix $(d_i a_{ij})$ is symmetric and positive definite, $a_{ii} = 2$ for all i , and $a_{ij} \leq 0$ for all $i \neq j$.

1.2. Let \mathfrak{g} be the corresponding finite-dimensional semisimple Lie algebra over the rational field \mathbb{Q} . As we know, \mathfrak{g} is the Lie algebra given by the generators e_i, f_i, h_i ($1 \leq i \leq n$) and relations

$$\begin{aligned} \text{(a)} \quad & [h_i, h_j] = 0, \quad [e_i, f_j] = \delta_{ij} h_i, \\ \text{(b)} \quad & [h_i, e_j] = a_{ij} e_j, \quad [h_i, f_j] = -a_{ij} f_j, \\ \text{(c)} \quad & (\text{ad } e_i)^{1-a_{ij}} e_j = 0 = (\text{ad } f_i)^{1-a_{ij}} f_j \quad (i \neq j). \end{aligned}$$

We fix the Cartan subalgebra \mathfrak{h} spanned by the h_i . Let Φ be the root system of \mathfrak{g} with respect to \mathfrak{h} , and let W be the Weyl group of Φ . We fix ordered bases $\Pi = \{\alpha_1, \dots, \alpha_n\} \subset \mathfrak{h}^*$, $\Pi^\vee = \{\alpha_1^\vee, \dots, \alpha_n^\vee\} \subset \mathfrak{h}$ such that $\langle \alpha_i^\vee, \alpha_j \rangle = a_{ij}$ for all i, j .

1.3. Let v be an indeterminate, and set $\mathcal{A} = \mathbb{Z}[v, v^{-1}]$, with quotient field $\mathbb{Q}(v)$. Set $v_i = v^{d_i}$. Given $a \in \mathbb{Z}$, $t \in \mathbb{N}$ set $[a]_i = (v_i^a - v_i^{-a}) / (v_i - v_i^{-1})$,

$$\text{(a)} \quad [t]_i! = \prod_{s=1}^t \frac{v_i^s - v_i^{-s}}{v_i - v_i^{-1}}, \quad \left[\begin{matrix} a \\ t \end{matrix} \right]_i = \frac{\prod_{s=0}^{t-1} (v_i^{a-s} - v_i^{-a+s})}{\prod_{s=1}^t (v_i^s - v_i^{-s})}.$$

The subscript i can be omitted when $d_i = 1$.

1.4. Let $X = \{\lambda \in \mathfrak{h}^* \mid \langle \alpha_i^\vee, \lambda \rangle \in \mathbb{Z}, \text{ all } i = 1, \dots, n\}$ (the weight lattice) and $X^+ = \{\lambda \in X \mid \langle \alpha_i^\vee, \lambda \rangle \geq 0 \text{ all } i = 1, \dots, n\}$ (the set of dominant weights). For $\lambda, \mu \in X$ write $\lambda \leq \mu$ if $\mu - \lambda \in \mathbb{N}\alpha_1 + \dots + \mathbb{N}\alpha_n$. This defines a partial order on X .

1.5. Let π be a finite ideal in the poset X^+ ($\lambda \leq \mu$ for $\mu \in \pi$, $\lambda \in X^+$ implies $\lambda \in \pi$). Such sets of weights are sometimes called *saturated*. We define an algebra (associative with 1) over $\mathbb{Q}(v)$ given by generators E_i, F_i ($1 \leq i \leq n$), 1_λ ($\lambda \in W\pi$) and relations

$$\begin{aligned} \text{(a)} \quad & 1_\lambda 1_\mu = \delta_{\lambda\mu} 1_\lambda, \quad \sum_{\lambda \in W\pi} 1_\lambda = 1, \\ \text{(b)} \quad & E_i F_j - F_j E_i = \delta_{ij} \sum_{\lambda \in W\pi} [\langle \alpha_i^\vee, \lambda \rangle]_i 1_\lambda, \end{aligned}$$

$$(c) \quad E_i 1_\lambda = \begin{cases} 1_{\lambda+\alpha_i} E_i & \text{if } \lambda + \alpha_i \in W\pi, \\ 0 & \text{otherwise,} \end{cases}$$

$$(d) \quad F_i 1_\lambda = \begin{cases} 1_{\lambda-\alpha_i} F_i & \text{if } \lambda - \alpha_i \in W\pi, \\ 0 & \text{otherwise,} \end{cases}$$

$$(e) \quad 1_\lambda E_i = \begin{cases} E_i 1_{\lambda-\alpha_i} & \text{if } \lambda - \alpha_i \in W\pi, \\ 0 & \text{otherwise,} \end{cases}$$

$$(f) \quad 1_\lambda F_i = \begin{cases} F_i 1_{\lambda+\alpha_i} & \text{if } \lambda + \alpha_i \in W\pi, \\ 0 & \text{otherwise,} \end{cases}$$

$$(g) \quad \sum_{s=0}^{1-a_{ij}} (-1)^s \begin{bmatrix} 1-a_{ij} \\ s \end{bmatrix}_i E_i^{1-a_{ij}-s} E_j E_i^s = 0 \quad (i \neq j),$$

$$(h) \quad \sum_{s=0}^{1-a_{ij}} (-1)^s \begin{bmatrix} 1-a_{ij} \\ s \end{bmatrix}_i F_i^{1-a_{ij}-s} F_j F_i^s = 0 \quad (i \neq j).$$

Denote this algebra by $\mathbf{S} = \mathbf{S}(\pi)$. It depends only on the Cartan matrix (a_{ij}) and the saturated set π .

2. RECOLLECTIONS

2.1. Let \mathbf{U} be the algebra (associative with 1) given by generators E_i, F_i, K_i, K_i^{-1} ($1 \leq i \leq n$) and relations

$$(a) \quad K_i K_j = K_j K_i, \quad K_i K_i^{-1} = 1 = K_i^{-1} K_i,$$

$$(b) \quad E_i F_j - F_j E_i = \delta_{ij} \frac{K_i - K_i^{-1}}{v_i - v_i^{-1}},$$

$$(c) \quad K_i E_j = v_i^{a_{ij}} E_j K_i, \quad K_i F_j = v_i^{-a_{ij}} F_j K_i,$$

$$(d) \quad \sum_{s=0}^{1-a_{ij}} (-1)^s \begin{bmatrix} 1-a_{ij} \\ s \end{bmatrix}_i E_i^{1-a_{ij}-s} E_j E_i^s = 0 \quad (i \neq j),$$

$$(e) \quad \sum_{s=0}^{1-a_{ij}} (-1)^s \begin{bmatrix} 1-a_{ij} \\ s \end{bmatrix}_i F_i^{1-a_{ij}-s} F_j F_i^s = 0 \quad (i \neq j).$$

Then \mathbf{U} is the Drinfeld-Jimbo quantized enveloping algebra corresponding to the Cartan matrix (a_{ij}) .

2.2. Let M be a finite-dimensional \mathbf{U} -module. Then $M = \bigoplus_{\lambda, \sigma} M_\sigma^\lambda$ for $\lambda \in X$, $\sigma : \{1, \dots, n\} \rightarrow \{1, -1\}$. Here

$$M_\sigma^\lambda = \{m \in M \mid K_i m = \sigma(i) v_i^{(\alpha_i, \lambda)} m, i = 1, \dots, n\}$$

is the (λ, σ) weight space. We write $M_\sigma = \bigoplus_{\lambda \in X} M_\sigma^\lambda$; then $M = \bigoplus M_\sigma$ where the sum is over all maps $\sigma : \{1, \dots, n\} \rightarrow \{1, -1\}$. One says that M has type σ if $M = M_\sigma$. In case $\sigma(i) = 1$ for all i , then one says that M has type $\mathbf{1}$.

It is known that every finite-dimensional \mathbf{U} -module is completely reducible. Moreover, for each σ as above there is an equivalence of categories between finite-dimensional \mathbf{U} -modules of type $\mathbf{1}$ and those of type σ . Thus we usually confine our attention to the type $\mathbf{1}$ modules.

Every type **1** simple \mathbf{U} -module is a highest weight module of highest weight λ , for some $\lambda \in X^+$. We denote this module by Λ_λ . For $\lambda \neq \lambda'$, Λ_λ is not isomorphic to $\Lambda_{\lambda'}$.

2.3. In Lusztig's book [Lu, Part IV], a "modified form" $\dot{\mathbf{U}}$ of \mathbf{U} was introduced, and it was shown how to extend the canonical basis from the plus part of \mathbf{U} to a canonical basis $\dot{\mathbf{B}} = \bigsqcup_{\lambda \in X^+} \dot{\mathbf{B}}[\lambda]$ on all of $\dot{\mathbf{U}}$. (See [Lu, 29.1.1] for the definition of $\dot{\mathbf{B}}[\lambda]$.) The \mathbf{U} -modules of type **1** having weight space decompositions can be regarded naturally as modules for $\dot{\mathbf{U}}$; on the other hand, more exotic \mathbf{U} -modules, such as those without weight decomposition, cannot be regarded as modules for $\dot{\mathbf{U}}$. The algebra $\dot{\mathbf{U}}$ in general does not have a unit element, but it does have a family $(1_\lambda)_{\lambda \in X}$ of orthogonal idempotents such that $\dot{\mathbf{U}} = \bigoplus_{\lambda, \lambda' \in X} 1_{\lambda'} \dot{\mathbf{U}} 1_\lambda$. In a sense, the family (1_λ) serves as a replacement for the identity. We shall need the following result from [Lu, 29.3.3], which Lusztig calls the "refined Peter-Weyl theorem."

Theorem 2.4. *Given $\lambda \in X^+$, define $\dot{\mathbf{U}}[\geq \lambda]$ (resp. $\dot{\mathbf{U}}[> \lambda]$) as the set of all $u \in \dot{\mathbf{U}}$ with the property: if $\lambda' \in X^+$ and u acts on $\Lambda_{\lambda'}$ by a nonzero linear map, then $\lambda' \geq \lambda$ (resp., $\lambda' > \lambda$). Then:*

(i) $\dot{\mathbf{U}}[\geq \lambda]$ and $\dot{\mathbf{U}}[> \lambda]$ are two-sided ideals of $\dot{\mathbf{U}}$, which are generated as vector spaces by their intersections with $\dot{\mathbf{B}}$. The quotient algebra $\dot{\mathbf{U}}[\geq \lambda]/\dot{\mathbf{U}}[> \lambda]$ is isomorphic (via the action of $\dot{\mathbf{U}}$ on Λ_λ) to the algebra $\text{End}(\Lambda_\lambda)$. Let $p : \dot{\mathbf{U}}[\geq \lambda] \rightarrow \dot{\mathbf{U}}[\geq \lambda]/\dot{\mathbf{U}}[> \lambda]$ be the natural projection.

(ii) There is a unique direct sum decomposition of $\dot{\mathbf{U}}[\geq \lambda]/\dot{\mathbf{U}}[> \lambda]$ into a direct sum of simple left $\dot{\mathbf{U}}$ -modules such that each summand is generated by its intersection with the basis $p(\dot{\mathbf{B}}[\lambda])$ of $\dot{\mathbf{U}}[\geq \lambda]/\dot{\mathbf{U}}[> \lambda]$.

(iii) There is a unique direct sum decomposition of $\dot{\mathbf{U}}[\geq \lambda]/\dot{\mathbf{U}}[> \lambda]$ into a direct sum of simple right $\dot{\mathbf{U}}$ -modules such that each summand is generated by its intersection with the basis $p(\dot{\mathbf{B}}[\lambda])$ of $\dot{\mathbf{U}}[\geq \lambda]/\dot{\mathbf{U}}[> \lambda]$.

(iv) Any summand in the decomposition (ii) and any summand in the decomposition (iii) have an intersection equal to a line consisting of all multiples of some element in the basis $p(\dot{\mathbf{B}}[\lambda])$. This gives a map from the set of pairs consisting of a summand in the decomposition (ii) and one in the decomposition (iii), to the set $p(\dot{\mathbf{B}}[\lambda])$. This map is a bijection.

2.5. For each $\lambda \in X^+$ let $M(\lambda)$ be any finite set in bijective correspondence with a basis of weight vectors for Λ_λ . We could simply take $M(\lambda)$ to be such a basis, for instance. Then the minimal left ideals in the decomposition (ii) above are indexed by elements of $M(\lambda)$; the same is true of the minimal right ideals in the decomposition (iii). According to part (iv) of the above theorem, there is a unique element $\mathbf{b}_{S,T}^\lambda$ of $\dot{\mathbf{B}}[\lambda]$ corresponding to the intersection of the minimal left ideal L_T indexed by T in the decomposition (ii) with the minimal right ideal R_S indexed by S in the decomposition (iii). Then in terms of this notation we have

$$(a) \quad \dot{\mathbf{B}}[\lambda] = \{\mathbf{b}_{S,T}^\lambda \mid S, T \in M(\lambda)\};$$

moreover, L_T and R_S are spanned, respectively, by the sets

$$(b) \quad \{b_{S,T}^\lambda \mid S \in M(\lambda)\}; \quad \{b_{S,T}^\lambda \mid T \in M(\lambda)\}.$$

These sets (each of cardinality $\dim \Lambda_\lambda$) are the left and right cells of [Lu, 29.4.1].

The sets $\dot{\mathbf{B}}[\lambda]$ are the two-sided cells.

2.6. We note that by [Lu, 23.1.6] there are maps $\sigma, \omega : \dot{\mathbf{U}} \rightarrow \dot{\mathbf{U}}$ (induced from corresponding maps $\mathbf{U} \rightarrow \mathbf{U}$) such that $\iota = \sigma\omega = \omega\sigma$ is a $\mathbb{Q}(v)$ -linear anti-involution on $\dot{\mathbf{U}}$. From the proof of [Lu, 29.3.3(e)] we see that $\iota(b_{S,T}^\lambda) = b_{T,S}^\lambda$ for all $S, T \in M(\lambda)$; in particular, ι induces an involution on the vector space $\dot{\mathbf{U}}[\geq \lambda]/\dot{\mathbf{U}}[> \lambda]$ which interchanges the minimal left and right ideals in the decompositions (ii), (iii) in the preceding theorem.

2.7. For $\lambda \in X^+$ and a fixed $T \in M(\lambda)$ the elements $p(b_{S,T}^\lambda)$ for $S \in M(\lambda)$ span the $\dot{\mathbf{U}}$ -module L_T ; thus for any $u \in \dot{\mathbf{U}}$ we have

$$(a) \quad u p(b_{S,T}^\lambda) = \sum_{S' \in M(\lambda)} r_u(S', S) p(b_{S',T}^\lambda)$$

where $r_u(S', S) \in \mathbb{Q}(v)$ does not depend on T . It follows that in $\dot{\mathbf{U}}$ we have

$$(b) \quad u b_{S,T}^\lambda = \sum_{S' \in M(\lambda)} r_u(S', S) b_{S',T}^\lambda \pmod{\dot{\mathbf{U}}[> \lambda]}.$$

It follows that the basis $\dot{\mathbf{B}} = \{b_{S,T}^\lambda\}$ is a cellular basis for the algebra $\dot{\mathbf{U}}$; more precisely, the datum $(X^+, M, \mathbf{b}, \iota)$ is a cell datum for $\dot{\mathbf{U}}$, in the sense of Graham and Lehrer [GL] (with respect to the opposite partial order on X^+). We note that this cell datum is of profinite type, in the sense of [RG, 2.1.2]; hence the algebra $\dot{\mathbf{U}}$ can be completed to a procellular algebra $\widehat{\mathbf{U}}$. The algebra $\widehat{\mathbf{U}}$ was studied in [BLM] in the type A case.

2.8. Let ${}_{\mathcal{A}}\dot{\mathbf{U}}$ be the set of all finite \mathcal{A} -linear combinations of elements of the canonical basis $\dot{\mathbf{B}}$. This is an \mathcal{A} -subalgebra of $\dot{\mathbf{U}}$; it is generated by all $E_i^{(m)} 1_\lambda, F_i^{(m)} 1_\lambda$, for various $\lambda \in X, 1 \leq i \leq n, m \geq 0$, where

$$(a) \quad E_i^{(m)} = E_i^m / ([m]_i!), F_i^{(m)} = F_i^m / ([m]_i!).$$

Lusztig [Lu, 25.4] has shown that, with respect to the canonical basis $\dot{\mathbf{B}}$, the structure constants of $\dot{\mathbf{U}}$ lie in \mathcal{A} . It follows that, for $u \in {}_{\mathcal{A}}\dot{\mathbf{U}}$,

$$(b) \quad \text{the elements } r_u(S', S) \text{ in 2.7(b) lie in } \mathcal{A}$$

for all $S, S' \in M(\lambda)$. Hence, ${}_{\mathcal{A}}\dot{\mathbf{U}}$ is a cellular algebra with the same cell datum $(X^+, M, \mathbf{b}, \iota)$ considered above.

3. GENERALIZED q -SCHUR ALGEBRAS

We show that the algebra $\mathbf{S} = \mathbf{S}(\pi)$ is (a quantization of) a generalized Schur algebra in the sense of Donkin [Do1].

3.1. For convenience, set $\lambda_i = \langle \alpha_i^\vee, \lambda \rangle$ for all $1 \leq i \leq n$, $\lambda \in X$. Note that $\lambda \in X$ is uniquely determined by its vector $(\lambda_1, \dots, \lambda_n)$ of values on the coroots α_i^\vee . In $\mathbf{S} = \mathbf{S}(\pi)$ we define elements

$$(a) \quad K_i = \sum_{\lambda \in W\pi} v_i^{\lambda_i} 1_\lambda; \quad K_i^{-1} = \sum_{\lambda \in W\pi} v_i^{-\lambda_i} 1_\lambda$$

for any $1 \leq i \leq n$. Then one verifies immediately from 1.5(a) and the definition that K_i, K_i^{-1} satisfy relation 2.1(a); in particular, the K_i all commute.

Lemma 3.2. *Let \mathbf{S}^0 be the subalgebra of \mathbf{S} generated by K_1, \dots, K_n .*

(i) *The idempotents 1_λ ($\lambda \in W\pi$) lie within \mathbf{S}^0 .*

(ii) *The K_i^{-1} lie in \mathbf{S}^0 .*

(iii) *The 1_λ ($\lambda \in W\pi$) form a basis of \mathbf{S}^0 .*

Proof. We set $\Gamma(i, \lambda) = \{\mu \in W\pi \mid \mu_i = \lambda_i\}$, and $J_i^\lambda = \prod_{\mu \in \Gamma(i, \lambda)} (K_i - v_i^{\mu_i})$ where the product is taken over all $\mu \in W\pi - \Gamma(i, \lambda)$. We have equalities

$$(a) \quad \begin{aligned} J_i^\lambda &= \prod_{\mu} \left(\sum_{\lambda' \in W\pi} v_i^{\lambda'_i} 1_{\lambda'} - v_i^{\mu_i} \sum_{\lambda' \in W\pi} 1_{\lambda'} \right) \\ &= \prod_{\mu} \left(\sum_{\lambda' \in W\pi} (v_i^{\lambda'_i} - v_i^{\mu_i}) 1_{\lambda'} \right) \\ &= \sum_{\lambda' \in W\pi} \prod_{\mu} (v_i^{\lambda'_i} - v_i^{\mu_i}) 1_{\lambda'} \end{aligned}$$

where all products are taken over $\mu \in W\pi - \Gamma(i, \lambda)$ and where we have used the idempotent orthogonality relations 1.5(a) to interchange the sum and product. Noting that the product in the sum on the last line above vanishes for any $\lambda' \in W\pi - \Gamma(i, \lambda)$, we obtain the expression

$$(b) \quad J_i^\lambda = \sum_{\lambda' \in \Gamma(i, \lambda)} \prod_{\mu} (v_i^{\lambda'_i} - v_i^{\mu_i}) 1_{\lambda'}$$

where the product in this sum is a nonzero *constant*, since $\lambda'_i = \lambda_i$ for all $\lambda' \in \Gamma(i, \lambda)$. This proves that J_i^λ is (up to a nonzero scalar) the sum of all idempotents $1_{\lambda'}$ for which $\lambda'_i = \lambda_i$. This property holds for all i . Thus it follows that the product $J_1^\lambda \cdots J_n^\lambda$ is, up to a nonzero scalar multiple, equal to 1_λ , since 1_λ is the unique idempotent appearing in each of the sums in the product.

By definition J_i^λ belongs to the subalgebra of $\mathbf{S}(\pi)$ generated by K_i , so the result of the previous paragraph shows that 1_λ (for any $\lambda \in W\pi$) lies within the subalgebra of $\mathbf{S}(\pi)$ generated by all K_1, \dots, K_n . This proves part (i).

Part (ii) follows from part (i) and the definition of K_i^{-1} .

By definition of the K_i we see that the subalgebra of \mathbf{S} generated by the 1_λ contains the K_i . By part (i) this subalgebra equals \mathbf{S}^0 . Part (iii) now follows from the fact that the 1_λ form a family of orthogonal idempotents. \square

3.3. The algebra $\mathbf{S}(\pi)$ is generated by all E_i, F_i, K_i, K_i^{-1} ($1 \leq i \leq n$). In fact, by Lemma 3.2(ii), it is generated by the E_i, F_i, K_i . We have already observed that

the generators satisfy relation 2.1(a). From 1.5(b) and the definitions we obtain equalities

$$\begin{aligned}
 E_i F_j - F_j E_i &= \delta_{ij} \sum_{\lambda \in W\pi} [(\alpha_i^\vee, \lambda)]_i 1_\lambda \\
 (a) \qquad \qquad &= \delta_{ij} \sum_{\lambda \in W\pi} \frac{v_i^{\langle \alpha_i^\vee, \lambda \rangle} - v_i^{-\langle \alpha_i^\vee, \lambda \rangle}}{v_i - v_i^{-1}} 1_\lambda \\
 &= \delta_{ij} \frac{K_i - K_i^{-1}}{v_i - v_i^{-1}},
 \end{aligned}$$

which shows that the elements K_i, K_i^{-1}, E_i, F_i must satisfy 2.1(b).

We set $1_\mu = 0$ for any $\mu \in X - W\pi$. This gives a meaning to the symbol 1_λ for all $\lambda \in X$. With this convention we can write the relations 1.5(c)–(f) more compactly, in the form

$$(b) \qquad \qquad E_i 1_\lambda = 1_{\lambda + \alpha_i} E_i, \quad F_i 1_\lambda = 1_{\lambda - \alpha_i} F_i$$

for all i and all $\lambda \in X$. Then we have

$$\begin{aligned}
 K_i E_j &= \sum_{\lambda \in X} v_i^{\langle \alpha_i^\vee, \lambda \rangle} 1_\lambda E_j \\
 &= \sum_{\lambda \in X} v_i^{\langle \alpha_i^\vee, \lambda \rangle} E_j 1_{\lambda - \alpha_j} \\
 (c) \qquad &= \sum_{\lambda \in X} v_i^{\langle \alpha_i^\vee, \lambda + \alpha_j \rangle} E_j 1_\lambda \\
 &= v_i^{\langle \alpha_i^\vee, \alpha_j \rangle} E_j \sum_{\lambda \in X} v_i^{\langle \alpha_i^\vee, \lambda \rangle} 1_\lambda \\
 &= v_i^{a_{ij}} E_j K_i
 \end{aligned}$$

which proves the first part of relation 2.1(c). The other part of 2.1(c) is proved by the analogous calculation. So the generators K_i, K_i^{-1}, E_i, F_i satisfy 2.1(c). They also satisfy relations 2.1(d) and (e) since those relations are identical with 1.5(g) and (h). We have proved the following result.

Proposition 3.4. *The algebra $\mathbf{S}(\pi)$ is a homomorphic image of \mathbf{U} , via the homomorphism sending the elements E_i, F_i, K_i, K_i^{-1} of \mathbf{U} to the corresponding elements of $\mathbf{S}(\pi)$ denoted by the same symbols.*

3.5. Set $p_i(X) = \prod_{\mu \in W\pi} (X - v_i^{\mu_i}) \in \mathbb{Q}(v)[X]$, where X is a formal indeterminate.

Lemma 3.6. *In the algebra $\mathbf{S}(\pi)$ we have:*

- (i) *The K_i satisfy the polynomial identity $p_i(K_i) = 0$ for $i = 1, \dots, n$.*
- (ii) *The E_i, F_i are nilpotent.*

Proof. The proof of (i) is similar to the proof of Lemma 3.2. We have (for μ, λ varying over $W\pi$)

$$\begin{aligned}
 p_i(K_i) &= \prod_{\mu} (K_i - v_i^{\mu_i}) \\
 &= \prod_{\mu} \left(\sum_{\lambda} v_i^{\lambda_i} 1_{\lambda} - v_i^{\mu_i} \sum_{\lambda} 1_{\lambda} \right) \\
 \text{(a)} \quad &= \prod_{\mu} \sum_{\lambda} (v_i^{\lambda_i} - v_i^{\mu_i}) 1_{\lambda} \\
 &= \sum_{\lambda} \prod_{\mu} (v_i^{\lambda_i} - v_i^{\mu_i}) 1_{\lambda} \\
 &= 0.
 \end{aligned}$$

This proves part (i).

Part (ii) follows immediately from the defining relations 1.5(c)–1.5(f), since if we choose m sufficiently large we must have $E_i^m 1_{\lambda} = 0$ and $F_i^m 1_{\lambda} = 0$ for all $\lambda \in W\pi$. (Note that $\lambda \pm m\alpha_i \notin W\pi$ for $m \gg 0$ since $W\pi$ is a finite set.) \square

3.7. We may regard every \mathbf{S} -module as a \mathbf{U} -module by composition with the quotient map $\mathbf{U} \rightarrow \mathbf{S}$. A simple \mathbf{S} -module must be simple as a \mathbf{U} -module.

Proposition 3.8. $\mathbf{S} = \mathbf{S}(\pi)$ is a finite-dimensional semisimple algebra.

Proof. We know that \mathbf{U} has a triangular decomposition $\mathbf{U} = \mathbf{U}^- \mathbf{U}^0 \mathbf{U}^+$ where \mathbf{U}^0 (resp., \mathbf{U}^- , \mathbf{U}^+) is the subalgebra of \mathbf{U} generated by the K_i, K_i^{-1} (resp., F_i, E_i). It follows that $\mathbf{S} = \mathbf{S}(\pi)$ has a similar decomposition $\mathbf{S} = \mathbf{S}^- \mathbf{S}^0 \mathbf{S}^+$ where $\mathbf{S}^0, \mathbf{S}^-, \mathbf{S}^+$ are defined as the homomorphic images of $\mathbf{U}^0, \mathbf{U}^-, \mathbf{U}^+$, respectively, under the quotient map $\mathbf{U} \rightarrow \mathbf{S}$ from Proposition 3.4. By the preceding lemma, we see that the algebras $\mathbf{S}^0, \mathbf{S}^-, \mathbf{S}^+$ are finite-dimensional. It follows that \mathbf{S} is finite-dimensional.

Moreover, \mathbf{S} is a \mathbf{U} -module (via the quotient map $\mathbf{U} \rightarrow \mathbf{S}$). Finite-dimensional \mathbf{U} -modules are semisimple. Thus \mathbf{S} is semisimple as a \mathbf{U} -module. Hence \mathbf{S} must be semisimple as an \mathbf{S} -module, which proves that \mathbf{S} is a semisimple algebra. \square

Lemma 3.9. If M is any finite-dimensional \mathbf{S} -module, then the decomposition $M = \bigoplus_{\lambda \in W\pi} 1_{\lambda} M$ is a weight space decomposition of M as a \mathbf{U} -module. Moreover, M has type $\mathbf{1}$ when regarded as a \mathbf{U} -module.

Proof. Let M be a finite-dimensional \mathbf{S} -module. As a \mathbf{U} -module, we have the weight space decomposition $M = \bigoplus_{\mu, \sigma} M_{\sigma}^{\mu}$ (see 2.2). On the other hand, in \mathbf{S} we have the equality $1 = \sum_{\lambda \in W\pi} 1_{\lambda}$, which implies that $M = \bigoplus_{\lambda \in W\pi} 1_{\lambda} M$. Clearly, we have inclusions $1_{\lambda} M \subset M_{\mathbf{1}}^{\lambda}$ for all $\lambda \in W\pi$, since

$$\text{(a)} \quad K_i 1_{\lambda} = v_i^{\lambda_i} 1_{\lambda} \quad (i = 1, \dots, n).$$

Thus M has type $\mathbf{1}$ (viewed as a \mathbf{U} -module).

Moreover, if $m \in M_{\mathbf{1}}^{\lambda}$, then by definition we have $K_i m = v_i^{\lambda_i} m$ for all i , so by multiplication by 1_{μ} we obtain

$$\text{(b)} \quad v_i^{\mu_i} 1_{\mu} m = v_i^{\lambda_i} 1_{\mu} m \quad (i = 1, \dots, n).$$

It follows that $1_{\mu} m = 0$ for any $\mu \neq \lambda$. Thus $m = 1 \cdot m = \sum_{\mu \in W\pi} 1_{\mu} m = 1_{\lambda} m$. This proves that $m \in 1_{\lambda} M$, which establishes the inclusion $M_{\mathbf{1}}^{\lambda} \subset 1_{\lambda} M$. Combining

this with the opposite inclusion from the previous paragraph, we conclude that $M_1^\lambda = 1_\lambda M$ for all $\lambda \in W\pi$.

Thus $M = \bigoplus_{\lambda \in W\pi} 1_\lambda M$ is the weight space decomposition of M . \square

Proposition 3.10. *The set $\{\Lambda_\lambda \mid \lambda \in \pi\}$ is the set of isomorphism classes of simple \mathbf{S} -modules, and $\dim \mathbf{S}(\pi) = \sum_{\lambda \in \pi} (\dim \Lambda_\lambda)^2$.*

Proof. The simple \mathbf{S} -modules are simple \mathbf{U} -modules of type **1**. Let $\lambda \in X^+ - \pi$. If Λ_λ was an \mathbf{S} -module, then by Lemma 3.9 Λ_λ would be a direct sum of the weight spaces Λ_λ^μ as μ varies over $W\pi$. This is a contradiction since $\Lambda_\lambda^\lambda \neq 0$ and $\lambda \notin W\pi$.

On the other hand, for every $\lambda \in \pi$, Λ_λ inherits a well-defined \mathbf{S} -module structure from its \mathbf{U} -module structure, just by defining the action on the generators of \mathbf{S} by the obvious formulas. The first part of the proposition is proved.

The second part of the proposition follows immediately by standard theory of finite-dimensional algebras. \square

3.11. Let π be an *arbitrary* subset of X^+ . We could define an algebra $\mathbf{S}(\pi)$ by the generators and relations given in §1. Then all of the results of this section, up to but not including the preceding proposition, remain valid for that algebra, since we do not use the assumption of saturation in any argument. However, examining the proof of the preceding proposition, we see that if π is not saturated, then the only Λ_λ which admit an $\mathbf{S}(\pi)$ -module structure are those of highest weight λ belonging to π' , the largest saturated subset of π . In effect, if π is not saturated, the algebra $\mathbf{S}(\pi)$ collapses to $\mathbf{S}(\pi')$. In particular, if π has no saturated subset, then $\mathbf{S}(\pi)$ is the zero algebra. That is why we assumed, at the outset, that π is saturated.

3.12. Following Donkin, we say that a \mathbf{U} -module *belongs to* π if all its composition factors are of the form Λ_λ for $\lambda \in \pi$.

Corollary 3.13. *The algebra $\mathbf{S}(\pi)$ is isomorphic with the quotient algebra \mathbf{U}/I where I is the ideal of \mathbf{U} consisting of all elements of \mathbf{U} which annihilate every \mathbf{U} -module belonging to π . Thus $\mathbf{S}(\pi)$ is a generalized q -Schur algebra in the sense of [Do1, §3.2].*

Proof. Let I be the kernel of the quotient mapping $\mathbf{U} \rightarrow \mathbf{S}$. From Proposition 3.10 it is clear that I annihilates every module belonging to π . On the other hand, if $x \in \mathbf{U}$ annihilates every \mathbf{U} -module belonging to π , then the algebra $\mathbf{U}/(x)$ has at least as many simple modules as does $\mathbf{S} = \mathbf{U}/I$, and thus there is a quotient map $\mathbf{U}/(x) \rightarrow \mathbf{U}/I$, so $(x) \subset I$. \square

4. THE ALGEBRA $\dot{\mathbf{U}}/\dot{\mathbf{U}}[P]$

4.1. Set $P = X^+ - \pi$. Then P is a cofinite coideal in the poset X^+ ($\lambda \leq \mu$ for $\lambda \in P$, $\mu \in X^+$ implies $\mu \in P$ and $X^+ - P$ is a finite set). Under precisely these assumptions on P , Lusztig [Lu, 29.2] showed that the subspace $\dot{\mathbf{U}}[P]$ of $\dot{\mathbf{U}}$ spanned by $\bigsqcup_{\lambda \in P} \dot{\mathbf{B}}[\lambda]$ is a two-sided ideal of $\dot{\mathbf{U}}$, and that the corresponding quotient algebra $\dot{\mathbf{U}}/\dot{\mathbf{U}}[P]$ is a finite-dimensional semisimple algebra of dimension $\sum_{\lambda \in \pi} (\dim \Lambda_\lambda)^2$. The following result provides a presentation by generators and relations for $\dot{\mathbf{U}}/\dot{\mathbf{U}}[P]$.

Theorem 4.2. *The algebra $\mathbf{S}(\pi)$ is isomorphic with Lusztig's algebra $\dot{\mathbf{U}}/\dot{\mathbf{U}}[P]$.*

Proof. First we show that $\dot{\mathbf{U}}/\dot{\mathbf{U}}[P]$ is a homomorphic image of $\mathbf{S}(\pi)$. By Lusztig [Lu, 23.2.2(c)] the algebra $\dot{\mathbf{U}}$ is generated by elements of the form $E_i^{(m)}1_\mu, F_i^{(m)}1_\mu$ for various $i = 1, \dots, n, \mu \in X, m \geq 0$. By [Lu, 29.2.1] the ideal $\dot{\mathbf{U}}[P]$ contains the idempotents 1_μ for $\mu \in WP = X - W\pi$. Write $\dot{1}_\mu$ for the image of 1_μ under the quotient map $\dot{\mathbf{U}} \rightarrow \dot{\mathbf{U}}/\dot{\mathbf{U}}[P]$. Thus $\dot{1}_\mu = 0$ for all $\mu \in WP$, and thus in the quotient we have $\sum_{\lambda \in X} \dot{1}_\lambda = \sum_{\lambda \in W\pi} \dot{1}_\lambda = 1$.

It follows that $\dot{\mathbf{U}}/\dot{\mathbf{U}}[P]$ is generated by all 1_λ ($\lambda \in W\pi$) and E_i, F_i ($1 \leq i \leq n$). The elements E_i, F_i are not elements of $\dot{\mathbf{U}}$ but their images make sense in the quotient since $E_i = \sum_{\lambda \in W\pi} E_i 1_\lambda$, and similarly for F_i .

We need to verify that these generators satisfy the defining relations of $\mathbf{S}(\pi)$. Relations 1.5(a) follow from [Lu, 23.1.1], 1.5(b)–(f) follow from [Lu, 23.1.3], and the quantized Serre relations 1.5(g) and (h) hold on the E_i, F_i since they are, by Lusztig's definition of $\dot{\mathbf{U}}$, images of the E_i, F_i in \mathbf{U} . This proves that $\dot{\mathbf{U}}/\dot{\mathbf{U}}[P]$ is a homomorphic image of $\mathbf{S}(\pi)$. By dimension comparison, the two algebras are isomorphic. The proof is complete. \square

Corollary 4.3. *The algebra $\mathbf{S}(\pi)$ has a canonical basis, formed by the nonzero elements in the image of the canonical basis $\dot{\mathbf{B}}$ of $\dot{\mathbf{U}}$.*

Proof. This follows immediately from [Lu, 29.2.3]. \square

4.4. Denote the canonical basis of $\mathbf{S}(\pi)$ by $\dot{\mathbf{B}}(\pi)$. It may be regarded naturally as a subset of $\dot{\mathbf{B}}$. It is the disjoint union of the various two-sided cells $\dot{\mathbf{B}}[\lambda]$ for $\lambda \in \pi$.

5. CHANGE OF BASE RING

5.1. Define ${}_{\mathcal{A}}\mathbf{S}$ to be the \mathcal{A} -subalgebra of \mathbf{S} generated by the elements 1_λ ($\lambda \in W\pi$) and $E_i^{(m)} = E_i^m / ([m]_i!)$, $F_i^{(m)} = F_i^m / ([m]_i!)$ ($1 \leq i \leq n, m \geq 0$). Then ${}_{\mathcal{A}}\mathbf{S}$ is the image of ${}_{\mathcal{A}}\mathbf{U}$ under the quotient map (see Proposition 3.4) $\mathbf{U} \rightarrow \mathbf{S}$; it is also the image of ${}_{\mathcal{A}}\dot{\mathbf{U}}$ under the map $\dot{\mathbf{U}} \rightarrow \dot{\mathbf{U}}/\dot{\mathbf{U}}[P] \simeq \mathbf{S}$. Since $\dot{\mathbf{B}}$ is an \mathcal{A} -basis for ${}_{\mathcal{A}}\dot{\mathbf{U}}$, it follows that $\dot{\mathbf{B}}(\pi)$ is a basis for ${}_{\mathcal{A}}\mathbf{S}$ as an \mathcal{A} -module.

Proposition 5.2. *${}_{\mathcal{A}}\mathbf{S}$ is a cellular algebra, with cell datum $(\pi, M, \mathbf{b}, \iota)$. Its canonical basis $\dot{\mathbf{B}}(\pi)$ is the cellular basis for this cell datum.*

Proof. The cellularity follows immediately from the refined Peter-Weyl theorem of Lusztig (see Theorem 2.4). It is immediate that the anti-involution ι on $\dot{\mathbf{U}}$ induces such a map on the quotient $\mathbf{S} \simeq \dot{\mathbf{U}}/\dot{\mathbf{U}}[P]$. The other properties are easily checked. \square

5.3. Given any homomorphism $\mathcal{A} \rightarrow R$ to a commutative ring R such that v maps to an invertible element of R , we may regard R as an \mathcal{A} -module. We define ${}_R\mathbf{S} = R \otimes_{\mathcal{A}} {}_{\mathcal{A}}\mathbf{S}$. Then ${}_R\mathbf{S}$ is a cellular algebra with essentially the same cell datum as ${}_{\mathcal{A}}\mathbf{S}$; in particular, the elements $1 \otimes \mathbf{b}_{S,T}^\lambda$ for $\lambda \in \pi, S, T \in M(\lambda)$ form a cellular basis over R . Note that in case $R = \mathbb{Q}(v)$ with $\mathcal{A} \rightarrow \mathbb{Q}(v)$ given by the obvious embedding, we have an isomorphism ${}_{\mathbb{Q}(v)}\mathbf{S} \simeq \mathbf{S}$.

Theorem 5.4. *If R is a field, then ${}_R\mathbf{S}$ is quasihereditary.*

Proof. We apply the theory of cellular algebras from [GL]. According to [GL, (3.10) Remark] it is enough to show that a certain bilinear form ϕ_λ is nonzero for each $\lambda \in \pi$; equivalently, it is enough to show that $\pi_0 = \pi$, where π_0 is the set of $\lambda \in \pi$ such that $\phi_\lambda \neq 0$. By [GL, (3.4) Theorem] the set π_0 is in bijective correspondence with the set of isomorphism classes of simple ${}_R\mathbf{S}$ -modules.

But one can easily construct a simple module for ${}_R\mathbf{S}$ by standard techniques, for each $\lambda \in \pi$. Set $\Delta(\lambda) = R \otimes_{\mathcal{A}} {}_{\mathcal{A}}\Lambda_\lambda$ where ${}_{\mathcal{A}}\Lambda_\lambda = {}_{\mathcal{A}}\mathbf{U}v^+ = {}_{\mathcal{A}}\dot{\mathbf{U}}v^+$ with $v^+ \neq 0$ a maximal vector ($\mathbf{U}^+v^+ = 0$) in Λ_λ . Then by standard arguments one sees that $\Delta(\lambda)$ is a highest weight module; its unique simple quotient is a highest weight module of highest weight λ . One obtains a simple module in this way for each $\lambda \in \pi$; these simple modules are pairwise non-isomorphic.

This proves that $\pi_0 = \pi$; hence ${}_R\mathbf{S}$ is quasihereditary. □

5.5. It is well-known that quasihereditary algebras have nice homological properties; for instance, from the above it follows that (when R is a field) ${}_R\mathbf{S}$ has finite global dimension and that its matrix of Cartan invariants has determinant 1 (see [KX, Theorem 1.1]).

6. THE CLASSICAL CASE

6.1. Let U be the associative algebra (with 1) on generators e_i, f_i, h_i ($1 \leq i \leq n$) with relations

- (a)
$$h_i h_j = h_j h_i, \quad e_i f_j - f_j e_i = \delta_{ij} h_i,$$
- (b)
$$h_i e_j - e_j h_i = a_{ij} e_j, \quad h_i f_j - f_j h_i = -a_{ij} f_j,$$
- (c)
$$\sum_{s=0}^{1-a_{ij}} (-1)^s \binom{1-a_{ij}}{s} e_i^{1-a_{ij}-s} e_j e_i^s = 0 \quad (i \neq j)$$
- (d)
$$\sum_{s=0}^{1-a_{ij}} (-1)^s \binom{1-a_{ij}}{s} f_i^{1-a_{ij}-s} f_j f_i^s = 0 \quad (i \neq j).$$

Then U is isomorphic with the universal enveloping algebra $U(\mathfrak{g})$ of the Lie algebra \mathfrak{g} . The relations are the same as the relations 1.2 defining \mathfrak{g} , with the Lie bracket $[x, y]$ given by $xy - yx$.

6.2. We define an algebra $S(\pi)$ over \mathbb{Q} , depending only on the saturated subset $\pi \subset X^+$ and the given Cartan matrix (a_{ij}) . The algebra $S(\pi)$ is the associative algebra (with 1) on generators e_i, f_i ($1 \leq i \leq n$), 1_λ ($\lambda \in W\pi$) with relations

- (a)
$$1_\lambda 1_\mu = \delta_{\lambda\mu} 1_\lambda, \quad \sum_{\lambda \in W\pi} 1_\lambda = 1,$$
- (b)
$$e_i f_j - f_j e_i = \delta_{ij} \sum_{\lambda \in W\pi} \langle \alpha_i^\vee, \lambda \rangle 1_\lambda,$$

$$\begin{aligned}
 \text{(c)} \quad e_i 1_\lambda &= \begin{cases} 1_{\lambda+\alpha_i} e_i & \text{if } \lambda + \alpha_i \in W\pi, \\ 0 & \text{otherwise,} \end{cases} \\
 \text{(d)} \quad f_i 1_\lambda &= \begin{cases} 1_{\lambda-\alpha_i} f_i & \text{if } \lambda - \alpha_i \in W\pi, \\ 0 & \text{otherwise,} \end{cases} \\
 \text{(e)} \quad 1_\lambda e_i &= \begin{cases} e_i 1_{\lambda-\alpha_i} & \text{if } \lambda - \alpha_i \in W\pi, \\ 0 & \text{otherwise,} \end{cases} \\
 \text{(f)} \quad 1_\lambda f_i &= \begin{cases} f_i 1_{\lambda+\alpha_i} & \text{if } \lambda + \alpha_i \in W\pi, \\ 0 & \text{otherwise,} \end{cases} \\
 \text{(g)} \quad \sum_{s=0}^{1-a_{ij}} (-1)^s \binom{1-a_{ij}}{s} e_i^{1-a_{ij}-s} e_j e_i^s &= 0 \quad (i \neq j), \\
 \text{(h)} \quad \sum_{s=0}^{1-a_{ij}} (-1)^s \binom{1-a_{ij}}{s} f_i^{1-a_{ij}-s} f_j f_i^s &= 0 \quad (i \neq j).
 \end{aligned}$$

We obtained these relations by replacing v by 1 in 1.5.

6.3. In the algebra $S = S(\pi)$ we define elements

$$\text{(a)} \quad h_i = \sum_{\lambda \in W\pi} \langle \alpha_i^\vee, \lambda \rangle 1_\lambda$$

for $1 \leq i \leq n$. From 6.2(a) it follows that the e_i, f_i, h_i must satisfy relation 6.1(a). In particular, the h_i commute with each other.

Lemma 6.4. *Let S^0 be the subalgebra of S generated by h_1, \dots, h_n .*

- (i) *The idempotents 1_λ ($\lambda \in W\pi$) lie within S^0 .*
- (ii) *The 1_λ ($\lambda \in W\pi$) form a basis for S^0 .*

Proof. (Compare with the proof of Lemma 3.2.) Again we set $\lambda_i = \langle \alpha_i^\vee, \lambda \rangle$ and write $\Gamma(i, \lambda) = \{\mu \in W\pi \mid \mu_i = \lambda_i\}$. We set $J_i^\lambda = \prod_{\mu \in \Gamma(i, \lambda)} (h_i - \mu_i)$ where the product is taken over all $\mu \in W\pi - \Gamma(i, \lambda)$. We have equalities

$$\begin{aligned}
 J_i^\lambda &= \prod_{\mu} \left(\sum_{\lambda' \in W\pi} \lambda'_i 1_{\lambda'} - \mu_i \sum_{\lambda' \in W\pi} 1_{\lambda'} \right) \\
 \text{(a)} \quad &= \prod_{\mu} \left(\sum_{\lambda' \in W\pi} (\lambda'_i - \mu_i) 1_{\lambda'} \right) \\
 &= \sum_{\lambda' \in W\pi} \prod_{\mu} (\lambda'_i - \mu_i) 1_{\lambda'}
 \end{aligned}$$

where all products are taken over $\mu \in W\pi - \Gamma(i, \lambda)$ and where we have used the idempotent orthogonality relations 6.2(a) to interchange the sum and product. Noting that the product in the sum on the last line above vanishes for any $\lambda' \in W\pi - \Gamma(i, \lambda)$, we obtain the expression

$$\text{(b)} \quad J_i^\lambda = \sum_{\lambda' \in \Gamma(i, \lambda)} \prod_{\mu} (\lambda'_i - \mu_i) 1_{\lambda'}$$

where the product in this sum is a nonzero *constant*, since $\lambda'_i = \lambda_i$ for all $\lambda' \in \Gamma(i, \lambda)$. This proves that J_i^λ is (up to a nonzero scalar) the sum of all idempotents $1_{\lambda'}$ for

which $\lambda'_i = \lambda_i$. This property holds for all i . Thus it follows that the product $J_1^\lambda \cdots J_n^\lambda$ is, up to a nonzero scalar multiple, equal to 1_λ , since 1_λ is the unique idempotent appearing in each of the sums in the product. This proves part (i).

Part (ii) follows from part (i) by the same argument given in the proof of Lemma 3.2. \square

Proposition 6.5. *The algebra $S = S(\pi)$ is a homomorphic image of U , via the map sending e_i, f_i, h_i to the corresponding elements of S .*

Proof. Similar to the proof of Proposition 3.4. From the lemma it follows that $S(\pi)$ is generated by the elements e_i, f_i, h_i ($1 \leq i \leq n$). These generators satisfy relations 6.1(a), as noted above. One easily verifies that they satisfy relations 6.1(b), by a calculation similar to those given above. They evidently satisfy 6.1(c) and (d). The proof is complete. \square

6.6. Set $p_i(X) = \prod_{\mu \in W\pi} (X - \mu_i) \in \mathbb{Q}[X]$, where X is a formal indeterminate.

Lemma 6.7. *In the algebra $S(\pi)$ we have:*

- (i) *The h_i all satisfy the polynomial identity $p_i(h_i) = 0$ for $i = 1, \dots, n$.*
- (ii) *The e_i, f_i are nilpotent.*

Proof. We have (for μ, λ varying over $W\pi$)

$$\begin{aligned} p_i(h_i) &= \prod_{\mu} (h_i - \mu_i) \\ &= \prod_{\mu} \left(\sum_{\lambda} \lambda_i 1_{\lambda} - \mu_i \sum_{\lambda} 1_{\lambda} \right) \\ &= \prod_{\mu} \sum_{\lambda} (\lambda_i - \mu_i) 1_{\lambda} \\ &= \sum_{\lambda} \prod_{\mu} (\lambda_i - \mu_i) 1_{\lambda} \\ &= 0. \end{aligned}$$

This proves part (i).

Part (ii) follows immediately from the defining relations 6.2(c)–(f), since if we choose m sufficiently large we must have $e_i^m 1_{\lambda} = 0$ and $f_i^m 1_{\lambda} = 0$ for all $\lambda \in W\pi$. \square

6.8. We denote by L_{λ} the simple U -module of highest weight $\lambda \in X^+$. We may regard every S -module as a U -module by composition with the quotient map $U \rightarrow S$. A simple S -module must be simple as a U -module.

Proposition 6.9. *$S = S(\pi)$ is a finite-dimensional semisimple algebra.*

Proof. The proof is virtually identical to the proof of Proposition 3.8. \square

Lemma 6.10. *If M is any finite-dimensional S -module, then the decomposition $M = \bigoplus_{\lambda \in W\pi} 1_{\lambda} M$ is a weight space decomposition of M as a U -module.*

Proof. The proof is similar to the proof of Lemma 3.9. The details are left to the reader. \square

Proposition 6.11. *The set $\{L_{\lambda} \mid \lambda \in \pi\}$ is the set of isomorphism classes of simple S -modules, and $\dim S(\pi) = \sum_{\lambda \in \pi} (\dim L_{\lambda})^2$.*

Proof. Again, the proof is similar to the proof given in the quantum case; see Proposition 3.10. \square

6.12. We say that a U -module belongs to π if all its composition factors are of the form L_λ for $\lambda \in \pi$.

Corollary 6.13. *The algebra $S(\pi)$ is isomorphic with the quotient algebra U/I where I is the ideal of U consisting of all elements of U which annihilate every U -module belonging to π . Thus $S(\pi)$ is a generalized Schur algebra in the sense of [Do1, §3.2].*

Proof. The proof is the same as the proof of Corollary 3.13. \square

6.14. The cell datum for $\mathbf{S}(\pi)$ gives rise to a corresponding cell datum for $S(\pi)$. Any element of ${}_{\mathcal{A}}\mathbf{S}(\pi)$ can be written in terms of an \mathcal{A} -linear combination of products of generators; such an expression corresponds to a \mathbb{Z} -linear combination of products of corresponding generators of $S(\pi)$, obtained from the \mathcal{A} -linear combination by setting v to 1. In particular, the canonical basis of $\mathbf{S}(\pi)$ determines a corresponding canonical basis of $S(\pi)$.

6.15. One can define ${}_{\mathbb{Z}}S(\pi)$ to be the subalgebra of $S(\pi)$ generated by the divided powers $e_i^m/(m!)$, $f_i^m/(m!)$ for various $1 \leq i \leq n$, $m \geq 0$. (It is the image of the Kostant \mathbb{Z} -form ${}_{\mathbb{Z}}U$ of U under the quotient map $U \rightarrow S(\pi)$.) The cell datum for $S(\pi)$ is a cell datum for ${}_{\mathbb{Z}}S(\pi)$.

For any ring R we set ${}_RS(\pi) = R \otimes_{\mathbb{Z}} {}_{\mathbb{Z}}S(\pi)$. Then it follows from Donkin's results that, when R is a field, ${}_RS(\pi)$ is quasihereditary. Alternatively, one can see this directly as a consequence of the cellular structure, as we did above in the quantum case. (The argument is the same.)

7. EXAMPLES

7.1. One natural and interesting class of examples can be constructed as follows, for an arbitrary Cartan matrix. Fix a finite-dimensional representation V of \mathbf{U} (resp., U) and for each $d \geq 0$ take $\pi = \pi(d)$ to be the set of dominant weights occurring in $V^{\otimes d}$. The resulting algebra ${}_R\mathbf{S}(\pi)$ (resp., ${}_RS(\pi)$) has connections with classical invariant theory.

A canonical special case of this construction is obtained by taking V to be the simple nontrivial module of *smallest* dimension. In types A , B , C , D this V is the natural module (see [Ja, 5A.1, 5A.2] for an explicit construction of this module in the quantum case). Denote the resulting algebra by ${}_R\mathbf{S}(d)$ (resp., ${}_RS(d)$). It is defined for every indecomposable Cartan matrix.

7.2. In type A_{n-1} the algebras ${}_R\mathbf{S}(d)$, ${}_RS(d)$ have been extensively studied; this is the motivating example. The algebra ${}_R\mathbf{S}(d)$ is isomorphic with the q -Schur algebra constructed by Dipper and James [DJ1, DJ2]; the same algebra was also considered by Jimbo [Ji] in the generic case. The algebra ${}_RS(d)$ is isomorphic with the classical Schur algebra constructed in [Gr] in order to study the polynomial representations of general linear groups (based on Schur's dissertation). The isomorphism follows from [DG, 1.4, 2.4]. (Note that one needs to replace q by $v = q^{1/2}$ in the Dipper-James construction.) The algebra ${}_{\mathcal{A}}\mathbf{S}(d)$ was constructed and studied from a geometric viewpoint in [BLM]; it is the algebra denoted \mathbf{K}_d there.

There is a vast literature concerning these algebras. In particular, ${}_{\mathcal{A}}\mathbf{S}(d)$ can be constructed as the space of linear endomorphisms of ${}_{\mathcal{A}}V^{\otimes d}$ commuting with a natural action of the Hecke algebra in type A ; see [Ji], [Du]. Similarly, ${}_{\mathbb{Z}}\mathbf{S}(d)$ is the space of linear endomorphisms of ${}_{\mathbb{Z}}V^{\otimes d}$ commuting with the action of the symmetric group, acting by place permutation. Here ${}_{\mathcal{A}}V$ (resp., ${}_{\mathbb{Z}}V$) are appropriate lattices in V .

7.3. In type C the algebra ${}_R\mathbf{S}(d)$ is isomorphic with the symplectic q -Schur algebra studied in [Oe]. Indeed, the two algebras in question have “the same” representation theory and the same dimension. Similarly, the algebra ${}_R S(d)$ (in type C) is isomorphic with the symplectic Schur algebra studied in [Do2], [D].

7.4. There is another construction, equally natural, which leads to a different generalization of Schur algebras. Again we fix a module V and consider $V^{\otimes d}$. The action of \mathbf{U} (resp., U) determines a representation $\mathbf{U} \rightarrow \text{End}(V^{\otimes d})$ (resp., $U \rightarrow \text{End}(V^{\otimes d})$) and one can study the finite-dimensional algebra (and its various specializations) obtained as the image of the representation. These types of algebras were called “enveloping” in Weyl’s book on the classical groups; in types A and C with V the natural module they coincide with the generalized Schur algebras considered in 7.2 and 7.3.

However, if one takes V to be the adjoint module for \mathbf{U} in type A_1 , and takes $d = 1$, then the image of $\mathbf{U} \rightarrow \text{End}(V)$ is a 9-dimensional algebra, while the generalized q -Schur algebra ${}_R\mathbf{S}(\pi)$ is a 10-dimensional algebra. (It is the q -Schur algebra in degree 2.) Thus the construction of this subsection is, in general, different from that considered in 7.1. Precisely the same difficulty occurs in the classical setting.

Algebras of the type constructed in this subsection were considered in [D] for types B, D (in the classical setting); I do not know whether they are generalized Schur algebras. To prove that they are, one needs to show that for every dominant weight of $V^{\otimes d}$ there is a simple direct summand of $V^{\otimes d}$ of that highest weight, where V is the natural module.

8. INTEGRAL FORMS

8.1. Consider the algebra (associative with 1) over \mathcal{A} generated by the symbols $E_i^{(a)}, F_i^{(b)}$ ($1 \leq i \leq n, a, b \geq 0$), 1_λ ($\lambda \in W\pi$) and satisfying the relations

- (a) $1_\lambda 1_\mu = \delta_{\lambda\mu} 1_\lambda, \quad \sum_{\lambda \in W\pi} 1_\lambda = 1,$
- (b) $E_i^{(a)} E_i^{(b)} = E_i^{(a+b)}, \quad F_i^{(a)} F_i^{(b)} = F_i^{(a+b)}, \quad E_i^{(0)} = F_i^{(0)} = 1,$
- (c) $E_i^{(a)} F_j^{(b)} = F_j^{(b)} E_i^{(a)} \quad (i \neq j),$
- (d) $E_i^{(a)} 1_{-\lambda} F_i^{(b)} = \sum_{t \geq 0} \begin{bmatrix} a+b - \langle \alpha_i^\vee, \lambda \rangle \\ t \end{bmatrix}_i F_i^{(b-t)} 1_{-\lambda + (a+b-t)\alpha_i} E_i^{(a-t)},$
- (e) $F_i^{(b)} 1_\lambda E_i^{(a)} = \sum_{t \geq 0} \begin{bmatrix} a+b - \langle \alpha_i^\vee, \lambda \rangle \\ t \end{bmatrix}_i E_i^{(a-t)} 1_{\lambda - (a+b-t)\alpha_i} F_i^{(b-t)},$
- (f) $E_i^{(a)} 1_\lambda = \begin{cases} 1_{\lambda + a\alpha_i} E_i^{(a)} & \text{if } \lambda + a\alpha_i \in W\pi, \\ 0 & \text{otherwise,} \end{cases}$

$$\begin{aligned}
 \text{(g)} \quad & F_i^{(b)} 1_\lambda = \begin{cases} 1_{\lambda - b\alpha_i} F_i^{(b)} & \text{if } \lambda - b\alpha_i \in W\pi, \\ 0 & \text{otherwise,} \end{cases} \\
 \text{(h)} \quad & 1_\lambda E_i^{(a)} = \begin{cases} E_i^{(a)} 1_{\lambda - a\alpha_i} & \text{if } \lambda - a\alpha_i \in W\pi, \\ 0 & \text{otherwise,} \end{cases} \\
 \text{(i)} \quad & 1_\lambda F_i^{(b)} = \begin{cases} F_i^{(b)} 1_{\lambda + b\alpha_i} & \text{if } \lambda + b\alpha_i \in W\pi, \\ 0 & \text{otherwise,} \end{cases} \\
 \text{(j)} \quad & \sum_{s=0}^{1-a_{ij}} (-1)^s E_i^{(1-a_{ij}-s)} E_j^{(1)} E_i^{(s)} = 0 \quad (i \neq j), \\
 \text{(k)} \quad & \sum_{s=0}^{1-a_{ij}} (-1)^s F_i^{(1-a_{ij}-s)} F_j^{(1)} F_i^{(s)} = 0 \quad (i \neq j).
 \end{aligned}$$

Denote this algebra by ${}_{\mathcal{A}}\mathbf{T}(\pi)$. It depends only on the Cartan matrix (a_{ij}) and the saturated set π . It follows from Theorem 8.3 below that $E_i^{(a)} = F_i^{(b)} = 0$ for a, b sufficiently large, so ${}_{\mathcal{A}}\mathbf{T}(\pi)$ is finitely generated.

8.2. We define $\mathbf{T}(\pi) = \mathbb{Q}(v) \otimes_{\mathcal{A}} {}_{\mathcal{A}}\mathbf{T}(\pi)$, where $\mathbb{Q}(v)$ is regarded as an \mathcal{A} -algebra via the natural embedding $\mathcal{A} \rightarrow \mathbb{Q}(v)$. We identify ${}_{\mathcal{A}}\mathbf{T}(\pi)$ with the \mathcal{A} -submodule of $\mathbf{T}(\pi)$ spanned by all elements of the form $1 \otimes X$, for $X \in {}_{\mathcal{A}}\mathbf{T}(\pi)$; in this way ${}_{\mathcal{A}}\mathbf{T}(\pi)$ may be regarded as an \mathcal{A} -subalgebra of $\mathbf{T}(\pi)$. In the algebra $\mathbf{T}(\pi)$ we write E_i for $E_i^{(1)}$, F_i for $F_i^{(1)}$.

Theorem 8.3. (i) *There is an isomorphism $\mathbf{S}(\pi) \xrightarrow{\cong} \mathbf{T}(\pi)$ of $\mathbb{Q}(v)$ -algebras sending E_i to E_i , F_i to F_i , and 1_λ to 1_λ .*

(ii) *The above isomorphism carries ${}_{\mathcal{A}}\mathbf{S}(\pi)$ onto ${}_{\mathcal{A}}\mathbf{T}(\pi)$.*

Proof. As a $\mathbb{Q}(v)$ -algebra, $\mathbf{T}(\pi)$ is the algebra generated by all $E_i^{(a)}$, $F_i^{(b)}$ ($1 \leq i \leq n, a, b \geq 0$), 1_λ ($\lambda \in W\pi$) and satisfying the relations 8.1. By 8.1(b) we have the equalities $E_i^{(a)} = E_i^a / ([a]_i!)$, $F_i^{(b)} = E_i^b / ([b]_i!)$ in $\mathbf{T}(\pi)$. Thus the elements E_i , F_i , 1_λ generate $\mathbf{T}(\pi)$ as a $\mathbb{Q}(v)$ -algebra. These generators satisfy relations 1.5. Indeed, all of those relations are immediate, except for 1.5(b), which requires a small calculation that we leave to the reader. Therefore, we know by Proposition 3.2 and Theorem 4.1 that $\mathbf{T}(\pi)$ is a homomorphic image of \mathbf{U} and also of $\dot{\mathbf{U}}$, so that all relations in those algebras also hold in $\mathbf{T}(\pi)$. Thus relations 8.1 are consequences of the relations 1.5. This follows from Lusztig [Lu, 23.1.3] and [Lu, 1.4.3]. Thus we see that $\mathbf{T}(\pi)$ is determined by relations 1.5, and so there is an isomorphism $\mathbf{S}(\pi) \rightarrow \mathbf{T}(\pi)$ of $\mathbb{Q}(v)$ -algebras, taking $E_i \rightarrow E_i$, $F_i \rightarrow F_i$, $1_\lambda \rightarrow 1_\lambda$. This proves part (i).

By definition, ${}_{\mathcal{A}}\mathbf{S}(\pi)$ is the subalgebra generated by all $E_i^{(a)}$, $F_i^{(b)}$ ($1 \leq i \leq n, a, b \geq 0$), 1_λ ($\lambda \in W\pi$). These map onto the corresponding defining generators for ${}_{\mathcal{A}}\mathbf{T}(\pi)$. This proves part (ii). \square

8.4. Consider the \mathbb{Z} -algebra generated by symbols $e_i^{(a)}$, $f_i^{(b)}$ ($1 \leq i \leq n, a, b \geq 0$), 1_λ ($\lambda \in W\pi$) and subject to the relations

$$\text{(a)} \quad 1_\lambda 1_\mu = \delta_{\lambda\mu} 1_\lambda, \quad \sum_{\lambda \in W\pi} 1_\lambda = 1,$$

- (b) $e_i^{(a)} e_i^{(b)} = e_i^{(a+b)}, \quad f_i^{(a)} f_i^{(b)} = f_i^{(a+b)}, \quad e_i^{(0)} = f_i^{(0)} = 1,$
- (c) $e_i^{(a)} f_j^{(b)} = f_j^{(b)} e_i^{(a)} \quad (i \neq j),$
- (d) $e_i^{(a)} 1_{-\lambda} f_i^{(b)} = \sum_{t \geq 0} \binom{a+b - \langle \alpha_i^\vee, \lambda \rangle}{t} f_i^{(b-t)} 1_{-\lambda + (a+b-t)\alpha_i} e_i^{(a-t)},$
- (e) $f_i^{(b)} 1_\lambda e_i^{(a)} = \sum_{t \geq 0} \binom{a+b - \langle \alpha_i^\vee, \lambda \rangle}{t} e_i^{(a-t)} 1_{\lambda - (a+b-t)\alpha_i} f_i^{(b-t)},$
- (f) $e_i^{(a)} 1_\lambda = \begin{cases} 1_{\lambda + a\alpha_i} e_i^{(a)} & \text{if } \lambda + a\alpha_i \in W\pi, \\ 0 & \text{otherwise,} \end{cases}$
- (g) $f_i^{(b)} 1_\lambda = \begin{cases} 1_{\lambda - b\alpha_i} f_i^{(b)} & \text{if } \lambda - b\alpha_i \in W\pi, \\ 0 & \text{otherwise,} \end{cases}$
- (h) $1_\lambda e_i^{(a)} = \begin{cases} e_i^{(a)} 1_{\lambda - a\alpha_i} & \text{if } \lambda - a\alpha_i \in W\pi, \\ 0 & \text{otherwise,} \end{cases}$
- (i) $1_\lambda f_i^{(b)} = \begin{cases} f_i^{(b)} 1_{\lambda + b\alpha_i} & \text{if } \lambda + b\alpha_i \in W\pi, \\ 0 & \text{otherwise,} \end{cases}$
- (j) $\sum_{s=0}^{1-a_{ij}} (-1)^s e_i^{(1-a_{ij}-s)} e_j^{(1)} e_i^{(s)} = 0 \quad (i \neq j),$
- (k) $\sum_{s=0}^{1-a_{ij}} (-1)^s f_i^{(1-a_{ij}-s)} f_j^{(1)} f_i^{(s)} = 0 \quad (i \neq j).$

Denote this algebra by ${}_{\mathbb{Z}}T(\pi)$. It depends only on the Cartan matrix (a_{ij}) and the saturated set π .

8.5. We define $T(\pi) = \mathbb{Q} \otimes_{\mathbb{Z}} {}_{\mathbb{Z}}T(\pi)$, where \mathbb{Q} is regarded as a \mathbb{Z} -algebra via the natural embedding $\mathbb{Z} \rightarrow \mathbb{Q}$. We identify ${}_{\mathbb{Z}}T(\pi)$ with the \mathbb{Z} -submodule of $T(\pi)$ spanned by all elements of the form $1 \otimes X$, for $X \in {}_{\mathbb{Z}}T(\pi)$; in this way ${}_{\mathbb{Z}}T(\pi)$ may be regarded as a \mathbb{Z} -subalgebra of $T(\pi)$. In the algebra $T(\pi)$ we write e_i for $e_i^{(1)}$, f_i for $f_i^{(1)}$.

Theorem 8.6. (i) *There is an isomorphism $S(\pi) \xrightarrow{\cong} T(\pi)$ of \mathbb{Q} -algebras sending e_i to e_i , f_i to f_i , and 1_λ to 1_λ .*

(ii) *The above isomorphism carries ${}_{\mathbb{Z}}S(\pi)$ onto ${}_{\mathbb{Z}}T(\pi)$.*

Proof. The proof is similar to the proof of the corresponding result in the quantum case, given above. We leave the details to the reader. \square

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