

## THE FOURIER-JACOBI MAP AND SMALL REPRESENTATIONS

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ABSTRACT. We study the “Fourier-Jacobi” functor on smooth representations of split, simple, simply-laced  $p$ -adic groups. This functor has been extensively studied on the symplectic group, where it provides the representation-theoretic analogue of the Fourier-Jacobi expansion of Siegel modular forms. Our applications are different from those studied classically with the symplectic group. In particular, we are able to describe the composition series of certain degenerate principal series. This includes the location of minimal and small (in the sense of the support of the local character expansion) representations as spherical subquotients.

### INTRODUCTION

We will be interested in the smooth representations of a split, simple, simply-laced (type A-D-E) group  $G$  over a  $p$ -adic field  $k$ . Our methods rely on the existence of two different parabolic subgroups  $P$  and  $Q$  having the following properties: the unipotent radical of  $P$  is isomorphic to a Heisenberg group, and the unipotent radical of  $Q$  is abelian. Fix Levi decompositions  $P = LH$  and  $Q = MN$  of these parabolics over  $k$ . If  $G_1$  denotes the derived subgroup of  $L$ , then the Fourier-Jacobi map we define will be a functor from the category of smooth representations of  $G$  to the category of smooth representations of  $G_1$ . The functorial properties of this map allow us to study representations inductively, by reducing representations of  $G$  to representations of the smaller group  $G_1$ .

This functor is only well behaved on representations that are already somewhat small. For instance, admissibility is only preserved on a restricted subcategory of representations of  $G$ . However, the Fourier-Jacobi map is well behaved on some degenerate principal series representations  $I(s)$  induced from  $Q$ . This allows us to explicitly determine the  $N$ -rank of spherical subquotients of these principal series.

We outline the contents of this paper here: in the first section, we describe the structure theory of the parabolics  $P$  and  $Q$  as above. This includes a classification of the groups  $G$  which we can study, together with parabolics  $P$  and  $Q$ . Section 2 reviews the  $p$ -adic analogue of the Stone-von Neumann theorem, completely describing the representations of the Heisenberg groups over a  $p$ -adic field.

In Section 3, the main definitions and functorial properties of the Fourier-Jacobi map are discussed. In Section 4, we describe the effect of the Fourier-Jacobi map

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on degenerate principal series. Section 5 contains a description of the composition series of the degenerate principal series. Finally, in Section 6, we study the effect of the Fourier-Jacobi map on the rank of representations, finding small representations in degenerate principal series.

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## 1. STRUCTURE THEORY

Fix a simple split Lie algebra  $\mathfrak{g}$  over a field  $k$  of characteristic 0, with Cartan subalgebra  $\mathfrak{t}$  and Borel subalgebra  $\mathfrak{b}$ . Suppose that  $\mathfrak{g}$  is simply-laced (type A, D, or E). Let  $\Phi$  be the set of roots for the adjoint action of  $\mathfrak{t}$  on  $\mathfrak{g}$ . For any root  $\alpha \in \Phi$ , let  $\mathfrak{g}_\alpha$  denote the corresponding one-dimensional root space. Then, viewing  $\Phi$  as a subset of the dual space  $\mathfrak{t}^\vee$ , the vector space  $E = \sum_{\alpha \in \Phi} \mathbb{Q}\alpha$  has dimension equal to the rank of  $\mathfrak{g}$ . The Weyl group  $W$  of  $\mathfrak{g}$ , generated by symmetries  $s_\alpha$ ,  $\alpha \in \Phi$ , is a finite group that acts irreducibly on  $E$ . There is a unique positive-definite  $W$ -invariant inner product  $\langle \cdot, \cdot \rangle$  on  $E$  up to scaling; we will soon normalize this inner product.

The choice of Borel subalgebra  $\mathfrak{b}$  corresponds to a partition of  $\Phi$  into positive and negative roots:  $\Phi = \Phi^+ \cup \Phi^-$ . If

$$\Delta = \{\alpha_1, \dots, \alpha_l\}$$

denotes the corresponding root basis for  $E$ , consisting of simple roots of  $\Phi^+$ , then every  $\alpha \in \Phi^+$  has a unique expression,

$$\alpha = \sum_{i=1}^l m_i(\alpha) \cdot \alpha_i,$$

where the  $m_i(\alpha)$  are nonnegative integers.

**1.1 The Heisenberg Parabolic.** Our study of the Heisenberg parabolic borrows heavily from the treatment in Gross and Wallach [G-W]. The highest root  $\beta$  in  $\Phi^+$  is characterized by  $m_i(\beta) \geq m_i(\alpha)$  for all  $\alpha \in \Phi^+$ . We now normalize the inner product on  $E$  by setting  $\langle \beta, \beta \rangle = 2$ . Let  $w_\beta$  denote the corresponding element of the Weyl group of  $\mathfrak{g}$ , so that

$$w_\beta(\alpha) = \alpha - \langle \alpha, \beta \rangle \beta.$$

We recall the following proposition from [G-W]:

**Proposition 1.1.1.** *If  $\alpha \in \Phi^+$ , then  $\langle \alpha, \beta \rangle \in \{0, 1, 2\}$ . If  $\langle \alpha, \beta \rangle = 2$ , then  $\alpha = \beta$ . If  $\Sigma^+ = \{\alpha \in \Phi^+ | \langle \alpha, \beta \rangle = 1\}$ , then  $\alpha \mapsto \beta - \alpha$  is a fixed-point free involution of  $\Sigma^+$ . The cardinality of  $\Sigma^+$  is an even integer  $2d$ .*

The highest coroot  $\beta^\vee = \langle \cdot, \beta \rangle$  corresponds to a standard parabolic subalgebra whose nilradical is of Heisenberg type. We define this Heisenberg parabolic,

$$\mathfrak{p} = \mathfrak{t} \oplus \bigoplus_{\alpha \in \Phi, \langle \alpha, \beta \rangle \geq 0} \mathfrak{g}_\alpha.$$

The Heisenberg parabolic has Levi decomposition  $\mathfrak{p} = \mathfrak{l} \oplus \mathfrak{h}$ , where

$$\begin{aligned} \mathfrak{l} &= \mathfrak{t} \oplus \bigoplus_{\alpha \in \Phi, \langle \alpha, \beta \rangle = 0} \mathfrak{g}_\alpha, \\ \mathfrak{h} &= \mathfrak{g}_\beta \oplus \bigoplus_{\alpha \in \Sigma^+} \mathfrak{g}_\alpha. \end{aligned}$$

The dimension of  $\mathfrak{h}$  is equal to  $2d + 1$ , and we call  $d$  the structure constant of  $\mathfrak{p}$ . Define  $\mathfrak{g}_1$  to be the derived subalgebra  $[\mathfrak{l}, \mathfrak{l}]$ .

Then the Weyl group  $W_L$  of  $\mathfrak{l}$  is precisely the stabilizer of  $\mathfrak{g}_\beta$ . We recall the further structure of this parabolic from [G-W]:

**Proposition 1.1.2.** *The algebra  $\mathfrak{h}$  is of Heisenberg type, with center  $\mathfrak{z} = \mathfrak{g}_\beta$ . The quotient  $\mathfrak{s} = \mathfrak{h}/\mathfrak{z}$  is abelian of dimension  $2d$ , and the Lie bracket induces a nondegenerate symplectic pairing  $\wedge^2 \mathfrak{s} \rightarrow \mathfrak{z}$ . The elements of  $\mathfrak{g}_1$  act trivially on  $\mathfrak{g}_\beta$ , and thus symplectically on the vector space  $\mathfrak{s}$ .*

**1.2. Parabolics with Abelian Nilradical.** Our study of parabolics with abelian nilradical follows the paper by Richardson, Röhrle, and Steinberg [RRS]. We assume here that  $\mathfrak{g} \neq \mathfrak{sl}_2$ . To any simple root  $\tau \in \Delta$ , we associate a maximal parabolic subalgebra  $\mathfrak{q}$  by setting

$$\mathfrak{q} = \mathfrak{t} \oplus \bigoplus_{\alpha \in \Phi, m_\tau(\alpha) \geq 0} \mathfrak{g}_\alpha.$$

We fix a Levi decomposition,  $\mathfrak{q} = \mathfrak{m} \oplus \mathfrak{n}$  with

$$\begin{aligned} \mathfrak{m} &= \mathfrak{t} \oplus \bigoplus_{\alpha \in \Phi, m_\tau(\alpha) = 0} \mathfrak{g}_\alpha, \\ \mathfrak{n} &= \bigoplus_{\alpha \in \Phi, m_\tau(\alpha) > 0} \mathfrak{g}_\alpha. \end{aligned}$$

Define

$$\rho_{\mathfrak{n}} = \frac{1}{2} \sum_{\alpha \in \Phi, m_\tau(\alpha) > 0} \alpha.$$

Then we call  $D = \langle \rho_{\mathfrak{n}}, \tau \rangle$  the structure constant of  $\mathfrak{n}$ . Now we have the following result from Richardson et al. [RRS]:

**Proposition 1.2.1.** *The algebra  $\mathfrak{q}$  associated to a simple root  $\tau$  has abelian nilradical if and only if the multiplicity  $m_\tau(\beta)$  of  $\tau$  in  $\beta$  is equal to 1.*

Hereafter, we fix  $\tau$  with  $m_\tau(\beta) = 1$ , so that  $\mathfrak{n}$  is abelian. Let  $\Psi^+$  (respectively  $\Psi^0, \Psi^-$ ) denote the set of roots  $\alpha$  satisfying  $m_\tau(\alpha) > 0$  (resp.,  $m_\tau(\alpha) = 0, m_\tau(\alpha) < 0$ ).

A sequence  $(\beta_0, \dots, \beta_{r-1})$  of roots is called orthogonal if it is pairwise orthogonal, i.e.,  $\langle \beta_i, \beta_j \rangle = 0$  for all  $i \neq j$ . Given an orthogonal sequence  $(\beta_0, \dots, \beta_{r-1})$ , we say that  $r$  is the length of the sequence. If  $W_M$  denotes the Weyl group of  $\mathfrak{m}$ , then we have the following useful result from [RRS]:

**Proposition 1.2.2.** *For all  $r \geq 1$ ,  $W_M$  acts transitively on the set of orthogonal sequences  $(\beta_0, \dots, \beta_{r-1})$  of length  $r$  in  $\Psi^+$ .*

From this proposition, it makes sense to define the rank of  $\mathfrak{n}$  to be the length of a maximal orthogonal sequence  $(\beta_0, \dots, \beta_{r-1})$  of roots in  $\Psi^+$ . Every such maximal orthogonal sequence is conjugate via the action of  $W_M$ , and thus has the same length.

**1.3. Interaction between Parabolics.** The structural foundation of this paper is the interaction between the two parabolics  $\mathfrak{p}$  and  $\mathfrak{q}$ . We recall that a complete polarization on the Heisenberg algebra  $\mathfrak{h}$  is a vector space decomposition:  $\mathfrak{h} = \mathfrak{a} \oplus \mathfrak{a}' \oplus \mathfrak{z}$ , such that  $\mathfrak{a}$  and  $\mathfrak{a}'$  are isotropic, i.e.,

$$[\mathfrak{a}, \mathfrak{a}] = [\mathfrak{a}', \mathfrak{a}'] = 0.$$

Then  $\mathfrak{a}$  and  $\mathfrak{a}'$  have the same dimension  $d$ . We have the following important structural fact:

**Proposition 1.3.1.** *Our choice of a parabolic subalgebra  $\mathfrak{q}$  with abelian unipotent radical allows us to canonically choose a complete polarization of  $\mathfrak{h}$ . Set  $\Sigma_{\mathfrak{a}}^+ = \Sigma^+ \cap \Psi^+$  and  $\Sigma_{\mathfrak{a}'}^+ = \Sigma^+ \cap \Psi^0$ . Then we define*

$$\mathfrak{a} = \bigoplus_{\alpha \in \Sigma_{\mathfrak{a}}^+} \mathfrak{g}_{\alpha},$$

$$\mathfrak{a}' = \bigoplus_{\alpha \in \Sigma_{\mathfrak{a}'}^+} \mathfrak{g}_{\alpha}.$$

*This gives a complete polarization of  $\mathfrak{h}$ .*

*Proof.* It is clear from the definition that  $\mathfrak{a}$  and  $\mathfrak{a}'$  are contained in  $\mathfrak{h}$ , and their intersection with  $\mathfrak{z}$  is trivial. Since every root in  $\Sigma^+$  is positive, and every positive root is contained in  $\Psi^+$  or  $\Psi^0$ , we see that  $\Sigma^+ = \Sigma_{\mathfrak{a}}^+ \cup \Sigma_{\mathfrak{a}'}^+$ . Corresponding to this, we have

$$\mathfrak{a} \oplus \mathfrak{a}' \oplus \mathfrak{z} = \mathfrak{h}.$$

It is left to show that  $\mathfrak{a}$  and  $\mathfrak{a}'$  are isotropic. But recall that the nilradical  $\mathfrak{n}$  is abelian, and  $\mathfrak{a} \subset \mathfrak{n}$ , so  $\mathfrak{a}$  is isotropic. Since  $\mathfrak{a}' = \mathfrak{m} \cap \mathfrak{h}$ , we know that  $[\mathfrak{a}', \mathfrak{a}'] \subset \mathfrak{m} \cap \mathfrak{z} = 0$ . Hence  $\mathfrak{a}'$  is isotropic.  $\square$

Just as the choice of a root  $\tau$  with  $m_{\tau}(\beta) = 1$  yields additional structure on the Heisenberg parabolic, the highest root yields additional structure on the parabolic  $\mathfrak{q}$ . Note that unless  $\mathfrak{p}$  is contained in  $\mathfrak{q}$ ,  $\mathfrak{q}_1 = \mathfrak{q} \cap \mathfrak{g}_1$  is a maximal parabolic subgroup of  $\mathfrak{g}_1$ .  $\mathfrak{g}_1$  is not necessarily simple, but there is a unique simple factor of  $\mathfrak{g}_1$  which is not contained in  $\mathfrak{q}$ . Let  $\beta_1$  denote the highest root of this simple factor, and  $\mathfrak{p}_1 = \mathfrak{l}_1 \oplus \mathfrak{h}_1$  the corresponding Heisenberg parabolic. Then  $\mathfrak{q}_1$  yields a complete polarization  $\mathfrak{a}_1 \oplus \mathfrak{a}'_1 \oplus \mathfrak{z}_1 = \mathfrak{h}_1$ . Note that  $\mathfrak{a}_1$  and  $\mathfrak{z}_1$  are contained in  $\mathfrak{q}_1$ .

Continuing this process inductively yields a decomposition of  $\mathfrak{n}$ :

$$\mathfrak{n} = \bigoplus_{i=0}^{r-1} (\mathfrak{a}_i \oplus \mathfrak{z}_i),$$

$$\mathfrak{z}_i = \mathfrak{g}_{\beta_i}.$$

Here  $\mathfrak{a}_0 = \mathfrak{a}$ , and  $\mathfrak{z}_0 = \mathfrak{z}$ . The sequence  $(\beta_0, \dots, \beta_{r-1})$  is a maximal orthogonal sequence of roots. Define  $d_i = \dim(\mathfrak{a}_i)$ . Then  $d = d_0$ , and, more generally,  $d_i$  is the structure constant of a Heisenberg parabolic of  $\mathfrak{g}_i$ . Define  $D_i = \langle \rho_{\mathfrak{n}_i}, \tau \rangle$ , where  $\rho_{\mathfrak{n}_i}$  denotes half the sum of the roots occurring in  $\mathfrak{n}_i$ .

<u>Cartan type of <math>\mathfrak{g}</math></u>	<u>Vertices for <math>\mathfrak{p}</math> circled</u>	<u>Vertex for <math>\mathfrak{q}</math> circled</u>
$A_n$		Any maximal parabolic
$D_n$		
$D_n$		
$E_6$		
$E_7$		

TABLE 1.4.1. Possible triples  $\mathfrak{g}, \mathfrak{p}, \mathfrak{q}$ .

1.4. **Classification of Groups and Algebras.** For future reference, we tabulate the possible algebras  $\mathfrak{g}, \mathfrak{p}, \mathfrak{q}$  by listing the Cartan type of  $\mathfrak{g}$ , and circling the “deleted simple root(s)” associated to the parabolics  $\mathfrak{p}$  and  $\mathfrak{q}$  in Table 1.4.1.

Now, suppose that  $k$  is a finite extension of  $\mathbb{Q}_p$  for a prime  $p$ . Let  $q$  denote the cardinality of the residue field of  $k$ . Let  $\mathcal{O}_k$  denote the ring of integers of  $k$ , with uniformizer  $\varpi$ . Normalize the valuation on  $k$  by setting  $v(\varpi) = 1$ , and  $|\varpi| = q^{-1}$ . We fix the Haar measure on  $k$  which gives  $\mathcal{O}_k$  measure 1.

Corresponding to the simply-laced simple Lie algebras classified here, let  $G$  denote a simply-laced, simply-connected, absolutely simple split group over  $k$ . Suppose, moreover, that  $G \neq SL_2$ . Let  $T$  be a maximal split  $k$ -torus of  $G$ , and  $B$  a  $k$ -Borel subgroup of  $G$  containing  $T$ . Let  $P$  and  $Q$  denote standard parabolic subgroups of  $G$  corresponding to a choice of subalgebras  $\mathfrak{p}$  and  $\mathfrak{q}$  as classified above. Fix Levi decompositions over  $k$ :  $P = LH$ ,  $Q = MN$ , and set  $G_1 = [L, L]$ . The Heisenberg group  $H$  has canonical polarization:  $H = AA'Z$ , where  $Z$  is the center of  $H$ , and  $AZ = H \cap N$ . We say that the triple  $(G, P, Q)$  is terminal if  $P$  is contained in  $Q$ , or equivalently if the rank  $r$  of  $N$  is equal to 1. Note that we have a

decomposition of  $N$ :

$$N = \prod_{i=0}^{r-1} A_i Z_i,$$

where  $A = A_0$ ,  $Z = Z_0$ . We list the possible triples  $(G, P, Q)$  below for future reference:

- (1) Type  $A_n$ . We consider the group  $G = SL_{n+1}$ , with  $n > 1$ . The Heisenberg parabolic  $P$  has unipotent radical  $H$  of dimension  $2n - 1$ .  $G_1$  is isomorphic to  $SL_{n-1}$ . Every maximal parabolic has abelian unipotent radical. Thus  $Q = MN$  may be any maximal parabolic, and (up to conjugacy)  $M$  may equal  $S(GL_r \times GL_{n+1-r})$  for any  $1 \leq r \leq \lfloor n/2 \rfloor$ . The rank of  $N$  is equal to  $r$ . The triple  $(G, P, Q)$  is terminal if  $r = 1$ .
- (2) Type  $D_n$ . Here there are two possibilities for the triple  $(G, P, Q)$ . We have  $G = Spin_{2n}(k)$ , with  $n > 3$ . Then  $G_1 = SL_2(k) \times_{\mu_2} Spin_{2n-4}(k)$ . There are two choices for the parabolic  $Q$  with abelian unipotent radical. In the first case, we may choose  $Q = MN$  with  $M = CSpin_{2n-2}(k)$ . The group  $CSpin_{2m}$  fits into a short exact sequence:

$$1 \longrightarrow GL_1 \longrightarrow CSpin_{2m} \longrightarrow SO_{2m} \longrightarrow 1.$$

The most convenient construction of  $CSpin_{2m}$  is via Clifford algebras; a good treatment can be found in a paper by Deligne [Del].

In the second case, we choose  $Q = MN$  with  $M$  a double cover of  $GL_n(k)$  coming from the square root of the determinant character.

- (3) Type  $E_6$ . We take  $G = E_6(k)$  to be the simply connected simple group of type  $E_6$ , in which  $G_1 = SL_6(k)$ . There are two maximal parabolics with abelian unipotent radical; both have Levi factor  $M$  isomorphic to  $CSpin_{10}(k)$ .
- (4) Type  $E_7$ . We take  $G = E_7(k)$  to be the usual split simply-connected group, in which  $G_1 = Spin_{12}(k)$ . Then  $Q = MN$  with  $M = CE_6(k)$  as its Levi factor.  $CE_6$  is defined to be the quotient of  $E_6 \times GL_1$  obtained by identifying the center  $\mathbb{Z}/3\mathbb{Z}$  of  $E_6$  with the subgroup  $\mu_3$  of third roots of unity in  $GL_1$ .

Let  $w_\beta$  denote the root reflection associated to  $\beta$  in the Weyl group of the pair  $G, T$ . Let  $\sigma_\beta$  denote a representative for  $w_\beta$  in the normalizer of  $T$ . Later, we will use the following structural result:

**Proposition 1.4.2.** *If  $a \in A$ , then  $\sigma_\beta a \sigma_\beta^{-1} \in M$ .*

*Proof.* The adjoint action of  $\sigma_\beta$  sends the root subgroup of  $\alpha \in \Phi$  to that of

$$w_\beta[\alpha] = \alpha - \langle \alpha, \beta \rangle \beta.$$

Hence if  $\alpha \in \Sigma_a^+$ , so  $m_\tau(\alpha) = 1$ , and  $\langle \alpha, \beta \rangle = 1$ , we have

$$\begin{aligned} m_\tau(w_\beta[\alpha]) &= m_\tau(\alpha - \langle \alpha, \beta \rangle \beta) \\ &= m_\tau(\alpha) - \langle \alpha, \beta \rangle m_\tau(\beta) \\ &= 1 - 1 \cdot 1 = 0. \end{aligned}$$

Thus  $w_\beta[\alpha] \in \Psi^0$ . The desired result follows directly.  $\square$

2. REPRESENTATIONS OF THE HEISENBERG GROUP

In this section, we study the irreducible smooth representations of the Heisenberg group  $H$  of dimension  $2d + 1$  over  $k$ , and, in particular, the  $p$ -adic analogue of the Stone-von Neumann theorem. This appears countless times in the literature, with many different applications. In [How], one finds applications to dual reductive pairs and the  $\Theta$ -distribution. One encounters a detailed treatment of the representation theory of  $H$  in the articles of Mœglin, Vigneras and Waldspurger [MVW], and van Dijk [Dij]. Here we review some mostly well-known results about the Heisenberg group and its representations. The proofs given are heavily influenced by existing literature, and personal communications with G. Savin.

Let  $\underline{H}$  denote a smooth group scheme over  $\mathcal{O}_k$ , with center  $\underline{Z}$ , such that  $H = \underline{H}(k)$  and  $Z = \underline{Z}(k)$ . Moreover, suppose  $\underline{H}$  is endowed with a complete polarization over  $\mathcal{O}_k$ :  $\underline{H} = \underline{A}\underline{A}'\underline{Z}$ . Let  $L = \underline{A}(\mathcal{O}_k)$ ,  $L' = \underline{A}'(\mathcal{O}_k)$ , and  $C = \underline{Z}(\mathcal{O}_k)$  denote the resulting lattices in  $A$ ,  $A'$ , and  $Z$ . Fix a nontrivial smooth (and thus unitary) character  $\psi$  of  $Z$ , such that  $\ker(\psi) = C$ . We study the smooth representations of  $H$  of central character  $\psi$ .

We begin with the representation theory of the abelian group  $AZ$ . The pairing  $A \times A' \rightarrow \mathbb{C}^\times$  given by  $(a, a') \mapsto \psi([a, a'])$  allows us to identify  $A'$  with the unitary dual  $\hat{A}$  of  $A$ . Since  $AZ \cong A \times Z$ , the set of characters of  $AZ$  which restrict to  $\psi$  on  $Z$  can be identified naturally with  $\hat{A} = A'$ . For any element  $a' \in A'$ , let  $\tilde{\psi}_{a'}$  denote the character of  $AZ$  whose restriction to  $Z$  is  $\psi$ , obtained by this identification. Thus  $\tilde{\psi}_1$  is the character of  $AZ \cong A \times Z$  given by  $1 \otimes \psi$ .

If  $\tilde{\psi}$  is any character of  $AZ$  extending  $\psi$  on  $Z$ , and  $V$  is any smooth representation of  $AZ$ , then we let  $V(AZ, \tilde{\psi})$  denote the linear span of the vectors  $pv - \tilde{\psi}(p)v$  for all  $p \in AZ$ ,  $v \in V$ . Let  $V_{AZ, \tilde{\psi}} = V/V(AZ, \tilde{\psi})$  denote the quotient space, which is the maximal quotient of  $V$  on which  $AZ$  acts via the character  $\tilde{\psi}$ . The following result is a consequence of sections 5.10 through 5.17 of Bernstein and Zelevinsky [B-Z]:

**Proposition 2.1.** *Let  $V$  be a smooth representation of  $AZ$ , on which  $Z$  acts via the character  $\psi$ . There exists a smooth sheaf (in the language of [B-Z] an  $l$ -sheaf)  $\mathcal{F}$  on  $A'$  whose stalk  $\mathcal{F}_{a'}$  over  $a' \in A'$  is isomorphic to the space of co-invariants  $V_{AZ, \tilde{\psi}_{a'}}$ . Moreover, as a representation of  $AZ$ ,  $V$  is isomorphic to the space of smooth sections of  $\mathcal{F}$  with compact support.*

Fix a representation  $\tilde{\psi}$  extending  $\psi$  to  $AZ$ . We study a representation of  $H$  obtained by smooth induction,

$$(\rho_{\tilde{\psi}}, W_{\tilde{\psi}}(A)) = \text{Ind}_{AZ}^H(\tilde{\psi}).$$

In other words,  $W_{\tilde{\psi}}(A)$  is the space of functions  $w: H \rightarrow \mathbb{C}$  satisfying the following conditions:

- (1) There exists a compact open subgroup  $K \subset H$  such that  $w(hk) = w(h)$  for all  $h \in H, k \in K$ .
- (2)  $w(azh) = \tilde{\psi}(az)w(h)$  for all  $a \in A, z \in Z$ .

In fact, the smoothness (the first of the above conditions) implies that functions  $w \in W_{\tilde{\psi}}(A)$  are compactly supported modulo  $AZ$ . The action  $\rho_{\tilde{\psi}}$  of  $H$  on  $W_{\tilde{\psi}}(A)$  is by right-translation. Then  $(\rho_{\tilde{\psi}}, W_{\tilde{\psi}}(A))$  is a smooth representation of  $H$ . We have the following application of Frobenius reciprocity from the beginning of Bernstein

and Zelevinsky [B-Z]:

$$\text{Hom}_H(V, W_{\tilde{\psi}}(A)) = \text{Hom}_{AZ}(V_{AZ, \tilde{\psi}}, \mathbb{C}_{\tilde{\psi}}).$$

It is easy to see that the representation  $W_{\tilde{\psi}}(A)$  is independent of the chosen extension  $\tilde{\psi}$  of  $\psi$ , up to isomorphism. Replacing the function  $w(h)$  by  $w(a'h)$  for various  $a' \in A'$  yields an isomorphism from  $W_{\tilde{\psi}_1}(A)$  to  $W_{\tilde{\psi}_{a'}}(A)$ . Therefore, it makes sense to write  $(\rho_\psi, W_\psi(A))$  for the representation, without specifying the extension  $\tilde{\psi}$  used. We can prove:

**Proposition 2.2.** *The representation  $(\rho_\psi, W_\psi(A))$  is the unique smooth irreducible representation of  $H$  with central character  $\psi$ .*

*Proof.* Suppose that  $V$  is a smooth irreducible representation of  $H$  with central character  $\psi$ . Then there exists a smooth character  $\tilde{\psi}$  of  $AZ$  extending  $\psi$  on  $Z$ , such that  $V_{AZ, \tilde{\psi}} \neq 0$  by Proposition 2.1. The action of  $A'$  on such characters  $\tilde{\psi}$  is transitive, so that  $V_{AZ, \tilde{\psi}} \neq 0$  for all  $\tilde{\psi}$ . It follows now from Frobenius reciprocity that  $V$  is a submodule of  $W_\psi(A)$ .

For the character  $\tilde{\psi}_{a'}$  extending  $\psi$ , the space  $W_\psi(A)(AZ, \tilde{\psi}_{a'})$  can be naturally identified with  $C_c^\infty(A' \setminus \{a'\})$ . Hence the co-invariants  $W_\psi(A)_{AZ, \tilde{\psi}_{a'}}$  form a one-dimensional vector space. Since  $V_{AZ, \tilde{\psi}} \neq 0$  for all  $\tilde{\psi}$ , we see that  $V_{AZ, \tilde{\psi}} = W_\psi(A)_{AZ, \tilde{\psi}}$ . If  $V' = W_\psi(A)/V$ , then by exactness of taking co-invariants, we see that  $V'_{AZ, \tilde{\psi}} = 0$  for all  $\tilde{\psi}$ . By Proposition 2.1, this implies that  $V' = 0$ , and we are done.  $\square$

From the version of Frobenius reciprocity introduced above, it follows that  $W_\psi$  is injective in the category of smooth representations of  $H$ . We introduce the lattice model of the representation  $W_\psi$  in order to prove that it is also projective. Define the subgroup  $K_0$  of  $H$  by

$$K_0 = LL'Z.$$

Then  $\psi$  extends to a character  $\psi_0$  of  $K_0$ , letting  $\psi_0(L) = \psi_0(L') = 1$ , since  $\psi$  is trivial on  $[L, L'] \subset C$ . We define the representation of  $H$ ,

$$W_\psi(K_0) = \mathbb{C}[H] \otimes_{\mathbb{C}[K_0]} \mathbb{C}_{\psi_0}.$$

By  $\mathbb{C}[H]$  and  $\mathbb{C}[K_0]$ , we mean the naïve group rings over  $\mathbb{C}$ . In the above tensor product, we view  $\mathbb{C}[H]$  as a left  $\mathbb{C}[K_0]$ -module, and view the entire space  $W_\psi(K_0)$  as an  $H$ -module via the right multiplication action of  $H$  on  $\mathbb{C}[H]$ . Thus  $W_\psi(K_0)$  is spanned by elements of the form  $\alpha h$  for  $\alpha \in \mathbb{C}, h \in H$ , subject to the identifications  $k_0 h = \psi_0(k_0)h$  for all  $k_0 \in K_0$ , and subject to the  $H$ -action:  $h'[\alpha h] = \alpha h h'$ . Then for any smooth  $H$ -module, we have the form of Frobenius reciprocity (cf. the change of rings formula in chapter 3.4 of Fulton and Harris [F-H]):

$$\text{Hom}_H(W_\psi(K_0), V) = \text{Hom}_{K_0}(\mathbb{C}_{\psi_0}, V).$$

We can now show the following:

**Proposition 2.3.** *The representation  $W_\psi(K_0)$  is isomorphic to  $W_\psi(A)$ .*

*Proof.* First, we note that the  $(K_0, \psi_0)$ -co-invariants of  $W_\psi(K_0)$  are one-dimensional. The kernel of the projection  $W_\psi(K_0) \rightarrow W_\psi(K_0)_{K_0, \psi_0}$  is by definition the

linear span:

$$\begin{aligned} \text{Span}\{hk_0 - \psi(k_0)h\} &= \text{Span}\{hk_0h^{-1}h - \psi(k_0)h\} \\ &= \text{Span}\{\psi(hk_0h^{-1})h - \psi(k_0)h\}. \end{aligned}$$

The projection annihilates  $h$  unless  $\psi(hk_0h^{-1}) = \psi(k_0)$  for all  $k \in K_0$ , in which case  $h \in K_0$ . Therefore, the image of the projection is precisely  $\mathbb{C}[K_0] \otimes_{\mathbb{C}[K_0]} \mathbb{C}_{\psi_0} = \mathbb{C}_{\psi_0}$ , and is one-dimensional.

The set of characters of  $K_0$  that extend  $\psi$  on  $Z$  can be identified with the discrete quotient  $H/K_0$ . If  $\bar{h} \in H/K_0$ , then we write  $\psi_{\bar{h}}$  for the corresponding character of  $K_0$ . Moreover, if  $V$  is a smooth representation of  $K_0$ , on which  $Z$  acts by  $\psi$ , then we may decompose  $V$  as a direct sum:

$$V = \bigoplus_{\bar{h} \in H/K_0} V_{K_0, \psi_{\bar{h}}}.$$

Since  $H$  acts transitively on  $H/K_0$ , we see that if  $V$  is a smooth representation of  $H$ , then the dimension of  $V_{K_0, \psi_{\bar{h}}}$  is independent of  $\bar{h}$ . Hence, as in the proof of Proposition 2.2, we see that  $W_{\psi}(K_0)$  is irreducible. By uniqueness, it is isomorphic to  $W_{\psi}(A)$ .  $\square$

Hereafter, we write  $W_{\psi}$  for the unique smooth irreducible representation of  $H$  of central character  $\psi$ , when we only care about the isomorphism class.

**Corollary 2.4.** *The category  $Rep(H, \psi)$  of smooth representations of  $H$  with central character  $\psi$  is equivalent to the category of (complex) vector spaces. Specifically, the functor  $Hom_H(W_{\psi}, \cdot)$  is exact on  $Rep(H, \psi)$ , and any object  $W$  of  $Rep(H, \psi)$  satisfies*

$$W \cong Hom_H(W_{\psi}, W) \otimes_{\mathbb{C}} W_{\psi}.$$

*Proof.* The functor  $Hom_H(W_{\psi}, \cdot)$  is isomorphic to the functor  $Hom_{K_0}(\mathbb{C}_{\psi_0}, \cdot)$  by Frobenius reciprocity. This functor is exact on the category of representations of central character  $\psi$  since  $K_0$  is compact modulo its center. Thus  $Hom_H(W_{\psi}, \cdot)$  is an exact functor from  $Rep(H, \psi)$  to the category of vector spaces.

Finally, consider the canonical map, defined on a spanning subset as follows:

$$\begin{aligned} Hom(W_{\psi}, W) \otimes_{\mathbb{C}} W_{\psi} &\rightarrow W \\ \sigma \otimes w &\mapsto \sigma(w). \end{aligned}$$

Upon taking the  $(K_0, \psi_0)$ -co-invariants, and applying Frobenius reciprocity, the above map yields a morphism of  $K_0$  representations:

$$Hom(\mathbb{C}_{\psi_0}, W_{K_0, \psi_0}) \otimes_{\mathbb{C}} \mathbb{C}_{\psi_0} \rightarrow W_{K_0, \psi_0}.$$

This is trivially an isomorphism. The same holds, replacing  $\psi_0$  by any other character of  $K_0$  extending  $\psi$ . Therefore, the original map is an isomorphism, by decomposing the source and target into its  $K_0$ -isotypic quotients.  $\square$

The representation  $W_{\psi}(A)$  of  $H$  has a natural unitary structure, which is most easily constructed with the Schrödinger realization. Namely, a function  $w \in W_{\psi}(A)$  is determined by its restriction to  $A' \subset H$ , which is a smooth compactly-supported function. Given two elements  $w, w' \in W_{\psi}(A)$ , we use this restriction to define an inner product (up to a choice of Haar measure):

$$\langle w, w' \rangle = \int_{A'} w(a') \overline{w'(a')} da'.$$

It is not difficult to show that the action of  $H$  respects this inner product, so  $W_\psi(A)$  is unitarizable.

Let  $(G, P, Q)$  be a triple as classified in the first section, so that  $G$  is simply-laced,  $G \neq SL_2$ , with Heisenberg parabolic  $P = LH$ , and  $G_1 = [L, L]$ . Let  $G^J = G_1H$ ,  $Q_1 = M_1N_1 = G_1 \cap Q$ ,  $M_1 = M \cap G_1$ ,  $N_1 = N \cap G_1$ . Then we quote the following result from Kazhdan and Savin [K-S]:

**Proposition 2.5.** *The representation  $(\rho_\psi, W_\psi(A))$  of  $H$  extends uniquely to a smooth representation  $(\tilde{\rho}_\psi, W_\psi)$  of  $G^J$ , which again respects the inner product above.*

This should be contrasted with the nonsimply-laced case; for instance, when  $G$  is a symplectic group, the representation  $W_\psi$  extends only after passing to a metaplectic cover of  $G_1$ . It is difficult to explicitly write down the extension  $\tilde{\rho}_\psi$ , however, it is possible to describe the action of  $\tilde{\rho}_\psi(q_1)$  on  $W_\psi(A)$  when  $q_1 \in G_1$  stabilizes the polarization  $H = AA'Z$ . In this case, there is a natural adjoint action of  $q_1$  on the vector space  $W_\psi(A)$  given by

$$[Ad(q_1)w](h) = w(q_1 h q_1^{-1}).$$

**Proposition 2.6.** *For any  $q_1 \in G_1$  that stabilizes the polarization  $AA'Z$ , the endomorphism  $Ad(q_1) \circ \tilde{\rho}_\psi(q_1)$  of  $W_\psi(A)$  is equal to the scalar  $\delta_1(q_1)^{1/2}$ , where  $\delta_1$  is the multiplier for the action of  $q_1$  on the Haar measure of  $A$  (or equivalently of  $A'$ ).*

*Proof.* First, we can see that  $Ad(q_1) \circ \tilde{\rho}_\psi(q_1)$  commutes with the action of  $H$ . So we have

$$\begin{aligned} [Ad(q_1) \circ \tilde{\rho}_\psi(q_1) \circ \tilde{\rho}_\psi(h)w](x) &= [Ad(q_1) \circ \tilde{\rho}_\psi(q_1 h q_1^{-1}) \circ \tilde{\rho}_\psi(q_1)w](x) \\ &= [\tilde{\rho}_\psi(q_1)w](q_1 x q_1^{-1} q_1 h q_1^{-1}) \\ &= [\tilde{\rho}_\psi(q_1)w](q_1 x h q_1^{-1}) \\ &= [\tilde{\rho}_\psi(h) \circ Ad(q_1) \circ \tilde{\rho}_\psi(q_1)w](x). \end{aligned}$$

Therefore,  $Ad(q_1) \circ \tilde{\rho}_\psi(q_1)$  is an endomorphism of  $W_\psi(A)$  that commutes with the action of  $H$ . By irreducibility, this implies that  $Ad(q_1) \circ \tilde{\rho}_\psi(q_1)$  is a scalar. To determine the scalar, it suffices to look at the action of  $Ad(q_1) \circ \tilde{\rho}_\psi(q_1)$  on the norm  $\langle w, w \rangle$  of a nonzero  $w \in W_\psi(A)$ . Since  $\tilde{\rho}_\psi(q_1)$  preserves the unitary structure, it therefore suffices to study the action of  $Ad(q_1)$  on the norm. It is easy to check that  $\langle Ad(q_1)w, Ad(q_1)w \rangle = \delta_1(q_1) \langle w, w \rangle$ . Hence the action of  $Ad(q_1) \circ \tilde{\rho}_\psi(q_1)$  is via a scalar  $\eta(q_1) \delta_1(q_1)^{1/2}$ , where  $\eta$  is a unitary character of  $Q_1$ . To determine the character  $\eta$ , we use a trick suggested by G. Savin.

There exists a simple root  $\gamma$  and an associated map  $\phi_\gamma$  from  $SL_2$  into  $G_1$  defined over  $k$ , such that  $M_1 = [M_1 M_1] T_\gamma$ , where  $T_\gamma$  is the image of the usual maximal torus of  $SL_2$  in  $G_1$ . Thus the character  $\eta$  of  $Q_1 = M_1 N_1$  is determined by its values on  $T_\gamma$ . Let  $S_\gamma$  denote the image of  $SL_2$  under  $\phi_\gamma$ . Then, following the techniques in the proof of Theorem 2 of [K-S], there exists a different polarization,  $H = A_\gamma A'_\gamma Z$ , which is preserved by  $S_\gamma$ . Corresponding to this polarization, there is another representation  $W(A_\gamma)$  of  $G_1 H$ , and an intertwining operator between  $W$  and  $W(A_\gamma)$  given by a (partial) Fourier transform. From this realization, we see that  $S_\gamma$  acts via an adjoint action on functions in  $W(A_\gamma)$  modulo a character again. However,  $S_\gamma$  is simple, so this character is trivial, and moreover, restricting this character to  $T_\gamma$  yields the character  $\eta$ , so  $\eta$  is trivial.  $\square$

3. THE FOURIER-JACOBI MAP

Let  $G$  be a simple, simply-laced, simply-connected split group over  $k$ , with parabolics  $P = LH, Q = MN$  as classified in the first section. Let  $Sm(G)$  (respectively  $Sm(G_1)$ ) denote the category of smooth representations of  $G$  (resp.,  $G_1$ ) on complex vector spaces. Suppose that  $(\pi, V) \in Sm(G)$ . Define  $G^J = G_1H$ . Since  $G^J$  centralizes  $Z$ ,  $G^J$  acts naturally on the space of co-invariants  $V_{Z,\psi}$ . Denote this representation by  $\pi_{Z,\psi}$ . Consider now the vector space

$$V_1 = \text{Hom}_H(W_\psi, V_{Z,\psi}).$$

Then  $G_1$  acts on elements  $\sigma$  of  $V_1$  by setting

$$[g\sigma](w) = \pi_{Z,\psi}(g)\sigma(g^{-1}w).$$

Note that we use the unique extension of  $(\rho_\psi, W_\psi)$  to a representation of  $G^J$ . This yields a representation  $(\pi_1, V_1)$  of  $G_1$ . We have constructed a map, which we call the Fourier-Jacobi map

$$FJ: Sm(G) \rightarrow Sm(G_1) \\ (\pi, V) \mapsto (\pi_1, V_1).$$

*Remark.* In the study of automorphic forms on symplectic groups, numerous authors have studied the ‘‘Fourier-Jacobi’’ expansion, for instance, Eichler and Zagier in [E-Z]. The representation theory underlying this expansion has been studied more recently, and one may consult Berndt and Schmidt in [B-S]. Our functor  $FJ$  is the representation theoretic analogue of the Fourier-Jacobi expansion, where we work with simply-laced groups rather than the symplectic group.

It is not difficult to prove the following:

**Proposition 3.1.**  *$FJ$  is an exact functor from  $Sm(G)$  to  $Sm(G_1)$ . It is independent of the choice of the nontrivial character  $\psi$ .*

*Proof.*  $FJ$  is an exact functor because it the composition of two exact functors. The exactness of taking co-invariants is well known (the theory of Jacquet modules). The exactness of the functor  $\text{Hom}(W_\psi, \cdot)$  is discussed in the previous section.

We may identify the characters of  $Z$  with elements of the opposite subgroup  $\bar{Z} \in G$ . Since we assume  $G \neq SL_2$ , the Levi component  $L$  of  $P$  acts transitively on the nontrivial elements of  $\bar{Z}$  (this is a consequence of the existence of a single bond to the highest root in the extended Dynkin diagram). Hence, if  $\psi_1, \psi_2$  are nontrivial characters of  $Z$ , there exists an element  $l \in L$  satisfying  $\psi_1(z) = \psi_2(lzl^{-1})$ . Now, if  $v = \pi(z)w - \psi_1(z) \cdot w \in V(Z, \psi_1)$ , we have

$$\begin{aligned} \pi(l)v &= \pi(l)\pi(z)w - \pi(l)\psi_1(z) \cdot w, \\ &= \pi(lzl^{-1})(\pi(l)w) - \psi_1(z)(\pi(l)w), \\ &= \pi(lzl^{-1})(\pi(l)w) - \psi_2(lzl^{-1})(\pi(l)w). \end{aligned}$$

Hence  $\pi(l)$  maps  $V(Z, \psi_1)$  isomorphically onto  $V(Z, \psi_2)$ . Therefore,  $\pi(l)$  yields a vector space isomorphism from  $V_{Z,\psi_1}$  to  $V_{Z,\psi_2}$ . Similarly,  $l$  yields a vector space isomorphism from  $W_{\psi_1}$  to  $W_{\psi_2}$  in such a way that we have an isomorphism

$$\text{Hom}_H(W_{\psi_1}, V_{Z,\psi_1}) \simeq \text{Hom}_H(W_{\psi_2}, V_{Z,\psi_2}).$$

□

The precise relation between  $(\pi_1, V_1)$  and the representation  $(\pi_{Z,\psi}, V_{Z,\psi})$  is given by the following:

**Proposition 3.2.** *The representation  $(\pi_{Z,\psi}, V_{Z,\psi})$  decomposes as a tensor product:*

$$(\pi_{Z,\psi}, V_{Z,\psi}) = (\pi_1, V_1) \otimes (\rho_\psi, W_\psi).$$

*Proof.* Consider the canonical map  $V_1 \otimes W_\psi \rightarrow V_{Z,\psi}$  given by  $(\sigma, w) \mapsto \sigma(w)$ . This is clearly a  $G^J$ -equivariant map, with the natural action of  $G^J$  on  $V_1 \otimes W_\psi$  via  $g(\sigma \otimes w) = \pi_1(g)\sigma \otimes \rho_\psi(g)w$  for all  $g \in G^J$ . Note that we are letting  $G^J$  act on  $V_1$  by having  $H$  act trivially. This canonical map is an isomorphism by Corollary 2.3. □

#### 4. DEGENERATE PRINCIPAL SERIES

In this section, we apply the Fourier-Jacobi map to degenerate principal series representations of the group  $G$ , induced from the parabolic  $Q$ . If  $(G, P, Q)$  is not terminal (i.e.,  $Q$  does not contain  $P$ ), then recall that  $Q_1 = M_1N_1 = G_1 \cap Q$  is a maximal parabolic subgroup of  $G_1$  with abelian unipotent radical. We let  $W, W_L, W_M$  denote the Weyl groups of  $G, L$ , and  $M$ . We choose a representative  $w_\beta$  in  $G$  for the root reflection in the highest root  $\beta$ .

**4.1. Character Identities.** Let  $T$  denote the maximal torus of  $G$  as before,  $T_1 = T \cap G_1$ , and let

$$X(T) = \text{Hom}(T, k^\times)$$

be the weight space. If  $\nu \in X(T)$  is  $W_M$ -invariant, then  $\nu$  yields a character  $\chi_\nu: M \rightarrow \mathbb{R}^\times$  satisfying

$$\chi_\nu(t) = |\nu(t)|.$$

The map  $\nu \mapsto \chi_\nu$  extends further to a homomorphism

$$(X(T) \otimes_{\mathbb{Z}} \mathbb{R})^{W_M} \rightarrow \text{Hom}(M, \mathbb{R}^\times),$$

by setting  $\chi_{s\nu}(m) = \chi_\nu(m)^s$  for all  $s \in \mathbb{R}, \nu \in X(T)^{W_M}$ . The same facts hold for  $M_1$  instead of  $M$ .

Let  $\rho$  (respectively  $\rho_1$ ) denote half of the sum of the roots occurring in  $N$  (resp.,  $N_1$ ). We recall that the structure constant  $D$  of  $N$  is given by  $D = \langle \rho, \tau \rangle$ , where  $\tau$  is the simple root associated to  $Q$ , and  $D_1 = \langle \rho_1, \tau \rangle$ . It is easy to see that

$$\rho \in (X(T) \otimes_{\mathbb{Z}} \mathbb{R})^{W_M},$$

and similarly for  $\rho_1$ . Let  $\nu = \langle \rho, \tau \rangle^{-1}\rho$  and  $\nu_1 = \langle \rho_1, \tau \rangle^{-1}\rho_1$ . Let  $\chi = \chi_\nu$  and  $\chi_1 = \chi_{\nu_1}$  be the associated characters of  $M$  and  $M_1$ . If  $t \in T_1$ , then it is not hard to show that

$$\chi(t_1) = \chi_1(t_1).$$

Hence  $\chi$  agrees with  $\chi_1$  on  $M_1$ .

Let  $\Delta: Q \rightarrow \mathbb{R}^\times$  denote the modular character corresponding to the adjoint action of  $Q$  on  $N$ . Similarly, define  $\Delta_1: Q_1 \rightarrow \mathbb{R}^\times$  for the adjoint action of  $Q_1$  on  $N_1$ .  $Q_1$  also acts by conjugation on the polarizing subgroup  $AZ$ . Let  $\delta_1: Q_1 \rightarrow \mathbb{R}^\times$

denote the corresponding modular character. We will use the following identities among characters:

$$\begin{aligned} \Delta(q_1) &= \Delta_1(q_1)\delta_1(q_1), \\ \chi(q_1) &= \chi_1(q_1), \\ \Delta(q)^{\frac{1}{2D}} &= \chi(q), \\ \Delta_1(q)^{\frac{1}{2D_1}} &= \chi_1(q_1). \end{aligned}$$

**4.2. Bruhat Filtration on Principal Series.** We now define the degenerate principal series:

**Definition 4.2.1.** Let  $I(s)$  denote the space of all functions  $f: G \rightarrow \mathbb{C}$  satisfying the following conditions:

- (1) There exists a compact open subgroup  $K \subset G$  such that  $f(gk) = f(g)$  for all  $g \in G, k \in K$ .
- (2) For all  $m \in M, n \in N, g \in G$ , we have  $f(mng) = \chi(m)^s \Delta(m)^{1/2} f(g)$ .

Then the action of  $G$  on  $I(s)$  by right-translations is a smooth representation of  $G$ . We define the smooth representation  $I_1(s)$  of  $G_1$  identically, inserting the subscript 1 wherever possible.

We will study the action of the Fourier-Jacobi map on these degenerate principal series in a few steps. Recall the Bruhat decomposition  $G = \bigsqcup_w QwP$  as  $w$  ranges over a set of representatives for the double cosets  $W_M \backslash W / W_L$ . Motivated by this decomposition, and the work of Bernstein and Zelevinsky [B-Z], we define a subspace of the degenerate principal series for any  $w \in W$ :

$$I^w(s) = \{f \in I(s) \mid \text{Supp}(f) \subset QwP, \\ f \text{ compactly supported modulo } Q\}.$$

By a dimension argument, we see that  $I^w(s) = 0$  unless  $w = w_\beta$ , in which case  $I^{w_\beta}(s)$  consists of those functions supported in the big cell  $Qw_\beta P$ , and compactly supported modulo  $Q$ . We require the following structural lemma:

**Lemma 4.2.2.** *The subgroup  $wZw^{-1}$  is not contained in  $Q$  if and only if  $W_M w W_L = W_M w_\beta W_L$ .*

*Proof.* Since  $w_\beta Z w_\beta$  corresponds to the root space  $\mathfrak{g}_{-\beta}$ ,  $w_\beta Z w_\beta^{-1}$  is not contained in  $Q$ . We are left to check that no other double coset in  $W_M \backslash W / W_L$  sends  $Z$  outside of  $Q$ . Recall that the Weyl group  $W_L$  of  $L$  is precisely the subgroup of the Weyl group that stabilizes  $Z$ . Hence a single coset  $wW_L$  is uniquely determined in  $W/W_L$  by  $wZw^{-1}$ , or equivalently by the root  $w(\beta)$ . Moreover, the Weyl group  $W_M$  of  $M$  acts transitively on the root spaces in the unipotent radical  $N$  and its opposite  $N^-$ . From this we see that if  $wZw^{-1}$  does not lie in  $Q$ , then there exists  $w' \in W_M$  such that  $w'w(\beta) = w_\beta(\beta)$ . Hence  $w$  is in the same double-coset in  $W_M \backslash W / W_L$  as  $w_\beta$ . This verifies uniqueness.  $\square$

From this lemma, we deduce the following useful result:

**Proposition 4.2.3.** *The co-invariants of  $I(s)$  are determined by those on the big cell,*

$$I(s)_{Z,\psi} = I^{w_\beta}(s)_{Z,\psi}.$$

*Proof.* Recall that  $I(s)(Z, \psi)$  is generated by functions of the form  $f'(g) = f(gz) - \psi(z)f(g)$  for functions  $f \in I(s)$ . Therefore, if  $f \in I(s)$ , then in the quotient  $I(s)_{Z, \psi}$ , we have

$$f(g) \equiv f(g) - \frac{1}{1 - \psi(z)} [f(gz) - \psi(z)f(g)].$$

Now, if  $wZw^{-1} \subset Q$ , then  $f(gz) = f(g)$  for all  $f \in I(s)$ ,  $z \in Z$ , and  $g \in QwP$ . By our last lemma, this implies that  $f(gz) = f(g)$  whenever  $g$  is not contained in the big cell  $Qw_\beta P$ . From this we see that the right-hand side of the above equation is supported on the big cell when  $z \notin \ker(\psi)$ . It is also compactly supported modulo  $Q$ , since  $f$  itself is. Therefore, every function  $f \in I(s)$  is congruent to a function in  $I^{w_\beta}(s)$  modulo  $I(s)(Z, \psi)$ . This proves our proposition.  $\square$

### 4.3. The Effect of Fourier-Jacobi.

**Theorem 4.3.1.** *The Fourier-Jacobi map sends  $I(s)$  to  $I_1(s)$ . In other words, there is a natural intertwining operator that identifies the representations  $(I(s))_1$  and  $I_1(s)$ .*

*Proof.* By previous results, it suffices to prove that  $I^{w_\beta}(s)_{Z, \psi} \simeq I_1(s) \otimes W_\psi$  as representations of  $G^J$ . We introduce a new representation of  $G^J$  by defining

$$\begin{aligned} Y = \{ & F \in C^\infty(G^J) \mid F(q_1 j) = \chi(q_1)^s \Delta(q_1)^{1/2} F(j), \\ & F(azj) = \psi(z)F(j), \\ & F(j) \text{ is compactly supported modulo } Q_1AZ \}. \end{aligned}$$

We prove our theorem now in two steps. First we show that  $I^{w_\beta}(s)_{Z, \psi} \simeq Y$ , by writing an explicit intertwining operator.

Begin with a function  $f \in I^{w_\beta}(s)$ . Then for fixed  $j \in G^J$ ,  $f(w_\beta j z)$  is compactly supported in the variable  $z \in Z$ . Therefore, we may define a function  $F$  on  $G^J$  by setting

$$F(j) = \int_Z f(w_\beta j z) \overline{\psi(z)} dz.$$

This yields a  $G^J$ -intertwining homomorphism from  $I^{w_\beta}(s)$  to  $Y$ . By standard techniques in the theory of Jacquet's functors, we see that the kernel of  $f \mapsto F$  is precisely the space  $I^{w_\beta}(s)(Z, \psi)$ . To see that the image is all of  $Y$ , note that one can recover  $f$  from  $F$  by setting

$$f(q_1 w_\beta j) = \chi(q_1)^s \Delta(q_1)^{1/2} F(j),$$

for all  $q_1 \in Q_1, j \in G^J$ . Note here that  $Qw_\beta G^J = Qw_\beta P$ , so this is sufficient. Therefore,  $I^{w_\beta}(s)_{Z, \psi}$  is isomorphic to  $Y$ .

Second, we show that  $I_1(s) \otimes W_\psi$  is also isomorphic to  $Y$ . Suppose that  $f_1 \in I_1(s)$  and  $w \in W_\psi$ , and define

$$F(g_1 h) = f_1(g_1) [\tilde{\rho}(g_1) w] (g_1 h g_1^{-1}),$$

for all  $g_1 \in G_1$  and  $h \in H$ . Then  $F$  is a function on  $G^J$ , and  $f_1 \otimes w \mapsto F$  is a  $G^J$ -intertwining homomorphism from  $I_1(s) \otimes W_\psi$  to  $Y$ . The fact that  $F \in Y$  follows from the character identities of the previous section, together with Proposition 2.6. It is not hard to check that  $f_1 \otimes w \mapsto F$  is an injective  $G^J$ -intertwining homomorphism. Surjectivity follows from the identification of  $C^\infty(G_1 \times H)$  with  $C^\infty(G_1) \otimes C^\infty(H)$ . Thus  $I_1(s) \otimes W_\psi$  is isomorphic to  $Y$  also, and we are done.  $\square$

## 5. REDUCIBILITY POINTS AND COMPOSITION SERIES

**5.1. Lengths of Degenerate Principal Series.** In this section, we consider the reducibility points of the degenerate principal series  $I(s)$ . The following lemma applies the Fourier-Jacobi map to study the length of  $I(s)$  at a reducibility point:

**Lemma 5.1.1.** *Suppose that the length of  $I(s)$  is greater than the length of  $I_1(s)$ . Then  $s = \pm D$ , the trivial representation occurs as a Jordan Hölder component of  $I(s)$ , and the length of  $I(s)$  is exactly one more than the length of  $I_1(s)$ .*

*Proof.* Let  $L, L_1$  denote the lengths of  $I(s)$  and  $I_1(s)$  respectively. If  $L > L_1$ , then the Fourier-Jacobi map must kill  $L - L_1$  components of  $I(s)$ . By Corollary 6.1.3 (which does not depend on this lemma), the trivial representation of  $G$  is the only one killed by the Fourier-Jacobi map. Hence the trivial representation occurs with multiplicity  $L - L_1$  in  $I(s)$ . Thus the  $U$ -co-invariants  $I(s)_{U,1}$  contain the trivial representation of  $T$  as a subquotient. But Bernstein and Zelevinsky [B-Z] precisely describe the  $U$ -co-invariants, to show that  $I(s)_{U,1}$  is “glued” from the components

$$F_w = (\chi^s \circ w)\Delta^{1/2},$$

where  $w$  ranges over the set  $W^{M,T} = W_M \setminus W$ . Recall that  $\Delta^{1/2} = \chi^D$  from section 4.1. In particular, we see that the trivial representation occurs only when  $s = \pm D$ , and only once at these values of  $s$ . Hence  $s = \pm D$ , and  $L - L_1 = 1$ .  $\square$

This lemma implies the following bound on the length of  $I(s)$ :

**Corollary 5.1.2.** *The length of  $I(s)$  is less than or equal to 2.*

*Proof.* We prove the statement of the corollary, and also that reducibility points occur for  $|s| \leq D$  inductively on the rank of  $N$ . As a base step, we suppose first that  $(G, P, Q)$  is a terminal triple, or equivalently that the rank of  $N$  is equal to 1. Then the Fourier-Jacobi map sends every  $I(s)$  to a character of  $G_1$ . Hence the only reducibility points of  $I(s)$  are when  $s = \pm D$ , and the length at these reducibility points is equal to 2 by the previous lemma. Now we inductively assume that our corollary is true for  $I_1(s)$ . Then all reducibility points of  $I(s)$  occur for values of  $s$  where  $I_1(s)$  is reducible, or when  $s = \pm D$ . Clearly,  $D_1 < D$ , so our previous lemma implies that the length at these reducibility points is less than or equal to 2.  $\square$

**5.2. Reducibility Points via L-Functions.** The Fourier-Jacobi map provides an upper bound on the length of degenerate principal series:  $I(s)$  has at most 2 Jordan-Hölder components. However, we would like to show that  $I(s)$  is reducible whenever allowed by the Fourier-Jacobi map. In other words, we would like to know that when  $I_1(s)$  is reducible, so is  $I(s)$ . In this section, we use the results of Shahidi and Muić [M-S] to completely relate the reducibility points of the degenerate principal series to the vanishing of local  $L$ -functions for the Steinberg representation. G. Savin has remarked that it is possible to locate the reducibility points of  $I(s)$  in a simpler, or at least more self-contained, fashion by using intertwining operators; this has been accomplished for  $GL_{2n}$  in the work of Kudla and Sweet [KuS]. However, the deep results of Shahidi and Muić yield an exceptional and efficient tool to study reducibility points.

The principal result we use is an immediate consequence of Proposition 3.3 and its subsequent remarks in Shahidi and Muić [M-S]:

**Proposition 5.2.1.** *Let  $G, Q, M, N$  satisfy our usual hypotheses:  $G$  is split, simple, simply-laced, and  $Q$  a maximal parabolic with abelian unipotent radical  $N$  and Levi  $M$ . Let  $M^L$  denote the dual Levi factor to  $M$  in the Langlands dual  $G^L$ , with its action on the unipotent radical  $N^L$  of the dual parabolic. Call the differentiated action  $r: M^L \rightarrow \text{Aut}(\mathfrak{n}^L)$ , so that  $r$  is irreducible (since  $N^L$  is abelian). Let  $St_M$  denote the Steinberg representation of  $M$ . Then for  $s > 0$ ,  $I(s)$  is reducible (and so is  $I(-s)$ ) if and only if*

$$L(1 - s, St_M, r)^{-1} = 0.$$

When  $s > 0$  and  $L(1 - s, St_M, r)$  has a pole as above, the intertwining operator from  $I(-s)$  to  $I(s)$  kills the spherical vector in  $I(-s)$ . Therefore,  $I(-s)$  has a proper spherical subrepresentation and  $I(s)$  has a proper spherical quotient.

To compute these local  $L$ -functions, we work on the side of representations of the Weil group  $W_k$  of  $k$  (one may consult the article of Tate [Tat] for details about Weil groups). Let  $\alpha: W_k \rightarrow SL_2$  be the homomorphism

$$\alpha(w) = \begin{pmatrix} \|w\|^{1/2} & 0 \\ 0 & \|w\|^{-1/2} \end{pmatrix}.$$

Let  $\phi: SL_2 \rightarrow M^L$  denote the principal homomorphism as in the paper of Gross [Gro]. Then  $\sigma = \phi \circ \alpha: W_k \rightarrow M^L$  is the Langlands parameter of the Steinberg representation  $St_M$ . More explicitly, we may compute in the split case

$$\sigma(w) = \rho_M(\|w\|),$$

where  $\rho_M$  denotes half the sum of the positive roots of  $M$ , viewed as coroots of  $M^L$ . Now let

$$R = r \circ \phi: SL_2 \longrightarrow GL(\mathfrak{n}^L)$$

denote the adjoint action of the principal  $SL_2$  on the Lie algebra  $\mathfrak{n}^L$  of  $N^L$ . The representation  $R \circ \alpha$  of the Weil group comes with the structure of a ‘‘Weil-Deligne’’ representation, i.e., it comes with a unipotent element  $u$  of  $GL(\mathfrak{n}^L)$ . This element is simply

$$u = R \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

Let  $\mathfrak{n}_u^L = \{X \in \mathfrak{n} \mid \text{Ad}(u)X = 0\}$ . Since

$$\alpha(Fr) = \begin{pmatrix} q^{-1/2} & 0 \\ 0 & q^{1/2} \end{pmatrix},$$

for a geometric Frobenius element  $Fr$  the local  $L$ -function  $L(s, St_M, r)$  may be explicitly written as

$$L(s, St_M, r) = \det \left( 1 - R \begin{pmatrix} q^{-1/2} & 0 \\ 0 & q^{1/2} \end{pmatrix} q^{-s} |_{\mathfrak{n}_u^L} \right)^{-1}.$$

This allows us to compute reducibility points via the following proposition:

**Proposition 5.2.2.** *Suppose that the representation  $R$  of the principal  $SL_2$  on  $\mathfrak{n}^L$  breaks up into irreducible representations as*

$$R = \bigoplus_i \text{Sym}^{t_i},$$

$G$	$M$	$r$	$R$
$SL_n$	$S(GL_i \times GL_j)$	$Stand_i \otimes Stand_j$	$Sym^{i-1} \otimes Sym^{j-1}$
$Spin_{2n}$	$CSpin_{2n-2}$	$Stand_{2n-2}$	$Sym^{2n-4} \oplus Sym^0$
$Spin_{2n}$	$\widetilde{GL}_n$	$\wedge^2(Stand_n)$	$\wedge^2(Sym^{n-1})$
$E_6$	$CSpin_{10}$	$\frac{1}{2}$ -spin representation	$Sym^{10} \oplus Sym^4$
$E_7$	$CE_6$	27-dimensional representation	$Sym^{16} \oplus Sym^8 \oplus Sym^0$

TABLE 5.2.3. Adjoint representations of the principal  $SL_2$

for some set of nonnegative integers  $t_i$  (here, of course,  $Sym^t$  denotes the  $t$ -th symmetric power of the standard representation). Then the local  $L$ -function may be computed:

$$L(s, St_M, r) = \prod_i \left(1 - q^{-s - \frac{t_i}{2}}\right)^{-1}.$$

Hence the reducibility points of the degenerate principal series  $I(s)$  occur precisely at the points

$$s = \pm\left(1 + \frac{t_i}{2}\right).$$

*Proof.* Let  $(Sym^t, S^t)$  denote the  $t$ -th symmetric power representation of  $SL_2$ . Then it is not hard to check that the space

$$S_u^t = \{X \in S^t \mid Sym^t \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} X = X\}$$

is one-dimensional. Moreover, one may check directly that this subspace is an eigenspace for the action of  $Sym^t \begin{pmatrix} q^{-1/2} & 0 \\ 0 & q^{1/2} \end{pmatrix}$ , with eigenvalue equal to  $q^{-t/2}$ .

Our proposition now follows directly from decomposing  $\mathfrak{n}^L$  into irreducible representations of the form  $S^{t_i}$ , and noting that  $\mathfrak{n}_u^L = \bigoplus_i S_u^{t_i}$ . The reducibility points can be computed from the previous proposition.  $\square$

For future reference, we tabulate many of the groups and representations considered above:

We can use this table together with our previous proposition to tabulate the reducibility points of  $I(s)$  for various triples  $(G, P, Q)$ . This is carried out in the appendix. For now, we just observe (checking case-by-case) that every reducibility point of  $I_1(s)$  is also a reducibility point of  $I(s)$ .

## 6. THE RANK OF REPRESENTATIONS

**6.1.  $N$ -Rank.** Fix a representation  $(\pi, V) \in Sm(G)$ , and consider a character  $\phi$  of  $N$ . Suppose  $N$  has rank  $r$ . The Levi component  $M$  of  $Q$  acts on such characters in the obvious manner; if  $m \in M$ , then we set  $[m\phi](n) = \phi(mnm^{-1})$ . From this action, we have vector space isomorphisms

$$V(N, \phi) \rightarrow V(N, m\phi), v \mapsto \pi(m)v$$

for any  $m \in M$ . Therefore, we see that the  $N, \phi$  co-invariants of  $V$  essentially depend only on the  $M$ -orbit of  $\phi$ ; explicitly, we have

$$V_{N,\phi} = 0 \iff V_{N,m\phi} = 0.$$

The  $M$ -orbits of characters of  $N$  correspond to the  $M$ -orbits of the opposite nilradical  $\bar{N}$ . Richardson et al. [RRS] show that over an algebraically closed field, there is exactly one orbit for each integer  $s \leq r$ . The integer  $s$  associated to an orbit is called the rank of the orbit. The rank of a character of  $N$  is defined to be the rank of its  $M(\bar{k})$ -orbit. In our case, we work with simply connected groups and over the base field  $k$ . We may describe the  $M$  orbits on  $\bar{N}$  with the following:

**Proposition 6.1.1.** *Though the orbit of  $M(\bar{k})$  on  $\bar{N}(\bar{k})$  of rank  $r$  may split into multiple orbits of  $M$  on  $\bar{N}$ , there is exactly one orbit of  $M$  on  $\bar{N}$  of each rank  $s < r$ . Equivalently, there is exactly one  $M$ -conjugacy class of characters of  $N$  of rank less than  $r$ .*

*Proof.* In Proposition 2.13 of [RRS], it is shown that every element  $\bar{n} \in \bar{N}$  is conjugate to an element of the form  $\prod_{i=0}^{s-1} u_{-\beta_i}$  for some  $s \leq r$ , where  $u_{-\beta_i}$  are contained in root subgroups associated to an orthogonal sequence  $(-\beta_0, \dots, -\beta_{s-1})$  of roots in  $\bar{N}$ . This particular statement and its proof in [RRS] carry over to our case without changes. Moreover, by conjugation by elements of the Weyl group of  $M$ , every element  $\bar{n} \in \bar{N}$  is conjugate to a product  $\prod_{i=0}^{s-1} u_{-\beta_i}$  for the canonical sequence of roots  $\beta_i$ . Thus every character of  $N$  is  $M$ -conjugate to a character of the form  $\prod_{i=0}^{s-1} \psi_i$  for nontrivial characters  $\psi_i$  of the root subgroup of  $Z_i$  of  $\beta_i$ . Here, we recall the decomposition of  $N$ ,

$$N = \prod_{i=0}^{r-1} A_i \times Z_i,$$

and consider  $\prod_{i=0}^{s-1} \psi_i$  as a character of  $N$  in the obvious way. Now, we apply induction on the rank of  $N$ . When  $N$  has rank 1, the assertion is trivial. Now suppose that  $\phi = \prod_{i=0}^{s-1} \psi_i$  is a character of rank  $s$  of  $N$ , and  $0 < s < r$ . Then  $\phi_1 = \prod_{i=1}^{s-1} \psi_i$  is a character of  $N_1$  of rank  $s - 1 < r - 1$ . Hence  $\phi_1$  is conjugate via  $M_1 = M \cap G_1$  to any given character of  $N_1$  of rank  $s - 1$  by our inductive hypothesis. Since  $G_1$  centralizes  $Z$ , conjugation of  $\phi$  by  $M_1$  does not change  $\psi_0$ . Since  $G$  is simply-laced and  $r > 1$ , so  $G \neq SL_2$ ,  $T$  acts transitively on the nontrivial characters of  $Z$ . Therefore, conjugation by  $TM_1$  is transitive on characters of the form  $\psi_0 \cdot \prod_{i=1}^{s-1} \psi_i$  and we are done.  $\square$

The following definition is a direct generalization of the work of Howe [How].

**Definition 6.1.2.** A smooth representation  $(\pi, V)$  of  $G$  has  $N$ -rank  $s$  if  $s < r$  and  $s$  is the maximal integer such that for some (and therefore every) character  $\phi_s$  of rank  $s$ ,

$$V_{N,\phi_s} \neq 0.$$

We follow the convention that if  $V = 0$ , then the  $N$ -rank of  $(\pi, V)$  is  $-1$ . If there exists a character  $\phi_r$  of rank  $r$  such that  $V_{N,\phi_r} \neq 0$ , then we say that the  $N$ -rank of  $(\pi, V)$  is  $r$ .

It is not too surprising from this definition that there is a close relationship between  $N$ -rank of a smooth representation and the Fourier-Jacobi map. A useful result is the following:

**Proposition 6.1.3.** *Let  $(\pi, V)$  be a smooth representation of  $G$ . Let  $(\pi_1, V_1)$  be the smooth representation of  $G_1$  obtained via the Fourier-Jacobi map. Then if  $s$  denotes the  $N_1$ -rank of  $(\pi_1, V_1)$ , then the  $N$ -rank of  $(\pi, V)$  is  $s + 1$ . In other words, the Fourier-Jacobi map decreases  $N$ -rank by 1.*

*Proof.* Fix nontrivial characters  $\psi_i$  of  $Z_i$  for  $0 \leq i \leq r - 1$ . We know from before that  $V_{Z, \psi_0} = V_1 \otimes W_{\psi_0}$ . If  $V_1$  has  $N_1$ -rank  $s$ , we know that

$$\begin{aligned} (V_1)_{\prod_{i=1}^t Z_i, \prod_{i=1}^t \psi_i} &= 0 \text{ if } t > s, \\ (V_1)_{\prod_{i=1}^t Z_i, \prod_{i=1}^t \psi_i} &\neq 0 \text{ if } t = s. \end{aligned}$$

Even if  $t = 0$ , the above expression makes sense, and just reads  $V_1 = 0$ . Here we note that  $(Z_1, \dots, Z_t)$  is a sequence of orthogonal root subgroups of length  $t$  in  $G_1 \cap N = N_1$ . Now the action of  $Z_i$  on  $W_\psi$  is trivial for all  $i \geq 1$ . Hence the  $(Z_i, \psi_i)$  co-invariants of  $W_\psi$  are trivial for all  $i \geq 1$ . We can deduce

$$\begin{aligned} V_{\prod_{i=0}^t Z_i, \prod_{i=0}^t \psi_i} &= (V_{Z, \psi})_{\prod_{i=1}^t Z_i, \prod_{i=1}^t \psi_i} \\ &= (V_1 \otimes W_\psi)_{\prod_{i=1}^t Z_i, \prod_{i=1}^t \psi_i} \end{aligned}$$

If  $t > s$ , then this last quantity is equal to zero, and if  $t = s$ , then this last quantity is nonzero. Therefore,  $V$  has  $N$ -rank  $s + 1$ . □

We may now completely characterize smooth representations of  $N$ -rank 0 as follows:

**Corollary 6.1.4.** *Given a smooth irreducible representation  $(\pi, V)$  of a split group  $G$ , the following conditions are equivalent:*

- (1)  $(\pi, V)$  has  $N$ -rank equal to 0.
- (2) The Fourier-Jacobi map kills  $(\pi, V)$ , i.e.,  $V_1 = 0$ .
- (3) The representation  $(\pi, V)$  is one-dimensional.

*Proof.* The equivalence of (1) and (2) follows directly from our previous proposition. So suppose that  $(\pi, V)$  is one-dimensional. Then the subgroup  $Z$  of  $G$  must act on  $V$  by some character  $\psi_0$ . Choose some  $\psi \neq \psi_0$ , so that

$$\begin{aligned} V(Z, \psi) &= \text{Span}\{\pi(z)v - \psi(z) \cdot v \mid \forall z \in Z, v \in V\}, \\ &= \text{Span}\{\psi_0(z) \cdot v - \psi(z) \cdot v \mid \forall z \in Z, v \in V\}, \\ &= V. \end{aligned}$$

We have  $V_{Z, \psi} = V/V(Z, \psi) = 0$ . Therefore, the Fourier-Jacobi map kills  $(\pi, V)$ .

Conversely, suppose that  $(\pi, V)$  is smooth and irreducible, and killed by the Fourier-Jacobi map. Then  $Z$  acts trivially on the vector space  $V$ . Now the Weyl group  $W$  of  $G$  acts transitively on the root subgroups, since we assume  $G$  is simply-laced. Hence if  $Z$  acts trivially on  $V$ , the representation  $(\pi, V)$  is irreducible even upon restriction to a maximal torus  $T$ . Irreducible smooth representations of  $T$  are all one-dimensional since  $T$  is abelian, so we are done. □

**6.2. Local Character Expansion.** Traditionally, the size of a representation of a  $p$ -adic group has been studied via the local character expansion. If  $(\pi, V)$  is a smooth admissible representation of  $G$ , then its character  $\Theta_\pi$  is an invariant distribution on  $G$ . Let  $\mathcal{O}$  denote the (finite) set of nilpotent orbits in the Lie algebra  $\mathfrak{g}$  of  $G$ . There exist canonical functions  $\hat{\mu}_\mathfrak{o}$  on the regular set for each  $\mathfrak{o} \in \mathcal{O}$

(independent of  $(\pi, V)$ ), and constants  $c_{\mathfrak{o}}$  for each  $\mathfrak{o} \in \mathcal{O}$  (depending on  $(\pi, V)$ ), such that there exists an open neighborhood  $\Xi$  of 0 in  $\mathfrak{g}$  on which

$$\Theta_{\pi}(\exp X) = \sum_{\mathfrak{o} \in \mathcal{O}} c_{\mathfrak{o}} \hat{\mu}_{\mathfrak{o}}(X),$$

for all  $X \in \Xi$ .

The set  $\mathcal{O}$  carries a natural partial ordering by setting  $\mathfrak{o} \leq \mathfrak{o}'$  if  $\mathfrak{o}$  is contained in the closure of  $\mathfrak{o}'$ . One way of measuring the size of the representation  $(\pi, V)$ , is by looking at the support of the constants  $c_{\mathfrak{o}}$ , called the wave front set:

$$WF(\pi, V) = \{\mathfrak{o} \in \mathcal{O} | c_{\mathfrak{o}} \neq 0\}.$$

For some representations  $(\pi, V)$ , the support of the local character expansion is automatically small. For instance, suppose that  $Q = MN$  is a maximal parabolic subgroup, and  $N$  is abelian as before. Let  $\mathfrak{o}_N$  denote the stable Richardson orbit, i.e., the intersection of the Richardson orbit over  $\bar{k}$  with  $\mathfrak{g}(k)$ , and suppose that  $\pi$  is a representation induced from a character of  $M$ . Then it follows that

$$WF(\pi, V) \subset \{\mathfrak{o} \in \mathcal{O} | \mathfrak{o} \subset \overline{\mathfrak{o}_N}\}.$$

In other words, the wave front set of  $(\pi, V)$  is contained in the closure of the stable Richardson orbit.

In this case, there is a close connection between  $N$ -rank and the local character expansion. Let  $\mathcal{O}_N \subset \mathcal{O}$  denote the partially ordered subset of  $\mathcal{O}$ , consisting of nilpotent orbits contained in the closure of the stable Richardson orbit  $\mathfrak{o}_N$ . Then  $\mathcal{O}_N$  is totally ordered, until the Richardson orbit; we can write

$$\mathcal{O}_N = \{0, \mathfrak{o}_1, \dots, \mathfrak{o}_{r-1}\} \cup \{\mathfrak{o} \subset \mathfrak{o}_N\}$$

where  $\mathfrak{o}_i < \mathfrak{o}_j$  precisely when  $i < j$ . The constant  $r$  is the aforementioned rank of  $N$ . The following result will be useful:

**Proposition 6.2.1.** *Suppose that  $(\pi, V)$  is a smooth representation of finite length, and  $WF(\pi, V)$  is contained in  $\mathcal{O}_N$ . Then  $WF(\pi_i, V_i)$  is contained in  $\mathcal{O}_N$  for every Jordan-Hölder component  $(\pi_i, V_i)$  of  $(\pi, V)$ .*

*Proof.* Suppose that, to the contrary, there exists a nilpotent orbit  $\mathfrak{o} \notin \mathcal{O}_N$ , such that  $\mathfrak{o}$  is contained in the wave front set of some component  $(\pi_i, V_i)$ . By the finiteness of  $\mathcal{O}$ , we may assume that  $\mathfrak{o}$  is maximal with this property. Let  $c_{\mathfrak{o}}^i$  denote the coefficient of  $\hat{\mu}_{\mathfrak{o}}$  in the character expansion of  $(\pi_i, V_i)$ . Then for any  $i$ , either  $c_{\mathfrak{o}}^i$  is equal to 0, or  $c_{\mathfrak{o}}^i$  is positive, since by maximality and results of Mœglin and Waldspurger [M-W], it is the dimension of a certain space of degenerate Whittaker models. At least one  $c_{\mathfrak{o}}^i$  must be nonzero, by definition of  $\mathfrak{o}$ ; therefore,

$$c_{\mathfrak{o}} = \sum_i c_{\mathfrak{o}}^i > 0.$$

This is a contradiction, since  $WF(\pi, V) \subset \mathcal{O}_N$ , and we are done.  $\square$

For representations whose wave front set is contained in  $\mathcal{O}_N$ , we can connect  $N$ -rank with the local character expansion of representations:

**Proposition 6.2.2.** *Suppose that  $(\pi, V)$  is an admissible representation, such that  $WF(\pi, V) \subset \mathcal{O}_N$ . Then  $(\pi, V)$  has  $N$ -rank equal to  $s$  with  $0 \leq s < r$  if and only if  $WF(\pi, V) \subset \{0, \mathfrak{o}_1, \dots, \mathfrak{o}_s\}$ .*

*Proof.* By a result of Mœglin and Waldspurger [M-W], we see that the maximal element of the wave front set of  $(\pi, V)$  is  $\mathfrak{o}_s$  if and only if  $V_{n, \phi_t} = 0$  for every character  $\phi_t$  of rank  $t > s$  (in their language, this is the nonexistence of a degenerate Whittaker model). Our proposition follows directly from this.  $\square$

**6.3. Small Representations at Reducibility Points.** The trivial representation occurs as a subrepresentation of  $I(s)$  at  $s = -D$ . Since  $s = -D$  is the smallest value of  $s$  for which  $I(s)$  is reducible, we call  $-D$  the first reducibility point. The next smallest value of  $s$  where  $I(s)$  is reducible is  $-D_1$ , and so  $-D_1$  is called the second reducibility point, and so on. We know from before that  $I(s)$  has length two at a reducibility point; the following result describes an irreducible subrepresentation:

**Proposition 6.3.1.** *If  $s < 0$  is the  $n$ -th reducibility point of  $I(s)$ , then there exists a unique irreducible spherical subrepresentation  $V_n$  of  $I(s)$ , and  $V_n$  has  $N$ -rank  $n - 1$ .*

*Proof.* At the  $n$ -th reducibility point, there is a proper subrepresentation  $V_n$  of  $I(s)$ , such that the Fourier-Jacobi map applied  $n - 1$  times sends  $V_n$  to the trivial representation. Also, there is a proper subrepresentation  $V'_n$  of  $I(s)$  which is spherical, by the comments after Proposition 5.2.1. Let  $I_{n-1}(s)$  denote the degenerate principal series representation obtained by applying the Fourier-Jacobi map  $n - 1$  times to  $I(s)$ . Then the trivial representation occurs as a subrepresentation of  $I_{n-1}(s)$ , but not as a quotient; in other words,  $I_{n-1}(s)$  does not split. This can be seen more easily by looking at  $I_{n-1}(-s)$ , where the trivial representation occurs as a quotient, but obviously not as a subrepresentation. The exactness of the Fourier-Jacobi map then implies that  $I(s)$  does not split into the direct sum of two irreducibles. Hence  $V_n = V'_n$  is the unique irreducible subrepresentation of  $I(s)$ . Since the Fourier-Jacobi map decreases  $N$ -rank by 1 each time, and the trivial representation has  $N$ -rank 0, the proposition follows.  $\square$

From the results of the previous section, we may re-interpret the above result in terms of the wave front set.

**Corollary 6.3.2.** *Let  $V_n$  denote the unique spherical subrepresentation of  $I(s)$  at the  $n$ -th reducibility point. Then the leading term of  $WF(V_n)$  is  $\mathfrak{o}_{n-1}$ . In particular,  $V_2$ , which appears as the unique spherical subrepresentation of  $I(-D_1)$ , is a minimal representation in the sense that the leading term of the wave front set is  $\mathfrak{o}_1$ .*

APPENDIX. FOURIER-JACOBI TOWERS

$G$	$M$	Reducibility points of $I(s)$
$A_{2n-1}$	$A_{n-1}^2$	<p>A horizontal line with arrows at both ends, labeled <math>s</math> at the right end. Six points are marked on the line with dots and labeled from left to right as <math>-n</math>, <math>1-n</math>, <math>-1</math>, <math>1</math>, <math>n-1</math>, and <math>n</math>. Ellipses between <math>1-n</math> and <math>-1</math>, and between <math>1</math> and <math>n-1</math> indicate intermediate points.</p>
$A_{2n-3}$	$A_{n-2}^2$	<p>A horizontal line with arrows at both ends, labeled <math>s</math> at the right end. Four points are marked on the line with dots and labeled from left to right as <math>1-n</math>, <math>-1</math>, <math>1</math>, and <math>n-1</math>. Ellipses between <math>1-n</math> and <math>-1</math>, and between <math>1</math> and <math>n-1</math> indicate intermediate points.</p>
$A_1$	$A_0^2$	<p>A horizontal line with arrows at both ends, labeled <math>s</math> at the right end. Two points are marked on the line with dots and labeled from left to right as <math>-1</math> and <math>1</math>.</p>

Fourier-Jacobi tower for  $G = SL_{2n}$ ,  $M = S(GL_n)^2$

$G$	$M$	Reducibility points of $I(s)$
$D_n$	$D_{n-1}^2$	<p>A horizontal line with arrows at both ends, labeled <math>s</math> at the right end. Four points are marked on the line with dots and labeled from left to right as <math>1-n</math>, <math>-1</math>, <math>1</math>, and <math>n-1</math>.</p>
$A_1 \times D_{n-2}$	$D_{n-2}$	<p>A horizontal line with arrows at both ends, labeled <math>s</math> at the right end. Two points are marked on the line with dots and labeled from left to right as <math>-1</math> and <math>1</math>.</p>

Fourier-Jacobi tower for  $G = Spin_{2n}$ ,  $M = CSpin_{2n-2}$

$G$	$M$	Reducibility points of $I(s)$
$D_n$	$A_{n-1}$	
$A_1 \times D_{n-2}$	$A_1 \times A_{n-3}$	
$A_1^{n/2-1} \times A_2$	$A_1^{n/2}$	

Fourier-Jacobi tower for  $G = Spin_{2n}$ ,  $M = \widetilde{GL}_n$

$G$	$M$	Reducibility points of $I(s)$
$E_6$	$D_5$	
$A_5$	$A_4$	

Fourier-Jacobi tower for  $G = E_6$ ,  $M = CSpin_{10}$

$G$	$M$	Reducibility points of $I(s)$
$E_7$	$E_6$	
$D_6$	$D_5$	
$A_1 \times D_4$	$D_4$	

Fourier-Jacobi tower for  $G = E_7$ ,  $M = CE_6$ 

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