ON SOME REPRESENTATIONS OF THE RATIONAL CHEREDNIK ALGEBRA

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Abstract. We study lowest weight representations of the rational Cherednik algebra attached to a complex reflection group $W$. In particular, we generalize a number of previous results due to Berest, Etingof, and Ginzburg.

1. Introduction

In this paper we study the structure of some lowest weight representations of the rational Cherednik algebra $H_c(W)$ attached to an irreducible complex reflection group $W$ ([EG, GGOR]).

The composition of the paper is as follows. In Section 2 we recall the definition and basic properties of the rational Cherednik algebra and its representations. Then we describe the rank 1 case, and prove a few general results. In particular, we show that for real reflection groups, a finite dimensional quotient of the polynomial representation is irreducible if and only if it is a Gorenstein algebra.

In Section 3, we consider the special case $W = S_n$. Let $M_k$ be the polynomial representation of $H_k(S_n)$. Dunkl showed in [Du] that if $r$ is a positive integer not divisible by $n$ and $k = r/n$, then $M_k$ contains a copy of the reflection representation of $S_n$ in degree $r$ which consists of singular vectors. Let $I_k$ be the $H_k(S_n)$-submodule in $M_k$ generated by these singular vectors. We compute the support of the module $M_k/I_k$ as a $\mathbb{C}[h]$-module. In particular, we show that the Gelfand-Kirillov dimension of $M_k/I_k$ is $d - 1$, where $d$ is the greatest common divisor of $r$ and $n$. In the special case $d = 1$, this implies that $M_k/I_k$ is finite dimensional. Using this fact and the results of Section 2, we give a simple proof of the result from [BEG] that the module $M_k/I_k$ has dimension $r^n - 1$ and is irreducible.

In Section 4, consider the case when $W$ is the complex reflection group $S_n \ltimes (\mathbb{Z}/l\mathbb{Z})^n$. We use a similar formula for singular vectors (due to Dunkl and Opdam, [DO]) to study finite dimensional representations of $H_c(W)$. More specifically, for each positive integer $r$ not divisible by $l$, we define a hyperplane $E_r$ in the space of the functions $c$, and for each $c \in E_r$ construct a quotient $Y_c$ of the polynomial representation $M_c$ of dimension $r^n$, which is generically irreducible. For $l = 2$, we use the results of Section 2 to obtain more precise information about the set of $c \in E_r$ for which $Y_c$ is irreducible.

Acknowledgments. The research of P.E. was partially supported by the NSF grant DMS-9988796, and was done in part for the Clay Mathematics Institute.
2. Representations of the rational Cherednik algebra
for a general complex reflection group

2.1. Definitions and notation. In this subsection we recall the standard theory of the rational Cherednik algebra \[ CG_{\mathcal{O}} \] [EG].

Let \( W \) be an irreducible complex reflection group with reflection representation \( \mathfrak{h} \) of dimension \( \ell \). Let \( c \) be a conjugation invariant complex function on the set \( S \) of complex reflections of \( W \). The rational Cherednik algebra \( \mathcal{H}_c(W) \) is generated by \( g \in W, x \in \mathfrak{h}^*, y \in \mathfrak{h} \), with defining relations

\[
\begin{align*}
g x g^{-1} &= x^g, g y g^{-1} = y^g, [x, x'] = 0, [y, y'] = 0, \\
[y, x] &= (y, x) - \sum_{s \in S} c_s(y, \alpha_s)(\alpha_s^\vee, x)s,
\end{align*}
\]

for \( x, x' \in \mathfrak{h}^*, y, y' \in \mathfrak{h}, g \in W \). Here \( \alpha_s \) is a nonzero linear function on \( \mathfrak{h} \) vanishing on the reflection hyperplane for \( s \), and \( \alpha_s^\vee \) is a linear function on \( \mathfrak{h}^* \) with the same property, such that \( (\alpha_s^\vee, \alpha_s) = 2 \).

For any irreducible representation \( \tau \) of \( W \), let \( M_c(\tau) \) be the standard representation of \( \mathcal{H}_c(W) \) with lowest weight \( \tau \); i.e., \( M_c(\tau) = \mathcal{H}_c(W) \otimes_{\mathbb{C}[W]} \mathbb{C}[\mathfrak{h}^*] \tau \), where \( \tau \) is the representation of \( CW \rtimes \mathbb{C}[\mathfrak{h}^*] \), in which \( y \in \mathfrak{h} \) act by 0. Thus, as a vector space, \( M_c(\tau) \) is naturally identified with \( \mathbb{C}[\mathfrak{h}] \otimes \tau \).

An important special case of \( M_c(\tau) \) is \( M_c(\mathbb{C}) \), the polynomial representation, corresponding to the case when \( \tau = \mathbb{C} \) is trivial. The polynomial representation can thus be naturally identified with \( \mathbb{C}[\mathfrak{h}] \). Elements of \( W \) and \( \mathfrak{h}^* \) act in this space in the obvious way, while elements \( y \in \mathfrak{h} \) act by Dunkl operators

\[
\partial_y - \sum_{s \in S} \frac{2c_s}{1 - \lambda_s} \frac{(\alpha_s, y)}{\alpha_s}(1 - s),
\]

where \( \lambda_s \) is the nontrivial eigenvalue of \( s \) in the dual reflection representation.

An important element in \( \mathcal{H}_c(W) \) is the element

\[
h = \sum_i x_i y_i + \frac{\ell}{2} - \sum_{s \in S} \frac{2c_s}{1 - \lambda_s}s,
\]

where \( y_i \) is a basis of \( \mathfrak{h} \) and \( x_i \) the dual basis of \( \mathfrak{h}^* \). This element is \( W \) invariant and satisfies the equations \( [h, x] = x \) and \( [h, y] = -y \). The category \( \mathcal{O} \) of \( \mathcal{H}_c(W) \)-modules is the category of \( \mathcal{H}_c(W) \)-modules \( \mathcal{V} \), such that \( V \) is the direct sum of finite dimensional generalized eigenspaces of \( h \), and the real part of the spectrum of \( h \) is bounded below. The standard representations \( M_c(\tau) \) and their irreducible quotients \( L_c(\tau) \) belong to \( \mathcal{O} \). The character of a module \( V \in \mathcal{O} \) is \( \chi_V(g, t) = \text{Tr}_V(g^h) \), \( g \in W \) (this is a series in \( t \)). For example, the character of \( M_c(\tau) \) is

\[
\chi_{M_c(\tau)}(g, t) = \frac{\chi_{\tau}(g) t^{h(\tau)}}{\det |_{\mathfrak{h}^*} (1 - gt)}.
\]

We note that if \( W \) is a real reflection group, then \( h \) can be included in an \( sl_2 \) triple \( h, E = \frac{1}{2} \sum x_i^2, F = \frac{1}{2} \sum y_i^2 \), where \( x_i, y_i \) are orthonormal bases of \( \mathfrak{h}^* \) and \( \mathfrak{h} \), respectively (see e.g. [BEG1], Section 3).

The module \( L_c(\tau) \) can be characterized in terms of the contragredient standard modules. Namely, let \( M_c(\tau^*) = \tau^* \otimes_{\mathbb{C}[W]} \mathbb{C}[\mathfrak{h}] H_c(W) \) be a right \( H_c(W) \)-module, and \( M_c(\tau)^\vee = \hat{M}_c(\tau)^* \) its restricted dual, which may be called the contragredient
standard module. Clearly, there is a natural morphism $\phi : M_c(\tau) \to M_c(\tau)'$. The module $L_\phi(\tau)$ is the image of $\phi$.

Note that the map $\phi$ can be viewed as a bilinear form $B : M_c(\tau) \otimes M_c(\tau) \to \mathbb{C}$. This form is analogous to the Shapovalov form in Lie theory.

If $W$ is a real reflection group, then we can fix an invariant inner product on $\mathfrak{h}$, and define an anti-involution of $H_c(W)$ by $x_i \to y_i$, $y_i \to x_i$, $g \to g^{-1}$ for $g \in W$ (where $x_i, y_i$ are orthonormal bases of $\mathfrak{h}^*$ and $\mathfrak{h}$ dual to each other). Under this anti-involution, the right module $M_c(\tau)$ turns into the left module $M_c(\tau^*)$, so the form $B$ is a (possibly degenerate) pairing $M_c(\tau^*) \otimes M_c(\tau) \to \mathbb{C}$ (note that since $W$ is a real reflection group, $\tau^*$ is always isomorphic to $\tau$). Moreover, it is clear that if $Y, Y'$ are any quotients of $M_c(\tau), M_c(\tau^*)$ respectively, then $B$ descends to a pairing $Y' \otimes Y \to \mathbb{C}$ (nondegenerate iff $Y, Y'$ are irreducible). This pairing satisfies the contravariance equations $B(a, x; b) = B(ya, b)$, $B(a, yb) = B(xa, b)$, and $B(ga, gb) = B(a, b)$.

2.2. The rank 1 case. One of the main problems in representation theory of the rational Cherednik algebra is to compute the multiplicities of $L_c(\sigma)$ in $M_c(\tau)$ or, equivalently, the characters of $L_c(\sigma)$. This problem is hard and open in general. However, in the rank one case ($\ell = 1$) this problem is trivial to solve. Nevertheless, it is an instructive example, and we will give the answer, omitting the proofs, which are straightforward.

In the rank 1 case, $W = \mathbb{Z}/l\mathbb{Z}$, and the reflection representation is $\mathbb{C}$, with the generator $s$ of $W$ acting by $\varepsilon$, where $\varepsilon$ is a primitive root of unity of degree $l$. The function $c$ is a collection of numbers $(c_1, ..., c_{l-1})$ (where $c_i = c_{ci}$), and the algebra $H_c(W)$ is generated by three generators $x, y, s$ with defining relations

$$sx = \varepsilon^{-1}xs, sy = \varepsilon ys, s^l = 1,$$

$$[y, x] = 1 - 2 \sum c_j s^j.$$

The irreducible representations of $W$ are $\eta^j$, where $\eta(s) = \varepsilon$.

Define the polynomial $f_c(\varepsilon) = \sum_{i=0}^{l-1} \frac{2c_i}{1 - \varepsilon^i} \varepsilon^i$. The lowest eigenvalue of $h$ on $M_c(\eta^j)$ is $\frac{1}{2} - f_c(\varepsilon^j)$.

**Theorem 2.1.** (i) The multiplicity of $L_c(\eta^m)$ in $M_c(\eta^p)$ is 1 if $f_c(\varepsilon^p) - f_c(\varepsilon^m)$ is a positive integer congruent to $p - m$ modulo $l$, and zero otherwise.

(ii) If $L_c(\eta^p) \neq M_c(\eta^p)$, then $L_c(\eta^p)$ is finite dimensional, and the character of $L_c(\eta^p)$ is

$$Tr(s^j t^h) = \varepsilon^{pjt} t^{\frac{1}{2} - f_c(\varepsilon^p) \frac{1 - p\varepsilon^{-bj}}{1 - \varepsilon^{-j}}}.$$

where $b$ is the smallest positive integer of the form $f_c(\varepsilon^p) - f_c(\varepsilon^m)$ congruent to $p - m$ modulo $l$.

2.3. The Gorenstein property. Any submodule $J$ of the polynomial representation $M_c = \mathbb{C}[h]$ is an ideal in $\mathbb{C}[h]$, so the quotient $A = M_c/J$ is a $\mathbb{Z}_+$-graded commutative algebra.

Now suppose that $W$ is a real reflection group. Recall that it was shown in [BEC], Proposition 1.13, that if $A$ is irreducible (i.e., $A = L_c$, the irreducible quotient of $M_c$), then $A$ is a Gorenstein algebra (see [E], pp. 529 and 532 for definitions). Here we prove the converse statement.
Theorem 2.2. If $A = M_c/J$ is finite dimensional and Gorenstein, then $A = L_c$ (i.e. $A$ is irreducible).

Proof. Since $A$ is Gorenstein, the highest degree component of $A$ is 1-dimensional, and the pairing $E : A \otimes A \to \mathbb{C}$ given by $E(a, b) := h.c.(ab)$ (where h.c. stands for the highest degree coefficient) is nondegenerate. This pairing obviously satisfies the condition $E(xa, b) = E(a, xb), x \in \mathfrak{h}^*$. Now recall that $H_c(W)$ and $A$ admits a natural action of the group $SL_2(\mathbb{C})$. Let $g = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \in SL_2(\mathbb{C})$; then $g(x_i) = y_i$ for orthonormal bases $x_i, y_i$ of $\mathfrak{h}^*$ and $\mathfrak{h}$ which are dual to each other. Thus the nondegenerate form $\tilde{B}(a, b) := E(a, gb)$ on $A$ satisfies the equations $\tilde{B}(a, x; b) = \tilde{B}(y, a, b)$. So for any $f_1, f_2 \in \mathbb{C}[\mathfrak{h}]$, one has $\tilde{B}(f_1(x)v, f_2(x)v) = \tilde{B}(f_2(y)f_1(x)v, v)$, where $v = 1$ is the lowest weight vector of $A$. This shows that $\tilde{B}$ coincides with the Shapovalov form $B$ on $A$. Thus $A$ is an irreducible representation of $H_c(W)$. □

Remark. It easy to see by considering the rank 1 case that for complex reflection groups Theorem 2.2 is, in general, false.

Theorem 2.3. Let $W$ be a complex reflection group, and $U \subset M_c$ be a $W$-subrepresentation of dimension $\ell = \dim(\mathfrak{h})$ sitting in degree $r$, consisting of singular vectors (i.e. those killed by $y \in \mathfrak{h}$). Let $J$ be the ideal generated by $U$. Assume that the quotient representation $A = M_c/J$ is finite dimensional. Then

(i) The algebra $A$ is Gorenstein.

(ii) The representation $A$ admits a BGG type resolution

$$A \leftarrow M_c(\mathbb{C}) \leftarrow M_c(U) \leftarrow M_c(\wedge^2 U) \leftarrow \ldots \leftarrow M_c(\wedge^\ell U) \leftarrow 0.$$ 

(iii) The character of $A$ is given by the formula

$$\chi_A(g, t) = t^r - \sum c_{\lambda} \det [U(1 - gt^r)] / \det h^*(1 - gt).$$

In particular, the dimension of $A$ is $r^\ell$.

(iv) If $W$ is a real reflection group, then $A$ is irreducible.

Proof. (i) Since Spec($A$) is a complete intersection, $A$ is Gorenstein ([E], p. 541).

(ii) Consider the subring $\mathbb{C}[U]$ in $\mathbb{C}[\mathfrak{h}]$. Then $\mathbb{C}[\mathfrak{h}]$ is a finitely generated $\mathbb{C}[U]$-module. A standard theorem of Serre [S] says that if $B = \mathbb{C}[t_1, ..., t_n], f_1, ..., f_n \in B$ are homogeneous, $A = \mathbb{C}[f_1, ..., f_n] \subset \tilde{B}$, and $B$ is a finitely generated module over $A$, then $B$ is a free module over $A$. Applying this in our situation, we see that $\mathbb{C}[\mathfrak{h}]$ is a free $\mathbb{C}[U]$-module. It is easy to see by computing the Hilbert series that the rank of this free module is $r^\ell$.

Consider the Koszul complex attached to the module $\mathbb{C}[\mathfrak{h}]$ over $\mathbb{C}[U]$. Since this module is free, the Koszul complex is exact (i.e. it is a resolution of the zero-fiber). At the level of $\mathbb{C}[\mathfrak{h}]$ modules, this resolution looks exactly as we want in (ii). So we need to show that the maps of the resolution are in fact morphisms of $H_c(W)$-modules and not only $\mathbb{C}[\mathfrak{h}]$-modules. This is easily established by induction (going from left to right); cf. proof of Proposition 2.2 in [BEG] and also [Go].

(iii) Follows from (ii) by the Euler-Poincare principle.

(iv) Follows from Theorem 2.2 □
3. Representations of the rational Cherednik algebra of type $A$

3.1. The results. Let $W = S_n$. In this case the function $c$ reduces to one number $k$. We will denote the rational Cherednik algebra $H_k(S_n)$ by $H_k(n)$. The polynomial representation $M_k$ of this algebra is the space of $\mathbb{C}[x_1, \ldots, x_n]^T$ of polynomials of $x_1, \ldots, x_n$, which are invariant under simultaneous translation $x_i \mapsto x_i + a$. In other words, it is the space of regular functions on $\mathfrak{h} = \mathbb{C}^n/\Delta$, where $\Delta$ is the diagonal.

**Proposition 3.1** (Du). Let $r$ be a positive integer not divisible by $n$, and $k = r/n$. Then $M_k$ contains a copy of the reflection representation $\mathfrak{h}$ of $S_n$, which consists of singular vectors (i.e., those killed by $y \in \mathfrak{h}$). This copy sits in degree $r$ and is spanned by the functions

$$f_i(x_1, \ldots, x_n) = \text{Res}_\infty[(z-x_1)(z-x_n)]^{r/n} \frac{dz}{z-x_i}$$

(the symbol $\text{Res}_\infty$ denotes the residue at infinity).

**Remark.** The space spanned by $f_i$ is $n-1$-dimensional, since $\sum_i f_i = 0$ (this sum is the residue of an exact differential).

**Proof.** This proposition can be proved by a straightforward computation. The functions $f_i$ are a special case of Jack polynomials. \hfill $\square$

Let $I_k$ be the submodule of $M_k$ generated by $f_i$. Consider the $H_k(n)$-module $V_k = M_k/I_k$, and regard it as a $\mathbb{C}[\mathfrak{h}]$-module.

Our result is

**Theorem 3.2.** Let $d = (r, n)$ denote the greatest common divisor of $r$ and $n$. Then the (set-theoretical) support of $V_k$ is the union of $S_n$-translates of the subspaces of $\mathbb{C}^n/\Delta$, defined by the equations

$$x_1 = x_2 = \cdots = x_{\frac{r}{d}};$$

$$x_{\frac{r}{d}+1} = \cdots = x_{2\frac{r}{d}};$$

$$\cdots$$

$$x_{(d-1)\frac{r}{d}+1} = \cdots = x_n.$$

In particular, the Gelfand-Kirillov dimension of $V_k$ is $d - 1$.

The theorem allows us to give a simple proof of the following result of [BEG] (without the use of the KZ functor and Hecke algebras).

**Corollary 3.3** (BEG). If $d = 1$, then the module $V_k := M_k/I_k$ is finite dimensional, irreducible, admits a BGG type resolution, and its character is

$$\chi_{V_k}(g, t) = t^{1-r(n-1)/2} \frac{\det |_{\mathfrak{h}}(1-g^T)}{\det |_{\mathfrak{h}}(1-g)}.$$

**Proof.** For $d = 1$ Theorem 3.2 says that the support of $M_k/I_k$ is $\{0\}$. This implies that $M_k/I_k$ is finite dimensional. The rest follows from Theorem 2.3. \hfill $\square$

3.2. Proof of Theorem 3.2. The support of $V_k$ is the zero-set of $I_k$, i.e., the common zero set of $f_i$. Fix $x_1, \ldots, x_n \in \mathbb{C}$. Then $f_i(x_1, \ldots, x_n) = 0$ for all $i$ iff
\[
\sum_{i=1}^{n} \lambda_i f_i = 0 \quad \text{for all } \lambda_i, \text{ i.e.,}
\]
\[
\text{Res}_{\infty} \left( \prod_{j=1}^{n} (z - x_j)^{\frac{r}{n}} \sum_{i=1}^{n} \frac{\lambda_i}{z - x_i} \right) dz = 0.
\]

Assume that \(x_1, \ldots, x_n\) take distinct values \(y_1, \ldots, y_p\) with positive multiplicities \(m_1, \ldots, m_p\). The previous equation implies that the point \((x_1, \ldots, x_n)\) is in the zero set iff
\[
\text{Res}_{\infty} \prod_{j=1}^{p} (z - y_j)^{m_j \frac{r}{p}-1} \left( \sum_{i=1}^{p} \nu_i (z - y_1) \ldots (z - y_i) \ldots (z - y_p) \right) dz = 0 \quad \forall \nu_i.
\]

Since \(\nu_i\) are arbitrary, this is equivalent to the condition
\[
\text{Res}_{\infty} \prod_{j=1}^{p} (z - y_j)^{m_j \frac{r}{p}-1} z^i dz = 0, \quad i = 0, \ldots, p - 1.
\]

We will now need the following lemma.

**Lemma 3.4.** Let \(a(z) = \prod_{j=1}^{p} (z - y_j)^{\mu_j}\), where \(\mu_j \in \mathbb{C}, \sum_j \mu_j \in \mathbb{Z}\) and \(\sum_j \mu_j > -p\). Suppose
\[
\text{Res}_{\infty} a(z) z^i dz = 0, \quad i = 0, 1, \ldots, p - 2.
\]
Then \(a(z)\) is polynomial.

**Proof.** Let \(g\) be a polynomial. Then
\[
0 = \text{Res}_{\infty} d(g(z) \cdot a(z)) = \text{Res}_{\infty} (g'(z) a(z) + a'(z) g(z)) dz
\]
and hence
\[
\text{Res}_{\infty} \left( g'(z) + \sum_i \frac{\mu_j}{z - y_j} g(z) \right) a(z) dz = 0.
\]

Let \(g(z) = z^{l} \prod_j (z - y_j)\). Then \(g'(z) + \sum_i \frac{\mu_j}{z - y_j} g(z)\) is a polynomial of degree \(l + p - 1\) with highest coefficient \(l + p + \sum \mu_j \neq 0\) (as \(\sum \mu_j > -p\)). This means that for every \(l \geq 0\), \(\text{Res}_{\infty} z^{l+p-1} a(z) dz\) is a linear combination of residues of \(z^q a(z) dz\) with \(q < l + p - 1\). By the assumption of the lemma, this implies by induction in \(l\) that all such residues are 0 and hence \(a\) is a polynomial.

In our case \(\sum (m_j \frac{n}{d} - 1) = r - p\) (since \(\sum m_j = n\)) and the conditions of the lemma are satisfied. Hence \((x_1, \ldots, x_n)\), is in the zero set of \(I_k\) iff \(\prod_{j=1}^{p} (z - y_j)^{m_j \frac{r}{p}-1}\) is a polynomial. This is equivalent to saying that all \(m_j\) are divisible by \(\frac{r}{d}\).

We have proved that \((x_1, \ldots, x_n)\) is in the zero set of \(I_k\) iff \((z - x_1) \ldots (z - x_n)\) is the \(n/d\)-th power of a polynomial of degree \(d\). This implies the theorem.
4. Representations of the rational Cherednik algebra for the complex reflection group $S_n \ltimes (\mathbb{Z}/l\mathbb{Z})^n$.

4.1. The formula for the singular vector. Let $l \geq 2$ be an integer. Consider the complex reflection group $W = S_n \ltimes (\mathbb{Z}/l\mathbb{Z})^n$ acting on the $n$-dimensional space. The set $S \subset W$ of complex reflections consists of the elements $s_i^m$ and $\sigma_{i,j}^{(m)}$ defined by

$$s_i^m(x_1, \ldots, x_n) = (x_1, \ldots, \varepsilon^m x_i, \ldots, x_n), \quad 1 \leq m < l;$$

$$\sigma_{i,j}^{(m)}(x_1, \ldots, x_i, \ldots, x_j, \ldots, x_n) = (x_1, \ldots, \varepsilon^m x_j, \ldots, \varepsilon^{-m} x_i, \ldots, x_n), \quad 0 \leq m < l$$

(here $\varepsilon$ is a primitive $l$-th root of unity). Consider the $W$-invariant function $c : S \rightarrow \mathbb{C}$ defined by $c(s_i^m) = c_m$ for all $i$ and $c(\sigma_{i,j}^{(m)}) = k$ for all $i, j, m$ (where $k$ and $\{c_m\}$ are fixed constants). Let $\mathcal{H}_c(n, l)$ denote the rational Cherednik algebra $\mathcal{H}_c(W)$ corresponding to $W$ and $c$. Let $M_c$ be the polynomial representation of this algebra.

Fix a positive integer $r$, which is not divisible by $l$. Thus $r = (p-1)l + q$, where $p$ is a positive integer, and $1 \leq q \leq l-1$ is an integer. Denote by $E_r$ the affine hyperplane of those functions $c$ for which $l(n-1)k + 2 \sum_{j=1}^{l-1} c_{j+1} \frac{1 - e^{-rj}}{1 - e^{-j}} = r$. Let $\mathfrak{h}_q$ be the representation of $W$ on $\mathbb{C}^n$ in which $S_n$ acts by permutations, and $s_i$ multiplies the $i$-th coordinate of a vector by $e^{-q}$ (thus the reflection representation corresponds to $q = l-1$ and the dual reflection representation to $q = 1$). Let $s$ be the largest integer which is $< p/n$.

We have the following analog of Proposition [31].

**Proposition 4.1.** For $c \in E_r$, the polynomial representation $M_c$ of $\mathcal{H}_c(n, l)$ contains a copy of representation $\mathfrak{h}_q$ of $W$ in degree $r$ consisting of singular vectors. This copy is spanned by the functions $f_i$, where

$$f_i(x_1, \ldots, x_n) = \text{Res}_x \frac{z^{(p-nk)l-1}}{(k-1) \ldots (k-s)} \prod_{j=1}^n \frac{x_j^q dz}{z^l - x_i^l}. $$

**Remark.** Clearly, if $k = 1, \ldots, s$, then

$$\text{Res}_x z^{(p-nk)l-1} \prod_{j=1}^n \frac{x_j^q dz}{z^l - x_i^l} = 0.$$ 

Thus the functions $f_i$ are polynomial in $k$. Moreover, they do not vanish identically for any $k$, as easily seen by computing the coefficients.

**Proof.** This proposition can be proved by direct computation. It can also be obtained as a simple consequence of the results of [DO], Section 3. \hfill $\square$

4.2. Finite dimensional representations of $\mathcal{H}_c(n, l)$. Let $U_r = \mathbb{C}[w]/(w^r)$. This is naturally a $\mathbb{Z}_r$-graded representation of $\mathbb{Z}/l\mathbb{Z}$, $m \circ w^j = \varepsilon^{-mj} w^j$. Thus $U_r \otimes \mathfrak{h}_q$ is a graded representation of $W$. The character of this representation is $\text{Tr} \psi_g \otimes \mathfrak{h}_q = \frac{\det \psi_g(x_1 - x_1')}{\det \psi_g(1 - x_1) \ldots (1 - x_s)}$, where $D$ is the grading operator.

**Theorem 4.2.** (i) For $c \in E_r$, there exists a lowest weight module $Y_c$ over $\mathcal{H}_c(n, l)$ with trivial lowest weight (i.e. a quotient of $M_c$) which is isomorphic to $U_r \otimes \mathfrak{h}_q$ as a graded $W$-module.

(ii) For generic $c \in E_r$, $Y_c$ is irreducible.
Proof. Let \( I_c \) be the submodule generated by \( f_i \), and \( \bar{Y}_c = M_c/I_c \).

If \( k = 0 \), a direct computation shows that \( f_i(x_1, \ldots, x_n) = C x_i^r \), where \( C \) is a nonzero constant. In this case, \( H_c(n, l) = \mathbb{C}[S_n] \times (H_{c_1}, \ldots, c_{l-1}) = (1, l)^{\otimes n} \), so \( \bar{Y}_c \) is simply \( U_r^{\otimes n} \), where \( U_r \) is the \( r \)-dimensional lowest weight module over \( H_{c_1}, \ldots, c_{l-1} \) (with trivial lowest weight) which exists when \( \sum_{j=1}^{l-1} c_j \prod_{i=1}^l (1-x_i)^{-c_j} = r \). It follows from Theorem 2.1 that for generic \( c_1, \ldots, c_{l-1} \) satisfying this equation, the module \( U_r \) is irreducible. Thus, for generic \( c \in E_r \cap \{ k = 0 \} \), the module \( \bar{Y}_c \) is irreducible (of dimension \( r^n \)), and the kernel of the Shapovalov form on \( M_c \) coincides with \( I_c \).

Using standard semicontinuity arguments, we conclude from this that for generic \( c \in E_r \), \( \dim(\bar{Y}_c) \leq r^n \).

On the other hand, let \( L_c \) denote the irreducible module over \( H_c(n, l) \) with trivial lowest weight, i.e., the quotient of \( M_c \) by the kernel of the Shapovalov form. The previous argument shows that for generic \( c \in E_r \), \( \dim L_c \geq r^n \). But \( L_c \) is clearly a quotient of \( \bar{Y}_c \). This implies that for generic \( c \in E_r \), \( L_c = \bar{Y}_c \), and in particular \( \bar{Y}_c \) is irreducible.

To finish the proof, observe that the submodule \( I_c \) really depends only on one parameter \( k \). Therefore, we can define \( Y_c \) by setting, for generic \( k \), \( Y_c := \bar{Y}_c = M_c/I_c \), and for special \( k \) (i.e. finitely many values), \( Y_c = M_c/I_c \), where \( I_c = \lim_{c \to c} I_c \) (the limit exists since any regular map from \( \mathbb{P}^l \setminus \{ k_1, \ldots, k_N \} \) to a Grassmannian can be extended to the whole \( \mathbb{P}^l \)). Clearly, \( Y_c \) is the lowest weight \( H_c(n, l) \)-module, satisfying the conditions of the theorem. Thus the theorem is proved. \( \square \)

Remark. Let \( l = 2 \). In this case our condition on \( r \) is that \( r \) is odd, and the equation of \( E_r \) has the form \( (n - 1)k + 2c_1 = r \). Thus, in this case we recover Theorem 6.1 from [BEG].

4.3. The dimension of \( \bar{Y}_c \). Let

\[
\Sigma_r = \{ \frac{P}{Q} \mid P, Q \text{ integers and } (P, Q) = 1, \ 1 \leq P \leq p - 1, \ 1 \leq Q \leq n \}
\]

(where \( r = (p - 1)l + q \)), and let \( \bar{Y}_c, Y_c \) be as defined in the previous subsection.

Theorem 4.3. (i) \( \bar{Y}_c \) is finite dimensional if and only if \( k \notin \Sigma_r \).

(ii) \( \bar{Y}_c \) is finite dimensional if and only if \( \bar{Y}_c = Y_c \).

(iii) If \( l = 2 \) and \( k \notin \Sigma_r \), then \( Y_c \) is irreducible.

Proof. (i) It suffices to prove that the support of \( \bar{Y}_c \) is \( \{ 0 \} \) iff \( k \notin \Sigma_r \). The proof will be analogous to that of Theorem 3.2.

The support of \( \bar{Y}_c \) is the common zero set of the functions \( f_i \). Fix \( x_1, \ldots, x_n \in \mathbb{C} \).

We have \( f_i(x_1, \ldots, x_n) = 0 \) for all \( i \) if \( \sum_{i=1}^n \lambda_i f_i(x_1, \ldots, x_n) = 0 \) for all \( \lambda_i \), i.e.,

\[
\text{Res}_z \left( z^a \prod_{j=1}^n (z^l - x_j^l) \right)^k \sum_{i=1}^n \frac{\lambda_i x_i^q}{z^l - x_i^l} \right) dz = 0,
\]

where \( a = (p - nk)l \).

Let \( d \geq 0 \) be the number of distinct nonzero numbers among \( x_1^l, \ldots, x_n^l \). More specifically, assume that \( x_1^l, \ldots, x_n^l \) take values \( y_0 = 0, y_1, \ldots, y_d \) with multiplicities \( m_0, m_1, \ldots, m_d \), such that \( m_j > 0 \) if \( j > 0 \) (so \( \sum_{i=0}^d m_j = n \)). From the above
equation we see that the point \((x_1, \ldots, x_n)\) is in the zero set if and only if for any \(\nu_i, i = 1, \ldots, d\)

\[
\text{Res}_{\infty} \left[ z^{a+\ell m_0} \prod_{j=1}^{d} (z^j - y_j)^{m_j k} \right] \left( \sum_{i=1}^{d} \nu_i (z^i - y_1) \cdots (z^i - y_i) \cdots (z^i - y_d) \right) z^{-1} dz = 0.
\]

This equation is equivalent to

\[
\text{Res}_{\infty} \left( z^{a+(m_0 k + i)d} \prod_{j=1}^{d} (z^j - y_j)^{m_j k} \right) z^{-1} dz = 0, \quad i = 0, \ldots, d - 1.
\]

Let’s make a change of variables \(w = z^j\). Then the equations above will take the form

\[
\text{Res}_{\infty} \left( (w - y_0)^{p-(n-m_0)k} \prod_{j=1}^{d} (w - y_j)^{m_j k} \right) dw = 0, \quad i = 0, \ldots, d - 1.
\]

Applying Lemma 3.3 (whose conditions are clearly satisfied) we get that these equations hold iff the function

\[
F(w) := w^{p-(n-m_0)k} \prod_{j=1}^{d} (w - y_j)^{m_j k}
\]

is a polynomial.

The function \(F(w)\) is a polynomial iff \(p - (n - m_0)k - 1\) and \(m_j k - 1\) for all \(1 \leq j \leq d\) are nonnegative integers.

Suppose \(k \not\in \mathbb{Q}\) or \(k = \frac{P}{Q}\) with \((P, Q) = 1\) and \(Q > n\). Then there is no integer \(1 \leq m \leq n\), such that \(mk - 1 \in \mathbb{Z}\). This means that for such \(k\) the function \(F\) is a polynomial only if \(d = 0\). Hence the support of \(\tilde{Y}_c\) is \(\{0\}\).

Suppose \(k = \frac{P}{Q}\) with \((P, Q) = 1\) and \(Q \leq n\). Let \(M_j = km_j\). The condition that \(F\) is a polynomial implies that \(M_j\) are positive integers and \(p - 1 - \sum_{j=1}^{d} M_j \geq 0\). This means that either \(d = 0\) or \(P \leq p - 1\), i.e., \(k \in \Sigma_r\). Thus, if the support of \(\tilde{Y}_c\) is nonzero, then \(k \in \Sigma_r\).

Conversely, let \(k = \frac{P}{Q} \in \Sigma_r\). Take \(d = 1, m_1 = Q, m_0 = n - Q\), and choose \(y_1 \neq 0\) arbitrarily. Then \(F\) is a polynomial. So the support of \(\tilde{Y}_c\) in this case is nonzero.

Thus, statement (i) is proved.

(ii) Clearly, \(\tilde{Y}_c\) is a quotient of \(\tilde{Y}_r\). The rest follows from Theorem 2.3.

(iii) If \(l = 2\), then \(W\) is a reflection group. Thus the result follows from Theorem 2.3.

Part (iii) of Theorem 4.3 generalizes Proposition 6.4 of [BEG].

Remark. Part (iii) of Theorem 4.3 fails for complex reflection groups, as seen from considering the rank 1 case: in this case \(\Sigma_r\) is empty, but \(Y_c\) is not always irreducible. Similarly, one cannot drop in part (iii) the assumption that \(k \not\in \Sigma_r\); this is demonstrated by Subsection 6.4 in [BEG].
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