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ON SOME REPRESENTATIONS OF THE RATIONAL CHEREDNIK ALGEBRA

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ABSTRACT. We study lowest weight representations of the rational Cherednik algebra attached to a complex reflection group W. In particular, we generalize a number of previous results due to Berest, Etingof, and Ginzburg.

1. INTRODUCTION

In this paper we study the structure of some lowest weight representations of the rational Cherednik algebra $H_c(W)$ attached to an irreducible complex reflection group W ([EG, GGOR]).

The composition of the paper is as follows. In Section 2 we recall the definition and basic properties of the rational Cherednik algebra and its representations. Then we describe the rank 1 case, and prove a few general results. In particular, we show that for real reflection groups, a finite dimensional quotient of the polynomial representation is irreducible if and only if it is a Gorenstein algebra.

In Section 3, we consider the special case $W = S_n$. Let M_k be the polynomial representation of $H_k(S_n)$. Dunkl showed in [Du] that if r is a positive integer not divisible by n and $k = r/n$, then M_k contains a copy of the reflection representation of S_n in degree r which consists of singular vectors. Let I_k be the $H_k(S_n)$-submodule in M_k generated by these singular vectors. We compute the support of the module M_k/I_k as a $\mathbb{C}[\mathfrak{h}]$-module. In particular, we show that the Gelfand-Kirillov dimension of M_k/I_k is $d-1$, where d is the greatest common divisor of r and n. In the special case $d = 1$, this implies that M_k/I_k is finite dimensional. Using this fact and the results of Section 2, we give a simple proof of the result from [BEG] that the module M_k/I_k has dimension r^{n-1} and is irreducible.

In Section 4, consider the case when W is the complex reflection group $S_n \ltimes (\mathbb{Z}/l\mathbb{Z})^n$. We use a similar formula for singular vectors (due to Dunkl and Opdam, [DO]) to study finite dimensional representations of $H_c(W)$. More specifically, for each positive integer r not divisible by l, we define a hyperplane E_r in the space of the functions c, and for each $c \in E_r$ construct a quotient Y_c of the polynomial representation M_c of dimension r^n, which is generically irreducible. For $l = 2$, we use the results of Section 2 to obtain more precise information about the set of $c \in E_r$ for which Y_c is irreducible.

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2. Representations of the rational Cherednik algebra for a general complex reflection group

2.1. **Definitions and notation.** In this subsection we recall the standard theory of the rational Cherednik algebra [GGOR, EG].

Let W be an irreducible complex reflection group with reflection representation $\mathfrak{h}$ of dimension ℓ. Let c be a conjugation invariant complex function on the set S of complex reflections of W. The rational Cherednik algebra $H_c(W)$ is generated by $g \in W$, $x \in \mathfrak{h}^*, y \in \mathfrak{h}$, with defining relations

$$gxg^{-1} = x^g, gyg^{-1} = y^g, [x, x'] = 0, [y, y'] = 0,$$
$$[y, x] = (y, x) - \sum_{s \in S} c_s (y, \alpha_s)(\alpha_s^\vee, x) s,$$

for $x, x' \in \mathfrak{h}^*$, $y, y' \in \mathfrak{h}$, $g \in W$. Here α_s is a nonzero linear function on $\mathfrak{h}$ vanishing on the reflection hyperplane for s, and $\alpha_s^\vee$ is a linear function on $\mathfrak{h}^*$ with the same property, such that $(\alpha_s^\vee, \alpha_s) = 2$.

For any irreducible representation τ of W, let $M_c(\tau)$ be the standard representation of $H_c(W)$ with lowest weight τ; i.e., $M_c(\tau) = H_c(W) \otimes_{\mathbb{C}[W] \ltimes \mathbb{C}[\mathfrak{h}^*]} \tau$, where τ is the representation of $\mathbb{C}W \ltimes \mathbb{C}[\mathfrak{h}^*]$, in which $y \in \mathfrak{h}$ act by 0. Thus, as a vector space, $M_c(\tau)$ is naturally identified with $\mathbb{C}[\mathfrak{h}] \otimes \tau$.

An important special case of $M_c(\tau)$ is $M_c = M_c(\mathbb{C})$, the polynomial representation, corresponding to the case when $\tau = \mathbb{C}$ is trivial. The polynomial representation can thus be naturally identified with $\mathbb{C}[\mathfrak{h}]$. Elements of W and $\mathfrak{h}^*$ act in this space in the obvious way, while elements $y \in \mathfrak{h}$ act by Dunkl operators

$$\partial_y - \sum_{s \in S} \frac{2c_s}{1 - \lambda_s} \frac{(\alpha_s, y)}{\alpha_s}(1 - s),$$

where λ_s is the nontrivial eigenvalue of s in the dual reflection representation.

An important element in $H_c(W)$ is the element

$$\mathbf{h} = \sum_i x_i y_i + \frac{\ell}{2} - \sum_{s \in S} \frac{2c_s}{1 - \lambda_s} s,$$

where y_i is a basis of $\mathfrak{h}$ and x_i the dual basis of $\mathfrak{h}^*$. This element is W-invariant and satisfies the equations $[\mathbf{h}, x] = x$ and $[\mathbf{h}, y] = -y$. The category $\mathcal{O}$ of $H_c(W)$-modules is the category of $H_c(W)$-modules V, such that V is the direct sum of finite dimensional generalized eigenspaces of $\mathbf{h}$, and the real part of the spectrum of $\mathbf{h}$ is bounded below. The standard representations $M_c(\tau)$ and their irreducible quotients $L_c(\tau)$ belong to $\mathcal{O}$. The character of a module $V \in \mathcal{O}$ is $\chi_V(g, t) = \mathrm{Tr}|_V(g t^{\mathbf{h}})$, $g \in W$ (this is a series in t). For example, the character of $M_c(\tau)$ is

$$\chi_{M_c(\tau)}(g, t) = \frac{\chi_\tau(g) t^{h(\tau)}}{\det|_{\mathfrak{h}^*}(1 - gt)},$$

where $h(\tau) := \ell/2 - \sum_s \frac{2c_s}{1-\lambda_s} s|_\tau$ is the lowest eigenvalue of $\mathbf{h}$ in $M_c(\tau)$.

We note that if W is a real reflection group, then $\mathbf{h}$ can be included in an sl_2 triple $\mathbf{h}, \mathbf{E} = \frac{1}{2}\sum x_i^2, \mathbf{F} = \frac{1}{2}\sum y_i^2$, where x_i, y_i are orthonormal bases of $\mathfrak{h}^*$ and $\mathfrak{h}$, respectively (see e.g. [BEG1], Section 3).

The module $L_c(\tau)$ can be characterized in terms of the contragredient standard modules. Namely, let $\hat{M}_c(\tau) = \tau^* \otimes_{\mathbb{C}[W] \ltimes \mathbb{C}[\mathfrak{h}]} H_c(W)$ be a right $H_c(W)$-module, and $M_c(\tau)^\vee = \hat{M}_c(\tau)^*$ its restricted dual, which may be called the contragredient standard module. Clearly, there is a natural morphism $\phi : M_c(\tau) \to M_c(\tau)^\vee$. The module $L_c(\tau)$ is the image of ϕ.

Note that the map ϕ can be viewed as a bilinear form $B : \hat{M}_c(\tau) \otimes M_c(\tau) \to \mathbb{C}$. This form is analogous to the Shapovalov form in Lie theory.

If W is a real reflection group, then we can fix an invariant inner product on $\mathfrak{h}$, and define an anti-involution of $H_c(W)$ by $x_i \to y_i$, $y_i \to x_i$, $g \to g^{-1}$ for $g \in W$ (where x_i, y_i are orthonormal bases of $\mathfrak{h}^*$ and $\mathfrak{h}$ dual to each other). Under this anti-involution, the right module $\hat{M}_c(\tau)$ turns into the left module $M_c(\tau^*)$, so the form B is a (possibly degenerate) pairing $M_c(\tau^*) \otimes M_c(\tau) \to \mathbb{C}$ (note that since W is a real reflection group, τ^* is always isomorphic to τ). Moreover, it is clear that if Y, Y' are any quotients of $M_c(\tau), M_c(\tau^*)$ respectively, then B descends to a pairing $Y' \otimes Y \to \mathbb{C}$ (nondegenerate iff Y, Y' are irreducible). This pairing satisfies the contravariance equations $B(a, x_i b) = B(y_i a, b)$, $B(a, y_i b) = B(x_i a, b)$, and $B(ga, gb) = B(a, b)$.

2.2. **The rank 1 case.** One of the main problems in representation theory of the rational Cherednik algebra is to compute the multiplicities of $L_c(\sigma)$ in $M_c(\tau)$ or, equivalently, the characters of $L_c(\sigma)$. This problem is hard and open in general. However, in the rank one case ($\ell = 1$) this problem is trivial to solve. Nevertheless, it is an instructive example, and we will give the answer, omitting the proofs, which are straightforward.

In the rank 1 case, $W = \mathbb{Z}/l\mathbb{Z}$, and the reflection representation is $\mathbb{C}$, with the generator s of W acting by ε, where ε is a primitive root of unity of degree l. The function c is a collection of numbers $(c_1, ..., c_{l-1})$ (where $c_i = c_{s^i}$), and the algebra $H_c(W)$ is generated by three generators x, y, s with defining relations

$$sx = \varepsilon^{-1} xs, sy = \varepsilon ys, s^l = 1,$$
$$[y, x] = 1 - 2 \sum c_j s^j.$$

The irreducible representations of W are η^j, where $\eta(s) = \varepsilon$.

Define the polynomial $f_c(z) = \sum \frac{2c_j}{1-\varepsilon^{-j}} z^j$. The lowest eigenvalue of $\mathbf{h}$ on $M_c(\eta^j)$ is $\frac{1}{2} - f_c(\varepsilon^j)$.

Theorem 2.1. *(i) The multiplicity of $L_c(\eta^m)$ in $M_c(\eta^p)$ is 1 if $f_c(\varepsilon^p) - f_c(\varepsilon^m)$ is a positive integer congruent to $p - m$ modulo l, and zero otherwise.*

(ii) If $L_c(\eta^p) \neq M_c(\eta^p)$, then $L_c(\eta^p)$ is finite dimensional, and the character of $L_c(\eta^p)$ is

$$Tr(s^j t^{\mathbf{h}}) = \varepsilon^{pj} t^{\frac{1}{2} - f(\varepsilon^p)} \frac{1 - t^b \varepsilon^{-bj}}{1 - t\varepsilon^{-j}},$$

where b is the smallest positive integer of the form $f_c(\varepsilon^p) - f_c(\varepsilon^m)$ congruent to $p - m$ modulo l.

2.3. **The Gorenstein property.** Any submodule J of the polynomial representation $M_c = \mathbb{C}[\mathfrak{h}]$ is an ideal in $\mathbb{C}[\mathfrak{h}]$, so the quotient $A = M_c/J$ is a $\mathbb{Z}_+$-graded commutative algebra.

Now suppose that W is a real reflection group. Recall that it was shown in [BEG], Proposition 1.13, that if A is irreducible (i.e., $A = L_c$, the irreducible quotient of M_c), then A is a Gorenstein algebra (see [E], pp. 529 and 532 for definitions). Here we prove the converse statement.

Theorem 2.2. *If $A = M_c/J$ is finite dimensional and Gorenstein, then $A = L_c$ (i.e. A is irreducible).*

Proof. Since A is Gorenstein, the highest degree component of A is 1-dimensional, and the pairing $E : A \otimes A \to \mathbb{C}$ given by $E(a, b) := \text{h.c.}(ab)$ (where h.c. stands for the highest degree coefficient) is nondegenerate. This pairing obviously satisfies the condition $E(xa, b) = E(a, xb), x \in \mathfrak{h}^*$. Now recall that $H_c(W)$ and A admits a natural action of the group $SL_2(\mathbb{C})$. Let $g = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \in SL_2(\mathbb{C})$; then $g(x_i) = y_i$ for orthonormal bases x_i, y_i of $\mathfrak{h}^*$ and $\mathfrak{h}$ which are dual to each other. Thus the nondegenerate form $\tilde{B}(a, b) := E(a, gb)$ on A satisfies the equations $\tilde{B}(a, x_i b) = \tilde{B}(y_i a, b)$. So for any $f_1, f_2 \in \mathbb{C}[\mathfrak{h}]$, one has $\tilde{B}(f_1(x)v, f_2(x)v) = \tilde{B}(f_2(y)f_1(x)v, v)$, where $v = 1$ is the lowest weight vector of A. This shows that $\widetilde{B}$ coincides with the Shapovalov form B on A. Thus A is an irreducible representation of $H_c(W)$. □

Remark. It easy to see by considering the rank 1 case that for complex reflection groups Theorem 2.2 is, in general, false.

Theorem 2.3. *Let W be a complex reflection group, and $U \subset M_c$ be a W-subrepresentation of dimension $\ell = \dim(\mathfrak{h})$ sitting in degree r, consisting of singular vectors (i.e. those killed by $y \in \mathfrak{h}$). Let J be the ideal generated by U. Assume that the quotient representation $A = M_c/J$ is finite dimensional. Then*

(i) The algebra A is Gorenstein.

(ii) The representation A admits a BGG type resolution

$$A \leftarrow M_c(\mathbb{C}) \leftarrow M_c(U) \leftarrow M_c(\wedge^2 U) \leftarrow \dots \leftarrow M_c(\wedge^\ell U) \leftarrow 0.$$

(iii) The character of A is given by the formula

$$\chi_A(g, t) = t^{\frac{\ell}{2} - \sum_s \frac{2c_s}{1-\lambda_s}} \frac{\det|_U(1 - gt^r)}{\det_{\mathfrak{h}^*}(1 - gt)}.$$

In particular, the dimension of A is r^ℓ.

(iv) If W is a real reflection group, then A is irreducible.

Proof. (i) Since $\text{Spec}(A)$ is a complete intersection, A is Gorenstein ([E], p. 541).

(ii) Consider the subring $\mathbb{C}[U]$ in $\mathbb{C}[\mathfrak{h}]$. Then $\mathbb{C}[\mathfrak{h}]$ is a finitely generated $\mathbb{C}[U]$-module. A standard theorem of Serre [S] says that if $B = \mathbb{C}[t_1, ..., t_n]$, $f_1, ..., f_n \in B$ are homogeneous, $A = \mathbb{C}[f_1, ..., f_n] \subset B$, and B is a finitely generated module over A, then B is a free module over A. Applying this in our situation, we see that $\mathbb{C}[\mathfrak{h}]$ is a free $\mathbb{C}[U]$-module. It is easy to see by computing the Hilbert series that the rank of this free module is r^ℓ.

Consider the Koszul complex attached to the module $\mathbb{C}[\mathfrak{h}]$ over $\mathbb{C}[U]$. Since this module is free, the Koszul complex is exact (i.e. it is a resolution of the zero-fiber). At the level of $\mathbb{C}[\mathfrak{h}]$ modules, this resolution looks exactly as we want in (ii). So we need to show that the maps of the resolution are in fact morphisms of $H_c(W)$-modules and not only $\mathbb{C}[\mathfrak{h}]$-modules. This is easily established by induction (going from left to right); cf. proof of Proposition 2.2 in [BEG] and also [Go].

(iii) Follows from (ii) by the Euler-Poincare principle.

(iv) Follows from Theorem 2.2. □

3. Representations of the rational Cherednik algebra of type A

3.1. **The results.** Let $W = S_n$. In this case the function c reduces to one number k. We will denote the rational Cherednik algebra $H_k(S_n)$ by $H_k(n)$. The polynomial representation M_k of this algebra is the space of $\mathbb{C}[x_1, ..., x_n]^T$ of polynomials of $x_1, ..., x_n$, which are invariant under simultaneous translation $x_i \mapsto x_i + a$. In other words, it is the space of regular functions on $\mathfrak{h} = \mathbb{C}^n/\Delta$, where Δ is the diagonal.

Proposition 3.1 ([Du]). *Let r be a positive integer not divisible by n, and $k = r/n$. Then M_k contains a copy of the reflection representation $\mathfrak{h}$ of S_n, which consists of singular vectors (i.e. those killed by $y \in \mathfrak{h}$). This copy sits in degree r and is spanned by the functions*

$$f_i(x_1, ..., x_n) = \mathrm{Res}_\infty[(z - x_1)...(z - x_n)]^{\frac{r}{n}} \frac{dz}{z - x_i}$$

(the symbol Res_∞ denotes the residue at infinity).

Remark. The space spanned by f_i is $n-1$-dimensional, since $\sum_i f_i = 0$ (this sum is the residue of an exact differential).

Proof. This proposition can be proved by a straightforward computation. The functions f_i are a special case of Jack polynomials. □

Let I_k be the submodule of M_k generated by f_i, Consider the $H_k(n)$-module $V_k = M_k/I_k$, and regard it as a $\mathbb{C}[\mathfrak{h}]$-module.

Our result is

Theorem 3.2. *Let $d = (r, n)$ denote the greatest common divisor of r and n. Then the (set-theoretical) support of V_k is the union of S_n-translates of the subspaces of $\mathbb{C}^n/\Delta$, defined by the equations*

$$x_1 = x_2 = \cdots = x_{\frac{n}{d}};$$
$$x_{\frac{n}{d}+1} = \cdots = x_{2\frac{n}{d}};$$
$$\cdots$$
$$x_{(d-1)\frac{n}{d}+1} = \cdots = x_n.$$

In particular, the Gelfand-Kirillov dimension of V_k is $d - 1$.

The theorem allows us to give a simple proof of the following result of [BEG] (without the use of the KZ functor and Hecke algebras).

Corollary 3.3 ([BEG]). *If $d = 1$, then the module $V_k := M_k/I_k$ is finite dimensional, irreducible, admits a BGG type resolution, and its character is*

$$\chi_{V_k}(g, t) = t^{(1-r)(n-1)/2} \frac{\det|_{\mathfrak{h}}(1 - gt^r)}{\det|_{\mathfrak{h}}(1 - gt)}.$$

Proof. For $d = 1$ Theorem 3.2 says that the support of M_k/I_k is $\{0\}$. This implies that M_k/I_k is finite dimensional. The rest follows from Theorem 2.3. □

3.2. **Proof of Theorem 3.2.** The support of V_k is the zero-set of I_k, i.e., the common zero set of f_i. Fix $x_1, ..., x_n \in \mathbb{C}$. Then $f_i(x_1, ..., x_n) = 0$ for all i iff

$\sum_{i=1}^{n} \lambda_i f_i = 0$ for all λ_i, i.e.,

$$\mathrm{Res}_\infty \left(\prod_{j=1}^{n} (z - x_j)^{\frac{r}{n}} \sum_{i=1}^{n} \frac{\lambda_i}{z - x_i} \right) dz = 0.$$

Assume that $x_1, \ldots, x_n$ take distinct values $y_1, \ldots, y_p$ with positive multiplicities $m_1, \ldots, m_p$. The previous equation implies that the point $(x_1, ..., x_n)$ is in the zero set iff

$$\mathrm{Res}_\infty \prod_{j=1}^{p} (z - y_j)^{m_j \frac{r}{n} - 1} \left(\sum_{i=1}^{p} \nu_i (z - y_1) \ldots \widehat{(z - y_i)} \ldots (z - y_p) \right) dz = 0 \quad \forall \nu_i.$$

Since ν_i are arbitrary, this is equivalent to the condition

$$\mathrm{Res}_\infty \prod_{j=1}^{p} (z - y_j)^{m_j \frac{r}{n} - 1} z^i dz = 0, \quad i = 0, \ldots, p - 1.$$

We will now need the following lemma.

Lemma 3.4. *Let* $a(z) = \prod_{j=1}^{p} (z - y_j)^{\mu_j}$*, where* $\mu_j \in \mathbb{C}$*,* $\sum_j \mu_j \in \mathbb{Z}$ *and* $\sum_j \mu_j > -p$*. Suppose*

$$\mathrm{Res}_\infty a(z) z^i dz = 0, \quad i = 0, 1, \ldots, p - 2.$$

Then $a(z)$ *is polynomial.*

Proof. Let g be a polynomial. Then

$$0 = \mathrm{Res}_\infty d(g(z) \cdot a(z)) = \mathrm{Res}_\infty (g'(z) a(z) + a'(z) g(z)) dz$$

and hence

$$\mathrm{Res}_\infty \left(g'(z) + \sum_i \frac{\mu_j}{z - y_j} g(z) \right) a(z) dz = 0.$$

Let $g(z) = z^l \prod_j (z - y_j)$. Then $g'(z) + \sum \frac{\mu_j}{z - y_j} g(z)$ is a polynomial of degree $l + p - 1$ with highest coefficient $l + p + \sum \mu_j \neq 0$ (as $\sum \mu_j > -p$). This means that for every $l \geq 0$, $\mathrm{Res}_\infty z^{l+p-1} a(z) dz$ is a linear combination of residues of $z^q a(z) dz$ with $q < l + p - 1$. By the assumption of the lemma, this implies by induction in l that all such residues are 0 and hence a is a polynomial. ◻

In our case $\sum (m_j \frac{r}{n} - 1) = r - p$ (since $\sum m_j = n$) and the conditions of the lemma are satisfied. Hence $(x_1, \ldots, x_n)$, is in the zero set of I_k iff $\prod_{j=1}^{p} (z - y_j)^{m_j \frac{r}{n} - 1}$ is a polynomial. This is equivalent to saying that all m_j are divisible by $\frac{n}{d}$.

We have proved that $(x_1, \ldots, x_n)$ is in the zero set of I_k iff $(z - x_1) \ldots (z - x_n)$ is the n/d-th power of a polynomial of degree d. This implies the theorem.

4. Representations of the rational Cherednik algebra for the complex reflection group $S_n \ltimes (\mathbb{Z}/l\mathbb{Z})^n$.

4.1. **The formula for the singular vector.** Let $l \geq 2$ be an integer. Consider the complex reflection group $W = S_n \ltimes (\mathbb{Z}/l\mathbb{Z})^n$ acting on the n-dimensional space. The set $S \subset W$ of complex reflections consists of the elements s_i^m and $\sigma_{i,j}^{(m)}$ defined by

$$s_i^m(x_1, \ldots, x_i, \ldots, x_n) = (x_1, \ldots, \varepsilon^m x_i, \ldots, x_n), \quad 1 \leq m < l;$$

$$\sigma_{i,j}^{(m)}(x_1, \ldots, x_i, \ldots, x_j, \ldots, x_n) = (x_1, \ldots, \varepsilon^m x_j, \ldots, \varepsilon^{-m} x_i, \ldots, x_n), \quad 0 \leq m < l$$

(here ε is a primitive l-th root of unity). Consider the W-invariant function $c : S \to \mathbb{C}$ defined by $c(s_i^m) = c_m$ for all i and $c(\sigma_{i,j}^{(m)}) = k$ for all i, j, m (where k and $\{c_m\}$ are fixed constants). Let $H_c(n, l)$ denote the rational Cherednik algebra $H_c(W)$ corresponding to W and c. Let M_c be the polynomial representation of this algebra.

Fix a positive integer r, which is not divisible by l. Thus $r = (p-1)l + q$, where p is a positive integer, and $1 \leq q \leq l-1$ is an integer. Denote by E_r the affine hyperplane of those functions c for which $l(n-1)k + 2\sum_{j=1}^{l-1} c_j \frac{1-\varepsilon^{-jq}}{1-\varepsilon^{-j}} = r$. Let $\mathfrak{h}_q$ be the representation of W on $\mathbb{C}^n$ in which S_n acts by permutations, and s_i multiplies the i-th coordinate of a vector by ε^{-q} (thus the reflection representation corresponds to $q = l-1$ and the dual reflection representation to $q = 1$). Let s be the largest integer which is $< p/n$.

We have the following analog of Proposition 3.1.

Proposition 4.1. *For $c \in E_r$ the polynomial representation M_c of $H_c(n, l)$ contains a copy of representation $\mathfrak{h}_q$ of W in degree r consisting of singular vectors. This copy is spanned by the functions f_i, where*

$$f_i(x_1, \ldots, x_n) = \mathrm{Res}_\infty \frac{z^{(p-nk)l-1}}{(k-1)\ldots(k-s)} \prod_{j=1}^n (z^l - x_j^l)^k \frac{x_i^q dz}{z^l - x_i^l}.$$

Remark. Clearly, if $k = 1, \ldots, s$, then

$$\mathrm{Res}_\infty z^{(p-nk)l-1} \prod_{j=1}^n (z^l - x_j^l)^k \frac{x_i^q dz}{z^l - x_i^l} = 0.$$

Thus the functions f_i are polynomial in k. Moreover, they do not vanish identically for any k, as easily seen by computing the coefficients.

Proof. This proposition can be proved by direct computation. It can also be obtained as a simple consequence of the results of [DO], Section 3. □

4.2. **Finite dimensional representations of $H_c(n, l)$.** Let $U_r = \mathbb{C}[u]/(u^r)$. This is naturally a $\mathbb{Z}_+$-graded representation of $\mathbb{Z}/l\mathbb{Z}$, $m \circ u^j = \varepsilon^{-mj} u^j$. Thus $U_r^{\otimes n}$ is a graded representation of W. The character of this representation is $\mathrm{Tr}|_{U_r}(gt^D) = \frac{\det|_{\mathfrak{h}_q}(1-t^r g)}{\det|_{\mathfrak{h}^*}(1-tg)}$, where D is the grading operator.

Theorem 4.2. *(i) For $c \in E_r$, there exists a lowest weight module Y_c over $H_c(n, l)$ with trivial lowest weight (i.e. a quotient of M_c) which is isomorphic to $U_r^{\otimes n}$ as a graded W-module.*

(ii) For generic $c \in E_r$, Y_c is irreducible.

Proof. Let I_c be the submodule generated by f_i, and $\widetilde{Y}_c = M_c/I_c$.

If $k = 0$, a direct computation shows that $f_i(x_1, ..., x_n) = Cx_i^r$, where C is a nonzero constant. In this case, $H_c(n, l) = \mathbb{C}[S_n] \ltimes (H_{c_1,...,c_{l-1}}(1, l)^{\otimes n})$, so $\widetilde{Y}_c$ is simply $U_r^{\otimes n}$, where U_r is the r-dimensional lowest weight module over $H_{c_1,...,c_{l-1}}(1, l)$ (with trivial lowest weight) which exists when $2\sum_{j=1}^{l-1} c_j \frac{1-\varepsilon^{-jq}}{1-\varepsilon^{-j}} = r$. It follows from Theorem 2.1 that for generic $c_1, ..., c_{l-1}$ satisfying this equation, the module U_r is irreducible. Thus, for generic $c \in E_r \cap \{k = 0\}$, the module $\widetilde{Y}_c$ is irreducible (of dimension r^n), and the kernel of the Shapovalov form on M_c coincides with I_c.

Using standard semicontinuity arguments, we conclude from this that for generic $c \in E_r$, $\dim(\widetilde{Y}_c) \le r^n$.

On the other hand, let L_c denote the irreducible module over $H_c(n, l)$ with trivial lowest weight, i.e., the quotient of M_c by the kernel of the Shapovalov form. The previous argument shows that for generic $c \in E_r$, $\dim L_c \ge r^n$. But L_c is clearly a quotient of $\widetilde{Y}_c$. This implies that for generic $c \in E_r$, $L_c = \widetilde{Y}_c$, and in particular $\widetilde{Y}_c$ is irreducible.

To finish the proof, observe that the submodule I_c really depends only on one parameter k. Therefore, we can define Y_c by setting, for generic k, $Y_c := \widetilde{Y}_c = M_c/I_c$, and for special k (i.e. finitely many values), $Y_c = M_c/\hat{I}_c$, where $\hat{I}_c = \lim_{c' \to c} I_{c'}$ (the limit exists since any regular map from $\mathbb{P}^1 \setminus \{k_1, ..., k_N\}$ to a Grassmannian can be extended to the whole $\mathbb{P}^1$). Clearly, Y_c is a lowest weight $H_c(n, l)$-module, satisfying the conditions of the theorem. Thus the theorem is proved. $\square$

Remark. Let $l = 2$. In this case our condition on r is that r is odd, and the equation of E_r has the form $2(n-1)k + 2c_1 = r$. Thus, in this case we recover Theorem 6.1 from [BEG].

4.3. **The dimension of $\widetilde{Y}_c$.** Let

$$\Sigma_r = \{\frac{P}{Q}| \quad P, Q \text{ integers and } \quad (P, Q) = 1, \quad 1 \le P \le p-1, \quad 1 \le Q \le n\}$$

(where $r = (p-1)l + q$), and let $\widetilde{Y}_c$, Y_c be as defined in the previous subsection.

Theorem 4.3. *(i) $\widetilde{Y}_c$ is finite dimensional if and only if $k \notin \Sigma_r$.*
(ii) $\widetilde{Y}_c$ is finite dimensional if and only if $\widetilde{Y}_c = Y_c$.
(iii) If $l = 2$ and $k \notin \Sigma_r$, then Y_c is irreducible.

Proof. (i) It suffices to prove that the support of $\widetilde{Y}_c$ is $\{0\}$ iff $k \notin \Sigma_r$. The proof will be analogous to that of Theorem 3.2.

The support of $\widetilde{Y}_c$ is the common zero set of the functions f_i. Fix $x_1, ..., x_n \in \mathbb{C}$. We have $f_i(x_1, \ldots, x_n) = 0$ for all i iff $\displaystyle\sum_{i=1}^n \lambda_i f_i(x_1, ..., x_n) = 0$ for all λ_i, i.e.,

$$\operatorname{Res}_\infty \left(z^a \prod_{j=1}^n (z^l - x_j^l)^k \sum_{i=1}^n \frac{\lambda_i x_i^q}{z^l - x_i^l} \right) dz = 0,$$

where $a = (p - nk)l$.

Let $d \ge 0$ be the number of distinct nonzero numbers among $x_1^l, ..., x_n^l$. More specifically, assume that $x_1^l, \ldots, x_n^l$ take values $y_0 = 0, y_1, \ldots, y_d$ with multiplicities $m_0, m_1, \ldots, m_d$, such that $m_j > 0$ if $j > 0$ (so $\sum_{i=0}^d m_j = n$). From the above equation we see that the point $(x_1, \dots, x_n)$ is in the zero set if and only if for any ν_i, $i = 1, \dots, d$

$$\mathrm{Res}_\infty \left[z^{a+lm_0} \prod_{j=1}^{d} (z^l - y_j)^{m_j k - 1} \left(\sum_{i=1}^{d} \nu_i (z^l - y_1) \dots \widehat{(z^l - y_i)} \dots (z^l - y_d) \right) \right] z^{-1} dz = 0.$$

This equation is equivalent to

$$\mathrm{Res}_\infty \left(z^{a+(m_0 k + i)l} \prod_{j=1}^{d} (z^l - y_j)^{m_j k - 1} \right) z^{-1} dz = 0, \quad i = 0, \dots, d-1.$$

Let's make a change of variables $w = z^l$. Then the equations above will take the form

$$\mathrm{Res}_\infty \left((w - y_0)^{p-(n-m_0)k-1} \prod_{j=1}^{d} (w - y_j)^{m_j k - 1} w^i \right) dw = 0, \quad i = 0, \dots, d-1.$$

Applying Lemma 3.4 (whose conditions are clearly satisfied) we get that these equations hold iff the function

$$F(w) := w^{p-(n-m_0)k-1} \prod_{j=1}^{d} (w - y_j)^{m_j k - 1}$$

is a polynomial.

The function $F(w)$ is a polynomial iff $p - (n - m_0)k - 1$ and $m_j k - 1$ for all $1 \le j \le d$ are nonnegative integers.

Suppose $k \notin \mathbb{Q}$ or $k = \frac{P}{Q}$ with $(P, Q) = 1$ and $Q > n$. Then there is no integer $1 \le m \le n$, such that $mk - 1 \in \mathbb{Z}$. This means that for such k the function F is a polynomial only if $d = 0$. Hence the support of $\widetilde{Y}_c$ is $\{0\}$.

Suppose $k = \frac{P}{Q}$ with $(P, Q) = 1$ and $Q \le n$. Let $M_j = km_j$. The condition that F is a polynomial implies that M_j are positive integers and $p - 1 - \sum_{j=1}^{d} M_j \ge 0$. This means that either $d = 0$ or $P \le p - 1$, i.e., $k \in \Sigma_r$. Thus, if the support of $\widetilde{Y}_c$ is nonzero, then $k \in \Sigma_r$.

Conversely, let $k = P/Q \in \Sigma_r$. Take $d = 1$, $m_1 = Q$, $m_0 = n - Q$, and choose $y_1 \neq 0$ arbitrarily. Then F is a polynomial. So the support of $\widetilde{Y}_c$ in this case is nonzero.

Thus, statement (i) is proved.

(ii) Clearly, Y_c is a quotient of $\widetilde{Y}_c$. The rest follows from Theorem 2.3.

(iii) If $l = 2$, then W is a reflection group. Thus the result follows from Theorem 2.3. □

Part (iii) of Theorem 4.3 generalizes Proposition 6.4 of [BEG].

Remark. Part (iii) of Theorem 4.3 fails for complex reflection groups, as seen from considering the rank 1 case: in this case Σ_r is empty, but Y_c is not always irreducible. Similarly, one cannot drop in part (iii) the assumption that $k \notin \Sigma_r$: this is demonstrated by Subsection 6.4 in [BEG].

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