REPRESENTATION THEORY
An Electronic Journal of the American Mathematical Society
Volume 7, Pages 651–680 (November 24, 2003)
S 1088-4165(03)00211-5

A GEOMETRIC APPROACH TO STANDARD MONOMIAL THEORY

M. BRION AND V. LAKSHMIBAI

Abstract. We obtain a geometric construction of a "standard monomial basis" for the homogeneous coordinate ring associated with any ample line bundle on any flag variety. This basis is compatible with Schubert varieties, opposite Schubert varieties, and unions of intersections of these varieties. Our approach relies on vanishing theorems and a degeneration of the diagonal; it also yields a standard monomial basis for the multi–homogeneous coordinate rings of flag varieties of classical type.

Introduction

Consider the Grassmannian X of linear subspaces of dimension r in k^n, where k is a field. We regard X as a closed subvariety of projective space $\mathbb{P}(\wedge^r k^n)$ via the Plücker embedding; let L be the corresponding very ample line bundle on X. Then the ring $\bigoplus_{m=0}^{\infty} H^0(X, L^{\otimes m})$ admits a nice basis, defined as follows.

Let $\{v_1, \ldots, v_n\}$ be the usual basis of k^n; then the $v_{i_1} \wedge \cdots \wedge v_{i_r}$, $1 \leq i_1 < \cdots < i_r \leq n$, form a basis of $\wedge^r k^n$. We put $I = (i_1, \ldots, i_r)$, $v_I = v_{i_1} \wedge \cdots \wedge v_{i_r}$, and we denote by $\{p_I\}$ the dual basis of the basis $\{v_I\}$; the p_I (regarded in $H^0(X, L)$) are the Plücker coordinates. Define a partial order on the set $\mathcal{I}$ of indices I by letting $I = (i_1, \ldots, i_r) \leq (j_1, \ldots, j_r) = J$ if and only if $i_1 \leq j_1, \ldots, i_r \leq j_r$. Then

(i) *The monomials* $p_{I_1} p_{I_2} \cdots p_{I_m}$ *where* $I_1, \ldots, I_m \in \mathcal{I}$ *satisfy* $I_1 \leq I_2 \leq \cdots \leq I_m$, *form a basis of* $H^0(X, L^{\otimes m})$.

(ii) *For any* $I, J \in \mathcal{I}$, *we have* $p_I p_J - \sum_{I', J',\, I' \leq I, J \leq J'} a_{I'J'} p_{I'} p_{J'} = 0$, *where* $a_{I'J'} \in k$.

The monomials in (i) are called the *standard monomials of degree* m, and the relations in (ii) are the *quadratic straightening relations*; they allow us to express any nonstandard monomial in the p_I as a linear combination of standard monomials.

Further, this *standard monomial basis* of the homogeneous coordinate ring of X is compatible with its Schubert subvarieties, in the following sense. For any $I \in \mathcal{I}$, let $X_I = \{V \in X \mid \dim(V \cap \operatorname{span}(v_1, \ldots, v_s)) \geq \#(j, i_j \leq s), 1 \leq s \leq r\}$ be the corresponding Schubert variety; then the restriction $p_J|_{X_I}$ is nonzero if and only if $J \leq I$. The monomial $p_{I_1} \cdots p_{I_m}$ will be called *standard on* X_I if $I_1 \leq \cdots \leq I_m \leq I$; equivalently, this monomial is standard and does not vanish identically on X_I. Now:

(iii) *The standard monomials of degree* m *on* X_I *restrict to a basis of* $H^0(X_I, L^{\otimes m})$. *The standard monomials of degree* m *that are not standard on* X_I, *form a basis of the kernel of the restriction map* $H^0(X, L^{\otimes m}) \to H^0(X_I, L^{\otimes m})$.

Received by the editors November 8, 2001 and, in revised form, September 12, 2003.
2000 *Mathematics Subject Classification.* Primary 14M15, 20G05, 14L30, 14L40.

These classical results go back to Hodge; see [5]. They have important geometric consequences, e.g., X is projectively normal in the Plücker embedding; its homogeneous ideal is generated by the quadratic straightening relations; the homogeneous ideal of any Schubert variety X_I is generated by these relations together with the p_J where $J \not\leq I$.

The purpose of *Standard Monomial Theory* (SMT) is to generalize Hodge's results to any flag variety $X = G/P$ (where G is a semisimple algebraic group over an algebraically closed field k, and P a parabolic subgroup) and to any effective line bundle L on X. SMT was developed by Lakshmibai, Musili, and Seshadri in a series of papers, culminating in [9] where it is established for all classical groups G. There the approach goes by ascending induction on the Schubert varieties, using their partial resolutions as projective line bundles over smaller Schubert varieties.

Further results concerning certain exceptional or Kac–Moody groups led to conjectural formulations of a general SMT; see [10]. These conjectures were then proved by Littelmann, who introduced new combinatorial and algebraic tools: the path model of representations of any Kac–Moody group, and Lusztig's Frobenius map for quantum groups at roots of unity (see [11, 12]).

In the present paper, we obtain a geometric construction of a SMT basis for $H^0(X, L)$, where $X = G/P$ is any flag variety and L is any ample line bundle on X. This basis is compatible with Schubert varieties (that is, with orbit closures in X of a Borel subgroup B of G) and also with opposite Schubert varieties (the orbit closures of an opposite Borel subgroup B^-); in fact, it is compatible with any intersection of a Schubert variety with an opposite Schubert variety. We call such intersections *Richardson varieties*, since they were first considered by Richardson in [17]. Our approach adapts to the case where L is an effective line bundle on a flag variety *of classical type* in the sense of [9]. This sharpens the results of [9] concerning the classical groups.

Our work may be regarded as one step towards a purely geometric proof of Littelmann's results concerning SMT. He constructed a basis of T–eigenvectors for $H^0(X, L)$ (where T is the maximal torus common to B and B^-) indexed by certain piecewise linear paths in the positive Weyl chamber, called *LS paths.* This basis turns out to be compatible with Richardson varieties; notice that these are T–invariant. In fact, the endpoints of the path indexing a basis vector parametrize the smallest Richardson variety where this vector does not vanish identically (see [8]). If L is associated with a weight of classical type, then the LS paths are just line segments: they are uniquely determined by their endpoints. This explains a posteriori why our geometric approach completes the program of SMT in that case.

In fact, our approach of SMT for an ample line bundle L on a flag variety X uses little of the rich geometry and combinatorics attached to X. Specifically, we only rely on vanishing theorems for unions of Richardson varieties (these being direct consequences of the existence of a Frobenius splitting of X, compatible with Schubert varieties and opposite Schubert varieties), together with the following property.

(iv) *The diagonal in* $X \times X$ *admits a flat* T*–invariant degeneration to the union of all products* $X_w \times X^w$*, where the* X_w *are the Schubert varieties and the* X^w *are the corresponding opposite Schubert varieties.*

The latter result follows from [2] (we provide a direct proof in Section 3). It plays an essential rôle in establishing generalizations of (i) and (iii); conversely, it turns out that the existence of a SMT basis implies (iv); see the Remark after Proposition 7.

It is worth noticing that (iv) is a stronger form of the fact that the classes of Schubert varieties form a free basis of the homology group (or Chow group) of X, the dual basis for the intersection pairing consisting of the classes of opposite Schubert varieties. This fact (in a different formulation) has been used by Knutson to establish an asymptotic version of the Littelmann character formula; see [7].

This paper is organized as follows. In the preliminary Section 1, we introduce notation and study the geometry of Richardson varieties. Vanishing theorems for cohomology groups of line bundles on Richardson varieties are established in Section 2, by slight generalizations of the methods of Frobenius splitting. In Section 3, we construct filtrations of the T–module $H^0(X,L)$ that are compatible with restrictions to Richardson varieties. Our SMT basis of $H^0(X,L)$ is defined in Section 4; it is shown to be compatible with all unions of Richardson varieties. In Section 5, we generalize statements (i) and (iii) above to any ample line bundle L on a flag variety G/P; then (ii) follows from (i) together with compatibility properties of our basis. The case where the homogeneous line bundle L is associated with a weight of classical type (e.g., a fundamental weight of a classical group) is considered in detail in Section 6. There we give a geometric characterization of the *admissible pairs* of [9] (these parametrize the weights of the T–module $H^0(X,L)$). The final Section 7 develops SMT for those effective line bundles that correspond to sums of weights of classical type.

Acknowledgements. The second author was partially supported by N.S.F. Grant DMS-9971295. She is grateful to the Institut Fourier for the hospitality extended during her visit in June 2001; it was there that this work originated.

1. Richardson varieties

The ground field k is algebraically closed, of arbitrary characteristic. Let G be a simply–connected semisimple algebraic group. Choose opposite Borel subgroups B and B^- of G, with common torus T; let $\mathcal{X}(T)$ be the group of characters of T, also called weights. In the root system R of (G,T), we have the subset R^+ of positive roots (that is, of roots of (B,T)), and the subset S of simple roots. For each $\alpha \in R$, let $\check{\alpha}$ be the corresponding coroot and let U_α be the corresponding additive one–parameter subgroup of G, normalized by T.

We also have the Weyl group W of (G,T); for each $\alpha \in R$, we denote by $s_\alpha \in W$ the corresponding reflection. Then the group W is generated by the simple reflections s_α, $\alpha \in S$; this defines the length function ℓ and the Bruhat order $\leq$ on W. Let w_o be the longest element of W, then $B^- = w_o B w_o$.

Let P be a parabolic subgroup of G containing B and let W_P be the Weyl group of (P,T), a parabolic subgroup of W; let $w_{o,P}$ be the longest element of W_P. Each right W_P–coset in W contains a unique element of minimal length; this defines the subset W^P of minimal representatives of the quotient W/W_P. This subset is invariant under the map $w \longmapsto w_o w w_{o,P}$; the induced bijection of W^P reverses the Bruhat order.

Each character λ of P defines a G–linearized line bundle on the homogeneous space G/P; we denote that line bundle by L_λ. The assignment $\lambda \longmapsto L_\lambda$ yields an isomorphism from the character group $\mathcal{X}(P)$ to the Picard group of G/P. Further, the line bundle L_λ is generated by its global sections if and only if λ (regarded as a character of T) is dominant; in that case, $H^0(G/P, L_\lambda)$ is a G–module with lowest weight $-\lambda$.

Let W_λ be the isotropy group of λ in W, and let P_λ be the parabolic subgroup of G generated by B and W_λ; then $W_\lambda \supseteq W_P$, $W^\lambda \subseteq W^P$, and $P_\lambda \supseteq P$. We shall identify W^λ with the W–orbit of the weight λ, and denote by $w(\lambda)$ the image of $w \in W$ in $W/W_\lambda \simeq W^\lambda$.

The *extremal weight vectors* $p_{w(\lambda)} \in H^0(G/P, L_\lambda)$ are the T–eigenvectors of weight $-w(\lambda)$ for some $w \in W^\lambda$. These vectors are uniquely defined up to scalars.

We say that λ is *P–regular* if $P_\lambda = P$. The ample line bundles on G/P are the L_λ where λ is dominant and P–regular; under these assumptions, L_λ is in fact very ample. We may then identify each $w \in W^P$ to $w(\lambda)$, and we put $p_w = p_{w(\lambda)}$.

The T–fixed points in G/P are the $e_w = wP/P$ ($w \in W/W_P$); we index them by W^P. The B–orbit $C_w = Be_w$ is a *Bruhat cell*, an affine space of dimension $\ell(w)$; its closure in G/P is the *Schubert variety* X_w. The complement $X_w - C_w$ is the *boundary* ∂X_w. We have

$$\partial X_w = \bigcup_{v \in W^P,\ v<w} X_v,$$

and the irreducible components of ∂X_w are the *Schubert divisors* X_v where $v \in W^P$, $v < w$ and $\ell(v) = \ell(w) - 1$. Then there exists $\beta \in R^+$ such that $v = ws_\beta$.

Let λ be a character of P and let f_w be the restriction to X_w of the natural map $G/P \longrightarrow G/P_\lambda$; then $f_w(X_w) = X_{w(\lambda)}$. The set

$$\partial_\lambda X_w := f_w^{-1}(\partial X_{w(\lambda)})$$

is called the *λ–boundary* of X_w; it is the union of the Schubert divisors X_{ws_β} where $\langle \lambda, \check\beta \rangle > 0$. If λ is dominant, then we have by Chevalley's formula:

$$\operatorname{div}(p_{w(\lambda)}|_{X_w}) = \sum \langle \lambda, \check\beta \rangle X_{ws_\beta}$$

(sum over all $\beta \in R^+$ such that X_{ws_β} is a divisor in X_w). In particular, the zero set of $p_{w(\lambda)}$ in X_w is $\partial_\lambda X_w$. If, in addition, λ is P–regular, then $\partial_\lambda X_w = \partial X_w$.

We shall also need the *opposite Bruhat cell* $C^w = B^- e_w$ of codimension $\ell(w)$ in G/P, the *opposite Schubert variety* X^w (the closure of C^w) and its boundary ∂X^w. Then $X^w = w_o X_{w_o w w_{o,P}}$ and

$$\partial X^w = \bigcup_{v \in W^P, v>w} X^v.$$

Recall that all Schubert varieties are normal and Cohen–Macaulay (thus, the same holds for all opposite Schubert varieties). Further, all scheme–theoretic intersections of unions of Schubert varieties and opposite Schubert varieties are reduced (see [14, 15, 16]).

Definition 1. Let v, w in W^P. We call the intersection

$$X_w^v := X_w \cap X^v$$

a Richardson variety in G/P. We define its boundaries by

$$(\partial X_w)^v := \partial X_w \cap X^v \text{ and } (\partial X^v)_w := X_w \cap \partial X^v.$$

Notice that X_w^v and its boundaries are closed reduced, T–stable subschemes of G/P. The X_w^v were considered by Richardson, who showed, e.g., that they are irreducible (see [17]; the intersections $C_w \cap C^v$ were analyzed by Deodhar, see [4]). We shall give another proof of this result, and obtain a little more.

Lemma 1. (1) *X_w^v is nonempty if and only if $v \leq w$; then X_w^v is irreducible of dimension $\ell(w) - \ell(v)$, and $(\partial X_w)^v$, $(\partial X^v)_w$ have pure codimension 1 in X_w^v. Further, X_w^v is normal and Cohen–Macaulay.*

(2) *The T–fixed points in X_w^v are the e_x where $x \in W^P$ and $v \leq x \leq w$.*

(3) *For x, y in W^P, we have $X_y^x \subseteq X_w^v \iff v \leq x \leq y \leq w$.*

Proof. (2) is evident; it implies (3) and the first assertion of (1). To prove the remaining assertions, we use a variant of the argument of [1], Lemma 2. Consider the fiber product $G \times^B X_w$ with projection map

$$p : G \times^B X_w \longrightarrow G/B,$$

a G–equivariant locally trivial fibration with fiber X_w. We also have the "multiplication" map

$$m : G \times^B X_w \longrightarrow G/P,\ (g, x) \longmapsto gx.$$

This is a G–equivariant map to G/P; thus, it is also a locally trivial fibration. Its fiber $m^{-1}(e_1)$ is isomorphic to $\overline{Pw^{-1}B}/B$ (a Schubert variety in G/B).

Next let $i : X^v \longrightarrow G/P$ be the inclusion and consider the cartesian product

$$Z = X^v \times_{G/P} (G \times^B X_w)$$

with projections ι to $G \times^B X_w$, μ to X^v and π to G/B, as displayed in the following commutative diagram:

$$\begin{tikzcd}
G/B \arrow[d, "id"'] & Z \arrow[l, "\pi"'] \arrow[r, "\mu"] \arrow[d, "\iota"'] & X^v \arrow[d, "i"'] \\
G/B & G \times^B X_w \arrow[l, "p"'] \arrow[r, "m"] & G/P
\end{tikzcd}$$

By definition, the square on the right is cartesian, so that μ is also a locally trivial fibration with fiber $\overline{Pw^{-1}B}/B$ and base X^v. Since Schubert varieties are irreducible, normal and Cohen–Macaulay, it follows that the same holds for Z. Further, we have

$$\dim(Z) = \dim(G \times^B X_w) + \dim(X^v) - \dim(G/P) = \dim(G/B) + \ell(w) - \ell(v).$$

Notice that the fiber of $\pi : Z \longrightarrow G/B$ at each gB/B identifies to the intersection $X^v \cap gX_w$; in particular, $\pi^{-1}(B/B) = X_w^v$. Notice also that $\iota : Z \longrightarrow G \times^B X_w$ is a closed immersion with B^-–stable image (since this holds for $i : X^v \longrightarrow G/P$). Thus, B^- acts on Z so that π is equivariant. Since $B^- B/B$ is an open neighborhood of B/B in G/B, isomorphic to U^-, its pullback under π is an open subset of Z, isomorphic to $U^- \times X_w^v$. Therefore, X_w^v is irreducible, normal and Cohen–Macaulay of dimension $\ell(w) - \ell(v)$. □

We also record the following easy result, to be used in Section 7.

Lemma 2. *Let $v \leq w$ in W^P, let λ be a dominant character of P and let $x(\lambda) \in W^\lambda$. Then the restriction of $p_{x(\lambda)}$ to X_w^v is nonzero if and only if $x(\lambda)$ admits a lift $x \in W^P$ such that $v \leq x \leq w$. Further, the ring*

$$\bigoplus_{n=0}^{\infty} H^0(X_w^v, L_{n\lambda})$$

is integral over its subring generated by the $p_{x(\lambda)}|_{X_w^v}$ where $x \in W^P$ and $v \leq x \leq w$.

Proof. Consider the natural map $G/P \to G/P_\lambda$ and its restriction $f : X_w^v \longrightarrow f(X_w^v)$. The open subset $(p_{x(\lambda)} \neq 0)$ of G/P_λ is affine, T–stable and contains $e_{x(\lambda)}$ as its unique closed T–orbit. Thus, $p_{x(\lambda)}|_{X_w^v} \neq 0$ if and only if $e_{x(\lambda)} \in f(X_w^v)$. By Borel's fixed point theorem, this amounts to the existence of a T–fixed point $e_x \in X_w^v$ such that $f(e_x) = e_{x(\lambda)}$. Now Lemma 1 (2) completes the proof of the first assertion.

By the preceding arguments, the sections $p_{x(\lambda)}|_{X_w^v}$, $x \in W^P$, $v \leq x \leq w$ do not vanish simultaneously at a T–fixed point of X_w^v. Since these sections are eigenvectors of T, it follows that they have no common zeroes. This implies the second assertion. ☐

Remark. The image of a Richardson variety X_w^v under a morphism $G/P \longrightarrow G/P_\lambda$ need not be another Richardson variety. Consider for example $G = SL(3)$ with simple reflections s_1, s_2. Let $P = B$, $w = s_2 s_1$, $v = s_2$ and $\lambda = \omega_1$ (the fundamental weight fixed by s_2). Then X_w^v is one–dimensional and mapped isomorphically to its image $f(X_w^v)$ in G/P_λ. Since the T–fixed points in $f(X_w^v)$ are e_{ω_1} and $e_{s_2 s_1(\omega_1)}$, it follows that $f(X_w^v)$ is not a Richardson variety.

2. Cohomology vanishing for Richardson varieties

In this section, we assume that the characteristic of k is $p > 0$. Let X be a scheme of finite type over k. Let $F : X \longrightarrow X$ be the absolute Frobenius morphism, that is, F is the identity map on the topological space of X, and $F^\# : \mathcal{O}_X \longrightarrow F_*\mathcal{O}_X$ is the p–th power map. Then X is called *Frobenius split* if the map $F^\#$ is split. We shall need a slight generalization of this notion, involving the composition $F^r = F \circ \cdots \circ F$ (r times), where r is any positive integer.

Definition 2. We say that X is split if there exists a positive integer r such that the map

$$(F^r)^\# : \mathcal{O}_X \longrightarrow F_*^r \mathcal{O}_X$$

splits, that is, there exists an $\mathcal{O}_X$–linear map

$$\varphi : F_*^r \mathcal{O}_X \longrightarrow \mathcal{O}_X$$

such that $\varphi \circ (F^r)^\#$ is the identity; then φ is called a splitting.

We shall also need a slight generalization of the notion of Frobenius splitting relative to an effective Cartier divisor (see [16]).

Definition 3. Let X be a normal variety and D an effective Weil divisor on X, with canonical section s. We say that X is D–split if there exist a positive integer r and an $\mathcal{O}_X$–linear map

$$\psi : F_*^r \mathcal{O}_X(D) \longrightarrow \mathcal{O}_X$$

such that the map

$$\varphi : F_*^r \mathcal{O}_X \longrightarrow \mathcal{O}_X, \ f \longmapsto \psi(fs)$$

is a splitting. Then ψ is called a D–splitting.

We say that a closed subscheme Y of X, with ideal sheaf $\mathcal{I}_Y$, is compatibly D–split if (a) no irreducible component of Y is contained in the support of D, and (b) $\varphi(F_*^r \mathcal{I}_Y) = \mathcal{I}_Y$.

Remarks. (i) Let U be an open subset of X such that $X - U$ has codimension at least 2 in X. Then X is D–split if and only if U is $D \cap U$–split (to see this, let $i : U \longrightarrow X$ be the inclusion, then $i_*\mathcal{O}_U = \mathcal{O}_X$ and $i_*\mathcal{O}_U(D \cap U) = \mathcal{O}_X(D)$ by normality of X).

Let Y be a closed subscheme of X such that $Y \cap U$ is dense in Y. Then Y is compatibly D–split if and only if $Y \cap U$ is compatibly $D \cap U$–split (this is checked by the arguments of [16], 1.4–1.7).

(ii) If X is split compatibly with an effective Weil divisor D, then X is $(p^r - 1)D$–split; to see this, one may assume that X is nonsingular, by (i). Let φ be a compatible splitting, then $\varphi(F^r_*\mathcal{O}_X(-D)) = \mathcal{O}_X(-D)$. Define $\psi : F^r_*\mathcal{O}_X(D) \longrightarrow \mathcal{O}_X$ by $\psi(f\sigma^{p^r-1}) = \sigma\varphi(f\sigma^{-1})$ for any local sections f of $\mathcal{O}_X$ and σ of $\mathcal{O}_X(D)$. Then one checks that ψ is well defined, $\mathcal{O}_X$–linear and satisfies $\psi(fs^{p^r-1}) = \varphi(f)$.

(iii) Let D and E be effective Weil divisors in X, such that $D - E$ is effective. If X is D–split, then it is E–split as well; if, in addition, a closed subscheme Y of X is compatibly D–split, then it is compatibly E–split (this follows from (i) together with [16], Remark 1.3 (ii)).

Lemma 3. *Let D, E be effective Weil divisors on a normal variety X, such that the support of D contains the support of E. If X is D–split, then X is E–split as well. If, moreover, a closed subscheme Y of X is compatibly D–split, then X is compatibly E–split.*

Proof. Let U be the set of those points of X at which D is a Cartier divisor. Then U is an open subset with complement of codimension at least 2 (since U contains the nonsingular locus of X). Moreover, $Y \cap U$ is dense in Y (since U contains the complement of the support of D). Thus, by Remark (i), we may replace X with U, and hence assume that D is a Cartier divisor.

Now let $\psi : F^r_*\mathcal{O}_X(D) \longrightarrow \mathcal{O}_X$ be a D–splitting. We regard ψ as an additive map $\mathcal{O}_X(D) \longrightarrow \mathcal{O}_X$ such that $\psi(s) = 1$, and $\psi(f^{p^r}\sigma) = f\psi(\sigma)$ for any local sections f of $\mathcal{O}_X$ and σ of $\mathcal{O}_X(D)$. For any positive integer n, we set

$$\mathbf{n} = p^{r(n-1)} + p^{r(n-2)} + \cdots + 1$$

(then $\mathbf{1} = 1$), and we define inductively a map

$$\psi^n : F^{rn\#}\mathcal{O}_X(\mathbf{n}D) \longrightarrow \mathcal{O}_X$$

by $\psi^1 = \psi$, and

$$\psi^n(f\sigma^{\mathbf{n}}) = \psi(\psi^{n-1}(f\sigma^{\mathbf{n}-1})\sigma)$$

for any local sections f of $\mathcal{O}_X$ and σ of $\mathcal{O}_X(D)$. Then one may check that ψ^n is well defined and is a $\mathbf{n}D$–splitting of X. If, moreover, a closed subscheme Y is compatibly D–split, then $\psi(F^r_*(\mathcal{I}_Y s)) = \mathcal{I}_Y$. By induction, it follows that $\psi^n(F^{rn}_*(\mathcal{I}_Y s^{\mathbf{n}})) = \mathcal{I}_Y$, so that Y is compatibly $\mathbf{n}D$–split.

Since the support of D contains the support of E, there exists a positive integer n such that $\mathbf{n}D - E$ is effective. Then X is $\mathbf{n}D$–split, so that it is E-split by Remark (ii). ☐

Lemma 4. *Let X be a normal projective variety endowed with an effective Weil divisor D and with a globally generated line bundle L; let Y be a closed subscheme of X. Assume that (a) X is D–split compatibly with Y, and (b) the support of D contains the support of an effective ample divisor. Then $H^i(X, L) = 0 = H^i(Y, L)$ for all $i \geq 1$, and the restriction map $H^0(X, L) \longrightarrow H^0(Y, L)$ is surjective.*

Proof. Choose an effective ample Cartier divisor E, with support contained in the support of D. Then X is E–split compatibly with Y by Lemma 3. Now the assertions follow from [16], 1.12 and 1.13. $\square$

We now apply this to Richardson varieties. By [16], 3.5, the variety G/P is split compatibly with all Schubert varieties and with all opposite Schubert varieties; as a consequence, G/P is split compatibly with all unions of Richardson varieties. By [16] 1.10, it follows that all scheme–theoretical intersections of unions of Richardson varieties are reduced; and using [16], 1.13, this also implies

Lemma 5. *Let λ be a regular dominant character of P and let Z be a union of Richardson varieties in G/P. Then the restriction map $H^0(G/P, L_\lambda) \to H^0(Z, L_\lambda)$ is surjective, and $H^i(Z, L_\lambda) = 0$ for all $i \geq 1$. As a consequence, $H^i(X, L_\lambda \otimes \mathcal{I}_Z) = 0$ for all $i \geq 1$.*

Remark. If we only assume that λ is dominant, then Lemma 5 extends to all unions of Schubert varieties (by [16]), but not to all unions of Richardson varieties. As a trivial example, take $G/P = \mathbb{P}^1$, the projective line with T–fixed points 0 and ∞, and $\lambda = 0$. Then $Z := \{0, \infty\}$ is a union of Richardson varieties, and the restriction map $H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}) \longrightarrow H^0(Z, \mathcal{O}_Z)$ is not surjective. As a less trivial example, take $G/P = \mathbb{P}^1 \times \mathbb{P}^1$, $Z = (\mathbb{P}^1 \times \{0, \infty\}) \cup (\{0, \infty\} \times \mathbb{P}^1)$, and $\lambda = 0$. Then Z is again a union of Richardson varieties, and one checks that $H^1(Z, \mathcal{O}_Z) \neq 0$.

However, Lemma 5 does extend to all dominant characters and to unions of Richardson varieties *with a common index.*

Proposition 1. *Let λ be a dominant character of P and let Z be a union of Richardson varieties X_w^v in G/P, all having the same w. Then the restriction $H^0(G/P, L_\lambda) \to H^0(Z, L_\lambda)$ is surjective, and $H^i(Z, L_\lambda) = 0$ for all $i \geq 1$.*

As a consequence, we have $H^i(X_w^v, L_\lambda(-Z)) = 0$ for all $i \geq 1$, where $v \leq w$ in W^P, and Z is a union of irreducible components of $(\partial X^v)_w$.

Proof. The Schubert variety X_w is split compatibly with the effective Weil divisor ∂X_w and with Z. By assumption, ∂X_w contains no irreducible component of Z. Using Remarks (i) and (ii), it follows that X_w is $(p-1)\partial X_w$–split compatibly with Z. Further, ∂X_w is the support of an ample effective divisor, as follows from Chevalley's formula. Thus, Lemma 4 applies and yields surjectivity of $H^0(X_w, L_\lambda) \longrightarrow H^0(Z, L_\lambda)$ together with vanishing of $H^i(Z, L_\lambda)$ for $i \geq 1$. Now surjectivity of $H^0(G/P, L_\lambda) \longrightarrow H^0(X_w, L_\lambda)$ completes the proof of the first assertion.

In particular, we have $H^i(X_w^v, L_\lambda) = H^i(Z, L_\lambda) = 0$ for all $i \geq 1$, and the restriction map $H^0(X_w^v, L_\lambda) \longrightarrow H^0(Z, L_\lambda)$ is surjective; this implies the second assertion. $\square$

We shall also need the following, more technical vanishing result.

Proposition 2. *Let λ be a dominant character of P and let v, w in W^P such that $v \leq w$. Then*

$$H^i(X_w^v, L_\lambda(-(\partial_\lambda X_w)^v - Z)) = 0$$

for any $i \geq 1$ and for any (possibly empty) union Z of irreducible components of $(\partial X^v)_w$.

Proof. We shall rely on the following result (see [13] Theorem 1). Let $\pi : X \longrightarrow Y$ be a proper morphism of schemes. Let D (resp. E) be a closed subscheme of X (resp. Y) and let i be a positive integer such that:

(i) $\pi^{-1}(E)$ is contained in D (as sets).

(ii) $R^i\pi_*(\mathcal{I}_D) = 0$ outside E.

(iii) X is split compatibly with D.

Then $R^i\pi_*(\mathcal{I}_D) = 0$ everywhere.

To apply this result, consider the restriction

$$f : X_w^v \longrightarrow f(X_w^v)$$

of the natural map $G/P \longrightarrow G/P_\lambda$. Then $L_\lambda = f^*M_\lambda$ for a very ample line bundle M_λ on $f(X_w^v)$. Let Y be the corresponding affine cone over $f(X_w^v)$, with vertex 0 and projection map

$$q : Y - \{0\} \longrightarrow f(X_w^v).$$

And let X be the total space of the line bundle $L_{-\lambda}$ (dual to L_λ), with projection map

$$p : X \longrightarrow X_w^v$$

and zero section X_0. Then the algebra

$$H^0(X, \mathcal{O}_X) = \bigoplus_{n=0}^{\infty} H^0(X_w^v, L_{n\,\lambda})$$

contains $H^0(Y, \mathcal{O}_Y)$ as the subalgebra generated by $H^0(f(X_w^v), M_\lambda)$. The algebra $H^0(X, \mathcal{O}_X)$ is finitely generated, and the corresponding morphism

$$X \longrightarrow \operatorname{Spec} H^0(X, \mathcal{O}_X)$$

is proper, since the line bundle L_λ is globally generated. Moreover, since L_λ is the pullback under f of the very ample line bundle M_λ, the algebra $H^0(X, \mathcal{O}_X)$ is a finite module over its subalgebra $H^0(Y, \mathcal{O}_Y)$. This defines a proper morphism $\pi : X \longrightarrow Y$, and we have $\pi^{-1}(0) = X_0$ (as sets). Moreover, the diagram

$$\begin{tikzcd}
X - X_0 \arrow[r, "\pi"] \arrow[d, "p"'] & Y - \{0\} \arrow[d, "q"'] \\
X_w^v \arrow[r, "f"] & f(X_w^v)
\end{tikzcd}$$

is cartesian, and the vertical maps are principal $\mathbb{G}_m$–bundles.

Now let $D = X_0 \cup p^{-1}((\partial_\lambda X_w)^v \cup Z)$; this is a closed subscheme of X with ideal sheaf

$$\mathcal{I}_D = p^*L_\lambda(-(\partial_\lambda X_w)^v - Z).$$

Let E be the affine cone over $f((\partial_\lambda X_w)^v)$; this is a closed subscheme of Y. We check that the conditions (i), (ii) and (iii) hold.

For (i), notice that

$$\pi^{-1}(E) = X_0 \cup p^{-1}f^{-1}f((\partial_\lambda X_w)^v) = X_0 \cup p^{-1}((\partial_\lambda X_w)^v)$$

(as sets), by the definition of $(\partial_\lambda X_w)^v$. In other words, $\pi^{-1}(E) \subseteq D$ as sets.

For (ii), observe that $\mathcal{I}_D = p^*L_\lambda(-Z)$ outside $\pi^{-1}(E)$. Thus, (ii) is equivalent to $R^i\pi_*(p^*L_\lambda(-Z)) = 0$ outside E. We show that $R^i\pi_*(p^*L_\lambda(-Z)) = 0$ outside 0. Using the cartesian square above, it suffices to check that $R^if_*(L_\lambda(-Z)) = 0$;

by the Leray spectral sequence and the Serre vanishing theorem, this amounts to $H^i(X^v_w, L_{n\lambda}(-Z)) = 0$ for large n. But this holds by Proposition 1.

For (iii), recall that X^v_w is split compatibly with $(\partial_\lambda X_w)^v \cup Z$. Let φ be a compatible splitting; then φ lifts uniquely to a splitting of X compatibly with X_0 and with $p^{-1}((\partial_\lambda X_w)^v \cup Z)$. It follows that X is split compatibly with D.

We thus obtain $R^i\pi_*(\mathcal{I}_D) = 0$ for all $i \geq 1$. Since Y is affine, this amounts to $H^i(X, \mathcal{I}_D) = 0$ for all $i \geq 1$. On the other hand, since the morphism $p : X \longrightarrow X^v_w$ is affine, we obtain that $H^i(X, \mathcal{I}_D)$ is isomorphic to

$$H^i(X^v_w, p_*p^*(L_\lambda(-(\partial_\lambda X_w)^v) - Z)) = H^i(X^v_w, L_\lambda(-(\partial_\lambda X_w)^v - Z) \otimes p_*\mathcal{O}_X).$$

Further, $p_*\mathcal{O}_X$ contains $\mathcal{O}_{X^v_w}$ as a direct factor. This yields

$$H^i(X^v_w, L_\lambda(-(\partial_\lambda X_w)^v - Z)) = 0$$

for all $i \geq 1$. □

Corollary 1. *With the above notations, the restriction map*

$$H^0(X^v_w, L_\lambda(-(\partial_\lambda X_w)^v)) \longrightarrow H^0(X^x_w, L_\lambda(-(\partial_\lambda X_w)^x))$$

is surjective for any $x \in W^P$ *such that* $v \leq x \leq w$.

Proof. We may reduce to the case that $\ell(x) = \ell(v) + 1$, that is, X^x_w is an irreducible component of $(\partial X^v)_w$. Then $H^1(X^v_w, L_\lambda(-(\partial_\lambda X_w)^v - X^x_w)) = 0$ by Proposition 2. Now the assertion follows from the exact sequence

$$0 \to L_\lambda|_{X^v_w}(-(\partial_\lambda X_w)^v - X^x_w) \to L_\lambda|_{X^v_w}(-(\partial_\lambda X_w)^v) \to L_\lambda|_{X^x_w}(-(\partial_\lambda X_w)^x) \to 0.$$

□

Notice finally that Lemma 5, Propositions 1 and 2, and Corollary 1 also hold in characteristic zero, as follows from the argument in [16], 3.7.

3. FILTRATIONS

In this section, we shall obtain natural filtrations of the T–modules $H^0(X^v_w, L_\lambda)$ and $H^0(X^v_w, L_\lambda(-(\partial_\lambda X_w)^v))$ (where X^v_w is a Richardson variety in G/P, and λ is a dominant character of P), and we shall describe their associated graded modules. For this, we shall construct a degeneration of X^v_w embedded diagonally in $G/P \times G/P$, to a union of products of Richardson varieties.

Such a degeneration was obtained in [2], Theorem 16 for $X^v_w = G/B$, by using the wonderful compactification of the adjoint group of G; it was extended to certain subvarieties in G/P, including Schubert varieties, in [1], Theorem 2. Here we follow a direct, self–contained approach, at the cost of repeating some of the arguments in [2] and [1]. We begin by establishing a Künneth decomposition of the class of the diagonal of G/P, in the Grothendieck group of $G/P \times G/P$; such a decomposition is deduced in [2] from a degeneration of the diagonal.

Let $K(G/P \times G/P)$ be the Grothendieck group of the category of coherent sheaves on $G/P \times G/P$. The class of a coherent sheaf $\mathcal{F}$ in this group will be denoted by $[\mathcal{F}]$.

Lemma 6. *We have in* $K(G/P \times G/P)$*:*

$$[\mathcal{O}_{\mathrm{diag}(G/P)}] = [\mathcal{O}_{\bigcup_{x \in W^P} X_x \times X^x}]$$
$$= \sum_{x \in W^P} [\mathcal{O}_{X_x}(-\partial X_x) \otimes \mathcal{O}_{X^x}] = \sum_{x \in W^P} [\mathcal{O}_{X_x} \otimes \mathcal{O}_{X^x}(-\partial X^x)].$$

Proof. Let $Z = \bigcup_{x\in W^P} X_x \times X^x$. We first claim that

$$[\mathcal{O}_Z] = \sum_{x\in W^P} [\mathcal{O}_{X_x}(-\partial X_x) \otimes \mathcal{O}_{X^x}].$$

Let $W^P = \{x_1, \ldots, x_N\}$ be an indexing such that $i \le j$ whenever $x_i \le x_j$. Then one obtains easily

$$(X_{x_i} \times X^{x_i}) \cap (\bigcup_{j<i} X_{x_j} \times X^{x_j}) = \partial X_{x_i} \times X^{x_i}.$$

Now let $\mathcal{O}_{Z,\ge i}$ be the subsheaf of $\mathcal{O}_Z$ consisting of those sections that vanish on $X_{x_j} \times X^{x_j}$ for each $j < i$. Then the $\mathcal{O}_{Z,\ge i}$ are a decreasing filtration of $\mathcal{O}_Z$, and

$$\mathcal{O}_{Z,\ge i}/\mathcal{O}_{Z,\ge i+1} \simeq \mathcal{O}_{Z,\ge i}|_{X_{x_i}\times X^{x_i}} \simeq \mathcal{O}_{X_i}(-\partial X_i) \otimes \mathcal{O}_{X^i}.$$

Further, $[\mathcal{O}_Z] = \sum_{i=1}^N [\mathcal{O}_{Z,\ge i}/\mathcal{O}_{Z,\ge i+1}]$ in $K(G/P \times G/P)$. This implies our claim.

One checks similarly that

$$[\mathcal{O}_Z] = \sum_{x\in W^P} [\mathcal{O}_{X_x} \otimes \mathcal{O}_{X^x}(-\partial X^x)],$$

using the increasing filtration of $\mathcal{O}_Z$ by the subsheaves $\mathcal{O}_{Z,\le i}$ consisting of those sections that vanish on $X_{x_j} \times X^{x_j}$ for each $j > i$.

To complete the proof, it suffices to check that

$$(*) \qquad [\mathcal{O}_{\mathrm{diag}(G/P)}] = \sum_{x\in W^P} [\mathcal{O}_{X_x} \otimes \mathcal{O}_{X^x}(-\partial X^x)].$$

For this, we recall some well–known facts on Grothendieck groups of flag varieties.

Since the Bruhat cells C_x, $x \in W^P$, form a cellular decomposition of G/P, the abelian group $K(G/P)$ is generated by the $[\mathcal{O}_{X_x}]$, $x \in W^P$. Likewise, it is generated by the $[\mathcal{O}_{X^y}]$, $y \in W^P$. Further, $K(G/P)$ is a ring for the product

$$[\mathcal{F}] \cdot [\mathcal{G}] = \sum_{i\ge 0} (-1)^i [Tor_i^{G/P}(\mathcal{F}, \mathcal{G})],$$

and the Euler characteristic of coherent sheaves yields an additive map

$$\begin{array}{cccc} \chi : & K(G/P) & \longrightarrow & \mathbb{Z} \\ & [\mathcal{F}] & \longmapsto & \chi(\mathcal{F}). \end{array}$$

Since X_x and X^y are Cohen–Macaulay and intersect properly in G/P, we have $Tor_i^{G/P}(\mathcal{O}_{X_x}, \mathcal{O}_{X^y}) = 0$ for all $i \ge 1$ (see [1], Lemma 1 for details). And since the intersection $X_x \cap X^y = X_x^y$ is reduced, we obtain

$$Tor_0^{G/P}(\mathcal{O}_{X_x}, \mathcal{O}_{X^y}) = \mathcal{O}_{X_x} \otimes_{\mathcal{O}_{G/P}} \mathcal{O}_{X^y} = \begin{cases} \mathcal{O}_{X_x^y} & \text{if } y \le x, \\ 0 & \text{otherwise.} \end{cases}$$

Together with Proposition 1, it follows that

$$\chi([\mathcal{O}_{X_x}] \cdot [\mathcal{O}_{X^y}]) = \begin{cases} 1 & \text{if } y \le x, \\ 0 & \text{otherwise.} \end{cases}$$

On the other hand, we have in $K(G/P)$,

$$[\mathcal{O}_{X^y}] = \sum_{z\in W^P,\ z\ge y} [\mathcal{O}_{X^z}(-\partial X^z)]$$

(more generally, for any union Z of opposite Schubert varieties, we have $[\mathcal{O}_Z] = \sum_{z\in W^P,\ X^z\subseteq Z}[\mathcal{O}_{X^z}(-\partial X^z)]$ by an easy induction, using the fact that intersections of unions of opposite Schubert varieties are reduced.) It follows that

$$\chi([\mathcal{O}_{X_x}]\cdot[\mathcal{O}_{X^y}(-\partial X^y)]) = \delta_{x,y}.$$

Thus, the $[\mathcal{O}_{X_x}]$, $x \in W^P$ form a basis for $K(G/P)$; further, the bilinear form $K(G/P)\times K(G/P) \longrightarrow \mathbb{Z}$, $(u,v)\longmapsto \chi(u\cdot v)$ is nondegenerate, and the dual basis of the $[\mathcal{O}_{X_x}]$ with respect to this pairing consists of the $[\mathcal{O}_{X^x}(-\partial X^x)]$.

It follows that a given class $u \in K(G/P\times G/P)$ is zero if and only if $\chi(u\cdot[\mathcal{O}_{X^y}(-\partial X^y)\otimes\mathcal{O}_{X_z}]) = 0$ for all $y,z\in W^P$. Further,

$$\chi([\mathcal{O}_{\mathrm{diag}(G/P)}]\cdot[\mathcal{O}_{X^y}(-\partial X^y)\otimes\mathcal{O}_{X_z}]) = \chi([\mathcal{O}_{X^y}(-\partial X^y)]\cdot[\mathcal{O}_{X_z}]) = \delta_{y,z};$$

whereas

$$\chi(\sum_{x\in W^P}[\mathcal{O}_{X_x}\otimes\mathcal{O}_{X^x}(-\partial X^x)]\cdot[\mathcal{O}_{X^y}(-\partial X^y)\otimes\mathcal{O}_{X_z}])$$
$$= \sum_{x\in W^P}\chi([\mathcal{O}_{X_x}]\cdot[\mathcal{O}_{X^y}(-\partial X^y)])\,\chi([\mathcal{O}_{X^x}(-\partial X^x)]\cdot[\mathcal{O}_{X_z}]) = \sum_{x\in W^P}\delta_{x,y}\delta_{x,z} = \delta_{y,z}.$$

This completes the proof of $(*)$, and hence of the lemma. ☐

We now construct a degeneration of the diagonal of any Richardson variety. Let $\theta:\mathbb{G}_m\longrightarrow T$ be a regular dominant one–parameter subgroup. Let $\mathcal{X}$ be the closure in $G/P\times G/P\times\mathbb{A}^1$ of the subset

$$\{(x,\theta(s)x,s)\mid x\in G/P,\ s\in k^*\}.$$

The variety $\mathcal{X}$ is invariant under the action of $\mathbb{G}_m\times T$ defined by

$$(s,t)(x,y,z) = (tx,\theta(s)ty,sz).$$

Consider the projections

$$p_1,p_2:\mathcal{X}\longrightarrow G/P,\ \pi:\mathcal{X}\longrightarrow\mathbb{A}^1.$$

Clearly, π is proper and flat, and its fibers identify with closed subschemes of $G/P\times G/P$ via $p_1\times p_2$; this identifies the fiber at 1 with $diag(G/P)\simeq G/P$. By equivariance, every "general" fiber $\pi^{-1}(z)$, where $z\neq 0$, is also isomorphic to G/P.

We shall denote the "special" (scheme–theoretical) fiber $\pi^{-1}(0)$ by F, with projections

$$q_1,q_2:F\longrightarrow G/P.$$

Next let v,w in W^P such that $v\le w$. Let $\mathcal{X}_w^v$ be the closure in $G/P\times G/P\times\mathbb{A}^1$ of the subset

$$\{(x,\theta(s)x,s)\mid x\in X_w^v,\ s\in k^*\}.$$

This is a subvariety of $\mathcal{X}\cap(X_w^v\times X_w^v\times\mathbb{A}^1)$, invariant under the action of $\mathbb{G}_m\times T$. We shall denote the restrictions of p_1, p_2, π to $\mathcal{X}_w^v$ by the same letters; then π is again proper and flat, and its "general" fibers are isomorphic to X_w^v. Let F_w^v be the "special" fiber, with projections q_1, q_2 to X_w^v.

Lemma 7. (1) *The schemes F and F_w^v are reduced. Further,*

$$F = \bigcup_{x\in W^P} X_x\times X^x \text{ and } F_w^v = \bigcup_{x\in W^P,\ v\le x\le w} X_x^v\times X_w^x.$$

(2) *Choose a total ordering* $\leq_t$ *of* W^P *such that* $x \leq_t y$ *whenever* $x \leq y$. *For* $x \in W^P$, *let* $\mathcal{O}_{F,\leq_t x}$ *(resp.* $\mathcal{O}_{F,\geq_t x}$*) be the subsheaf of* $\mathcal{O}_F$ *consisting of those sections that vanish identically on* $X_y \times X^y$ *for each* $y >_t x$ *(resp.* $y <_t x$*). Then the* $\mathcal{O}_{F,\leq_t x}$ *(resp.* $\mathcal{O}_{F,\geq_t x}$*) are an ascending (resp. descending) filtration of* $\mathcal{O}_F$, *with associated graded*

$$\bigoplus_{x\in W^P} \mathcal{O}_{X_x} \otimes \mathcal{O}_{X^x}(-\partial X^x), \text{ resp. } \bigoplus_{x\in W^P} \mathcal{O}_{X_x}(-\partial X_x) \otimes \mathcal{O}_{X^x}.$$

The induced filtrations on the structure sheaf $\mathcal{O}_{F_w^v}$ *have associated graded*

$$\bigoplus_{x\in W^P,\, v\leq x\leq w} \mathcal{O}_{X_x^v} \otimes \mathcal{O}_{X_w^x}(-(\partial X^x)_w), \text{ resp. } \bigoplus_{x\in W^P,\, v\leq x\leq w} \mathcal{O}_{X_x^v}(-(\partial X_x)^v) \otimes \mathcal{O}_{X_w^x}.$$

The induced map

$$\mathcal{O}_{F_w^v,\leq_t} \longrightarrow \operatorname{gr}_{\leq_t} \mathcal{O}_{F_w^v}$$

is just the restriction to $X_x^v \times X_w^x$; *the same holds for the induced map*

$$\mathcal{O}_{F_w^v,\geq_t} \longrightarrow \operatorname{gr}_{\geq_t} \mathcal{O}_{F_w^v}.$$

Proof. (1) Let $x \in W^P$. We claim that

$$C_x \times C^x \subseteq F.$$

To check this, consider the subset xC^1 of G/P. This is an open T–stable neighborhood of e_x in G/P, isomorphic to affine space where T acts linearly with weights the $\alpha \in x(R^- - R_P)$. Choose corresponding coordinate functions z_α on xC^1, then C_x (resp. C^x) is the closed subset of xC^1 where $z_\alpha = 0$ whenever $\alpha \in R^-$ (resp. $\alpha \in R^+$). Let $z = (z_\alpha) \in xC^1$, then

$$\theta(s)z = (s^{\langle \alpha,\theta\rangle} z_\alpha).$$

Denote by z_+ (resp. z_-) the point of C_x (resp. C^x) with coordinates z_α, $\alpha \in R^+$ (resp. $\alpha \in R^-$). Let $z'(s)$ be the point of xC^1 with α–coordinate z_α if $\alpha \in R^+$, and $\theta(s^{-1})z_\alpha$ otherwise. Since θ is regular dominant, we obtain

$$\lim_{s\to 0}(z'(s), \theta(s)z'(s), s) = (z_+, z_-, 0).$$

Since z_+ (resp. z_-) is an arbitrary point of C_x (resp. C^x), this proves our claim.

The claim implies that F contains $\bigcup_{x\in W^P} X_x \times X^x$ as a reduced closed subscheme. Let $\mathcal{I}$ be the ideal sheaf of this closed subscheme in $\mathcal{O}_F$; we regard $\mathcal{I}$ as a coherent sheaf on $G/P \times G/P$. Then we have in $K(G/P \times G/P)$:

$$[\mathcal{I}] = [\mathcal{O}_F] - [\mathcal{O}_{\bigcup_{x\in W^P} X_x\times X^x}] = [\mathcal{O}_{\operatorname{diag}(G/P)}] - [\mathcal{O}_{\bigcup_{x\in W^P} X_x\times X^x}] = 0,$$

where the first equality follows from the definition of $\mathcal{I}$, the second one from the fact that $\pi : \mathcal{X} \to \mathbb{A}^1$ is flat with fibers F and $\operatorname{diag}(G/P)$, and the third one from Lemma 6. As a consequence, $\mathcal{I}$ is trivial (e.g., since its Hilbert polynomial is zero); this completes the proof for F.

In the case of F_w^v, notice that

$$F_w^v \subseteq F \cap (X^v \times X_w) = (\bigcup_{x\in W^P} X_x \times X^x) \cap (X^v \times X_w) = \bigcup_{x\in W^P, v\leq x\leq w} X_x^v \times X_w^x$$

as schemes, since all involved scheme–theoretic intersections are reduced. Further, we have in the Chow ring of $G/P \times G/P$:

$$\begin{aligned}[F_w^v] = [\mathrm{diag}(X_w^v)] = [\mathrm{diag}(G/P) \cap (X^v \times X_w)] = [\mathrm{diag}(G/P)] \cdot [X^v \times X_w] \\ = [F] \cdot [X^v \times X_w] = [F \cap (X^v \times X_w)] = \sum_{x \in W^P,\ v \le x \le w} [X_x^v \times X_w^x],\end{aligned}$$

since all involved intersections are proper and reduced. It follows that F_w^v equals $\bigcup_{x \in W^P,\ v \le x \le w} X_x^v \times X_w^x$.

(2) has been established in the case of F, at the beginning of the proof of Lemma 6. The general case is similar. ☐

Next let λ be a dominant character of P. This yields T–linearized line bundles $q_2^* L_\lambda$ on F and on F_w^v, together with "adjunction" maps $H^0(G/P, L_\lambda) \longrightarrow H^0(F, q_2^* L_\lambda)$ and $H^0(X_w^v, L_\lambda) \longrightarrow H^0(F_w^v, q_2^* L_\lambda)$.

Proposition 3. (1) *These maps are isomorphisms, and the restriction map*

$$H^0(F, q_2^* L_\lambda) \longrightarrow H^0(F_w^v, q_2^* L_\lambda)$$

is surjective.

(2) *The ascending filtration of $\mathcal{O}_F$ yields an ascending filtration of the T–module $H^0(F, q_2^* L_\lambda)$, with associated graded*

$$\bigoplus_{x \in W^P} H^0(X^x, L_\lambda(-\partial X^x)).$$

(3) *The image of this filtration under restriction to F_w^v has associated graded*

$$\bigoplus_{x \in W^P,\ v \le x \le w} H^0(X_w^x, L_\lambda(-(\partial X^x)_w)).$$

Hence this is the associated graded of an ascending filtration

$$H^0(X_w^v, L_\lambda)_{\le_t x}, \ \ v \le x \le w$$

of $H^0(X_w^v, L_\lambda)$, compatible with the T–action and restrictions to smaller Richardson varieties.

(4) *The subspace*

$$H^0(X_w^v, L_\lambda)_{\le_t x} \subseteq H^0(X_w^v, L_\lambda)$$

consists of those sections that vanish identically on X_w^y for all $y >_t x$. Further, the map

$$H^0(X_w^v, L_\lambda)_{\le_t x} \longrightarrow \mathrm{gr}_x H^0(X_w^v, L_\lambda) = H^0(X_w^x, L_\lambda(-(\partial X^x)_w))$$

is just the restriction to X_w^x.

Proof. (1) We have

$$H^0(F, q_2^* L_\lambda) = H^0(G/P, q_{2*} q_2^* L_\lambda) = H^0(G/P, L_\lambda \otimes q_{2*} \mathcal{O}_F)$$

by the projection formula. Further, the associated graded of the descending filtration of $\mathcal{O}_F$ is acyclic for q_{2*}; indeed, $H^i(X_x, \mathcal{O}_{X_x}(-\partial X_x)) = 0$ for all $i \ge 1$ and all $x \in W^P$, by Proposition 1. Notice also that $H^0(X_x, \mathcal{O}_{X_x}(-\partial X_x)) = 0$ for all $x \ne 1$, since ∂X_x is a nonempty subscheme of the complete variety X_x. It follows that the natural map $\mathcal{O}_{G/P} \longrightarrow q_{2*}\mathcal{O}_F$ is an isomorphism. Hence the same holds for the map $H^0(G/P, L_\lambda) \longrightarrow H^0(F, q_2^* L_\lambda)$.

Likewise, the map $H^0(X_w^v, L_\lambda) \longrightarrow H^0(F_w^v, q_2^* L_\lambda)$ is an isomorphism as well. Since the restriction map $H^0(G/P, L_\lambda) \longrightarrow H^0(X_w^v, L_\lambda)$ is surjective by Proposition 1, the same holds for $H^0(F, q_2^* L_\lambda) \longrightarrow H^0(F_w^v, q_2^* L_\lambda)$.

(2) By Lemma 7 again, the ascending filtration of $\mathcal{O}_F$ yields one on $q_2^* L_\lambda$, with associated graded

$$\bigoplus_{x \in W^P} \mathcal{O}_{X_x} \otimes L_\lambda|_{X^x}(-\partial X^x).$$

The latter is acyclic by Proposition 1. It follows that $H^0(F, q_2^* L_\lambda)$ has an ascending filtration with associated graded as claimed.

(3) is checked similarly.

(4) We have

$$\begin{aligned} H^0(X_w^v, L_\lambda)_{\le_t x} = H^0(F_w^v, q_2^* L_\lambda)_{\le_t x} &= H^0(F_w^v, q_2^* L_\lambda \otimes \mathcal{I}_{\cup_{y >_t x} X_y^v \times X_w^y}) \\ &= H^0(X_w^v, L_\lambda \otimes q_{2*} \mathcal{I}_{\cup_{y >_t x} X_y^v \times X_w^y}) \end{aligned}$$

by the projection formula. Further, $q_{2*}\mathcal{O}_{F_w^v} = \mathcal{O}_{X_w^v}$ as seen in the proof of (1). It follows that

$$q_{2*} \mathcal{I}_{\cup_{y >_t x} X_y^v \times X_w^y} = \mathcal{I}_{\cup_{y >_t x} X_w^y}.$$

This implies our statement. ☐

We now construct a similar filtration of the T–submodule

$$H^0(X_w^v, L_\lambda(-(\partial_\lambda X_w)^v)) \subseteq H^0(X_w^v, L_\lambda).$$

For this, we define a sheaf on F_w^v by

$$q_2^* L_\lambda(-(\partial_\lambda X_w)^v) = (q_2^* L_\lambda) \otimes_{\mathcal{O}_{F_w^v}} \mathcal{I}_{q_2^{-1}((\partial_\lambda X_w)^v)}.$$

This is a subsheaf of $q_2^* L_\lambda$; it may differ from the pullback sheaf of $L_\lambda(-(\partial_\lambda X_w)^v)$ under q_2. We also have an "adjunction" map

$$H^0(X_w^v, L_\lambda(-(\partial_\lambda X_w)^v)) \longrightarrow H^0(F_w^v, q_2^* L_\lambda(-(\partial_\lambda X_w)^v)).$$

In particular, we obtain a map

$$H^0(X_w, L_\lambda(-\partial_\lambda X_w)) \longrightarrow H^0(F_w, q_2^* L_\lambda(-\partial_\lambda X_w)).$$

Proposition 4. (1) *These maps are isomorphisms, and the restriction map*

$$H^0(F_w, q_2^* L_\lambda(-\partial_\lambda X_w)) \longrightarrow H^0(F_w^v, q_2^* L_\lambda(-(\partial_\lambda X_w)^v))$$

is surjective.

(2) *The ascending filtration of $\mathcal{O}_{F_w}$ yields an ascending filtration of the T–module $H^0(F_w, q_2^* L_\lambda(-\partial_\lambda X_w))$, with associated graded*

$$\bigoplus_{x \in W^P,\ x \le w} H^0(X_w^x, L_\lambda(-(\partial_\lambda X_w)^x - (\partial X^x)_w)).$$

(3) *The image of this filtration under restriction to F_w^v has associated graded*

$$\bigoplus_{x \in W^P,\ v \le x \le w} H^0(X_w^x, L_\lambda(-(\partial_\lambda X_w)^x - (\partial X^x)_w)).$$

Hence this is also the associated graded of an ascending filtration

$$H^0(X_w^v, L_\lambda(-(\partial_\lambda X_w)^v))_{\le_t x},\ v \le x \le w$$

of $H^0(X_w^v, L_\lambda(-(\partial_\lambda X_w)^v))$, compatible with the T–action and with restrictions to smaller Richardson varieties.

(4) *The subspace*

$$H^0(X_w^v, L_\lambda(-(\partial_\lambda X_w)^v))_{\leq_t x} \subseteq H^0(X_w^v, L_\lambda)$$

consists of those sections that vanish identically on $(\partial_\lambda X_w)^v$ *and on* X_w^y *for all* $y >_t x$. *Further, the map*

$$\begin{aligned} H^0(X_w^v, L_\lambda(-(\partial_\lambda X_w)^v))_{\leq_t x} \longrightarrow \operatorname{gr}_x H^0(X_w^v, L_\lambda(-(\partial_\lambda X_w)^v)) \\ = H^0(X_w^x, L_\lambda(-(\partial_\lambda X_w)^x - (\partial X^x)_w)) \end{aligned}$$

is just the restriction to X_w^x.

Proof. It follows from Lemma 7 that the sheaf $q_2^* L_\lambda(-(\partial_\lambda X_w)^v)$ on F_w^v admits a descending filtration with associated graded

$$\bigoplus_{x \in W^P, v \leq x \leq w} \mathcal{O}_{X_x^v}(-(\partial X_x)^v) \otimes L_\lambda|_{X_w^x}(-(\partial_\lambda X_w)^v),$$

and an ascending filtration with associated graded

$$\bigoplus_{x \in W^P, v \leq x \leq w} \mathcal{O}_{X_x^v} \otimes L_\lambda|_{X_w^x}(-(\partial_\lambda X_w)^v - (\partial X_x)^v).$$

As in the proof of Proposition 3, the associated graded of the first filtration is acyclic for q_{2*}; it follows that the adjunction map is an isomorphism. Further, the restriction map

$$H^0(X_w, L_\lambda(-\partial_\lambda X_w)) \longrightarrow H^0(X_w^v, L_\lambda(-(\partial_\lambda X_w)^v))$$

is surjective by Corollary 1. Finally, the associated graded of the second filtration is acyclic by Proposition 2. These facts imply our statements, as in the proof of Proposition 3. □

Remarks. (1) By Proposition 3, the $H^0(G/P, L_\lambda)_{\leq_t x}$ are B^-–submodules of $H^0(G/P, L_\lambda)$. Likewise, the descending filtration of $\mathcal{O}_F$ yields a descending filtration of $H^0(G/P, L_\lambda)$ by B–submodules $H^0(G/P, L_\lambda)_{\geq_t x}$, consisting of those sections that vanish on X_y whenever $y <_t x$.

(2) We may have defined directly the preceding filtrations by Propositions 3 (4) and 4 (4), without using the degeneration of the diagonal constructed in Lemma 7. In fact, this alternative definition suffices for the construction of a standard basis in the next section. But the degeneration of the diagonal will play an essential rôle in the section on standard products.

4. CONSTRUCTION OF A STANDARD BASIS

In this section, we fix a dominant weight λ and we consider Richardson varieties in G/P, where $P = P_\lambda$. We shall construct a basis of $H^0(G/P, L_\lambda)$ adapted to the filtrations of Propositions 3 and 4. We first prove the key

Lemma 8. *Let* $v \leq w \in W^\lambda$. *Then any element of* $H^0(X_w^v, L_\lambda(-(\partial X_w)^v - (\partial X^v)_w))$ *can be lifted to an element of* $H^0(G/P, L_\lambda)$ *that vanishes identically on all Schubert varieties* X_y, $y \not\geq w$, *and on all opposite Schubert varieties* X^x, $x \not\leq v$.

Proof. Put

$$X = X_w^v \text{ and } Y = (\bigcup_{y \not\geq w} X_y) \cup (\bigcup_{x \not\leq v} X^x).$$

Notice that

$$X \cap Y = (\partial X_w)^v \cup (\partial X^v)_w$$

(as schemes), since any intersection of unions of Richardson varieties is reduced. This yields an exact sequence

$$0 \longrightarrow \mathcal{I}_{X \cup Y} \longrightarrow \mathcal{I}_Y \longrightarrow \mathcal{I}_Y \otimes_{\mathcal{O}_{G/P}} \mathcal{O}_X \simeq \mathcal{O}_X(-X \cap Y) \longrightarrow 0.$$

Tensoring by L_λ and taking the associated long exact sequence of cohomology groups yields an exact sequence

$$H^0(G/P, L_\lambda \otimes \mathcal{I}_Y) \longrightarrow H^0(X, L_\lambda(-X \cap Y)) \longrightarrow H^1(X, L_\lambda \otimes \mathcal{I}_{X \cup Y}).$$

Further, $H^1(X, L_\lambda \otimes \mathcal{I}_{X \cup Y}) = 0$ by Lemma 5; this completes the proof. □

Definition 4. For any $v \leq w \in W^\lambda$, let

$$H_w^v(\lambda) = H^0(X_w^v, L_\lambda(-(\partial X_w)^v - (\partial X^v)_w))$$

and

$$\chi_w^v(\lambda) = \{\text{the weights of the } T\text{–module } H_w^v(\lambda)\},$$

these weights being counted with multiplicity. Let

$$\{p_{w,v}^\xi,\ \xi \in \chi_w^v(\lambda)\}$$

be a basis for $H_w^v(\lambda)$, where each $p_{w,v}^\xi$ is a T–eigenvector of weight ξ.

For any triple (w, v, ξ) as above, let p_π be a lift of $p_{w,v}^\xi$ in $H^0(G/P, L_\lambda)$ such that:

p_π is a T–eigenvector of weight ξ, and
p_π vanishes identically on all X_y, $y \not\geq w$ and on all X^x, $x \not\leq v$.

(The existence of such lifts follows from Lemma 8.) If $v = w$, then X_w^v consists of the point e_w, and hence $\chi_w^v(\lambda)$ consists of the weight $-w(\lambda)$. We then denote the unique $p_{w,v}^\xi$ by p_w. Its lift to $H^0(G/P, L_\lambda)$ is unique; it is the extremal weight vector p_w.

Definition 5. Let $\pi = (w, v, \xi)$ be as in Definition 4. We set $i(\pi) = w$, $e(\pi) = v$, and call them respectively the initial and end elements of π.

By construction of the p_π and Lemma 1, we obtain:

Lemma 9. *With notation as above, we have for $x, y \in W^\lambda$,*

$$p_\pi|_{X_y^x} \neq 0 \Longleftrightarrow X_{i(\pi)}^{e(\pi)} \subseteq X_y^x \Longleftrightarrow x \leq e(\pi) \leq i(\pi) \leq y.$$

Proposition 5. *The restrictions to X_v^w of the p_π where $i(\pi) = w$, $e(\pi) \geq v$ form a basis for the T–module $H^0(X_w^v, L_\lambda(-(\partial X_w)^v))$, adapted to its ascending filtration $\leq_t$ of Proposition 4.*

Proof. By construction, p_π vanishes identically on X^x for any $x \not\leq e(\pi)$, and hence for any $x >_t e(\pi)$. Thus, $p_\pi \in H^0(G/P, L_\lambda)_{\leq_t e(\pi)}$ by Proposition 3. Further, the image of p_π in the associated graded is just its restriction to $X^{e(\pi)}$.

Together with Lemma 9, it follows that the restrictions of the p_π to X_w^v belong to $H^0(X_w^v, L_\lambda(-(\partial X_w)^v))_{\leq_t e(\pi)}$, and that their images in the associated graded $H_w^{e(\pi)}(\lambda)$ are the restrictions of the p_π to $X_w^{e(\pi)}$; by construction, these images form a basis of $H_w^{e(\pi)}(\lambda)$. □

Now the T–module $H^0(X_w^v, L_\lambda)$ has a descending filtration by the submodules

$$H^0(X_w^v, L_\lambda(-(\partial X_w)^v))_{\geq_t x}$$

consisting of those sections that vanish identically on X_y^v whenever $y <_t x$. Also, as in Proposition 3, the associated graded is

$$\bigoplus_{x \in W^P, v \leq x \leq w} H^0(X_x^v, L_\lambda(-(\partial X_x)^v)).$$

Further, we may check as in the proof of Proposition 5 that

$$p_\pi|_{X_w^v} \in H^0(X_w^v, L_\lambda(-(\partial X_w)^v))_{\geq_t i(\pi)}$$

whenever $i(\pi) \geq w$, and the image of p_π in the associated graded is just its restriction to $X_{i(\pi)}^v$. Together with Proposition 5, this implies

Proposition 6. *The restrictions to X_w^v of the p_π where $v \leq e(\pi) \leq i(\pi) \leq w$ form a basis of $H^0(X_w^v, L_\lambda)$; the p_π where $v \not\leq e(\pi)$ or $i(\pi) \not\leq w$ form a basis of the kernel of the restriction map $H^0(G/P, L_\lambda) \longrightarrow H^0(X_w^v, L_\lambda)$.*

In view of Proposition 6, the restriction to p_π to X_y^x, where $x \leq e(\pi) \leq i(\pi) \leq y$, will be denoted by just p_π.

Definition 6. Set

$$\Pi(\lambda) = \{(v, w, \xi) \mid v, w \in W^\lambda,\ v \leq w,\ \xi \in \chi_w^v(\lambda)\}.$$

For any $v, w \in W^\lambda, v \leq w$, set

$$\Pi_w^v(\lambda) := \{\pi \mid v \leq e(\pi) \leq i(\pi) \leq w\}.$$

In view of Lemma 9, we have, $\Pi_w^v(\lambda) = \{\pi \in \Pi(\lambda) \mid p_\pi|_{X_w^v} \neq 0\}$.

More generally, for a union Z of Richardson varieties, define

$$\Pi_Z(\lambda) = \{\pi \in \Pi(\lambda) \mid p_\pi|_Z \neq 0\}.$$

Theorem 1. *Let Z be a union of Richardson varieties. Then $\{p_\pi|_Z, \pi \in \Pi_Z(\lambda)\}$ is a basis for $H^0(Z, L_\lambda)$, and $\{p_\pi, \pi \in \Pi(\lambda) - \Pi_Z(\lambda)\}$ is a basis for the kernel of the restriction map $H^0(G/P, L_\lambda) \longrightarrow H^0(Z, L_\lambda)$.*

Proof. By definition of $\Pi_Z(\lambda)$, it suffices to prove the first assertion. Let $Z = \bigcup_{i=1}^r X_{w_i}^{v_i}$. We shall prove the result by induction on r and $\dim Z$. Write $Z = X \cup Y$ where $X = X_{w_i}^{v_i}$ for some i, and $\dim X = \dim Z$. Then $X \cap Y$ is a union of Richardson varieties of dimension $< \dim Z$. Consider the exact sequence

$$(*) \qquad 0 \to \mathcal{O}_Z = \mathcal{O}_{X \cup Y} \to \mathcal{O}_X \oplus \mathcal{O}_Y \to \mathcal{O}_{X \cap Y} \to 0.$$

Tensoring by L_λ, taking global sections and using the vanishing of $H^1(Z, L_\lambda)$ (Lemma 5), we obtain the exact sequence

$$0 \longrightarrow H^0(Z, L_\lambda) \longrightarrow H^0(X, L_\lambda) \oplus H^0(Y, L_\lambda) \longrightarrow H^0(X \cap Y, L_\lambda) \longrightarrow 0.$$

In particular, denoting $\dim H^0(Z, L_\lambda)$ by $h^0(Z, L_\lambda)$ etc., we obtain,

$$h^0(Z, L_\lambda) = h^0(X, L_\lambda) + h^0(Y, L_\lambda) - h^0(X \cap Y, L_\lambda).$$

We have by hypothesis (and induction hypothesis), $h^0(X, L_\lambda) = \#\Pi_X(\lambda)$, $h^0(Y, L_\lambda) = \#\Pi_Y(\lambda)$, $h^0(X \cap Y, L_\lambda) = \#\Pi_{X \cap Y}(\lambda)$. Thus we obtain,

$$(1) \qquad h^0(Z, L_\lambda) = \#\Pi_X(\lambda) + \#\Pi_Y(\lambda) - \#\Pi_{X \cap Y}(\lambda).$$

On the other hand, we have

$$\Pi_Z(\lambda) = (\Pi_X(\lambda) \mathbin{\dot\cup} \Pi_Y(\lambda)) \setminus \Pi_{X\cap Y}(\lambda). \tag{2}$$

From (1) and (2), we obtain, $h^0(Z, L_\lambda) = \#\Pi_Z(\lambda)$. Further, the $p_\pi|_Z$, $\pi \in \Pi_Z(\lambda)$, span $H^0(Z, L_\lambda)$ (since the p_π, $\pi \in \Pi(\lambda)$, span $H^0(G/P, L_\lambda)$, and the restriction map $H^0(G/P, L_\lambda) \to H^0(Z, L_\lambda)$ is surjective). Thus, the $p_\pi|_Z$, $\pi \in \Pi_Z(\lambda)$, are a basis of $H^0(Z, L_\lambda)$. □

5. Standard monomials

Let λ, μ be dominant weights such that $P_\lambda = P_\mu := P$. Consider the product map

$$H^0(G/P, L_\lambda) \otimes H^0(G/P, L_\mu) \longrightarrow H^0(G/P, L_{\lambda+\mu}).$$

This map is surjective by [16] 2.2 and 3.5. Using Proposition 1, it follows that the product map

$$H^0(X_w^v, L_\lambda) \otimes H^0(X_w^v, L_\mu) \longrightarrow H^0(X_w^v, L_{\lambda+\mu})$$

is also surjective, for any $v \le w$ in W^P. We shall construct a basis for $H^0(X_w^v, L_{\lambda+\mu})$ from the bases of $H^0(X_w^v, L_\lambda)$, $H^0(X_w^v, L_\mu)$ obtained in Theorem 1. For this, we need the following

Definition 7. Let $v, w \in W^P, v \le w$. Let $\varphi \in \Pi_w^v(\lambda)$ and $\psi \in \Pi_w^v(\mu)$. The pair (φ, ψ) is called standard on X_w^v if

$$v \le e(\psi) \le i(\psi) \le e(\varphi) \le i(\varphi) \le w.$$

Then the product $p_\varphi p_\psi \in H^0(G/P, L_{\lambda+\mu})$ is called standard on X_w^v as well.

Clearly, we have

Lemma 10. *Let $p_\varphi p_\psi$ be a standard product on G/P and let $v \le w \in W^P$. Then*

$$p_\varphi p_\psi|_{X_w^v} \ne 0 \Longleftrightarrow v \le e(\psi) \le i(\varphi) \le w.$$

Proposition 7. *The standard products on X_w^v form a basis of $H^0(X_w^v, L_{\lambda+\mu})$. The standard products on G/P that are not standard on X_w^v form a basis of the kernel of the restriction map $H^0(G/P, L_{\lambda+\mu}) \longrightarrow H^0(X_w^v, L_{\lambda+\mu})$.*

Proof. Consider the T–linearized invertible sheaf $q_1^* L_\mu \otimes q_2^* L_\lambda$ on F_w^v. By Lemma 7, the ascending filtration of $\mathcal{O}_{F_w^v}$ yields one of that sheaf, with associated graded

$$\bigoplus_{x \in W^P, v \le x \le w} L_\mu|_{X_x^v}(-(\partial X_x)^v) \otimes L_\lambda|_{X_w^x}.$$

By Proposition 1, the latter sheaf is acyclic. This yields an ascending filtration of the T–module $H^0(F_w^v, q_1^* L_\mu \otimes q_2^* L_\lambda)$, with associated graded

$$\bigoplus_{x \in W^P, v \le x \le w} H^0(X_x^v, L_\mu(-(\partial X_x)^v)) \otimes H^0(X_w^x, L_\lambda);$$

it also follows that $H^i(F_w^v, q_1^* L_\mu \otimes q_2^* L_\lambda) = 0$ for all $i \ge 1$.

By Proposition 3, we may identify $H^0(X_w^v, L_\lambda)$ with $H^0(F_w^v, q_2^* L_\lambda)$; likewise, we may identify $H^0(X_w^v, L_\mu)$ with $H^0(F_w^v, q_1^* L_\mu)$. Using the multiplication map

$$H^0(F_w^v, q_1^* L_\mu) \otimes H^0(F_w^v, q_2^* L_\lambda) \longrightarrow H^0(F_w^v, q_1^* L_\mu \otimes q_2^* L_\lambda),$$

this defines "dot products" in $H^0(F_w^v, q_1^* L_\mu \otimes q_2^* L_\lambda)$.

Let $x \in W^P$ such that $v \leq x \leq w$. Recall that the p_ψ, $v \leq e(\psi) \leq i(\psi) = x$, are a basis of $H^0(X_x^v, L_\mu(-(\partial X_x)^v))$. Further, the p_φ, $x \leq e(\varphi) \leq i(\varphi) \leq w$, are a basis of $H^0(X_w^x, L_\lambda)$. Thus, the dot products $p_\psi \cdot p_\varphi$, where there exists $x \in W^P$ such that

$$v \leq e(\psi) \leq i(\psi) = x \text{ and } x \leq e(\varphi) \leq i(\varphi) \leq w,$$

restrict to a basis of $H^0(X_x^v, L_\mu(-(\partial X_x)^v)) \otimes H^0(X_w^x, L_\lambda)$. By construction of the filtration of $H^0(F_w^v, q_1^* L_\mu \otimes q_2^* L_\lambda)$, it follows that the standard dot products are a basis of that space.

Consider now the T–linearized invertible sheaf $p_1^* L_\mu \otimes p_2^* L_\lambda$ on $\mathcal{X}_w^v$. This sheaf is flat on $\mathbb{A}^1$; by vanishing of $H^1(F_w^v, q_1^* L_\mu \otimes q_2^* L_\lambda)$ and semicontinuity, it follows that the restriction

$$H^0(\mathcal{X}_w^v, p_1^* L_\mu \otimes p_2^* L_\lambda) \longrightarrow H^0(F_w^v, q_1^* L_\mu \otimes q_2^* L_\lambda)$$

is surjective, and that $H^0(\mathcal{X}_w^v, p_1^* L_\mu \otimes p_2^* L_\lambda)$ is a free module over $H^0(\mathbb{A}^1, \mathcal{O}_{\mathbb{A}_1}) = k[z]$, generated by any lift of its quotient space $H^0(F_w^v, q_1^* L_\mu \otimes q_2^* L_\lambda)$.

We now construct such a lift, as follows. Consider the adjunction maps

$$H^0(X_w^v, L_\lambda) \longrightarrow H^0(\mathcal{X}_w^v, p_2^* L_\lambda) \text{ and } H^0(X_w^v, L_\mu) \longrightarrow H^0(\mathcal{X}_w^v, p_1^* L_\mu).$$

These yield dot products $p_\psi \cdot p_\varphi$ in $H^0(\mathcal{X}_w^v, p_1^* L_\mu \otimes p_2^* L_\lambda)$ which lift the corresponding products in $H^0(F_w^v, q_1^* L_\mu \otimes q_2^* L_\lambda)$. Since the latter standard products are a basis of that space, the standard dot products $p_\psi \cdot p_\varphi$ are a basis of $H^0(\mathcal{X}_w^v, p_1^* L_\mu \otimes p_2^* L_\lambda)$ over $k[z]$. Therefore, they restrict to a basis of the space of sections of $p_1^* L_\mu \otimes p_2^* L_\lambda$ over any fiber of π. But the fiber at 1 is $diag(X_w^v)$, and the restriction of $p_1^* L_\mu \otimes p_2^* L_\lambda$ to that fiber is just $L_{\lambda+\mu}$ whereas the restrictions of the dot products are just the usual products. We have proved that the standard products on X_w^v form a basis of $H^0(X_w^v, L_{\lambda+\mu})$.

To complete the proof, notice that any standard product on G/P that is not standard on X_w^v vanishes identically on that subvariety by Lemma 10. □

Remark. The proof of Proposition 7 relies on the fact that the special fiber F_w^v of the flat family $\pi : \mathcal{X}_w^v \to \mathbb{A}^1$ equals $\bigcup_{x \in W^P, v \leq x \leq w} X_x^v \times X_w^x$. Conversely, this fact can be recovered from Proposition 7, as follows.

We have the equalities of Euler characteristics:

$$\chi(F, q_1^* L_\mu \otimes q_2^* L_\lambda) = \chi(G/P, L_{\lambda+\mu}) = \sum_{x \in W^P} \chi(X_x, L_\mu(-\partial X_x))\, \chi(X^x, L_\lambda),$$

where the first equality holds by flatness of π, and the second one by Propositions 5, 6 and 7. It follows that

$$\chi(F, q_1^* L_\mu \otimes q_2^* L_\lambda) = \chi(\bigcup_{x \in W^P} X_x \times X^x, q_1^* L_\mu \otimes q_2^* L_\lambda).$$

Since F contains $\bigcup_{x \in W^P} X_x \times X^x$ by the first claim in the proof of Lemma 7, and λ, μ are arbitrary dominant P–regular weights, it follows that $F = \bigcup_{x \in W^P} X_x \times X^x$ (e.g., since both have the same Hilbert polynomial). Now the argument of Lemma 7 yields $F_w^v = \bigcup_{x \in W^P, v \leq x \leq w} X_x^v \times X_w^x$.

We now extend Proposition 7 to unions of Richardson varieties.

Definition 8. Let $\Pi(\lambda,\mu)$ be the set of all standard pairs (φ,ψ) where $\varphi\in\Pi(\lambda)$ and $\psi\in\Pi(\mu)$. For $v\le w\in W^P$, let $\Pi_w^v(\lambda,\mu)$ be the subset of standard pairs on X_w^v. In view of Lemma 10, we have

$$\Pi_w^v(\lambda,\mu)=\{(\varphi,\psi)\in\Pi(\lambda,\mu)\mid p_\varphi p_\psi|_{X_w^v}\neq 0\}.$$

Finally, for a union Z of Richardson varieties, let

$$\Pi_Z(\lambda,\mu)=\{(\varphi,\psi)\in\Pi(\lambda,\mu)\mid p_\varphi p_\psi|_Z\neq 0\}.$$

Now arguing as in the proof of Theorem 1, we obtain

Theorem 2. *Let Z be a union of Richardson varieties in G/P. Then the products $p_\varphi p_\psi$, where $(\varphi,\psi)\in\Pi_Z(\lambda,\mu)$, form a basis of $H^0(Z,L_{\lambda+\mu})$. The products $p_\varphi p_\psi$, where $(\varphi,\psi)\in\Pi(\lambda,\mu)-\Pi_Z(\lambda,\mu)$, form a basis of the kernel of the restriction map $H^0(G/P,L_{\lambda+\mu})\longrightarrow H^0(Z,L_{\lambda+\mu})$.*

Corollary 2. *For any $\varphi\in\Pi(\lambda)$ and $\psi\in\Pi(\mu)$, the product $p_\varphi p_\psi\in H^0(G/P,L_{\lambda+\mu})$ is a linear combination of standard products $p_{\varphi'}p_{\psi'}$ where $i(\varphi')\ge i(\varphi)$ and $e(\psi')\le e(\psi)$.*

Proof. Notice that $p_\varphi p_\psi$ vanishes identically on all X_y where $y\not\ge i(\varphi)$, and on all X^x where $x\not\le e(\psi)$. By Theorem 2, it follows that $p_\varphi p_\psi$ is a linear combination of standard products $p_{\varphi'}p_{\psi'}$, where $i(\varphi)'\not\le y$ whenever $y\not\ge i(\varphi)$, and $e(\psi)'\not\ge x$ whenever $x\not\le e(\psi)$. But this means exactly that $i(\varphi')\ge i(\varphi)$ and $e(\psi')\le e(\psi)$. ☐

Next we consider a family of dominant weights $\lambda_1,\ldots,\lambda_m$ such that $P=P_{\lambda_1}=\cdots=P_{\lambda_m}$. For any union Z of Richardson varieties in G/P, we shall construct a basis of $H^0(Z,L_{\lambda_1+\cdots+\lambda_m})$, in terms of *standard monomials of degree m*. These are defined as follows.

Definition 9. Let $\pi_i\in\Pi(\lambda_i)$ for $1\le i\le m$. Then the sequence $\underline{\pi}:=(\pi_1,\pi_2,\ldots,\pi_m)$ is standard if

$$e(\pi_m)\le i(\pi_m)\le\cdots\le e(\pi_1)\le i(\pi_1).$$

Further, let $v,w\in W^P$ such that $v\le w$; then $\underline{\pi}$ is standard on X_w^v if

$$v\le e(\pi_m)\le i(\pi_m)\le\cdots\le e(\pi_1)\le i(\pi_1)\le w.$$

Finally, $\underline{\pi}$ is standard on $Z=\bigcup X_{w_i}^{v_i}$ if it is standard on $X_{w_i}^{v_i}$ for some i.

Set

$$\Pi_w^v(\lambda_1,\ldots,\lambda_m)=\{\underline{\pi}=(\pi_1,\pi_2,\ldots,\pi_m)\,|\,\underline{\pi}\text{ is standard on }X_w^v\},$$
$$\Pi_Z(\lambda_1,\ldots,\lambda_m)=\{\underline{\pi}=(\pi_1,\pi_2,\ldots,\pi_m)\,|\,\underline{\pi}\text{ is standard on }Z\}.$$

Definition 10. Given $\underline{\pi}=(\pi_1,\pi_2,\ldots,\pi_m)$, set $p_{\underline{\pi}}:=p_{\pi_1}\cdots p_{\pi_m}$.

Note that $p_{\underline{\pi}}\in H^0(G/P,L_{\lambda_1+\cdots+\lambda_m})$. If $\underline{\pi}$ is standard, then we call $p_{\underline{\pi}}$ a standard monomial on G/P. If $\underline{\pi}$ is standard on X_w^v (resp. Z) , then we call $p_{\underline{\pi}}$ a standard monomial on X_w^v (resp. Z).

By Theorem 2 and induction on m, we obtain

Corollary 3. *Let Z be a union of Richardson varieties in G/P and let $\lambda_1,\ldots,\lambda_m$ be dominant weights such that $P=P_{\lambda_1}=\cdots=P_{\lambda_m}$. Then the monomials $p_{\underline{\pi}}$ where $\underline{\pi}\in\Pi_Z(\lambda_1,\ldots,\lambda_m)$ form a basis of $H^0(Z,L_{\lambda_1+\cdots+\lambda_m})$. Further, the monomials $p_{\underline{\pi}}$ where $\underline{\pi}\in\Pi(\lambda_1,\ldots,\lambda_m)-\Pi_Z(\lambda_1,\ldots,\lambda_m)$, form a basis of the kernel of the restriction map $H^0(G/P,L_{\lambda_1+\cdots+\lambda_m})\longrightarrow H^0(Z,L_{\lambda_1+\cdots+\lambda_m})$.*

As an application, we determine the equations of unions of Richardson varieties in their projective embeddings given by very ample line bundles on G/P. Let λ be a dominant P–regular weight. For any $\pi_1, \pi_2 \in \Pi(\lambda)$, we have in $H^0(G/P, L_{2\lambda})$:

$$p_{\pi_1} p_{\pi_2} - \sum a_{\pi'_1,\pi'_2} p_{\pi'_1} p_{\pi'_2} = 0,$$

where $a_{\pi'_1,\pi'_2} \in k$ and the sum is over those standard pairs $(\pi'_1, \pi'_2) \in \Pi(\lambda, \lambda)$ such that $i(\pi'_1) \geq i(\pi_1)$ and $e(\pi'_2) \leq e(\pi_2)$ (as follows from Corollary 2).

Definition 11. The preceding elements $p_{\pi_1} p_{\pi_2} - \sum a_{\pi'_1,\pi'_2} p_{\pi'_1} p_{\pi'_2}$ when regarded in $S^2 H^0(G/P, L_\lambda)$, will be called the quadratic straightening relations.

Corollary 4. *Let λ be a regular dominant character of P.*

(1) *The multiplication map*

$$\bigoplus_{m=0}^{\infty} S^m H^0(G/P, L_\lambda) \longrightarrow \bigoplus_{m=0}^{\infty} H^0(G/P, L_{m\lambda})$$

is surjective, and its kernel is generated as an ideal by the quadratic straightening relations.

(2) *For any union Z of Richardson varieties in G/P, the restriction map*

$$\bigoplus_{m=0}^{\infty} H^0(G/P, L_{m\lambda}) \longrightarrow \bigoplus_{m=0}^{\infty} H^0(Z, L_{m\lambda})$$

is surjective. Its kernel is generated as an ideal by the p_π, $\pi \in \Pi(\lambda) - \Pi_Z(\lambda)$ together with the standard products $p_{\pi_1} p_{\pi_2}$ where $i(\pi_1) \not\leq w$ or $e(\pi_2) \not\geq v$ whenever X_w^v is an irreducible component of Z. If, in addition, Z is a union of Richardson varieties X_w^v all having the same w, then the p_π, $\pi \in \Pi(\lambda) - \Pi_Z(\lambda)$ suffice.

Proof. (1) By [16], Theorem 3.11, the multiplication map is surjective, and its kernel is generated as an ideal by the kernel K of the map $S^2 H^0(G/P, L_\lambda) \longrightarrow H^0(G/P, L_{2\lambda})$. Let J be the subspace of $S^2 H^0(G/P, L_\lambda)$ generated by all quadratic straightening relations. Then $J \subseteq K$, and the quotient space $S^2 H^0(G/P, L_\lambda)/J$ is spanned by the images of the standard products. Further, their images in $S^2 H^0(G/P, L_\lambda)/K \simeq H^0(G/P, L_{2\lambda})$ form a basis, by Proposition 7. It follows that $J = K$.

(2) The first assertion follows from Lemma 5. Consider a standard monomial $p_{\underline{\pi}} = p_{\pi_1} \cdots p_{\pi_m} \in H^0(G/P, L_{m\lambda})$. By Corollary 3, $p_{\underline{\pi}}$ vanishes identically on Z if and only if $i(\pi_1) \not\leq w$ or $e(\pi_m) \not\geq v$ for all irreducible components X_w^v. This amounts to $p_{\pi_1} p_{\pi_m}$ vanishes identically on Z. If, in addition, w is independent of the component, then p_{π_1} or p_{π_m} vanishes identically on Z; further, $p_{\pi_1} p_{\pi_m}$ is a standard product on G/P. This implies the remaining assertions, since the kernel of $H^0(G/P, L_{m\lambda}) \longrightarrow H^0(Z, L_{m\lambda})$ is spanned by those standard monomials on G/P that are not standard on Z (Corollary 3). □

Remark. In particular, the p_π, where $\pi \in \Pi(\lambda) - \Pi_Z(\lambda)$, generate the homogeneous ideal of Z in G/P, whenever Z is a union of Schubert varieties (or a union of opposite Schubert varieties). But this does not extend to arbitrary unions of Richardson varieties, as shown by the obvious example where $G/P = \mathbb{P}^1$, $Z = \{0, \infty\}$ and $L_\lambda = \mathcal{O}(1)$; then $\Pi(\lambda) = \Pi_Z(\lambda)$.

6. Weights of classical type

In this section, we shall determine the "building blocks"

$$H_w^v(\lambda) = H^0(X_w^v, L_\lambda(-(\partial X_w)^v - (\partial X^v)_w))$$

in the case where the dominant weight λ is of classical type (as introduced in [9], cf. the next definition). Along the way, we shall retrieve the results of loc. cit., using our basis $\{p_\pi\}$. In particular, we shall give a geometric characterization of "admissible pairs" of loc. cit. (cf. Definition 16 below).

Definition 12. Let λ be a dominant weight. We say, λ is of classical type if $\langle \lambda, \beta^\vee \rangle \leq 2$, for all $\beta \in R^+$.

Remarks. (1) Any dominant weight of classical type is either fundamental, or a sum of two minuscule fundamental weights.
(2) G is classical if and only if all fundamental weights of G are of classical type.

For the rest of this section, we fix a dominant weight λ of classical type.

Proposition 8. *Let $v, w \in W^\lambda, v \leq w$. Then the T–module $H_w^v(\lambda)$ is at most one–dimensional; further, if nonzero, then it has the weight $-\frac{1}{2}(w(\lambda) + v(\lambda))$.*

As a consequence, the weights of the T–module $H^0(X_w, L_\lambda(-\partial X_w))$ are among the $-\frac{1}{2}(w(\lambda) + x(\lambda))$ where $x \leq w$, and the corresponding weight spaces are one–dimensional.

Proof. Let $p \in H_w^v(\lambda)$. Then p^2 belongs to $H^0(X_w^v, L_{2\lambda})$, and vanishes of order ≥ 2 along each component of the whole boundary $(\partial X_w)^v \cup (\partial X^v)_w$. On the other hand, the product $p_w p_v$ also belongs to $H^0(X_w^v, L_{2\lambda})$ and satisfies, by Chevalley's formula,

$$\operatorname{div}(p_w p_v) = \sum_\beta \langle \lambda, \beta^\vee \rangle X_{ws_\beta}^v + \sum_\gamma \langle \lambda, \gamma^\vee \rangle X_w^{vs_\gamma},$$

where X_{ws_β} (resp. X^{vs_γ}) runs over all the components X_x (resp. X^y) of ∂X_w (resp. ∂X^v) such that $x \geq v$ (resp. $y \leq w$). Hence, $p_w p_v$ vanishes of order at most 2 along each component of $(\partial X_w)^v \cup (\partial X^v)_w$ (since λ is of classical type), and nowhere else. Thus, $\frac{p^2}{p_w p_v}$ (a rational function on X_w^v) has no poles. It follows that $p^2 = c p_w p_v, c \in k$, and hence that p is unique up to scalars; further, p is either zero or has weight $\frac{1}{2}(\text{weight } p_w + \text{weight } p_v) = -\frac{1}{2}(w(\lambda) + v(\lambda))$. □

As a corollary to the proof of the above proposition, we have

Lemma 11. *Let $v, w \in W^\lambda, v \leq w$. Further, let $H_w^v(\lambda)$ be nonzero. Then for each divisor X_{ws_β} (resp. X^{vs_γ}) of X_w (resp. X^v) such that $ws_\beta \geq v$ (resp. $vs_\gamma \leq w$), β (resp. γ) being in R^+, we have, $\langle \lambda, \beta^\vee \rangle$ (resp. $\langle \lambda, \gamma^\vee \rangle$) $= 2$.*

We shall denote by $p_{w,v}$ the unique $p_{w,v}^\xi$, if nonzero (then $p_{w,w} = p_w$). By Proposition 8, $p_{w,v}$ lifts to a unique T–eigenvector in $H^0(X_w, L_\lambda(-\partial X_w))$; we still denote that lift by $p_{w,v}$. The nonzero $p_{w,v}$, where $v \leq w$, form a basis of $H^0(X_w, L_\lambda(-\partial X_w))$.

Notice that $\frac{p_{w,v}^2}{p_w}$ is a rational section of L_λ on X_w, eigenvector of T with weight $-v(\lambda)$, and without poles by the argument of Proposition 8. This implies

Lemma 12. *With notation as above, we have* $p_{w,v}^2 = p_w p_v$ *on* X_w*, up to a nonzero scalar.*

We now aim at characterizing those pairs (v, w) such that $p_{w,v} \neq 0$. For this, we recall some definitions and lemmas from [9].

Definition 13. Let X_v be a Schubert divisor in X_w; further, let $v = s_\alpha w$ where $\alpha \in R^+$. If α is simple, then we say, X_v is a moving divisor in X_w, moved by α.

Lemma 13 ([9], Lemma 1.5)**.** *Let* X_v *be a moving divisor in* X_w*, moved by* α*. Let* X_u *be any Schubert subvariety of* X_w*. Then either,* $X_u \subseteq X_v$ *or* $X_{s_\alpha u} \subseteq X_v$*.*

Definition 14. Let $v, w \in W^\lambda, v \leq w, \ell(v) = \ell(w) - 1$; further let $v = ws_\beta = s_\gamma w$, for some positive roots β, γ. We denote the positive integer $\langle \lambda, \beta^\vee \rangle (= \langle v(\lambda), \gamma^\vee \rangle = -\langle w(\lambda), \gamma^\vee \rangle)$ by $m_\lambda(v, w)$, and refer to it as the Chevalley multiplicity of X_v in X_w (see [3]).

Lemma 14 ([9] Lemma 2.5)**.** *Let* $v, w \in W^\lambda$ *such that* X_v *is a moving divisor in* X_w*, moved by* α*. Let* X_u *be another Schubert divisor in* X_w*. Then* $X_{s_\alpha u}$ *is a divisor in* X_v*, and* $m_\lambda(s_\alpha u, v) = m_\lambda(u, w)$*.*

Definition 15. Let $v, w \in W^\lambda$ such that X_v is a divisor in X_w. If $m_\lambda(v, w) = 2$, then we shall refer to X_v as a double divisor in X_w.

By Lemma 11, if $p_{w,v} \neq 0$, then all Schubert divisors in X_w that meet X^v are double divisors.

Lemma 15 ([9], Lemma 2.6.)**.** *Let* $u, w \in W^\lambda$ *such that* X_u *is a double divisor in* X_w*. Then* X_u *is a moving divisor in* X_w*.*

A geometric characterization of Admissible pairs. Recall (cf. [9]):

Definition 16. A pair (v, w) in W^λ is called admissible if either $v = w$ (in which case, it is called a trivial admissible pair), or there exists a sequence $w = w_1 > w_2 > \cdots > w_r = v$, such that $X_{w_{i+1}}$ is a double divisor in X_{w_i}, i.e., $m_\lambda(w_{i+1}, w_i) = 2$. We shall refer to such a chain as a double chain.

We shall give a geometric characterization of admissible pairs (cf. Proposition 9 below). First we prove some preparatory lemmas.

Lemma 16. *Let* $p_{w,v} \neq 0$*, then*

(1) *For any double divisor* $X_{s_\alpha w}$ *in* X_w *meeting* X^v*, we have*

$$p_{s_\alpha w,v} = e_{-\alpha} p_{w,v} \text{ and } e_{-\alpha}^2 p_{w,v} = 0,$$

where $e_{-\alpha}$ *is a generator of the Lie algebra of* $U_{-\alpha}$*. Further,* $p_{s_\alpha w,v} \neq 0$*.*

(2) *Likewise, for any double divisor* $X^{s_\alpha v}$ *in* X^v *meeting* X_w*, we have*

$$p_{w,s_\alpha v} = e_\alpha p_{w,v} \text{ and } e_\alpha^2 p_{w,v} = 0,$$

where e_α *is a generator of the Lie algebra of* U_α*. Further,* $p_{w,s_\alpha v} \neq 0$*.*

(3) *The pair* (v, w) *is admissible.*

Proof. (1) Consider the T–module $H^0(X_w^v, L_\lambda(-(\partial X^v)_w))$. By Proposition 8, it has a basis $\{p_{x,v} | \, v \leq x \leq w\}$ with corresponding weights $-\frac{1}{2}(x(\lambda) + v(\lambda))$. Notice that X_w is invariant under $U_{-\alpha}$ (since $s_\alpha w < w$); hence X_w^v and $(\partial X^v)_w$ are also $U_{-\alpha}$–invariant. Thus, $U_{-\alpha}$ acts on $H^0(X_w^v, L_\lambda(-(\partial X^v)_w))$, compatibly with the T–action. The $U_{-\alpha}$–submodule M generated by $p_{w,v}$ is T–invariant, with weights of the form $-\frac{1}{2}(w(\lambda)+v(\lambda))-m\alpha$ for some nonnegative integers m. But if $x(\lambda)=w(\lambda)+2m\alpha$, then either $x=w$ and $m=0$, or $x=s_\alpha w$ and $m=1$ (by Lemma 11). Hence M is either spanned by $p_{w,v}$, or by $p_{w,v}$ and $p_{s_\alpha w,v}$. Further, $e_{-\alpha}^2 p_{w,v}=0$.

To complete the proof, it suffices to show that $U_{-\alpha}$ does not fix $p_{w,v}$. Otherwise, the zero locus of $p_{w,v}$ in X_w^v is $U_{-\alpha}$–invariant, and hence so is $(\partial X_w)^v$. Thus,

$$\overline{U_{-\alpha}e_{s_\alpha w}}\subseteq(\partial X_w)^v.$$

But $e_w\in\overline{U_{-\alpha}e_{s_\alpha w}}$ (since $s_\alpha w<w$) and $e_w\notin(\partial X_w)^v$, a contradiction.

(2) is checked similarly. (3) follows from (1) together with Lemma 11 by induction on $\ell(w)$. □

Lemma 17. *Let (v,w) be an admissible pair, then $p_{w,v}\neq 0$.*

Proof. We argue by induction on $\ell(w)$. We may chose a simple root α such that $w>s_\alpha w\geq v$ and that X_{ws_α} is a double divisor in X_w. Then $\langle w(\lambda),\check{\alpha}\rangle=-2$, and also $p_{s_\alpha w,v}\neq 0$ by the induction hypothesis. The weight of this vector is

$$-\frac{1}{2}(s_\alpha w(\lambda)+v(\lambda))=-\frac{1}{2}(w(\lambda)+v(\lambda))-\alpha.$$

The scalar product of this weight with $\check{\alpha}$ being integral, $\langle v(\lambda),\check{\alpha}\rangle$ is an even integer. Since λ is of classical type, it follows that

$$\langle v(\lambda),\check{\alpha}\rangle\in\{2,0,-2\}.$$

We now distinguish the following three cases:

Case 1: $(v(\lambda),\alpha^\vee)=2$. Then $w\geq s_\alpha w, s_\alpha v>v$. As a first step, we find a relation between $H^0(X_w^v,L_\lambda(-(\partial X_w)^v))$ and $H^0(X_{s_\alpha w}^v,L_\lambda(-(\partial X_{s_\alpha w})^v))$.

Let G_α be the subgroup of G generated by U_α, $U_{-\alpha}$ and T; let $B_\alpha=G_\alpha\cap B$. Then the derived subgroup of G_α is isomorphic to $SL(2)$ or to $PSL(2)$, and G_α/B_α is isomorphic to the projective line $\mathbb{P}^1$. For a B_α–module M, we shall denote the associated G_α–linearized locally free sheaf on G_α/B_α by $\underline{M}$.

Notice that X_w, X^v and hence X_w^v are invariant under G_α, and $(\partial X_w)^v$ is invariant under B_α; we have

$$(\partial X_w)^v=X_{s_\alpha w}^v\cup G_\alpha(\partial X_{s_\alpha w})^v.$$

Consider the fiber product $G_\alpha\times^{B_\alpha}X_{s_\alpha w}^v$ with projection

$$p:G_\alpha\times^{B_\alpha}X_{s_\alpha w}^v\longrightarrow G_\alpha/B_\alpha\simeq\mathbb{P}^1$$

and "multiplication" map

$$\psi:G_\alpha\times^{B_\alpha}X_{s_\alpha w}^v\longrightarrow X_w^v.$$

Then ψ is birational (since it is an isomorphism at e_w). Further, we have

$$(\partial X_w)^v=\psi(X_{s_\alpha w}^v\cup G_\alpha\times^{B_\alpha}(\partial X_{s_\alpha w})^v)$$

where $X_{s_\alpha w}^v$ is the fiber of p at B_α/B_α. By the projection formula, it follows that

$$L_\lambda(-(\partial X_w)^v)=\psi_*\psi^*L_\lambda(-X_{s_\alpha w}^v-G_\alpha\times^{B_\alpha}(\partial X_{s_\alpha w})^v).$$

This yields an isomorphism

$$H^0(X_w^v,L_\lambda(-(\partial X_w)^v))\cong H^0(G_\alpha/B_\alpha,p_*\psi^*L_\lambda(-X_{s_\alpha w}^v-G_\alpha\times^{B_\alpha}(\partial X_{s_\alpha w})^v)).$$

Further, we may identify the G_α–linearized sheaf $p_*\psi^* L_\lambda(-G_\alpha \times^{B_\alpha} (\partial X_{s_\alpha w})^v))$ on G_α/B_α, to the sheaf $\underline{H^0(X^v_{s_\alpha w}, L_\lambda(-(\partial X_{s_\alpha w})^v))}$. Therefore, we obtain an exact sequence of B_α–modules

$$0 \longrightarrow H^0(X^v_w, L_\lambda(-(\partial X_w)^v)) \longrightarrow H^0(G_\alpha/B_\alpha, \underline{H^0(X^v_{s_\alpha w}, L_\lambda(-(\partial X_{s_\alpha w})^v))}) \\ \longrightarrow H^0(X^v_{s_\alpha w}, L_\lambda(-(\partial X_{s_\alpha w})^v)) \longrightarrow 0,$$

where the map on the right is the "evaluation" map (its surjectivity follows e.g. from Corollary 1).

Next we analyse the B_α–module $H^0(X^v_{s_\alpha w}, L_\lambda(-(\partial X_{s_\alpha w})^v))$. By Proposition 8, its weights have multiplicity one; they are among the $-\frac{1}{2}(s_\alpha w(\lambda) + x(\lambda))$, where $v \le x \le s_\alpha w$, and the weight $-\frac{1}{2}(w(\lambda) + v(\lambda)) - \alpha$ occurs, since $p_{s_\alpha w, v} \neq 0$; its α–weight (the scalar product with $\check{\alpha}$) is -2.

If $s_\alpha v \not\le s_\alpha w$, then the span M of $p_{s_\alpha w, v}$ is invariant under B_α. Thus, the T–module $H^0(G_\alpha/B_\alpha, \underline{M})$ has weights $-\frac{1}{2}(w(\lambda) + v(\lambda)) + (m-1)\alpha$, $m = 0, 1, 2$, each of them having multiplicity one. Further, the kernel of the evaluation map $H^0(G_\alpha/B_\alpha, \underline{M}) \longrightarrow M$ contains an element of weight $-\frac{1}{2}(w(\lambda) + v(\lambda))$. By the exact sequence above, this weight occurs in $H^0(X^v_w, L_\lambda(-(\partial X_w)^v))$; using Proposition 8 again, it follows that $p_{w,v} \neq 0$.

On the other hand, if $s_\alpha v \le s_\alpha w$, then Lemma 16 (2) applied to $(s_\alpha w, v)$ yields

$$p_{s_\alpha w, s_\alpha v} = e_\alpha p_{s_\alpha w, v} \neq 0.$$

Hence the span M of $p_{s_\alpha w, v}$ and $p_{s_\alpha w, s_\alpha v}$ is a nontrivial B_α-module with α-weights -2 and 0 (note that $e_\alpha p_{s_\alpha w, s_\alpha v} = 0$ by weight considerations). Thus, we have an isomorphism of B_α–modules

$$M \cong M_1 \otimes M_2,$$

where M_1 is a one–dimensional B_α–module with α–weight -1, and M_2 is the standard two–dimensional G_α–module. It follows that the weights of the T–module

$$H^0(G_\alpha/B_\alpha, \underline{M}) \cong H^0(G_\alpha/B_\alpha, \underline{M_1}) \otimes M_2$$

are exactly $-\frac{1}{2}(w(\lambda) + v(\lambda)) - \alpha$, $-\frac{1}{2}(w(\lambda) + v(\lambda)) + \alpha$ (both of multiplicity one) and $-\frac{1}{2}(w(\lambda) + v(\lambda))$ (of multiplicity two). Thus, the kernel of the evaluation map $H^0(G_\alpha/B_\alpha, \underline{M}) \longrightarrow M$ contains an element of weight $-\frac{1}{2}(w(\lambda) + v(\lambda))$, and we conclude as above.

Case 2: $\langle v(\lambda), \check{\alpha} \rangle = 0$. Then $w > s_\alpha w \ge v = s_\alpha v$, so that X_w, X^v and X^v_w are again invariant under G_α, whereas $(\partial X_w)^v$ is B_α–invariant. Arguing as in Case 1, we obtain the same relation between $H^0(X^v_w, L_\lambda(-(\partial X_w)^v))$ and $H^0(X^v_{s_\alpha w}, L_\lambda(-(\partial X_{s_\alpha w})^v))$; but now the latter B_α–module contains the span M of $p_{s_\alpha w, v}$, as a B_α-submodule of α–weight -1. As in Case 1, it follows that $p_{w,v} \neq 0$.

Case 3: $\langle v(\lambda), \check{\alpha} \rangle = -2$. Then $w > s_\alpha w \ge v > s_\alpha v$, and X^v is a double divisor in $X^{s_\alpha v}$. Therefore, the pair $(s_\alpha v, s_\alpha w)$ is admissible. By the induction hypothesis, we have, $p_{s_\alpha v, s_\alpha w} \neq 0$. Then Case 1 applies to the pair $(s_\alpha v, w)$ and yields $p_{w, s_\alpha v} \neq 0$. Further, X^v is a double divisor in $X^{s_\alpha v}$. Hence by Lemma 16 (2) applied to $(w, s_\alpha v)$, we obtain $p_{w,v} \neq 0$. □

Now combining Lemmas 11, 16 and 17, we obtain

Proposition 9. *Let $v, w \in W^\lambda, v \le w$. Then the pair (v, w) is admissible if and only if $p_{w,v}$ is nonzero. In this case, every chain from v to w is a double chain.*

7. Standard monomials for sums of weights of classical type

In this section, we obtain a standard monomial basis for $H^0(X_w^v, L_{\lambda_1+\cdots+\lambda_m})$, where X_w^v is a Richardson variety in G/P, and $\lambda_1, \ldots, \lambda_m$ are dominant characters of classical type of P (in the sense of Definition 12).

We begin with the case where $m = 1$; we shall need a definition, and a result of Deodhar ([9], Lemmas 4.4 and 4.4') on the Bruhat ordering.

Definition 17. Let $w \in W^P$ and let λ be a dominant character of P. We say that $x \in W^P$ is λ–maximal in w (resp. λ–minimal on w) if $xy \leq x$ for any $y \in W_\lambda$ such that $xy \in W^P$ and $xy \leq w$ (resp. if $xy \geq x$ for any $y \in W_\lambda$ such that $xy \in W^P$ and $xy \geq w$).

Lemma 18. *Let $w \in W$ and $x \in W^\lambda$ such that $x \leq w(\lambda)$ (resp. $x \geq v(\lambda)$). Then the set $\{y \in W_\lambda \mid xy \leq w\}$ (resp. $\{y \in W_\lambda \mid w \leq xy\}$ admits a unique maximal (resp. minimal) element.*

We shall also need the following consequences of this result.

Lemma 19. (1) *Let $w \in W^P$ and $x \in W^\lambda$ such that $x \leq w(\lambda)$ (resp. $x \geq w(\lambda)$). Then $x \in W/W_\lambda$ admits a unique lift $\tilde{x} \in W^P$ such that $\tilde{x}$ is λ–maximal in w (resp. λ–minimal on w).*

(2) *Let $v \leq w \in W^P$, then v is λ–maximal in w (resp. w is λ–minimal in v) if and only if $(\partial_\lambda X^v)_w = (\partial X^v)_w$ (resp. $(\partial_\lambda X_w)^v = (\partial X_w)^v$).*

Proof. (1) Let $x \leq w(\lambda)$. By Lemma 18, the set $\{y \in W_\lambda \mid xy \leq w\}$ admits a unique maximal element that we still denote by y. Let $\tilde{x}$ be the representative in W^P of $xy \in W$, then $\tilde{x}(\lambda) = xy(\lambda) = x(\lambda)$. Further, if we have $\tilde{x}z \leq w$ for some $z \in W_\lambda$ such that $\tilde{x}z \in W^P$, then we can write $\tilde{x}z = xu$ where $u \in W_\lambda$. Since $xu \leq w$, we have $u \leq y$ and hence $xu \leq xy$ (since $x \in W^\lambda$ and $u, y \in W_\lambda$). But $xu = \tilde{x}z \in W^P$, so that $\tilde{x}z \leq \tilde{x}$. This proves the assertion concerning λ–maximal elements, and hence the dual assertion concerning λ–minimal elements.

(2) If $(\partial_\lambda X^v)_w \neq (\partial X^v)_w$, then there exists $y \in W_\lambda$ such that $v < vy \leq w$ and $\ell(vy) = \ell(v) + 1$. Thus, v is not λ–maximal in w.

Conversely, if v is not λ–maximal in w, then $v < \tilde{v} \leq w$ where $\tilde{v} \in vW_\lambda$ is λ–maximal in w. Hence there exists $y \in W_\lambda$ such that $v < vy \leq \tilde{v} \leq w$ and $\ell(vy) = \ell(v) + 1$. Now X_w^{vy} is contained in $(\partial X^v)_w$ but not in $(\partial_\lambda X^v)_w$. □

Now we consider the T–module $H^0(X_w^v, L_\lambda)$, where $v \leq w \in W^P$ and λ is a dominant character of P, not necessarily P–regular. Notice that the diagram

$$\begin{tikzcd}
H^0(X_{w(\lambda)}, L_\lambda) \arrow[r] \arrow[d] & H^0(X_{w(\lambda)}^{v(\lambda)}, L_\lambda) \arrow[d] \\
H^0(X_w, L_\lambda) \arrow[r] & H^0(X_w^v, L_\lambda)
\end{tikzcd}$$

is commutative, where the horizontal (resp. vertical) maps are restrictions (resp. pull–backs). Further, both restrictions are surjective by Proposition 1, and the pull–back on the left is an isomorphism, since the natural map $f : X_w \to X_{w(\lambda)}$ satisfies $f_*\mathcal{O}_{X_w} = \mathcal{O}_{X_{w(\lambda)}}$. Thus, we may regard the T–module $H^0(X_w^v, L_\lambda)$ as a quotient of $H^0(X_{w(\lambda)}^{v(\lambda)}, L_\lambda)$.

Likewise, by using the commutative diagram

$$\begin{tikzcd}
H^0(X_{w(\lambda)}, L_\lambda(-\partial X_{w(\lambda)})) \arrow[r] \arrow[d] & H^0(X_{w(\lambda)}^{v(\lambda)}, L_\lambda(-(\partial X_{w(\lambda)})^{v(\lambda)})) \arrow[d] \\
H^0(X_w, L_\lambda(-\partial_\lambda X_w)) \arrow[r] & H^0(X_w^v, L_\lambda(-(\partial_\lambda X_w)^v))
\end{tikzcd}$$

and Corollary 1, we may regard the T–module $H^0(X_w^v, L_\lambda(-(\partial_\lambda X_w)^v))$ as a quotient of $H^0(X_{w(\lambda)}^{v(\lambda)}, L_\lambda(-(\partial X_{w(\lambda)})^{v(\lambda)}))$. The latter has been described in Section 6, in the case that λ is of classical type: it has a basis consisting of the $p_{w(\lambda),x(\lambda)}$ where $x(\lambda) \in W^\lambda$, $v(\lambda) \le x(\lambda) \le w(\lambda)$ and the pair $(x(\lambda), w(\lambda))$ is admissible. From that description we shall deduce

Proposition 10. *Let $v \le w \in W^P$ and let λ be a dominant character of classical type of P.*

1. *The space $H^0(X_w^v, L_\lambda(-(\partial_\lambda X_w)^v - (\partial X^v)_w))$ is spanned by $p_{w(\lambda),v(\lambda)}$, if v is λ–maximal in w; otherwise, this space is zero.*
2. *The $p_{w(\lambda),x(\lambda)}$ where $x \in W^P$ and $v \le x \le w$, form a basis of the space $H^0(X_w^v, L_\lambda(-(\partial_\lambda X_w)^v))$.*
3. *The $p_{w(\lambda),x(\lambda)}$ where $x \in W^P$ is λ–minimal on v, and w is λ–minimal on x, form a basis of $H^0(X_w^v, L_\lambda(-(\partial X_w)^v))$.*
4. *The $p_{y(\lambda),x(\lambda)}$ where $x, y \in W^P$ and $v \le x \le y \le w$, form a basis of $H^0(X_w^v, L_\lambda)$.*

Proof. (1) Assume that $H^0(X_w^v, L_\lambda(-(\partial_\lambda X_w)^v - (\partial X^v)_w))$ contains a nonzero element p. Then, by the argument of Proposition 8, $\frac{p^2}{p_{w(\lambda)}p_{v(\lambda)}}$ is a rational function on X_w^v, without poles; further, it vanishes identically on $(\partial X^v)_w - (\partial_\lambda X^v)_w$, since the zero locus of $p_{v(\lambda)}$ is $(\partial_\lambda X^v)_w$. It follows that p^2 is a constant multiple of $p_{w(\lambda)}p_{v(\lambda)}$, and that $(\partial X^v)_w = (\partial_\lambda X^v)_w$. Hence p is a constant multiple of $p_{w(\lambda),v(\lambda)}$, and v is λ–maximal in w (by Lemma 19).

Conversely, let v be λ–maximal in w; then $(\partial_\lambda X^v)_w = (\partial X^v)_w$. Thus, $p_{w(\lambda),v(\lambda)}$ vanishes identically on $(\partial_\lambda X_w)^v \cup (\partial X^v)_w$. Further, $p_{w(\lambda),v(\lambda)} \neq 0$ on X_w^v, since $p_{w(\lambda),v(\lambda)}^2 = p_{w(\lambda)}p_{v(\lambda)}$ on X_w (by Lemma 12).

(2) By Proposition 8, the space $H^0(X_w^v, L_\lambda(-(\partial_\lambda X_w)^v))$ is spanned by the images of the $p_{w(\lambda),x(\lambda)}$ where $v(\lambda) \le x(\lambda) \le w(\lambda)$. Further, $p_{w(\lambda),x(\lambda)}^2 = p_{w(\lambda)}p_{x(\lambda)}$ on X_w. Using Lemma 2, we see that $p_{w(\lambda),x(\lambda)}$ is nonzero on X_w^v if and only if $x(\lambda)$ has a representative $x \in W^P$ such that $v \le x \le w$.

(3) $H^0(X_w^v, L_\lambda(-(\partial X_w)^v))$ is a T–stable subspace of $H^0(X_w^v, L_\lambda(-(\partial_\lambda X_w)^v))$; thus, it is spanned by certain $p_{w(\lambda),x(\lambda)}$ where $v \le x \le w$. By Lemma 12, the zero locus $(p_{w(\lambda),x(\lambda)} = 0)$ in X_w^v equals $(p_{x(\lambda)} = 0) \cup (\partial_\lambda X_w)^v$. Hence $p_{w(\lambda),x(\lambda)}$ belongs to $H^0(X_w^v, L_\lambda(-(\partial X_w)^v))$ if and only if $p_{x(\lambda)}$ vanishes identically on $(\partial X_w)^v - (\partial_\lambda X_w)^v$. By Lemma 2, this amounts to: $x(\lambda)$ admits no lift x' such that $v \le x' \le wy$ for some $y \in W_\lambda$, $wy < y$, $\ell(wy) = \ell(w) - 1$. Let $\tilde{x}$ be the lift of $x(\lambda)$ that is λ–minimal on v, then the preceding condition means that w is λ–minimal on $\tilde{x}$.

(4) By Proposition 3, we obtain a basis of the space $H^0(X_w^v, L_\lambda)$ by choosing a basis of $H^0(X_w^x, L_\lambda(-(\partial X^x)_w))$ for each $x \in W^P$ such that $v \le x \le w$, and lifting this basis to $H^0(X_w^v, L_\lambda)$ under the (surjective) restriction map $H^0(X_w^v, L_\lambda) \to H^0(X_w^x, L_\lambda)$. Together with (3), it follows that a basis of $H^0(X_w^v, L_\lambda)$ consists of the $p_{y(\lambda),x(\lambda)}$, where y is λ–maximal in w, x is λ–maximal in y, and $v \le x$. But

given any $x', y' \in W^P$ such that $v \le x' \le y' \le w$, we have $v \le x \le y \le w$ and $p_{y'(\lambda),x'(\lambda)} = p_{y(\lambda),x(\lambda)}$, where x (resp. y) is the representative of $x(\lambda)$ that is λ–maximal in w (resp. x). □

Definition 18. Let $\lambda_1, \dots, \lambda_m$ be dominant characters of classical type of P. Let $\pi_i = (w_i, v_i)$ where $v_i \le w_i \in W^{\lambda_i}$ for $1 \le i \le m$. Then the sequence $\underline{\pi} = (\pi_1, \dots, \pi_m)$ is standard if there exist lifts $\tilde{w}_i, \tilde{v}_i$ in W^P for $1 \le i \le m$, such that

$$\tilde{v}_m \le \tilde{w}_m \le \cdots \le \tilde{v}_1 \le \tilde{w}_1.$$

The monomial

$$p_{\underline{\pi}} = p_{w_1(\lambda_1),v_1(\lambda_1)} \cdots p_{w_m(\lambda_m),v_m(\lambda_m)} \in H^0(G/P, L_{\lambda_1+\cdots+\lambda_m})$$

is called standard as well.

Further, let $v, w \in W^P$ such that $v \le w$; then $\underline{\pi}$ is standard on X_w^v if there exist lifts as above, such that

$$v \le \tilde{v}_m \le \tilde{w}_m \le \cdots \le \tilde{v}_1 \le \tilde{w}_1 \le w.$$

The restriction of $p_{\underline{\pi}}$ to X_w^v is called a standard monomial on X_w^v; it is a T–eigenvector in $H^0(X_w^v, L_{\lambda_1+\cdots+\lambda_m})$.

Notice that there is no loss of generality in assuming that $\tilde{v}_m$ is λ_m–minimal on v, and that $\tilde{w}_m$ is λ_m–minimal on $\tilde{v}_m$.

Now the argument of Proposition 7, together with Proposition 10 and induction on m, yields the following partial generalization of Corollary 3.

Theorem 3. *Let $v \le w \in W^P$ and let $\lambda_1, \dots, \lambda_m$ be dominant characters of P. If $\lambda_1, \dots, \lambda_m$ are of classical type, then the standard monomials on X_w^v form a basis for $H^0(X_w^v, L_{\lambda_1+\cdots+\lambda_m})$.*

Remarks. (1) In particular, Theorem 3 applies to $P = B$ if all fundamental weights are of classical type, that is, if G is classical. Thereby, we retrieve all results of [9].

(2) The second assertion of Corollary 3 does not generalize to this setting; that is, there are examples of standard monomials on G/P which are not standard on X_w^v, but which restrict nontrivially to that subvariety.

Specifically, let $G = SL(3)$ with simple reflections s_1, s_2 and fundamental weights ω_1, ω_2. Then one may check that the monomial

$$p_{s_1(\omega_1)} p_{s_2(\omega_2)} \in H^0(G/B, L_{\omega_1+\omega_2})$$

is standard on G/B and restricts nontrivially to $X_{s_2 s_1}$, but is not standard there.

References

[1] M. Brion: Positivity in the Grothendieck group of complex flag varieties, J. Algebra **258** (2002), 137–159. MR **2003m:**14017

[2] M. Brion and P. Polo: Large Schubert varieties, *Represent. Theory* **4** (2000), 97–126. MR **2001j:**14066

[3] C. Chevalley: Sur les décompositions cellulaires des espaces G/B (with a foreword by A. Borel), *Proc. Sympos. Pure Math.* **56**, Part 1, Algebraic Groups and their Generalizations: Classical Methods (University Park, PA; 1991), *Amer. Math. Soc.*, Providence, RI (1994), 1–23. MR **95e:**14041

[4] V. V. Deodhar: On some geometric aspects of Bruhat orderings. I. A finer decomposition of Bruhat cells. *Invent. Math.* **79** (1985), 499–511. MR **86f:**20045

[5] W. V. D. Hodge: Some enumerative results in the theory of forms, *Proc. Camb. Phil. Soc.* **39** (1943), 22-30. MR **4:**184e

[6] B. Kostant and S. Kumar: T-equivariant K-theory of generalized flag varieties, *J. Differential Geom.* **32** (1990), 549–603. MR **92c:**19006

[7] A. Knutson: A Littelmann-type formula for Duistermaat-Heckman measures, *Invent. Math.* **135** (1999), 185–200. MR **2000d:**53127

[8] V. Lakshmibai and P. Littelmann: Richardson varieties and equivariant K-theory, J. Algebra **260** (2003), 230-260.

[9] V. Lakshmibai and C.S. Seshadri: Geometry of G/P-V, *J. Alg*, **100** (1986), 462–557. MR **87k:**14059

[10] V. Lakshmibai and C.S. Seshadri: Standard monomial theory, *Proc. Hyderabad Conference on Algebraic Groups*, (S. Ramanan et al., eds.), Manoj Prakashan, Madras (1991), 279–322. MR **92k:**14053

[11] P. Littelmann: A Littlewood-Richardson rule for symmetrizable Kac-Moody algebras, *Invent. Math.* **116** (1994), 329–346. MR **95f:**17023

[12] P. Littelmann: Contracting modules and standard monomial theory, *J. Amer. Math. Soc.* **11** (1998), 551–567. MR **99d:**17027

[13] V. Mehta and W. van der Kallen: On a Grauert-Riemenschneider theorem for Frobenius split varieties in characteristic p, *Invent. Math.* **108** (1992), 11–13. MR **93a:**14017

[14] S. Ramanan and A. Ramanathan: Projective normality of flag varieties and Schubert varieties, *Invent. Math.* **79** (1985), 217–234. MR **86j:**14051

[15] A. Ramanathan: Schubert varieties are arithmetically Cohen-Macaulay, *Invent. Math.* **80** (1985), 283–294. MR **87d:**14044

[16] A. Ramanathan: Equations defining Schubert varieties and Frobenius splitting of diagonals, *Pub. Math. IHES* **65** (1987), 61–90. MR **88k:**14032

[17] R. W. Richardson: Intersections of double cosets in algebraic groups, *Indag. Math. (N. S.)* **3** (1992), 69–77. MR **93b:**20081

Institut Fourier, UMR 5582 du CNRS, F-38402 Saint-Martin d'Hères Cedex
E-mail address: `Michel.Brion@ujf-grenoble.fr`

Department of Mathematics, Northeastern University, Boston, Massachusetts 02115-5096
E-mail address: `lakshmibai@neu.edu`