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REPRESENTATIONS OF REDUCTIVE GROUPS OVER FINITE RINGS

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Abstract. In this paper we construct a family of irreducible representations of a Chevalley group over a finite ring R of truncated power series over a field $\mathbf{F}_q$. This is done by a cohomological method extending that of Deligne and the author in the case $R = \mathbf{F}_q$.

Introduction

0.1. In [L, Sec.4] a cohomological construction was given (without proof) for certain representations of a Chevalley group over a finite ring R (arising from the ring of integers in a non-archimedean local field by reduction modulo a power of the maximal ideal); that construction was an extension of the construction of the virtual representations R_T^θ in [DL] for groups over a finite field. One of the aims of this paper is to provide the missing proof. For simplicity we will assume that $R = \mathbf{F}_{q,r} = \mathbf{F}_q[[\epsilon]]/(\epsilon^r)$ (ϵ is an indeterminate, $\mathbf{F}_q$ is a finite field with q elements and $r \geq 1$). A similar method applies in the case where R is a finite quotient of a ring of Witt vectors. On the other hand, we treat possibly twisted groups.

Let $\mathbf{F}$ be an algebraic closure of $\mathbf{F}_q$. Let G be a connected reductive algebraic group defined over $\mathbf{F}$ with a given $\mathbf{F}_q$-rational structure with associated Frobenius map $F : G \to G$.

Using a cohomological method, extending that of [DL], we will construct a family of irreducible representations of the finite group $G(\mathbf{F}_{q,r}), r \geq 1$, attached to a "maximal torus" and a character of it in general position. In the case where $r \geq 2$ and G is split over $\mathbf{F}_q$, the representations that we construct are likely to be the same as those found by Gérardin [G] by a non-cohomological method (induction from a subgroup if r is even; induction from a subgroup in combination with a use of a Weil representation, if r is odd, ≥ 3). In any case, since for $r = 1$, the cohomological construction is the only known construction of almost all representations, it seems natural to seek a cohomological construction which works uniformly for all $r \geq 1$; this is what we do in this paper.

0.2. Notation. Let ϵ be an indeterminate. If X is an affine algebraic variety over $\mathbf{F}$ and $r \geq 1$, we set $X_r = X(\mathbf{F}[[\epsilon]]/(\epsilon^r))$. Thus, if X is the set of common

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zeroes of the polynomials $f_i : \mathbf{F}^N \to \mathbf{F}(i = 1, \dots, m)$, then X_r is the set of all $(x_1, x_2, \dots, x_N) \in (\mathbf{F}[[\epsilon]]/(\epsilon^r))^N$ such that $f_i(x_1, x_2, \dots, x_N)$ (a priori an element of $\mathbf{F}[[\epsilon]]/(\epsilon^r)$) is equal to 0 for $i = 1, \dots, m$. We have $X_1 = X$. For $r = 0$ we set $X_r =$ point. Then $X \mapsto X_r$ is a functor from the category of algebraic varieties over $\mathbf{F}$ into itself. If X' is a closed subvariety of X, then X'_r is a closed subvariety of X_r. If X is irreducible of dimension d, then X_r is irreducible of dimension dr. For any $r \ge r' \ge 0$ we have a canonical morphism $\phi_{r,r'} : X_r \to X_{r'}$. If $r \ge 1$, we have naturally $X \subset X_r$ (using $\mathbf{F} \subset \mathbf{F}[[\epsilon]]/(\epsilon^r)$). If G is an algebraic group over $\mathbf{F}$, then G_r is naturally an algebraic group over $\mathbf{F}$. For any $r \ge r' \ge 0$, $\phi_{r,r'} : G_r \to G_{r'}$ is a homomorphism of algebraic groups hence its kernel, $G_r^{r'}$, is a normal subgroup of G_r. For $r \ge 1$ we have naturally $G \subset G_r$. We have

$$\{1\} = G_r^r \subset G_r^{r-1} \subset \dots \subset G_r^1 \subset G_r^0 = G_r.$$

For $r > r' \ge 0$, we set $G_r^{r',*} = G_r^{r'} - G_r^{r'+1}$. We have a partition

$$G_r = G_r^{0,*} \sqcup G_r^{1,*} \sqcup \dots \sqcup G_r^{r-1,*} \sqcup \{1\}.$$

We fix a prime number l invertible in $\mathbf{F}$. If X is an algebraic variety over $\mathbf{F}$ we write $H_c^j(X)$ instead of $H_c^j(X, \bar{\mathbf{Q}}_l)$.

For a finite group Γ let $\hat{\Gamma} = \mathrm{Hom}(\Gamma, \bar{\mathbf{Q}}_l^*)$.

0.3. If T is a commutative algebraic group over $\mathbf{F}$ with a fixed $\mathbf{F}_q$-structure and with Frobenius map $F : T \to T$, we have a norm map

$$N_F^{F^n} : T^{F^n} \to T^F, t \mapsto tF(t)F^2(t) \dots F^{n-1}(t).$$

1. Lemmas

Lemma 1.1. *Let $\mathcal{T}, \mathcal{T}'$ be two commutative, connected algebraic groups over $\mathbf{F}$ with fixed $\mathbf{F}_q$-rational structures with Frobenius maps $F : \mathcal{T} \to \mathcal{T}, F : \mathcal{T}' \to \mathcal{T}'$. Let $f : \mathcal{T} \xrightarrow{\sim} \mathcal{T}'$ be an isomorphism of algebraic groups over $\mathbf{F}$. Let $n \ge 1$ be such that $F^n f = fF^n : \mathcal{T} \to \mathcal{T}'$; thus $f : \mathcal{T}^{F^n} \xrightarrow{\sim} \mathcal{T}'^{F^n}$. Let*

$$H = \{(t, t') \in \mathcal{T} \times \mathcal{T}'; f(F(t)^{-1}t) = F(t')^{-1}t'\}.$$

(A subgroup of $\mathcal{T} \times \mathcal{T}'$ containing $\mathcal{T}^F \times \mathcal{T}'^F$.) Let $\theta \in \widehat{\mathcal{T}^F}$, $\theta' \in \widehat{\mathcal{T}'^F}$ be such that $\theta^{-1} \boxtimes \theta'$ is trivial on $(\mathcal{T}^F \times \mathcal{T}'^F) \cap H^0$. Then $\theta N_F^{F^n} = \theta' N_F^{F^n} f \in \widehat{\mathcal{T}^{F^n}}$.

Setting $t_1 = tF(t) \dots F^{n-1}(t) \in \mathcal{T}$, $\quad t_2 = f(t)F(f(t)) \dots F^{n-1}(f(t)) \in \mathcal{T}'$ for $t \in \mathcal{T}$, we have

$$f(F(t_1)^{-1}t_1) = f(tF^n(t)^{-1}) = f(t)f(F^n(t))^{-1} = f(t)F^n(f(t))^{-1} = F(t_2)^{-1}t_2,$$

so that $(t_1, t_2) \in H$. Now $t \mapsto (t_1, t_2)$ is a morphism $\mathcal{T} \to H$ of algebraic varieties and $\mathcal{T}$ is connected; hence the image of this morphism is contained in H^0. In particular, if $t \in \mathcal{T}^{F^n}$, we have $(N_F^{F^n}(t), N_F^{F^n}(f(t))) \in (\mathcal{T}^F \times \mathcal{T}'^F) \cap H^0$ hence, by assumption, $\theta^{-1}(N_F^{F^n}(t))\theta'(N_F^{F^n}(f(t))) = 1$ for all $t \in \mathcal{T}^{F^n}$. The lemma is proved.

1.2. Let G be a connected reductive algebraic group over $\mathbf{F}$ with a fixed $\mathbf{F}_q$-rational structure with Frobenius map $F : G \to G$. If $r \ge 1$, then $F : G \to G$ induces a homomorphism $F : G_r \to G_r$ which is the Frobenius map for a $\mathbf{F}_q$-rational structure on G_r.

Let T, T' be two F-stable maximal tori of G and let U (resp. U') be the unipotent radical of a Borel subgroup of G that contains T (resp. T'). Note that U, U' are not necessarily defined over $\mathbf{F}_q$. Let $r \geq 2$. Let $\mathcal{T} = T_r^{r-1}, \mathcal{T}' = T_r'^{r-1}$,

$$\Sigma = \{(x, x', y) \in F(U_r) \times F(U'_r) \times G_r; xF(y) = yx'\}.$$

Let $N(T, T') = \{\nu \in G; \nu^{-1} T \nu = T'\}$. Then T acts on $N(T, T')$ by left multiplication and T' acts on $N(T, T')$ by right multiplication. The orbits of T are the same as the orbits of T'; we set $W(T, T') = T \backslash N(T, T') = N(T, T')/T'$ (a finite set). For each $w \in W(T, T')$ we choose a representative $\dot{w}$ in $N(T, T')$. We have $G = \bigsqcup_{w \in W(T,T')} G_w$ where $G_w = UT\dot{w}U' = U\dot{w}T'U'$.

Let $G_{w,r}$ be the inverse image of G_w under $\phi_{r,1} : G_r \to G$ and let $\Sigma_w = \{(x, x', y) \in \Sigma; y \in G_{w,r}\}$.

Now $T_r^F \times T_r'^F$ acts on Σ by $(t, t') : (x, x', y) \mapsto (txt^{-1}, t'x't'^{-1}, tyt'^{-1})$. This restricts to an action of $T_r^F \times T_r'^F$ on Σ_w for any $w \in W$.

If $\theta \in \widehat{T_r^F}, \theta' \in \widehat{T_r'^F}$ and M is a $T_r^F \times T_r'^F$-module, we shall write $M_{\theta^{-1},\theta'}$ for the subspace of M on which $T_r^F \times T_r'^F$ acts according to $\theta^{-1} \boxtimes \theta'$.

Lemma 1.3. *Assume that $r \geq 2$. Let $w \in W(T, T')$. Let $\theta \in \widehat{T_r^F}$, $\theta' \in \widehat{T_r'^F}$. Assume that $H_c^j(\Sigma_w)_{\theta^{-1},\theta'} \neq 0$ for some $j \in \mathbf{Z}$. Let $g = F(\dot{w})^{-1}$ and let $n \geq 1$ be such that $g \in G^{F^n}$. Then $\mathrm{Ad}(g)$ carries $\mathcal{T}^{F^n}$ onto $\mathcal{T}'^{F^n}$ and $\theta|_{\mathcal{T}^F} \circ N_F^{F^n} \in \widehat{\mathcal{T}^{F^n}}$ to $\theta'|_{\mathcal{T}'^F} \circ N_F^{F^n} \in \widehat{\mathcal{T}'^{F^n}}$.*

By the definition of $G_{w,r}$, the map $U_r \times G_r^1 \times (T_r\dot{w}) \times U'_r \to G_{w,r}$ given by $(u, k, \nu, u') \mapsto uk\nu u'$ is a locally trivial fibration with all fibres isomorphic to a fixed affine space. Hence the map

$$\tilde{\Sigma}_w = \{(x, x', u, u', k, \nu) \in F(U_r) \times F(U'_r) \times U_r \times U'_r \times G_r^1 \times T_r\dot{w};$$
$$xF(u)F(k)F(\nu)F(u') = uk\nu u'x'\} \to \Sigma_w$$

given by $(x, x', u, u', k, \nu) \mapsto (x, x', uk\nu u')$, is a locally trivial fibration with all fibres isomorphic to a fixed affine space. This map is compatible with the $T_r^F \times T_r'^F$ actions where $T_r^F \times T_r'^F$ acts on $\tilde{\Sigma}_w$ by

(a) $\quad (t, t') : (x, x', u, u', k, \nu) \mapsto (txt^{-1}, t'x't'^{-1}, tut^{-1}, t'u't'^{-1}, tkt^{-1}, t\nu t'^{-1}).$

Hence there exists $j' \in \mathbf{Z}$ such that $H_c^{j'}(\tilde{\Sigma}_w)_{\theta^{-1},\theta'} \neq 0$. By the substitution $xF(u) \mapsto x, x'F(u')^{-1} \mapsto x'$, the variety $\tilde{\Sigma}_w$ is rewritten as

(b) $$\{(x, x', u, u', k, \nu) \in F(U_r) \times F(U'_r) \times U_r \times U'_r \times G_r^1 \times T_r\dot{w};$$
$$xF(k)F(\nu) = uk\nu u'x'\};$$

in these coordinates, the action of $T_r^F \times T_r'^F$ is still given by (a). Let

$$H = \{(t, t') \in \mathcal{T} \times \mathcal{T}'; t'F(t')^{-1} = F(\dot{w})^{-1}tF(t)^{-1}F(\dot{w})\}.$$

(A closed subgroup of $T_r \times T'_r$.) It acts on the variety (b) by the same formula as in (a). (We use the fact that $hk = kh$ for any $h \in G_r^{r-1}, k \in G_r^1$.) By [DL, 6.5], the induced action of H on $H_c^{j'}(\tilde{\Sigma}_w)$ is trivial when restricted to H^0. In particular, the intersection $(T_r^F \times T_r'^F) \cap H^0$ acts trivially on $H_c^{j'}(\tilde{\Sigma}_w)$. Since $H_c^{j'}(\tilde{\Sigma}_w)_{\theta^{-1},\theta'} \neq 0$, it follows that $\theta^{-1} \boxtimes \theta'$ is trivial on $(T_r^F \times T_r'^F) \cap H^0$. Let $g = F(\dot{w})^{-1}$ and let $n \geq 1$ be such that $g \in G^{F^n}$. Then $\mathrm{Ad}(g)$ carries $\mathcal{T}^{F^n}$ onto $\mathcal{T}'^{F^n}$ and (by Lemma 1.1 with $f = \mathrm{Ad}(g)$) it carries $\theta|_{\mathcal{T}^F} \circ N_F^{F^n}$ to $\theta'|_{\mathcal{T}'^F} \circ N_F^{F^n}$. The lemma is proved.

Lemma 1.4. *Assume that $r \geq 2$. Let $\theta \in \widehat{T_r^F}$, $\theta' \in \widehat{T_r'^F}$ be such that*

(a) $$H_c^j(\Sigma)_{\theta^{-1},\theta'} \neq 0$$

for some $j \in \mathbf{Z}$. There exists $n \geq 1$ and $g \in N(T',T)^{F^n}$ such that $\mathrm{Ad}(g)$ carries $\theta|_{\mathcal{T}^F} \circ N_F^{F^n} \in \widehat{\mathcal{T}^{F^n}}$ to $\theta'|_{\mathcal{T}'^F} \circ N_F^{F^n} \in \widehat{\mathcal{T}'^{F^n}}$.

The subvarieties G_w of G have the following property: for some ordering $\leq$ of $W(T,T')$, the unions $\bigcup_{w' \leq w} G_{w'}$ are closed in G. It follows that the unions $\bigcup_{w' \leq w} G_{w',r}$ are closed in G_r and the unions $\bigcup_{w' \leq w} \Sigma_{w'}$ are closed in Σ. The spectral sequence associated to the filtration of Σ by these unions, together with (a), shows that there exists $w \in W(T,T')$ and $j \in \mathbf{Z}$ such that $H_c^j(\Sigma_w)_{\theta^{-1},\theta'} \neq 0$. We can therefore apply Lemma 1.3. The lemma follows.

1.5. Let Φ be the set of characters $\alpha : T \to \mathbf{F}^*$ such that $\alpha \neq 1$ and T acts on some line $L_\alpha \subset \mathrm{Lie}\, G$ via α (in the adjoint action); for such α, let G^α be the one-dimensional unipotent subgroup of G such that $\mathrm{Lie}\, G^\alpha = L_\alpha$. For $\alpha \in \Phi$ there is a unique 1-dimensional torus T^α in T such that T^α is contained in the subgroup of G generated by $G^\alpha, G^{\alpha^{-1}}$. Let $\mathcal{T}^\alpha = (T^\alpha)_r^{r-1}$ (a one-dimensional subgroup of $\mathcal{T} = T_r^{r-1}$).

Let $\chi \in \widehat{\mathcal{T}^F}$. We say that χ is *regular* if for any $\alpha \in \Phi$ and any $n \geq 1$ such that $F^n(\mathcal{T}^\alpha) = \mathcal{T}^\alpha$, the restriction of $\chi \circ N_F^{F^n} : \mathcal{T}^{F^n} \to \bar{\mathbf{Q}}_l^*$ to $(\mathcal{T}^\alpha)^{F^n}$ is non-trivial. (It is enough to check that $\chi \circ N_F^{F^n}|_{(\mathcal{T}^\alpha)^{F^n}}$ is non-trivial for any α and for just one n such that $F^n(\mathcal{T}^\alpha) = \mathcal{T}^\alpha$ for all α.)

Let $\theta \in \widehat{T^F}$. We say that θ is *regular* if $\theta|_{\mathcal{T}^F}$ is regular.

1.6. Let T be an F-stable maximal torus of G. Let $U, \tilde{U}, V, \tilde{V}$ be unipotent radicals of Borel subgroups containing T such that $U \cap V = \tilde{U} \cap \tilde{V} = \{1\}$. Let Φ be as in 1.5. Let

$$\Phi^+ = \{\alpha \in \Phi; G^\alpha \subset \tilde{V}\}, \Phi^- = \{\alpha \in \Phi; G^\alpha \subset \tilde{U}\}.$$

Then $\Phi = \Phi^+ \sqcup \Phi^-$ and $\Phi^- = \{\alpha^{-1}; \alpha \in \Phi^+\}$.

For $\alpha \in \Phi^+$ let $ht(\alpha)$ be the largest integer $n \geq 1$ such that $\alpha = \alpha_1\alpha_2 \dots \alpha_n$ with $\alpha_i \in \Phi^+$.

Let $x \in (G^\alpha)_r^b, x' \in (G^{\alpha'})_r^c$ where $\alpha, \alpha' \in \Phi$ and $b, c \in [0, r]$.

(a) If $b + c \geq r$, then $xx' = x'x$.

(b) If $b + c \leq r$ and $\alpha\alpha' \neq 1$, then $xx' = x'xu$ where u is of the form $\prod_{i,i' \geq 1; \alpha^i\alpha'^{i'} \in \Phi} u_{i,i'}$ with $u_{i,i'} \in (G^{\alpha^i\alpha'^{i'}})_r^{b+b'}$.

(The factors in the last product are written in some fixed order. In the special case where $b + c = r - 1$, these factors commute with each other by (a), since $r - 1 + r - 1 \geq r$.)

(c) If $b + c \geq r - 1, b + 2c \geq r$ and $\alpha\alpha' = 1$, then $xx' = x'x\tau_{x,x'}u$ where $\tau_{x,x'} \in \mathcal{T}^\alpha$ and $u \in (G^\alpha)_r^{r-1}$ are uniquely determined.

Lemma 1.7. *We fix an order on Φ^+. For any $z \in \tilde{V}_r, \beta \in \Phi^+$, define $x_\beta^z \in G_r^\beta$ by $z = \prod_{\beta \in \Phi^+} x_\beta^z$ (factors written using the given order on Φ^+). Let $\alpha \in \Phi^-, a \in [1, r-1]$. Let $z \in \tilde{V}_r^a$ be such that $x_\beta^z \in (G^\beta)_r^{a+1}$ for all $\beta \in \Phi^+$ with $ht(\beta) > ht(\alpha^{-1})$. Let $\xi \in (G^\alpha)_r^{r-a-1}$. Then $\xi z = z\xi\tau_{\xi,z}\omega_{\xi,z}$ where $\tau_{\xi,z} \in \mathcal{T}^\alpha$ and $\omega_{\xi,z} \in \tilde{U}_r^{r-1}$.*

We argue by induction on $N_z = \sharp(\beta \in \Phi^+; x^z_\beta \neq 1)$. If $N_z = 0$, the result is clear. Assume now that $N_z = 1$ so that $z \in G^\beta_r$ with $\beta \in \Phi^+$. If $\alpha\beta = 1$, the result follows from 1.6(c). If $\alpha\beta \neq 1$ and $ht(\beta) > ht(\alpha^{-1})$, then $z \in (G^\beta)^{a+1}_r$ and $\xi z = z\xi$ by 1.6(b). If $\alpha\beta \neq 1$ and $ht(\beta) \le ht(\alpha^{-1})$, then by 1.6(b) we have $\xi z = z\xi u$ where $u = \prod_{i,i' \ge 1; \alpha^i\beta^{i'} \in \Phi} u_{i,i'}$ with $u_{i,i'} \in (G^{\alpha^i\beta^{i'}})^{r-1}_r$; it is enough to show that if $i, i' \ge 1$, we cannot have $\alpha^i\beta^{i'} \in \Phi^+$. (If $\alpha^i\beta^{i'} \in \Phi^+$ for some $i, i' \ge 1$, then $\alpha\beta \in \Phi^+$ hence $ht(\beta) > ht(\alpha^{-1})$, contradiction.)

Assume now that $N_z \ge 2$. We can write $z = z'z''$ where $z', z'' \in \tilde V^a_r$, $N_{z'} < N_z, N_{z''} < N_z$. Using the induction hypothesis we have

$$\xi z = \xi z'z'' = z'\xi\tau_{\xi,z'}\omega_{\xi,z'}z''$$

where $\tau_{\xi,z'} \in \mathcal{T}^\alpha$, $\omega_{\xi,z'} \in \tilde U^{r-1}_r$. We have $\omega_{\xi,z'}z'' = z''\omega_{\xi,z'}$ and $\tau_{\xi,z'}z'' = z''\tau_{\xi,z'}$. Using again the induction hypothesis, we have

$$\begin{aligned} z'\xi\tau_{\xi,z'}\omega_{\xi,z'}z'' &= z'\xi\tau_{\xi,z'}z''\omega_{\xi,z'} = z'\xi z''\tau_{\xi,z'}\omega_{\xi,z'} \\ &= z'z''\xi\tau_{\xi,z''}\omega_{\xi,z''}\tau_{\xi,z'}\omega_{\xi,z'} = z\xi\tau_{\xi,z'}\tau_{\xi,z''}\omega_{\xi,z'}\omega_{\xi,z''}. \end{aligned}$$

Thus, $\xi z = z\xi\tau_{\xi,z}\omega_{\xi,z}$ where

$$\tau_{\xi,z} = \tau_{\xi,z'}\tau_{\xi,z''}, \omega_{\xi,z} = \omega_{\xi,z'}\omega_{\xi,z''}.$$

The lemma is proved.

1.8. In the setup of 1.6, let $Z = V \cap \tilde V$. Let $\Phi' = \{\beta \in \Phi; G^\beta \subset Z\}$. We have $\Phi' \subset \Phi^+$. Let $\mathcal{X}$ be the set of all subsets $I \subset \Phi'$ such that $I \neq \emptyset$ and $ht : \Phi^+ \to \mathbf{N}$ is constant on I.

To any $z \in Z^1_r - \{1\}$ we associate a pair (a, I_z) where $a \in [1, r-1]$ and $I_z \in \mathcal{X}$ as follows. We define a by the condition that $z \in Z^{a,*}_r$. If $x^z_\beta \in G^\beta$ are defined as in 1.8 in terms of a fixed order on Φ^+, then $x^z_\beta \in (G^\beta)^a_r$ for all $\beta \in \tilde\Phi$ and $x^z_\beta = 1$ for all $\beta \in \Phi^+ - \tilde\Phi$. Let I_z be the set of all $\alpha' \in \tilde\Phi$ such that $x^z_{\alpha'} \in (G^{\alpha'})^{a,*}_r$ and $x^z_\beta \in (G^\beta)^{a+1}_r$ for all $\beta \in \Phi^+$ such that $ht(\beta) > ht(\alpha')$. It is easy to see, using 1.6(a),(b), that the definition of I_z does not depend on the choice of an order on Φ^+. For $a \in [1, r-1]$ and $I \in \mathcal{X}$ let $Z^{a,*,I}_r$ be the set of all $z \in Z^1_r - \{1\}$ such that $z \in Z^{a,*}_r, I = I_z$. Thus we have a partition

$$\text{(a)} \qquad Z^1_r - \{1\} = \bigsqcup_{a \in [1,r-1], I \in \mathcal{X}} Z^{a,*,I}_r.$$

Lemma 1.9. *Let $T, T', U, U', r, \mathcal{T}, \mathcal{T}'$ be as in 1.2. Let $\theta \in \widehat{T^F_r}, \theta' \in \widehat{T'^F_r}$. Assume that $\theta'|_{\mathcal{T}^F} = \chi$ is regular. Let Σ be as in 1.2. Then $\sum_{j \in \mathbf{Z}}(-1)^j \dim H^j_c(\Sigma)_{\theta^{-1},\theta'}$ is equal to the number of $w \in W(T, T')^F$ such that $Ad(\dot w) : T'^F_r \to T^F_r$ carries θ to θ'.*

Using the partition $\Sigma = \bigsqcup_{w \in W(T,T')} \Sigma_w$ we see that it is enough to prove that $\sum_{j \in \mathbf{Z}}(-1)^j \dim H^j_c(\Sigma_w)_{\theta^{-1},\theta'}$ is equal to 1 if $F(w) = w$ and $Ad(\dot w) : T'^F_r \to T^F_r$ carries θ to θ' and equals 0, otherwise. We now fix $w \in W(T, T')$. We have

$$\begin{aligned} \Sigma_w = \{&(x, x', y) \in F(U_r) \times F(U'_r) \times G_r; xF(y) \\ &= yx', y \in U_r G^1_r \dot w T'_r U'_r = U_r Z^1_r \dot w T'_r U'_r\} \end{aligned}$$

where $Z = V \cap \dot{w}V'\dot{w}^{-1}$. Here V (resp. V') is the unipotent radical of a Borel subgroup containing T (resp. T') such that $U \cap V = \{1\}$ (resp. $U' \cap V' = \{1\}$. Let

$$\hat{\Sigma}_w = \{(x, x', u, u', z, \tau') \in F(U_r) \times F(U'_r) \times U_r \times U'_r \times Z^1_r \times T'_r; \\ xF(u)F(z)F(\dot{w})F(\tau')F(u') = uz\dot{w}\tau'u'x'\}.$$

The map $\hat{\Sigma}_w \to \Sigma_w$ given by $(x, x', u, u', z, \tau') \mapsto (x, x', uz\dot{w}\tau'u')$ is a locally trivial fibration with all fibres isomorphic to a fixed affine space. This map is compatible with the $T_r^F \times T_r'^F$-actions where $T_r^F \times T_r'^F$ acts on $\hat{\Sigma}_w$ by

(a) $$(t, t') : (x, x', u, u', z, \tau') \\ \mapsto (txt^{-1}, t'x't'^{-1}, tut^{-1}, t'u't'^{-1}, tzt^{-1}, \dot{w}^{-1}t\dot{w}\tau t'^{-1}).$$

Hence it is enough to show that

$\sum_{j\in\mathbf{Z}}(-1)^j \dim H^j_c(\hat{\Sigma}_w)_{\theta^{-1},\theta'}$ is equal to 1 if $F(w) = w$ and $Ad(\dot{w}) : T_r'^F \to T_r^F$ carries θ to θ' and equals 0, otherwise.

By the change of variable $xF(u) \mapsto x, x'F(u')^{-1} \mapsto x'$ we may rewrite $\hat{\Sigma}_w$ as

$$\hat{\Sigma}_w = \{(x, x', u, u', z, \tau') \in F(U_r) \times F(U'_r) \times U_r \times U'_r \times Z^1_r \times T'_r; \\ xF(z)F(\dot{w})F(\tau') = uz\dot{w}\tau'u'x'\}$$

with the $T_r^F \times T_r'^F$-action still given by (a). We have a partition $\hat{\Sigma}_w = \hat{\Sigma}'_w \sqcup \hat{\Sigma}''_w$ where

$$\hat{\Sigma}'_w = \{(x, x', u, u', z, \tau') \in F(U_r) \times F(U'_r) \times U_r \times U'_r \times (Z^1_r - \{1\}) \times T'_r; \\ xF(z)F(\dot{w})F(\tau') = uz\dot{w}\tau'u'x'\},$$

$$\hat{\Sigma}''_w = \{(x, x', u, u', 1, \tau') \in F(U_r) \times F(U'_r) \times U_r \times U'_r \times \{1\} \times T'_r; \\ xF(\dot{w})F(\tau') = u\dot{w}\tau'u'x'\},$$

are stable under the $T_r^F \times T_r'^F$-action. It is then enough to show that

(b) $\sum_{j\in\mathbf{Z}}(-1)^j \dim H^j_c(\hat{\Sigma}''_w)_{\theta^{-1},\theta'}$ is equal to 1 if $F(w) = w$ and $Ad(\dot{w}) : T_r'^F \to T_r^F$ carries θ to θ' and equals 0, otherwise.

(c) $H^j_c(\hat{\Sigma}'_w)_{\theta^{-1},\theta'} = 0$ for all j.

We first prove (c). If M is a $\mathcal{T}'^F$-module we shall write $M_{(\chi)}$ for the subspace of M on which $\mathcal{T}'^F$ acts according to χ. Now $\mathcal{T}'^F$ acts on $\hat{\Sigma}'_w$ by

$$t' : (x, x', u, u', z, \tau') \mapsto (x, t'x't'^{-1}, u, t'u't'^{-1}, z, \tau't'^{-1}).$$

Hence $H^j_c(\hat{\Sigma}'_w)$ becomes a $\mathcal{T}'^F$-module. It is enough to show that $H^j_c(\hat{\Sigma}'_w)_{(\chi)} = 0$.

We shall use the definitions and results in 1.6–1.8 relative to $U, \tilde{U}, V, \tilde{V}$ where $\tilde{U} = \dot{w}U'\dot{w}^{-1}$, $\tilde{V} = \dot{w}V'\dot{w}^{-1}$. The partition 1.8(a) gives rise to a partition $\hat{\Sigma}'_w = \bigsqcup_{a,I} \hat{\Sigma}^{a,I}_w$ indexed by $a \in [0, r-1], I \in \mathcal{X}$ where

$$\hat{\Sigma}^{a,I}_w = \{(x, x', u, u', z, \tau') \in \hat{\Sigma}'_w; z \in Z^{a,*,I}_r\}.$$

It is easy to see that there is a total order on the set of indices (a, I) such that the union of the $\hat{\Sigma}^{a,I}_w$ for (a, I) less than or equal to some given (a^0, I^0) is closed in $\hat{\Sigma}'_w$. Since the subsets $\hat{\Sigma}^{a,I}_w$ are stable under the action of $\mathcal{T}'^F$, we see that, in order to prove (c), it is enough to show that

(d) $$H^j_c(\hat{\Sigma}^{a,I}_w)_{(\chi)} = 0$$

for any fixed a, I as above. We choose $\alpha' \in I$. Let $\alpha = \alpha'^{-1}$. Then $G_r^\alpha \subset U_r \cap \dot{w} U_r' \dot{w}^{-1}$.

For any $z \in Z_r^{a,*}, \xi \in (G^\alpha)_r^{r-a-1}$ we have

$$\xi z = z \xi \tau_{\xi,z} \omega_{\xi,z}$$

where $\tau_{\xi,z} \in \mathcal{T}^\alpha, \omega(\xi, z) \in \dot{w} U_r'^{r-1} \dot{w}^{-1}$ are uniquely determined. (See 1.7.) Moreover, the map $(G^\alpha)_r^{r-a-1} \to \mathcal{T}^\alpha, \xi \mapsto \tau(\xi, z)$ factors through an isomorphism

$$\lambda_z : (G^\alpha)_r^{r-a-1}/(G^\alpha)_r^{r-a} \xrightarrow{\sim} \mathcal{T}^\alpha.$$

Let $\pi : (G^\alpha)_r^{r-a-1} \to (G^\alpha)_r^{r-a-1}/(G^\alpha)_r^{r-a}$ be the canonical homomorphism. We can find a morphism of algebraic varieties

$$\psi : (G^\alpha)_r^{r-a-1}/(G^\alpha)_r^{r-a} \to (G^\alpha)_r^{r-a-1}$$

such that $\pi\psi = 1$ and $\psi(1) = 1$. Let

$$\mathcal{H}' = \{t' \in \mathcal{T}'; t'^{-1} F(t') \in \dot{w}^{-1} \mathcal{T}^\alpha \dot{w}\}.$$

This is a closed subgroup of $\mathcal{T}'$. For any $t' \in \mathcal{H}'$ we define $f_{t'} : \hat{\Sigma}_w^{a,I} \to \hat{\Sigma}_w^{a,I}$ by

$$f_{t'}(x, x', u, u', z, \tau') = (xF(\xi), \hat{x}', u, F(t')^{-1} u' F(t'), z, \tau' F(t'))$$

where

$$\xi = \psi \lambda_z^{-1} (\dot{w} F(t')^{-1} t' \dot{w}^{-1}) \in (G^\alpha)_r^{r-a-1} \subset U_r \cap \dot{w} U_r' \dot{w}^{-1}$$

and $\hat{x}' \in G_r$ is defined by the condition that

$$xF(\xi)F(z)F(\dot{w})F(\tau' F(t')) = uz\dot{w}\tau' F(t') F(t')^{-1} u' F(t') \hat{x}'.$$

In order for this to be well defined we must check that $\hat{x}' \in F(U_r')$. Thus we must show that

$$xF(\xi)F(z)F(\dot{w})F(\tau' F(t')) \in uz\dot{w}\tau' u' F(t') F(U_r')$$

or that

$$xF(z)F(\xi)F(\tau_{\xi,z})F(\omega_{\xi,z})F(\dot{w})F(\tau' F(t')) \in uz\dot{w}\tau' u' F(t') F(U_r').$$

Since $xF(z) = uz\dot{w}\tau' u' x' F(\tau')^{-1} F(\dot{w}^{-1})$, it is enough to show that

$$\begin{aligned} &uz\dot{w}\tau' u' x' F(\tau')^{-1} F(\dot{w}^{-1}) F(\xi) F(\tau_{\xi,z}) F(\omega_{\xi,z}) F(\dot{w}) F(\tau' F(t')) \\ &\quad \in uz\dot{w}\tau' u' F(t') F(U_r') \end{aligned}$$

or that

$$x' F(\tau')^{-1} F(\dot{w}^{-1}) F(\xi) F(\tau_{\xi,z}) F(\omega_{\xi,z}) F(\dot{w}) F(\tau' F(t')) \in F(t') F(U_r').$$

Since $x' \in F(U_r'), F(\dot{w}^{-1}) F(\omega_{\xi,z}) F(\dot{w}) \in F(U_r')$, it is enough to check that

$$F(\tau')^{-1} F(\dot{w}^{-1}) F(\xi) F(\tau_{\xi,z}) F(\dot{w}) F(\tau' F(t')) \in F(t') F(U_r').$$

Since $F(\dot{w}^{-1}) F(\xi) F(\dot{w}) \in F(U_r')$ it is enough to check that

$$F(\tau')^{-1} F(\dot{w}^{-1}) F(\tau_{\xi,z}) F(\dot{w}) F(\tau' F(t')) \in F(t') F(U_r')$$

or that

$$F(\dot{w}^{-1}) F(\tau_{\xi,z}) F(\dot{w}) F(F(t')) = F(t')$$

or that $\dot{w}^{-1} \tau_{\xi,z} \dot{w} = F(t')^{-1} t'$ or that $\lambda_z(\pi_z(\xi)) = \dot{w} F(t')^{-1} t' \dot{w}^{-1}$. But this is clear.

Thus, $f_{t'} : \hat{\Sigma}_w^{a,I} \to \hat{\Sigma}_w^{a,I}$ is well defined for $t' \in \mathcal{H}'$. It is clearly an isomorphism for any $t' \in \mathcal{H}'$. In particular, it is a well-defined isomorphism for any $t' \in \mathcal{H}'^0$. By general principles, the induced map $f_{t'}^* : H_c^j(\hat{\Sigma}_w^{a,I}) \to H_c^j(\hat{\Sigma}_w^{a,I})$ is constant when t' varies in $\mathcal{H}'^0$. In particular, it is constant when t' varies in $\mathcal{T}'^F \cap \mathcal{H}'^0$. Now

$\mathcal{T}'^F \subset \mathcal{H}'$ and for $t' \in \mathcal{T}'^F$, the map $f_{t'}$ coincides with the action of t' in the $\mathcal{T}'^F$-action on $\hat{\Sigma}_w^{a,I}$. (We use that $\psi(1) = 1$.) We see that the induced action of $\mathcal{T}'^F$ on $H_c^j(\hat{\Sigma}_w^{a,I})$ is trivial when restricted to $\mathcal{T}'^F \cap \mathcal{H}'^0$.

We can find $n \geq 1$ such that $F^n(\dot{w}^{-1}\mathcal{T}^\alpha \dot{w}) = \dot{w}^{-1}\mathcal{T}^\alpha \dot{w}$. Then

$$t' \mapsto t'F(t')F^2(t') \dots F^{n-1}(t')$$

is a well-defined morphism $\dot{w}^{-1}\mathcal{T}^\alpha \dot{w} \to \mathcal{H}'$. Its image is a connected subgroup of $\mathcal{H}'$ hence is contained in $\mathcal{H}'^0$. If $t' \in (\dot{w}^{-1}\mathcal{T}^\alpha \dot{w})^{F^n}$, then $N_F^{F^n}(t') \in \mathcal{T}'^F$; thus, $N_F^{F^n}(t') \in \mathcal{T}'^F \cap \mathcal{H}'^0$. We see that the action of $N_F^{F^n}(t') \in \mathcal{T}'^F$ on $H_c^j(\hat{\Sigma}_w^{a,I})$ is trivial for any $t' \in (\dot{w}^{-1}\mathcal{T}^\alpha \dot{w})^{F^n}$.

If we assume that $H_c^j(\hat{\Sigma}_w^{a,I})_{(\chi)} \neq 0$, it follows that $t' \mapsto \chi(N_F^{F^n}(t'))$ is the trivial character of $(\dot{w}^{-1}\mathcal{T}^\alpha \dot{w})^{F^n}$. This contradicts our assumption that χ is regular. Thus, (d) holds. Hence (c) holds.

We now prove (b). Let

$$\tilde{H} = \{(t, t') \in T_r \times T_r'; tF(t)^{-1} = F(\dot{w})t'F(t')^{-1}F(\dot{w}^{-1})\}.$$

This is a closed subgroup of $T_r \times T_r'$ containing $T_r^F \times T_r'^F$. Now the action of $T_r^F \times T_r'^F$ on $\hat{\Sigma}_w''$ extends to an action of $\tilde{H}$ given by the same formula. To see this consider $(t, t') \in \tilde{H}$ and $(x, x', u, u', 1, \tau') \in \hat{\Sigma}_w''$. We must show that

$$(txt^{-1}, t'x't'^{-1}, tut^{-1}, t'u't'^{-1}, 1, \dot{w}^{-1}t\dot{w}\tau't'^{-1}) \in \hat{\Sigma}_w'',$$

that is,

$$txt^{-1}F(\dot{w})F(\dot{w}^{-1})F(t)F(\dot{w})F(\tau')F(t'^{-1}) = tut^{-1}\dot{w}\dot{w}^{-1}t\dot{w}\tau't'^{-1}t'u't'^{-1}t'x't'^{-1}$$

or that

$$xt^{-1}F(t)F(\dot{w})F(\tau')F(t'^{-1}) = u\dot{w}\tau'u'x't'^{-1}$$

or that

$$xt^{-1}F(t)F(\dot{w})F(\tau')F(t'^{-1}) = xF(\dot{w})F(\tau')t'^{-1}$$

or that $t^{-1}F(t)F(\dot{w})F(t'^{-1}) = F(\dot{w})t'^{-1}$; this is clear. Let T_*, T_*' be the reductive part of T_r, T_r' (thus T_* is a torus isomorphic to T). Let $\tilde{H}_* = \tilde{H} \cap (T_* \times T_*')$. Then $\tilde{H}_*^0$ is a torus acting on $\hat{\Sigma}_w''$ by restriction of the $\tilde{H}$-action. The fixed point set $(\hat{\Sigma}_w'')^{\tilde{H}_*^0}$ of the $\tilde{H}_*^0$-action is stable under the action of $T_r^F \times T_r'^F$ and by general principles we have

$$\sum_{j \in \mathbf{Z}} (-1)^j \dim H_c^j(\hat{\Sigma}_w'')_{\theta^{-1}, \theta'} = \sum_{j \in \mathbf{Z}} (-1)^j \dim H_c^j((\hat{\Sigma}_w'')^{\tilde{H}_*^0})_{\theta^{-1}, \theta'}.$$

It is then enough to show that

(e) $\sum_{j \in \mathbf{Z}} (-1)^j \dim H_c^j((\hat{\Sigma}_w'')^{\tilde{H}_*^0})_{\theta^{-1}, \theta'}$ is equal to 1 if $F(w) = w$ and $Ad(\dot{w}) : T_r'^F \to T_r^F$ carries θ to θ' and equals 0, otherwise.

Let $(x, x', u, u', 1, \tau) \in (\hat{\Sigma}_w'')^{\tilde{H}_*^0}$. By Lang's theorem the first projection $\tilde{H}_* \to T_*$ is surjective. It follows that the first projection $\tilde{H}_*^0 \to T_*$ is surjective. Similarly the second projection $\tilde{H}_*^0 \to T_*'$ is surjective. Hence for any $t \in T_*, t' \in T_*'$ we have

$$txt^{-1} = x, t'x't'^{-1} = x', tut^{-1} = u, t'u't'^{-1} = u'$$

hence $x = x' = u = u' = 1$. Thus, $(\hat{\Sigma}_w'')^{\tilde{H}_*^0}$ is contained in

(f) $\{(1, 1, 1, 1, 1, \tau'); \tau' \in T_r', F(\dot{w}\tau') = \dot{w}\tau'\}$.

The set (f) is clearly contained in the fixed point set of $\tilde{H}$. Note that (f) is empty unless $F(w) = w$. We can therefore assume that $F(w) = w$. In this case, (f) is stable under the action of $\tilde{H}$. In particular, it is stable under the action of $\tilde{H}^0_*$. Since (f) is finite and $\tilde{H}^0_*$ is connected, we see that $\tilde{H}^0_*$ must act trivially on (f). Thus, (f) is exactly the fixed point set of $\tilde{H}^0_*$. Hence this fixed point can be identified with $(\dot{w}T'_r)^F$. From this (e) follows easily. The lemma is proved.

2. The main results

2.1. Let G, F be as in 1.2. Let T be an F-stable maximal torus in G and let U be the unipotent radical of a Borel subgroup of G that contains T. (Note that U is not necessarily F-stable.) Let $r \geq 1$. Let $\mathcal{R}(G_r^F)$ be the group of virtual representations of G_r^F over $\bar{\mathbf{Q}}_l$. Let $\langle , \rangle$ be the standard inner product $\mathcal{R}(G_r^F) \times \mathcal{R}(G_r^F) \to \mathbf{Z}$. Let

$$S_{T,U} = \{g \in G_r; g^{-1}F(g) \in F(U_r)\}.$$

The finite group $G_r^F \times T_r^F$ acts on $S_{T,U}$ by $(g_1, t) : g \mapsto g_1 g t^{-1}$. For any $i \in \mathbf{Z}$ we have an induced action of $G_r^F \times T_r^F$ on $H_c^i(S_{T,U})$. For $\theta \in \widehat{T_r^F}$, we denote by $H_c^i(S_{T,U})_\theta$ the subspace of $H_c^i(S_{T,U})$ on which T_r^F acts according to θ. This is a G_r^F-submodule of $H_c^i(S_{T,U})$. Let

$$R^\theta_{T_r,U_r} = \sum_{i \in \mathbf{Z}} (-1)^i H_c^i(S_{T,U})_\theta \in \mathcal{R}(G_r^F).$$

Proposition 2.2. *Assume that $r \geq 2$. Let (T', U', θ') be another triple like T, U, θ. Let $\mathcal{T} = T_r^{r-1}, \mathcal{T}' = T_r'^{r-1}$.*

(a) *Let i, i' be integers. Assume that there exists an irreducible G_r^F-module that appears in the G_r^F-module $(H_c^i(S_{T,U})_{\theta^{-1}})^*$ (dual of $H_c^i(S_{T,U})_{\theta^{-1}}$) and in the G_r^F-module $H_c^{i'}(S_{T',U'})_{\theta'}$. There exists $n \geq 1$ and $g \in N(T', T)^{F^n}$ such that $\mathrm{Ad}(g)$ carries $\theta \circ N_F^{F^n}|_{\mathcal{T}^{F^n}} \in \widehat{\mathcal{T}^{F^n}}$ to $\theta' \circ N_F^{F^n}|_{\mathcal{T}'^{F^n}} \in \widehat{\mathcal{T}'^{F^n}}$.*

(b) *Assume that there exists an irreducible G_r^F-module that appears in the virtual G_r^F-module $\sum_i (-1)^i H_c^i(S_{T,U})_\theta$ and in the virtual G_r^F-module $\sum_i (-1)^i H_c^i(S_{T',U'})_{\theta'}$. There exists $n \geq 1$ and $g \in N(T', T)^{F^n}$ such that $\mathrm{Ad}(g)$ carries $\theta \circ N_F^{F^n}|_{\mathcal{T}^{F^n}} \in \widehat{\mathcal{T}^{F^n}}$ to $\theta' \circ N_F^{F^n}|_{\mathcal{T}'^{F^n}} \in \widehat{\mathcal{T}'^{F^n}}$.*

We prove (a). Consider the free G_r^F-action on $S_{T,U} \times S_{T',U'}$ given by $g_1 : (g, g') \mapsto (g_1 g, g_1 g')$. The map

$$(g, g') \mapsto (x, x', y), x = g^{-1}F(g), x' = g'^{-1}F(g'), y = g^{-1}g'$$

defines an isomorphism of $G_r^F \backslash (S_{T,U} \times S_{T',U'})$ onto Σ (as in 1.2).

The action of $T_r^F \times T_r'^F$ on $S_{T,U} \times S_{T',U'}$ given by right multiplication by t^{-1} on the first factor and by t'^{-1} on the second factor becomes an action of $T_r^F \times T_r'^F$ on Σ given by $(x, x', y) \mapsto (txt^{-1}, t'x't'^{-1}, tyt'^{-1})$. Our assumption implies that the G_r^F-module $H_c^i(S_{T,U})_{\theta^{-1}} \otimes H_c^{i'}(S_{T',U'})_{\theta'}$ contains the unit representation with non-zero multiplicity. Hence the subspace of $H_c^{i+i'}(G_r^F \backslash (S_{T,U} \times S_{T',U'}))$ on which $T_r^F \times T_r'^F$ acts according to $\theta^{-1} \boxtimes \theta'$ is non-zero. Equivalently, $H_c^{i+i'}(\Sigma)_{\theta^{-1},\theta'} \neq 0$. We now use Lemma 1.4; (a) follows.

We prove (b). By general principles we have

$$\sum_i (-1)^i (H_c^i(S_{T,U})_{\theta^{-1}})^* = \sum_i (-1)^i H_c^i(S_{T,U})_\theta.$$

Hence the assumption of (b) implies that the assumption of (a) holds. Hence the conclusion of (a) holds. The proposition is proved.

Proposition 2.3. *We preserve the setup of 2.2. Assume that θ or θ' is regular (see 1.5). The inner product $\langle R^\theta_{T_r,U_r}, R^{\theta'}_{T'_r,U'_r}\rangle$ is equal to the number of $w \in W(T,T')^F$ such that $Ad(\dot{w}) : T'^F_r \to T^F_r$ carries θ to θ'.*

We may assume that θ' is regular. As in the proof of 2.2, we have

$$\begin{aligned}\langle R^\theta_{T_r,U_r}, R^{\theta'}_{T'_r,U'_r}\rangle & \\ &= \sum_{i,i'\in\mathbf{Z}} (-1)^{i+i'} \dim(H^i_c(S_{T,U})_{\theta^{-1}} \otimes H^{i'}_c(S_{T',U'})_\theta)^{G^F_r} \\ &= \sum_{j\in\mathbf{Z}} (-1)^j \dim H^j_c(G^F_r\backslash(S_{T,U} \times S_{T',U'}))_{\theta^{-1},\theta'} \\ &= \sum_{j\in\mathbf{Z}} (-1)^j \dim H^j_c(\Sigma)_{\theta^{-1},\theta'}\end{aligned}$$

where $()^{G^F_r}$ denotes the space of G^F_r-invariants. It remains to use 1.9.

Corollary 2.4. *Assume that $r \geq 2$. Let T, U be as in 2.1. Assume that $\theta \in \widehat{T^F_r}$ is regular.*

(a) *$R^\theta_{T_r,U_r}$ is independent of the choice of U.*

(b) *Assume also that the stabilizer of θ in $W(T,T)^F$ is $\{1\}$. Then $R^\theta_{T_r,U_r}$ is $\pm$ an irreducible G^F_r-module.*

We prove (a). Let U' be the unipotent radical of another Borel subgroup of G containing T. Let $R = R^\theta_{T_r,U_r}$, $R' = R^\theta_{T_r,U'_r}$. By 2.3 we have

$$\langle R, R\rangle = \langle R, R'\rangle = \langle R', R\rangle = \langle R', R'\rangle.$$

Hence $\langle R - R', R - R'\rangle = 0$, so that $R = R'$. This proves (a). In the setup of (b), we see from 2.3 that $\langle R^\theta_{T_r,U_r}, R^\theta_{T_r,U_r}\rangle = 1$. This proves (b).

2.5. Assume that $r \geq 2$. Let T be as in 2.1. Assume that $\theta \in \widehat{T^F_r}$ is regular. We set

$$R^\theta_{T_r} = R^\theta_{T_r,U_r}$$

where U is chosen as in 2.1. (By 2.4(a), this is independent of the choice of U.)

3. An example

I wish to thank A. Stasinski for pointing out an error in an earlier version of this section.

3.1. Let $A = \mathbf{F}[[\epsilon]]/(\epsilon^2)$. Define $F : A \to A$ by $F(a_0 + \epsilon a_1) = a_0^q + \epsilon a_1^q$ where $a_0, a_1 \in \mathbf{F}$. Let V be a 2-dimensional $\mathbf{F}$-vector space with a fixed $\mathbf{F}_q$-rational structure with Frobenius map $F : V \to V$. Let $G = SL(V)$. Then G has an $\mathbf{F}_q$-rational structure with Frobenius map $F : G \to G$ such that $F(gv) = F(g)F(v)$ for all $g \in G, v \in V$. Let $V_2 = A \otimes_{\mathbf{F}} V$. Then G_2 (see 0.2) may be identified with the group of all automorphisms of the free A-module V_2 with determinant 1. We regard V as a subset of V_2 by $v \mapsto 1 \otimes v$. Any element of V_2 can be written uniquely in the form $v_0 + \epsilon v_1$ where $v_0, v_1 \in V$. The Frobenius map $F : V_2 \to V_2$ satisfies $F(v_0 + \epsilon v_1) = F(v_0) + \epsilon F(v_1)$ for $v_0, v_1 \in V$.

Let $\widehat{G_2^F}$ be the set of isomorphism classes of irreducible representations of G_2^F over $\bar{\mathbf{Q}}_l$. The objects of $\widehat{G_2^F}$ can be classified by Mackey's method using the fact that G_2^F is a semidirect product of G^F and $\mathbf{F}_q^3$.

The table below shows the number of representations in $\widehat{G_2^F}$ of various dimensions assuming that q is odd; the first column indicates the dimension, the second column indicates the number of representations of that dimension. (See also [S] for the closely related case of PGL_2.)

dim	$\sharp$
1	1
q	1
$q+1$	$(q-3)/2$
$(q+1)/2$	2
$q-1$	$(q-1)/2$
$(q-1)/2$	2
q^2+q	$(q-1)^2/2$
q^2-q	$(q^2-1)/2$
$(q^2-1)/2$	$4q$

The analogous table in the case where q is a power of 2 is

dim	$\sharp$
1	1
q	1
$q+1$	$(q-2)/2$
$q-1$	$q/2$
q^2+q	$(q-1)(q-2)/2$
$(q^2+q)/2$	$2(q-1)$
q^2-q	$(q^2-q)/2$
$(q^2-q)/2$	$2(q-1)$
q^2-1	q

3.2. Let $\mathcal{B}$ be the set of all A-submodules $L \subset V_2$ such that L is a direct summand of V_2 and L is free of rank 1. Now G_2 acts transitively on $\mathcal{B}$. If $L \in \mathcal{B}$, then $F(L) \in \mathcal{B}$. Thus we obtain a map $F : \mathcal{B} \to \mathcal{B}$, the Frobenius map of a $\mathbf{F}_q$-rational structure on $\mathcal{B}$. Let

$$X = \{L \in \mathcal{B}; L \cap F(L) = 0\}.$$

Then X is a G_2^F-stable subvariety of $\mathcal{B}$. We now define a finite covering of X as follows. Let e, e' be an $\mathbf{F}$-basis of V such that $F(e) = e, F(e') = e'$. Let $\underline{T}$ be the subgroup of G consisting of the automorphisms $e \mapsto ae, e' \mapsto a^{-1}e'$ with $a \in \mathbf{F}^*$. (An F-stable maximal torus of G.) Let $\underline{U}$ be the subgroup of G consisting of the automorphisms

$$e \mapsto e + be', e' \mapsto e' \quad \text{with } b \in \mathbf{F}^*.$$

Let $\nu \in G$ be such that $\nu(e) = e', \nu(e') = -e$. Let

$$\tilde{X} = \{g \in G_2; g^{-1}F(g) \in \nu \underline{U}_2\}.$$

(We use the action of $\underline{U}_2$ on G_2 by right translation.) Then $g \mapsto Age'$ is a well defined morphism $\tilde{X} \to X$. This is a finite principal covering with group Γ (acting by right translation) where

$$\Gamma = \{t \in \underline{T}_2; F(t) = t^{-1}\} \quad (\text{of order } q^2 + q).$$

For any variety Y with an action of a finite (abelian) group and any character ω of that finite group, let $H_c^j(Y)_\omega$ denote the subspace of $H_c^j(Y)$ on which the finite group acts according to ω. Thus, for $\omega \in \hat{\Gamma}$, $H_c^j(\tilde{X})_\omega$ is well defined.

3.3. Let

$$\mathfrak{S}_0 = \{x_0 \in V; x_0 \wedge F(x_0) = e \wedge e'\}, \quad \mathfrak{S}_{00} = \{x_0 \in \mathfrak{S}_0; F^2(x_0) = -x_0\}.$$

Now G^F acts on $\mathfrak{S}_0$ (restriction of the G-action on V). This restricts to a G^F-action on $\mathfrak{S}_{00}$. We show that this action is simply transitive. If $g \in G^F$ keeps fixed some $x_0 \in \mathfrak{S}_{00}$, then it also keeps fixed $F(x_0)$ hence it must be 1 (recall that $x_0, F(x_0)$ form a basis of V). Thus the G^F-action on $\mathfrak{S}_{00}$ has trivial isotropy. We may identify $\mathfrak{S}_{00}$ with $\{(a,b) \in \mathbf{F}^2; ab^q - a^q b = 1, a^{q^2} = -a, b^{q^2} = -b\}$. For such (a,b) we have automatically $a \neq 0$. We make a change of variable $(a,b) \mapsto (a,c)$ where $c = b/a$. Then $\mathfrak{S}_{00}$ becomes

$$\{(a,c) \in \mathbf{F}^2; a^{q+1}(c^q - c) = 1, a^{q^2} = -a, c^{q^2} = c\}.$$

The second projection maps this to $\{c \in \mathbf{F}; c^{q^2} = c, c^q \neq c\}$ which has $q^2 - q$ elements. The fibre at c is $\{a \in \mathbf{F}; a^{q+1} = (c^q - c)^{-1}\}$. (For such a we have automatically $a^{q^2} = -a$ since $c^{q^2} = c$.) This fibre has exactly $q+1$ elements since $(c^q - c)^{-1} \neq 0$. We see that $\sharp(\mathfrak{S}_{00}) = (q+1)(q^2-q) = \sharp(G^F)$. It follows that the G^F-action on $\mathfrak{S}_{00}$ is indeed simply transitive.

3.4. We now analyze $\tilde{X}$. Let

$$\mathfrak{S} = \{x \in V_2; x \wedge F(x) = e \wedge e'\}.$$

Now G_2^F acts on $\mathfrak{S}$ by $g_1 : x \mapsto g_1 x$. The map $g \mapsto g(e')$ defines an isomorphism

$$\iota : \tilde{X} \xrightarrow{\sim} \mathfrak{S}.$$

We check that this is a well-defined bijection. Let $g \in \tilde{X}$. Then $F(g) = g\nu u$ for some $u \in \underline{U}_2$. Let $x = ge'$. Then for some $u \in \underline{U}_2$ we have

$$\begin{aligned} x \wedge F(x) &= (ge') \wedge F(ge') = (ge') \wedge F(g)e' = e' \wedge g^{-1}F(g)e' = e' \wedge \nu u e' \\ &= e' \wedge \nu e' = e' \wedge (-e) = e \wedge e', \end{aligned}$$

hence $x \in \mathfrak{S}$ and ι is well defined. Now let $x \in \mathfrak{S}$. We can find $g \in G_2$ such that $ge' = x$. Then

$$e \wedge e' = x \wedge F(x) = (ge') \wedge F(ge') = (ge') \wedge F(g)e' = e' \wedge g^{-1}F(g)e'.$$

Hence $g^{-1}F(g)e' = -e + be'$ for some $b \in A$. It follows that $g^{-1}F(g) = u'\nu u$ where $u, u' \in \underline{U}_2$. Then $(gu')^{-1}F(gu') = \nu u F(u')$ hence $gu' \in \tilde{X}$. Clearly, $\iota(gu') = x$ so that ι is surjective. Now assume that $g, g' \in \tilde{X}$ satisfy $\iota(g) = \iota(g')$, that is, $ge' = g'e'$. Then $g' = gu'$, $u' \in \underline{U}_2$. We have $g'^{-1}F(g') = \nu u$ with $u \in \underline{U}_2$, hence $u'^{-1}g^{-1}F(g)F(u') = \nu u$. Also, $g^{-1}F(g) = \nu\tilde{u}$ with $\tilde{u} \in \underline{U}_2$, hence $u'^{-1}\nu\tilde{u}F(u') = \nu u$ so that $u' \in \nu\underline{U}_2\nu^{-1}$. Thus, $u' \in \underline{U}_2 \cap (\nu\underline{U}_2\nu^{-1}) = \{1\}$, hence $u' = 1$ and $g' = g$. Thus, ι is injective hence bijective. It commutes with the G_2^F-actions.

Now $\mathfrak{S}$ consists of the elements $x_0 + \epsilon x_1$, with $x_0, x_1 \in V$ such that

$$(x_0 + \epsilon x_1) \wedge (F(x_0) + \epsilon F(x_1)) = e \wedge e',$$

that is,

$$x_0 \wedge F(x_0) = e \wedge e' \quad \text{and} \quad x_1 \wedge F(x_0) + x_0 \wedge F(x_1) = 0.$$

We have a morphism

$$\kappa : \mathfrak{S} \to \mathfrak{S}_0, x_0 + \epsilon x_1 \mapsto x_0.$$

If $x_0 \in \mathfrak{S}_0$, then $\kappa^{-1}(x_0)$ may be identified with

$$\{x_1 \in V; x_1 \wedge F(x_0) + x_0 \wedge F(x_1) = 0\}.$$

Note that $x_0, F(x_0)$ form a basis of V hence $F^2(x_0) = c_0 x_0 + c_1 F(x_0)$ with $c_0, c_1 \in \mathbf{F}$. Since $x_0 \wedge F(x_0) = e \wedge e'$ is F-stable, we have $x_0 \wedge F(x_0) = F(x_0) \wedge F^2(x_0)$ hence $c_0 = -1$. Let $\mathfrak{S}_{01} = \mathfrak{S}_0 - \mathfrak{S}_{00}$. We have a partition $\mathfrak{S} = \mathfrak{S}_* \cup \mathfrak{S}_{**}$ where $\mathfrak{S}_* = \kappa^{-1}(\mathfrak{S}_{00}), \mathfrak{S}_{**} = \kappa^{-1}(\mathfrak{S}_{01})$ are G_2^F-stable. If $x_0 \in \mathfrak{S}_0$, then any $x_1 \in V$ can be written uniquely in the form

$$x_1 = a_0 x_0 + a_1 F(x_0)$$

with $a_0, a_1 \in \mathbf{F}$. The condition that $x_0 + \epsilon x_1 \in \kappa^{-1}(x_0)$ is

$$(a_0 x_0 + a_1 F(x_0)) \wedge F(x_0) + x_0 \wedge (a_0^q F(x_0) + a_1^q F^2(x_0)) = 0,$$

that is,

$$a_0 x_0 \wedge F(x_0) + x_0 \wedge (a_0^q F(x_0) - a_1^q x_0 + a_1^q c_1 F(x_0)) = 0,$$

that is,

$$a_0 x_0 \wedge F(x_0) + a_0^q x_0 \wedge F(x_0) + a_1^q c_1 x_0 \wedge F(x_0) = 0,$$

or

$$a_0 + a_0^q + a_1^q c_1 = 0.$$

Thus we may identify $\kappa^{-1}(x_0)$ with $\{(a_0, a_1) \in \mathbf{F}^2; a_0 + a_0^q + a_1^q c_1 = 0\}$. If $c_1 \neq 0$ (that is, if $x_0 \in \mathfrak{S}_{01}$), this is isomorphic to the affine line. Thus, κ restricts to an affine line bundle $\mathfrak{S}_{**} \to \mathfrak{S}_{01}$.

Now the action of Γ on $\tilde{X}$ corresponds under ι to the action of $\{\lambda \in A; \lambda F(\lambda) = 1\}$ on $\mathfrak{S}$ by scalar multiplication. Hence the action of $\{t \in \Gamma; t \in T_2^1\}$ on $\tilde{X}$ corresponds to the action of $A' = \{\lambda \in A; \lambda F(\lambda) = 1, \lambda \in 1 + \epsilon A\}$ on $\mathfrak{S}$ by scalar multiplication. The action of $1 + \epsilon\lambda_1 \in A'$ (with $\lambda_1 \in \mathbf{F}$) in the coordinates (x_0, a_0, a_1) is $(x_0, a_0, a_1) \mapsto (x_0, a_0 + \lambda_1, a_1)$. Thus it preserves each fibre of κ.

Now $\mathfrak{S}_{**}$ is stable under the action of $\{\lambda \in A; \lambda F(\lambda) = 1\}$ and the restriction of this action to A' preserves each fibre of $\mathfrak{S}_{**} \to \mathfrak{S}_{01}$ (an affine line); hence this group acts trivially on $H_c^j()$ of each such fibre, hence it also acts trivially on $H_c^j(\mathfrak{S}_{**})$. Thus, $H_c^j(\mathfrak{S}) \to H_c^j(\mathfrak{S}_*)$ is an isomorphism on the part where $\sum_{\lambda \in A'} \lambda$ acts as 0.

We now study $H_c^j(\mathfrak{S}_*)$. If $x_0 \in \mathfrak{S}_{00}$, then $\kappa^{-1}(x_0)$ may be identified with $\{(a_0, a_1) \in \mathbf{F}^2; a_0 + a_0^q = 0\}$. Thus, $\mathfrak{S}_*$ is an affine line bundle over

$$\mathfrak{S}_{00} \times \{a_0 \in \mathbf{F}; a_0 + a_0^q = 0\}$$

which is a transitive permutation representation of G_2^F that is explicitly known from 3.3. It follows that $H_c^j(\mathfrak{S}_*) = 0$ for $j \neq 2$ and the part of $H_c^2(\mathfrak{S}_*)$ where $\sum_{\lambda \in A'} \lambda$ acts as 0 is the direct sum of the irreducible representations of degree $q^2 - q$ (each one with multiplicity 2) and of degree $(q^2 - q)/2$ (each one with multiplicity 1); note that the latter representations occur only when q is a power of 2.

We now study the part of $H_c^j(\mathfrak{S})$ where A' acts as 1. This is the same as $H_c^j(A' \backslash \mathfrak{S})$. The map $(x_0, a_0, a_1) \mapsto (x_0, \tilde{a}_0, a_1), \tilde{a}_0 = a_0 + a_0^q$ is an isomorphism of $A' \backslash \mathfrak{S}$ with the set of all $(x_0, \tilde{a}_0, a_1) \in \mathfrak{S}_0 \times \mathbf{F} \times \mathbf{F}$ such that $\tilde{a}_0 + a_1^q c_1 = 0$. (Here c_1 is determined by x_0 as above.) Hence the map $(x_0, a_0, a_1) \mapsto (x_0, a_1)$ is an isomorphism $A' \backslash \mathfrak{S} \xrightarrow{\sim} \mathfrak{S}_0 \times \mathbf{F}$. Thus, $H_c^j(A' \backslash \mathfrak{S}) = H_c^{j-2}(\mathfrak{S}_0)$. Thus, G_2^F acts on $H_c^j(A' \backslash \mathfrak{S})$ through its quotient G^F and that action is explicitly known from the representation theory of G^F.

We see that $H^4_c(\tilde X)$ is the 1-dimensional representation; $H^3_c(\tilde X)$ is the direct sum of all irreducible representations of degree $q-1$ (each one with multiplicity 2) and those of degree $(q-1)/2, q$ (each one with multiplicity 1); $H^2_c(\tilde X)$ is the direct sum of all irreducible representations of degree q^2-q (each one with multiplicity 2) and of degree $(q^2-q)/2$ (each one with multiplicity 1); $H^j_c(\tilde X)=0$ for $j\notin\{2,3,4\}$; note that the representations of degree $(q-1)/2$ occur only for q odd, while those of degree $(q^2-q)/2$ occur only for q a power of 2.

More precisely, if $\omega\in\hat\Gamma$ and q is odd, then:

$H^4_c(\tilde X)_\omega$ is irreducible of degree 1 if $\omega=1$ and is 0 otherwise;
$H^3_c(\tilde X)_\omega$ is irreducible of degree $q-1$ if $\omega|_{\Gamma\cap T^1_2}=1$, $\omega^2\ne 1$; it is the direct sum of two irreducible representations of degree $(q-1)/2$ if $\omega|_{\Gamma\cap T^1_2}=1$, $\omega^2=1,\omega\ne 1$; it is irreducible of degree q if $\omega=1$; it is 0 if $\omega|_{\Gamma\cap T^1_2}\ne 1$;
$H^2_c(\tilde X)_\omega$ is irreducible of degree q^2-q if $\omega|_{\Gamma\cap T^1_2}\ne 1$ and is 0 otherwise.

Similarly, if $\omega\in\hat\Gamma$ and q is a power of 2, then:

$H^4_c(\tilde X)_\omega$ is irreducible of degree 1 if $\omega=1$ and is 0 otherwise;
$H^3_c(\tilde X)_\omega$ is irreducible of degree $q-1$ if $\omega|_{\Gamma\cap T^1_2}=1$, $\omega\ne 1$; it is irreducible of degree q if $\omega=1$; it is 0 if $\omega|_{\Gamma\cap T^1_2}\ne 1$;
$H^2_c(\tilde X)_\omega$ is irreducible of degree q^2-q if $\omega|_{\Gamma\cap T^1_2}\ne 1$, $\omega^2\ne 1$; it is the direct sum of two irreducible representations of degree $(q^2-q)/2$ if $\omega^2=1,\omega\ne 1$; it is 0 otherwise.

3.5. Let $\gamma\in G$ be such that $\gamma^{-1}F(\gamma)=\nu$. We set $T=\gamma\underline{T}\gamma^{-1}$, $U=\gamma\underline{U}\gamma^{-1}$. Then T is an F-stable maximal torus of G and U is the unipotent radical of a Borel subgroup of G containing T. Hence $S_{T,U}$ is defined (with $r=2$). Now $g\mapsto g\gamma^{-1}$ defines an isomorphism

$$\tilde X \xrightarrow{\sim} S_{T,U}$$

and an isomorphism

$$\Gamma\xrightarrow{\sim} T^F_2.$$

Also $G^F_2\times\Gamma$ acts on $\tilde X$ by $(g_1,t): g\mapsto g_1gt^{-1}$. This action is compatible with the $G^F_2\times T^F_2$-action on $S_{T,U}$ via the isomorphisms above. We see that the virtual representations $\sum_{j\in\mathbf Z}(-1)^jH^j_c(\tilde X)_\omega$ of G^F_2 are the same as the virtual representations $R^\theta_{T,U}$.

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