RESOLUTIONS AND HILBERT SERIES
OF THE UNITARY HIGHEST WEIGHT MODULES
OF THE EXCEPTIONAL GROUPS

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ABSTRACT. We give a sufficient criterion on a highest weight module of a semisimple Lie algebra to admit a resolution in terms of sums of modules induced from a parabolic subalgebra. In particular, we show that all unitary highest weight modules admit such a resolution. As an application of our results we compute (minimal) resolutions and explicit formulas for the Hilbert series of the unitary highest weight modules of the exceptional groups.

1. INTRODUCTION

1.1. Bernstein, Gelfand and Gelfand [BGG] gave a resolution of each finite dimensional representation F of a semisimple Lie algebra $\mathfrak{g}$ in terms of sums of representations induced from one-dimensional representations of a Borel subalgebra (the Verma modules). This result was extended by Lepowsky [L] to give resolutions of finite dimensional representations in terms of sums of representations induced from parabolic subalgebras (the generalized Verma modules). In this article we extend these results by replacing the finite dimensional representation F by an irreducible highest weight representation L. We give a sufficient criterion on L for such a resolution to exist. Our criterion takes the form of a condition on the $\mathfrak{u}$-cohomology of L, where $\mathfrak{u}$ is the nilradical of the parabolic subalgebra. These resolutions we will call generalized BGG resolutions.

1.2. As an application of our results we turn to the exceptional groups with unitary highest weight representations. We prove that every unitary highest weight representation has a generalized BGG resolution. For classical groups this result is in [EW]. We note that in the case of unitary highest weight representations, i.e., in the Hermitian symmetric setting, the BGG resolutions can be interpreted as graded free resolutions in the sense of commutative algebra as follows. Let (G, K) be an irreducible Hermitian symmetric pair and let $(\mathfrak{g}, \mathfrak{k})$ be the corresponding pair of complexified Lie algebras. We have the usual decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}^+ \oplus \mathfrak{p}^-$ of $\mathfrak{g}$ as a $\mathfrak{k}$-module. Let $S = S(\mathfrak{p}^-)$ be the symmetric algebra of $\mathfrak{p}^-$. Then every highest weight module L of $(\mathfrak{g}, \mathfrak{k})$ is also a finitely generated graded S-module. As such, L admits a minimal free resolution. It was proved in [EH] by the authors of this paper that the generalized BGG resolutions of unitary highest weight representations are minimal free resolutions.

Furthermore, we can associate a Hilbert series $h_L(t)$ to the graded S-module L. It is standard that $h_L(t)$ can be written as a rational function in the form

$$h_L(t) = \frac{P(t)}{(1-t)^d} \ ,$$

where $P(t)$ is polynomial with integer coefficients such that $P(1) \neq 0$. The number d is equal to the Gelfand-Kirillov dimension of L and the number $P(1)$ is equal to the Bernstein degree of L. In this paper we also compute explicitly the Hilbert series of the unitary highest weight representations of the exceptional groups.

1.3. Let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{k}$ and hence also $\mathfrak{g}$. In [DES] it was observed that the set of all $\lambda \in \mathfrak{h}^*$ that arise as a highest weight of some unitary highest weight representation of $\mathfrak{g}$ that is not induced (from a finite dimensional representation of the parabolic subalgebra $\mathfrak{k} \oplus \mathfrak{p}^+$) can be written as a disjoint union of translated integral cones. This is called the *cone decomposition*. In this paper we propose a finer decomposition by subdividing each cone into what we call *unitary strata*. We show, by using Zuckerman translation, that the Gelfand-Kirillov dimension as well as other indices associated with the generalized BGG resolution of a unitary highest weight representation L with highest weight λ are constant for λ varying over a single stratum.

In this paper we determine explicitly all unitary strata for the exceptional groups. It turns out that there are 17 unitary strata for type E_6 and 20 unitary strata for type E_7. Each unitary stratum has a unique minimal vertex. We explicitly compute the generalized BGG resolution and the Hilbert series of the unitary highest weight representation corresponding to these vertices. The variety of numerator polynomials $P(t)$ of the Hilbert series that arise is quite surprising. Of the 37 polynomials only 6 have nonnegative coefficients and five of these lie in the Wallach set and two are the trivial representations of E_6 and E_7.

1.4. Organization of the paper. In section 2 we recall basic definitions and the generalized BGG resolution due to Lepowsky [L] for finite dimensional representations. Following this we consider hypotheses under which a highest weight representation (not necessarily finite dimensional) admits a generalized BGG resolution. In section 3 we summarize what is known about resolutions of unitary highest weight representations. We then extend these results to the setting of this article to prove that all unitary highest weight representations admit a generalized BGG resolution. In section 4 we recall the cone decomposition from [DES] and introduce the notion of unitary strata. The section ends with tables describing some of the invariants associated with the stratified cone decomposition for E_6 and E_7. In section 5 several examples are given to indicate how various parameter root subspaces are calculated. In sections 6 and 7 explicit formulas for the generalized BGG resolutions and the Hilbert series of the unitary highest weight representations are given for vertices of the unitary strata for E_6 and E_7.

1.5. Remarks and acknowledgements. This article may be regarded as an extension of the article [EH] and as such the overlap of topics with [NOTYK] applies here as well. In particular, for the Wallach representations the Gelfand-Kirillov dimension and the Bernstein degree are computed in [NOTYK]. However, the full Hilbert series are not determined there. We also thank Nolan Wallach for several helpful discussions regarding cohomology and characters.

2. Generalized BGG resolutions

2.1. In this section we recall some basic definitions and constructions related to highest weight modules that will be used throughout the rest of the paper. In particular, we recall the generalized BGG resolution of a finite dimensional representation by generalized Verma modules. Then we consider the corresponding results resolving highest weight representations which are not finite dimensional.

2.2. Notation. We consider the general setting of a parabolic subalgebra of $\mathfrak{g}$. Let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$ and let Δ be the root system of the pair $(\mathfrak{g}, \mathfrak{h})$. For $\alpha \in \Delta$, let $\mathfrak{g}_\alpha$ be the α-root subspace of $\mathfrak{g}$. Let S be a proper subset of the simple roots of a positive system Δ^+ of Δ. Let Δ_S equal the subroot system generated by S. Let $\Delta_S^+ = \Delta_S \cap \Delta^+$ and write Δ^+ as a disjoint union $\Delta^+ = \Delta_S^+ \cup \Delta_{\mathfrak{u}}$. Put $\mathfrak{u}^\pm = \sum_{\alpha \in \Delta_{\mathfrak{u}}} \mathfrak{g}_{\pm\alpha}$ and $\mathfrak{r} = \mathfrak{h} \oplus \sum_{\alpha \in \Delta_S} \mathfrak{g}_\alpha$. Then $\mathfrak{q} = \mathfrak{r} \oplus \mathfrak{u}^+$ is the parabolic subalgebra associated with S and $\mathfrak{g} = \mathfrak{u}^- \oplus \mathfrak{q}$.

Let W be the Weyl group of the pair $(\mathfrak{g}, \mathfrak{h})$ and let W_S be the Weyl group of the pair $(\mathfrak{r}, \mathfrak{h})$ which we may identify with the subgroup of W that is generated by the reflections s_α with $\alpha \in \Delta_S$. Let $\rho = \frac{1}{2} \sum_{\alpha \in \Delta^+} \alpha$ and define

$$W^S = \{w \in W \mid w\rho \text{ is } \Delta_S^+\text{-dominant}\} = \{w \in W \mid w\Delta^+ \supset \Delta_S^+\}.$$

Then $W = W_S \cdot W^S$ and $W_S \cap W^S = \{e\}$. In particular, W^S is a set of coset representatives for $W_S \backslash W$. Let ℓ denote the usual length function on W.

Fix a dominant element $\mu \in \mathfrak{h}^*$ and define T as the subset of all simple roots α for which $(\mu, \alpha) = 0$. Define Δ_T, Δ_T^+ and W_T as above with S replaced by T. The involution $x \to x^{-1}$ changes the Weyl group decomposition from right cosets to left cosets giving

$$^TW = \{w \in W \mid w^{-1}\rho \text{ is } \Delta_T^+\text{-dominant}\} = \{w \in W \mid w^{-1}\Delta^+ \supset \Delta_T^+\}.$$

Then $W = {}^TW \cdot W_T$ and $W_T \cap {}^TW = \{e\}$. In particular, ^TW is a set of coset representatives for W/W_T.

2.3. Generalized Verma modules. Let $\lambda \in \mathfrak{h}^*$ be Δ_S^+-dominant integral and let F_λ be the irreducible finite dimensional $\mathfrak{r}$-module with highest weight λ. We may consider F_λ as a $\mathfrak{q} = \mathfrak{q}_S = \mathfrak{r} \oplus \mathfrak{u}^+$-module by letting $\mathfrak{u}^+$ act by zero. We then define the generalized Verma module N_λ with highest weight λ as

$$N_\lambda = U(\mathfrak{g}) \otimes_{U(\mathfrak{q})} F_\lambda .$$

Here $U(\mathfrak{g})$ and $U(\mathfrak{q})$ denote the universal enveloping algebras of $\mathfrak{g}$ and $\mathfrak{q}$, respectively. Let L_λ denote the unique irreducible quotient of N_λ. By the PBW theorem

$$N_\lambda \simeq S(\mathfrak{u}^-) \otimes F_\lambda \quad \text{as an } \mathfrak{r}\text{-module} .$$

Here as $\mathfrak{r}$-modules we have identified the universal enveloping algebra $U(\mathfrak{u}^-)$ with the symmetric algebra $S(\mathfrak{u}^-)$. Sometimes we will also write $N_{\mathfrak{g},\lambda}$ and $F_{\mathfrak{r},\lambda}$ instead of N_λ and F_λ.

2.4. The relative category $\mathcal{O}_S$. We denote by $\mathcal{O}_S = \mathcal{O}(\mathfrak{g}, \mathfrak{q}_S)$ the category of $\mathfrak{g}$-modules M that satisfy the following three conditions:

   (i) M is finitely generated as a $U(\mathfrak{g})$-module,
   (ii) M is locally finite as a $U(\mathfrak{q})$-module,
   (iii) M is completely reducible as a $U(\mathfrak{r})$-module.

Let $\chi_{\lambda+\rho}$ denote the infinitesimal character of N_λ. We denote by $\mathcal{O}(\mathfrak{g}, \mathfrak{q}_S)_{\lambda+\rho}$ the full subcategory of $\mathcal{O}(\mathfrak{g}, \mathfrak{q}_S)$ consisting of modules with generalized infinitesimal character $\chi_{\lambda+\rho}$. The simple modules in the category $\mathcal{O}(\mathfrak{g}, \mathfrak{q}_S)_{\lambda+\rho}$ are parametrized by a special collection of double cosets which we now define.

2.5. Double cosets and highest weights. Choose μ dominant with $\lambda+\rho \in W\mu$ and let T be defined as in 2.2. Then W/W_T parametrizes the W-orbit of μ. For $x \in W$, the module $N_{x\mu-\rho}$ is nonzero if and only if $x\mu - \rho$ is Δ_S^+-dominant integral if and only if $x\mu$ is Δ_S^+-regular dominant integral. Consider the double cosets $W_S \backslash W / W_T$. Then our category $\mathcal{O}(\mathfrak{g}, \mathfrak{q}_S)_\mu$ is nonempty precisely when there exist double cosets of maximal size, $|W_S||W_T|$. We shall assume that this is the case. Set $\mathcal{D}$ equal to the set of all such double cosets of maximal size. Each double coset $\delta \in \mathcal{D}$ contains a unique element x of minimal length and we write $\delta = W_S x W_T$. Let δ_x denote this double coset. These elements x lie in $^T W \cap W^S$. Let D denote the subset of $^T W \cap W^S$ which is the image of $\mathcal{D}$ under $\delta_x \to x$. Summarizing we have:

Lemma. *The highest weights of the irreducible modules in $\mathcal{O}(\mathfrak{g}, \mathfrak{q}_S)_{\lambda+\rho}$ are the elements $x\mu - \rho$, for $x \in D$.*

2.6. The generalized BGG resolution. Define $W^{S,i} = \{w \in W^S \mid \ell(w) = i\}$, where ℓ denotes the usual length function of W.

Theorem [L]. *Let $\lambda \in \mathfrak{h}^*$ be Δ^+-dominant integral and let E_λ be the irreducible finite dimensional $\mathfrak{g}$-module with highest weight λ. Then E_λ has a resolution as a $\mathfrak{g}$-module of the form*

$$0 \to N_p \to \cdots \to N_1 \to N_0 \to E_\lambda \to 0 \ ,$$

where

$$N_i = \bigoplus_{w \in W^{S,i}} N_{w(\lambda+\rho)-\rho} \ .$$

2.7. Generalized Kostant modules. The resolution given in Theorem 2.6 is our model. We wish to generalize the result by replacing E_λ by an arbitrary unitary irreducible highest weight representation. Note that when the infinitesimal character is regular, the setup is more transparent than in the general case. The reader may wish to restrict to this case for a first reading.

Fix an irreducible $\mathfrak{g}$-module L having infinitesimal character χ_μ with μ dominant. Let D be the set that was defined in 2.5. Suppose $X = \bigcup_{0 \le i \le t} X^i$ is a graded subset of D. We call the grading a *Kostant grading* if for every $1 \le i \le t$, any two distinct elements $x, y \in X^i \subseteq D$ are not comparable in the Bruhat order, i.e., $x \not\prec y$ and $y \not\prec x$. For $0 \le i \le t$, set $N_i = \bigoplus_{w \in X^i} N_{w\mu-\rho}$. We say that L is a *generalized Kostant module* for data X if the following hold:

(i) $X \subset D$ has a Kostant grading $X = \bigcup_{0 \le i \le t} X^i$.

(ii) There exist $\mathfrak{g}$-module maps $d_i : N_i \to N_{i-1}$, $1 \le i \le t$, so that

$$0 \to N_t \overset{d_t}{\to} N_{t-1} \overset{d_{t-1}}{\to} \cdots \overset{d_1}{\to} N_0 \overset{\epsilon}{\to} L \to 0$$

is complex; i.e., $d_i \circ d_{i-1} = 0$.

(iii) For all i and $w \in X^i$, the restriction of d_i to $N_{w\mu-\rho}$ is nonzero.

(iv) For $0 \le i \le t$, $\quad H^i(\mathfrak{u}^+, L) \simeq \bigoplus_{w \in X^i} F_{w\mu-\rho}$.

Remarks.

(1) If L is a generalized Kostant module, then the set X and its grading are uniquely determined by (iv).

(2) The Killing form induces a nondegenerate pairing of $S(\mathfrak{u})$ and $S(\mathfrak{u}^-)$. This gives a contragradient pairing of $H^i(\mathfrak{u}^+, L) = \operatorname{Ext}^i_{\mathfrak{u}}(\mathbb{C}, A)$ and $\operatorname{Tor}^{\mathfrak{u}^-}_i(\mathbb{C}, A^\vee)$, where $A^\vee$ denotes the contragradient dual to A. Note that each simple module here is equal to its contragradient dual. Therefore, the identity 2.7(iv) can be interpreted as the corresponding identity for $\operatorname{Tor}^{\mathfrak{u}^-}_i(\mathbb{C}, L)$.

(3) In [C] the notion of a Kostant module is defined via the homology of the "Iwasawa" Borel subalgebra $\mathfrak{b}^\sim$ and its nilradical $\mathfrak{n}^\sim$. Following this lead we call $L = L_\lambda$ a *Kostant module* if μ is regular integral and if the data $X = \bigcup_{0 \le i \le t} X^i$ is given by $\lambda + \rho = w_0\mu$ and $X^i = \{w \in W^S \mid w \prec w_0 \text{ and } \ell(w) - \ell(w_0) = i\}$.

2.8. Theorem. *Suppose L is a generalized Kostant module. Then the complex in 2.7(ii) is a resolution of L.*

We will prove the theorem in 2.11. The proof is a modification of the argument in Lepowsky [L, Theorem 4.3] which proves Theorem 2.6 above. We will prove exactness at each term in the complex 2.7(ii) by induction on the index. Set $\mathfrak{q}^- = \mathfrak{p}^- \oplus \mathfrak{k}$.

2.9. Lemma [L, Lemma 4.8(ii)]. *Let X be a module in the category $\mathcal{O}_S$ and Y any $\mathfrak{q}^-$-module which is a weight module. Then any $\mathfrak{q}^-$-module homomorphism $\varphi : Y \to X$ is surjective if and only if the induced map $\overline{\varphi} : Y/\mathfrak{u}^- Y \to X/\mathfrak{u}^- X$ is surjective.*

2.10. Lemma. *Suppose $B \subset C$ are $\mathfrak{g}$-modules in the category $\mathcal{O}(\mathfrak{g}, \mathfrak{q}_S)_\mu$.*

(a) *There exist $x_i \in W$, $1 \le i \le t$, and a $\mathfrak{g}$-module filtration of C of the form $C = C_t \supset C_{t-1} \supset \cdots \supset C_1 \supset C_0 = 0$ such that*

$$C/\mathfrak{u}^- C \simeq \sum_{1 \le i \le t} F(\mathfrak{r}, x_i\mu - \rho)$$

and C_i/C_{i-1} is a quotient of $N(x_i\mu - \rho)$ for all $1 \le i \le t$.

(b) *Suppose $B \subset \mathfrak{u}^- C$ and L_ξ is a Jordan-Holder component of B. Then there exist $x, y \in D$ such that $\xi = x\mu - \rho$, $x \ne y$, $x \prec y$ and $L_{y\mu-\rho}$ is a Jordan-Holder component of C.*

Proof. We first prove (a). Choose x_1 such that $x_1\mu - \rho$ is a highest weight of C with highest weight vector c_1. Set C_1 (resp. F_1) equal to the $\mathfrak{g}$-module (resp. $\mathfrak{r}$-module) generated by c_1. Then $C_1 = F_1 \oplus (C_1 \cap \mathfrak{u}^- C)$. Let $j \ge 1$ and assume that C_j and F_j have been defined with vector $c_j \in C_j$ such that

$$F_j = U(\mathfrak{r})c_j \,, \quad C_j = C_{j-1} + U(\mathfrak{u}^-)F_j \quad \text{and} \quad C_j/(C_j \cap \mathfrak{u}^- C) \simeq \sum_{1 \le i \le j} F_i \,.$$

Now set c_{j+1} equal to any weight vector in C that projects to a highest weight vector in C/C_j and generates an irreducible $\mathfrak{r}$-module. Let $F_{j+1} = U(\mathfrak{r})c_{j+1}$ and let C_{j+1} equal the $\mathfrak{g}$-module generated by F_{j+1} and C_j. Assertion (a) follows from this.

We now prove (b). Set $L = L_\xi$ and choose a submodule $B_1 \subset B$ with L a submodule of B/B_1. Now apply part (a) to $A = C/B_1$, obtaining the filtration

A_i and the $\mathfrak{r}$-modules F_i. Choose j maximal with L not contained in A_j. Then L injects into A_{j+1}/A_j which is a quotient of $N(x_{j+1}\mu - \rho)$. Since $L \subset \mathfrak{u}^- A$, $x_{j+1}\mu - \rho \neq \xi$ and so $x \neq x_{j+1}$. Setting $y = x_{j+1}$ completes the proof. $\qquad\square$

2.11. *Proof of Theorem* 2.8. For any module A we write $\mathrm{Tor}_i(A)$ in place of the more precise $\mathrm{Tor}_i^{\mathfrak{u}^-}(\mathbb{C}, A)$. Then $\mathrm{Tor}_0(A) \simeq A/\mathfrak{u}^- A$. Set A equal to the kernel of ϵ and consider the short exact sequences

(*) $\qquad\qquad 0 \to A \to N_0 \overset{\epsilon}{\to} L \to 0 \quad$ and $\quad 0 \to K \to N_1 \overset{d_1}{\to} I \to 0$,

where I equals the image of N_1 and K is the kernel. Then $I/\mathfrak{u}^- I = \mathrm{Tor}_0(I)$ is the image of $\mathrm{Tor}_0(N_1) \simeq \sum_{x \in X^1} F_{x\mu - \rho}$. Using 2.7(iii) and $K \subset \mathfrak{u}^- N_1$, $\mathrm{Tor}_0(K)$ and $\mathrm{Tor}_0(N_1)$ have no $\mathfrak{r}$-modules in common. Thus we have an isomorphism

$$\mathrm{Tor}_0(I) \simeq \mathrm{Tor}_0(N_1) \simeq \sum_{x \in X^1} F_{x\mu - \rho} .$$

The first of the short exact sequences in (*) induces an exact sequence

$$0 \to \mathrm{Tor}_1(L) \to \mathrm{Tor}_0(A) \to \mathrm{Tor}_0(N_0) \to \mathrm{Tor}_0(L) \to 0 .$$

Since $\mathrm{Tor}_0(N_0) \simeq \mathbb{C}_\lambda$, this resolution splits into two isomorphisms:

(**) $\qquad\qquad \mathrm{Tor}_1(L) \simeq \mathrm{Tor}_0(A) \quad$ and $\quad \mathbb{C}_\lambda \simeq \mathrm{Tor}_0(N_0) \simeq \mathrm{Tor}_0(L)$.

This proves that $\mathrm{Tor}_0(I)$ and $\mathrm{Tor}_0(A)$ are isomorphic.

Now $I \subset A \subset N_0$ and $A \subset \mathfrak{u}^- N_0$. We claim that the inclusion of I in A induces an inclusion $I/\mathfrak{u}^- I \to A/\mathfrak{u}^- A$, i.e., $I \cap \mathfrak{u}^- A = \mathfrak{u}^- I$. Suppose not. Then by Lemma 2.10(a) there exists $x \in X^1$ with $F_{x\mu - \rho}$ contained in $\mathrm{Tor}_0(I)$ and $L_{x\mu - \rho}$ occurring as a Jordan-Holder component of I which is contained in $\mathfrak{u}^- A$. Applying Lemma 2.10(b), we obtain $y \neq x, x \prec y$ with $L_{y\mu - \rho}$ occurring as a Jordan-Holder component of A. If $L_{y\mu - \rho}$ is not contained in $\mathfrak{u}^- A$, then we have $x \prec y$ with both contained in X^1. If $L_{y\mu - \rho}$ is contained in $\mathfrak{u}^- A$, then repeat the argument to obtain $z \in D$ with $y \prec z$ and $L_{z\mu - \rho}$ occurring as a Jordan Holder component of A. Repeat this process until we obtain $w \in D$ with $x \prec w$ and $L_{w\mu - \rho}$ is not contained in $\mathfrak{u}^- A$. So $x \prec w$ and both are contained in X^1. This contradicts 2.7(i) and proves the claim.

We now prove $A = I$. By the previous paragraph the induced map $\mathrm{Tor}_0(I) \to \mathrm{Tor}_0(A)$ is injective and hence by (**) is an isomorphism. Then Lemma 2.9 implies that the inclusion of I into A must also be an isomorphism, $A = I$. This proves exactness at N_0.

We now consider the general case. The argument is similar. Assume $j > 0$ and the complex is exact at N_i for $0 \leq i \leq j - 1$. Set A equal to the kernel of d_j and I equal to the image of d_{j+1}. We now prove $I = A$. Consider the resolutions

$$0 \to A \to N_j \overset{d_j}{\to} N_{j-1} \overset{d_{j-1}}{\to} \cdots \overset{d_1}{\to} N_0 \overset{\epsilon}{\to} L \to 0$$

and

$$0 \to K \to N_{j+1} \overset{d_{j+1}}{\to} I \to 0 .$$

From the set of associated short exact sequences and the freeness of each N_i , we obtain that $\mathrm{Tor}_0(A) \simeq \mathrm{Tor}_{j+1}(L)$. From the second resolution and 2.7(iii) and (iv), the inclusion $K \subset \mathfrak{u}^- N_{j+1}$ implies $\mathrm{Tor}_0(I) \simeq \mathrm{Tor}_0(N_{j+1}) \simeq \mathrm{Tor}_{j+1}(L)$. This proves $\mathrm{Tor}_0(A)$ and $\mathrm{Tor}_0(I)$ are isomorphic.

Next we argue as above that the induced map $I/\mathfrak{u}^- I \to A/\mathfrak{u}^- A$ is an injection. Suppose this is not the case. Then by Lemma 2.10(b) we obtain $x, y \in D$ with $F_{x\mu-\rho} \subset \mathfrak{u}^- A \cap I$ and not contained in $\mathfrak{u}^- I$, $x \neq y, x \prec y$ and $L_{y\mu-\rho}$ occurring as a Jordan-Holder component of A. Then as above we may repeat this process to obtain w with $x \neq w, x \prec w$ and $F_{w\mu-\rho}$ contained in $\mathrm{Tor}_0(A)$. Then both x and w lie in X^{j+1} which contradicts 2.7(i). This proves the map $\mathrm{Tor}_0(I)$ into $\mathrm{Tor}_0(A)$ is injective and since they are isomorphic $\mathfrak{r}$-modules, the map is an isomorphism. Then Lemma 2.9 implies that $I = A$. This completes the induction step and the proof. $\qquad\square$

3. RESOLUTIONS OF UNITARY HIGHEST WEIGHT MODULES

3.1. The reduced Hermitian symmetric pair. In [E] a formula for the $\mathfrak{p}^+$-cohomology of unitary highest weight modules is given for Hermitian symmetric pairs of classical type. In [EW] these formulas are used to give a generalized BGG resolution in the classical cases. Here we formulate these results for arbitrary Hermitian symmetric pairs and extend the proof to the exceptional groups. In the language of the previous section, we prove that each unitary highest weight representation is a generalized Kostant module with data X. We describe the sets X and their Kostant gradings. Our answer is given in terms of the reduced Hermitian symmetric pair which we now define.

3.2. Definition. Let $\lambda \in \mathfrak{h}^*$ and let Ψ_λ be the set of roots in Δ that are orthogonal to $\lambda + \rho$. Following [E] we then define a root system Δ_λ as follows. Let W_λ be the subgoup of W that is generated by the reflections s_β with β satisfying the following conditions:

(i) $\beta \in \Delta_n^+$ and $(\lambda + \rho, \beta^\vee)$ is a positive integer;
(ii) β is orthogonal to Ψ_λ;
(iii) β is short if Ψ_λ contains a long root.

Let Δ_λ be the subset of roots $\alpha \in \Delta$ with $s_\alpha \in W_\lambda$. Then Δ_λ is an abstract root system and W_λ is the associated Weyl group. Let $\Delta_{\lambda,c} = \Delta_c \cap \Delta_\lambda$, $\Delta_\lambda^+ = \Delta^+ \cap \Delta_\lambda$ and $\Delta_{\lambda,c}^+ = \Delta_c^+ \cap \Delta_\lambda$. Let $\mathfrak{g}_\lambda$ and $\mathfrak{k}_\lambda$ be the complex Lie algebras with Cartan subalgebra $\mathfrak{h}$ and root systems Δ_λ and $\Delta_{\lambda,c}$. So $\mathfrak{g}_\lambda = \mathfrak{h} \oplus \sum_{\alpha \in \Delta_\lambda} \mathfrak{g}^\alpha$ and $\mathfrak{k}_\lambda = \mathfrak{h} \oplus \sum_{\alpha \in \Delta_{\lambda,c}} \mathfrak{g}^\alpha$. Note that these Lie algebras are the complexified Lie algebras of a Hermitian symmetric pair (G_λ, K_λ). Also note that in general $\mathfrak{g}_\lambda$ is not a subalgebra of $\mathfrak{g}$. We call both (G_λ, K_λ) and $(\mathfrak{g}_\lambda, \mathfrak{k}_\lambda)$ the reduced Hermitian symmetric pair associated to λ.

3.3. Put $\rho_\lambda = \frac{1}{2} \sum_{\alpha \in \Delta_\lambda^+} \alpha$ and define $W_\lambda^c = \{w \in W_\lambda \mid w\Delta_\lambda^+ \supseteq \Delta_{\lambda,c}^+\}$. Note that $W_{\lambda,c} \subseteq W_c$ but $W_\lambda^c \nsubseteq W^c$ in general. Let ℓ_λ denote the length function on W_λ. Set

$$W_\lambda^{c,i} = \{w \in W_\lambda^c \mid \ell_\lambda(w) = i\} \ .$$

Comparing with 2.2, note that in general ℓ_λ does not equal the restriction of ℓ to W_λ. For any $\lambda \in \mathfrak{h}^*$ with $(\lambda, \alpha^\vee) \in \mathbb{R}$ for all $\alpha \in \Delta$, let λ^+ denote the unique element in the W_c-orbit of λ which is Δ_c^+-dominant.

3.4. Now we make the connections with the graded subsets of D which parametrize the BGG resolutions in section two. Recall the double cosets in 2.5. Choose μ dominant and $w_0 \in W$ with $\lambda + \rho = w_0\mu$.

Lemma. *For $x \in W_\lambda^c$, the double coset $W_c x w_0 W_T$ is of maximal size $|W_c||W_T|$ and contains a unique element $\delta(x)$ of minimal length.*

Proof. Each element of Ψ_λ is fixed by x. Suppose that $(x(\lambda + \rho), \alpha) = 0$ for some compact root α. Then $x^{-1}\alpha \in \Psi_\lambda$ which implies $\alpha \in \Psi_\lambda$. This is a contradiction. Hence the double coset has maximal size. An elementary tensoring argument proves the existence of the minimal length representative. $\qquad\square$

3.5. Recall from 2.5 the subset D of representatives of maximal double cosets in $W_c \backslash W / W_T$. Then D is a subset of $^T W \cap W^c$. Define the graded set $X_\lambda = \bigcup X_\lambda^i$ by setting X_λ^i equal to the image of $W_\lambda^{c,i}$ under the inclusion

$$x \to \delta(x), \qquad W_\lambda^c \hookrightarrow {}^T W \cap W^c.$$

We now turn to the verification that L with data X is a generalized Kostant module. Let Δ_{int} denote the integral root system for $\lambda + \rho$. Then $\Delta_c \subset \Delta_{\mathrm{int}} \subset \Delta$. Then replacing Δ by Δ_{int} we obtain the length function ℓ_{int} and the Weyl group decomposition $W_{\mathrm{int}} \simeq W_c \cdot W_{\mathrm{int}}^c$ and the grading $W_{\mathrm{int}}^{c,i}$.

3.6. Lemma [L, Proposition 3.7]. *Let μ be dominant and regular and α a simple root. Suppose $w, w' \in W^c$ satisfy conditions $w = w's_\alpha$ and $\ell_{\mathrm{int}}(w) = \ell_{\mathrm{int}}(w') + 1$. Then the standard map $N(\mathfrak{g}, w\mu - \rho) \to N(\mathfrak{g}, w'\mu - \rho)$ is nonzero.*

Proof. In [L], the proposition is stated only for integral points but the proof given there applies to the more general case as well. $\qquad\square$

3.7. Theorem [DES, Theorem 4.1]. *Suppose that L_λ is unitary. Then for $i \geq 0$,*

$$H^i(\mathfrak{p}^+, L_\lambda) \simeq \bigoplus_{w \in W_\lambda^{c,i}} F_{\delta(w)\mu}.$$

3.8. Corollary. *Suppose that L_λ is a unitary highest weight module. Then L_λ has a resolution of the form*

$$0 \to N_{p_\lambda} \to \cdots \to N_1 \to N_0 \to L_\lambda \to 0$$

with

$$N_i = \bigoplus_{w \in W_\lambda^{c,i}} N_{\delta(w)\mu}.$$

The length p_λ of the resolution is equal to $\dim \mathfrak{p}_\lambda^+$.

Proof. Two proofs are sketched in [EW] which cover the classical cases. Here we will complete the result by giving a proof for the exceptional cases. Since the argument of Lepowsky even for the classical cases must be modified at several points and the details are not available elsewhere we begin with some comments on the proofs in the classical cases.

Suppose we are in one of the five classical cases, $L = L_\lambda$ and $\lambda + \rho$ is dominant and regular. If in addition $\lambda + \rho$ is integral, then L is finite dimensional and Corollary 3.8 reduces to Theorem 2.6. If $\lambda + \rho$ is dominant and regular but not integral, let Δ_{int} denote integral root system. Then we are in one of two cases. Either $\mathfrak{g} \simeq \mathfrak{so}(2, 2n - 1)$ and the category $\mathcal{O}(\mathfrak{g})_{\lambda+\rho}$ is equivalent to the category $\mathcal{O}$ for $\mathfrak{sl}(2)$ or $\mathfrak{g} \simeq \mathfrak{sp}(n)$ and L is one of the two components of the Weil representation. In the first case one can verify Corollary 3.8 directly. So suppose we are in the second case. Here $\mathfrak{g} \simeq \mathfrak{sp}(n)$, W_λ is the Weyl group for type D_n and $\delta(x) = x$ for all W_λ^c.

Using the constants associated with this Weyl group [BGG] and the maps induced by the standard maps we find that L satisfies 2.7(i) and (ii). Then Lemma 3.6 implies 2.7(iii) and Theorem 3.7 gives 2.7(iv) and so the corollary now follows from Theorem 2.8. This is the first application of what we referred to as the Lepowsky argument.

The next step is a reduction by equivalence of category following the arguments in [E]. Suppose $\lambda + \rho$ is not regular dominant. Choose a Δ^+-dominant μ with $\lambda + \rho = w_0\mu$. Then there is an equivalence of category T_λ defined from the category $\mathcal{C} = \mathcal{O}_{S \cap \Delta_\lambda}(\mathfrak{g}_\lambda)_{\lambda+\rho}$ to a subcategory $\mathcal{C}' \subset \mathcal{O}_S(\mathfrak{g})_{\lambda+\rho}$. This equivalence is defined case by case in [E] and has the essential property, for all $x \in W_\lambda^c$,

$$T_\lambda : N(\mathfrak{g}_\lambda, x(\lambda + \rho) - \rho_\lambda) \longmapsto N(\mathfrak{g}, \delta(x)\mu - \rho).$$

Since $\lambda + \rho$ is dominant regular for Δ_λ^+, we may apply the cases above to obtain the resolution for $L(\mathfrak{g}_\lambda, \lambda)$. Applying the equivalence we obtain Corollary 3.8 for L.

Now we turn to the exceptional groups E_6 and E_7. Here all the reduction points are integral. So first assume $\lambda + \rho$ is regular integral. From [C, Figures 2 and 3], $\lambda + \rho = x\mu$ where x is one of the vertices $\{w_0, w_1, w_2, w_3, w_4, w_5, w_7, w_{12}, , w_{26}\}$ for E_6 and vertices $\{w_0, w_1, w_2, w_3, w_4, w_5, w_6, w_8, w_{16}, w_{55}\}$ for E_7. The last vertex corresponds to the trivial representation in each case. There the corollary follows from Theorem 2.6. So we assume $x \neq w_{26}$ or w_{55}. $\qquad\square$

3.9. Lemma. *Suppose $\mathfrak{g}$ is of type E_6 or E_7, L_λ is unitary and $\lambda + \rho$ is regular. Then the map $x \to \delta(x)$ is an order preserving injection from the poset W_λ^c onto its image $D' \subset W^c$. Moreover, its inverse is also order preserving. The map is an isomorphism of the Bruhat order on W_λ^c onto the restriction to D' of the Bruhat order on W^c.*
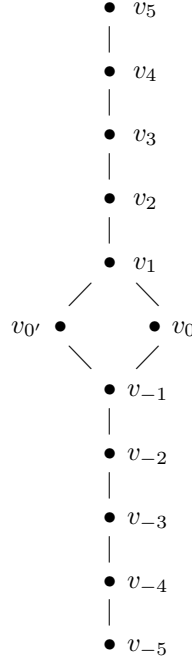
Proof. This is a straightforward verification. For details see [C]. $\qquad\square$

Combining this lemma and Lemma 3.6, we find that the data X_λ^i as defined in 3.5 is a Kostant grading which satisfies 2.7(i), (ii) and (iii). Then Theorem 3.7 implies 2.7(iv) and so L is a generalized Kostant module. Theorem 2.8 now finishes the proof of Corollary 3.8 for regular points.

Now we turn to the singular points. We begin with the E_6 case. Recall BGG reciprocity [ES1, Proposition 2.2]. Let P_ν be the projective cover of the simple module L_ν. Then P_ν has a generalized Verma flag and the multiplicity of N_ν in a generalized Verma flag of P_ξ is equal to the Jordan-Holder multiplicity of L_ξ in N_ν. Then [ES2, Proposition 2.3] gives the Verma flag multiplicities. In turn this gives with BGG reciprocity the Jordan-Holder series of each generalized Verma module. Identify the semi-regular highest weights plus rho with the linearly ordered set $v_0, v_1, v_2, v_3, v_4, v_5$ and let N_{v_i} (resp. L_{v_i}) denote the module $N(\mathfrak{g}, v_i(\lambda + \rho) - \rho)$ (resp. $L(\mathfrak{g}, v_i(\lambda + \rho) - \rho)$). We find N_{v_0} is simple and for $1 \leq i \leq 5, N_{v_i}$ has Jordan-Holder components L_{v_i} and $L_{v_{i-1}}$. Moreover, the latter is the socle. From this we can directly construct the generalized BGG resolution for any of the vertices in this case. All singular reduction points are of this form. Therefore, the corollary holds for E_6.

For E_7 the singular reduction points are either semi-regular or have an $A_1 \times A_1$ singularity. By [ES2, Lemma 3.5] the category for an $A_1 \times A_1$ singularity looks like the $\mathfrak{sl}(2)$ category with two simple modules. Both have the desired cohomology formulas and we see directly that both have generalized BGG resolutions. For

the semi-regular points, [ES2, Proposition 3.8] determines the generalized Verma module flags of the projective covers. In turn with BGG reciprocity we can compute the Jordan-Holder series for each generalized Verma module. The semi-regular category corresponds to the following diagram:

$$
\begin{array}{c}
\bullet \; v_5 \\
| \\
\bullet \; v_4 \\
| \\
\bullet \; v_3 \\
| \\
\bullet \; v_2 \\
| \\
\bullet \; v_1 \\
\diagup \quad \diagdown \\
v_{0'} \; \bullet \qquad \bullet \; v_0 \\
\diagdown \quad \diagup \\
\bullet \; v_{-1} \\
| \\
\bullet \; v_{-2} \\
| \\
\bullet \; v_{-3} \\
| \\
\bullet \; v_{-4} \\
| \\
\bullet \; v_{-5}
\end{array}
$$

The Jordan-Holder components and socles of generalized Verma modules in this category are given in the following table:

generalized Verma module	Jordan-Holder components	socle
N_5	$L_5,\ L_4,\ L_{-5}$	L_{-5}
N_4	$L_4,\ L_3,\ L_{-4},\ L_{-5}$	L_{-4}
N_3	$L_3,\ L_2,\ L_{-3},\ L_{-4}$	L_{-3}
N_2	$L_2,\ L_1,\ L_{-2},\ L_{-3}$	L_{-2}
N_1	$L_1,\ L_0,\ L_{0'},\ L_{-1},\ L_{-2}$	L_{-1}
N_0	$L_0,\ L_{-1}$	L_{-1}
$N_{0'}$	$L_{0'},\ L_{-1}$	L_{-1}
N_{-1}	$L_{-1},\ L_{-2}$	L_{-2}
N_{-2}	$L_{-2},\ L_{-3}$	L_{-3}
N_{-3}	$L_{-3},\ L_{-4}$	L_{-4}
N_{-4}	$L_{-4},\ L_{-5}$	L_{-5}
N_{-5}	L_{-5}	L_{-5}

By construction (cf. [ES2]) the projective covers $P_{v_{-5}}, P_{v_{-4}}, P_{v_{-3}}, P_{v_{-2}}$ and $P_{v_{-1}}$ are self dual. This implies that Socle $N_j=$ Socle $P_{-j} = L_{-j}$ for $1 \leq j \leq 5$. From the Jordan-Holder table we obtain the other socle information. This is listed in the far right column. Directly, as a result of these tables, if $\lambda + \rho$ corresponds to one of the vertices $\{v_{-5}, v_{-4}, v_{-3}, v_{-2}, v_{-1}, v_0, v_{0'}, v_5\}$, then we can define maps d_j so that

2.7(i)–(iii) all hold. All semiregular $\lambda + \rho$ for E_7 with L_λ unitary correspond to these vertices. Then using Theorem 2.8 and Theorem 3.7 we obtain Corollary 3.8 for E_7. This completes the entire proof. $\square$

3.10. Corollary to Proof. *Suppose $\mathfrak{g}$ is of type E_6 and $\lambda + \rho$ is singular integral. Then L_λ is a generalized Kostant module.*

4. The cone decomposition and unitary strata

4.1. The classification of unitary highest weight modules. We review some of the notation from the paper [EHW] on the classification of unitary highest weight modules and the cone decomposition. Let β denote the unique maximal root in Δ^+ and let ζ be the unique fundamental root $(\zeta, \beta^\vee) = 1$. Let λ_0 be a Δ_c^+-dominant integral element of $\mathfrak{h}^*$ such that $(\lambda_0 + \rho, \beta^\vee) = 0$. If L_λ is unitary, then there exists a unique λ_0 as above such that $\lambda = \lambda_0 + z\zeta$ with $z \in \mathbb{R}$. For fixed λ_0 the set of all $z \in \mathbb{R}$ such that the irreducible highest weight module $L_{\lambda_0 + z\zeta}$ is unitary is as follows:
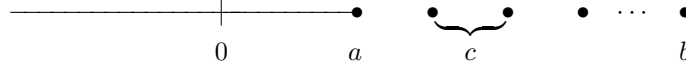
The constants a and b depend on λ_0, whereas the constant c depends only on the Hermitian symmetric pair.

Since in this article we consider resolutions and Hilbert series of unitary highest weight modules L_λ, we are interested in those highest weights λ for which N_λ is reducible (and hence L_λ has a nontrivial resolution). We call the set of such highest weights the *set of reduction points*. In the picture above, the reduction points correspond to the equally spaced points between a and b. The highest weight corresponding to $z = a$ is called the first reduction point; the highest weight corresponding to $z = b$ is called the last reduction point.

For simplicity, in what follows we only consider Hermitian symmetric pairs with simply laced Dynkin diagram, i.e., root systems of ADE-type. In this case the constants a and b mentioned above have some simple interpretations in terms of a certain root sytem, Q_{λ_0}, attached to the line $\lambda_0 + \mathbb{R}\zeta$.

4.2. The root system Q_{λ_0}. We begin with a general observation on Dynkin diagrams of Hermitian symmetric pairs (which is also valid in the nonsimply laced cases). Consider the extended Dynkin diagram that is obtained from the Dynkin diagram of Δ by adding an extra node corresponding to $-\beta$ that is connected to the other nodes using the usual rules. For example, if Δ is the Dynkin diagram of E_6, then the extended diagram is:
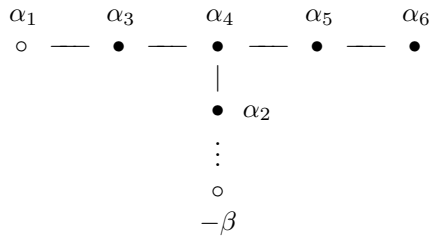
Now delete the node corresponding to the unique simple noncompact root. Then the resulting diagram is again the Dynkin diagram of the root system of a Hermitian

symmetric pair of the same type with $-\beta$ being the node corresponding to the simple noncompact root. For example, in the case of E_6:

$$
\begin{array}{ccccccc}
\alpha_3 & & \alpha_4 & & \alpha_5 & & \alpha_6 \\
\bullet & \!\!-\!\! & \bullet & \!\!-\!\! & \bullet & \!\!-\!\! & \bullet \\
& & | & & & & \\
& & \bullet & \alpha_2 & & & \\
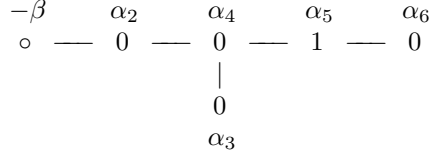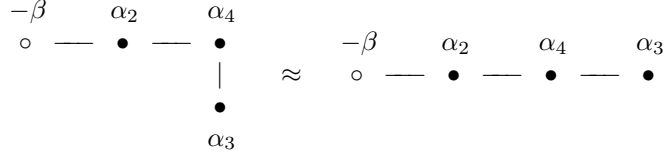& & | & & & & \\
& & \circ & & & & \\
& & -\beta & & & &
\end{array}
$$

$$
\approx \quad
\begin{array}{ccccccccc}
-\beta & & \alpha_2 & & \alpha_4 & & \alpha_5 & & \alpha_6 \\
\circ & \!\!-\!\! & \bullet & \!\!-\!\! & \bullet & \!\!-\!\! & \bullet & \!\!-\!\! & \bullet \\
& & & & | & & & & \\
& & & & \bullet & \alpha_3 & & &
\end{array}
$$
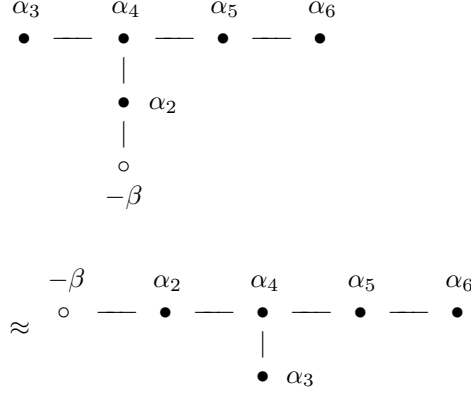
As in [EHW], we attach to λ a root system, Q_λ, as follows: the diagram of Q_λ is the maximal connected subdiagram of the diagram above containing $-\beta$ and having the property that every compact simple root is orthogonal to λ. For example, consider the case of E_6 and $\lambda = \omega_5 - 9\omega_1$. To find Q_λ we first label the nodes corresponding the simple compact roots by the dot products $(\lambda, \alpha_i^\vee)$:

$$
\begin{array}{ccccccccc}
-\beta & & \alpha_2 & & \alpha_4 & & \alpha_5 & & \alpha_6 \\
\circ & \!\!-\!\! & 0 & \!\!-\!\! & 0 & \!\!-\!\! & 1 & \!\!-\!\! & 0 \\
& & & & | & & & & \\
& & & & 0 & & & & \\
& & & & \alpha_3 & & & &
\end{array}
$$

Thus Q_λ is of type $SU(1,4)$:

$$
\begin{array}{ccccc}
-\beta & & \alpha_2 & & \alpha_4 \\
\circ & \!\!-\!\! & \bullet & \!\!-\!\! & \bullet \\
& & & & | \\
& & & & \bullet \\
& & & & \alpha_3
\end{array}
\qquad \approx \qquad
\begin{array}{ccccccc}
-\beta & & \alpha_2 & & \alpha_4 & & \alpha_3 \\
\circ & \!\!-\!\! & \bullet & \!\!-\!\! & \bullet & \!\!-\!\! & \bullet
\end{array}
$$

Note that if $\lambda = \lambda_0 + z\zeta$ as in 4.1, then $Q_\lambda = Q_{\lambda_0}$ since ζ is orthogonal to all compact roots.

The constants a and b from 4.1 (in the simply laced cases) can be expressed in terms of $Q = Q_{\lambda_0}$ as follows. Let r be the split rank of Q. Then

$$
b = 2(\rho_{Q,c}, \beta^\vee) + 1 \qquad \text{and} \qquad a = b - (r-1)c
$$

Here $\rho_{Q,c}$ denotes half the sum of all the positive compact roots of Q.

4.3. The cone decomposition. The set of all reduction points can be written as a (disjoint) union of certain translated integral cones. This cone decomposition was explicitly introduced in [DES]. Here, by a *translated integral cone* in $\mathfrak{h}^*$ with vertex ν, we mean a subset of $\mathfrak{h}^*$ of the form $\mathcal{C} = \{\nu + \sum_{j=1}^m a_j \pi_j \mid a_j \in \mathbb{Z}_+\}$, where ν and π_j are Δ_c-integral.

Let Q be a root system of the form $Q = Q_\lambda$ for some reduction point $\lambda = \lambda_0 + z\zeta$. Fixing the root system Q and value z we define a subset $\mathcal{C}_{Q,z}$ of the set of all reduction points by

$$\mathcal{C}_{Q,z} = \{\lambda = \lambda_0 + z\zeta \mid \lambda \text{ is a reduction point and } Q_\lambda = Q\} .$$

It was observed in [DES] that the set $\mathcal{C}_{Q,z}$ is a translated integral cone. More precisely, $\mathcal{C}_{Q,z} = \{\nu + \sum_{j=1}^{m} a_j \pi_j \mid a_j \in \mathbb{Z}_+\}$, where the vertex ν and the generators π_j are given as follows. Let ω_i denote the fundamental weight corresponding to the simple root α_i. Then $\nu = \sum_i c_i \omega_i$ is such that $(\nu + \rho, \beta^\vee) = z$ and the coefficients $c_i = (\nu, \alpha_i^\vee)$ for compact α_i are equal to 1 if α_i does not belong to Q but the corresponding node in the extended Dynkin diagram is connected to some node of the diagram of Q and is equal to 0 otherwise. The generators π_j are the weights of the form $\omega_i - (\omega_i, \beta^\vee)\zeta$, where α_i is a compact root that does not belong to Q.

4.4. A finer decomposition of the cones. Let Σ denote the set of simple roots in Δ^+ and for $\alpha \in \Sigma$ let H_α denote the hyperplane in $\mathfrak{h}_\mathbb{R}^*$ that is perpendicular to α. (Here we identify $\mathfrak{h}^*$ with $\mathfrak{h}$ via the Killing form.) For a subset $T \subseteq \Sigma$ define

$$H_T = \bigcap_{\alpha \in T} H_\alpha$$

Let $\mathfrak{C}_T^+$ denote the interior in H_T of the intersection $H_T \cap \overline{\mathfrak{C}^+}$. We then can write $\mathfrak{h}_\mathbb{R}^*$ as a disjoint union as follows:

$$\mathfrak{h}_\mathbb{R}^* = \bigcup_{T \subseteq \Sigma} \bigcup_{w \in {}^T W} w\mathfrak{C}_T^+.$$

The following lemma can be verified by a case by case analysis.

Lemma. *Suppose the set $\{\lambda \in \mathcal{C}_{Q,z} \mid \lambda + \rho \in w\mathfrak{C}_T^+\}$ is nonempty. Then $\{\lambda \in \mathcal{C}_{Q,z} \mid \lambda + \rho \in w\mathfrak{C}_T^+\}$ is itself a translated integral cone with a unique vertex.*

Definition. We will call $\{\lambda \in \mathcal{C}_{Q,z} \mid \lambda + \rho \in w\mathfrak{C}_T^+\}$, provided it is nonempty, the *unitary stratum* with parameters (w, T). Its vertex is denoted $\nu_{w,T}$. At the end of this section we will explicitly list all the unitary strata for the exceptional groups.

4.5. Posets for E_6 and E_7. Many of the calculations to come later are direct consequences of the poset structure of W^c. The partially ordered set W^c is given

by the following figures for E_6 and E_7 respectively:
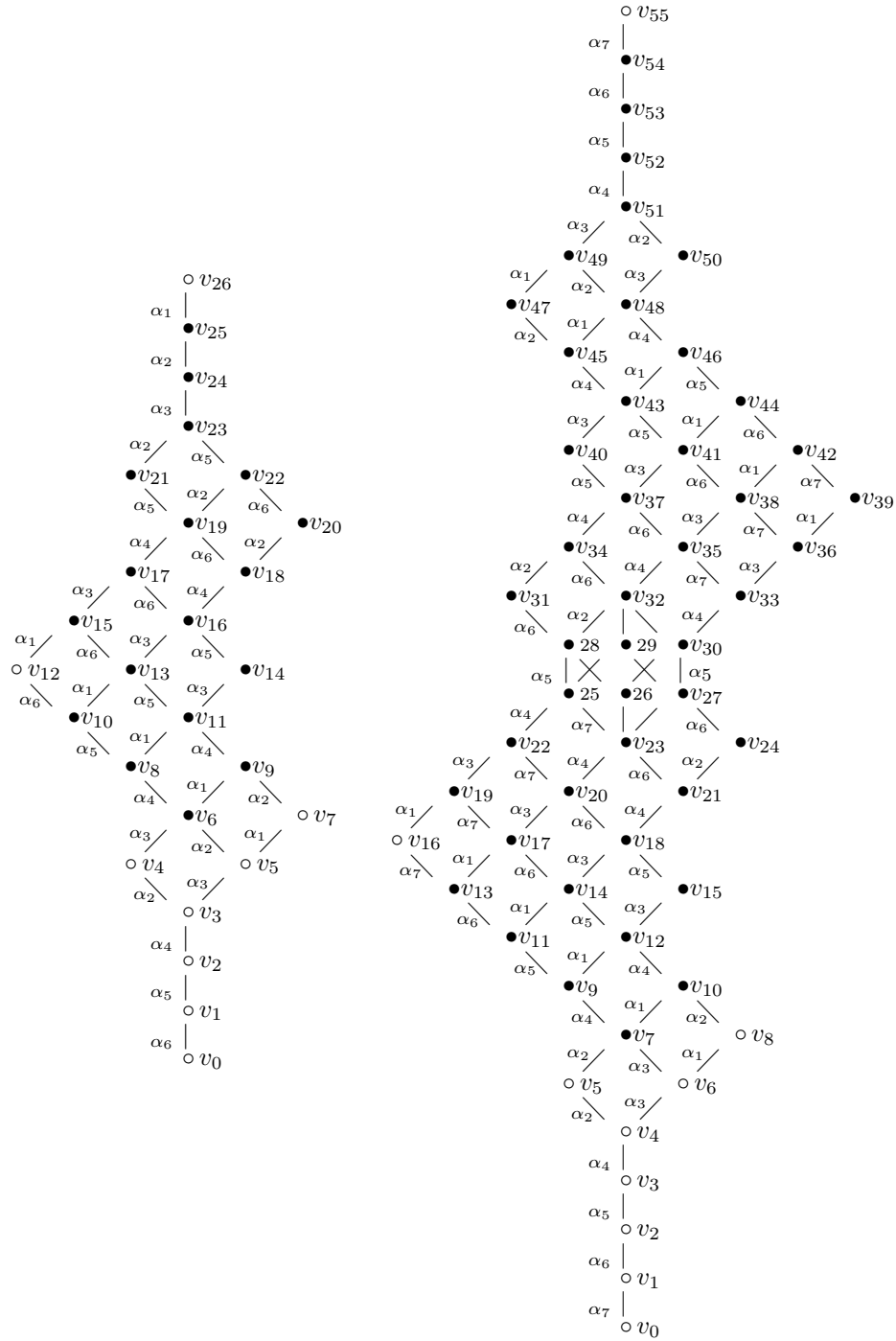
4.6. Root orderings for Δ_n^+ of E_6 and E_7. The positive noncompact roots and their Dynkin diagrams are given as follows.

Simple roots and extended Dynkin diagram for E_6:

$$
\begin{array}{ccccccccc}
\alpha_1 & & \alpha_3 & & \alpha_4 & & \alpha_5 & & \alpha_6 \\
\circ & \!\!-\!\! & \bullet & \!\!-\!\! & \bullet & \!\!-\!\! & \bullet & \!\!-\!\! & \bullet
\end{array}
$$

$$
\begin{array}{ll}
\alpha_1 = \tfrac{1}{2} & (\ +1,\ -1,\ -1,\ -1,\ -1,\ -1,\ -1,\ +1\ ) \\
\alpha_2 = & (\ +1,\ +1,\ \ \ 0,\ \ \ 0,\ \ \ 0,\ \ \ 0,\ \ \ 0,\ \ \ 0\ ) \\
\alpha_3 = & (\ -1,\ +1,\ \ \ 0,\ \ \ 0,\ \ \ 0,\ \ \ 0,\ \ \ 0,\ \ \ 0\ ) \\
\alpha_4 = & (\ \ \ 0,\ -1,\ +1,\ \ \ 0,\ \ \ 0,\ \ \ 0,\ \ \ 0,\ \ \ 0\ ) \\
\alpha_5 = & (\ \ \ 0,\ \ \ 0,\ -1,\ +1,\ \ \ 0,\ \ \ 0,\ \ \ 0,\ \ \ 0\ ) \\
\alpha_6 = & (\ \ \ 0,\ \ \ 0,\ \ \ 0,\ -1,\ +1,\ \ \ 0,\ \ \ 0,\ \ \ 0\ )
\end{array}
$$

where the diagram below α_4 reads $\bullet\ \alpha_2$, then a vertical dotted line, then $\circ\ -\beta_0$.

Noncompact positive roots for E_6:

$$
\begin{array}{ll}
\beta_0 & = \tfrac{1}{2}\ (\ +1,\ +1,\ +1,\ +1,\ +1,\ -1,\ -1,\ +1\ ) \\
\beta_1 & = \tfrac{1}{2}\ (\ -1,\ -1,\ +1,\ +1,\ +1,\ -1,\ -1,\ +1\ ) \\
\beta_2 & = \tfrac{1}{2}\ (\ -1,\ +1,\ -1,\ +1,\ +1,\ -1,\ -1,\ +1\ ) \\
\beta_3 & = \tfrac{1}{2}\ (\ +1,\ -1,\ -1,\ +1,\ +1,\ -1,\ -1,\ +1\ ) \\
\beta_4 & = \tfrac{1}{2}\ (\ -1,\ +1,\ +1,\ -1,\ +1,\ -1,\ -1,\ +1\ ) \\
\beta_5 & = \tfrac{1}{2}\ (\ +1,\ -1,\ +1,\ -1,\ +1,\ -1,\ -1,\ +1\ ) \\
\beta_6 & = \tfrac{1}{2}\ (\ -1,\ +1,\ +1,\ +1,\ -1,\ -1,\ -1,\ +1\ ) \\
\beta_7 & = \tfrac{1}{2}\ (\ +1,\ +1,\ -1,\ -1,\ +1,\ -1,\ -1,\ +1\ ) \\
\beta_8 & = \tfrac{1}{2}\ (\ +1,\ -1,\ +1,\ +1,\ -1,\ -1,\ -1,\ +1\ ) \\
\beta_9 & = \tfrac{1}{2}\ (\ -1,\ -1,\ -1,\ -1,\ +1,\ -1,\ -1,\ +1\ ) \\
\beta_{10} & = \tfrac{1}{2}\ (\ +1,\ +1,\ -1,\ +1,\ -1,\ -1,\ -1,\ +1\ ) \\
\beta_{11} & = \tfrac{1}{2}\ (\ -1,\ -1,\ -1,\ +1,\ -1,\ -1,\ -1,\ +1\ ) \\
\beta_{12} & = \tfrac{1}{2}\ (\ +1,\ +1,\ +1,\ -1,\ -1,\ -1,\ -1,\ +1\ ) \\
\beta_{13} & = \tfrac{1}{2}\ (\ -1,\ -1,\ +1,\ -1,\ -1,\ -1,\ -1,\ +1\ ) \\
\beta_{14} & = \tfrac{1}{2}\ (\ -1,\ +1,\ -1,\ -1,\ -1,\ -1,\ -1,\ +1\ ) \\
\beta_{15} & = \tfrac{1}{2}\ (\ +1,\ -1,\ -1,\ -1,\ -1,\ -1,\ -1,\ +1\ )
\end{array}
$$

The ordering diagram (Hasse diagram) of the β_i with edges labelled by simple compact roots:

$$
\begin{array}{c}
\bullet\,\beta_0 \\
\alpha_2\ | \\
\bullet\,\beta_1 \\
\alpha_4\ | \\
\bullet\,\beta_2 \\
{}^{\alpha_3}\diagup\ \ {}^{\alpha_5}\diagdown \\
\bullet\,\beta_3 \qquad \bullet\,\beta_4 \\
{}_{\alpha_5}\diagdown\ {}^{\alpha_3}\diagup\ {}^{\alpha_6}\diagdown \\
\bullet\,\beta_5 \qquad \bullet\,\beta_6 \\
{}^{\alpha_4}\diagup\ {}_{\alpha_6}\diagdown\ {}^{\alpha_3}\diagup \\
\bullet\,\beta_7 \qquad \bullet\,\beta_8 \\
{}^{\alpha_2}\diagup\ {}_{\alpha_6}\diagdown\ {}^{\alpha_4}\diagup \\
\bullet\,\beta_9 \qquad \bullet\,\beta_{10} \\
{}_{\alpha_6}\diagdown\ {}^{\alpha_2}\diagup\ {}_{\alpha_5}\diagdown \\
\bullet\,\beta_{11} \qquad \bullet\,\beta_{12} \\
{}_{\alpha_5}\diagdown\ {}^{\alpha_2}\diagup \\
\bullet\,\beta_{13} \\
\alpha_4\ | \\
\bullet\,\beta_{14} \\
\alpha_3\ | \\
\bullet\,\beta_{15}
\end{array}
$$

Remark. Here two roots β_a above β_b are connected by an edge α if $\beta_a = \beta_b + \alpha$ with α a simple compact root.
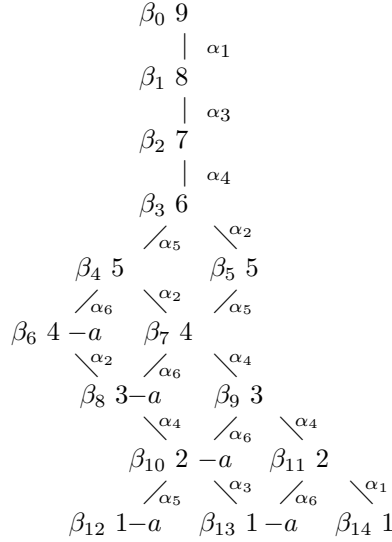
Simple roots and extended Dynkin diagram for E_7:

$$
\begin{array}{ccccccccccccc}
-\beta_0 & & \alpha_1 & & \alpha_3 & & \alpha_4 & & \alpha_5 & & \alpha_6 & & \alpha_7 \\
\circ & \cdots & \bullet & \!\!-\!\! & \bullet & \!\!-\!\! & \bullet & \!\!-\!\! & \bullet & \!\!-\!\! & \bullet & \!\!-\!\! & \circ
\end{array}
$$

with a vertical bond from α_4 down to $\bullet\ \alpha_2$.

$$
\begin{array}{ll}
\alpha_1 = \tfrac{1}{2} & (\ +1,\ -1,\ -1,\ -1,\ -1,\ -1,\ -1,\ +1\ ) \\
\alpha_2 = & (\ +1,\ +1,\ \ \ 0,\ \ \ 0,\ \ \ 0,\ \ \ 0,\ \ \ 0,\ \ \ 0\ ) \\
\alpha_3 = & (\ -1,\ +1,\ \ \ 0,\ \ \ 0,\ \ \ 0,\ \ \ 0,\ \ \ 0,\ \ \ 0\ ) \\
\alpha_4 = & (\ \ \ 0,\ -1,\ +1,\ \ \ 0,\ \ \ 0,\ \ \ 0,\ \ \ 0,\ \ \ 0\ ) \\
\alpha_5 = & (\ \ \ 0,\ \ \ 0,\ -1,\ +1,\ \ \ 0,\ \ \ 0,\ \ \ 0,\ \ \ 0\ ) \\
\alpha_6 = & (\ \ \ 0,\ \ \ 0,\ \ \ 0,\ -1,\ +1,\ \ \ 0,\ \ \ 0,\ \ \ 0\ ) \\
\alpha_7 = & (\ \ \ 0,\ \ \ 0,\ \ \ 0,\ \ \ 0,\ -1,\ +1,\ \ \ 0,\ \ \ 0\ )
\end{array}
$$

Noncompact positive roots for E_7:

$$\beta_0 = (\quad 0, \quad 0, \quad 0, \quad 0, \quad 0, \quad 0, -1, \quad 1)$$
$$\beta_1 = \tfrac{1}{2}(-1, \quad 1, \quad 1, \quad 1, \quad 1, \quad 1, -1, \quad 1)$$
$$\beta_2 = \tfrac{1}{2}(\quad 1, -1, \quad 1, \quad 1, \quad 1, \quad 1, -1, \quad 1)$$
$$\beta_3 = \tfrac{1}{2}(\quad 1, \quad 1, -1, \quad 1, \quad 1, \quad 1, -1, \quad 1)$$
$$\beta_4 = \tfrac{1}{2}(\quad 1, \quad 1, \quad 1, -1, \quad 1, \quad 1, -1, \quad 1)$$
$$\beta_5 = \tfrac{1}{2}(-1, -1, -1, \quad 1, \quad 1, \quad 1, -1, \quad 1)$$
$$\beta_6 = \tfrac{1}{2}(\quad 1, \quad 1, \quad 1, \quad 1, -1, \quad 1, -1, \quad 1)$$
$$\beta_7 = \tfrac{1}{2}(-1, -1, \quad 1, -1, \quad 1, \quad 1, -1, \quad 1)$$
$$\beta_8 = \tfrac{1}{2}(-1, -1, \quad 1, \quad 1, -1, \quad 1, -1, \quad 1)$$
$$\beta_9 = \tfrac{1}{2}(-1, \quad 1, -1, -1, \quad 1, \quad 1, -1, \quad 1)$$
$$\beta_{10} = \tfrac{1}{2}(-1, \quad 1, -1, \quad 1, -1, \quad 1, -1, \quad 1)$$
$$\beta_{11} = \tfrac{1}{2}(\quad 1, -1, -1, -1, \quad 1, \quad 1, -1, \quad 1)$$
$$\beta_{12} = \tfrac{1}{2}(-1, \quad 1, \quad 1, -1, -1, \quad 1, -1, \quad 1)$$
$$\beta_{13} = \tfrac{1}{2}(\quad 1, -1, -1, \quad 1, -1, \quad 1, -1, \quad 1)$$
$$\beta_{14} = (\quad 0, \quad 0, \quad 0, \quad 0, \quad 1, \quad 1, \quad 0, \quad 0)$$
$$\beta_{15} = \tfrac{1}{2}(\quad 1, -1, \quad 1, -1, -1, \quad 1, -1, \quad 1)$$
$$\beta_{16} = (\quad 0, \quad 0, \quad 0, \quad 1, \quad 0, \quad 1, \quad 0, \quad 0)$$
$$\beta_{17} = \tfrac{1}{2}(\quad 1, \quad 1, -1, -1, -1, \quad 1, -1, \quad 1)$$
$$\beta_{18} = (\quad 0, \quad 0, \quad 1, \quad 0, \quad 0, \quad 1, \quad 0, \quad 0)$$
$$\beta_{19} = \tfrac{1}{2}(-1, -1, -1, -1, -1, \quad 1, -1, \quad 1)$$
$$\beta_{20} = (\quad 0, \quad 1, \quad 0, \quad 0, \quad 0, \quad 1, \quad 0, \quad 0)$$
$$\beta_{21} = (-1, \quad 0, \quad 0, \quad 0, \quad 0, \quad 1, \quad 0, \quad 0)$$
$$\beta_{22} = (\quad 1, \quad 0, \quad 0, \quad 0, \quad 0, \quad 1, \quad 0, \quad 0)$$
$$\beta_{23} = (\quad 0, -1, \quad 0, \quad 0, \quad 0, \quad 1, \quad 0, \quad 0)$$
$$\beta_{24} = (\quad 0, \quad 0, -1, \quad 0, \quad 0, \quad 1, \quad 0, \quad 0)$$
$$\beta_{25} = (\quad 0, \quad 0, \quad 0, -1, \quad 0, \quad 1, \quad 0, \quad 0)$$
$$\beta_{26} = (\quad 0, \quad 0, \quad 0, \quad 0, -1, \quad 1, \quad 0, \quad 0)$$

The poset diagram of noncompact positive roots (nodes β_0 through β_{26} connected by edges labeled $\alpha_1, \ldots, \alpha_6$):

β_0
α_1
β_1
α_3
β_2
α_4
β_3
α_5 $\quad$ α_2
β_4 $\quad$ β_5
α_6 $\quad$ α_2 $\quad$ α_5
β_6 $\quad$ β_7
α_2 $\quad$ α_6 $\quad$ α_4
β_8 $\quad$ β_9
α_4 $\quad$ α_6 $\quad$ α_3
β_{10} $\quad$ β_{11}
α_5 $\quad$ α_3 $\quad$ α_6 $\quad$ α_1
β_{12} $\quad$ β_{13} $\quad$ β_{14}
α_3 $\quad$ α_5 $\quad$ α_1 $\quad$ α_6
β_{15} $\quad$ β_{16}
α_4 $\quad$ α_1 $\quad$ α_5
β_{17} $\quad$ β_{18}
α_2 $\quad$ α_1 $\quad$ α_4
β_{19} $\quad$ β_{20}
α_1 $\quad$ α_2 $\quad$ α_3
β_{21} $\quad$ β_{22}
α_3 $\quad$ α_2
β_{23}
α_4
β_{24}
α_5
β_{25}
α_6
β_{26}

4.7. Example of decomposition into strata. Consider the case of E_7 when $Q = SO(2,10)$ and $z = 9$. This is the most difficult case. In this case the points in the cone $\mathcal{C}_{Q,z}$ are given by $\lambda = a\omega_6 + (-2a - 8)\omega_7$ with $a \geq 1$. We now evaluate $(\lambda + \rho, \beta^\vee)$ for all $\beta \in \Delta_n^+$. We display these values in a diagram as follows. Start with the poset diagram of Δ_n^+. Then label each node of the diagram with the corresponding number $(\lambda + \rho, \beta^\vee)$. We are only interested in the nonnegative values and so we ignore the part of the diagram for which the values $(\lambda + \rho, \beta^\vee)$ are negative. What we obtain is a diagram as follows:

$$
\begin{array}{c}
\beta_0\ 9 \\
\mid\ \alpha_1 \\
\beta_1\ 8 \\
\mid\ \alpha_3 \\
\beta_2\ 7 \\
\mid\ \alpha_4 \\
\beta_3\ 6 \\
{}^{\alpha_5}\diagup\quad\diagdown{}^{\alpha_2} \\
\beta_4\ 5 \qquad\qquad \beta_5\ 5 \\
{}^{\alpha_6}\diagup\quad\diagdown{}^{\alpha_2}\quad\diagup{}^{\alpha_5} \\
\beta_6\ 4-a \qquad \beta_7\ 4 \\
\diagdown{}^{\alpha_2}\quad\diagup{}^{\alpha_6}\quad\diagdown{}^{\alpha_4} \\
\beta_8\ 3-a \qquad \beta_9\ 3 \\
\diagdown{}^{\alpha_4}\quad\diagup{}^{\alpha_6}\quad\diagdown{}^{\alpha_4} \\
\beta_{10}\ 2-a \qquad \beta_{11}\ 2 \\
\diagup{}^{\alpha_5}\quad\diagdown{}^{\alpha_3}\quad\diagup{}^{\alpha_6}\quad\diagdown{}^{\alpha_1} \\
\beta_{12}\ 1-a \qquad \beta_{13}\ 1-a \qquad \beta_{14}\ 1
\end{array}
$$

Our calculations will rely on the following lemma.

Lemma. *The following are equivalent:*

   (a) $\lambda + \rho$ *lies in the closed chamber* $w\overline{\mathfrak{C}^+}$;

   (b) $(\lambda + \rho, w\beta^\vee) \geq 0$ *for all* $\beta \in \Delta_n^+$;

   (c) $w\Delta_n^+ \cap \Delta_n^+ = \{\beta \in \Delta_n^+ \mid (\lambda + \rho, \beta^\vee) \geq 0\}$.

We obtain five different cases depending on the value of a, namely $a = 1$, 2, 3, 4 and ≥ 5. For $a \geq 5$, the parameter $\lambda + \rho$ is integral regular and lies in the chamber $w\mathfrak{C}^+$, where w is the unique element of W^c with $w\Delta_n^+ \cap \Delta_n^+ = \{\beta_0, \beta_1, \beta_2, \beta_3, \beta_4, \beta_5, \beta_7, \beta_9, \beta_{11}, \beta_{14}\} = \Theta$. From the figure in 4.5 we verify that $w = v_{16}$. Thus the parameters of the stratum $\{a\omega_6 + (-2a - 8)\omega_7 \mid a \geq 5\}$ are $(w, T) = (v_{16}, \varnothing)$. Next consider $a = 4$. Then $w \in W^c$ is the unique element with $w\Delta_n^+ \cap \Delta_n^+ = \Theta \cup \{\beta_6\}$, which implies by the figure in 4.5 that $w = v_{19}$. One also checks that $\beta_6 = v_{16}\alpha_1$ and hence the parameters of the stratum $\{4\omega_6 - 16\omega_7\}$ are $(w, T) = (v_{19}, \{\alpha_1\})$. Similarly, for $a = 3$ and 2 we obtain parameters $(w, T) = (v_{22}, \{\alpha_3\})$ and $(v_{22}, \{\alpha_4\})$, respectively. Finally, for $a = 1$ we have $w\Delta_n^+ \cap \Delta_n^+ = \{\beta_1, \beta_2, \ldots, \beta_{14}\}$. Here there are two singular roots: β_{12} and β_{13}. With $w = v_{32}$ we have $\beta_{12} = v_{32}\alpha_2$, $\beta_{13} = v_{32}\alpha_5$. Thus the parameters of the stratum $\{\omega - 10\omega_7\}$ are $(w, T) = (v_{32}, \{\alpha_2, \alpha_5\})$.

4.8. Invariance on stratum. Now we determine invariance properties on strata. Suppose ξ and ν lie in the positive chamber, have integral difference $\xi - \nu$ and equal stabilizers in W, i.e., $\{w \in W \mid w\xi = \xi\} = \{w \in W \mid w\nu = \nu\}$. Applying standard techniques of Zuckerman translation we have an equivalence of categories $\Theta : \mathcal{O}_\xi \xrightarrow{\sim} \mathcal{O}_\nu$. Suppose $w \in W^c$ and assume both $w\xi$ and $w\nu$ are Δ_c^+-dominant integral and regular. Then $\Theta N_{w\xi} \simeq N_{w\nu}$ and $\Theta L_{w\xi} \simeq L_{w\nu}$. Applying this equivalence to our modules and resolutions we obtain:

Lemma. *The Gelfand-Kirilov dimension, the length of a minimal free resolution and the degree of the numerator polynomial in the Hilbert series are constant on each stratum.*

4.9. Table of strata decomposition. The following tables summarize the decomposition of cones into strata. Each line corresponds to a strata. For each stratum we give the vertex of the stratum, the Gelfand-Kirillov dimension of L_λ with $\lambda + \rho$ in the stratum, the length p of the generalized BGG resolution and the degree e of the numerator polynomial $P(t)$.

Unitary strata for E_6:

Q	z	w	T	vertex $\nu_{w,T}$	GKdim	p	e
$SU(1,1)$	1	v_1	$\varnothing$	$\omega_2 - 12\omega_1$	16	1	1
$SU(1,2)$	2	v_2	$\varnothing$	$\omega_4 - 12\omega_1$	16	2	2
$SU(1,3)$	3	v_3	$\varnothing$	$\omega_3 + \omega_5 - 12\omega_1$	16	3	3
$SU(1,4)^{(a)}$	4	v_4	$\varnothing$	$2\omega_5 - 11\omega_1$	16	4	4
		v_6	$\{\alpha_3\}$	$\omega_5 - 9\omega_1$	16	1	1
$SU(1,4)^{(b)}$	4	v_5	$\varnothing$	$2\omega_3 + \omega_6 - 12\omega_1$	16	4	4
		v_6	$\{\alpha_2\}$	$\omega_3 + \omega_6 - 10\omega_1$	16	1	1
$SU(1,5)$	5	v_7	$\varnothing$	$3\omega_3 - 12\omega_1$	16	5	5
		v_9	$\{\alpha_2\}$	$2\omega_3 - 10\omega_1$	16	2	2
		v_{11}	$\{\alpha_4\}$	$\omega_3 - 8\omega_1$	16	2	3
$SO(2,8)$	4	v_8	$\{\alpha_4\}$	$\omega_6 - 8\omega_1$	16	1	2
	7	v_{12}	$\varnothing$	$4\omega_6 - 8\omega_1$	16	8	8
		v_{15}	$\{\alpha_1\}$	$3\omega_6 - 7\omega_1$	16	4	4
		v_{17}	$\{\alpha_3\}$	$2\omega_6 - 6\omega_1$	16	4	5
		v_{19}	$\{\alpha_4\}$	$\omega_6 - 5\omega_1$	16	4	6
E_6	8	v_{24}	$\{\alpha_4\}$	$-3\omega_1$	11	5	3
	11	v_{26}	$\varnothing$	0	1	16	0

The two cases with $Q = SU(1,4)$ correspond to the following Dynkin diagrams:

$$SU(1,4)^{(a)} : \quad \underset{-\beta}{\circ} \!—\! \underset{\alpha_2}{\bullet} \!—\! \underset{\alpha_4}{\bullet} \!—\! \underset{\alpha_3}{\bullet}$$

$$SU(1,4)^{(b)} : \quad \underset{-\beta}{\circ} \!—\! \underset{\alpha_2}{\bullet} \!—\! \underset{\alpha_4}{\bullet} \!—\! \underset{\alpha_5}{\bullet}$$

Unitary strata for E_7:

Q	z	w	T	vertex $\nu_{w,T}$	GKdim	p	e
$SU(1,1)$	1	v_1	$\varnothing$	$\omega_1 - 18\omega_7$	27	1	1
$SU(1,2)$	2	v_2	$\varnothing$	$\omega_3 - 18\omega_7$	27	2	2
$SU(1,3)$	3	v_3	$\varnothing$	$\omega_4 - 18\omega_7$	27	3	3
$SU(1,4)$	4	v_4	$\varnothing$	$\omega_2 + \omega_5 - 18\omega_7$	27	4	4
$SU(1,5)^{(a)}$	5	v_5	$\varnothing$	$2\omega_5 - 18\omega_7$	27	5	5
		v_7	$\{\alpha_3\}$	$\omega_5 - 15\omega_7$	27	1	1
$SU(1,5)^{(b)}$	5	v_6	$\varnothing$	$2\omega_2 + \omega_6 - 18\omega_7$	27	5	5
		v_7	$\{\alpha_2\}$	$\omega_2 + \omega_6 - 16\omega_7$	27	1	1
$SU(1,6)$	6	v_8	$\varnothing$	$3\omega_2 - 17\omega_7$	27	6	6
		v_{10}	$\{\alpha_2\}$	$2\omega_2 - 15\omega_7$	27	2	2
		v_{12}	$\{\alpha_4\}$	$\omega_2 - 13\omega_7$	27	2	3
$SO(2,10)$	5	v_9	$\{\alpha_4\}$	$\omega_6 - 14\omega_7$	27	1	2
	9	v_{16}	$\varnothing$	$5\omega_6 - 18\omega_7$	26	10	9
		v_{19}	$\{\alpha_1\}$	$4\omega_6 - 16\omega_7$	26	5	4
		v_{22}	$\{\alpha_3\}$	$3\omega_6 - 14\omega_7$	26	5	5
		v_{25}	$\{\alpha_4\}$	$2\omega_6 - 12\omega_7$	26	5	6
		v_{32}	$\{\alpha_2, \alpha_5\}$	$\omega_6 - 10\omega_7$	26	1	0
E_7	9	v_{37}	$\{\alpha_4, \alpha_6\}$	$-8\omega_7$	26	1	2
	13	v_{52}	$\{\alpha_4\}$	$-4\omega_7$	17	10	5
	17	v_{55}	$\varnothing$	0	1	27	0

The two cases with $Q = SU(1,5)$ correspond to the following Dynkin diagrams:

$$SU(1,5)^{(a)} : \quad \underset{-\beta}{\circ} \ -\!\!-\ \underset{\alpha_1}{\bullet} \ -\!\!-\ \underset{\alpha_3}{\bullet} \ -\!\!-\ \underset{\alpha_4}{\bullet} \ -\!\!-\ \underset{\alpha_2}{\bullet}$$

$$SU(1,5)^{(b)} : \quad \underset{-\beta}{\circ} \ -\!\!-\ \underset{\alpha_1}{\bullet} \ -\!\!-\ \underset{\alpha_3}{\bullet} \ -\!\!-\ \underset{\alpha_4}{\bullet} \ -\!\!-\ \underset{\alpha_5}{\bullet}$$

5. Computing reduced root systems

5.1. In sections 6 and 7 we give formulas for the generalized BGG resolutions, the Hilbert series, Bernstein degree, etc. of the unitary highest weight modules of the exceptional groups of E_6 and E_7, respectively. In order to find the explicit

resolution of the unitary highest weight module L_λ with highest weight λ one has to determine the reduced root system Δ_λ, which was introduced in [E]. In the following we describe an algorithm to find Δ_λ.

5.2. An algorithm to find Δ_λ. Begin with the set Δ_n^+ of positive noncompact roots described as an ordered set in 4.6. Given the highest weight λ of the unitary irreducible highest weight module we compute the scalar product $(\lambda + \rho, \beta^\vee)$ for all $\beta \in \Delta_n^+$. The $\beta \in \Delta_n^+$ for which $(\lambda + \rho, \beta^\vee) = 0$ are the singular (positive) roots for the parameter $\lambda + \rho$. This gives $\Psi_\lambda \cap \Delta^+$. We next determine the noncompact positive roots orthogonal to Ψ_λ and with positive integral inner product with $\lambda + \rho$. The set of all these roots is the set $\Delta_{n,\lambda}^+$. It inherits a partial ordering from the partial ordering of Δ_n^+. The simple roots of the root system Δ_λ are the roots that arise as differences of successive elements of Δ_n^+.

5.3. Two examples of the Δ_λ^+ calculation. We again consider the case of E_7 when $Q = SO(2,10)$ and $z = 9$ as in 4.7. In this case the points in the cone $\mathcal{C}_{Q,z}$ are given by $\lambda = a\omega_6 + (-2a - 8)\omega_7$ with $a \geq 1$. We now evaluate $(\lambda + \rho, \beta^\vee)$ for all $\beta \in \Delta_n^+$. We display these values in a diagram as follows. Start with the poset diagram of Δ_n^+. Then label each node of the diagram with the corresponding number $(\lambda + \rho, \beta^\vee)$. We are only interested in the nonnegative values and so we ignore the part of the diagram for which the values $(\lambda + \rho, \beta^\vee)$ are negative. What we obtain is a diagram as follows:

$$
\begin{array}{c}
\beta_0\ 9 \\
|\quad \alpha_1 \\
\beta_1\ 8 \\
|\quad \alpha_3 \\
\beta_2\ 7 \\
|\quad \alpha_4 \\
\beta_3\ 6 \\
\diagup^{\alpha_5}\quad \diagdown^{\alpha_2} \\
\beta_4\ 5 \qquad \beta_5\ 5 \\
\diagup^{\alpha_6}\ \diagdown^{\alpha_2}\ \diagup^{\alpha_5} \\
\beta_6\ 4\ -a \quad \beta_7\ 4 \\
\diagdown^{\alpha_2}\ \diagup^{\alpha_6}\ \diagdown^{\alpha_4} \\
\beta_8\ 3-a \qquad \beta_9\ 3 \\
\diagdown^{\alpha_4}\ \diagup^{\alpha_6}\ \diagdown^{\alpha_3} \\
\beta_{10}\ 2-a \quad \beta_{11}\ 2 \\
\diagup^{\alpha_5}\ \diagdown^{\alpha_3}\ \diagup^{\alpha_6}\ \diagdown^{\alpha_1} \\
\beta_{12}\ 1-a \quad \beta_{13}\ 1-a \quad \beta_{14}\ 1
\end{array}
$$
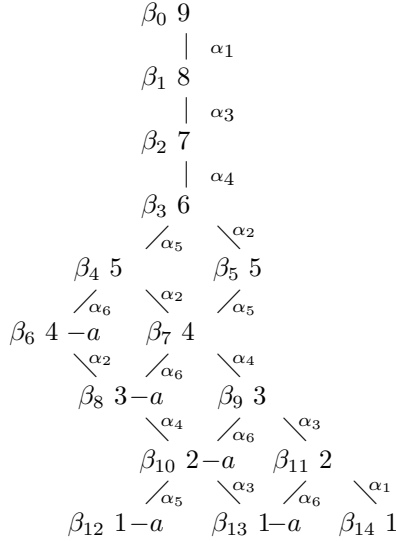
Let us first consider the regular strata corresponding to $a \geq 5$. Then Ψ_λ is empty and $\Delta_{\lambda,n}^+ = \{\beta_0, \beta_1, \beta_2, \beta_3, \beta_4, \beta_5, \beta_7, \beta_9, \beta_{11}, \beta_{14}\}$. Taking differences for adjacent β we find that the simple roots of Δ_λ^+ are $-\beta_0, \alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5$ which give the following Dynkin diagram:

$$
\begin{array}{ccccccccc}
-\beta_0 & & \alpha_1 & & \alpha_3 & & \alpha_4 & & \alpha_5 \\
\circ & \cdots & \bullet & - & \bullet & - & \bullet & - & \bullet \quad - \\
& & & & & & | & & \\
& & & & & & \bullet & & \\
& & & & & & \alpha_2 & &
\end{array}
$$
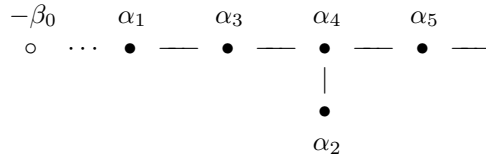
This is the diagram for $SO(2,10)$ and so Δ_λ is of type $SO(2,10)$.

Next we consider a different stratum of the same cone, say the stratum corresponding to $a = 2$. Then $\Psi_\lambda = \{\pm\beta_{10}\}$. From the data given in 4.6 we find that the noncompact roots positive at $\lambda + \rho$ and orthogonal to β_{10} are:

$$
\begin{array}{ll}
\beta_2 \;\bullet & \\
\quad| & \alpha_4 + \alpha_5 \\
\beta_4 \;\bullet & \\
\quad| & \alpha_2 \\
\beta_7 \;\bullet & \\
\quad| & \alpha_3 + \alpha_4 \\
\beta_{11} \;\bullet & \\
\quad| & \alpha_1 \\
\beta_{14} \;\bullet &
\end{array}
$$

Taking differences for adjacent β we obtain the following Dynkin diagram for Δ_λ^+:

$$
\begin{array}{ccccccccc}
-\beta_2 & & \alpha_4+\alpha_5 & & \alpha_2 & & \alpha_3+\alpha_4 & & \alpha_1 \\
\circ & \!\!\!\text{———}\!\!\! & \bullet & \!\!\!\text{———}\!\!\! & \bullet & \!\!\!\text{———}\!\!\! & \bullet & \!\!\!\text{———}\!\!\! & \bullet
\end{array}
$$

This is the diagram for $SU(1,4)$ and so Δ_λ is of type $SU(1,4)$ when $a = 2$.

6. Resolutions and Hilbert series for E_6

6.1. In this section we give explicit resolutions and Hilbert series for the unitary highest weight representation of E_6. We list the representations by cones and their strata as in table 4.9. In each case we first give the general form of a highest weight λ in the cone. The nonnegative root numbers, i.e., the values $(\lambda + \rho, \beta_i^\vee)$ for β_i noncompact, are given in diagramatic form as in 4.7. Below this diagram we then list the cases by strata. In each case we first give the singularity type of $\lambda + \rho$. More precisely, we give the parameters (w, T) for the stratum as explained in 4.4. We also list the vertex $v_{w,T}$ as explained in 4.4. Next we give the reduced root system Δ_λ which is from the root numbers as in section 5. We then give the minimal resolution of $L = L_\lambda$ by generalized Verma modules as in Corollary 3.8. We write N_i for the generalized Verma module of highest weights η_i. All highest weight are explicitly given. Furthermore, we give the degrees of the maps in the resolution. For example,

$$
0 \to N_4 \xrightarrow{+2} N_3 \xrightarrow{+1} N_2 \xrightarrow{+2} N_1 \xrightarrow{+1} N_0 \to L \to 0
$$

would mean that the map from N_1 to N_0 is homogenous of degree 1, the map from N_2 to N_1 is homogeneous of degree 2, etc. Here we consider the modules N_i and L as graded $S = S(\mathfrak{p}^-)$ modules. (For more details see [EH].) We then give the Hilbert series expressed as a rational function $h(t) = P(t)/(1 - t)^d$ with $P(t) \in \mathbb{Z}[t]$ such that $P(1) \neq 0$. When an explicit expression is too complicated to write down, we express the coefficients simply in terms of the dimensions of the finite dimensional irreducible $\mathfrak{k}$-modules F_{η_i}. We write d_i for $\dim(F_{\eta_i})$. The Gelfand-Kirilov dimension of the representation L is the degree of the denominator of the Hilbert series; the Bernstein degree is the (positive) integer $P(1)$. We give the explicit Hilbert series and Bernstein degree of the module $L = L_\lambda$ when $\lambda = \nu_{v,T}$ is the vertex of the stratum.

6.2. $Q_\lambda \simeq SU(1,1)$, $z = 1$.

Highest weight: $\lambda = \sum_{i=2}^{6} a_i\omega_i + (-2a_2 - 2a_3 - 3a_4 - 2a_5 - a_6 - 10)\omega_1$ with $a_2 \geq 1$ and $a_i \geq 0$ for $i \neq 2$.

Stratum: $(v, T) = (v_1, \varnothing)$ ($\lambda + \rho$ regular)   Vertex: $\omega_2 - 12\omega_1$.

Reduced root system: $\Delta_\lambda \simeq SU(1,1)$, $\Delta_{\lambda,n}^+ = \{\beta_0\}$.

Resolution:
$$0 \to N_1 \xrightarrow{+1} N_0 \to L \to 0.$$

$\eta_0 = \lambda$,

$\eta_1 = \lambda - \beta_0 = \lambda - \omega_2$.

Hilbert series: $h(t) = (d_0 - d_1 t)/(1 - t)^{16}$.

Gelfand-Kirillov dimension: 16.

Bernstein degree: $d_0 - d_1$.

Hilbert series at vertex: $h(t) = (16 - t)/(1 - t)^{16}$.

Bernstein degree at vertex: 15.

6.3. $Q_\lambda \simeq SU(1,2)$, $z = 2$.

Highest weight: $\lambda = a\omega_3 + b\omega_4 + c\omega_5 + d\omega_6 + (-2a - 3b - 2c - d - 9)\omega_1$ with $b \geq 1$ and $a, c, d \geq 0$.

Stratum: $(v, T) = (v_2, \varnothing)$ ($\lambda + \rho$ regular)   Vertex: $\omega_4 - 12\omega_1$.

Reduced root system: $\Delta_\lambda \simeq SU(1,2)$, $\Delta_{\lambda,n}^+ = \{\beta_0, \beta_1\}$.

Resolution:
$$0 \to N_2 \xrightarrow{+1} N_1 \xrightarrow{+1} N_0 \to L \to 0.$$

$\eta_0 = \lambda = a\omega_3 + b\omega_4 + c\omega_5 + d\omega_6 + (-2a - 3b - 2c - d - 9)\omega_1$,

$\eta_1 = \lambda - \beta_1 = \omega_2 + a\omega_3 + (b-1)\omega_4 + c\omega_5 + d\omega_6 + (-2a - 3b - 2c - d - 9)\omega_1$,

$\eta_2 = \lambda - \beta_1 - \beta_0 = a\omega_3 + (b-1)\omega_4 + c\omega_5 + d\omega_6 + (-2a - 3b - 2c - d - 9)\omega_1$.

Hilbert series: $h(t) = (d_0 - d_1 t + d_2 t^2)/(1 - t)^{16}$.

Gelfand-Kirillov dimension: 16.

Bernstein degree: $d_0 - d_1 + d_2$.

Hilbert series at vertex: $h(t) = (120 - 16t + t^2)/(1 - t)^{16}$.

Bernstein degree at vertex: 105.

6.4. $Q_\lambda \simeq SU(1,3)$, $z = 3$.

Highest weight: $\lambda = a\omega_3 + b\omega_5 + c\omega_6 + (-2a - 2b - c - 8)\omega_1$ with $a, b \geq 1$ and $c \geq 0$.

Root numbers:
$$\begin{array}{cl} \beta_0 & 3 \\ & |\,\alpha_2 \\ \beta_1 & 2 \\ & |\,\alpha_4 \\ \beta_2 & 1 \end{array}$$

Stratum: $(v, T) = (v_3, \varnothing)$ ($\lambda + \rho$ regular)   Vertex: $\omega_3 + \omega_5 - 12\omega_1$.

Reduced root system: $\Delta_\lambda \simeq SU(1,3)$, $\Delta_{\lambda,n}^+ = \{\beta_0, \beta_1, \beta_2\}$.

Resolution:
$$0 \to N_3 \xrightarrow{+1} N_2 \xrightarrow{+1} N_1 \xrightarrow{+1} N_0 \to L \to 0.$$

$\eta_0 = \lambda = a\omega_3 + b\omega_5 + c\omega_6 + (-2a - 2b - c - 8)\omega_1$,

$\eta_1 = \lambda - \beta_2 = (a-1)\omega_3 + \omega_4 + (b-1)\omega_5 + c\omega_6 + (-2a - 2b - c - 8)\omega_1$,

$\eta_2 = \lambda - \beta_2 - \beta_1 = \omega_2 + (a-1)\omega_3 + (b-1)\omega_5 + c\omega_6 + (-2a - 2b - c - 8)\omega_1$,

$\eta_3 = \lambda - \beta_2 - \beta_1 - \beta_0 = (a-1)\omega_3 + (b-1)\omega_5 + c\omega_6 + (-2a - 2b - c - 8)\omega_1$.

Hilbert series: $h(t) = (d_0 - d_1 t + d_2 t^2 - d_3 t^3)/(1-t)^{16}$.
Gelfand-Kirillov dimension: 16.
Bernstein degree: $d_0 - d_1 + d_2 - d_3$.
Hilbert series at vertex: $h(t) = (560 - 120t + 16t^2 - t^3)/(1-t)^{16}$.
Bernstein degree at vertex: 455.

6.5. $Q_\lambda \simeq SU(1,4)^{(a)}$, $z = 4$.
Highest weight: $\lambda = a\omega_5 + b\omega_6 + (-2a - b - 7)\omega_1$ with $a \geq 1$ and $b \geq 0$.
Root numbers:

$$
\begin{array}{c}
\beta_0 \ 4 \\
| \ \alpha_2 \\
\beta_1 \ 3 \\
| \ \alpha_4 \\
\beta_2 \ 2 \\
{}^{\displaystyle \swarrow \alpha_3} \quad {}^{\displaystyle \searrow \alpha_5} \\
\beta_3 \ 1 \qquad \beta_4 \ 1 \ -a
\end{array}
$$

$a > 1$:
Stratum: $(v, T) = (v_4, \varnothing)$ ($\lambda + \rho$ regular)   Vertex: $2\omega_5 - 11\omega_1$.
Reduced root system: $\Delta_\lambda \simeq SU(1,4)$, $\Delta^+_{\lambda,n} = \{\beta_0, \beta_1, \beta_2, \beta_3\}$.
Resolution:

$$0 \to N_4 \xrightarrow{+1} N_3 \xrightarrow{+1} N_2 \xrightarrow{+1} N_1 \xrightarrow{+1} N_0 \to L \to 0.$$

$\eta_0 = \lambda = a\omega_5 + b\omega_6 + (-2a - b - 7)\omega_1$,
$\eta_1 = \lambda - \beta_3 = \omega_3 + (a-1)\omega_5 + b\omega_6 + (-2a - b - 8)\omega_1$,
$\eta_2 = \lambda - \beta_3 - \beta_2 = \omega_4 + (a-2)\omega_5 + b\omega_6 + (-2a - b - 8)\omega_1$,
$\eta_3 = \lambda - \beta_3 - \beta_2 - \beta_1 = \omega_2 + (a-2)\omega_5 + b\omega_6 + (-2a - b - 8)\omega_1$,
$\eta_4 = \lambda - \beta_2 - \beta_1 - \beta_1 - \beta_0 = (a-2)\omega_5 + b\omega_6 + (-2a - b - 8)\omega_1$.
Hilbert series: $h(t) = (\sum_{i=0}^4 (-1)^i d_i t^i)/(1-t)^{16}$.
Gelfand-Kirillov dimension: 16.
Bernstein degree: $\sum_{i=0}^4 (-1)^i d_i$.
Hilbert series at vertex: $h(t) = (770 - 560 + 120t^2 - 16t^3 + t^4)/(1-t)^{16}$.
Bernstein degree at vertex: 315.
$a = 1$:
Stratum: $(v, T) = (v_6, \{\alpha_3\})$ ($\lambda + \rho$ semi-regular)   Vertex: $\omega_5 - 9\omega_1$.
Reduced root system: $\Delta_\lambda \simeq SU(1,1)$, $\Delta^+_{\lambda,n} = \{\beta_3\}$.
Resolution:

$$0 \to N_1 \xrightarrow{+1} N_0 \to L \to 0.$$

$\eta_0 = \lambda = \omega_5 + b\omega_6 + (-b - 9)\omega_1$,
$\eta_1 = \lambda - \beta_3 = \omega_3 + b\omega_6 + (-b - 10)\omega_1$,
Hilbert series: $h(t) = (d_0 - d_1 t)/(1-t)^{27}$.
Gelfand-Kirillov dimension: 16.
Bernstein degree: $d_0 - d_1 = (4b+29)(b+7)(b+6)(b+5)(b+4)(b+3)(b+1)/2520$.
Hilbert series at vertex: $h(t) = (45 - 16t)/(1-t)^{16}$.
Bernstein degree at vertex: 29.

6.6. $Q_\lambda \simeq SU(1,4)^{(b)}$, $z = 4$.
Highest weight: $\lambda = a\omega_3 + b\omega_6 + (-2a - b - 7)\omega_1$ with $a, b \geq 1$.

Root numbers:

$$
\begin{array}{c}
\beta_0\ 4 \\
|\ \ \alpha_2 \\
\beta_1\ 3 \\
|\ \ \alpha_4 \\
\beta_2\ 2 \\
{}^{\alpha_3}\diagup\ \ \diagdown^{\alpha_5} \\
\beta_3\ 1\ -a \qquad \beta_4\ 1
\end{array}
$$

$a > 1$:

    Stratum: $(v, T) = (v_5, \varnothing)$ $(\lambda + \rho$ regular$)$ Vertex: $2\omega_3 + \omega_6 - 12\omega_1$.

    Reduced root system: $\Delta_\lambda \simeq SU(1,4)$, $\Delta_{\lambda,n}^+ = \{\beta_0, \beta_1, \beta_2, \beta_4\}$.

    Resolution:

$$
0 \to N_4 \xrightarrow{+1} N_3 \xrightarrow{+1} N_2 \xrightarrow{+1} N_1 \xrightarrow{+1} N_0 \to L \to 0.
$$

    $\eta_0 = \lambda = a\omega_3 + b\omega_6 + (-2a - b - 7)\omega_1$,
    $\eta_1 = \lambda - \beta_4 = (a-1)\omega_3 + \omega_5 + (b-1)\omega_6 + (-2a - b - 7)\omega_1$,
    $\eta_2 = \lambda - \beta_4 - \beta_2 = (a-2)\omega_3 + \omega_4 + (b-1)\omega_6 + (-2a - b - 7)\omega_1$,
    $\eta_3 = \lambda - \beta_4 - \beta_2 - \beta_1 = \omega_2 + (a-2)\omega_3 + (b-1)\omega_6 + (-2a - b - 7)\omega_1$,
    $\eta_4 = \lambda - \beta_4 - \beta_2 - \beta_1 - \beta_0 = (a-2)\omega_3 + (b-1)\omega_6 + (-2a - b - 7)\omega_1$.

    Hilbert series: $h(t) = (\sum_{i=0}^5 (-1)^i d_i t^i)/(1-t)^{16}$.

    Gelfand-Kirillov dimension: 16.

    Bernstein degree: $\sum_{i=0}^4 (-1)^i d_i$.

    Hilbert series at vertex: $h(t) = (1050 - 560t + 120t^2 - 16t^3 + t^4)/(1-t)^{16}$.

    Bernstein degree at vertex: 595.

$a = 1$:

    Stratum: $(v, T) = (v_6, \{\alpha_2\})$. $(\lambda + \rho$ semi-regular$)$ Vertex: $\omega_3 + \omega_6 - 10\omega_1$.

    Reduced root system: $\Delta_\lambda \simeq SU(1,1)$, $\Delta_{\lambda,n}^+ = \{\beta_4\}$.

    Resolution:

$$
0 \to N_1 \xrightarrow{+1} N_0 \to L \to 0.
$$

    $\eta_0 = \lambda = \omega_3 + b\omega_6 + (-b - 9)\omega_1$,
    $\eta_1 = \lambda - \beta_4 = \omega_5 + (b-1)\omega_6 + (-b - 9)\omega_1$.

    Hilbert series: $h(t) = (d_0 - d_1 t)/(1-t)^{16}$.

    Gelfand-Kirillov dimension: 16.

    Bernstein degree: $d_0 - d_1 = (4b+7)(b+8)(b+6)(b+5)(b+4)(b+3)(b+2)/2520$.

    Hilbert series at vertex: $h(t) = (144 - 45t)/(1-t)^{16}$.

    Bernstein degree at vertex: 99.

6.7. $Q_\lambda \simeq SU(1,5)$, $z = 5$.

    Highest weight: $\lambda = a\omega_3 + (-2a - 6)\omega_1$ with $a \geq 1$.

    Root numbers:

$$
\begin{array}{c}
\beta_0\ 5 \\
|\ \ \alpha_2 \\
\beta_1\ 4 \\
|\ \ \alpha_4 \\
\beta_2\ 3 \\
{}^{\alpha_3}\diagup\ \ \diagdown^{\alpha_5} \\
\beta_3\ 2\ -a \qquad \beta_4\ 2 \\
\diagdown^{\alpha_5}\ \ {}^{\alpha_3}\diagup\ \ \diagdown^{\alpha_6} \\
\beta_5\ 1\ -a \qquad \beta_6\ 1
\end{array}
$$

$a > 2$:

Stratum: $(v, T) = (v_5, \varnothing)$ $(\lambda + \rho$ regular$)$ Vertex: $3\omega_3 - 12\omega_1$.

Reduced root system: $\Delta_\lambda \simeq SU(1,5)$, $\Delta^+_{\lambda,n} = \{\beta_0, \beta_1, \beta_2, \beta_4, \beta_6\}$.

Resolution:

$$0 \to N_5 \xrightarrow{+1} N_4 \xrightarrow{+1} N_3 \xrightarrow{+1} N_2 \xrightarrow{+1} N_1 \xrightarrow{+1} N_0 \to L \to 0.$$

$\eta_0 = \lambda = a\omega_3 + (-2a - 6)\omega_1$,

$\eta_1 = \lambda - \beta_6 = (a-1)\omega_3 + \omega_6 + (-2a - 6)\omega_1$,

$\eta_2 = \lambda - \beta_6 - \beta_4 = (a-2)\omega_3 + \omega_5 + (-2a - 6)\omega_1$,

$\eta_3 = \lambda - \beta_6 - \beta_4 - \beta_2 = (a-3)\omega_3 + \omega_4 + (-2a - 6)\omega_1$,

$\eta_4 = \lambda - \beta_6 - \beta_4 - \beta_2 - \beta_1 = \omega_2 + (a-3)\omega_3 + (-2a - 6)\omega_1$,

$\eta_5 = \lambda - \beta_6 - \beta_4 - \beta_2 - \beta_1 - \beta_0 = (a-3)\omega_3 + (-2a - 6)\omega_1$.

Hilbert series: $h(t) = (\sum_{i=0}^{5}(-1)^i d_i t^i)/(1-t)^{16}$.

Gelfand-Kirillov dimension: 16.

Bernstein degree: $\sum_{i=0}^{6}(-1)^i d_i = (2a+5)(a+4)(a+3)(a+2)(a+1)/120$.

Hilbert series at vertex: $h(t) = (672 - 1050t + 560t^2 - 120t^3 + 16t^4 - t^5)/(1-t)^{16}$.

Bernstein degree at vertex: 77.

$a = 2$:

Stratum: $(v, T) = (v_9, \{\alpha_2\})$ $(\lambda + \rho$ semi-regular$)$ Vertex: $2\omega_3 - 10\omega_1$.

Reduced root system: $\Delta_\lambda \simeq SU(1,2)$, $\Delta^+_{\lambda,n} = \{\beta_4, \beta_6\}$.

Resolution:

$$0 \to N_2 \xrightarrow{+1} N_1 \xrightarrow{+1} N_0 \to L \to 0.$$

$\eta_0 = \lambda = 2\omega_3 - 10\omega_1$,

$\eta_1 = \lambda - \beta_6 = \omega_3 + \omega_6 - 10\omega_1$,

$\eta_2 = \lambda - \beta_6 - \beta_4 = \omega_5 - 10\omega_1$.

Hilbert series: $h(t) = (126 - 144t + 45t^2)/(1-t)^{16}$.

Gelfand-Kirillov dimension: 16.

Bernstein degree: $126 - 144 + 45 = 27$.

$a = 1$:

Stratum: $(v, T) = (v_{11}, \{\alpha_4\})$ $(\lambda + \rho$ semi-regular$)$ Vertex: $\omega_3 - 8\omega_1$.

Reduced root system: $\Delta_\lambda \simeq SU(1,2)$, $\Delta^+_{\lambda,n} = \{\beta_2, \beta_6\}$.

Resolution:

$$0 \to N_2 \xrightarrow{+2} N_1 \xrightarrow{+1} N_0 \to L \to 0.$$

$\eta_0 = \lambda = \omega_3 - 8\omega_1$,

$\eta_1 = \lambda - \beta_6 = \omega_6 - 8\omega_1$,

$\eta_2 = s_3 s_5(\lambda + \rho - \beta_6 - 2\beta_2) - \rho = -9\omega_1$.

Hilbert series: $h(t) = (16 - 10t + t^3)/(1-t)^{16}$.

Gelfand-Kirillov dimension: 16.

Bernstein degree: $16 - 10 + 1 = 7$.

6.8. $Q_\lambda \simeq SO(2,8)$, $z = 4$ (first reduction point).

Highest weight: $\lambda = a\omega_6 + (-a - 7)\omega_1$ with $a \geq 1$.

Root numbers:

$$
\begin{array}{c}
\beta_0 \ 4 \\
| \ \alpha_2 \\
\beta_1 \ 3 \\
| \ \alpha_4 \\
\beta_2 \ 2 \\
{}^{\alpha_3}\diagup \quad \diagdown^{\alpha_5} \\
\beta_3 \ 1 \qquad \beta_4 \ 1 \\
\diagdown^{\alpha_5} \quad \diagup_{\alpha_3} \\
\beta_5 \ 0
\end{array}
$$

Stratum: $(v,T) = (v_8, \{\alpha_4\})$ $(\lambda + \rho$ semi-regular$)$ Vertex: $\omega_6 - 8\omega_1$.
Reduced root system: $\Delta^+_{\lambda,n} = \{\beta_2\}$, $\Delta_\lambda \simeq SU(1,1)$.
Resolution:

$$0 \to N_1 \xrightarrow{+2} N_0 \to L \to 0.$$

$\eta_0 = \lambda = a\omega_6 + (-a-7)\omega_1,$
$\eta_1 = s_3 s_5(\lambda + \rho - 2\beta_2) - \rho = (a-1)\omega_6 + (-a-8)\omega_1,$
Hilbert series: $h(t) = (d_0 - d_1 t^2)/(1-t)^{16}$.
Gelfand-Kirillov dimension: 16.
Bernstein degree: $d_0 - d_1 = (2a+7)(a+6)(a+5)(a+4)(a+3)(a+2)(a+1)/5040$.
Hilbert series at vertex: $h(t) = (10 - t^2)/(1-t)^{16}$.
Bernstein degree at vertex: 9.

6.9. $Q_\lambda \simeq SO(2,8)$, $z = 7$ (second reduction point).
Highest weight: $\lambda = a\,\omega_6 + (-a-4)\omega_1$ with $a \geq 1$.
Root numbers:

$$
\begin{array}{c}
\beta_0 \ 7 \\
| \ \alpha_2 \\
\beta_1 \ 6 \\
| \ \alpha_4 \\
\beta_2 \ 5 \\
{}^{\alpha_3}\diagup \quad \diagdown^{\alpha_5} \\
\beta_3 \ 4 \qquad \beta_4 \ 4 \\
\diagdown^{\alpha_5} \quad {}^{\alpha_3}\diagup \quad \diagdown^{\alpha_6} \\
\beta_5 \ 3 \qquad \beta_6 \ 3 \ -a \\
{}^{\alpha_4}\diagup \quad \diagdown^{\alpha_6} \quad {}^{\alpha_3}\diagup \\
\beta_7 \ 2 \qquad \beta_8 \ 2 \ -a \\
{}^{\alpha_2}\diagup \quad \diagdown^{\alpha_6} \quad {}^{\alpha_4}\diagup \\
\beta_9 \ 1 \qquad \beta_{10} \ 1 \ -a
\end{array}
$$

$a > 3$:

Stratum: $(v,T) = (v_{12}, \varnothing)$ $(\lambda + \rho$ regular$)$ Vertex: $4\omega_6 - 8\omega_1$.
Reduced root system: $\Delta_\lambda \simeq SU(2,8)$, $\Delta^+_{\lambda,n} = \{\beta_0, \beta_1, \beta_2, \beta_3, \beta_4, \beta_5, \beta_7, \beta_9\}$.
Resolution:

$$0 \to N_8 \xrightarrow{+1} N_7 \xrightarrow{+1} \cdots \xrightarrow{+1} N_5 \xrightarrow{+1} \overset{N_4}{\underset{N_4'}{\bigoplus}} \xrightarrow{+1} N_3 \xrightarrow{+1} \cdots \xrightarrow{+1} N_0 \to L \to 0.$$

$\eta_0 = \lambda = a\omega_6 + (-a-4)\omega_1,$
$\eta_1 = \lambda - \beta_9 = \omega_2 + (a-1)\omega_6 + (-a-5)\omega_1,$
$\eta_2 = \lambda - \beta_9 - \beta_7 = \omega_4 + (a-2)\omega_6 + (-a-6)\omega_1,$

$$\eta_3 = \lambda - \beta_9 - \beta_7 - \beta_5 = \omega_2 + \omega_5 + (a-3)\omega_6 + (-a-7)\omega_1,$$
$$\eta_4 = \lambda - \beta_9 - \beta_7 - \beta_5 - \beta_4 = 2\omega_5 + (a-4)\omega_6 + (-a-7)\omega_1,$$
$$\eta_4' = \lambda - \beta_9 - \beta_7 - \beta_5 - \beta_3 = 2\omega_2 + (a-3)\omega_6 + (-a-8)\omega_1,$$
$$\eta_5 = \lambda - \beta_9 - \beta_7 - \beta_5 - \beta_4 - \beta_3 = \omega_2 + \omega_5 + (a-4)\omega_6 + (-a-8)\omega_1,$$
$$\eta_6 = \lambda - \beta_9 - \beta_7 - \beta_5 - \beta_4 - \beta_3 - \beta_2 = \omega_4 + (a-4)\omega_6 + (-a-8)\omega_1,$$
$$\eta_7 = \lambda - \beta_9 - \beta_7 - \beta_5 - \beta_4 - \beta_3 - \beta_2 - \beta_1 = \omega_2 + (a-4)\omega_6 + (-a-8)\omega_1,$$
$$\eta_8 = \lambda - \beta_9 - \beta_7 - \beta_5 - \beta_4 - \beta_3 - \beta_2 - \beta_1 - \beta_0 = (a-4)\omega_6 + (-a-8)\omega_1.$$

Hilbert series: $h(t) = (d_0 - d_1 t + \cdots + (d_4 + d_4')t^4 - d_5 t^5 + \cdots + d_8 t^8)/(1-t)^{16}.$

Gelfand-Kirillov dimension: 16.

Bernstein degree: 1.

Hilbert series at vertex:
$$h(t) = (600 - 2640t + 4312t^2 - 3696t^3 + 1820t^4$$
$$- 560t^5 + 120t^6 - 16t^7 + t^8)/(1-t)^{16}.$$

$a = 3$:

   Stratum: $(v, T) = (v_{15}, \{\alpha_1\})$ ($\lambda + \rho$ semi-regular)   Vertex: $3\omega_6 - 7\omega_1$.

   Reduced root system: $\Delta_\lambda \simeq SU(1,4)$, $\Delta_{\lambda,n}^+ = \{\beta_3, \beta_5, \beta_7, \beta_9\}$.

   Resolution:
$$0 \to N_4 \xrightarrow{+1} N_3 \xrightarrow{+1} N_2 \xrightarrow{+1} N_1 \xrightarrow{+1} N_0 \to L \to 0.$$

$$\eta_0 = \lambda = 3\omega_6 - 7\omega_1,$$
$$\eta_1 = \lambda - \beta_9 = \omega_2 + 2\omega_6 - 8\omega_1,$$
$$\eta_2 = \lambda - \beta_9 - \beta_7 = \omega_4 + \omega_6 - 9\omega_1,$$
$$\eta_3 = \lambda - \beta_9 - \beta_7 - \beta_5 = \omega_2 + \omega_5 - 10\omega_1,$$
$$\eta_4 = \lambda - \beta_9 - \beta_7 - \beta_5 - \beta_3 = 2\omega_2 - 11\omega_1.$$

Hilbert series: $h(t) = (210 - 720t + 945t^2 - 560t^3 + 126t^4)/(1-t)^{16}.$

Gelfand-Kirillov dimension: 16.

Bernstein degree: $210 - 720 + 945 - 560 + 126 = 1.$

$a = 2$:

   Stratum: $(v, T) = (v_{17}, \{\alpha_3\})$ ($\lambda + \rho$ semi-regular)   Vertex: $2\omega_6 - 6\omega_1$.

   Reduced root system: $\Delta_\lambda \simeq SU(1,4)$, $\Delta_{\lambda,n}^+ = \{\beta_2, \beta_4, \beta_7, \beta_9\}$.

   Resolution:
$$0 \to N_4 \xrightarrow{+1} N_3 \xrightarrow{+2} N_2 \xrightarrow{+1} N_1 \xrightarrow{+1} N_0 \to L \to 0.$$

$$\eta_0 = \lambda = 2\omega_6 - 6\omega_1,$$
$$\eta_1 = \lambda - \beta_9 = \omega_2 + \omega_6 - 7\omega_1,$$
$$\eta_2 = \lambda - \beta_9 - \beta_7 = \omega_4 - 8\omega_1,$$
$$\eta_3 = s_3 s_6(\lambda + \rho - \beta_9 - \beta_7 - 2\beta_4) - \rho = \omega_5 - 9\omega_1,$$
$$\eta_4 = s_3 s_6(\lambda + \rho - \beta_9 - \beta_7 - 2\beta_4 - \beta_2) - \rho = \omega_2 - 10\omega_1.$$

Hilbert series: $h(t) = (54 - 144t + 120t^2 - 45t^4 + 16t^5)/(1-t)^{16}.$

Gelfand-Kirillov dimension: 16.

Bernstein degree: $54 - 144 + 120 - 45 + 16 = 1.$

$a = 1$:

   Stratum: $(v, T) = (v_{19}, \{\alpha_4\})$ ($\lambda + \rho$ semi-regular)   Vertex: $\omega_6 - 5\omega_1$.

   Reduced root system: $\Delta_\lambda \simeq SU(1,4)$, $\Delta_{\lambda,n}^+ = \{\beta_1, \beta_4, \beta_5, \beta_9\}$.

   Resolution:
$$0 \to N_4 \xrightarrow{+2} N_3 \xrightarrow{+1} N_2 \xrightarrow{+2} N_1 \xrightarrow{+1} N_0 \to L \to 0.$$

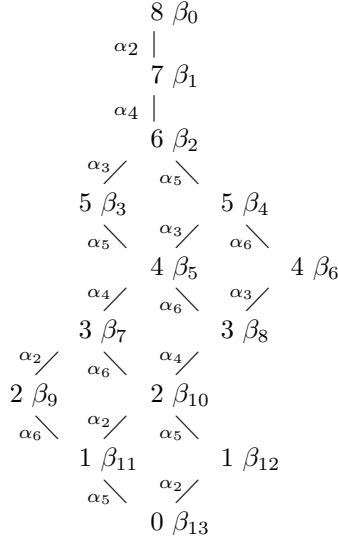$$\eta_0 = \lambda = \omega_6 - 5\omega_1,$$

$\eta_1 = \lambda - \beta_9 = \omega_2 - 6\omega_1,$
$\eta_2 = s_4 s_6(\lambda + \rho - \beta_9 - 2\beta_5) - \rho = \omega_3 - 8\omega_1,$
$\eta_3 = s_4 s_6(\lambda + \rho - \beta_9 - 2\beta_5 - \beta_4) - \rho = \omega_6 - 8\omega_1,$
$\eta_4 = s_3 s_5 s_4 s_6(\lambda + \rho - \beta_9 - 2\beta_5 - \beta_4 - 2\beta_1) - \rho = -9\omega_1.$
Hilbert series: $h(t) = (10 - 16t + 16t^3 - 10t^4 + t^6)/(1-t)^{16}.$
Gelfand-Kirillov dimension: 16.
Bernstein degree: $10 - 16 + 16 - 10 + 1 = 1.$

6.10. $Q_\lambda \simeq E_6$, $z = 8$.
  Highest weight: $\lambda = -3\omega_1.$
  Root numbers:

$$
\begin{array}{c}
8\ \beta_0 \\
\alpha_2\ | \\
7\ \beta_1 \\
\alpha_4\ | \\
6\ \beta_2 \\
\alpha_3 \diagup \quad \alpha_5 \diagdown \\
5\ \beta_3 \qquad\qquad 5\ \beta_4 \\
\alpha_5 \diagdown \quad \alpha_3 \diagup \quad \alpha_6 \diagdown \\
4\ \beta_5 \qquad\qquad 4\ \beta_6 \\
\alpha_4 \diagup \quad \alpha_6 \diagdown \quad \alpha_3 \diagup \\
3\ \beta_7 \qquad\qquad 3\ \beta_8 \\
\alpha_2 \diagup \quad \alpha_6 \diagdown \quad \alpha_4 \diagup \\
2\ \beta_9 \qquad\qquad 2\ \beta_{10} \\
\alpha_6 \diagdown \quad \alpha_2 \diagup \quad \alpha_5 \diagdown \\
1\ \beta_{11} \qquad\qquad 1\ \beta_{12} \\
\alpha_5 \diagdown \quad \alpha_2 \diagup \\
0\ \beta_{13}
\end{array}
$$

Stratum: $(v, T) = (v_{24}, \{\alpha_4\})$ ($\lambda + \rho$ semi-regular)   Vertex: $-3\omega_1.$
Reduced root system: $\Delta_\lambda \simeq SU(1,5)$, $\Delta_{\lambda,n}^+ = \{\beta_0, \beta_2, \beta_3, \beta_7, \beta_{10}\}.$
Resolution:

$$0 \to N_5 \xrightarrow{+2} N_4 \xrightarrow{+1} N_3 \xrightarrow{+2} N_2 \xrightarrow{+1} N_1 \xrightarrow{+2} N_0 \to L \to 0.$$

$\eta_0 = \lambda = -3\omega_1,$
$\eta_1 = s_5 s_2(\lambda + \rho - 2\beta_{10}) - \rho = \omega_6 - 5\omega_1,$
$\eta_2 = s_5 s_2(\lambda + \rho - \beta_6 - \beta_4) - \rho = \omega_2 - 6\omega_1,$
$\eta_3 = s_6 s_4 s_5 s_2(\lambda + \rho - \beta_6 - \beta_4 - \beta_2) - \rho = \omega_3 - 8\omega_1,$
$\eta_4 = s_6 s_4 s_5 s_2(\lambda + \rho - \beta_6 - \beta_4 - \beta_2 - \beta_1) - \rho = \omega_6 - 8\omega_1,$
$\eta_5 = s_5 s_3 s_6 s_4 s_5 s_2(\lambda + \rho - \beta_6 - \beta_4 - \beta_2 - \beta_1 - \beta_0) - \rho = -9\omega_1.$
Hilbert series:

$$h(t) = (1 - 10t^2 + 16t^3 - 16t^5 + 10t^6 - t^8)/(1-t)^{16}$$
$$= (1 + 5t + 5t^2 + t^3)/(1-t)^{11}.$$

Gelfand-Kirillov dimension: 11.
Bernstein degree: 12.
Special feature: $\lambda + \rho$ is quasi-dominant (Wallach representation).

7. Resolutions and Hilbert series for E_7

7.1. The same remarks apply as in 6.1.

7.2. $Q_\lambda \simeq SU(1,1)$, $z = 1$.

Highest weight $\lambda = \sum_{i=1}^{6} a_i\omega_i + (-2a_1 - 2a_2 - 3a_3 - 4a_4 - 3a_5 - 2a_6 - 16)\omega_7$
with $a_1 \geq 1$ and $a_i \geq 0$ for $i \neq 1$.
Stratum: $(v, T) = (v_1, \varnothing)$ $(\lambda + \rho$ regular$)$ Vertex: $\omega_1 - 18\omega_7$.
Reduced root system: $\Delta_\lambda \simeq SU(1,1)$, $\Delta_{\lambda,n}^+ = \{\beta_0\}$.
Resolution:

$$0 \to N_1 \xrightarrow{+1} N_0 \to L \to 0.$$

$\eta_0 = \lambda$,
$\eta_1 = \lambda - \beta_0 = \lambda - \omega_1$.
Hilbert series: $h(t) = (d_0 - d_1 t)/(1 - t)^{27}$.
Gelfand-Kirillov dimension: 27.
Bernstein degree: $d_0 - d_1$.
Hilbert series at vertex: $h(t) = (27 - t)/(1 - t)^{27}$.
Bernstein degree at vertex: 26.

7.3. $Q_\lambda \simeq SU(1,2)$, $z = 2$. .

Highest weight: $\lambda = \sum_{i=2}^{6} a_i\omega_i + (-2a_2 - 3a_3 - 4a_4 - 3a_5 - 2a_6 - 15)\omega_7$ with
$a_3 \geq 1$ and $a_i \geq 0$ for $i \neq 3$.
Stratum: $(v, T) = (v_2, \varnothing)$ $(\lambda + \rho$ regular$)$ Vertex: $\omega_3 - 18\omega_7$.
Reduced root system: $\Delta_\lambda \simeq SU(1,2)$, $\Delta_{\lambda,n}^+ = \{\beta_0, \beta_1\}$.
Resolution:

$$0 \to N_2 \xrightarrow{+1} N_1 \xrightarrow{+1} N_0 \to L \to 0.$$

$\eta_0 = \lambda$,
$\eta_1 = \lambda - \beta_1 = \lambda + \omega_1 - \omega_3$,
$\eta_2 = \lambda - \beta_1 - \beta_0 = \lambda - \omega_3$.
Hilbert series: $h(t) = (d_0 - d_1 t + d_2 t^2)/(1 - t)^{27}$.
Gelfand-Kirillov dimension: 27.
Bernstein degree: $d_0 - d_1 + d_2$.
Hilbert series at vertex: $h(t) = (351 - 27t + t^2)/(1 - t)^{27}$.
Bernstein degree at vertex: 325.

7.4. $Q_\lambda \simeq SU(1,3)$, $z = 3$.

Highest weight: $\lambda = a\omega_2 + b\omega_4 + c\omega_5 + d\omega_6 + (-2a - 4b - 3c - 2d - 14)\omega_7$ with
$b \geq 1$ and $a, c, d \geq 0$.
Stratum: $(v, T) = (v_3, \varnothing)$ $(\lambda + \rho$ regular$)$ Vertex: $\omega_4 - 18\omega_7$.
Reduced root system: $\Delta_\lambda \simeq SU(1,3)$, $\Delta_{\lambda,n}^+ = \{\beta_0, \beta_1, \beta_2\}$.
Resolution :

$$0 \to N_3 \xrightarrow{+1} N_2 \xrightarrow{+1} N_1 \xrightarrow{+1} N_0 \to L \to 0.$$

$\eta_0 = \lambda$,
$\eta_1 = \lambda - \beta_2 = \lambda + \omega_3 - \omega_4$,
$\eta_2 = \lambda - \beta_2 - \beta_1 = \lambda + \omega_1 - \omega_4$,
$\eta_3 = \lambda - \beta_2 - \beta_1 - \beta_0 = \lambda - \omega_4$.
Hilbert series: $h(t) = (d_0 - d_1 t + d_2 t^2 + d_3 t^3)/(1 - t)^{27}$.
Gelfand-Kirillov dimension: 27.
Bernstein degree: $d_0 - d_1 + d_2 - d_3$.
Hilbert series at vertex: $h(t) = (2925 - 351t + 27t^2 + t^3)/(1 - t)^{27}$.
Bernstein degree at vertex: 2600.

7.5. $Q_\lambda \simeq SU(1,4)$, $z = 4$.

　　Highest weight: $\lambda = a\omega_2 + b\omega_5 + c\omega_6 + (-2a - 3b - 2c - 13)\omega_7$ with $a, b \geq 1$ and
　　$c \geq 0$.

　　Stratum: $(v, T) = (v_4, \varnothing)$　$(\lambda + \rho$ regular$)$　　　Vertex: $\omega_2 + \omega_5 - 18\omega_7$.

　　Reduced root system: $\Delta_\lambda \simeq SU(1,4)$, $\Delta_{\lambda,n}^+ = \{\beta_0, \beta_1, \beta_2, \beta_3\}$.

　　Resolution:

$$0 \to N_4 \xrightarrow{+1} N_3 \xrightarrow{+1} N_2 \xrightarrow{+1} N_1 \xrightarrow{+1} N_0 \to L \to 0.$$

　　$\eta_0 = \lambda$,

　　$\eta_1 = \lambda - \beta_3 = \lambda - \omega_2 + \omega_4 - \omega_5$,

　　$\eta_2 = \lambda - \beta_3 - \beta_2 = \lambda - \omega_2 + \omega_3 - \omega_5$,

　　$\eta_3 = \lambda - \beta_3 - \beta_2 - \beta_1 = \lambda + \omega_1 - \omega_2 - \omega_5$,

　　$\eta_4 = \lambda - \beta_3 - \beta_2 - \beta_1 - \beta_0 = \lambda - \omega_2 - \omega_5$.

　　Hilbert series: $h(t) = (d_0 - d_1 t + d_2 t^2 - d_3 t^3 + d_4 t^4)/(1 - t)^{27}$.

　　Gelfand-Kirillov dimension: 27.

　　Bernstein degree: $d_0 - d_1 + d_2 - d_3 + d_4$.

　　Hilbert series at vertex: $h(t) = (17550 - 2925t + 351t^2 - 27t^3 + t^4)/(1 - t)^{27}$.

　　Bernstein degree at vertex: 14950.

7.6. $Q_\lambda \simeq SU(1,5)^{(b)}$, $z = 5$.

　　Highest weight: $\lambda = a\omega_2 + b\omega_6 + (-2a - 2b - 12)\omega_7$ with $a, b \geq 1$.

　　Root numbers:

$$
\begin{array}{c}
\beta_0 \ 5 \\
| \ \alpha_1 \\
\beta_1 \ 4 \\
| \ \alpha_3 \\
\beta_2 \ 3 \\
| \ \alpha_4 \\
\beta_3 \ 2 \\
{}^{\diagup \alpha_5} \quad {}^{\diagdown \alpha_2} \\
\beta_4 \ 1 \qquad \beta_5 \ 1 \ -a
\end{array}
$$

$a > 1$:

　　Stratum: $(v, T) = (v_6, \varnothing)$　$(\lambda + \rho$ regular$)$　　　Vertex: $2\omega_2 + \omega_6 - 18\omega_7$.

　　Reduced root system: $\Delta_\lambda \simeq SU(1,5)$, $\Delta_{\lambda,n}^+ = \{\beta_0, \beta_1, \beta_2, \beta_3, \beta_4\}$.

　　Resolution:

$$0 \to N_5 \xrightarrow{+1} N_4 \xrightarrow{+1} N_3 \xrightarrow{+1} N_2 \xrightarrow{+1} N_1 \xrightarrow{+1} N_0 \to L \to 0.$$

　　$\eta_0 = \lambda = a\omega_2 + b\omega_6 + (-2a - 2b - 12)\omega_7$,

　　$\eta_1 = \lambda - \beta_4 = (a - 1)\omega_2 + \omega_5 + (b - 1)\omega_6 + (-2a - 2b - 12)\omega_7$,

　　$\eta_2 = \lambda - \beta_4 - \beta_3 = (a - 2)\omega_2 + +\omega_4 + (b - 1)\omega_6 + (-2a - 2b - 12)\omega_7$,

　　$\eta_3 = \lambda - \beta_4 - \beta_3 - \beta_2 = (a - 2)\omega_2 + \omega_3 + (b - 1)\omega_6 + (-2a - 2b - 12)\omega_7$,

　　$\eta_4 = \lambda - \beta_4 - \beta_3 - \beta_2 - \beta_1 = \omega_1 + (a - 2)\omega_2 + (b - 1)\omega_6 + (-2a - 2b - 12)\omega_7$,

　　$\eta_5 = \lambda - \beta_4 - \beta_3 - \beta_2 - \beta_1 - \beta_0 = (a - 2)\omega_2 + (b - 1)\omega_6 + (-2a - 2b - 12)\omega_7$.

　　Hilbert series: $h(t) = (\sum_{i=0}^{5}(-1)^i d_i t^i)/(1 - t)^{27}$.

　　Gelfand-Kirillov dimension: 27.

　　Bernstein degree: $\sum_{i=0}^{5}(-1)^i d_i$.

　　Hilbert series at vertex:

$$h(t) = (46332 - 17550t + 2925t^3 - 351t^3 + 27t^4 - t^5)/(1 - t)^{27}.$$

Bernstein degree at vertex: 31382.

$a = 1$:

    Stratum: $(v, T) = (v_7, \{\alpha_2\})$ $(\lambda + \rho$ semi-regular $)$ Vertex: $\omega_2 + \omega_6 - 16\omega_7$.

    Reduced root system: $\Delta_\lambda \simeq SU(1,1)$, $\Delta_{\lambda,n}^+ = \{\beta_4\}$.

    Resolution:

$$0 \to N_1 \xrightarrow{+1} N_0 \to L \to 0.$$

    $\eta_0 = \lambda = \omega_2 + b\omega_6 + (-2b - 14)\omega_7,$

    $\eta_1 = \lambda - \beta_4 = \omega_5 + (b-1)\omega_6 + (-2b - 14)\omega_7.$

    Hilbert series: $h(t) = (d_0 - d_1 t)/(1 - t)^{27}$.

    Gelfand-Kirillov dimension: 27.

    Bernstein degree: $d_0 - d_1$.

    Hilbert series at vertex: $h(t) = (1728 - 351t)/(1 - t)^{27}$.

    Bernstein degree at vertex: 1377.

7.7. $Q_\lambda \simeq SU(1,5)^{(a)}$, $z = 5$.

    Highest weight: $\lambda = a\omega_5 + b\omega_6 + (-3a - 2b - 12)\omega_7$ with $a \geq 1$ and $b \geq 0$.

    Root numbers:

$$
\begin{array}{c}
\beta_0 \; 5 \\
\mid \; \alpha_1 \\
\beta_1 \; 4 \\
\mid \; \alpha_3 \\
\beta_2 \; 3 \\
\mid \; \alpha_4 \\
\beta_3 \; 2 \\
{}^{\alpha_5}\diagup \quad \diagdown^{\alpha_2} \\
\beta_4 \; 1 \; -a \qquad \beta_5 \; 1
\end{array}
$$

$a > 1$:

    Stratum: $(v, T) = (v_5, \varnothing)$ $(\lambda + \rho$ regular$)$ Vertex: $2\omega_5 - 18\omega_7$.

    Reduced root system: $\Delta_\lambda \simeq SU(1,5)$, $\Delta_{\lambda,n}^+ = \{\beta_0, \beta_1, \beta_2, \beta_3, \beta_5\}$.

    Resolution:

$$0 \to N_5 \xrightarrow{+1} N_4 \xrightarrow{+1} N_3 \xrightarrow{+1} N_2 \xrightarrow{+1} N_1 \xrightarrow{+1} N_0 \to L \to 0.$$

    $\eta_0 = \lambda = a\omega_5 + b\omega_6 + (-3a - 2b - 12)\omega_7,$

    $\eta_1 = \lambda - \beta_5 = \omega_2 + (a-1)\omega_5 + b\omega_6 + (-3a - 2b - 12)\omega_7,$

    $\eta_2 = \lambda - \beta_5 - \beta_3 = \omega_4 + (a-2)\omega_5 + b\omega_6 + (-3a - 2b - 12)\omega_7,$

    $\eta_3 = \lambda - \beta_5 - \beta_3 - \beta_2 = \omega_3 + (a-2)\omega_5 + b\omega_6 + (-3a - 2b - 12)\omega_7,$

    $\eta_4 = \lambda - \beta_5 - \beta_3 - \beta_2 - \beta_1 = \omega_1 + (a-2)\omega_5 + b\omega_6 + (-3a - 2b - 12)\omega_7,$

    $\eta_5 = \lambda - \beta_5 - \beta_3 - \beta_2 - \beta_1 - \beta_0 = (a-2)\omega_5 + b\omega_6 + (-3a - 2b - 12)\omega_7.$

    Hilbert series: $h(t) = (\sum_{i=0}^5 (-1)^i d_i t^i)/(1 - t)^{27}$.

    Gelfand-Kirillov dimension: 27.

    Bernstein degree: $\sum_{i=0}^5 (-1)^i d_i$.

    Hilbert series at vertex:

$$h(t) = (34398 - 17550t + 2925t^3 - 351t^3 + 27t^4 - t^5)/(1 - t)^{27}.$$

    Bernstein degree at vertex: 19448.

$a = 1$:

    Stratum: $(v, T) = (v_7, \{\alpha_3\})$ $(\lambda + \rho$ semi-regular$)$ Vertex: $\omega_5 - 15\omega_7$.

    Reduced root system: $\Delta_\lambda \simeq SU(1,1)$, $\Delta_{\lambda,n}^+ = \{\beta_5\}$.

Resolution:
$$0 \to N_1 \xrightarrow{+1} N_0 \to L \to 0.$$

$\eta_0 = \lambda = \omega_5 + b\omega_6 + (-2b - 15)\omega_7,$
$\eta_1 = \lambda - \beta_5 = \omega_2 + b\omega_6 + (-2b - 15)\omega_7.$
Hilbert series: $h(t) = (d_0 - d_1 t)/(1 - t)^{27}.$
Gelfand-Kirillov dimension: 27.
Bernstein degree:

$$d_0 - d_1 = (b + 13)(b + 11)_9(b + 9)(b + 7)^2(b + 5)(b + 1)/(11)_9(7)_3.$$

Hilbert series at vertex: $h(t) = (351 - 78t)/(1 - t)^{27}.$
Bernstein degree at vertex: 273.

7.8. $Q_\lambda \simeq SU(1, 6)$, $z = 6$.
Highest weight: $\lambda = a\omega_2 + (-2a - 11)\omega_7$ with $a \geq 1$.
Root numbers:

$$
\begin{array}{c}
\beta_0 \; 6 \\
| \; \alpha_1 \\
\beta_1 \; 5 \\
| \; \alpha_3 \\
\beta_2 \; 4 \\
| \; \alpha_4 \\
\beta_3 \; 3 \\
{}^{\alpha_5}\!\diagup \quad \diagdown^{\alpha_2} \\
\beta_4 \; 2 \qquad \beta_5 \; 2 \; -a \\
{}^{\alpha_6}\!\diagup \quad \diagdown^{\alpha_2} \quad \diagup^{\alpha_5} \\
\beta_6 \; 1 \qquad \beta_7 \; 1 \; -a
\end{array}
$$

$a > 2$:
Stratum: $(v, T) = (v_8, \varnothing)$ $\quad (\lambda + \rho$ regular$)$ $\qquad$ Vertex: $3\omega_2 - 17\omega_7$.
Reduced root system: $\Delta_\lambda \simeq SU(1, 6)$, $\Delta_{\lambda,n}^+ = \{\beta_0, \beta_1, \beta_2, \beta_3, \beta_4, \beta_6\}$.
Resolution:

$$0 \to N_6 \xrightarrow{+1} N_5 \xrightarrow{+1} \cdots \xrightarrow{+1} N_1 \xrightarrow{+1} N_0 \to L \to 0.$$

$\eta_0 = \lambda = a\omega_2 + (-2a - 11)\omega_7,$
$\eta_1 = \lambda - \beta_6 = (a - 1)\omega_2 + \omega_6 + (-2a - 12)\omega_7,$
$\eta_2 = \lambda - \beta_6 - \beta_4 = (a - 2)\omega_2 + \omega_5 + (-2a - 12)\omega_7,$
$\eta_3 = \lambda - \beta_6 - \beta_4 - \beta_3 = (a - 3)\omega_2 + \omega_4 + (-2a - 12)\omega_7,$
$\eta_4 = \lambda - \beta_6 - \beta_4 - \beta_3 - \beta_2 = (a - 3)\omega_2 + \omega_3 + (-2a - 12)\omega_7,$
$\eta_5 = \lambda - \beta_6 - \beta_4 - \beta_3 - \beta_2 - \beta_1 = \omega_1 + (a - 3)\omega_2 + (-2a - 12)\omega_7,$
$\eta_6 = \lambda - \beta_6 - \beta_4 - \beta_3 - \beta_2 - \beta_1 - \beta_0 = (a - 3)\omega_2 + (-2a - 12)\omega_7.$
Hilbert series: $h(t) = (\sum_{i=0}^{6}(-1)^i d_i t^i)/(1 - t)^{27}.$
Gelfand-Kirillov dimension: 27
Bernstein degree:

$$\sum_{i=0}^{6}(-1)^i d_i = (2a + 11)(2a + 9)(2a + 7)(2a + 5)(a + 7)(a + 6)$$
$$\cdot (a + 5)^2(a + 4)^3(a + 3)^2(a + 2)(a + 1)/(11!)(7)(5)(3).$$

Hilbert series at vertex:

$$h(t) = (43758 - 46332t + 17550t^2 - 2925t^3 + 351t^4 - 27t^5 + t^6)/(1 - t)^{27}.$$

Bernstein degree at vertex: 12376.

$a = 2$:

Stratum: $(v, T) = (v_{10}, \{\alpha_2\})$ ($\lambda + \rho$ semi-regular)   Vertex: $2\omega_2 - 15\omega_7$.
Reduced root system: $\Delta_\lambda \simeq SU(1,2)$, $\Delta_{\lambda,n}^+ = \{\beta_4, \beta_6\}$.
Resolution:
$$0 \to N_2 \xrightarrow{+1} N_1 \xrightarrow{+1} N_0 \to L \to 0.$$

$\eta_0 = \lambda = 2\omega_2 - 15\omega_7$,
$\eta_1 = \lambda - \beta_6 = \omega_2 + \omega_6 - 16\omega_7$,
$\eta_2 = \lambda - \beta_6 - \beta_4 = \omega_5 - 16\omega_7$.
Hilbert series: $h(t) = (2430 - 1728t + 351t^2)/(1-t)^{27}$.
Gelfand-Kirillov dimension: 27.
Bernstein degree: $2430 - 1728 + 351 = 1053$.

$a = 1$:

Stratum: $(v, T) = (v_{12}, \{\alpha_4\})$ ($\lambda + \rho$ semi-regular)   Vertex: $\omega_2 - 13\omega_7$.
Reduced root system: $\Delta_\lambda \simeq SU(1,2)$, $\Delta_{\lambda,n}^+ = \{\beta_3, \beta_6\}$.
Resolution:
$$0 \to N_2 \xrightarrow{+2} N_1 \xrightarrow{+1} N_0 \to L \to 0.$$

$\eta_0 = \lambda = \omega_2 - 13\omega_7$,
$\eta_1 = \lambda - \beta_6 = \omega_6 - 14\omega_7$,
$\eta_2 = s_2 s_5(\lambda + \rho - \beta_6 - 2\beta_3) - \rho = -14\omega_7$.
Hilbert series: $h(t) = (78 - 27t + t^3)/(1-t)^{27}$.
Gelfand-Kirillov dimension: 27.
Bernstein degree: $78 - 27 + 1 = 52$.

7.9. $Q_\lambda \simeq SO(2,10)$, $z = 5$ (first reduction point).
Highest weight: $\lambda = a\omega_6 + (-2a - 12)\omega_7$ with $a \geq 1$.
Root numbers:

$$
\begin{array}{c}
\beta_0 \; 5 \\
\mid \alpha_1 \\
\beta_1 \; 4 \\
\mid \alpha_3 \\
\beta_2 \; 3 \\
\mid \alpha_4 \\
\beta_3 \; 2 \\
{}^{\alpha_5}\diagup \quad \diagdown^{\alpha_2} \\
\beta_4 \; 1 \qquad \beta_5 \; 1 \\
\diagdown^{\alpha_2} \quad \diagup^{\alpha_5} \\
\beta_7 \; 0
\end{array}
$$

Stratum: $(v, T) = (v_9, \{\alpha_4\})$ ($\lambda + \rho$ semi-regular)   Vertex: $\omega_6 - 14\omega_7$.
Reduced root system: $\Delta_\lambda \simeq SU(1,1)$, $\Delta_{\lambda,n}^+ = \{\beta_3\}$.
Resolution:
$$0 \to N_1 \xrightarrow{+2} N_0 \to L \to 0.$$

$\eta_0 = \lambda = a\omega_6 + (-2a - 12)\omega_7$,
$\eta_1 = s_2 s_5(\lambda + \rho - 2\beta_3) - \rho = (a-1)\omega_6 + (-2a - 12)\omega_7$.
Hilbert series: $h(t) = (d_0 - d_1 t^2)/(1-t)^{27}$.
Gelfand-Kirillov dimension: 27.
Bernstein degree: $d_0 - d_1 = (2a+11)(a+10)_{10}(a+7)_4/(11)_{10}(7)_4$.
Hilbert series at vertex: $h(t) = (27 - t^2)/(1-t)^{27}$.
Bernstein degree at vertex: 26.

7.10. $Q_\lambda \simeq SO(2,10)$, $z = 9$ (second reduction point).
Highest weight: $\lambda = a\omega_6 + (-2a - 8)\omega_7$ with $a \geq 1$.
Root numbers:

$$
\begin{array}{c}
\beta_0 \ 9 \\
| \quad \alpha_1 \\
\beta_1 \ 8 \\
| \quad \alpha_3 \\
\beta_2 \ 7 \\
| \quad \alpha_4 \\
\beta_3 \ 6 \\
{}^{\alpha_5}\diagup \qquad \diagdown^{\alpha_2} \\
\beta_4 \ 5 \qquad\qquad \beta_5 \ 5 \\
{}^{\alpha_6}\diagup \quad \diagdown^{\alpha_2} \quad {}^{\alpha_5}\diagup \\
\beta_6 \ 4 \ -a \qquad \beta_7 \ 4 \\
\diagdown^{\alpha_2} \quad {}^{\alpha_6}\diagup \quad \diagdown^{\alpha_4} \\
\beta_8 \ 3 \ -a \qquad \beta_9 \ 3 \\
\diagdown^{\alpha_4} \quad {}^{\alpha_6}\diagup \quad \diagdown^{\alpha_4} \\
\beta_{10} \ 2 \ -a \qquad \beta_{11} \ 2 \\
{}^{\alpha_5}\diagup \quad \diagdown^{\alpha_3} \quad {}^{\alpha_6}\diagup \quad \diagdown^{\alpha_1} \\
\beta_{12} \ 1 \ -a \qquad \beta_{13} \ 1 \ -a \qquad \beta_{14} \ 1
\end{array}
$$

$a > 4$:
Stratum: $(v, T) = (v_{16}, \varnothing)$ $(\lambda + \rho$ regular$)$ Vertex: $5\omega_6 - 18\omega_7$.
Reduced root system: $\Delta_\lambda \simeq SO(2,10)$,
$\Delta_{\lambda,n}^+ = \{\beta_0, \beta_1, \beta_2, \beta_3, \beta_4, \beta_5, \beta_7, \beta_9, \beta_{11}, \beta_{14}\}$.
Resolution:

$$0 \to N_{10} \xrightarrow{+1} N_9 \xrightarrow{+1} N_8 \xrightarrow{+1} N_7 \xrightarrow{+1} N_6 \xrightarrow{(+1,+1)} N_5' \oplus N_5'' \xrightarrow{(+1,+1)}$$
$$N_4 \xrightarrow{+1} N_3 \xrightarrow{+1} N_2 \xrightarrow{+1} N_1 \xrightarrow{+1} N_0 \to L \to 0.$$

$\eta_0 = \lambda = a\omega_6 + (-a - 8)\omega_7$,
$\eta_1 = \lambda - \beta_{14} = \omega_1 + (a - 1)\omega_6 + (-a - 8)\omega_7$,
$\eta_2 = \lambda - \beta_{14} - \beta_{11} = \omega_3 + (a - 2)\omega_6 + (-a - 8)\omega_7$,
$\eta_3 = \lambda - \beta_{14} - \beta_{11} - \beta_9 = \omega_4 + (a - 3)\omega_6 + (-a - 8)\omega_7$,
$\eta_4 = \lambda - \beta_{14} - \beta_{11} - \beta_9 - \beta_7 = \omega_2 + \omega_5 + (a - 4)\omega_6 + (-a - 8)\omega_7$,
$\eta_5' = \lambda - \beta_{14} - \beta_{11} - \beta_9 - \beta_7 - \beta_5 = 2\omega_2 + (a - 4)\omega_6 + (-a - 8)\omega_7$,
$\eta_5'' = \lambda - \beta_{14} - \beta_{11} - \beta_9 - \beta_7 - \beta_4 = 2\omega_5 + (a - 5)\omega_6 + (-a - 8)\omega_7$,
$\eta_6 = \lambda - \beta_{14} - \beta_{11} - \beta_9 - \beta_7 - \beta_5 - \beta_4 = \omega_2 + \omega_5 + (a - 5)\omega_6 + (-a - 8)\omega_7$,
$\eta_7 = \lambda - \beta_{14} - \beta_{11} - \cdots - \beta_7 - \beta_5 - \beta_4 - \beta_3 = \omega_4 + (a - 5)\omega_6 + (-a - 8)\omega_7$,
$\eta_8 = \lambda - \beta_{14} - \beta_{11} - \cdots - \beta_5 - \beta_4 - \beta_3 - \beta_2 = \omega_3 + (a - 5)\omega_6 + (-a - 8)\omega_7$,
$\eta_9 = \lambda - \beta_{14} - \beta_{11} - \cdots - \beta_4 - \beta_3 - \beta_2 - \beta_1 = \omega_1 + (a - 5)\omega_6 + (-a - 8)\omega_7$,
$\eta_{10} = \lambda - \beta_{14} - \beta_{11} - \cdots - \beta_3 - \beta_2 - \beta_1 - \beta_0 = (a - 5)\omega_6 + (-a - 8)\omega_7$.
Hilbert series:

$$h(t) = \left(\sum_{i=0}^{10} (-1)^i d_i t^i\right)/(1 - t)^{27} = \left(\sum_{i=0}^{9} e_i t^i\right)/(1 - t)^{26},$$

where $d_i = \dim(F_{\eta_i})$ for $i \neq 5$ and $d_5 = \dim(F_{\eta_5'}) + \dim(F_{\eta_5''})$.
Gelfand-Kirillov dimension: 26.
Bernstein degree:

$$\sum_{i=0}^{9} e_i = (2a^2 + 9a + 7)(a + 6)(a + 5)(a + 4)(a + 3)(a + 2)/1680.$$

Hilbert series at vertex:

$$h(t) = (100386 - 371800t + 579150t^2 - 494208t^3 + 252252t^4 - 80730t^5$$
$$+ 17550t^6 - 2925t^7 + 351t^8 - 27t^9 + t^{10})/(1-t)^{27}$$
$$= (100386 - 271414t + 307736t^2 - 186472t^3 + 65780t^4 - 14950t^5$$
$$+ 2600t^6 - 325t^7 + 26t^8 - t^9)/(1-t)^{26}.$$

Bernstein degree at vertex: 3366.

$a = 4$:

Stratum: $(v, T) = (v_{19}, \{\alpha_1\})$ ($\lambda + \rho$ semi-regular) Vertex: $4\omega_6 - 16\omega_7$.
Reduced root system: $\Delta_\lambda \simeq SU(1,5)$, $\Delta_{\lambda,n}^+ = \{\beta_5, \beta_7, \beta_9, \beta_{11}, \beta_{14}\}$.
Resolution:

$$0 \to N_5 \xrightarrow{+1} N_4 \xrightarrow{+1} N_3 \xrightarrow{+1} N_2 \xrightarrow{+1} N_1 \xrightarrow{+1} N_0 \to L \to 0.$$

$\eta_0 = \lambda = 4\omega_6 - 16\omega_7,$
$\eta_1 = \lambda - \beta_{14} = \omega_1 + 3\omega_6 - 16\omega_7,$
$\eta_2 = \lambda - \beta_{14} - \beta_{11} = \omega_3 + 2\omega_6 - 16\omega_7,$
$\eta_3 = \lambda - \beta_{14} - \beta_{11} - \beta_9 = \omega_4 + \omega_6 - 16\omega_7,$
$\eta_4 = \lambda - \beta_{14} - \beta_{11} - \beta_9 - \beta_7 = \omega_2 + \omega_5 - 16\omega_7,$
$\eta_5 = \lambda - \beta_{14} - \beta_{11} - 2\beta_9 - \beta_7 - \beta_5 = 2\omega_2 - 16\omega_7.$

Hilbert series:

$$h(t) = (19305 - 61425t + 78975t^2 - 51975t^3 + 17550t^4 - 2430t^5)/(1-t)^{27}$$
$$= (19305 - 42120t + 36855t^2 - 15120t^3 + 2430t^4)/(1-t)^{26}.$$

Gelfand-Kirillov dimension: 26.
Bernstein degree: $19305 - 42120 + 36855 - 15120 + 2430 = 1350$.

$a = 3$:

Stratum: $(v, T) = (v_{22}, \{\alpha_3\})$ ($\lambda + \rho$ semi-regular) Vertex: $3\omega_6 - 14\omega_7$.
Reduced root system: $\Delta_\lambda \simeq SU(1,5)$, $\Delta_{\lambda,n}^+ = \{\beta_3, \beta_4, \beta_9, \beta_{11}, \beta_{14}\}$.
Resolution:

$$0 \to N_5 \xrightarrow{+1} N_4 \xrightarrow{+2} N_3 \xrightarrow{+1} N_2 \xrightarrow{+1} N_1 \xrightarrow{+1} N_0 \to L \to 0$$

$\eta_0 = \lambda = 3\omega_6 - 14\omega_7,$
$\eta_1 = \lambda - \beta_{14} = \omega_1 + 2\omega_6 - 14\omega_7,$
$\eta_2 = \lambda - \beta_{14} - \beta_{11} = \omega_3 + \omega_6 - 14\omega_7,$
$\eta_3 = \lambda - \beta_{14} - \beta_{11} - \beta_9 = \omega_4 - 14\omega_7,$
$\eta_4 = s_2 s_6(\lambda + \rho - \beta_{14} - \cdots - 2\beta_4) - \rho = \omega_5 - 15\omega_7,$
$\eta_5 = s_2 s_6(\lambda + \rho - \beta_{14} - \cdots - \beta_3) - \rho = \omega_2 - 15\omega_7.$

Hilbert series:

$$h(t) = (3003 - 7722t + 7371t^2 - 2925t^3 + 351t^5 - 78t^6)/(1-t)^{27}$$
$$= (3003 - 4719t + 2652t^2 - 273t^3 - 273t^4 + 78t^5)/(1-t)^{26}.$$

Gelfand-Kirillov dimension: 26.
Bernstein degree: $3003 - 4719 + 2652 - 273 - 273 + 78 = 468$.

$a = 2$:

Stratum: $(v, T) = (v_{25}, \{\alpha_4\})$ ($\lambda + \rho$ semi-regular) Vertex: $2\omega_6 - 12\omega_7$.
Reduced root system: $\Delta_\lambda \simeq SU(1,5)$. , $\Delta_{\lambda,n}^+ = \{\beta_2, \beta_4, \beta_7, \beta_{11}, \beta_{14}\}$.
Resolution:

$$0 \to N_5 \xrightarrow{+2} N_4 \xrightarrow{+1} N_3 \xrightarrow{+2} N_2 \xrightarrow{+1} N_1 \xrightarrow{+1} N_0 \to L \to 0.$$

$$\eta_0 = \lambda = 2\omega_6 - 12\omega_7,$$
$$\eta_1 = \lambda - \beta_{14} = \omega_1 + \omega_6 - 12\omega_7,$$
$$\eta_2 = \lambda - \beta_{14} - \beta_{11} = \omega_3 - 12\omega_7,$$
$$\eta_3 = s_4 s_6(\lambda + \rho - \beta_{14} - \beta_{11} - 2\beta_7) - \rho = \omega_2 - 13\omega_7,$$
$$\eta_4 = s_4 s_6(\lambda + \rho - \beta_{14} - \beta_{11} - 2\beta_7 - \beta_4) - \rho = \omega_6 - 14\omega_7,$$
$$\eta_5 = s_3 s_5 s_4 s_6(\lambda + \rho - \beta_{14} - \beta_{11} - \cdots - 2\beta_2) - \rho = -14\omega_7.$$

Hilbert series:

$$h(t) = (351 - 650t + 351t^2 - 78t^4 + 27t^5 - t^7)/(1-t)^{27}$$
$$= (351 - 299t + 52t^2 + 52t^3 - 26t^4 + t^5 + t^6)/(1-t)^{26}.$$

Gelfand-Kirillov dimension: 26.
Bernstein degree: $351 - 299 + 52 + 52 - 26 + 1 + 1 = 132$.

$a = 1$:

Stratum: $(v, T) = (v_{32}, \{\alpha_2, \alpha_5\})$ ($A_1 \times A_1$ singularity)   Vertex: $\omega_6 - 10\omega_7$.
Reduced root system: $\Delta_\lambda \simeq SU(1,1)$, $\Delta_{\lambda,n}^+ = \{\beta_{14}\}$.
Resolution:

$$0 \to N_1 \xrightarrow{+1} N_0 \to L \to 0.$$

$$\eta_0 = \lambda = \omega_6 - 10\omega_7,$$
$$\eta_1 = \lambda - \beta_3 = \omega_1 - 10\omega_7.$$

Hilbert series: $h(t) = (27 - 27t)/(1-t)^{27} = 27/(1-t)^{26}$.
Gelfand-Kirillov dimension: 26.
Bernstein degree: 27.
Special feature: $\lambda + \rho$ is quasi-dominant (as defined in [EHW, 4.4]).

7.11. $Q_\lambda \simeq E_7$, $z = 9$ (first reduction point).
Highest weight: $\lambda = -8\omega_7$.
Stratum: $(v, T) = (v_{37}, \{\alpha_4, \alpha_6\})$ ($A_1 \times A_1$ singularity)   Vertex $-8\omega_7$.
Reduced root system: $\Delta_\lambda \simeq SU(1,1)$, $\Delta_{\lambda,n}^+ = \{\beta_9\}$.
Resolution:

$$N_1 \xrightarrow{+3} N_0 \to L \to 0.$$

Hilbert series: $h(t) = (1 - t^3)/t^{27} = (1 + t + t^2)/(1-t)^{26}$.
Gelfand-Kirillov dimension: 26.
Bernstein degree: 3.
Special feature: $\lambda + \rho$ is quasi-dominant (Wallach representation).

7.12. $Q_\lambda \simeq E_7$, $z = 13$ (second reduction point).
Highest weight: $\lambda = -4\omega_7$.
$\lambda + \rho$ semi-regular with singular root β_{23}.
Stratum: $(v, T) = (v_{52}, \{\alpha_4\})$ ($\lambda + \rho$ semi-regular)   Vertex: $-4\omega_7$.
Reduced root system: $\Delta_\lambda \simeq SO(2, 10)$,
$\Delta_{\lambda,n}^+ = \{\beta_0, \beta_1, \beta_3, \beta_4, \beta_6, \beta_9, \beta_{10}, \beta_{12}, \beta_{17}, \beta_{20}\}$.
Resolution:

$$0 \to N_{10} \xrightarrow{+2} N_9 \xrightarrow{+1} N_8 \xrightarrow{+2} N_7 \xrightarrow{+1} N_6 \xrightarrow{(+1,+2)} N_5' \oplus N_5'' \xrightarrow{(+2,+1)}$$
$$N_4 \xrightarrow{+1} N_3 \xrightarrow{+2} N_2 \xrightarrow{+1} N_1 \xrightarrow{+2} N_0 \to L \to 0.$$

Hilbert series:

$$h(t) = (1 - 27t^2 + 78t^3 - 351t^5 + 650t^6 - 351t^7 - 351t^8$$
$$+ 650t^9 - 351t^{10} + 78t^{12} - 27t^{13} + t^{15})/(1-t)^{27}$$
$$= (1 + 10t + 28t^2 + 28t^3 + 10t^4 + t^5)/(1-t)^{17}.$$

Gelfand-Kirillov dimension: 17.

Bernstein degree: 78.

Special feature: $\lambda + \rho$ is quasi-dominant (Wallach representation).

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