

TOTAL POSITIVITY IN THE DE CONCINI-PROCESI COMPACTIFICATION

XUHUA HE

ABSTRACT. We study the nonnegative part $\overline{G_{>0}}$ of the De Concini-Procesi compactification of a semisimple algebraic group G , as defined by Lusztig. Using positivity properties of the canonical basis and parametrization of flag varieties, we will give an explicit description of $\overline{G_{>0}}$. This answers the question of Lusztig in *Total positivity and canonical bases*, Algebraic groups and Lie groups (ed. G.I. Lehrer), Cambridge Univ. Press, 1997, pp. 281-295. We will also prove that $\overline{G_{>0}}$ has a cell decomposition which was conjectured by Lusztig.

0. INTRODUCTION

Let G be a connected split semisimple algebraic group of adjoint type over \mathbf{R} . We identify G with the group of its \mathbf{R} -points. In [DP], De Concini and Procesi defined a compactification \bar{G} of G and decomposed it into strata indexed by the subsets of a finite set I . We will denote these strata by $\{Z_J \mid J \subset I\}$. Let $G_{>0}$ be the set of strictly totally positive elements of G and $G_{\geq 0}$ be the set of totally positive elements of G (see [L1]). We denote by $\overline{G_{>0}}$ the closure of $G_{>0}$ in \bar{G} . The main goal of this paper is to give an explicit description of $\overline{G_{>0}}$ (see 3.14). This answers the question in [L4, 9.4]. As a consequence, I will prove in 3.17 that $\overline{G_{>0}}$ has a cell decomposition which was conjectured by Lusztig.

To achieve our goal, it is enough to understand the intersection of $\overline{G_{>0}}$ with each stratum. We set $Z_{J,\geq 0} = \overline{G_{>0}} \cap Z_J$. Note that $Z_I = G$ and $Z_{I,\geq 0} = G_{\geq 0}$. We define $Z_{J,>0}$ as a certain subset of $Z_{J,\geq 0}$ analogous to $G_{>0}$ for $G_{\geq 0}$ (see 2.6). When G is simply-laced, we will prove in 2.7 a criterion for $Z_{J,>0}$ in terms of its image in certain representations of G , which is analogous to the criterion for $G_{>0}$ in [L4, 5.4]. As Lusztig pointed out in [L2], although the definition of total positivity was elementary, many of the properties were proved in a non-elementary way, using canonical bases and their positivity properties. Our Theorem 2.7 is an example of this phenomenon. As a consequence, we will see in 2.9 that $Z_{J,\geq 0}$ is the closure of $Z_{J,>0}$ in Z_J .

Note that Z_J is a fiber bundle over the product of two flag manifolds. Then understanding $Z_{J,\geq 0}$ is equivalent to understanding the intersection of $Z_{J,\geq 0}$ with each fiber. In 3.5, we will give a characterization of $Z_{J,\geq 0}$ which is analogous to the elementary fact that $G_{\geq 0} = \bigcap_{g \in G_{>0}} g^{-1}G_{>0}$. It allows us to reduce our problem to the problem of understanding certain subsets of some unipotent groups. Using the

Received by the editors October 3, 2003 and in revised form, March 10, 2004.
2000 *Mathematics Subject Classification*. Primary 20G20; Secondary 14M15.

©2004 American Mathematical Society

parametrization of the totally positive part of the flag varieties (see [MR]), we will give an explicit description of the subsets of G (see 3.7). Thus our main theorem can be proved.

1. PRELIMINARIES

1.1. We will often identify a real algebraic variety with the set of its \mathbf{R} -rational points. Let G be a connected semisimple adjoint algebraic group defined and split over \mathbf{R} , with a fixed épinglage $(T, B^+, B^-, x_i, y_i; i \in I)$ (see [L1, 1.1]). Let U^+, U^- be the unipotent radicals of B^+, B^- . Let X (resp. Y) be the free abelian group of all homomorphism of algebraic groups $T \rightarrow \mathbf{R}^*$ (resp. $\mathbf{R}^* \rightarrow T$) and $\langle, \rangle : Y \times X \rightarrow \mathbf{Z}$ be the standard pairing. We write the operation in these groups as addition. For $i \in I$, let $\alpha_i \in X$ be the simple root such that $tx_i(a)t^{-1} = x_i(a)^{\alpha_i(t)}$ for all $a \in \mathbf{R}, t \in T$ and let $\alpha_i^\vee \in Y$ be the simple coroot corresponding to α_i . For any root α , we denote by U_α the root subgroup corresponding to α .

There is a unique isomorphism $\psi : G \xrightarrow{\sim} G^{\text{opp}}$ (the opposite group structure) such that $\psi(x_i(a)) = y_i(a)$, $\psi(y_i(a)) = x_i(a)$ for all $i \in I, a \in \mathbf{R}$ and $\psi(t) = t$, for all $t \in T$.

If P is a subgroup of G and $g \in G$, we write gP instead of gPg^{-1} .

For any algebraic group H , we denote the Lie algebra of H by $\text{Lie}(H)$ and the center of H by $Z(H)$.

For any variety X and an automorphism σ of X , we denote the fixed point set of σ on X by X^σ .

For any group, We will write 1 for the identity element of the group.

For any finite set X , we will write $|X|$ for the cardinal of X .

1.2. Let $N(T)$ be the normalizer of T in G and $s_i = x_i(-1)y_i(1)x_i(-1) \in N(T)$ for $i \in I$. Set $W = N(T)/T$ and s_i to be the image of s_i in W . Then W together with $(s_i)_{i \in I}$ is a Coxeter group.

Define an expression for $w \in W$ to be a sequence $\mathbf{w} = (w_{(0)}, w_{(1)}, \dots, w_{(n)})$ in W , such that $w_{(0)} = 1, w_{(n)} = w$ and for any $j = 1, 2, \dots, n, w_{(j-1)}^{-1}w_{(j)} = 1$ or s_i for some $i \in I$. An expression $\mathbf{w} = (w_{(0)}, w_{(1)}, \dots, w_{(n)})$ is called reduced if $w_{(j-1)} < w_{(j)}$ for all $j = 1, 2, \dots, n$. In this case, we will set $l(w) = n$. It is known that $l(w)$ is independent of the choice of the reduced expression. Note that if \mathbf{w} is a reduced expression of w , then for all $j = 1, 2, \dots, n, w_{(j-1)}^{-1}w_{(j)} = s_{i_j}$ for some $i_j \in I$. Sometimes we will simply say that $s_{i_1}s_{i_2} \dots s_{i_n}$ is a reduced expression of w .

For $w \in W$, set $\dot{w} = s_{i_1}s_{i_1} \dots s_{i_n}$ where $s_{i_1}s_{i_2} \dots s_{i_n}$ is a reduced expression of w . It is well known that \dot{w} is independent of the choice of the reduced expression $s_{i_1}s_{i_2} \dots s_{i_n}$ of w .

Assume that $\mathbf{w} = (w_{(0)}, w_{(1)}, \dots, w_{(n)})$ is a reduced expression of w and $w_{(j)} = w_{(j-1)}s_{i_j}$ for all $j = 1, 2, \dots, n$. Suppose that $v \leq w$ for the standard partial order in W . Then there is a unique sequence $\mathbf{v}_+ = (v_{(0)}, v_{(1)}, \dots, v_{(n)})$ such that $v_{(0)} = 1, v_{(n)} = v, v_{(j)} \in \{v_{(j-1)}, v_{(j-1)}s_{i_j}\}$ and $v_{(j-1)} < v_{(j-1)}s_{i_j}$ for all $j = 1, 2, \dots, n$ (see [MR, 3.5]). \mathbf{v}_+ is called the positive subexpression of \mathbf{w} . We define

$$J_{\mathbf{v}_+}^+ = \{j \in \{1, 2, \dots, n\} \mid v_{(j-1)} < v_{(j)}\},$$

$$J_{\mathbf{v}_+}^\circ = \{j \in \{1, 2, \dots, n\} \mid v_{(j-1)} = v_{(j)}\}.$$

Then by the definition of \mathbf{v}_+ , we have $\{1, 2, \dots, n\} = J_{\mathbf{v}_+}^+ \sqcup J_{\mathbf{v}_+}^\circ$.

1.3. Let \mathcal{B} be the variety of all Borel subgroups of G . For B, B' in \mathcal{B} , there is a unique $w \in W$, such that (B, B') is in the G -orbit on $\mathcal{B} \times \mathcal{B}$ (diagonal action) that contains $(B^+, {}^w B^+)$. Then we write $\text{pos}(B, B') = w$. By the definition of pos , $\text{pos}(B, B') = \text{pos}({}^g B, {}^g B')$ for any $B, B' \in \mathcal{B}$ and $g \in G$.

For any subset J of I , let W_J be the subgroup of W generated by $\{s_j \mid j \in J\}$ and let w_0^J be the unique element of maximal length in W_J . (We will simply write w_0^J as w_0 .) We denote by P_J the subgroup of G generated by B^+ and by $\{y_j(a) \mid j \in J, a \in \mathbf{R}\}$ and denote by \mathcal{P}^J the variety of all parabolic subgroups of G conjugated to P_J . It is easy to see that for any parabolic subgroup P , $P \in \mathcal{P}^J$ if and only if $\{\text{pos}(B_1, B_2) \mid B_1, B_2 \text{ are Borel subgroups of } P\} = W_J$.

1.4. For any parabolic subgroup P of G , define U_P to be the unipotent radical of P and H_P to be the inverse image of the connected center of P/U_P under $P \rightarrow P/U_P$. If B is a Borel subgroup of G , then so is

$$P^B = (P \cap B)U_P.$$

It is easy to see that for any $g \in H_P$, we have ${}^g(P^B) = P^B$. Moreover, P^B is the unique Borel subgroup B' in P such that $\text{pos}(B, B') \in W^J$, where W^J is the set of minimal length coset representatives of W/W_J (see [L5, 3.2(a)]).

Let P, Q be parabolic subgroups of G . We say that P, Q are opposed if their intersection is a common Levi of P, Q . (We then write $P \bowtie Q$.) It is easy to see that if $P \bowtie Q$, then for any Borel subgroup B of P and B' of Q , we have $\text{pos}(B, B') \in W_J w_0$.

For any subset J of I , define $J^* \subset I$ by $\{Q \mid Q \bowtie P \text{ for some } P \in \mathcal{P}^J\} = \mathcal{P}^{J^*}$. Then we have $(J^*)^* = J$. Let Q_J be the subgroup of G generated by B^- and by $\{x_j(a) \mid j \in J, a \in \mathbf{R}\}$. We have $Q_J \in \mathcal{P}^{J^*}$ and $P_J \bowtie Q_J$. Moreover, for any $P \in \mathcal{P}^J$, we have $P = {}^g P_J$ for some $g \in G$. Thus $\psi(P) = \psi^{(g)^{-1}} Q_J \in \mathcal{P}^{J^*}$.

1.5. Recall the following definitions from [L1].

For any $w \in W$, assume that $w = s_{i_1} s_{i_2} \cdots s_{i_n}$ is a reduced expression of w . Define $\phi^\pm : R_{\geq 0}^n \rightarrow U^\pm$ by

$$\begin{aligned} \phi^+(a_1, a_2, \dots, a_n) &= x_{i_1}(a_1)x_{i_2}(a_2) \cdots x_{i_n}(a_n), \\ \phi^-(a_1, a_2, \dots, a_n) &= y_{i_1}(a_1)y_{i_2}(a_2) \cdots y_{i_n}(a_n). \end{aligned}$$

Let $U_{w, \geq 0}^\pm = \phi^\pm(R_{\geq 0}^n) \subset U^\pm$, $U_{w, > 0}^\pm = \phi^\pm(R_{> 0}^n) \subset U^\pm$. Then $U_{w, \geq 0}^\pm$ and $U_{w, > 0}^\pm$ are independent of the choice of the reduced expression of w . We will simply write $U_{w_0, \geq 0}^\pm$ as $U_{\geq 0}^\pm$ and $U_{w_0, > 0}^\pm$ as $U_{> 0}^\pm$.

$T_{> 0}$ is the submonoid of T generated by the elements $\chi(a)$ for $\chi \in Y$ and $a \in \mathbf{R}_{> 0}$.

$G_{\geq 0}$ is the submonoid $U_{\geq 0}^+ T_{> 0} U_{\geq 0}^- = U_{\geq 0}^- T_{> 0} U_{\geq 0}^+$ of G .

$G_{> 0}$ is the submonoid $U_{> 0}^+ T_{> 0} U_{> 0}^- = U_{> 0}^- T_{> 0} U_{> 0}^+$ of $G_{\geq 0}$.

$\mathcal{B}_{> 0}$ is the subset $\{{}^u B^- \mid u \in U_{> 0}^+\} = \{{}^u B^+ \mid u \in U_{> 0}^-\}$ of \mathcal{B} and $\mathcal{B}_{\geq 0}$ is the closure of $\mathcal{B}_{> 0}$ in the manifold \mathcal{B} .

For any subset J of I , $\mathcal{P}_{> 0}^J = \{P \in \mathcal{P}^J \mid \exists B \in \mathcal{B}_{> 0}, \text{ such that } B \subset P\}$ and $\mathcal{P}_{\geq 0}^J = \{P \in \mathcal{P}^J \mid \exists B \in \mathcal{B}_{\geq 0}, \text{ such that } B \subset P\}$ are subsets of \mathcal{P}^J .

1.6. For any $w, w' \in W$, define

$$\mathcal{R}_{w, w'} = \{B \in \mathcal{B} \mid \text{pos}(B^+, B) = w', \text{pos}(B^-, B) = w_0 w\}.$$

It is known that $\mathcal{R}_{w,w'}$ is nonempty if and only if $w \leq w'$ for the standard partial order in W (see [KL]). Now set

$$\mathcal{R}_{w,w',>0} = \mathcal{B}_{\geq 0} \cap \mathcal{R}_{w,w'}.$$

Then $\mathcal{R}_{w,w',>0}$ is a connected component of $\mathcal{R}_{w,w'}$ and is a semi-algebraic cell (see [R2, 2.8]). Furthermore, $\mathcal{B} = \bigsqcup_{w \leq w'} \mathcal{R}_{w,w'}$ and $\mathcal{B}_{\geq 0} = \bigsqcup_{w \leq w'} \mathcal{R}_{w,w',>0}$. Moreover, for any $u \in U_{w^{-1},>0}^+$, we have ${}^u\mathcal{R}_{w,w',>0} \subset \mathcal{R}_{1,w',>0}$ (see [R2, 2.2]).

Let J be a subset of I . Define $\pi^J : \mathcal{B} \rightarrow \mathcal{P}^J$ to be the map which sends a Borel subgroup to the unique parabolic subgroup in \mathcal{P}^J that contains the Borel subgroup. For any $w, w' \in W$ such that $w \leq w'$ and $w' \in W^J$, set $\mathcal{P}_{w,w'}^J = \pi^J(\mathcal{R}_{w,w'})$ and $\mathcal{P}_{w,w',>0}^J = \pi^J(\mathcal{R}_{w,w',>0})$. We have $\mathcal{P}_{\geq 0}^J = \bigsqcup_{w \leq w', w' \in W^J} \mathcal{P}_{w,w',>0}^J$ and $\pi^J|_{\mathcal{R}_{w,w',>0}}$ maps $\mathcal{R}_{w,w',>0}$ bijectively onto $\mathcal{P}_{w,w',>0}^J$ (see [R1, Chapter 4, 3.2]). Hence, for any $u \in U_{w^{-1},>0}^+$, we have ${}^u\mathcal{P}_{w,w',>0}^J = \pi^J({}^u\mathcal{R}_{w,w',>0}) \subset \pi^J(\mathcal{P}_{1,w',>0}^J)$.

1.7. Define $\pi_T : B^-B^+ \rightarrow T$ by $\pi_T(utu') = t$ for $u \in U^-, t \in T, u' \in U^+$. Then for $b_1 \in B^-, b_2 \in B^-B^+, b_3 \in B^+$, we have $\pi_T(b_1b_2b_3) = \pi_T(b_1)\pi_T(b_2)\pi_T(b_3)$.

Let J be a subset of I . We denote by Φ_J^+ the set of roots that are a linear combination of $\{\alpha_j \mid j \in J\}$ with nonnegative coefficients. We will simply write Φ_J^+ as Φ^+ and we will call a root α positive if $\alpha \in \Phi^+$. In this case, we will simply write $\alpha > 0$. Define U_J^+ to be the subgroup of U^+ generated by $\{U_\alpha \mid \alpha \in \Phi_J^+\}$ and $'U_J^+$ to be the subgroup of U^+ generated by $\{U_\alpha \mid \alpha \in \Phi^+ - \Phi_J^+\}$. Then $U^- \times T \times 'U_J^+ \times U_J^+$ is isomorphic to B^-B^+ via $(u, t, u_1, u_2) \mapsto utu_1u_2$. Now define $\pi_{U_J^+} : B^-B^+ \rightarrow U_J^+$ by $\pi_{U_J^+}(utu_1u_2) = u_2$ for $u \in U^-, t \in T, u_1 \in 'U_J^+$ and $u_2 \in U_J^+$. (We will simply write $\pi_{U_J^+}$ as π_{U^+} .) Note that $U^-T \cdot U^-T'U_J^+ = U^-T'U_J^+$. Thus it is easy to see that for any $a, b \in G$ such that $a, ab \in B^-B^+$, we have $\pi_{U_J^+}(ab) = \pi_{U_J^+}(\pi_{U^+}(a)b)$. Since $'U_J^+$ is a normal subgroup of U^+ , $\pi_{U_J^+}|_{U^+}$ is a homomorphism of U^+ onto U_J^+ . Moreover, we have

$$\pi_{U_J^+}(x_i(a)) = \begin{cases} x_i(a), & \text{if } i \in J; \\ 1, & \text{otherwise.} \end{cases}$$

Thus $\pi_{U_J^+}(U_{>0}^+) = U_{w_0^J, >0}^+$ and $\pi_{U_J^+}(U_{\geq 0}^+) = U_{w_0^J, \geq 0}^+$.

Let U_J^- be the subgroup of U^- generated by $\{U_{-\alpha} \mid \alpha \in \Phi_J^+\}$ and $'U_J^-$ to be the subgroup of U^- generated by $\{U_{-\alpha} \mid \alpha \in \Phi^+ - \Phi_J^+\}$. Then we define $\pi_{U_J^-} : U^- \rightarrow U_J^-$ by $\pi_{U_J^-}(u_1u_2) = u_1$ for $u_1 \in U_J^-$, $u_2 \in 'U_J^-$. (We will simply write $\pi_{U_J^-}$ as π_{U^-} .) We have $\pi_{U_J^-}(U_{>0}^-) = U_{w_0^J, >0}^-$ and $\pi_{U_J^-}(U_{\geq 0}^-) = U_{w_0^J, \geq 0}^-$.

1.8. For any vector space V and a nonzero element v of V , we denote the image of v in $P(V)$ by $[v]$.

If (V, ρ) is a representation of G , we denote by (V^*, ρ^*) the dual representation of G . Then we have the standard isomorphism $St_V : V \otimes V^* \xrightarrow{\cong} \text{End}(V)$ defined by $St_V(v \otimes v^*)(v') = v^*(v')v$ for all $v, v' \in V, v^* \in V^*$. Now we have the $G \times G$ action on $V \otimes V^*$ by $(g_1, g_2) \cdot (v \otimes v^*) = (g_1v) \otimes (g_2v^*)$ for all $g_1, g_2 \in G, v \in V, v^* \in V^*$ and the $G \times G$ action on $\text{End}(V)$ by $((g_1, g_2) \cdot f)(v) = g_1(f(g_2^{-1}v))$ for all $g_1, g_2 \in G, f \in \text{End}(V), v \in V$. The standard isomorphism between $V \otimes V^*$ and $\text{End}(V)$ commutes with the $G \times G$ action. We will identify $\text{End}(V)$ with $V \otimes V^*$ via the standard isomorphism.

2. THE STRATA OF THE DE CONCINI-PROCESI COMPACTIFICATION

2.1. Let \mathcal{V}_G be the projective variety whose points are the $\dim(G)$ -dimensional Lie subalgebras of $\text{Lie}(G \times G)$. For any subset J of I , define

$$Z_J = \{(P, Q, \gamma) \mid P \in \mathcal{P}^J, Q \in \mathcal{P}^{J^*}, \gamma = H_P g U_Q, P \bowtie^g Q\}$$

with the $G \times G$ action by $(g_1, g_2) \cdot (P, Q, H_P g U_Q) = ({}^{g_1}P, {}^{g_2}Q, H_{{}^{g_1}P}({}^{g_1}g {}^{g_2}g_2^{-1}) U_{{}^{g_2}Q})$.

For $(P, Q, \gamma) \in Z_J$ and $g \in \gamma$, we set

$$H_{P,Q,\gamma} = \{(l + u_1, \text{Ad}(g^{-1})l + u_2) \mid l \in \text{Lie}(P \cap {}^gQ), u_1 \in \text{Lie}(U_P), u_2 \in \text{Lie}(U_Q)\}.$$

Then $H_{P,Q,\gamma}$ is independent of the choice of g (see [L6, 12.2]) and is an element of \mathcal{V}_G (see [L6, 12.1]). Moreover, $(P, Q, \gamma) \rightarrow H_{P,Q,\gamma}$ is an embedding of $Z_J \subset \mathcal{V}_G$ (see [L6, 12.2]). We will identify Z_J with the subvariety of \mathcal{V}_G defined above. Then we have $\bar{G} = \bigsqcup_{J \subset I} Z_J$, where \bar{G} is the De Concini-Procesi compactification of G (see [L6, 12.3]). We will call $\{Z_J \mid J \subset I\}$ the strata of \bar{G} and Z_I (resp. Z_\emptyset) the highest (resp. lowest) stratum of \bar{G} . It is easy to see that Z_I is isomorphic to G and Z_\emptyset is isomorphic to $\mathcal{B} \times \mathcal{B}$.

Set $z_J^\circ = (P_J, Q_J, H_{P_J} U_{Q_J})$. Then $z_J^\circ \in Z_J$ (see 1.4) and $Z_J = (G \times G) \cdot z_J^\circ$.

Since G is adjoint, we have an isomorphism $\chi : T \xrightarrow{\cong} (\mathbf{R}^*)^I$ defined by $\chi(t) = (\alpha_i(t)^{-1})_{i \in I}$. We denote the closure of T in \bar{G} by \bar{T} . We have $H_{P_J, Q_J, H_{P_J} U_{Q_J}} = \{(l + u_1, l + u_2) \mid l \in \text{Lie}(P_J \cap Q_J), u_1 \in U_{P_J}, u_2 \in U_{Q_J}\}$. Moreover, for any $t \in Z(P_J \cap Q_J)$, H_t is the subspace of $\text{Lie}(G) \times \text{Lie}(G)$ spanned by the elements $(l, l), (u_1, \text{Ad}(t^{-1})u_1), (\text{Ad}(t)u_2, u_2)$, where $l \in \text{Lie}(P_J \cap Q_J), u_1 \in U_{P_J}, u_2 \in U_{Q_J}$. Thus it is easy to see that $z_J^\circ = \lim_{\substack{t_j=1, \forall j \in J \\ t_j \rightarrow 0, \forall j \notin J}} \chi^{-1}((t_i)_{i \in I}) \in \bar{T}$.

Proposition 2.2. *The automorphism ψ of the variety G (see 1.1) can be extended in a unique way to an automorphism $\bar{\psi}$ of \bar{G} . Moreover, $\bar{\psi}(P, Q, \gamma) = (\psi(Q), \psi(P), \psi(\gamma)) \in Z_J$ for $J \subset I$ and $(P, Q, \gamma) \in Z_J$.*

Proof. The map $\psi : G \rightarrow G$ induces a bijective map $\psi : \text{Lie}(G) \rightarrow \text{Lie}(G)$. Moreover, we have $\psi(\text{Ad}(g)v) = \text{Ad}(\psi(g)^{-1})\psi(v)$ and $\psi(v + v') = \psi(v) + \psi(v')$ for $g \in G, v, v' \in \text{Lie}(G)$. Now define $\delta : \text{Lie}(G) \times \text{Lie}(G) \rightarrow \text{Lie}(G) \times \text{Lie}(G)$ by $\delta(v, v') = (\psi(v'), \psi(v))$ for $v, v' \in \text{Lie}(G)$. Then δ induces a bijection $\bar{\psi} : \mathcal{V}_G \rightarrow \mathcal{V}_G$.

Note that for any $g \in G$, we have $H_g = \{(v, \text{Ad}(g)v) \mid v \in \text{Lie}(G)\}$ and $\bar{\psi}(H_g) = \{(\text{Ad}(\psi(g)^{-1})\psi(v), \psi(v)) \mid v \in \text{Lie}(G)\} = H_{\psi(g)}$. Thus $\bar{\psi}$ is an extension of the automorphism ψ of G into \mathcal{V}_G .

Now for any $(P, Q, \gamma) \in Z_J$ and $g \in \gamma$, we have $\psi(P) \in \mathcal{P}^{J^*}, \psi(Q) \in \mathcal{P}^J$ and $\psi(Q) \bowtie^{\psi(g)} \psi(P)$ (see 1.4). Thus $(\psi(Q), \psi(P), \psi(\gamma)) \in Z_J$. Moreover,

$$\begin{aligned} \bar{\psi}(H_{P,Q,\gamma}) &= \{(\text{Ad}(\psi(g))\psi(l) + \psi(u_2), \psi(l) + \psi(u_1)) \mid l \in \text{Lie}(P \cap {}^gQ), \\ &\quad u_1 \in \text{Lie}(U_P), u_2 \in \text{Lie}(U_Q)\} \\ &= \{(l + u_2, \text{Ad}(\psi(g)^{-1})l + u_1) \mid l \in \text{Lie}(\psi(Q) \cap {}^{\psi(g)}\psi(P)), \\ &\quad u_1 \in \text{Lie}(\psi(U_P)), u_2 \in \text{Lie}(\psi(U_Q))\} \\ &= H_{\psi(Q), \psi(P), \psi(\gamma)}. \end{aligned}$$

Thus $\bar{\psi}|_{\bar{G}}$ is an automorphism of \bar{G} . Moreover, since \bar{G} is the closure of G , $\bar{\psi}|_{\bar{G}}$ is the unique automorphism of \bar{G} that extends the automorphism ψ of G .

The proposition is proved. \square

2.3. For any $\lambda \in X$, set $\text{supp}(\lambda) = \{i \in I \mid \langle \alpha_i^\vee, \lambda \rangle \neq 0\}$.

In the rest of the section, I will fix a subset J of I and $\lambda_1, \lambda_2 \in X^+$ with $\text{supp}(\lambda_1) = I - J, \text{supp}(\lambda_2) = J$. Let (V_{λ_1}, ρ_1) (resp. (V_{λ_2}, ρ_2)) be the irreducible representation of G with the highest weight λ_1 (resp. λ_2). Assume that $\dim V_{\lambda_1} = n_1, \dim V_{\lambda_2} = n_2$ and $\{v_1, v_2, \dots, v_{n_1}\}$ (resp. $\{v'_1, v'_2, \dots, v'_{n_2}\}$) is the canonical basis of (V_{λ_1}, ρ_1) (resp. (V_{λ_2}, ρ_2)), where v_1 and v'_1 are the highest weight vectors. Moreover, after reordering $\{2, 3, \dots, n_2\}$, we could assume that there exists some integer $n_0 \in \{1, 2, \dots, n_2\}$ such that for any $i \in \{1, 2, \dots, n_2\}$, the weight of v'_i is of the form $\lambda_2 - \sum_{j \in J} a_j \alpha_j$ if and only if $i \leq n_0$.

Define $i_J : G \rightarrow P(\text{End}(V_{\lambda_1})) \times P(\text{End}(V_{\lambda_2}))$ by $i_J(g) = ([\rho_1(g)], [\rho_2(g)])$. Then since $\lambda_1 + \lambda_2$ is a dominant and regular weight, the closure of the image of i_J in $P(\text{End}(V_{\lambda_1})) \times P(\text{End}(V_{\lambda_2}))$ is isomorphic to the De Concini-Procesi compactification of G (See [DP, 4.1]). We will use i_J as the embedding of \bar{G} into $P(\text{End}(V_{\lambda_1})) \times P(\text{End}(V_{\lambda_2}))$. We will also identify \bar{G} with its image under i_J .

2.4. Now with respect to the canonical basis of V_{λ_1} and V_{λ_2} , we will identify $\text{End}(V_{\lambda_1})$ with $gl(n_1)$ and $\text{End}(V_{\lambda_2})$ with $gl(n_2)$. Thus we will regard $\rho_1(g), \rho_1^*(g)$ as $n_1 \times n_1$ matrices and $\rho_2(g), \rho_2^*(g)$ as $n_2 \times n_2$ matrices. It is easy to see that (in terms of matrices) for any $g \in G, \rho_1^*(g) = {}^t \rho_1(g^{-1})$ and $\rho_2^*(g) = {}^t \rho_2(g^{-1})$, where ${}^t M$ is the transpose of the matrix M . Now for any $g_1, g_2 \in G, M_1 \in gl(n_1), M_2 \in gl(n_2), (g_1, g_2) \cdot M_1 = \rho_1(g_1)M_1\rho_1(g_2^{-1})$ and $(g_1, g_2) \cdot M_2 = \rho_2(g_1)M_2\rho_2(g_2^{-1})$.

Set $L = P_J \cap Q_J$. Then L is a reductive algebraic group with the épinglage $(T, B^+ \cap L, B^- \cap L, x_j, y_j; j \in J)$. Now let V_L be the subspace of V_{λ_2} spanned by $\{v'_1, v'_2, \dots, v'_{n_0}\}$ and $I_L = (a_{ij}) \in gl(n_2)$, where

$$a_{ij} = \begin{cases} 1, & \text{if } i = j \in \{1, 2, \dots, n_0\}; \\ 0, & \text{otherwise.} \end{cases}$$

Then V_L is an irreducible representation of L with the highest weight λ_2 and canonical basis $\{v'_1, v'_2, \dots, v'_{n_0}\}$. Moreover, λ_2 is a dominant and regular weight for L . Now set $I_1 = \text{diag}(1, 0, 0, \dots, 0) \in gl(n_1), I_2 = \text{diag}(1, 0, 0, \dots, 0) \in gl(n_2)$. Then

$$i_J(z_J^\circ) = \lim_{\substack{t_j=1, \forall j \in J \\ t_j \rightarrow 0, \forall j \notin J}} i_J\left(\chi^{-1}((t_i)_{i \in I})\right) = \left([v_1 \otimes v_1^*], \left[\sum_{i=1}^{n_0} v'_i \otimes v_i'^*\right]\right) = ([I_1], [I_L]),$$

where $\{v_1^*, v_2^*, \dots, v_{n_1}^*\}$ (resp. $\{v_1'^*, v_2'^*, \dots, v_{n_2}'^*\}$) is the dual basis in $(V_{\lambda_1})^*$ (resp. $(V_{\lambda_2})^*$).

2.5. Recall that $\text{supp}(\lambda_1) = I - J$. Thus for any $P \in \mathcal{P}^J$, there is a unique P -stable line $L_{\rho_1(P)}$ in (V_{λ_1}, ρ_1) and $P \mapsto L_{\rho_1(P)}$ is an embedding of \mathcal{P}^J into $P(V_{\lambda_1})$. Similarly, for any $Q \in \mathcal{P}^{J^*}$, there is a unique Q -stable line $L_{\rho_1^*(Q)}$ in $(V_{\lambda_1}^*, \rho_1^*)$ and $Q \mapsto L_{\rho_1^*(Q)}$ is an embedding of \mathcal{P}^{J^*} into $P(V_{\lambda_1}^*)$. It is easy to see $L_{\rho_1(P_J)} = [v_1], L_{\rho_1^*(Q_J)} = [v_1^*]$ and $L_{\rho_1(gP)} = \rho_1(g)L_{\rho_1(P)}, L_{\rho_1^*(gQ)} = \rho_1^*(g)L_{\rho_1^*(Q)}$ for $P \in \mathcal{P}^J, Q \in \mathcal{P}^{J^*}, g \in G$.

There are projections $p_1 : P(\text{End}(V_{\lambda_1})) \times P(\text{End}(V_{\lambda_2})) \rightarrow P(\text{End}(V_{\lambda_1}))$ and $p_2 : P(\text{End}(V_{\lambda_1})) \times P(\text{End}(V_{\lambda_2})) \rightarrow P(\text{End}(V_{\lambda_2}))$. It is easy to see that $p_1|_{Z_J}, p_2|_{Z_J}$ commute with the $G \times G$ action and $p_1(z_J^\circ) = [v_1 \otimes v_1^*] = [L_{\rho_1(P_J)} \otimes L_{\rho_1^*(Q_J)}]$.

Now for any $g_1, g_2 \in G$, we have

$$p_1((g_1, g_2) \cdot z_J^\circ) = [\rho_1(g_1)L_{\rho_1(P)} \otimes \rho_1^*(g_2)L_{\rho_1^*(Q)}] = [L_{\rho_1(g_1P)} \otimes L_{\rho_1^*(g_2Q)}].$$

In other words, $p_1(z) = [L_{\rho_1(P)} \otimes L_{\rho_1^*(Q)}]$ for $z = (P, Q, \gamma) \in Z_J$.

2.6. Let $\overline{G_{>0}}$ be the closure of $G_{>0}$ in \bar{G} . Then $\overline{G_{>0}}$ is also the closure of $G_{\geq 0}$ in \bar{G} . We have $z_J^\circ \in \overline{G_{>0}}$ (see 2.1). Now set

$$Z_{J, \geq 0} = Z_J \cap \overline{G_{>0}},$$

$$Z_{J, > 0} = \{(g_1, g_2^{-1}) \cdot z_J^\circ \mid g_1, g_2 \in G_{>0}\}.$$

Since $\psi(G_{>0}) = G_{>0}$, we have $\bar{\psi}(\overline{G_{>0}}) = \overline{G_{>0}}$. Moreover, $\bar{\psi}(Z_J) = Z_J$ (see 2.2). Therefore $\bar{\psi}(Z_{J, \geq 0}) = Z_{J, \geq 0}$. Similarly, $(g_1, g_2^{-1}) \cdot Z_{J, \geq 0} \subset Z_{J, \geq 0}$ for any $g_1, g_2 \in G_{>0}$. Thus $Z_{J, > 0} \subset Z_{J, \geq 0}$. Moreover, it is easy to see that $\bar{\psi}(Z_{J, > 0}) = Z_{J, > 0}$.

Note that for any $u_1, u_4 \in U_{>0}^-$, $u_2, u_3 \in U_{>0}^+$, $t, t' \in T_{>0}$, we have

$$\begin{aligned} (u_1 u_2 t, u_3^{-1} u_4^{-1} t') \cdot z_J^\circ &= (u_1 u_2, u_3^{-1} u_4^{-1}) \cdot (P_J, Q_J, H_{P_J} t t' U_{Q_J}) \\ &= (u_1, u_3^{-1}) \cdot (P_J, Q_J, H_{P_J} \pi_{U_J^+}(u_2) t t' \pi_{U_J^-}(u_4) U_{Q_J}). \end{aligned}$$

Thus

$$\begin{aligned} Z_{J, > 0} &= \{(u_1, u_2^{-1}) \cdot (P_J, Q_J, H_{P_J} l U_{Q_J}) \mid u_1 \in U_{>0}^-, u_2 \in U_{>0}^+, l \in L_{>0}\} \\ &= \{(u_1' t, u_2'^{-1}) \cdot z_J^\circ \mid u_1' \in U_{>0}^-, u_2' \in U_{>0}^+, t \in T_{>0}\}. \end{aligned}$$

Moreover, for any $u_1, u_1' \in U^-$, $u_2, u_2' \in U^+$ and $t, t' \in T$, it is easy to see that $(u_1 t, u_2) \cdot z_J^\circ = (u_1' t', u_2') \cdot z_J^\circ$ if and only if $(u_1 t)^{-1} u_1' t' \in l H_{P_J} \cap B^- \subset l Z(L)$ and $u_2^{-1} u_2' \in l^{-1} H_{Q_J} \cap U^+ \subset l Z(L)$ for some $l \in L$, that is, $l \in Z(L)$, $u_1 = u_1'$, $u_2 = u_2'$ and $t \in t' Z(L)$. Thus, $Z_{J, > 0} \cong U_{>0}^- \times U_{>0}^+ \times T_{>0} / (T_{>0} \cap Z(L)) \cong R_{>0}^{2l(w_0) + |J|}$.

Now I will prove a criterion for $Z_{J, > 0}$.

Theorem 2.7. *Assume that G is simply-laced. Let $z \in Z_{J, \geq 0}$. Then $z \in Z_{J, > 0}$ if and only if z satisfies the condition:*

$$\begin{aligned} (*) \quad i_J(z) &= ([M_1], [M_2]) \text{ and } i_J(\bar{\psi}(z)) = ([M_3], [M_4]) \text{ for some matrices} \\ &M_1, M_3 \in gl(n_1) \text{ and } M_2, M_4 \in gl(n_2) \text{ with all the entries in } \mathbf{R}_{>0}. \end{aligned}$$

Proof. If $z \in Z_{J, > 0}$, then $z = (g_1, g_2^{-1}) \cdot z_J^\circ$, for some $g_1, g_2 \in G_{>0}$. Assume that $g_1 \cdot v_1 = \sum_{i=1}^{n_1} a_i v_i$ and $g_2^{-1} \cdot v_1^* = \sum_{i=1}^{n_1} b_i v_i^*$. Then for any $i = 1, 2, \dots, n_1$, $a_i, b_i > 0$. Set $a_{ij} = a_i b_j$. Then $p_1(z) = [\rho_1(g_1) I_1 \rho_1(g_2)] = [(a_{ij})]$ is a matrix with all the entries in $\mathbf{R}_{>0}$.

We have $p_2(z) = [\rho_2(g_1) I_L \rho_2(g_2)] = [\rho_2(g_1) I_2 \rho_2(g_2) + \rho_2(g_1) (I_L - I_2) \rho_2(g_2)]$. Note that $\rho_2(g_1) I_2 \rho_2(g_2)$ is a matrix with all the entries in $\mathbf{R}_{>0}$ and $\rho_2(g_1), \rho_2(g_2), (I_L - I_2)$ are matrices with all the entries in $\mathbf{R}_{\geq 0}$. Thus $\rho_2(g_1) (I_L - I_2) \rho_2(g_2)$ is a matrix with all its entries in $\mathbf{R}_{\geq 0}$. So $\rho_2(g_1) I_L \rho_2(g_2)$ is a matrix with all the entries in $\mathbf{R}_{>0}$.

Similarly, $i_J(\bar{\psi}(z)) = ([M_3], [M_4])$ for some matrices M_3, M_4 with all their entries in $\mathbf{R}_{>0}$.

On the other hand, assume that z satisfies the condition (*). Suppose that $z = (P, Q, \gamma)$ and $L_{\rho_1(P)} = [\sum_{i=1}^{n_1} a_i v_i]$, $L_{\rho_1^*(Q)} = [\sum_{i=1}^{n_1} b_i v_i^*]$. We may also assume that $a_{i_0} = b_{i_1} = 1$ for some integers $i_0, i_1 \in \{1, 2, \dots, n_1\}$.

Set $M = (a_{ij}) \in gl(n_1)$, where $a_{ij} = a_i b_j$ for $i, j \in \{1, 2, \dots, n_1\}$. Then $p_1(z) = [L_{\rho_1(P)} \otimes L_{\rho_1^*(Q)}] = [M]$. By the condition (*) and since $a_{i_0, i_1} = a_{i_0} b_{i_1} = 1$,

we have that M is a matrix with all its entries in $\mathbf{R}_{>0}$. In particular, for any $i \in \{1, 2, \dots, n_1\}$, $a_{i,i} = a_i > 0$. Therefore $L_{\rho_1(P)} = [\sum_{i=1}^{n_1} a_i v_i]$, where $a_i > 0$ for all $i \in \{1, 2, \dots, n_1\}$. By [R1, 5.1] (see also [L3, 3.4]), $P \in \mathcal{P}_{>0}^J$. Similarly, $\psi(Q) \in \mathcal{P}_{>0}^J$. Thus there exist $u_1 \in U_{>0}^-, u_2 \in U_{>0}^+$ and $l \in L$, such that $z = (u_1, u_2^{-1}) \cdot (P_J, Q_J, H_{P_J} U_{Q_J})$.

We can express u_1, u_2 in a unique way as $u_1 = u'_1 u''_1$, for some $u'_1 \in U_J^-, u''_1 \in U_J^-$ and $u_2 = u''_2 u'_2$, for some $u'_2 \in U_J^+, u''_2 \in U_J^+$ (see 1.7).

Recall that V_L is the subspace of V_{λ_2} spanned by $\{v'_1, v'_2, \dots, v'_{n_0}\}$. Let V'_L be the subspace of V_{λ_2} spanned by $\{v'_{n_0+1}, v'_{n_0+2}, \dots, v'_{n_2}\}$. Then $u \cdot v - v \in V'_L$ and $u \cdot V'_L \subset V'_L$, for all $v \in V_L$, $\alpha \notin \Phi_J^+$ and $u \in U_{-\alpha}$. Thus $u \cdot v - v \in V'_L$ and $u \cdot V'_L \subset V'_L$, for all $v \in V_L$ and $u \in U_J^-$.

Similarly, let V_L^* be the subspace of $V_{\lambda_2}^*$ spanned by $\{v_1^*, v_2^*, \dots, v_{n_0}^*\}$ and $V_L'^*$ be the subspace of $V_{\lambda_2}^*$ spanned by $\{v_{n_0+1}^*, v_{n_0+2}^*, \dots, v_{n_2}^*\}$. Then for any $v^* \in V_L^*$ and $u \in U_J^+$, we have $u \cdot v - v \in V_L'^*$ and $u V_L'^* \subset V_L'^*$.

We define a map $\pi_L : gl(n_2) \rightarrow gl(n_0)$ by

$$\pi_L((a_{ij})_{i,j \in \{1,2,\dots,n_2\}}) = (a_{ij})_{i,j \in \{1,2,\dots,n_0\}}.$$

Then for any $u \in U_J^-, u' \in U_J^+$ and $M \in gl(n_2)$, we have $\pi_L((u, u') \cdot M) = \pi_L(M)$. Set $M_2 = \rho_2(u_1 l) I_L \rho_2(u_2)$ and $l' = u''_1 l u''_2 \in L$. Then

$$\begin{aligned} \pi_L(M_2) &= \pi_L\left((u_1, u_2^{-1}) \cdot (\rho_2(l) I_L)\right) = \pi_L\left((u'_1, u''_2^{-1}) \cdot \left((u''_1, u''_2^{-1}) \cdot (\rho_2(l) I_L)\right)\right) \\ &= \pi_L\left((u''_1, u''_2^{-1}) \cdot (\rho_2(l) I_L)\right) = \pi_L(\rho_2(l') I_L) = \rho_L(l'). \end{aligned}$$

Since $p_2(z) = [M_2]$, M_2 is a matrix with all its entries nonzero. Therefore $\rho_L(l') = \pi_L(M_2)$ is a matrix with all its entries nonzero. Thus $l' = l_1 t_1 l_2$, for some $l_1 \in U^- \cap L, l_2 \in U^+ \cap L, t_1 \in T$.

Set $\tilde{u}_1 = u'_1 l_1$ and $\tilde{u}_2 = u'_2 l_2$. Then $\tilde{u}_1 P_J = u_1 (u''_1^{-1} l_1) P_J = u_1 P_J$. Similarly, we have $\tilde{u}_2^{-1} Q_J = u_2^{-1} Q_J$. So $z = (\tilde{u}_1, \tilde{u}_2^{-1}) \cdot (P_J, Q_J, H_{P_J} t_1 U_{Q_J})$.

Now for any $i_0, j_0 \in \{1, 2, \dots, n_1\}$, define a map $\pi_{i_0, j_0}^1 : gl(n_1) \rightarrow \mathbf{R}$ by

$$\pi_{i_0, j_0}^1((a_{ij})_{i,j \in \{1,2,\dots,n_1\}}) = a_{i_0, j_0}$$

and for any $i_0, j_0 \in \{1, 2, \dots, n_2\}$, define a map $\pi_{i_0, j_0}^2 : gl(n_2) \rightarrow \mathbf{R}$ by

$$\pi_{i_0, j_0}^2((a_{ij})_{i,j \in \{1,2,\dots,n_2\}}) = a_{i_0, j_0}.$$

Now $z = (\tilde{u}_1 t_1, \tilde{u}_2^{-1}) \cdot z_J^\circ$ and $\bar{\psi}(z) = (\psi(\tilde{u}_2) t_1, \psi(\tilde{u}_1)^{-1}) \cdot z_J^\circ$.

Set

$$\begin{aligned} \tilde{M}_1 &= \rho_1(\tilde{u}_1 t_1) I_1 \rho_1(\tilde{u}_2), & \tilde{M}_3 &= \rho_1(\psi(\tilde{u}_2) t_1) I_1 \rho_1(\psi(\tilde{u}_1)^{-1}), \\ \tilde{M}_2 &= \rho_2(\tilde{u}_1 t_1) I_L \rho_2(\tilde{u}_2), & \tilde{M}_4 &= \rho_2(\psi(\tilde{u}_2) t_1) I_L \rho_2(\psi(\tilde{u}_1)^{-1}). \end{aligned}$$

We have $\tilde{u}_1 \cdot v_1 = \sum_{i=1}^{n_1} \frac{\pi_{i,1}^1(\tilde{M}_1)}{\pi_{1,1}^1(\tilde{M}_1)} v_i$ and $\psi(\tilde{u}_2) \cdot v_1 = \sum_{i=1}^{n_1} \frac{\pi_{i,1}^1(\tilde{M}_3)}{\pi_{1,1}^1(\tilde{M}_3)} v_i$.

Moreover, let V_0 be the subspace of V_{λ_2} spanned by $\{v'_2, v'_3, \dots, v'_{n_2}\}$ and V_0^* be the subspace of $V_{\lambda_2}^*$ spanned by $\{v_2^*, v_3^*, \dots, v_{n_2}^*\}$. Then we have $u \cdot V_0 \subset V_0$, for all $u \in U^-$ and $u' \cdot V_0^* \subset V_0^*$, for all $u' \in U^+$.

Thus for all $i = 1, 2, \dots, n_2$,

$$\begin{aligned}\pi_{i,1}^2(M_2) &= \pi_{i,1}^2(\rho_2(\widetilde{u}_1 t_1) I_2 \rho_2(\widetilde{u}_2)) + \pi_{i,1}^2(\rho_2(\widetilde{u}_1 t_1)(I_L - I_2) \rho_2(\widetilde{u}_2)) \\ &= \pi_{i,1}^2(\rho_2(\widetilde{u}_1 t_1) I_2 \rho_2(\widetilde{u}_2)).\end{aligned}$$

So $\widetilde{u}_1 \cdot v'_1 = \sum_{i=1}^{n_2} \frac{\pi_{i,1}^2(\widetilde{M}_2)}{\pi_{i,1}^2(M_2)} v'_i$ and $\psi(\widetilde{u}_2) \cdot v'_1 = \sum_{i=1}^{n_2} \frac{\pi_{i,1}^2(\widetilde{M}_4)}{\pi_{i,1}^2(M_4)} v'_i$. By [L2, 5.4], we have $\widetilde{u}_1, \psi(\widetilde{u}_2) \in U_{>0}^-$. Therefore to prove that $z \in Z_{J,>0}$, it is enough to prove that $t_1 \in T_{>0} Z(L)$, where $Z(L)$ is the center of L .

For any $g \in (U^-, U^+) \cdot \bar{T}$, g can be expressed in a unique way as $g = (u_1, u_2) \cdot t$, for some $u_1 \in U^-$, $u_2 \in U^+$, $t \in \bar{T}$. Now define $\pi_{\bar{T}} : (U^-, U^+) \cdot \bar{T} \rightarrow \bar{T}$ by $\pi_{\bar{T}}((u_1, u_2) \cdot t) = t$ for all $u_1 \in U^-, u_2 \in U^+, t \in \bar{T}$. Note that $(U^-, U^+) \cdot \bar{T} \cap \overline{G_{>0}}$ is the closure of $G_{>0}$ in $(U^-, U^+) \cdot \bar{T}$. Then $\pi_{\bar{T}}((U^-, U^+) \cdot \bar{T} \cap \overline{G_{>0}})$ is contained in the closure of $T_{>0}$ in \bar{T} . In particular, $\pi_{\bar{T}}(z) = t_1 t_J$ is contained in the closure of $T_{>0}$ in \bar{T} . Therefore for any $j \in J$, $\alpha_j(t_1) > 0$. Now let t_2 be the unique element in T such that

$$\alpha_j(t_2) = \begin{cases} \alpha_j(t_1), & \text{if } j \in J; \\ \alpha_j(t_1)^2, & \text{if } j \notin J. \end{cases}$$

Then $t_2 \in T_{>0}$ and $t_2^{-1} t_1 \in Z(L)$. The theorem is proved. \square

Remark. Theorem 2.7 is analogous to the following statement in [L4, 5.4]: Assume that G is simply laced and V is the irreducible representation of G with the highest weight λ , where λ is a dominant and regular weight of G . For any $g \in G$, let $M(g)$ be the matrix of $g : V \rightarrow V$ with respect to the canonical basis of V . Then for any $g \in G$, $g \in G_{>0}$ if and only if $M(g)$ and $M(\psi(g))$ are matrices with all the entries in $\mathbf{R}_{>0}$.

2.8. Before proving Corollary 2.9, I will introduce some technical tools.

Since G is adjoint, there exists (in an essentially unique way) \tilde{G} with the épinglage $(\tilde{T}, \tilde{B}^+, \tilde{B}^-, \tilde{x}_{\tilde{i}}, \tilde{y}_{\tilde{i}}; \tilde{i} \in \tilde{I})$ and an automorphism $\sigma : \tilde{G} \rightarrow \tilde{G}$ (over \mathbf{R}) such that the following conditions are satisfied.

- (a) \tilde{G} is connected semisimple adjoint algebraic group defined and split over \mathbf{R} .
- (b) \tilde{G} is simply laced.
- (c) σ preserves the épinglage, that is, $\sigma(\tilde{T}) = \tilde{T}$ and there exists a permutation $\tilde{i} \rightarrow \sigma(\tilde{i})$ of \tilde{I} , such that $\sigma(\tilde{x}_{\tilde{i}}(a)) = \tilde{x}_{\sigma(\tilde{i})}(a)$, $\sigma(\tilde{y}_{\tilde{i}}(a)) = \tilde{y}_{\sigma(\tilde{i})}(a)$ for all $\tilde{i} \in \tilde{I}$ and $a \in \mathbf{R}$.
- (d) If $\tilde{i}_1 \neq \tilde{i}_2$ are in the same orbit of $\sigma : \tilde{I} \rightarrow \tilde{I}$, then \tilde{i}_1, \tilde{i}_2 do not form an edge of the Coxeter graph.
- (e) \tilde{i} and $\sigma(\tilde{i})$ are in the same connected component of the Coxeter graph, for any $\tilde{i} \in \tilde{I}$.
- (f) There exists an isomorphism $\phi : \tilde{G}^\sigma \rightarrow G$ (as algebraic groups over \mathbf{R}) which is compatible with the épinglage of G and the épinglage $(\tilde{T}^\sigma, \tilde{B}^{+\sigma}, \tilde{B}^{-\sigma}, \tilde{x}_p, \tilde{y}_p; p \in \bar{I})$ of \tilde{G}^σ , where \bar{I} is the set of orbit of $\sigma : \tilde{I} \rightarrow \tilde{I}$ and $\tilde{x}_p(a) = \prod_{\tilde{i} \in p} \tilde{x}_{\tilde{i}}(a)$, $\tilde{y}_p(a) = \prod_{\tilde{i} \in p} \tilde{y}_{\tilde{i}}(a)$ for all $p \in \bar{I}$ and $a \in \mathbf{R}$.

Let λ be a dominant and regular weight of \tilde{G} and (V, ρ) be the irreducible representation of \tilde{G} with highest weight λ . Let $\overline{\tilde{G}}$ be the closure of $\{[\rho(\tilde{g})] \mid \tilde{g} \in \tilde{G}\}$ in $P(\text{End}(V))$ and $\overline{\tilde{G}^\sigma}$ be the closure of $\{[\rho(\tilde{g})] \mid \tilde{g} \in \tilde{G}^\sigma\}$ in $P(\text{End}(V))$. Then since λ is a dominant and regular weight of \tilde{G} and $\lambda|_{\bar{T}^\sigma}$ is a dominant and regular weight

of \tilde{G}^σ , we have that $\overline{\tilde{G}}$ is the De Concini-Procesi compactification of \tilde{G} and $\overline{\tilde{G}^\sigma}$ is the De Concini-Procesi compactification of \tilde{G}^σ . Since $\overline{\tilde{G}}$ is closed in $P(\text{End}(V))$, $\overline{\tilde{G}^\sigma}$ is the closure of $\{[\rho(\tilde{g})] \mid \tilde{g} \in \tilde{G}^\sigma\}$ in $\overline{\tilde{G}}$.

We have $\overline{\tilde{G}} = \bigsqcup_{\tilde{J} \subset \tilde{I}} \tilde{Z}_{\tilde{J}} = \bigsqcup_{\tilde{J} \subset \tilde{I}} (\tilde{G} \times \tilde{G}) \cdot \tilde{z}_{\tilde{J}}^\circ$ and $\overline{\tilde{G}^\sigma} = \bigsqcup_{\tilde{J} \subset \tilde{I}, \sigma \tilde{J} = \tilde{J}} (\tilde{G}^\sigma \times \tilde{G}^\sigma) \cdot \tilde{z}_{\tilde{J}}^\circ$. Moreover, σ can be extended in a unique way to an automorphism $\bar{\sigma}$ of $\overline{\tilde{G}}$. Since $\overline{\tilde{G}^\sigma} = \bigsqcup_{\tilde{J} \subset \tilde{I}, \sigma \tilde{J} = \tilde{J}} (\tilde{Z}_{\tilde{J}})^\sigma$ is a closed subset of $\overline{\tilde{G}}$ containing \tilde{G}^σ , we have $\overline{\tilde{G}^\sigma} \subset \bigsqcup_{\tilde{J} \subset \tilde{I}, \sigma \tilde{J} = \tilde{J}} (\tilde{Z}_{\tilde{J}})^\sigma$.

By the condition (f), there exists a bijection ϕ between \tilde{I} and I , such that $\phi(\tilde{x}_p(a)) = x_{\phi(p)}(a)$, for all $p \in \tilde{I}, a \in \mathbf{R}$. Moreover, the isomorphism ϕ from \tilde{G}^σ to G can be extended in a unique way to an isomorphism $\bar{\phi} : \overline{\tilde{G}^\sigma} \rightarrow \overline{G}$. It is easy to see that for any $\tilde{J} \subset \tilde{I}$ with $\sigma \tilde{J} = \tilde{J}$, we have $\bar{\phi}((\tilde{G}^\sigma \times \tilde{G}^\sigma) \cdot \tilde{z}_{\tilde{J}}^\circ) = Z_{\phi \circ \pi(\tilde{J})}$, where $\pi : \tilde{I} \rightarrow I$ is the map sending element of \tilde{I} into the σ -orbit that contains it.

Corollary 2.9. $Z_{J, \geq 0} = \bigcap_{g_1, g_2 \in G_{>0}} (g_1^{-1}, g_2) \cdot Z_{J, >0}$ is the closure of $Z_{J, >0}$ in Z_J . As a consequence, $Z_{J, \geq 0}$ and $\overline{G}_{>0}$ are contractible.

Proof. I will prove that $Z_{J, \geq 0} \subset \bigcap_{g_1, g_2 \in G_{>0}} (g_1^{-1}, g_2) \cdot Z_{J, >0}$.

First, assume that G is simply laced.

For any $g \in G_{>0}$, $i_J(g) = ([\rho_1(g)], [\rho_2(g)])$, where $\rho_1(g)$ and $\rho_2(g)$ are matrices with all the entries in $\mathbf{R}_{>0}$. Then for any $z \in Z_{J, \geq 0}$, we have $i_J(z) = ([M_1], [M_2])$ for some matrices with all the entries in $\mathbf{R}_{\geq 0}$. Similarly, $i_J(\bar{\psi}(z)) = ([M_3], [M_4])$ for some matrices with all their entries in $\mathbf{R}_{\geq 0}$.

Note that for any $M'_1, M'_2, M'_3 \in gl(n)$ such that M'_1, M'_3 are matrices with all their entries in $\mathbf{R}_{>0}$ and M'_2 is a nonzero matrix with all the entries in $\mathbf{R}_{\geq 0}$, we have that $M'_1 M'_2 M'_3$ is a matrix with all the entries in $\mathbf{R}_{>0}$. Thus for any $g_1, g_2 \in G_{>0}$, we have that $(g_1, g_2^{-1}) \cdot z$ satisfies the condition (*) in 2.7. Moreover, $(g_1, g_2^{-1}) \cdot z \in Z_{J, \geq 0}$. Therefore by 2.7, $(g_1, g_2^{-1}) \cdot z \in Z_{J, >0}$ for all $g_1, g_2 \in G_{>0}$.

In the general case, we will keep the notation of 2.8. Since the isomorphism $\phi : \tilde{G}^\sigma \rightarrow G$ is compatible with the épinglages, we have $\phi((\tilde{U}_{>0}^\pm)^\sigma) = U_{>0}^\pm$, $\phi((\tilde{T}_{>0})^\sigma) = T_{>0}$ and $\phi((\tilde{G}_{>0})^\sigma) = G_{>0}$. Now for any $z \in Z_{J, \geq 0}$, z is contained in the closure of $G_{>0}$ in \tilde{G} . Thus $\bar{\phi}^{-1}(z)$ is contained in the closure of $(\tilde{G}_{>0})^\sigma$ in $\overline{\tilde{G}^\sigma}$, hence contained in the closure of $(\tilde{G}_{>0})^\sigma$ in $\overline{\tilde{G}}$. Therefore, $\bar{\phi}^{-1}(z) \in \tilde{Z}_{\tilde{J}, \geq 0}$, where $\tilde{J} = \pi^{-1} \circ \phi^{-1}(J)$.

For any $\tilde{g}_1, \tilde{g}_2 \in (\tilde{G}_{>0})^\sigma$, we have $(\tilde{g}_1, \tilde{g}_2^{-1}) \cdot \bar{\phi}^{-1}(z) = (\tilde{u}_1 \tilde{t}, \tilde{u}_2^{-1}) \cdot \tilde{z}_{\tilde{J}}^\circ$ for some $\tilde{u}_1 \in \tilde{U}_{>0}^-, \tilde{u}_2 \in \tilde{U}_{>0}^+, \tilde{t} \in \tilde{T}_{>0}$. Since $\bar{\phi}^{-1}(z) \in (\overline{\tilde{G}})^\sigma$, we have $(\tilde{g}_1, \tilde{g}_2^{-1}) \cdot \bar{\phi}^{-1}(z) \in (\tilde{Z}_{\tilde{J}, >0})^\sigma$. Then

$$\begin{aligned} \bar{\sigma}((\tilde{u}_1 \tilde{t}, \tilde{u}_2^{-1}) \cdot \tilde{z}_{\tilde{J}}^\circ) &= (\sigma(\tilde{u}_1 \tilde{t}), \sigma(\tilde{u}_2^{-1})) \cdot \bar{\sigma}(\tilde{z}_{\tilde{J}}^\circ) = (\sigma(\tilde{u}_1) \sigma(\tilde{t}), \sigma(\tilde{u}_2^{-1})) \cdot \tilde{z}_{\tilde{J}}^\circ \\ &= (\tilde{u}_1 \tilde{t}, \tilde{u}_2^{-1}) \cdot \tilde{z}_{\tilde{J}}^\circ. \end{aligned}$$

Thus $\sigma(\tilde{u}_1) = \tilde{u}_1$ and $\sigma(\tilde{u}_2) = \tilde{u}_2$. Moreover, $(\tilde{t}, 1) \cdot \tilde{z}_{\tilde{J}}^\circ = (\sigma(\tilde{t}), 1) \cdot \tilde{z}_{\tilde{J}}^\circ$, that is, $\tilde{\alpha}_{\tilde{J}}(\tilde{t}) = \tilde{\alpha}_{\tilde{J}}(\sigma(\tilde{t})) = \tilde{\alpha}_{\sigma(\tilde{J})}(\tilde{t})$ for all $\tilde{J} \in \tilde{\mathcal{J}}$, where $\{\tilde{\alpha}_{\tilde{I}} \mid \tilde{I} \in \tilde{\mathcal{I}}\}$ is the set of simple

roots of \tilde{G} . Let \tilde{t}' be the unique element in \tilde{T} such that

$$\tilde{\alpha}_{\tilde{j}}(\tilde{t}') = \begin{cases} \tilde{\alpha}_{\tilde{j}}(\tilde{t}), & \text{if } \tilde{j} \in \tilde{J}; \\ 1, & \text{otherwise.} \end{cases}$$

Then $\tilde{t}' \in (\tilde{T}_{>0})^\sigma$ and $(\tilde{t}, 1) \cdot \tilde{z}_{\tilde{j}}^\circ = (\tilde{t}', 1) \cdot \tilde{z}_{\tilde{j}}^\circ$. Thus $(\tilde{g}_1, \tilde{g}_2^{-1}) \cdot \bar{\phi}^{-1}(z) = (\tilde{u}_1 \tilde{t}', \tilde{u}_2^{-1}) \cdot \tilde{z}_{\tilde{j}}^\circ$. We have

$$\begin{aligned} (\phi(\tilde{g}_1), \phi(\tilde{g}_2)^{-1}) \cdot z &= \bar{\phi}((\tilde{g}_1, \tilde{g}_2^{-1}) \cdot \bar{\phi}^{-1}(z)) = \bar{\phi}((\tilde{u}_1 \tilde{t}', \tilde{u}_2^{-1}) \cdot \tilde{z}_{\tilde{j}}^\circ) \\ &= (\phi(\tilde{u}_1) \phi(\tilde{t}'), \phi(\tilde{u}_2^{-1})) \cdot z_{\tilde{j}}^\circ \in Z_{J, >0}. \end{aligned}$$

Since $\phi((\tilde{G}_{>0})^\sigma) = G_{>0}$, we have $Z_{J, \geq 0} \subset \bigcap_{g_1, g_2 \in G_{>0}} (g_1^{-1}, g_2) \cdot Z_{J, >0}$.

Note that $(1, 1)$ is contained in the closure of $\{(g_1, g_2^{-1}) \mid g_1, g_2 \in G_{>0}\}$. Hence, for any $z \in \bigcap_{g_1, g_2 \in G_{>0}} (g_1^{-1}, g_2) \cdot Z_{J, >0}$, z is contained in the closure of $Z_{J, >0}$. On the other hand, $Z_{J, \geq 0}$ is a closed subset in Z_J . $Z_{J, \geq 0}$ contains $Z_{J, >0}$, hence contains the closure of $Z_{J, >0}$ in Z_J . Therefore, $Z_{J, \geq 0} = \bigcap_{g_1, g_2 \in G_{>0}} (g_1^{-1}, g_2) \cdot Z_{J, >0}$ is the closure of $Z_{J, >0}$ in Z_J .

Now set $g_r = \exp(r \sum_{i \in I} (e_i + f_i))$, where e_i and f_i are the Chevalley generators related to our épinglage by $x_i(1) = \exp(e_i)$ and $y_i(1) = \exp(f_i)$. Then $g_r \in G_{>0}$ for $r \in \mathbf{R}_{>0}$ (see [L1, 5.9]). Define $f : R_{\geq 0} \times Z_{J, \geq 0} \rightarrow Z_{J, \geq 0}$ by $f(r, z) = (g_r, g_r^{-1}) \cdot z$ for $r \in R_{\geq 0}$ and $z \in Z_{J, \geq 0}$. Then $f(0, z) = z$ and $f(1, z) \in Z_{J, >0}$ for all $z \in Z_{J, \geq 0}$. Using the fact that $Z_{J, >0}$ is a cell (see 2.6), it follows that $Z_{J, \geq 0}$ is contractible.

Similarly, define $f' : R_{\geq 0} \times \overline{G_{>0}} \rightarrow \overline{G_{>0}}$ by $f'(r, z) = (g_r, g_r^{-1}) \cdot z$ for $r \in R_{\geq 0}$ and $z \in \overline{G_{>0}}$. Then $f'(0, z) = z$ and $f'(1, z) \in \bigsqcup_{K \subset I} Z_{K, >0}$ for all $z \in \overline{G_{>0}}$. Note that $\bigsqcup_{K \subset I} Z_{K, >0} = (U_{>0}^-, (U_{>0}^+)^{-1}) \cdot \bigsqcup_{K \subset I} (T_{>0}, 1) \cdot z_K^\circ \cong U_{>0}^- \times U_{>0}^+ \times \bigsqcup_{K \subset I} (T_{>0}, 1) \cdot z_K^\circ$ (see 2.6). Moreover, by [DP, 2.2], we have $\bigsqcup_{K \subset I} (T_{>0}, 1) \cdot z_K^\circ \cong R_{\geq 0}^I$. Thus $\bigsqcup_{K \subset I} Z_{K, >0} \cong R_{>0}^{2l(w_0)} \times R_{\geq 0}^I$ is contractible. Therefore $\overline{G_{>0}}$ is contractible. \square

3. THE CELL DECOMPOSITION OF $Z_{J, \geq 0}$

3.1. For any $P \in \mathcal{P}^J, Q \in \mathcal{P}^{J^*}, B \in \mathcal{B}$ and $g_1 \in H_P, g_2 \in U_Q, g \in G$, we have $\text{pos}(P^B, g_1 g g_2 (Q^B)) = \text{pos}(g_1^{-1} (P^B), g g_2 (Q^B)) = \text{pos}(P^B, g (Q^B))$. If moreover, $P \bowtie^g Q$, then $\text{pos}(P^B, g (Q^B)) = w w_0$ for some $w \in W_J$ (see 1.4). Therefore, for any $v, v' \in W, w, w' \in W^J$ and $y, y' \in W_J$ with $v \leq w$ and $v' \leq w'$, Lusztig introduced the subset $Z_J^{v, w, v', w'; y, y'}$ and $Z_{J, >0}^{v, w, v', w'; y, y'}$ of Z_J which are defined as follows:

$$\begin{aligned} Z_J^{v, w, v', w'; y, y'} &= \{(P, Q, H_P g U_Q) \in Z_J \mid P \in \mathcal{P}_{v, w}^J, \psi(Q) \in \mathcal{P}_{v', w'}^J, \\ &\quad \text{pos}(P^{B^+}, g (Q^{B^+})) = y w_0, \text{pos}(P^{B^-}, g (Q^{B^-})) = y' w_0\} \end{aligned}$$

and

$$Z_{J, >0}^{v, w, v', w'; y, y'} = Z_J^{v, w, v', w'; y, y'} \cap Z_{J, \geq 0}.$$

Then

$$Z_J = \bigsqcup_{\substack{v, v' \in W, w, w' \in W^J, y, y' \in W_J \\ v \leq w, v' \leq w'}} Z_J^{v, w, v', w'; y, y'},$$

$$Z_{J, \geq 0} = \bigsqcup_{\substack{v, v' \in W, w, w' \in W^J, y, y' \in W_J \\ v \leq w, v' \leq w'}} Z_{J, > 0}^{v, w, v', w'; y, y'}.$$

Lusztig conjectured that for any $v, v' \in W, w, w' \in W^J, y, y' \in W_J$ such that $v \leq w, v' \leq w', Z_{J, > 0}^{v, w, v', w'; y, y'}$ is either empty or a semi-algebraic cell. If it is nonempty, then it is also a connected component of $Z_J^{v, w, v', w'; y, y'}$.

In this section, we will prove this conjecture. Moreover, we will show exactly when $Z_{J, > 0}^{v, w, v', w'; y, y'}$ is nonempty and we will give an explicit description of $Z_{J, > 0}^{v, w, v', w'; y, y'}$.

First, I will prove some elementary facts about the total positivity of G .

Proposition 3.2.

$$\bigcap_{u \in U_{> 0}^\pm} u^{-1} U_{> 0}^\pm = \bigcap_{u \in U_{> 0}^\pm} U_{> 0}^\pm u^{-1} = \bigcap_{u \in U_{> 0}^\pm} u^{-1} U_{\geq 0}^\pm = \bigcap_{u \in U_{> 0}^\pm} U_{\geq 0}^\pm u^{-1} = U_{\geq 0}^\pm,$$

$$\bigcap_{g \in G_{> 0}} g^{-1} G_{> 0} = \bigcap_{g \in G_{> 0}} G_{> 0} g^{-1} = \bigcap_{g \in G_{> 0}} g^{-1} G_{\geq 0} = \bigcap_{g \in G_{> 0}} G_{\geq 0} g^{-1} = G_{\geq 0}.$$

Proof. I will only prove $\bigcap_{u \in U_{> 0}^+} u^{-1} \cdot U_{> 0}^+ = U_{\geq 0}^+$. The rest of the equalities could be proved in the same way.

Note that $uu_1 \in U_{> 0}^+$ for all $u_1 \in U_{\geq 0}^+, u \in U_{> 0}^+$. Thus $u_1 \in \bigcap_{u \in U_{> 0}^+} u^{-1} \cdot U_{> 0}^+$. On the other hand, assume that $u_1 \in \bigcap_{u \in U_{> 0}^+} u^{-1} \cdot U_{> 0}^+$. Then $uu_1 \in U_{> 0}^+$ for all $u \in U_{> 0}^+$. We have $u_1 = \lim_{\substack{u \in U_{> 0}^+ \\ u \rightarrow 1}} uu_1$ is contained in the closure of $U_{> 0}^+$ in U^+ , that is, $u_1 \in U_{\geq 0}^+$. So $\bigcap_{u \in U_{> 0}^+} u^{-1} \cdot U_{> 0}^+ = U_{\geq 0}^+$. \square

For any $v, v' \in W, w, w' \in W^J$ such that $v \leq w, v' \leq w'$, set $Z_J^{v, w, v', w'} = \bigsqcup_{y, y' \in W_J} Z_J^{v, w, v', w'; y, y'}$ and $Z_{J, > 0}^{v, w, v', w'} = \bigsqcup_{y, y' \in W_J} Z_{J, > 0}^{v, w, v', w'; y, y'}$. We will give a characterization of $z \in Z_{J, > 0}^{v, w, v', w'}$ in 3.5.

Lemma 3.3. *For any $w \in W, u \in U_{\geq 0}^-, \{\pi_{U^+}(u_1 u) \mid u_1 \in U_{w, > 0}^+\} = U_{w, > 0}^+$.*

Proof. The following identities hold (see [L1, 1.3]):

- (a) $tx_i(a) = x_i(\alpha_i(t)a)t, ty_i(a) = y_i(\alpha_i(t)^{-1}a)t$ for all $i \in I, t \in T, a \in \mathbf{R}$.
- (b) $y_{i_1}(a)x_{i_2}(b) = x_{i_2}(b)y_{i_1}(a)$ for all $a, b \in \mathbf{R}$ and $i_1 \neq i_2 \in I$.
- (c) $x_i(a)y_i(b) = y_i(\frac{b}{1+ab})\alpha_i(\frac{1}{1+ab})x_i(\frac{a}{1+ab})$ for all $a, b \in \mathbf{R}_{> 0}, i \in I$.

Thus $U_{w, > 0}^+ U_{\geq 0}^- \subset U_{\geq 0}^- T_{> 0} U_{w, > 0}^+$ for $w \in W$. So we only need to prove that $U_{w, > 0}^+ \subset \{\pi_{U^+}(u_1 u) \mid u_1 \in U_{w, > 0}^+\}$. Now I will prove the following statement:

$$\{\pi_{U^+}(u_1 y_i(a)) \mid u_1 \in U_{w, > 0}^+\} = U_{w, > 0}^+ \quad \text{for } i \in I, a \in \mathbf{R}_{> 0}.$$

We argue by induction on $l(w)$. It is easy to see that the statement holds for $w = 1$. Now assume that $w \neq 1$. Then there exist $j \in I$ and $w_1 \in W$ such that $w = s_j w_1$ and $l(w_1) = l(w) - 1$. For any $u_1 \in U_{w, > 0}^+$, we have $u_1 = u_2' u_3'$ for some $u_2' \in U_{s_j, > 0}^+$

and $u'_3 \in U_{w_1, >0}^+$. By induction hypothesis, there exists $u_3 \in U_{w_1, >0}^+$, $u' \in U^-$ and $t \in T$ such that $u_3 y_i(a) = u' t u'_3$. Since $U_{w, >0}^+ U_{s_i, >0}^- \subset U_{s_i, >0}^- T_{>0} U_{w, >0}^+$, we have $u' \in U_{s_i, >0}^-$ and $t \in T_{>0}$.

Now by (a), we have $tu'_2 t^{-1} \in U_{s_j, >0}^+$. So by (b) and (c), there exists $u_2 \in U_{s_j, >0}^+$ such that $\pi_{U^+}(u_2 u') = tu'_2 t^{-1}$. Thus

$$\begin{aligned} \pi_{U^+}(u_2 u_3 y_i(a)) &= \pi_{U^+}((u_2 u')(u'^{-1} u_3 y_i(a))) = \pi_{U^+}(\pi_{U^+}(u_2 u') u'^{-1} u_3 y_i(a)) \\ &= \pi_{U^+}(tu'_2 t^{-1} tu'_3) = \pi_{U^+}(tu'_2 u'_3) = u'_1. \end{aligned}$$

So $u'_1 \in \{\pi_{U^+}(u_1 y_i(a)) \mid u_1 \in U_{w, >0}^+\}$. The statement is proved.

Now assume that $u \in U_{w', >0}^-$. I will prove the lemma by induction on $l(w')$. It is easy to see that the lemma holds for $w' = 1$. Now assume that $w' \neq 1$. Then there exist $i \in I$ and $w'_1 \in W$ such that $l(w'_1) = l(w') - 1$ and $w' = s_i w'_1$. We have $u = y_i(a) u'$ for some $a \in \mathbf{R}_{>0}$ and $u' \in U_{w'_1, >0}^-$. So

$$\begin{aligned} \{\pi_{U^+}(u_1 u) \mid u_1 \in U_{w, >0}^+\} &= \{\pi_{U^+}(u_1 y_i(a) u') \mid u_1 \in U_{w, >0}^+\} \\ &= \{\pi_{U^+}(\pi_{U^+}(u_1 y_i(a) u)) \mid u_1 \in U_{w, >0}^+\} \\ &= \{\pi_{U^+}(u'_1 u') \mid u'_1 \in U_{w, >0}^+\}. \end{aligned}$$

By induction hypothesis, we have

$$\{\pi_{U^+}(u_1 u) \mid u_1 \in U_{w, >0}^+\} = \{\pi_{U^+}(u'_1 u') \mid u'_1 \in U_{w, >0}^+\} = U_{w, >0}^+.$$

□

Lemma 3.4. Set $Z_{J, >0}^1 = \{(g_1, g_2^{-1}) \cdot z_J^\circ \mid g_1 \in U_{\geq 0}^- T_{>0}, g_2 \in U_{\geq 0}^+\}$. Then

$$\begin{aligned} \text{(a)} \quad Z_{J, \geq 0} &= \bigcap_{u_1 \in U_{>0}^+, u_2^{-1} \in U_{>0}^-} (u_1^{-1}, u_2) \cdot Z_{J, >0}^1. \\ \text{(b)} \quad Z_{J, >0}^1 &= \bigsqcup_{w_1, w_2 \in W^J} \{(u_1 P_J, u_2^{-1} Q_J, u_1 H_{P_J} l U_{Q_J} u_2) \mid u_1 \in U_{w_1, >0}^-, \\ &\quad u_2 \in U_{w_2, >0}^+, l \in L_{\geq 0}\} \\ &= \{(P, Q, \gamma) \in Z_{J, \geq 0} \mid P =^{u_1} P_J, \psi(Q) =^{u_2} P_J \text{ for some } u_1, u_2 \in U_{\geq 0}^-\}. \end{aligned}$$

Proof. (a) By 2.9 and 3.2, we have

$$\begin{aligned} Z_{J, \geq 0} &= \bigcap_{g_1, g_2 \in G_{>0}} (g_1^{-1}, g_2) \cdot Z_{J, >0} = \bigcap_{\substack{t_1, t_2 \in T_{>0} \\ u_1, u_2 \in U_{>0}^+, u_3, u_4 \in U_{>0}^-}} (u_1^{-1} u_3^{-1} t_1^{-1}, u_4 u_2 t_2) \cdot Z_{J, >0} \\ &= \bigcap_{u_1 \in U_{>0}^+, u_4 \in U_{>0}^-} (u_1^{-1}, u_4) \cdot \bigcap_{u_2 \in U_{>0}^+, u_3 \in U_{>0}^-} (u_2^{-1}, u_3) \cdot \bigcap_{t_1, t_2 \in T_{>0}} (t_1^{-1}, t_2) \cdot Z_{J, >0} \\ &= \bigcap_{u_1 \in U_{>0}^+, u_4 \in U_{>0}^-} (u_1^{-1}, u_4) \cdot \bigcap_{u_2 \in U_{>0}^+, u_3 \in U_{>0}^-} (u_2^{-1}, u_3) \cdot Z_{J, >0} \\ &= \bigcap_{u_1 \in U_{>0}^+, u_4 \in U_{>0}^-} (u_1^{-1}, u_4) \cdot \bigcap_{u_2 \in U_{>0}^+, u_3 \in U_{>0}^-} (u_2^{-1} U_{>0}^- T_{>0}, (U_{>0}^+ u_3^{-1})^{-1}) \cdot z_J^\circ \\ &= \bigcap_{u_1 \in U_{>0}^+, u_2^{-1} \in U_{>0}^-} (u_1^{-1}, u_2) \cdot \left((U_{\geq 0}^- T_{>0}, (U_{\geq 0}^+)^{-1}) \cdot z_J^\circ \right). \end{aligned}$$

(b) For any $u \in U_{\geq 0}^-, v \in U_{\geq 0}^+, t \in T_{>0}$, there exist $w_1, w_2 \in W^J, w_3, w_4 \in W_J$, such that $u = u_1 u_3$ for some $u_1 \in U_{w_1, > 0}^-, u_3 \in U_{w_3, > 0}^-$ and $v = u_4 u_2$ for some $u_2 \in U_{w_2, > 0}^+, u_4 \in U_{w_4, > 0}^+$. Then $(ut, v^{-1}) \cdot z_J^\circ = ({}^{u_1}P_J, {}^{u_2^{-1}}Q_J, u_1 H_{P_J} u_3 t u_4 U_{Q_J} u_2)$. On the other hand, assume that $l \in L_{\geq 0}$, then $l = u_3 t u_4$ for some $u_3 \in U_{\geq 0}^-, u_4 \in U_{\geq 0}^+, t \in T_{>0}$. Thus for any $u_1 \in U_{\geq 0}^-, u_2 \in U_{\geq 0}^+$, we have

$$({}^{u_1}P_J, {}^{u_2^{-1}}Q_J, u_1 H_{P_J} l U_{Q_J} u_2) = (u_1 u_3 t, u_2^{-1} u_4^{-1}) \cdot z_J^\circ \in Z_{J, > 0}^1.$$

Therefore,

$$\begin{aligned} Z_{J, > 0}^1 &= \bigsqcup_{w_1, w_2 \in W^J} \{({}^{u_1}P_J, {}^{u_2^{-1}}Q_J, u_1 H_{P_J} l U_{Q_J} u_2) \mid u_1 \in U_{w_1, > 0}^-, \\ &\quad u_2 \in U_{w_2, > 0}^+, l \in L_{\geq 0}\} \\ &\subset \{(P, Q, \gamma) \in Z_{J, \geq 0} \mid P = {}^{u_1}P_J, \psi(Q) = {}^{u_2}P_J \text{ for some } u_1, u_2 \in U_{\geq 0}^-\}. \end{aligned}$$

Note that $\{{}^uP_J \mid u \in U_{\geq 0}^-\} = \bigsqcup_{w \in W^J} \{{}^uP_J \mid u \in U_{w, > 0}^-\}$. Now assume that $z = ({}^{u_1}P_J, \psi(u_2)^{-1} Q_J, u_1 H_{P_J} l U_{Q_J} \psi(u_2))$ for some $w_1, w_2 \in W^J$ and $u_1 \in U_{w_1, > 0}^-, u_2 \in U_{w_2, > 0}^-, l \in L$. To prove that $z \in Z_{J, > 0}^1$, it is enough to prove that $l \in L_{\geq 0} Z(L)$. By part (a), for any $u_3, u_4 \in U_{> 0}^+$,

$$(u_3, \psi(u_4)^{-1}) \cdot z = ({}^{u_3 u_1}P_J, \psi(u_4 u_2)^{-1} Q_J, u_3 u_1 H_{P_J} l U_{Q_J} \psi(u_4 u_2)) \in Z_{J, > 0}^1.$$

Note that $u_3 u_1 = u'_1 t_1 \pi_{U^+}(u_3 u_1)$ for some $u'_1 \in U_{w_1, > 0}^-, t_1 \in T_{>0}$ and $u_4 u_2 = u'_2 t_2 \pi_{U^+}(u_4 u_2)$ for some $u'_2 \in U_{w_2, > 0}^-, t_2 \in T_{>0}$. So we have ${}^{u_3 u_1}P_J = {}^{u'_1}P_J, \psi(u_4 u_2)^{-1} Q_J = \psi(u'_2)^{-1} Q_J$ and

$$\begin{aligned} u_3 u_1 H_{P_J} l U_{Q_J} \psi(u_4 u_2) &= u'_1 t_1 \pi_{U^+}(u_3 u_1) H_{P_J} l U_{Q_J} \psi(\pi_{U^+}(u_4 u_2)) t_2 \psi(u'_2) \\ &= u'_1 H_{P_J} t_1 \pi_{U^+}(u_3 u_1) l \psi(\pi_{U^+}(u_4 u_2)) t_2 U_{Q_J} \psi(u'_2). \end{aligned}$$

Then $t_1 \pi_{U^+}(u_3 u_1) l \psi(\pi_{U^+}(u_4 u_2)) t_2 \in L_{\geq 0} Z(L)$. Since $t_1, t_2 \in T_{>0}$, we have $\pi_{U^+}(u_3 u_1) l \psi(\pi_{U^+}(u_4 u_2)) \in L_{\geq 0} Z(L)$ for all $u_3, u_4 \in U_{> 0}^+$. By 1.8 and 3.3,

$$\pi_{U^+}(U_{> 0}^+ u_1) = \pi_{U^+}(\pi_{U^+}(U_{> 0}^+ u_1)) = \pi_{U^+}(U_{> 0}^+) = U_{w'_0, > 0}^+.$$

Similarly, we have $\pi_{U^+}(U_{> 0}^+ u_2) = U_{w'_0, > 0}^+$. Thus

$$\begin{aligned} l &\in \bigcap_{u_3, u_4 \in U_{w'_0, > 0}^+} u_3^{-1} U_{w'_0, \geq 0}^+ T_{> 0} Z(L) U_{w'_0, \geq 0}^- \psi(u_4)^{-1} \\ &= U_{w'_0, \geq 0}^+ T_{> 0} Z(L) U_{w'_0, \geq 0}^- = L_{\geq 0} Z(L). \end{aligned}$$

The lemma is proved. □

Proposition 3.5. *Let $z \in Z_J^{v, w, v', w'}$, then $z \in Z_{J, > 0}^{v, w, v', w'}$ if and only if for any $u_1 \in U_{v^{-1}, > 0}^+, u_2 \in U_{v', > 0}^+, (u_1, \psi(u_2^{-1})) \cdot z \in Z_{J, > 0}^1$.*

Proof. Assume that $z \in \bigcap_{u_1 \in U_{v^{-1}, > 0}^+, u_2 \in U_{v', > 0}^+} (u_1^{-1}, \psi(u_2)) Z_{J, > 0}^1$. Then we have $z = \lim_{u_1, u_2 \rightarrow 1} (u_1, \psi(u_2)^{-1}) \cdot z$ is contained in the closure of $Z_{J, > 0}^1$ in Z_J . Note that $Z_{J, > 0} \subset Z_{J, \geq 0}^1 \subset Z_{J, \geq 0}$. Thus by 2.9, $Z_{J, \geq 0}$ is the closure of $Z_{J, > 0}^1$ in Z_J . Therefore, z is contained in $Z_{J, \geq 0}$.

On the other hand, assume that $z = (P, Q, \gamma) \in Z_{J, > 0}^{v, w, v', w'}$. By 3.4(a), for any $u_1 \in U_{v^{-1}, > 0}^+$, $u_2 \in U_{v'^{-1}, > 0}^+$, we have $(u_1, \psi(u_2^{-1})) \cdot z \in Z_{J, \geq 0}$. Moreover, we have ${}^{u_1}P = {}^{u'_1}P_J$ for some $u'_1 \in U_{w, > 0}^-$ (see 1.6). Similarly, we have $\psi(\psi(u_2^{-1})Q) = {}^{u_2}\psi(Q) = {}^{u'_2}P_J$ for some $u'_2 \in U_{w', > 0}^-$. By 3.4(b), $(u_1, \psi(u_2^{-1})) \cdot z \in Z_{J, > 0}^1$. \square

3.6. Now I will fix $w \in W^J$ and a reduced expression $\mathbf{w} = (w_{(0)}, w_{(1)}, \dots, w_{(n)})$ of w . Assume that $w_{(j)} = w_{(j-1)}s_{i_j}$ for all $j = 1, 2, \dots, n$. Let $v \leq w$ and let $\mathbf{v}_+ = (v_{(0)}, v_{(1)}, \dots, v_{(n)})$ be the positive subexpression of \mathbf{w} .

Define

$$G_{\mathbf{v}_+, \mathbf{w}} = \left\{ g = g_1 g_2 \cdots g_k \begin{cases} g_j = y_{i_j}(a_j) \text{ for } a_j \in \mathbf{R} - \{0\}, & \text{if } v_{(j-1)} = v_{(j)} \\ g_j = s_{i_j}, & \text{if } v_{(j-1)} < v_{(j)} \end{cases} \right\},$$

$$G_{\mathbf{v}_+, \mathbf{w}, > 0} = \left\{ g = g_1 g_2 \cdots g_k \begin{cases} g_j = y_{i_j}(a_j) \text{ for } a_j \in \mathbf{R}_{> 0}, & \text{if } v_{(j-1)} = v_{(j)} \\ g_j = s_{i_j}, & \text{if } v_{(j-1)} < v_{(j)} \end{cases} \right\}.$$

Marsh and Rietsch have proved that the morphism $g \mapsto {}^g B^+$ maps $G_{\mathbf{v}_+, \mathbf{w}}$ into $\mathcal{R}_{v, w}$ (see [MR, 5.2]) and $G_{\mathbf{v}_+, \mathbf{w}, > 0}$ bijectively onto $\mathcal{R}_{v, w, > 0}$ (see [MR, 11.3]).

The following proposition is a technical tool needed in the proof of the main theorem.

Proposition 3.7. *For any $g \in G_{\mathbf{v}_+, \mathbf{w}, > 0}$, we have*

$$\bigcap_{u \in U_{v^{-1}, > 0}^+} (\pi_{U_J^+}(ug))^{-1} \cdot U_{w_0^J, \geq 0}^+ = \begin{cases} U_{w_0^J, \geq 0}^+, & \text{if } v \in W^J; \\ \emptyset, & \text{otherwise.} \end{cases}$$

The proof will be given in 3.13.

Lemma 3.8. *Suppose α_{i_0} is a simple root such that $v_1^{-1}\alpha_{i_0} > 0$ for $v \leq v_1 \leq w$. Then for all $g \in G_{\mathbf{v}_+, \mathbf{w}, > 0}$ and $a \in \mathbf{R}$, we have $x_{i_0}(a)g = gtg'$ for some $t \in T_{> 0}$ and $g' \in \prod_{\alpha \in R(v)} U_\alpha \cdot (\dot{v}^{-1}x_{i_0}(a)\dot{v})$, where $R(v) = \{\alpha \in \Phi^+ \mid v\alpha \in -\Phi^+\}$.*

Proof. Marsh and Rietsch proved in [MR, 11.8] that g is of the form

$$g = \left(\prod_{j \in J_{\mathbf{v}_+}^+} y_{v_{(j-1)}\alpha_{i_j}}(t_j) \right) \dot{v}$$

and $v_{(j-1)}\alpha_{i_1} \neq \alpha_{i_0}$, for all $j = 1, 2, \dots, n$. Thus $g = g_1\dot{v}$ for some

$$g_1 \in \prod_{\alpha \in \Phi^+ - \{\alpha_{i_0}\}} U_{-\alpha}.$$

Set $T_1 = \{t \in T \mid \alpha_{i_0}(t) = 1\}$, then $T_1 \prod_{\alpha \in \Phi^+ - \{\alpha_{i_0}\}} U_{-\alpha}$ is a normal subgroup of $\psi(P_{\{i_0\}})$. Now set $x = x_{i_0}(a)$, then $xg_1x^{-1} \in B^-$. We may assume that $xg_1x^{-1} = u_1t_1$ for some $u_1 \in U^-$ and $t_1 \in T$. Now $xg = xg_1\dot{v} = (xg_1x^{-1})x\dot{v} = u_1\dot{v}(\dot{v}^{-1}t_1\dot{v})(\dot{v}^{-1}x\dot{v})$. Moreover, by [MR, 11.8], $xg \in gB^+$. Thus $xg = g_1\dot{v}t_2g_2g_3 = g_1(\dot{v}t_2g_2t_2^{-1}\dot{v}^{-1})\dot{v}t_2g_3$, for some $t_2 \in T$, $g_2 \in \prod_{\alpha \in R(v)} U_\alpha$ and $g_3 \in \prod_{\alpha \in \Phi^+ - R(v)} U_\alpha$. Note that $g_1(\dot{v}t_2g_2t_2^{-1}\dot{v}^{-1})$, $u_1 \in U^-$, $t_2, \dot{v}^{-1}t_1\dot{v} \in T$ and $g_3, \dot{v}^{-1}x\dot{v} \in \prod_{\alpha \in \Phi^+ - R(v)} U_\alpha$. Thus $g_1(\dot{v}t_2g_2t_2^{-1}\dot{v}^{-1}) = u_1$, $t_2 = \dot{v}^{-1}t_1\dot{v}$ and $g_3 = \dot{v}^{-1}x\dot{v}$. Note that $g^{-1}x_{i_0}(b)g \in B^+$ for $b \in \mathbf{R}$ (see [MR, 11.8]). We have that $\{\pi_T(g^{-1}x_{i_0}(b)g) \mid b \in \mathbf{R}\}$ is connected and contains $\pi_T(g^{-1}x_{i_0}(0)g) = 1$. Hence $\pi_T(g^{-1}x_{i_0}(b)g) \in T_{> 0}$ for $b \in \mathbf{R}$.

In particular, $\pi_T(g^{-1}xg) = t_2 \in T_{>0}$. Therefore $xg = gt_2g'$ with $t_2 \in T_{>0}$ and $g' = g_2g_3 \in \prod_{\alpha \in R(v)} U_\alpha \cdot (\dot{v}^{-1}x\dot{v})$. \square

Remark. In [MR, 11.9], Marsh and Rietsch pointed out that for any $j \in J_{\mathbf{V}_+}^+$, we have $u^{-1}\alpha_{i_j} > 0$ for all $v_{(j)}^{-1}v \leq u \leq w_{(j)}^{-1}w$.

3.9. Suppose that $J_{\mathbf{V}_+}^+ = \{j_1, j_2, \dots, j_k\}$, where $j_1 < j_2 < \dots < j_k$ and $g = g_1g_2 \cdots g_n$, where

$$g_j = \begin{cases} y_{i_j}(a_j) \text{ for } a_j \in \mathbf{R}_{>0}, & \text{if } j \in J_{\mathbf{V}_+}^{\circ}; \\ s_{i_j}, & \text{if } j \in J_{\mathbf{V}_+}^+. \end{cases}$$

For any $m = 1, \dots, k$, define $v_m = v_{(j_m)}^{-1}v$, $g_{(m)} = g_{j_m+1}g_{j_m+2} \cdots g_n$ and $f_m(a) = g_{(m)}^{-1}x_{i_{j_m}}(-a)g_{(m)} \in B^+$ (see [MR, 11.8]). Now I will prove the following lemma.

Lemma 3.10. *Keep the notation in 3.9. Then*

- (a) For any $u \in U_{v^{-1}, >0}^+$, $ug = g'tu'$ for some $g' \in U_{w, >0}^-$, $t \in T_{>0}$ and $u' \in U^+$.
- (b) $\pi_{U^+}(U_{v^{-1}, >0}^+g) = \{\pi_{U^+}(f_k(a_k)f_{k-1}(a_{k-1}) \cdots f_1(a_1)) \mid a_1, a_2, \dots, a_k \in \mathbf{R}_{>0}\}$.

Proof. I will prove the lemma by induction on $l(v)$. It is easy to see that the lemma holds when $v = 1$. Now assume that $v \neq 1$.

For any $u \in U_{v^{-1}, >0}^+$, since ${}^gB^+ \in \mathcal{R}_{v, w, >0}$, we have ${}^{ug}B^+ \in \mathcal{R}_{1, w, >0}$. Thus $ug = g'tu'$ for some $g' \in U_{w, >0}^-$, $t \in T$ and $u' \in U^+$. Set $y = g_{i_1}g_{i_2} \cdots g_{i_{j_1-1}}$. Note that $y \in U_{\geq 0}^-$, we have $uy = y'tu'$ for some $y' \in U^-$, $u' \in U_{v^{-1}, >0}^+$ and $t \in T_{>0}$. Hence $\pi_T(ug) = \pi_T(uy s_{i_{j_1}} g_{(1)}) = \pi_T(y'tu' s_{i_{j_1}} g_{(1)}) \in T_{>0}\pi_T(u' s_{i_{j_1}} g_{(1)})$. To prove that $\pi_T(U_{v^{-1}, >0}^+g) \subset T_{>0}$, it is enough to prove that $\pi_T(us_{i_{j_1}}g_{(1)}) \in T_{>0}$ for all $u \in U_{v^{-1}, >0}^+$.

For any $u \in U_{v^{-1}, >0}^+$, we have $u = u_1x_{i_{j_1}}(a)$ for some $u_1 \in U_{v^{-1}s_{i_{j_1}}, >0}^+$ and $a \in \mathbf{R}_{>0}$. It is easy to see that $x_{i_{j_1}}(a)s_{i_{j_1}}g_{(1)} = \alpha_{i_{j_1}}^\vee(a)y_{i_{j_1}}(a)x_{i_{j_1}}(-a^{-1})g_{(1)}$. Note that $\alpha_{i_{j_1}}^\vee(a) \in T_{>0}$ and by 3.8, $g_{(1)}^{-1}x_{i_{j_1}}(-a^{-1})g_{(1)} \in T_{>0}U^+$. Hence by 1.7, we have

$$\begin{aligned} \pi_T(us_{i_{j_1}}g_{(1)}) &= \pi_T\left(u_1\alpha_{i_{j_1}}^\vee(a)y_{i_{j_1}}(a)g_{(1)}(g_{(1)}^{-1}x_{i_{j_1}}(-a^{-1})g_{(1)})\right) \\ &\in T_{>0}\pi_T(U_{v^{-1}s_{i_{j_1}}, >0}^+y_{i_{j_1}}(a)g_{(1)})T_{>0}. \end{aligned}$$

Set

$$\begin{aligned} \mathbf{w}' &= (1, w_{(j_1-1)}^{-1}w_{(j_1)}, \dots, w_{(j_1-1)}^{-1}w_{(n)}), \\ \mathbf{v}'_+ &= (1, s_{i_{j_1}}v_{(j_1)}, s_{i_{j_1}}v_{(j_1+1)}, \dots, s_{i_{j_1}}v_{(n)}). \end{aligned}$$

Then \mathbf{w}' is a reduced expression of $w_{(j_1-1)}^{-1}w_{(n)}$ and \mathbf{v}'_+ is a positive subexpression of \mathbf{w}' . For any $a \in \mathbf{R}_{>0}$, $y_{i_{j_1}}(a)g_{(1)} \in G_{\mathbf{v}'_+, \mathbf{w}', >0}$. Thus by induction hypothesis, for any $a \in \mathbf{R}_{>0}$, $\pi_T(U_{v^{-1}s_{i_{j_1}}, >0}^+y_{i_{j_1}}(a)g_{(1)}) \subset T_{>0}$. Therefore, $\pi_T(ug) \in T_{>0}$. Part (a) is proved.

We have

$$\begin{aligned}
\pi_{U^+}(U_{v^{-1}, > 0}^+ g) &= \pi_{U^+}(U_{v^{-1}, > 0}^+ y s_{i_{j_1}}^{\dot{}} g(1)) = \pi_{U^+}(\pi_{U^+}(U_{v^{-1}, > 0}^+ y) s_{i_{j_1}}^{\dot{}} g(1)) \\
&= \pi_{U^+}(U_{v^{-1}, > 0}^+ s_{i_{j_1}}^{\dot{}} g(1)) = \bigcup_{a \in \mathbf{R}_{> 0}} \pi_{U^+}(U_{v^{-1} s_{i_{j_1}}, > 0}^+ x_{i_{j_1}}(a^{-1}) s_{i_{j_1}}^{\dot{}} g(1)) \\
&= \bigcup_{a \in \mathbf{R}_{> 0}} \pi_{U^+}(U_{v^{-1} s_{i_{j_1}}, > 0}^+ \alpha_{i_{j_1}}^{\vee}(a^{-1}) y_{i_{j_1}}(a^{-1}) g(1) f_1(a)) \\
&= \bigcup_{a \in \mathbf{R}_{> 0}} \pi_{U^+}(\pi_{U^+}(U_{v^{-1} s_{i_{j_1}}, > 0}^+ \alpha_{i_{j_1}}^{\vee}(a^{-1}) y_{i_{j_1}}(a^{-1})) g(1) f_1(a)) \\
&= \bigcup_{a \in \mathbf{R}_{> 0}} \pi_{U^+}(U_{v^{-1} s_{i_{j_1}}, > 0}^+ g(1) f_1(a)) \\
&= \bigcup_{a \in \mathbf{R}_{> 0}} \pi_{U^+}(\pi_{U^+}(U_{v^{-1} s_{i_{j_1}}, > 0}^+ g(1)) f_1(a)).
\end{aligned}$$

By induction hypothesis,

$$\pi_{U^+}(U_{v^{-1} s_{i_{j_1}}, > 0}^+ g(1)) = \{\pi_{U^+}(f_k(a_k) f_{k-1}(a_{k-1}) \cdots f_2(a_2)) \mid a_2, a_3, \dots, a_k \in \mathbf{R}_{> 0}\}.$$

Thus

$$\begin{aligned}
\pi_{U^+}(U_{v^{-1}, > 0}^+ g) &= \bigcup_{a \in \mathbf{R}_{> 0}} \pi_{U^+}(\pi_{U^+}(U_{v^{-1} s_{i_{j_1}}, > 0}^+ g(1)) f_1(a)) \\
&= \{\pi_{U^+}(f_k(a_k) f_{k-1}(a_{k-1}) \cdots f_1(a_1)) \mid a_1, a_2, \dots, a_k \in \mathbf{R}_{> 0}\}.
\end{aligned}$$

□

Remark. The referee pointed out to me that the assertion $t \in T_{> 0}$ of 3.10(a) could also be proved using generalized minors.

Lemma 3.11. *Assume that α is a positive root and $u \in U_\alpha$, $u' \in U^+$ such that $u^n u' \in U_{\geq 0}^+$ for all $n \in \mathbf{N}$. Then $u = x_i(a)$ for some $i \in I$ and $a \in \mathbf{R}_{\geq 0}$.*

Proof. There exists $t \in T_{> 0}$, such that $\alpha_i(t) = 2$ for all $i \in I$. Then $tut^{-1} = u^{\alpha(t)} = u^m$ for some $m \in \mathbf{N}$. By assumption, $t^n u t^{-n} u' \in U_{\geq 0}^+$ for all $n \in \mathbf{N}$. Thus $u(t^{-n} u' t^n) = t^{-n} (t^n u t^{-n} u') t^n \in U_{\geq 0}^+$. Moreover, it is easy to see that $\lim_{n \rightarrow \infty} t^{-n} u' t^n = 1$. Since $U_{\geq 0}^+$ is a closed subset of U^+ , $\lim_{n \rightarrow \infty} u t^{-n} u' t^n = u \in U_{\geq 0}^+$. Thus $u = x_i(a)$ for some $i \in I$ and $a \in \mathbf{R}_{\geq 0}$. □

Lemma 3.12. *Assume that $w \in W$ and $i, j \in I$ such that $w^{-1} \alpha_i = \alpha_j$. Then there exists $c \in \mathbf{R}_{> 0}$, such that $\dot{w}^{-1} x_i(a) \dot{w} = x_j(ca)$ for all $a \in \mathbf{R}$.*

Proof. There exist $c, c' \in \mathbf{R} - \{0\}$, such that $y_i(a) \dot{w} = \dot{w} y_j(c'a)$ and $x_i(a) \dot{w} = \dot{w} x_j(ca)$ for $a \in \mathbf{R}$. Since ${}^{\dot{w}} B^- \in \mathcal{B}_{\geq 0}$, we have ${}^{y_i(1)\dot{w}} B^+ = {}^{\dot{w} y_j(c')} B^+ \in \mathcal{B}_{\geq 0}$. By 3.6, $c' \geq 0$. Thus $c' > 0$. Moreover, since $w \alpha_j = \alpha_i > 0$, we have $ws_j w^{-1} = s_i$ and $l(ws_j) = l(s_i w) = l(w) + 1$. Hence, setting $w' = ws_j = s_i w$, we have $\dot{w}' = \dot{w} s_j = \dot{s}_i \dot{w}$, that is $\dot{w} x_i(-1) y_i(1) x_i(-1) = x_j(-c) y_j(c') x_i(-c) \dot{w} = x_j(-1) y_j(1) x_j(-1) \dot{w}$. Therefore, $c = c'^{-1} > 0$. □

3.13. Proof of Proposition 3.7. If $v \in W^J$, then $v\alpha > 0$ for $\alpha \in \Phi_J^+$. So $\pi_{U_J^+}(\prod_{\alpha \in R(v)} U_\alpha) = \{1\}$. By 3.8, $f_m(a) \in T(\prod_{\alpha \in R(v_m)} U_\alpha) \cdot U_{v_m^{-1}\alpha_{i_{j_m}}}$ for all $m \in \{1, 2, \dots, k\}$. Note that $v\alpha \in -\Phi^+$ for all $a \in R(v_m)$ and $vv_m^{-1}\alpha_{i_{j_m}} = v_{(j_m)}\alpha_{i_{j_m}} \in -\Phi^+$. So $f_m(a) \in T\prod_{\alpha \in R(v)} U_\alpha$ and $f_k(a_k)f_{k-1}(a_{k-1}) \cdots f_1(a_1) \in T\prod_{\alpha \in R(v)} U_\alpha$. Hence by 3.10(b), $\pi_{U_J^+}(ug) = 1$ for all $u \in U_{v^{-1}, >0}^+$. Therefore $\bigcap_{u \in U_{v^{-1}, >0}^+} (\pi_{U_J^+}(ug))^{-1} \cdot U_{w_0^J, \geq 0}^+ = U_{w_0^J, \geq 0}^+$.

If $v \notin W^J$, then there exists $\alpha \in \Phi_J^+$ such that $v\alpha \in -\Phi_J^+$, that is, $v_m^{-1}\alpha_{i_{j_m}} \in \Phi_J^+$ for some $m \in \{1, 2, \dots, k\}$. Set $k_0 = \max\{m \mid v_m^{-1}\alpha_{i_{j_m}} \in \Phi_J^+\}$. Then since $R(v_{k_0}) = \{v_m^{-1}\alpha_{i_{j_m}} \mid m > k_0\}$, we have that $v_{k_0}\alpha > 0$ for $\alpha \in \Phi_J^+$. Hence by 3.8, $\pi_{U_J^+}(f_{k_0}(a)) = v_{k_0}^{-1}x_{i_{j_{k_0}}}(-a)v_{k_0}$. If $u' \in \bigcap_{u \in U_{v^{-1}, >0}^+} (\pi_{U_J^+}(ug))^{-1} \cdot U_{w_0^J, \geq 0}^+$, then $\pi_{U_J^+}(f_k(a_k)f_{k-1}(a_{k-1}) \cdots f_1(a_1))u' \in U_{w_0^J, \geq 0}^+$ for all $a_1, a_2, \dots, a_k \in \mathbf{R}_{>0}$. Since $U_{w_0^J, \geq 0}^+$ is a closed subset of G , $\pi_{U_J^+}(f_k(a_k)f_{k-1}(a_{k-1}) \cdots f_1(a_1))u' \in U_{w_0^J, \geq 0}^+$ for all $a_1, a_2, \dots, a_k \in \mathbf{R}_{\geq 0}$. Now take $a_m = 0$ for $m \in \{1, 2, \dots, k\} - \{k_0\}$, then $\pi_{U_J^+}(f_{k_0}(a))u' \in U_{w_0^J, \geq 0}^+$ for all $a \in \mathbf{R}_{>0}$. Set $u_1 = v_{k_0}^{-1}x_{i_{j_{k_0}}}(-1)v_{k_0}$. Then $u_1^n u' \in U_{w_0^J, \geq 0}^+$ for all $n \in \mathbf{N}$. Thus by 3.11, $v_{k_0}^{-1}\alpha_{i_{j_{k_0}}} = \alpha_{j'}$ for some $j' \in J$ and $u_1 \in U_{w_0^J, \geq 0}^+$. By 3.12, $u_1 = x_{j'}(-c)$ for some $c \in \mathbf{R}_{>0}$. That is a contradiction. The proposition is proved. \square

Let me recall that $L = P_J \cap Q_J$ (see 2.4). Now I will prove the main theorem.

Theorem 3.14. For any $v, w, v', w' \in W^J$ such that $v \leq w, v' \leq w'$, set

$$\tilde{Z}_{J, >0}^{v, w, v', w'} = \left\{ ({}^g P_J, \psi(g')^{-1} Q_J, gH_{P_J} l U_{Q_J} \psi(g')) \mid \begin{array}{l} g \in G_{\mathbf{v}_+, \mathbf{w}, >0}, \quad g' \in G_{\mathbf{v}'_+, \mathbf{w}', >0} \\ \text{and } l \in L_{\geq 0} \end{array} \right\}.$$

Then

$$Z_{J, >0}^{v, w, v', w'} = \begin{cases} \tilde{Z}_{J, >0}^{v, w, v', w'}, & \text{if } v, w, v', w' \in W^J, v \leq w, v' \leq w'; \\ \emptyset, & \text{otherwise.} \end{cases}$$

Proof. Note that $\{(P, Q, \gamma) \in Z_J \mid P \in \mathcal{P}_{\geq 0}^J, \psi(Q) \in \mathcal{P}_{\geq 0}^J\}$ is a closed subset containing $Z_{J, >0}$. Hence it contains $Z_{J, \geq 0}$. Now fix $g \in G_{\mathbf{v}_+, \mathbf{w}, >0}, g' \in G_{\mathbf{v}'_+, \mathbf{w}', >0}$ and $l \in L$. By 3.10 (a), for any $u \in U_{v^{-1}, >0}^+$, $ug = at\pi_{U^+}(ug)$ for some $a \in U_{w, >0}^-$ and $t \in T_{>0}$. Similarly, for any $u' \in U_{v'^{-1}, >0}^+$, $u'g' = a't'\pi_{U^+}(u'g')$ for some $a' \in U_{w', >0}^-$ and $t' \in T_{>0}$. Set $z = ({}^g P_J, \psi(g')^{-1} Q_J, gH_{P_J} l U_{Q_J} \psi(g'))$. We have

$$\begin{aligned} (u, \psi(u')^{-1}) \cdot z &= \left({}^a P_J, \psi(a')^{-1} Q_J, at\pi_{U^+}(ug)H_{P_J} l U_{Q_J} \psi(\pi_{U^+}(u'g'))t'\psi(a') \right) \\ &= \left({}^a P_J, \psi(a')^{-1} Q_J, aH_{P_J} t\pi_{U_J^+}(ug)l\psi(\pi_{U_J^+}(u'g'))t'U_{Q_J} \psi(a') \right). \end{aligned}$$

Then $(u, \psi(u')^{-1}) \cdot z \in Z_{J, >0}^1$ if and only if $t\pi_{U_J^+}(ug)l\psi(\pi_{U_J^+}(u'g'))t' \in L_{\geq 0}Z(L)$, that is,

$$\begin{aligned} l &\in \pi_{U_J^+}(ug)^{-1}L_{\geq 0}Z(L)\psi(\pi_{U_J^+}(u'g'))^{-1} \\ &= (\pi_{U_J^+}(ug)^{-1}U_{w_0^J, \geq 0}^+)T_{>0}Z(L)\psi(\pi_{U_J^+}(u'g'))^{-1}U_{w_0^J, \geq 0}^+. \end{aligned}$$

So by 3.5, $z \in Z_{J, \geq 0}$ if and only if

$$\begin{aligned} l &\in \bigcap_{\substack{u \in U_{v^{-1}, > 0}^+ \\ u' \in U_{v'^{-1}, > 0}^+}} (\pi_{U_J^+}(ug)^{-1}U_{w_0^J, \geq 0}^+)T_{> 0}Z(L)\psi(\pi_{U_J^+}(u'g')^{-1}U_{w_0^J, \geq 0}^+) \\ &= \bigcap_{u \in U_{v^{-1}, > 0}^+} (\pi_{U_J^+}(ug)^{-1}U_{w_0^J, \geq 0}^+)T_{> 0}Z(L)\psi\left(\bigcap_{u' \in U_{v'^{-1}, > 0}^+} \pi_{U_J^+}(u'g')^{-1}U_{w_0^J, \geq 0}^+\right). \end{aligned}$$

By 3.7, $z \in Z_{J, \geq 0}$ if and only if $v, v' \in W^J$ and $l \in L_{\geq 0}Z(L)$. The theorem is proved. \square

3.15. It is known that $G_{\geq 0} = \bigsqcup_{w, w' \in W} U_{w, > 0}^- T_{> 0} U_{w', > 0}^+$, where for any $w, w' \in W$, $U_{w, > 0}^- T_{> 0} U_{w', > 0}^+$ is a semi-algebraic cell (see [L1, 2.11]) and is a connected component of $B^+ \dot{w} B^+ \cap B^- \dot{w}' B^-$ (see [FZ]). Moreover, Rietsch proved in [R2, 2.8] that $\mathcal{B}_{\geq 0} = \bigsqcup_{v \leq w} \mathcal{R}_{v, w, > 0}$, where for any $v, w \in W$ such that $v \leq w$, $\mathcal{R}_{v, w, > 0}$ is a semi-algebraic cell and is a connected component of $\mathcal{R}_{v, w}$.

The following result generalizes these facts.

Corollary 3.16. $\overline{G_{> 0}} = \bigsqcup_{J \subset I} \bigsqcup_{\substack{v, w, v', w' \in W^J \\ v \leq w, v' \leq w'}} \bigsqcup_{y, y' \in W_J} Z_{J, > 0}^{v, w, v', w'; y, y'}$. Moreover,

for any $v, w, v', w' \in W^J$, $y, y' \in W_J$ with $v \leq w$, $v' \leq w'$, $Z_{J, > 0}^{v, w, v', w'; y, y'}$ is a connected component of $Z_J^{v, w, v', w'; y, y'}$ and is a semi-algebraic cell which is isomorphic to $\mathbf{R}_{> 0}^d$, where $d = l(w) + l(w') + 2l(w_0^J) + |J| - l(v) - l(v') - l(y) - l(y')$.

Proof. $\mathcal{P}_{v, w, > 0}^J$ (resp. $\mathcal{P}_{v', w', > 0}^J$) is a connected component of $\mathcal{P}_{v, w}^J$ (resp. $\mathcal{P}_{v', w'}^J$) (see [L3]). Thus $\{(P, Q, \gamma) \in Z_J^{v, w, v', w'; y, y'} \mid P \in \mathcal{P}_{v, w, > 0}^J, \psi(Q) \in \mathcal{P}_{v', w', > 0}^J\}$ is open and closed in $Z_J^{v, w, v', w'; y, y'}$. To prove that $Z_{J, > 0}^{v, w, v', w'; y, y'}$ is a connected component of $Z_J^{v, w, v', w'; y, y'}$, it is enough to prove that $Z_{J, > 0}^{v, w, v', w'; y, y'}$ is a connected component of $\{(P, Q, \gamma) \in Z_J^{v, w, v', w'; y, y'} \mid P \in \mathcal{P}_{v, w, > 0}^J, \psi(Q) \in \mathcal{P}_{v', w', > 0}^J\}$.

Assume that $g \in G_{\mathbf{v}_+, \mathbf{w}, > 0}$, $g' \in G_{\mathbf{v}'_+, \mathbf{w}', > 0}$ and $l \in L$. We have that $({}^g P_J)^{B^+}$ is the unique element $B \in \mathcal{R}_{v, w}$ that is contained in ${}^g P_J$ (see 1.4). Therefore $({}^g P_J)^{B^+} = {}^g B^+$. Similarly, $({}^g P_J)^{B^-} = {}^g \dot{w}_0^J B^+$, $(\psi(g'^{-1})Q_J)^{B^+} = \psi(g'^{-1})\dot{w}_0^J B^-$ and $(\psi(g'^{-1})Q_J)^{B^-} = \psi(g')^{-1} B^-$. Thus $\text{pos}\left(({}^g P_J)^{B^+}, {}^{gl}\psi(g')\left((\psi(g'^{-1})Q_J)^{B^+}\right)\right) = \text{pos}(B^+, {}^{l\dot{w}_0^J} B^-)$ and $\text{pos}\left(({}^g P_J)^{B^-}, {}^{gl}\psi(g')\left((\psi(g'^{-1})Q_J)^{B^-}\right)\right) = \text{pos}(\dot{w}_0^J B^+, {}^l B^-)$. Therefore we have that $({}^g P_J, \psi(g')^{-1} Q_J, gH_{P_J} l U_{Q_J} \psi(g')) \in Z_J^{v, w, v', w'; y, y'}$ if and only if $l \in B^+ \dot{y} \dot{w}_0 B^+ \dot{w}_0 \dot{w}_0^J \cap \dot{w}_0^J B^+ \dot{y}' \dot{w}_0 B^+ \dot{w}_0 = B^+ \dot{y} B^- \dot{w}_0^J \cap \dot{w}_0^J B^+ \dot{y}' B^-$.

Note that $L \cap B^+ \subset \dot{w}_0^J B^-$. Thus for any $x \in W_J$, $(L \cap B^+) \dot{x} (L \cap B^+) \subset B^+ \dot{x} \dot{w}_0^J B^- \dot{w}_0^J$. Therefore,

$$\begin{aligned} L \cap B^+ \dot{y} B^- \dot{w}_0^J &= \bigsqcup_{x \in W_J} (L \cap B^+) \dot{x} (L \cap B^+) \cap B^+ \dot{y} B^- \dot{w}_0^J \\ &= (L \cap B^+) \dot{y} \dot{w}_0^J (L \cap B^+). \end{aligned}$$

Similarly, $L \cap \dot{w}_0^J B^+ \dot{y}' B^- = (L \cap B^-) \dot{w}_0^J \dot{y}' (L \cap B^-)$.

Then $\{(P, Q, \gamma) \in Z_J^{v, w, v', w'; y, y'} \mid P \in \mathcal{P}_{v, w, > 0}^J, \psi(Q) \in \mathcal{P}_{v', w', > 0}^J\}$ is isomorphic to $G_{v, w, > 0} \times G_{v', w', > 0} \times ((L \cap B^+) \dot{y} \dot{w}_0^J (L \cap B^+) \cap (L \cap B^-) \dot{w}_0^J \dot{y}' (L \cap B^-)) / Z(L)$. Note

that $((L \cap B^+) \dot{y} \dot{w}_0^J (L \cap B^+) \cap (L \cap B^-) \dot{w}_0^J \dot{y}' (L \cap B^-)) \cap L_{\geq 0} = U_{yw_0^J, > 0}^- T_{> 0} U_{w_0^J y', > 0}^+$.

Therefore

$$\begin{aligned} Z_{J, > 0}^{v, w, v', w', y, y'} &\cong G_{v, w, > 0} \times G_{v', w', > 0} \times U_{yw_0^J, > 0}^- T_{> 0} U_{w_0^J y', > 0}^+ / (Z(L) \cap T_{> 0}) \\ &\cong \mathbf{R}_{> 0}^{l(w) + l(w') + 2l(w_0^J) + |J| - l(v) - l(v') - l(y) - l(y')}. \end{aligned}$$

By 3.15, we have that $U_{yw_0^J, > 0}^- T_{> 0} U_{w_0^J y', > 0}^+ / (Z(L) \cap T_{> 0})$ is a connected component of $((L \cap B^+) \dot{y} \dot{w}_0^J (L \cap B^+) \cap (L \cap B^-) \dot{w}_0^J \dot{y}' (L \cap B^-)) / Z(L)$. The corollary is proved. \square

ACKNOWLEDGEMENTS

I thank George Lusztig for suggesting the problem and for many helpful discussions. I also thank the referee for pointing out several mistakes in the original manuscript and for some useful comments, especially concerning 3.8, 3.10 and 3.15.

REFERENCES

- [DP] De Concini, C., and C. Procesi, *Complete symmetric varieties*, Invariant theory (Montecatini 1982), Lect. Notes Math., vol. 996, Springer, 1983, pp. 1-44. MR **85e**:14070
- [FZ] Fomin, S., and A. Zelevinsky, *Double Bruhat cells and total positivity*, J. Amer. Math. Soc. **12** (1999), no. 2, 335–380. MR **2001f**:20097
- [MR] Marsh, R., and K. Rietsch, *Parametrizations of flag varieties*, RT/0307017.
- [KL] Kazhdan, D., and G. Lusztig, *Representations of Coxeter groups and Hecke algebras*, Invent. Math. **53** (1979), 165-184. MR **81j**:20066
- [L1] Lusztig, G., *Total positivity in reductive groups*, Lie Theory and Geometry: in honor of Bertram Kostant, Progress in Math. **123** (1994), 531-568. MR **96m**:20071
- [L2] Lusztig, G., *Total positivity and canonical bases*, Algebraic groups and Lie groups (ed. G.I. Lehrer), Cambridge Univ. Press, 1997, pp. 281-295. MR **2000j**:20089
- [L3] Lusztig, G., *Total positivity in partial flag manifolds*, Represent. Theory **2** (1998), 70-78 (electronic). MR **2000b**:20060
- [L4] Lusztig, G., *Introduction to total positivity*, Positivity in Lie theory: open problems (eds. J. Hilgert, J.D. Lawson, K.H. Neeb, E.B. Vinberg), de Gruyter Berlin, 1998, pp. 133-145. MR **99h**:20077
- [L5] Lusztig, G., *Parabolic character sheaves, I*, RT/0302151.
- [L6] Lusztig, G., *Parabolic character sheaves, II*, RT/0302317.
- [R1] Rietsch, K., *Total positivity and real flag varieties*, MIT thesis (1998).
- [R2] Rietsch, K., *An algebraic cell decomposition of the nonnegative part of a flag variety*, J. Algebra **213** (1999), no. 1, 144-154. MR **2000e**:14086

DEPARTMENT OF MATHEMATICS, MASSACHUSETTS INSTITUTE OF TECHNOLOGY, CAMBRIDGE, MASSACHUSETTS 02139

E-mail address: hugo@math.mit.edu