TOTAL POSITIVITY IN THE DE CONCINI-PROCESI COMPACTIFICATION

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Abstract. We study the nonnegative part $G^>_{\geq 0}$ of the De Concini-Procesi compactification of a semisimple algebraic group $G$, as defined by Lusztig. Using positivity properties of the canonical basis and parametrization of flag varieties, we will give an explicit description of $G^>_{\geq 0}$. This answers the question of Lusztig in Total positivity and canonical bases, Algebraic groups and Lie groups (ed. G.I. Lehrer), Cambridge Univ. Press, 1997, pp. 281-295. We will also prove that $G^>_{\geq 0}$ has a cell decomposition which was conjectured by Lusztig.

0. Introduction

Let $G$ be a connected split semisimple algebraic group of adjoint type over $\mathbb{R}$. We identify $G$ with the group of its $\mathbb{R}$-points. In [DP], De Concini and Procesi defined a compactification $\overline{G}$ of $G$ and decomposed it into strata indexed by the subsets of a finite set $I$. We will denote these strata by $\{Z_J \mid J \subset I\}$. Let $G_{>0}$ be the set of strictly totally positive elements of $G$ and $G_{\geq 0}$ be the set of totally positive elements of $G$ (see [L1]). We denote by $G^>_{\geq 0}$ the closure of $G_{>0}$ in $G$. The main goal of this paper is to give an explicit description of $G^>_{\geq 0}$ (see 3.14). This answers the question in [L4, 9.4]. As a consequence, I will prove in 3.17 that $G^>_{\geq 0}$ has a cell decomposition which was conjectured by Lusztig.

To achieve our goal, it is enough to understand the intersection of $G^>_{\geq 0}$ with each stratum. We set $Z_{J,>0} = G^>_{\geq 0} \cap Z_J$. Note that $Z_I = G$ and $Z_{I,\geq 0} = G_{\geq 0}$. We define $Z_{J,>0}$ as a certain subset of $Z_{J,\geq 0}$ analogous to $G_{>0}$ for $G_{\geq 0}$ (see 2.6). When $G$ is simply-laced, we will prove in 2.7 a criterion for $Z_{J,>0}$ in terms of its image in certain representations of $G$, which is analogous to the criterion for $G_{>0}$ in [L4] 5.4. As Lusztig pointed out in [L2], although the definition of total positivity was elementary, many of the properties were proved in a non-elementary way, using canonical bases and their positivity properties. Our Theorem 2.7 is an example of this phenomenon. As a consequence, we will see in 2.9 that $Z_{J,>0}$ is the closure of $Z_{J,\geq 0}$ in $Z_J$.

Note that $Z_J$ is a fiber bundle over the product of two flag manifolds. Then understanding $Z_{J,>0}$ is equivalent to understanding the intersection of $Z_{J,\geq 0}$ with each fiber. In 3.5, we will give a characterization of $Z_{J,>0}$ which is analogous to the elementary fact that $G_{>0} = \bigcap_{g \in G_{>0}} g^{-1}G_{>0}$. It allows us to reduce our problem to the problem of understanding certain subsets of some unipotent groups. Using the
parametrization of the totally positive part of the flag varieties (see [MR]), we will give an explicit description of the subsets of $G$ (see 3.7). Thus our main theorem can be proved.

1. Preliminaries

1.1. We will often identify a real algebraic variety with the set of its $R$-rational points. Let $G$ be a connected semisimple adjoint algebraic group defined and split over $R$, with a fixed épingle $(T, B^+, B^-, x_i, y_i; i \in I)$ (see [LI] 1.1)). Let $U^+, U^-$ be the unipotent radicals of $B^+, B^-$. Let $X$ (resp. $Y$) be the free abelian group of all homomorphism of algebraic groups $T \rightarrow \mathbb{R}^*$ (resp. $\mathbb{R}^* \rightarrow T$) and $(,): Y \times X \rightarrow Z$ be the standard pairing. We write the operation in these groups as addition. For $i \in I$, let $\alpha_i \in X$ be the simple root such that $tx_i(a)t^{-1} = x_i(a)^{\alpha_i(t)}$ for all $a \in R, t \in T$ and let $\alpha_i^+ \in Y$ be the simple coroot corresponding to $\alpha_i$. For any root $\alpha$, we denote by $U_\alpha$ the root subgroup corresponding to $\alpha$.

There is a unique isomorphism $\psi: G \xrightarrow{\sim} G^{opp}$ (the opposite group structure) such that $\psi(x_i(a)) = y_i(a), \psi(y_i(a)) = x_i(a)$ for all $i \in I$, $a \in R$ and $\psi(t) = t$, for all $t \in T$.

If $P$ is a subgroup of $G$ and $g \in G$, we write $gP$ instead of $gPg^{-1}$.

For any algebraic group $H$, we denote the Lie algebra of $H$ by $\text{Lie}(H)$ and the center of $H$ by $Z(H)$.

For any variety $X$ and an automorphism $\sigma$ of $X$, we denote the fixed point set of $\sigma$ on $X$ by $X^\sigma$.

For any group, We will write $1$ for the identity element of the group.

For any finite set $X$, we will write $|X|$ for the cardinal of $X$.

1.2. Let $N(T)$ be the normalizer of $T$ in $G$ and $s_i = x_i(-1)y_i(1)x_i(-1) \in N(T)$ for $i \in I$. Set $W = N(T)/T$ and $s_i$ to be the image of $s_i$ in $W$. Then $W$ together with $(s_i)_{i \in I}$ is a Coxeter group.

Define an expression for $w \in W$ to be a sequence $w = (w_{(0)}, w_{(1)}, \ldots, w_{(n)})$ in $W$, such that $w_{(0)} = 1, w_{(n)} = w$ and for any $j = 1, 2, \ldots, n$, $w_{(j-1)}^{-1}w_{(j)} = 1$ or $s_i$ for some $i \in I$. An expression $w = (w_{(0)}, w_{(1)}, \ldots, w_{(n)})$ is called reduced if $w_{(j-1)} < w_{(j)}$ for all $j = 1, 2, \ldots, n$. In this case, we will set $l(w) = n$. It is known that $l(w)$ is independent of the choice of the reduced expression. Note that if $w$ is a reduced expression of $w$, then for all $j = 1, 2, \ldots, n$, $w_{(j-1)}^{-1}w_{(j)} = s_i$ for some $i \in J$. Sometimes we will simply say that $s_i, s_{i_{2}} \ldots s_{i_{n}}$ is a reduced expression of $w$.

For $w \in W$, set $w = s_{i_{1}}s_{i_{2}} \ldots s_{i_{n}}$ where $s_{i_{1}}s_{i_{2}} \ldots s_{i_{n}}$ is a reduced expression of $w$. It is well known that $w$ is independent of the choice of the reduced expression $s_{i_{1}}s_{i_{2}} \ldots s_{i_{n}}$ of $w$.

Assume that $w = (w_{(0)}, w_{(1)}, \ldots, w_{(n)})$ is a reduced expression of $w$ and $w_{(j)} = w_{(j-1)}s_{i_{j}}$ for all $j = 1, 2, \ldots, n$. Suppose that $v \leq w$ for the standard partial order in $W$. Then there is a unique sequence $v = (v_{(0)}, v_{(1)}, \ldots, v_{(n)})$ such that $v_{(0)} = 1, v_{(n)} = v, v_{(j)} \in \{v_{(j-1)}s_{i_{j}}, v_{(j-1)}s_{i_{j}}\}$ and $v_{(j-1)} < v_{(j-1)}s_{i_{j}}$ for all $j = 1, 2, \ldots, n$ (see [MR] 3.5). $v$ is called the positive subexpression of $w$. We define

$$J_{\mathbf{v}}^+ = \{ j \in \{1, 2, \ldots, n\} \mid v_{(j-1)} < v_{(j)} \},$$

$$J_{\mathbf{v}}^- = \{ j \in \{1, 2, \ldots, n\} \mid v_{(j-1)} = v_{(j)} \}.$$ 

Then by the definition of $v$, we have $\{1, 2, \ldots, n\} = J_{\mathbf{v}}^+ \cup J_{\mathbf{v}}^-$. 

1.3. Let $\mathcal{B}$ be the variety of all Borel subgroups of $G$. For $B, B'$ in $\mathcal{B}$, there is a unique $w \in W$, such that $(B, B')$ is in the $G$-orbit on $\mathcal{B} \times \mathcal{B}$ (diagonal action) that contains $(B^+, w B^+)$. Then we write $\text{pos}(B, B') = w$. By the definition of pos, $\text{pos}(B, B') = \text{pos}(g B, g B')$ for any $B, B' \in \mathcal{B}$ and $g \in G$.

For any subset $J$ of $I$, let $W_J$ be the subgroup of $W$ generated by $\{s_j \mid j \in J\}$ and let $w_0$ be the unique element of maximal length in $W_J$. (We will simply write $w_0$ as $w_0$.) We denote by $P_J$ the subgroup of $G$ generated by $B^+$ and by $\{y_j(a) \mid j \in J, a \in R\}$ and denote by $\mathcal{P}^J$ the variety of all parabolic subgroups of $G$ conjugated to $P_J$. If and only if $\{\text{pos}(B_1, B_2) \mid B_1, B_2 \text{ are Borel subgroups of } P\} = W_J$.

1.4. For any parabolic subgroup $P$ of $G$, define $U_P$ to be the unipotent radical of $P$ and $H_P$ to be the inverse image of the connected center of $P/U_P$ under $P \to P/U_P$.

If $B$ is a Borel subgroup of $G$, then so is $P^B = (P \cap B)U_P$.

It is easy to see that for any $g \in H_P$, we have $g(P^B) = P^B$. Moreover, $P^B$ is the unique Borel subgroup $B'$ in $P$ such that $\text{pos}(B, B') \in W_J$, where $W_J$ is the set of minimal length coset representatives of $W/W_J$ (see [L5 3.2(a)]).

Let $P, Q$ be parabolic subgroups of $G$. We say that $P, Q$ are opposed if their intersection is a common Levi of $P, Q$. (We then write $P \cong Q$.) It is easy to see that if $P \cong Q$, then for any parabolic subgroup $B$ of $P$ and $B'$ of $Q$, we have $\text{pos}(B, B') \in W_J w_0$.

For any subset $J$ of $I$, define $J^* \subset I$ by $\{Q \mid Q \cong P \text{ for some } P \in \mathcal{P}^J\} = \mathcal{P}^{J^*}$. Then we have $(J^*)^* = J$. Let $Q_J$ be the subgroup of $G$ generated by $B^-$ and by $\{x_j(a) \mid j \in J, a \in R\}$. We have $Q_J \in \mathcal{P}^{J^*}$ and $P_J \cong Q_J$. Moreover, for any $P \in \mathcal{P}^J$, we have $P = g P_J$ for some $g \in G$. Thus $\psi(P) = \psi(g)^{-1} Q_J \in \mathcal{P}^{J^*}$.

1.5. Recall the following definitions from [L1].

For any $w \in W$, assume that $w = s_{i_1}s_{i_2}\cdots s_{i_n}$ is a reduced expression of $w$. Define $\phi^\pm : R_{>0}^n \to U^\pm$ by

$$\phi^+(a_1, a_2, \ldots, a_n) = x_{i_1}(a_1)x_{i_2}(a_2)\cdots x_{i_n}(a_n),$$
$$\phi^-(a_1, a_2, \ldots, a_n) = y_{i_1}(a_1)y_{i_2}(a_2)\cdots y_{i_n}(a_n).$$

Let $U_{w,>0}^\pm = \phi^\pm(R_{>0}^n) \subset U^\pm$, $U_{w,0}^\pm = \phi^\pm(R_{>0}^n) \subset U^\pm$. Then $U_{w,>0}^\pm$ and $U_{w,0}^\pm$ are independent of the choice of the reduced expression of $w$. We will simply write $U_{w,>0}^\pm$ as $U^\pm_{>0}$ and $U_{w,0}^\pm$ as $U^\pm_0$.

Let $T_{>0}$ be the submonoid of $T$ generated by the elements $\chi(a)$ for $\chi \in Y$ and $a \in R_{>0}$.

$G_{>0}$ is the submonoid $U_{>0}^\pm T_{>0} U_{>0} = U_{>0}^\pm T_{>0} U_{>0}^\pm$ of $G$.

$G_{\geq 0}$ is the submonoid $U_{\geq 0}^\pm T_{>0} U_{>0} = U_{\geq 0}^\pm T_{>0} U_{>0}^\pm$ of $G_{>0}$.

$B_{>0}$ is the subset $\{u B^- \mid u \in U_{>0}^+\} = \{u B^+ \mid u \in U_{>0}\}$ of $B$ and $B_{\geq 0}$ is the closure of $B_{>0}$ in the manifold $B$.

For any subset $J$ of $I$, $\mathcal{P}^J_{>0} = \{P \in \mathcal{P}^J \mid \exists B \in B_{>0}, \text{ such that } B \subset P\}$ and $\mathcal{P}^J_{\geq 0} = \{P \in \mathcal{P}^J \mid \exists B \in B_{\geq 0}, \text{ such that } B \subset P\}$ are subsets of $\mathcal{P}^J$.

1.6. For any $w, w' \in W$, define

$$\mathcal{R}_{w, w'} = \{B \in \mathcal{B} \mid \text{pos}(B^+, B) = w', \text{pos}(B^-, B) = w_0 w\}.$$
It is known that $R_{w,w'}$ is nonempty if and only if $w \leq w'$ for the standard partial order in $W$ (see [KL]). Now set
\[ R_{w,w',>0} = B_{>0} \cap R_{w,w'}. \]
Then $R_{w,w',>0}$ is a connected component of $R_{w,w'}$ and is a semi-algebraic cell (see [R2] 2.8). Furthermore, $B = \bigcup_{w \leq w'} R_{w,w'}$ and $B_{\geq 0} = \bigcup_{w \leq w'} R_{w,w',>0}$. Moreover, for any $u \in U_{w,-1,>0}$, we have $uR_{w,w',>0} \subset R_{1,w',>0}$ (see [R2] 2.2).

Let $J$ be a subset of $I$. Define $\pi_J : B \to \mathcal{P}_J$ to be the map which sends a Borel subgroup to the unique parabolic subgroup in $\mathcal{P}_J$ that contains the Borel subgroup. For any $w, w' \in W$ such that $w \leq w'$ and $w' \in W_J$, set $\mathcal{P}_{w,w'}^J = \pi_J(R_{w,w'})$ and $\mathcal{P}_{w,w',>0}^J = \pi_J(R_{w,w',>0})$. We have $\mathcal{P}_{\geq 0}^J = \bigcup_{w \leq w'} \mathcal{P}_{w,w',>0}^J$ and $\mathcal{P}_J |_{R_{w,w',>0}}$ maps $R_{w,w',>0}$ bijectively onto $\mathcal{P}_{w,w',>0}^J$ (see [R1] Chapter 4, 3.2). Hence, for any $u \in U_{w,-1,>0}$, we have $u\mathcal{P}_{w,w',>0}^J = \mathcal{P}_{w,w',>0}^J(uR_{w,w',>0}) \subset \mathcal{P}_J |_{\mathcal{P}_{w,w',>0}}$.

**1.7.** Define $\pi_T : B \to T$ by $\pi_T(utu') = t$ for $u \in U^-, t \in T, u' \in U^+$. Then for $b_j \in B^-, b_j' \in B^+ \cdot B^+, b_j \in B^+$, we have $\pi_T(b_j b_j') = \pi_T(b_j) \pi_T(b_j') \pi_T(b_j)$.

Let $J$ be a subset of $I$. We denote by $\Phi_J^+$ the set of roots that are a linear combination of $\{\alpha_j | j \in J\}$ with nonnegative coefficients. We will simply write $\Phi_J^+$ as $\Phi^+$ and we will call a root $\alpha$ positive if $\alpha \in \Phi^+$. In this case, we will simply write $\alpha > 0$. Define $U_J^+$ to be the subgroup of $U^+$ generated by $\{U_\alpha | \alpha \in \Phi_J^+\}$ and $U_J^-$ to be the subgroup of $U^+$ generated by $\{U_\alpha | \alpha \in \Phi^+ - \Phi_J^+\}$. Then $U^- \cdot T \cdot U_J^+ \cdot U_J^- = U^- \cdot T \cdot U_J^+$. Thus it is easy to see that for any $a, b \in G$ such that $a, ab \in B^- \cdot B^+$, we have $\pi_{U_J^+}(ab) = \pi_{U_J^+}(\pi_{U^+}(a)b)$. Since $U_J^+$ is a normal subgroup of $U^+$, $\pi_{U_J^+} |_{U^+}$ is a homomorphism of $U^+$ onto $U_J^+$. Moreover, we have
\[ \pi_{U_J^+}(x_i(a)) = \begin{cases} x_i(a), & \text{if } i \in J; \\ 1, & \text{otherwise.} \end{cases} \]

Thus $\pi_{U_J^+}(U_{\alpha,-}^+) = U_{\alpha,-}^+$ and $\pi_{U_J^+}(U_{\alpha,>0}^+) = U_{\alpha,>0}^+$.

Let $U_{J}^-$ be the subgroup of $U^-$ generated by $\{U_{-\alpha} | \alpha \in \Phi_J^+\}$ and $U_{J}^-$ to be the subgroup of $U^-$ generated by $\{U_{-\alpha} | \alpha \in \Phi^+ - \Phi_J^+\}$. Then we define $\pi_{U_{J}^-} : U^- \to U_{J}^-$ by $\pi_{U_{J}^-}(u_1 u_2) = u_1$ for $u_1 \in U_{J}, u_2 \in U_{J}^-$. (We will simply write $\pi_{U_{J}^-}$ as $\pi_{U^-}$.) We have $\pi_{U_{J}^-}(U_{\alpha,-}^+) = U_{\alpha,-}^+$ and $\pi_{U_{J}^-}(U_{\alpha,>0}^-) = U_{\alpha,>0}^-$.  

**1.8.** For any vector space $V$ and a nonzero element $v$ of $V$, we denote the image of $v$ in $P(V)$ by $[v]$.

If $(V, \rho)$ is a representation of $G$, we denote by $(V^*, \rho^*)$ the dual representation of $G$. Then we have the standard isomorphism $St_V : V \otimes V^* \cong \End(V)$ defined by $St_V(v \otimes v^*)(v) = \rho^*(v')v$ for all $v, v' \in V, v^* \in V^*$. Now we have the $G \times G$ action on $V \otimes V^*$ by $(g_1, g_2) : (v \otimes v^*) = (g_1 v) \otimes (g_2 v^*)$ for all $g_1, g_2 \in G, v \in V, v^* \in V^*$ and the $G \times G$ action on $\End(V)$ by $[(g_1, g_2) : f](v) = g_1 f(g_2^{-1} v)$ for all $g_1, g_2 \in G, f \in \End(V), v \in V$. The standard isomorphism between $V \otimes V^*$ and $\End(V)$ commutes with the $G \times G$ action. We will identify $\End(V)$ with $V \otimes V^*$ via the standard isomorphism.
2. The strata of the De Concini-Procesi Compactification

2.1. Let $\mathcal{V}_G$ be the projective variety whose points are the dim($G$)-dimensional Lie subalgebras of Lie($G \times G$). For any subset $J$ of $I$, define

$$Z_J = \{(P, Q, \gamma) \mid P \in \mathcal{P}^J, Q \in \mathcal{P}^{J^*}, \gamma = H_P g U_Q, P \cong Q\}$$

with the $G \times G$ action by $(g_1, g_2) \cdot (P, Q, H_P g U_Q) = (g_1 P, g_2 Q, H_{g_1} P (g_1 g_2^{-1}) U_{g_2} Q)$.

For $(P, Q, \gamma) \in Z_J$ and $g \in G$, we set

$$H_{P, Q, \gamma} = \{(l + u_1, \text{Ad}(g^{-1}) l + u_2) \mid l \in \text{Lie}(P \cap Q), u_1 \in \text{Lie}(U_P), u_2 \in \text{Lie}(U_Q)\}$$

Then $H_{P, Q, \gamma}$ is independent of the choice of $g$ (see [L6, 12.2]) and is an element of $\mathcal{V}_G$ (see [L6, 12.1]). Moreover, $(P, Q, \gamma) \rightarrow H_{P, Q, \gamma}$ is an embedding of $Z_J \subset \mathcal{V}_G$ (see [L6, 12.2]). We will identify $Z_J$ with the subvariety of $\mathcal{V}_G$ defined above. Then we have $\tilde{G} = \bigsqcup_{J \subset I} Z_J$, where $\tilde{G}$ is the De Concini-Procesi compactification of $G$ (see [L6, 12.3]). We will call $\{Z_J \mid J \subset I\}$ the strata of $\tilde{G}$ and $Z_I$ (resp. $Z_\emptyset$) the highest (resp. lowest) stratum of $\tilde{G}$. It is easy to see that $Z_I$ is isomorphic to $G$ and $Z_\emptyset$ is isomorphic to $\mathcal{B} \times \mathcal{B}$.

Set $z_J^g = (P_J, Q_J, H_{P_J, Q_J})$. Then $z_J^g \in Z_J$ (see 1.4) and $Z_J = (G \times G) \cdot z_J^g$.

Since $G$ is adjoint, we have an isomorphism $\chi : T \overset{\cong}{\rightarrow} (\mathbb{R}^*)^I$ defined by $\chi(t) = (\alpha_i(t)^{-1})_{i \in J}$. We denote the closure of $T$ in $\tilde{G}$ by $\tilde{T}$. We have $H_{P_J, Q_J, H_{P_J, Q_J}} = \{(l + u_1, l + u_2) \mid l \in \text{Lie}(P_J \cap Q_J), u_1 \in U_P, u_2 \in U_Q\}$. Moreover, for any $t \in Z(P_J \cap Q_J)$, $H_t$ is the subspace of $\text{Lie}(G) \times \text{Lie}(G)$ spanned by the elements $(l, (l, \text{Ad}(t^{-1}) u_1), (\text{Ad}(t) u_2, u_2))$, where $l \in \text{Lie}(P_J \cap Q_J), u_1 \in U_P, u_2 \in U_Q$. Thus it is easy to see that $z_J^g = \lim_{t_i \rightarrow t, \forall j \notin J} \chi^{-1}((t_i)_{i \in I}) \in T$.

Proposition 2.2. The automorphism $\psi$ of the variety $G$ (see 1.1) can be extended in a unique way to an automorphism $\tilde{\psi}$ of $\tilde{G}$. Moreover, $\tilde{\psi}(P, Q, \gamma) = (\psi(Q), \psi(P), \psi(\gamma)) \in Z_J$ for $J \subset I$ and $(P, Q, \gamma) \in Z_J$.

Proof. The map $\psi : G \rightarrow G$ induces a bijective map $\psi : \text{Lie}(G) \rightarrow \text{Lie}(G)$. Moreover, we have $\psi(\text{Ad}(g)v) = \text{Ad}(\psi(g)^{-1}) \psi(v)$ and $\psi(v + v') = \psi(v) + \psi(v')$ for $g \in G, v, v' \in \text{Lie}(G)$. Now define $\delta : \text{Lie}(G) \times \text{Lie}(G) \rightarrow \text{Lie}(G) \times \text{Lie}(G)$ by $\delta(v, v') = (\psi(v'), \psi(v))$ for $v, v' \in \text{Lie}(G)$. Then $\delta$ induces a bijection $\tilde{\psi} : \mathcal{V}_G \rightarrow \mathcal{V}_G$.

Note that for any $g \in G$, we have $H_g = \{(v, \text{Ad}(g)v) \mid v \in \text{Lie}(G)\}$ and $\tilde{\psi}(H_g) = \{(\text{Ad}(\psi(g)^{-1}) v, \psi(v)) \mid v \in \text{Lie}(G)\} = H_{\psi(g)}$. Thus $\tilde{\psi}$ is an extension of the automorphism $\psi$ of $G$ into $\mathcal{V}_G$.

Now for any $(P, Q, \gamma) \in Z_J$ and $g \in G$, we have $\psi(P) \in \mathcal{P}^J$, $\psi(Q) \in \mathcal{P}^{J^*}$ and $\psi(Q) \cong \psi(g) \psi(P)$ (see 1.4). Thus $(\psi(Q), \psi(P), \psi(\gamma)) \in Z_J$. Moreover,

$$\tilde{\psi}(H_{P, Q, \gamma}) = \{(\text{Ad}(\psi(g)) \psi(l) + \psi(u_2), \psi(l) + \psi(u_1)) \mid l \in \text{Lie}(P \cap Q), u_1 \in \text{Lie}(U_P), u_2 \in \text{Lie}(U_Q)\}$$

$$= \{(l + u_2, \text{Ad}(\psi(g)^{-1}) l + u_1) \mid l \in \text{Lie}(\psi(Q) \cap \psi(g) \psi(P)), u_1 \in \text{Lie}(\psi(U_P)), u_2 \in \text{Lie}(\psi(U_Q))\}$$

Therefore $\tilde{\psi} |_{\tilde{G}}$ is an automorphism of $\tilde{G}$. Moreover, since $\tilde{G}$ is the closure of $G$, $\tilde{\psi} |_{\tilde{G}}$ is the unique automorphism of $\tilde{G}$ that extends the automorphism $\psi$ of $G$.

The proposition is proved.

\hfill $\Box$
2.3. For any \( \lambda \in X \), set \( \text{supp}(\lambda) = \{ i \in I \mid \langle \alpha_i^\vee, \lambda \rangle \neq 0 \} \).

In the rest of the section, I will fix a subset \( J \) of \( I \) and \( \lambda_1, \lambda_2 \in X^+ \) with \( \text{supp}(\lambda_1) = I - J, \text{supp}(\lambda_2) = J \). Let \( (V_{\lambda_1}, \rho_1) \) (resp. \( (V_{\lambda_2}, \rho_2) \)) be the irreducible representation of \( G \) with the highest weight \( \lambda_1 \) (resp. \( \lambda_2 \)). Assume that \( \dim V_{\lambda_1} = n_1, \dim V_{\lambda_2} = n_2 \) and \( \{ v_1, v_2, \ldots, v_{n_1} \} \) (resp. \( \{ v'_1, v'_2, \ldots, v'_{n_2} \} \)) is the canonical basis of \( (V_{\lambda_1}, \rho_1) \) (resp. \( (V_{\lambda_2}, \rho_2) \)), where \( v_1 \) and \( v'_1 \) are the highest weight vectors. Moreover, after reordering \( \{ 2, 3, \ldots, n_2 \} \), we could assume that there exists some integer \( n_0 \in \{ 1, 2, \ldots, n_2 \} \) such that for any \( i \in \{ 1, 2, \ldots, n_2 \} \), the weight of \( v'_i \) is of the form \( \lambda_2 - \sum_{j \in J} a_{ij} \alpha_j \) if and only if \( i \leq n_0 \).

Define \( i_J : G \to P(\text{End}(V_{\lambda_1})) \times P(\text{End}(V_{\lambda_2})) \) by \( i_J(g) = ([\rho_1(g)], [\rho_2(g)]) \).
Then since \( \lambda_1 + \lambda_2 \) is a dominant and regular weight, the closure of the image of \( i_J \) in \( P(\text{End}(V_{\lambda_1})) \times P(\text{End}(V_{\lambda_2})) \) is isomorphic to the De Concini-Procesi compactification of \( G \) (See [DP] 4.1). We will use \( i_J \) as the embedding of \( G \) into \( P(\text{End}(V_{\lambda_1})) \times P(\text{End}(V_{\lambda_2})) \). We will also identify \( \bar{G} \) with its image under \( i_J \).

2.4. Now with respect to the canonical basis of \( V_{\lambda_1} \) and \( V_{\lambda_2} \), we will identify \( \text{End}(V_{\lambda_1}) \) with \( gl(n_1) \) and \( \text{End}(V_{\lambda_2}) \) with \( gl(n_2) \). Thus we will regard \( \rho_1(g), \rho_2(g) \) as \( n_1 \times n_1 \) matrices and \( \rho_2(g), \rho_2'(g) \) as \( n_2 \times n_2 \) matrices. It is easy to see that (in terms of matrices) for any \( g \in G, \rho_1(g) = (M)^t \rho_1(g^{-1}) \) and \( \rho_2'(g) = (M)^t \rho_2(g^{-1}) \), where \( M \) is the transpose of the matrix \( M \). Now for any \( g_1, g_2 \in G, M_1 \in gl(n_1), M_2 \in gl(n_2) \), \( (g_1, g_2) \cdot M = \rho_1(g_1)M_1 \rho_2(g_2^{-1}) \) and \( (g_1, g_2) \cdot M_2 = \rho_2(g_1)M_2 \rho_2(g_2^{-1}) \).

Set \( L = P_J \cap Q_J \). Then \( L \) is a reductive algebraic group with the épingle \((T, B^+ \cap L, B^- \cap L, x_j, y_j; j \in J)\). Now let \( V_L \) be the subspace of \( V_{\lambda_2} \) spanned by \( \{ v'_1, v'_2, \ldots, v'_{n_2} \} \) and \( I_L = (a_{ij}) \in gl(n_2) \), where
\[
a_{ij} = \begin{cases} 1, & \text{if } i = j \in \{ 1, 2, \ldots, n_0 \}; \\ 0, & \text{otherwise}. \end{cases}
\]

Then \( V_L \) is an irreducible representation of \( L \) with the highest weight \( \lambda_2 \) and canonical basis \( \{ v'_1, v'_2, \ldots, v'_{n_2} \} \). Moreover, \( \lambda_2 \) is a dominant and regular weight for \( L \). Now set \( I_1 = \text{diag}(1, 0, 0, \ldots, 0) \in gl(n_1), I_2 = \text{diag}(1, 0, 0, \ldots, 0) \in gl(n_2) \).
Then
\[
 i_J(z_j^o) = \lim_{t_j \to -0, v_j \in J} i_J \left( \chi^{-1}((t_j)_{v_j \in I}) \right) = \left( [v_1 \otimes v'_1], \left[ \sum_{i=1}^{n_0} v'_i \otimes v^*_i \right] \right) = \left( [I_1], [I_L] \right),
\]
where \( \{ v_1^*, v_2^*, \ldots, v_{n_1}^* \} \) (resp. \( \{ v'_1^*, v'_2^*, \ldots, v'_{n_2}^* \} \)) is the dual basis in \( (V_{\lambda_1})^* \) (resp. \( (V_{\lambda_2})^* \)).

2.5. Recall that \( \text{supp}(\lambda_1) = I - J \). Thus for any \( P \in \mathcal{P}^J \), there is a unique \( P \)-stable line \( L_{\rho_1}(p) \) in \( (V_{\lambda_1}, \rho_1) \) and \( P \mapsto L_{\rho_1}(p) \) is an embedding of \( \mathcal{P}^J \) into \( P(V_{\lambda_1}) \). Similarly, for any \( Q \in \mathcal{P}^{J^*} \), there is a unique \( Q \)-stable line \( L_{\rho_1^*(Q)} \) in \( (V_{\lambda_1}^*, \rho_1^*) \) and \( Q \mapsto L_{\rho_1^*(Q)} \) is an embedding of \( \mathcal{P}^{J^*} \) into \( P(V_{\lambda_1}^*) \). It is easy to see \( L_{\rho_1^*(Q)} \mid [v_1], L_{\rho_1^*(Q)} \mid [v_1^*] \) and \( L_{\rho_1^*(Q)} = \rho_1(g) L_{\rho_1(p)} L_{\rho_1^*(Q)} = \rho_1^*(g) L_{\rho_1^*(Q)} \) for \( P \in \mathcal{P}^J, Q \in \mathcal{P}^{J^*}, g \in G \).

There are projections \( p_1 : P(\text{End}(V_{\lambda_1})) \times P(\text{End}(V_{\lambda_2})) \to P(\text{End}(V_{\lambda_1})) \) and \( p_2 : P(\text{End}(V_{\lambda_1})) \times P(\text{End}(V_{\lambda_2})) \to P(\text{End}(V_{\lambda_2})) \). It is easy to see that \( p_1 \mid z_j, p_2 \mid z_j \) commute with the \( G \times G \) action and \( p_1(z_j^o) = [v_1 \otimes v_1^*] = [L_{\rho_1(p_j)} \otimes L_{\rho_1^*(Q_j)}] \).
Now for any $g_1, g_2 \in G$, we have
\[ p_1((g_1, g_2) \cdot z_J^p) = [p_1(g_1)L_{p_1(P)} \otimes p_1^*(g_2)L_{p_1^*(Qj)}] = [L_{p_1(v_1P)} \otimes L_{p_1^*(v_2Qj)}]. \]
In other words, $p_1(z) = [L_{p_1(P)} \otimes L_{p_1^*(Qj)}]$ for $z = (P, Q, \gamma) \in Z_J$.

2.6. Let $G_{>0}$ be the closure of $G_0$ in $G$. Then $G_{>0}$ is also the closure of $G_0$ in $G$. We have $z_J^p \in G_{>0}$ (see 2.1). Now set
\[ Z_{J,>0} = Z_J \cap G_{>0}. \]

Since $\psi(G_{>0}) = G_{>0}$, we have $\psi(G_{>0}) = G_{>0}$. Moreover, $\psi(Z_J) = Z_J$ (see 2.2).

Therefore $\psi(Z_{J,>0}) = Z_{J,>0}$. Similarly, $(g_1, g_2^{-1}) \cdot Z_{J,>0} \subset Z_{J,>0}$ for any $g_1, g_2 \in G_0$. Thus $Z_{J,>0} \subset Z_{J,>0}$. Moreover, it is easy to see that $\psi(Z_{J,>0}) = Z_{J,>0}$.

Note that for any $u_1, u_2 \in U_{>0}, u_3, u_4 \in U_{>0}, t, t' \in T_{>0}$, we have
\[ (u_1u_2t, u_3^{-1}u_4^{-1}t') \cdot z_J^p = (u_1u_2, u_3^{-1}u_4^{-1}) \cdot (P_J, Q_J, H_{P_J}t't'U_{Q_J}). \]

Thus
\[ Z_{J,>0} = \{(u_1, u_2^{-1}) \cdot (P_J, Q_J, H_{P_J}t't'U_{Q_J}) \mid u_1 \in U_{>0}, u_2 \in U_{>0}, l \in L_{>0}\} = \{(u'_1t, u'_{2}^{-1}) \cdot z_J^p \mid u'_1 \in U_{>0}, u'_2 \in U_{>0}, t \in T_{>0}\}. \]

Moreover, for any $u_1, u'_1 \in U^-, u_2, u'_2 \in U^+$ and $t, t' \in T$, it is easy to see that
\[ (u_1t, u_2) \cdot z_J^p = (u'_1t', u'_2) \cdot z_J^p \text{ if and only if } (u_1t)^{-1}u'_1t' \in l^{-1}H_{Q_J} \cap U^+ \subset IZ(L) \] and \[ u_1^{-1}u'_2 \in l^{-1}H_{Q_J} \cap U^+ \subset IZ(L) \text{ for some } l \in L, \text{ that is, } l = Z(L), u_1 = u'_1, u_2 = u'_2 \] and $t \in t'Z(L)$. Thus, $Z_{J,>0} \cong U_{>0} \times U_{>0} \times T_{>0}/(T_{>0} \cap Z(L)) \cong R^2_{>0}$. \(\text{Proof.} \)

Now I will prove a criterion for $Z_{J,>0}$.

**Theorem 2.7.** Assume that $G$ is simply-laced. Let $z \in Z_{J,>0}$. Then $z \in Z_{J,>0}$ if and only if $z$ satisfies the condition (*):
\[ (*) \quad i_J(z) = \left( [M_1], [M_2] \right) \text{ and } i_J(\psi(z)) = \left( [M_3], [M_4] \right) \text{ for some matrices } \]
\[ M_1, M_3 \in gl(n_1) \text{ and } M_2, M_4 \in gl(n_2) \text{ with all the entries in } R_{>0}. \]

**Proof.** If $z \in Z_{J,>0}$, then $z = (g_1, g_2^{-1}) \cdot z_J^p$, for some $g_1, g_2 \in G_{>0}$. Assume that $g_1 \cdot v_1 = \sum_{i=1}^{n_1} a_i v_i$ and $g_2^{-1} \cdot v_1^* = \sum_{i=1}^{n_1} b_i v_i^*$. Then for any $i = 1, 2, \ldots, n_1$, $a_i, b_i > 0$. Set $a_{ij} = a_i b_j$. Then $p_1(z) = [p_1(g_1)v_1, p_1(g_2)] = [(a_{ij})]$ is a matrix with all the entries in $R_{>0}$.

We have $p_2(z) = [p_2(g_1)I_L p_2(g_2)] = [p_2(g_1)I_{h_1} p_2(g_2)]$. Note that $p_2(g_1)I_{h_1} p_2(g_2)$ is a matrix with all the entries in $R_{>0}$ and $p_2(g_1), p_2(g_2)$, $(I_{L} - I_2)$ are matrices with all the entries in $R_{>0}$. Thus $p_2(g_1)(I_{L} - I_2) p_2(g_2)$ is a matrix with all its entries in $R_{>0}$. So $p_2(g_1)I_L p_2(g_2)$ is a matrix with all the entries in $R_{>0}$.

Similarly, $i_J(z) = \left( [M_3], [M_4] \right)$ for some matrices $M_3, M_4$ with all their entries in $R_{>0}$.

On the other hand, assume that $z$ satisfies the condition (*). Suppose that $z = (P, Q, \gamma)$ and $L_{p_1(P)} = \sum_{i=1}^{n_1} a_i v_i$, $L_{p_1^*(Q)} = \sum_{i=1}^{n_1} b_i v_i^*$. We may also assume that $a_{i_0} = b_{i_1} = 1$ for some integers $i_0, i_1 \in \{1, 2, \ldots, n_1\}$.

Set $M = (a_{ij}) \in gl(n_1)$, where $a_{ij} = a_i b_j$ for $i, j \in \{1, 2, \ldots, n_1\}$. Then $p_1(z) = [L_{p_1(P)} \otimes L_{p_1^*(Q)}] = [M]$. By the condition (*) and since $a_{i_0i_1} = a_{i_0} b_{i_1} = 1,$
we have that $M$ is a matrix with all its entries in $\mathbb{R}_{>0}$. In particular, for any $i \in \{1, 2, \ldots, n\}$, $a_{i^1} = a_i > 0$. Therefore $L_{P_i}(P) = \sum_{i=1}^{n_1} a_i v_i$, where $a_i > 0$ for all $i \in \{1, 2, \ldots, n\}$. By [11 5.1] (see also [13 3.4]), $P \in \mathcal{P}^{J}_{>0}$. Similarly, $\psi(Q) \in \mathcal{P}^{J}_{>0}$, so $L^{(j)}_U \psi(Q) P$ is the subspace of $V_{\mathcal{Q}}$ spanned by $\{v_1, v_2, \ldots, v_{n_0}^0\}$. By [R1, 5.1] (see also [L3, 3.4]), $P \in \mathcal{P}^{J}_{>0}$. Thus there exist $u_1 \in U_{>0}$, $u_2 \in U_{>0}$ and $l \in L^0$, such that $z = (u_1, u_2^{-1}) \cdot (P_l, Q_l, H_{P_l} l U_{Q_l})$.

We can express $u_1, u_2$ in a unique way as $u_1 = u_1' u_1''$, for some $u_1' \in U^- J^0$, $u_1'' \in U^+ J$ and $u_2 = u_2'' u_2'$, for some $u_2' \in U^- J^0$, $u_2'' \in U^+ J$ (see 1.7).

Recall that $V_L$ is the subspace of $V_{\mathcal{L}}$ spanned by $\{v_1', v_2', \ldots, v_{n_0}^0\}$. Let $V_L'$ be the subspace of $V_{\mathcal{L}}$ spanned by $\{v_{n_0+1}', v_{n_0+2}', \ldots, v_{n_2}'\}$. Then $u \cdot v - v \in V^{J}_L$ and $u \cdot V^J_L \subset V^J_L$, for all $v \in V_L$, $\alpha \notin \Phi_J$ and $u \in U^- \alpha$. Thus $u \cdot v - v \in V^{J}_L$ and $u \cdot V^J_L \subset V^J_L$, for all $v \in V_L$ and $u \notin U^- J$.

Similarly, let $V^*_L$ be the subspace of $V^*_{\mathcal{L}}$ spanned by $\{v_1^*, v_2^*, \ldots, v_{n_0}^*\}$ and $V^*_L$ be the subspace of $V^*_{\mathcal{L}}$ spanned by $\{v_{n_0+1}^*, v_{n_0+2}^*, \ldots, v_{n_2}^*\}$. Then for any $v^* \in V^*_L$ and $u \notin U^- J$, we have $u \cdot v - v \in V^*_{L}$ and $u V^*_L \subset V^*_{L}$.

We define a map $\pi_L : gl(n_0) \to gl(n_0)$ by

$$\pi_L((a_{ij})_{i,j \in \{1, 2, \ldots, n_2\}}) = (a_{ij})_{i,j \in \{1, 2, \ldots, n_2\}}.$$

Then for any $u \in U^- J$, $u' \in U^- J$ and $M \in gl(n_2)$, we have $\pi_L((u, u') \cdot M) = \pi_L(M)$. Set $M_2 = \rho_2(u_1 l U_L, \rho_2(u_2))$ and $l' = u_1'' l_1 u_2'' \in L$. Then

$$\pi_L(M_2) = \pi_L((u_1, u_2^{-1}) \cdot (\rho_2(l) l_1)) = \pi_L((u_1', u_2'^{-1}) \cdot (\rho_2(l) l_1))$$

$$= \pi_L((u_1', u_2'^{-1}) \cdot (\rho_2(l) l_1)) = \pi_L(\rho_2(l') l_1) = \rho_L(l').$$

Since $p_2(z) = [M_2]$, $M_2$ is a matrix with all its entries nonzero. Therefore $\rho_L(l') = \pi_L(M_2)$ is a matrix with all its entries nonzero. Thus $l' = t_1 l_1 t_2$, for some $l_1 \in U^- \cap L, l_2 \in U^+ \cap L, t_1 \in T$.

Set $\tilde{u}_1 = u_1' l_1$ and $\tilde{u}_2 = u_2'' l_2$. Then $\tilde{u}_1 p_{J_1} = u_1 u_{n_0}^{-1} l_1, P_j = u_1, P_j$. Similarly, we have $u_{n_0}^{-1} Q_j = u_2^{-1} Q_j$. So $z = (\tilde{u}_1, \tilde{u}_2^{-1}) \cdot (P_J, Q_J, H_{P_J} t_1 U_{Q_J})$.

Now for any $i_0, j_0 \in \{1, 2, \ldots, n_1\}$, define a map $\pi^{i_0, j_0}_1 : gl(n_0) \to \mathbb{R}$ by

$$\pi^{i_0, j_0}_1((a_{ij})_{i,j \in \{1, 2, \ldots, n_1\}}) = a_{i_0, j_0}$$

and for any $i_0, j_0 \in \{1, 2, \ldots, n_2\}$, define a map $\pi^{i_0, j_0}_2 : gl(n_0) \to \mathbb{R}$ by

$$\pi^{i_0, j_0}_2((a_{ij})_{i,j \in \{1, 2, \ldots, n_2\}}) = a_{i_0, j_0}.$$

Now $z = (\tilde{u}_1 t_1, \tilde{u}_2^{-1}) \cdot z^o_j$ and $\psi(z) = (\psi(\tilde{u}_2) t_1, \psi(\tilde{u}_1)^{-1}) \cdot z^o_j$.

Set

$$\tilde{M}_1 = \rho_1(\tilde{u}_1 t_1) I_1 \rho_1(\tilde{u}_2), \quad \tilde{M}_3 = \rho_1(\psi(\tilde{u}_2) t_1) I_1 \rho_1(\psi(\tilde{u}_1)),$$

$$\tilde{M}_2 = \rho_2(\tilde{u}_1 t_1) I_1 \rho_2(\tilde{u}_2), \quad \tilde{M}_4 = \rho_2(\psi(\tilde{u}_2) t_1) I_1 \rho_2(\psi(\tilde{u}_1)).$$

We have $\tilde{u}_1 - v_1 = \sum_{i=1}^{n_1} \frac{\pi^{i_0, j_0}_1(\tilde{M}_1)}{\pi^{i_0, j_0}_1(\tilde{M}_3)} v_i$ and $\tilde{u}_2 - v_1 = \sum_{i=1}^{n_1} \frac{\pi^{i_0, j_0}_1(\tilde{M}_2)}{\pi^{i_0, j_0}_1(\tilde{M}_4)} v_i$.

Moreover, let $V_0$ be the subspace of $V_{\mathcal{L}}$ spanned by $\{v_2', v_3', \ldots, v_{n_2}'\}$ and $V_{0^*}'$ be the subspace of $V_{\mathcal{L}}^*$ spanned by $\{v_2^*, v_3^*, \ldots, v_{n_2}^*\}$. Then we have $u \cdot V_0 \subset V_0^*$ for all $u \in U^- \cup U^+$ and $u' \in V_{0^*}^*$ for all $u' \in U^+$. 

Thus for all $i = 1, 2, \ldots, n_2$,
\[
\pi^2_{i,1}(M_2) = \pi^2_{i,1}(\rho_2(\tilde{u}_1 t_1) I_2 \rho_2(\tilde{u}_2)) + \pi^2_{i,1}(\rho_2(\tilde{u}_1 t_1)(I_L - I_2)\rho_2(\tilde{u}_2))
= \pi^2_{i,1}(\rho_2(\tilde{u}_1 t_1) I_2 \rho_2(\tilde{u}_2)).
\]

So $\tilde{u}_1 \cdot v'_1 = \sum_{i=1}^{n_2} \pi^2_{i,1}(M_2) v'_i$ and $\psi(\tilde{u}_2) \cdot v'_1 = \sum_{i=1}^{n_2} \pi^2_{i,1}(M_2) v'_i$. By [L2, 5.4], we have $\tilde{u}_1, \psi(\tilde{u}_2) \in U_{>0}$. Therefore to prove that $z \in Z_{J,>0}$, it is enough to prove that $t_1 \in T_{>0}Z(L)$, where $Z(L)$ is the center of $L$.

For any $g \in (U^-, U^+) \cdot \hat{T}$, $g$ can be expressed in a unique way as $g = (u_1, u_2) \cdot t$, for some $u_1 \in U^-$, $u_2 \in U^+$, $t \in \hat{T}$. Now define $\pi_T : (U^-, U^+) \cdot \hat{T} \to \hat{T}$ by $\pi_T((u_1, u_2) \cdot t) = t$ for all $u_1 \in U^-$, $u_2 \in U^+$, $t \in \hat{T}$. Note that $(U^-, U^+) \cdot \hat{T} \cap G_{>0}$ is the closure of $G_{>0}$ in $(U^-, U^+) \cdot \hat{T}$. Then $\pi_T((U^-, U^+) \cdot \hat{T} \cap G_{>0})$ is contained in the closure of $T_{>0} \cap \hat{T}$. In particular, $\pi_T(z) = t_1 t_J$ is contained in the closure of $T_{>0}$ in $\hat{T}$. Therefore for any $j \in J$, $\alpha_j(t_1) > 0$. Now let $t_2$ be the unique element in $T$ such that
\[
\alpha_j(t_2) = \begin{cases} 
\alpha_j(t_1), & \text{if } j \in J; \\
\alpha_j(t_1)^2, & \text{if } j \notin J.
\end{cases}
\]

Then $t_2 \in T_{>0}$ and $t_2^{-1} t_1 \in Z(L)$. The theorem is proved. \qed

Remark. Theorem 2.7 is analogous to the following statement in [L4, 5.4]: Assume that $G$ is simply laced and $V$ is the irreducible representation of $G$ with the highest weight $\lambda$, where $\lambda$ is a dominant and regular weight of $G$. For any $g \in G$, let $M(g)$ be the matrix of $g : V \to V$ with respect to the canonical basis of $V$. Then for any $g \in G$, $g \in G_{>0}$ if and only if $M(g)$ and $M(\psi(g))$ are matrices with all the entries in $R_{>0}$.

2.8. Before proving Corollary 2.9, I will introduce some technical tools.

Since $G$ is adjoint, there exists (in an essentially unique way) $\hat{G}$ with the epinglage $(\hat{T}, \hat{B}^+, \hat{B}^-, \hat{x}_i, \hat{y}_i; i \in \hat{I})$ and an automorphism $\sigma : \hat{G} \to G$ (over $R$) such that the following conditions are satisfied.

(a) $\hat{G}$ is connected semisimple adjoint algebraic group defined and split over $R$.
(b) $\hat{G}$ is simply laced.
(c) $\sigma$ preserves the epinglage, that is, $\sigma(\hat{T}) = \hat{T}$ and there exists a permutation $\hat{i} \to \sigma(\hat{i})$ of $\hat{I}$, such that $\sigma(\hat{x}_i(a)) = \hat{x}_{\sigma(i)}(a), \sigma(\hat{y}_i(a)) = \hat{y}_{\sigma(i)}(a)$ for all $i \in \hat{I}$ and $a \in R$.
(d) If $\hat{i}_1 \neq \hat{i}_2$ are in the same orbit of $\sigma : \hat{I} \to \hat{I}$, then $\hat{i}_1, \hat{i}_2$ do not form an edge of the Coxeter graph.
(e) $\hat{i}$ and $\sigma(\hat{i})$ are in the same connected component of the Coxeter graph, for any $\hat{i} \in \hat{I}$.
(f) There exists an isomorphism $\phi : \hat{G}^\sigma \to G$ (as algebraic groups over $R$) which is compatible with the epinglage of $G$ and the epinglage $(\hat{T}^\sigma, \hat{B}^+, \hat{B}^-, \hat{x}_p, \hat{y}_p; p \in \hat{I})$ of $\hat{G}^\sigma$, where $\hat{I}$ is the set of orbit of $\sigma : \hat{I} \to \hat{I}$ and $\hat{x}_p(a) = \prod_{i \in p} \hat{x}_i(a), \hat{y}_p(a) = \prod_{i \in p} \hat{y}_i(a)$ for all $p \in \hat{I}$ and $a \in R$.

Let $\lambda$ be a dominant and regular weight of $\hat{G}$ and $(V, \rho)$ be the irreducible representation of $\hat{G}$ with highest weight $\lambda$. Let $\tilde{G}$ be the closure of $\{[\rho(g)] \mid \hat{g} \in \hat{G}\}$ in $P(\text{End}(V))$ and $\tilde{G}^\sigma$ be the closure of $\{[\rho(\hat{g})] \mid \hat{g} \in \hat{G}^\sigma\}$ in $P(\text{End}(V))$. Then since $\lambda$ is a dominant and regular weight of $\tilde{G}$ and $\lambda |_{T^\sigma}$ is a dominant and regular weight
of $\tilde{G}^\sigma$, we have that $\tilde{G}$ is the De Concini-Procesi compactification of $\tilde{G}$ and $\tilde{G}^\sigma$ is the De Concini-Procesi compactification of $\tilde{G}^\sigma$. Since $\tilde{G}$ is closed in $P(\text{End}(V))$, $\tilde{G}^\sigma$ is the closure of $\{ [\rho(\tilde{g})] \mid \tilde{g} \in \tilde{G} \}$ in $\tilde{G}$.

We have $\tilde{G} = \bigcup_{J \subseteq I} \tilde{Z}_J = \bigcup_{J \subseteq I} (\tilde{G} \times \tilde{G}) \cdot \tilde{z}_J^0$ and $\tilde{G}^\sigma = \bigcup_{J \subseteq I, \sigma J = J} (\tilde{G}^\sigma \times \tilde{G}^\sigma) \cdot \tilde{z}_J^0$.

Moreover, $\sigma$ can be extended in a unique way to an automorphism $\tilde{\sigma}$ of $\tilde{G}$. Since $\tilde{G}^\sigma = \bigcup_{J \subseteq I, \sigma J = J} (\tilde{Z}_J)^\sigma$ is a closed subset of $\tilde{G}$ containing $\tilde{G}^\sigma$, we have $\tilde{G}^\sigma \subseteq \bigcup_{J \subseteq I, \sigma J = J} (\tilde{Z}_J)^\sigma$.

By the condition (f), there exists a bijection $\phi$ between $\bar{I}$ and $I$, such that $\phi(\tilde{x}_p(a)) = x_{\phi(p)}(a)$, for all $p \in \bar{I}, a \in \mathbb{R}$. Moreover, the isomorphism $\phi$ from $\tilde{G}^\sigma$ to $G$ can be extended in a unique way to an isomorphism $\tilde{\phi} : \tilde{G}^\sigma \to \tilde{G}$. It is easy to see that for any $J \subseteq \bar{I}$ with $\sigma J = J$, we have $\tilde{\phi}((\tilde{G}^\sigma \times \tilde{G}^\sigma) \cdot \tilde{z}_J^0) = Z_{\phi \sigma(J)}$, where $\pi : \bar{I} \to I$ is the map sending element of $\bar{I}$ into the $\sigma$-orbit that contains it.

**Corollary 2.9.** $Z_{J, > 0} = \bigcap_{g_1, g_2 \in G_{>, 0}} (g_1^{-1}, g_2) \cdot Z_{J, > 0}$ is the closure of $Z_{J, > 0}$ in $Z_J$.

As a consequence, $Z_{J, > 0}$ and $\tilde{G}_{>, 0}$ are contractible.

**Proof.** I will prove that $Z_{J, > 0} \subseteq \bigcap_{g_1, g_2 \in G_{>, 0}} (g_1^{-1}, g_2) \cdot Z_{J, > 0}$.

First, assume that $G$ is simply laced.

For any $g \in G_{>, 0}$, $i_J(g) = ([\rho_1(g)], [\rho_2(g)])$, where $\rho_1(g)$ and $\rho_2(g)$ are matrices with all the entries in $\mathbb{R}_{>, 0}$. Then for any $z \in Z_{J, > 0}$, we have $i_J(z) = [[M_1], [M_2]]$ for some matrices with all the entries in $\mathbb{R}_{>, 0}$. Similarly, $i_J(\tilde{z}(z)) = [[M_3], [M_4]]$ for some matrices with all their entries in $\mathbb{R}_{>, 0}$.

Note that for any $M_1', M_2', M_3' \in gl(n)$ such that $M_1', M_3'$ are matrices with all their entries in $\mathbb{R}_{>, 0}$ and $M_2'$ is a nonzero matrix with all the entries in $\mathbb{R}_{>, 0}$, we have that $M_1' M_2' M_3'$ is a matrix with all the entries in $\mathbb{R}_{>, 0}$. Thus for any $g_1, g_2 \in G_{>, 0}$, we have that $(g_1, g_2^{-1}) \cdot z$ satisfies the condition (*) in 2.7. Moreover, $(g_1, g_2^{-1}) \cdot z \in Z_{J, > 0}$, therefore by 2.7, $(g_1, g_2^{-1}) \cdot z \in Z_{J, > 0}$ for all $g_1, g_2 \in G_{>, 0}$.

In the general case, we will keep the notation of 2.8. Since the isomorphism $\phi : \tilde{G}^\sigma \to G$ is compatible with the épínglages, we have $\phi((\tilde{U}_{>, 0})^\sigma) = U_{>, 0}, \phi((\tilde{T}_{>, 0})^\sigma) = T_{>, 0}$ and $\phi((\tilde{G}_{>, 0})^\sigma) = G_{>, 0}$. Now for any $z \in Z_{J, > 0}$, $z$ is contained in the closure of $G_{>, 0}$ in $\tilde{G}$. Thus $\tilde{\phi}^{-1}(z)$ is contained in the closure of $(\tilde{G}_{>, 0})^\sigma$ in $\tilde{G}$, hence contained in the closure of $(\tilde{G}_{>, 0})^\sigma$ in $\tilde{G}$. Therefore, $\tilde{\phi}^{-1}(z) \in \tilde{Z}_{J, > 0}$, where $J = \pi^{-1} \circ \phi^{-1}(J)$.

For any $\tilde{g}_1, \tilde{g}_2 \in (\tilde{G}_{>, 0})^\sigma$, we have $(\tilde{g}_1, \tilde{g}_2^{-1}) \cdot \tilde{\phi}^{-1}(z) = (\tilde{u}_1 \tilde{t}, \tilde{u}_2^{-1}) \cdot \tilde{z}_J^0$ for some $\tilde{u}_1 \in \tilde{U}_{>, 0}, \tilde{u}_2 \in \tilde{U}_{>, 0}, \tilde{t} \in \tilde{T}_{>, 0}$. Since $\tilde{\phi}^{-1}(z) \in (\tilde{G})^\sigma$, we have $(\tilde{g}_1, \tilde{g}_2^{-1}) \cdot \tilde{\phi}^{-1}(z) \in (\tilde{Z}_{J, > 0})^\sigma$. Then

$$\tilde{\sigma}((\tilde{u}_1 \tilde{t}, \tilde{u}_2^{-1}) \cdot \tilde{z}_J^0) = (\sigma(\tilde{u}_1 \tilde{t}), \sigma(\tilde{u}_2^{-1})) \cdot \tilde{\sigma}(\tilde{z}_J^0) = (\sigma(\tilde{u}_1)\sigma(\tilde{t}), \sigma(\tilde{u}_2^{-1})) \cdot \tilde{z}_J^0 = (\tilde{u}_1 \tilde{t}, \tilde{u}_2^{-1}) \cdot \tilde{z}_J^0.$$
roots of \( \tilde{G} \). Let \( \tilde{t}' \) be the unique element in \( \tilde{T} \) such that

\[
\tilde{\alpha}_j(\tilde{t}') = \begin{cases} 
\tilde{\alpha}_j(\tilde{t}), & \text{if } \tilde{j} \in \tilde{J}; \\
1, & \text{otherwise}.
\end{cases}
\]

Then \( \tilde{t}' \in (\tilde{T}_c)_0 \) and \((\tilde{t},1) \cdot \tilde{z}_j^0 = (\tilde{t}',1) \cdot \tilde{z}_j^0 \). Thus \((\tilde{g}_1, \tilde{g}_2^{-1}) \cdot \tilde{\phi}^{-1}(z) = (\tilde{u}_1 \tilde{t}', \tilde{u}_2^{-1}) \cdot \tilde{z}_j^0 \). We have

\[
(\phi(\tilde{g}_1), \phi(\tilde{g}_2)^{-1}) \cdot z = \tilde{\phi}((\tilde{g}_1, \tilde{g}_2^{-1}) \cdot \tilde{\phi}^{-1}(z)) = \tilde{\phi}((\tilde{u}_1 \tilde{t}', \tilde{u}_2^{-1}) \cdot \tilde{z}_j^0) = (\phi(\tilde{u}_1)\tilde{\phi}(\tilde{t}'), \phi(\tilde{u}_2^{-1})) \cdot \tilde{z}_j^0 \in Z_{J,>0}.
\]

Since \( \phi((\tilde{G}_c)_0) = G_{>0} \), we have \( Z_{J,>0} \subset \bigcap_{g_1,g_2 \in G_{>0}} (g_1^{-1}, g_2) \cdot Z_{J,>0} \).

Note that \((1,1)\) is contained in the closure of \(\{ (g_1, g_2^{-1}) | g_1, g_2 \in G_{>0} \} \). Hence, for any \( z \in \bigcap_{g_1,g_2 \in G_{>0}} (g_1^{-1}, g_2) \cdot Z_{J,>0}, \) \( z \) is contained in the closure of \( Z_{J,>0} \). On the other hand, \( Z_{J,>0} \) is a closed subset in \( Z_J \). \( Z_{J,>0} \) contains \( Z_{J,>0} \), hence contains the closure of \( Z_{J,>0} \) in \( Z_J \). Therefore, \( Z_{J,>0} = \bigcap_{g_1,g_2 \in G_{>0}} (g_1^{-1}, g_2) \cdot Z_{J,>0} \) is the closure of \( Z_{J,>0} \) in \( Z_J \).

Now set \( g_r = \exp(\sum_{i \in \mathcal{J}} (e_i + f_i)) \), where \( e_i \) and \( f_i \) are the Chevalley generators related to our epinglage by \( x_i(1) = \exp(e_i) \) and \( y_i(1) = \exp(f_i) \). Then \( g_r \in G_{>0} \) for \( r \in R_{>0} \) (see [Li 5.9]). Define \( f : R_{>0} \times Z_{J,>0} \to Z_{J,>0} \) by \( f(r,z) = (g_r, g_r^{-1}) \cdot z \) for \( r \in R_{>0} \) and \( z \in Z_{J,>0} \). Then \( f(0,z) = z \) and \( f(1,z) \in Z_{J,>0} \) for all \( z \in Z_{J,>0} \).

Using the fact that \( Z_{J,>0} \) is a cell (see 2.6), it follows that \( Z_{J,>0} \) is contractible.

Similarly, define \( f' : R_{>0} \times \bar{G}_{>0} \to \bar{G}_{>0} \) by \( f'(r,z) = (g_r, g_r^{-1}) \cdot z \) for \( r \in R_{>0} \) and \( z \in \bar{G}_{>0} \). Then \( f'(0,z) = z \) and \( f'(1,z) \in \bigcup_{K \subset I} Z_{K,>0} \) for all \( z \in \bar{G}_{>0} \). Note that \( \bigcup_{K \subset I} Z_{K,>0} = (U_{>0} \cdot (U_{>0}^{1\cdot 1})) \cdot \bigcup_{K \subset I} (T_{>0},1) \cdot (z_K^{>0} \cong U_{>0} \times U_{>0}^{1\cdot 1}) \). Moreover, by [DP 2.2], we have \( \bigcup_{K \subset I} (T_{>0},1) \cdot z_K^{>0} \cong R_{>0}^1 \). Thus \( \bigcup_{K \subset I} Z_{K,>0} \cong R_{>0}^1 \cdot R_{>0}^1 \) is contractible. Therefore \( \bar{G}_{>0} \) is contractible.

3. THE CELL DECOMPOSITION OF \( Z_{J,>0} \)

3.1. For any \( P \in \mathcal{P}_J, Q \in \mathcal{P}_J^\star, B \in B \) and \( g_1 \in H_P, g_2 \in U_Q, g \in G \), we have \( \text{pos}(P^B, g_1g_2g (Q^B)) = \text{pos}(g_1^{-1}(P^B), g_2g (Q^B)) = \text{pos}(P^B, g (Q^B)) \). If moreover, \( P \models g Q \), then \( \text{pos}(P^B, g (Q^B)) = wu_0 \) for some \( w \in W_J \) (see 1.4). Therefore, for any \( v, v' \in W, w, w' \in W^J \) and \( y, y' \in W_J \) with \( v \leq w \) and \( v' \leq w' \), Lustig introduced the subset \( Z_{J,v,w,v',w',y,y'} \) and \( Z_{J,>0,v,w,v',w',y,y'} \) of \( Z_J \) which are defined as follows:

\( Z_{J,v,w,v',w',y,y'} = \{(P,Q,H_PgU_Q) \in Z_J | \ P \in \mathcal{P}_v,w,w', \psi(Q) \in \mathcal{P}_v,w',w', \text{pos}(P^{B_v^w}, g (Q^{B_v^w})) = yw_0, \text{pos}(P^{B_v^w}, g (Q^{B_v^w})) = y'w_0 \} \)

and

\( Z_{J,>0,v,w,v',w',y,y'} = Z_{J,v,w,v',w',y,y'} \cap Z_{J,>0} \).
Then
\[ Z_J = \bigcup_{v, v' \in W, w, w' \in W^J, y, y' \in W_J} Z_{J, > 0}^{v, w, v', w', y, y'}, \]
\[ Z_{J, > 0} = \bigcup_{v, v' \in W, w, w' \in W^J, y, y' \in W_J} Z_{J, > 0}^{v, w, v', w', y, y'}. \]

Lusztig conjectured that for any \( v, v' \in W, w, w' \in W^J, y, y' \in W_J \) such that \( v \leq w, v' \leq w' \), \( Z_{J, > 0}^{v, w, v', w', y, y'} \) is either empty or a semi-algebraic cell. If it is nonempty, then it is also a connected component of \( Z_{J, > 0}^{v, w, v', w', y, y'} \).

In this section, we will prove this conjecture. Moreover, we will show exactly when \( Z_{J, > 0}^{v, w, v', w', y, y'} \) is nonempty and we will give an explicit description of \( Z_{J, > 0}^{v, w, v', w', y, y'} \).

First, I will prove some elementary facts about the total positivity of \( G \).

**Proposition 3.2.** For any \( v, v' \in W, w, w' \in W^J \) such that \( v \leq w, v' \leq w' \), set
\[ Z_J^{v, w, v', w', y, y'} = \bigcup_{y, y' \in W_J} Z_{J, > 0}^{v, w, v', w', y, y'}, \]
and \( Z_{J, > 0}^{v, w, v', w', y, y'} = \bigcup_{y, y' \in W_J} Z_{J, > 0}^{v, w, v', w', y, y'} \). We will give a characterization of \( z \in Z_{J, > 0}^{v, w, v', w', y, y'} \) in 3.5.

**Lemma 3.3.** For any \( w \in W, u \in U_{\geq 0}, \{ \pi_U u_1 u \mid u_1 \in U_{w, > 0}^+ \} = U_{w, > 0}^+ \).

**Proof.** The following identities hold (see [21. 1.3]):
(a) \( t x_i(a) = x_i(a_i(t)a)t, t y_i(a) = y_i(a_i(t)^{-1}a)t \) for all \( i, t \in T, a \in R \).
(b) \( y_i(a)x_{i+1}(b) = x_{i+1}(b)y_i(a) \) for all \( a, b \in R \) and \( i_1 \neq i_2 \).
(c) \( x_i(a)y_i(b) = y_i(b)(\frac{b}{1-ta})a_i(y_i(\frac{a}{1-ta}))x_i(a)(\frac{a}{1-ta}) \) for all \( a, b \in R_{\geq 0}, i \in I \).

Thus \( U_{w, > 0}^+ U_{w, > 0}^- \subset U_{w, > 0}^+ T_{w, > 0} U_{w, > 0}^+ \) for \( w \in W \). So we only need to prove that
\[ U_{w, > 0}^+ \subset \{ \pi_U u_1 u \mid u_1 \in U_{w, > 0}^+ \}. \]
Now I will prove the following statement:
\[ \{ \pi_U u_1 u_1(a) \mid u_1 \in U_{w, > 0}^+ \} = U_{w, > 0}^+ \text{ for } i \in I, a \in R_{\geq 0}. \]

We argue by induction on \( l(w) \). It is easy to see that the statement holds for \( w = 1 \). Now assume that \( w \neq 1 \). Then there exist \( j \in I \) and \( w_1 \in W \) such that \( w = s_j w_1 \) and \( l(w_1) = l(w) - 1 \). For any \( u_1' \in U_{w, > 0}^+ \), we have \( u_1' = u_{2}u_{3} \) for some \( u_{2} \in U_{s_j, > 0}^+ \).
and $u'_1 \in U^+_{w_1,>0}$. By induction hypothesis, there exists $u_3 \in U^+_{w_1,>0}$, $u' \in U^-$ and $t \in T$ such that $u_3 y_i(a) = u'tu'_3$. Since $U^+_{w,>0}U^-_{s,>0} \subset U^-_{s,>0}T_0U^+_{w,>0}$, we have $u' \in U^-_{s,>0}$ and $t \in T_{>0}$.

Now by (a), we have $tu'_3t^{-1} \in U^+_{s_1,>0}$. So by (b) and (c), there exists $u_2 \in U^+_{s_1,>0}$ such that $\pi_{U^+}(u_2u') = tu'_3t^{-1}$. Thus

$$\pi_{U^+}(u_2u_3y_i(a)) = \pi_{U^+}(u_2u')(u'u_3y_i(a)) = \pi_{U^+}(\pi_{U^+}(u_2u')u'u_3y_i(a)) = \pi_{U^+}(tu'_3t^{-1}u_3) = \pi_{U^+}(tu'_2u'_3) = u'_1.$$

So $u'_1 \in \{\pi_{U^+}(u_1y_i(a)) \mid u_1 \in U^+_{w,>0}\}$. The statement is proved.

Now assume that $u \in U^-_{w',>0}$. I will prove the lemma by induction on $l(u')$. It is easy to see that the lemma holds for $u' = 1$. Now assume that $u' \neq 1$. Then there exist $i \in I$ and $w'_1 \in W$ such that $l(w'_1) = l(u') - 1$ and $w'_1 = s_iw'_1$. We have $u = y_i(a)u'$ for some $a \in \mathbb{R}_{>0}$ and $u' \in U^-_{w_1,>0}$. So

$$\{\pi_{U^+}(u_1u) \mid u_1 \in U^+_{w,>0}\} = \{\pi_{U^+}(u_1y_i(a)u') \mid u_1 \in U^+_{w,>0}\} = \{\pi_{U^+}(\pi_{U^+}(u_1y_i(a)))u' \mid u_1 \in U^+_{w,>0}\} = \{\pi_{U^+}(u'_1u') \mid u'_1 \in U^+_{w,>0}\}.$$

By induction hypothesis, we have

$$\{\pi_{U^+}(u_1u) \mid u_1 \in U^+_{w,>0}\} = \{\pi_{U^+}(u'_1u') \mid u'_1 \in U^+_{w,>0}\} = U^+_{w,>0}.$$

\[\square\]

**Lemma 3.4.** Set $Z^1_{J,>0} = \{(g_1, g_2^{-1}) \cdot z^1_j \mid g_1 \in U^-_{>0}T_{>0}, g_2 \in U^+_{>0}\}$. Then

(a) $Z_{J,>0} = \bigcup_{u_1 \in U^+_{w_0}, u_2 \in U^-_{w_0}} (u_1^{-1}, u_2) \cdot Z^1_{J,>0}$.

(b) $Z_{J,>0} = \bigcup_{u_1, u_2 \in W_J} \{(u_1P_J, u_2) \cdot Q_J, u_1HP_JUQ_Ju_2 \mid u_1 \in U^+_{w_1,>0}, u_2 \in U^+_{w_2,>0}, l \in L_{>0}\} = \{(P, Q, \gamma) \in Z_{J,>0} \mid P = u_1P_J, \psi(Q) = u_2P_J \text{ for some } u_1, u_2 \in U^-_{>0}\}$.

**Proof.** (a) By 2.9 and 3.2, we have

$$Z_{J,>0} = \bigcup_{g_1, g_2 \in G_{>0}} (g_1^{-1}, g_2) \cdot Z_{J,>0} = \bigcup_{t_1, t_2 \in T_{>0}} \bigcup_{u_1, u_2 \in U^+_{w_0}} (u_1^{-1}u_3^{-1}t_1^{-1}, u_4u_2t_2) \cdot Z_{J,>0} = \bigcup_{u_1 \in U^+_{w_0}, u_4 \in U^-_{w_0}} (u_1^{-1}, u_4) \cdot \bigcup_{u_2 \in U^+_{w_0}, u_3 \in U^-_{w_0}} (u_2^{-1}, u_3) \cdot \bigcup_{t_1, t_2 \in T_{>0}} (t_1^{-1}, t_2) \cdot Z_{J,>0} = \bigcup_{u_1 \in U^+_{w_0}, u_4 \in U^-_{w_0}} (u_1^{-1}, u_4) \cdot \bigcup_{u_2 \in U^+_{w_0}, u_3 \in U^-_{w_0}} (u_2^{-1}, u_3) \cdot Z_{J,>0} = \bigcup_{u_1 \in U^+_{w_0}, u_4 \in U^-_{w_0}} (u_1^{-1}, u_4) \cdot \bigcup_{u_2 \in U^+_{w_0}, u_3 \in U^-_{w_0}} (u_2^{-1}u_3^{-1}t_1^{-1}, u_4u_2t_2) \cdot Z_{J,>0} = \bigcup_{u_1 \in U^+_{w_0}, u_4 \in U^-_{w_0}} (u_1^{-1}, u_4) \cdot \bigcup_{u_2 \in U^+_{w_0}, u_3 \in U^-_{w_0}} (u_2^{-1}U^-_{>0}T_{>0}, u_3^{-1}) \cdot Z_{J,>0} = \bigcup_{u_1 \in U^+_{w_0}, u_4 \in U^-_{w_0}} (u_1^{-1}, u_4) \cdot \bigcup_{u_2 \in U^+_{w_0}, u_3 \in U^-_{w_0}} (u_2^{-1}U^-_{>0}T_{>0}, U^+_{>0}u_3^{-1}) \cdot z^0_j = \bigcup_{u_1 \in U^+_{w_0}, u_4 \in U^-_{w_0}} (u_1^{-1}, u_4) \cdot \bigcup_{u_2 \in U^+_{w_0}, u_3 \in U^-_{w_0}} (U^-_{>0}T_{>0}, U^+_{>0}u_3^{-1}) \cdot z^0_j.$$
(b) For any \( u \in U^-_\infty, v \in U^+_\infty, t \in T_\infty \), there exist \( w_1, w_2 \in W^J, w_3, w_4 \in W^J \), such that \( u = u_1 u_3 \) for some \( u_1 \in U^-_\infty, u_3 \in U^-_\infty \) and \( v = u_4 u_2 \) for some \( u_2 \in U^+_w, u_4 \in U^-_w \). Then \((u t, v^{-1}) \cdot z^o_j = (u_1 P_J, u_2^{-1} Q_J, u_1 H_{P_J}, u_3 u_4 U_{Q_J}, u_2)\). On the other hand, assume that \( l \in L_\infty \), then \( l = u_3 u_4 \) for some \( u_3 \in U^-_\infty, u_4 \in U^+_\infty, t \in T_\infty \). Thus for any \( u_1 \in U^-_\infty, u_2 \in U^+_\infty \), we have

\[
(u_1 P_J, u_2^{-1} Q_J, u_1 H_{P_J}, u_3 u_4 U_{Q_J}, u_2) = (u_1 u_3 l, u_2^{-1} u_4^{-1}) \cdot z^o_j \in Z^1_{J,>0}.
\]

Therefore,

\[
Z^1_{J,>0} = \bigcup_{u_1, u_2 \in W^J} \{(u_1 P_J, u_2^{-1} Q_J, u_1 H_{P_J}, u_3 u_4 U_{Q_J}, u_2) \mid u_1 \in U^-_{J,>0}, u_2 \in U^+_{J,>0}, l \in L_\infty\}
\]

Note that \( \{u P_J \mid u \in U^-_\infty\} = \bigcup_{w \in W^J} \{u P_J \mid u \in U^-_w\} \). Now assume that \( z = (u_1 P_J, u_2^{-1} Q_J, u_1 H_{P_J}, u_3 u_4 U_{Q_J}, u_2) \) for some \( w_1, w_2 \in W^J \) and \( u_1 \in U^-_{w_1}, u_2 \in U^-_{w_2}, u_2 \in U^+_{w_2}, l \in L \). To prove that \( z \in Z^1_{J,>0} \), it is enough to prove that \( l \in L_\infty Z(L) \).

By part (a), for any \( u_3, u_4 \in U^+_{J,>0} \),

\[
(u_3, \psi(u_4)^{-1}) \cdot z = (u_3 u_1 P_J, \psi(u_2 u_4)^{-1} Q_J, u_3 u_1 H_{P_J}, u_3 U_{Q_J}, \psi(u_4 u_2)) \in Z^1_{J,>0}.
\]

Note that \( u_3 u_1 = u'_1 t_1 \pi_{U^+J}(u_3 u_1) \) for some \( u'_1 \in U^+_{w_1,>0}, t_1 \in T_\infty \) and \( u_4 u_2 = u'_2 t_2 \pi_{U^+J}(u_4 u_2) \) for some \( u'_2 \in U^{-w_2,>0}, t_2 \in T_\infty \). So we have \( u_3 u_1 P_J = u'_1 P_J, u_4 u_2 \) \( \psi(u_2 u_4)^{-1} Q_J = \psi(u_2)^{-1} Q_J \) and

\[
u_3 u_1 H_{P_J}, u_3 U_{Q_J}, \psi(u_4 u_2) = u'_1 \pi_{U^+J}(u_3 u_1) H_{P_J}, u_3 U_{Q_J}, \psi(\pi_{U^+J}(u_4 u_2)) t_2 \psi(u_2)\]

\[
= u'_1 H_{P_J}, t_1 \pi_{U^+J}(u_3 u_1) \psi(\pi_{U^+J}(u_4 u_2)) t_2 U_{Q_J}, \psi(u_2).
\]

Then \( t_1 \pi_{U^+J}(u_3 u_1) \psi(\pi_{U^+J}(u_4 u_2)) t_2 \in L_\infty Z(L) \). Since \( t_1, t_2 \in T_\infty \), we have \( \pi_{U^+J}(u_3 u_1) \psi(\pi_{U^+J}(u_4 u_2)) \in L_\infty Z(L) \) for all \( u_3, u_4 \in U^+_{J,>0} \). By 1.8 and 3.3, \( \pi_{U^+J}(u_3 u_1) \psi(\pi_{U^+J}(u_4 u_2)) = \pi_{U^+J}(u_3 u_1) U_{w_2,>0} \psi(u_4 u_2) \). Thus

\[
\pi_{U^+J}(U_{w_2,>0}) = \pi_{U^+J}(u_3 u_1) U_{w_2,>0} \psi(u_4 u_2) = U_{w_2,>0} \psi(u_4 u_2).
\]

Similarly, we have \( \pi_{U^+J}(U_{w_2,>0} u_2) = U_{w_2,>0} \psi(u_4 u_2) \).

The lemma is proved.

\begin{proposition}
Let \( z \in Z^1_{J,>0} \), then \( z \in Z^1_{J,>0} \) if and only if for any \( u_1 \in U_{v^{-1},>0}, u_2 \in U_{v^{-1},>0} \), \( z \in Z^1_{J,>0} \).
\end{proposition}

\begin{proof}
Assume that \( z \in \bigcap_{u_1, u_2 \in U_{v^{-1},>0} \in U_{v^{-1},>0}} (u_1, \psi(u_2)^{-1}) \cdot z \in Z^1_{J,>0} \).
\end{proof}
On the other hand, assume that $z = (P, Q, \gamma) \in Z_t^{v, w, v', w'}$. By 3.4(a), for any $u_1 \in U_{v_1, v', w'}^+ \cap U_{w_1, w, v}^+$ we have $(u_1, \psi(u_2^{-1})) \cdot z \in Z_{J, \geq 0}$. Moreover, we have $u_1 P = u_1' P_j$ for some $u_1' \in U_{w, > 0}$ (see 1.6). Similarly, we have $\psi(u_2^{-1}) Q = u_2' \psi(Q) = u_1' P_j$ for some $u_2' \in U_{w, > 0}$. By 3.4(b), $(u_1, \psi(u_2^{-1})) \cdot z \in Z_{J, \geq 0}$. □

3.6. Now I will fix $w \in W$ and a reduced expression $w = (w_0, w_1, \ldots, w_n)$ of $w$. Assume that $w_{(j)} = w_{(j-1)} s_{(j)}$ for all $j = 1, 2, \ldots, n$. Let $v \leq w$ and let $v_+ = (v_0, v_1, \ldots, v(n))$ be the positive subexpression of $w$.

Define

$$G_v, w = \left\{ g = g_1 g_2 \cdots g_k \mid g_j = y_i(a_j) \text{ for } a_j \in R - \{0\}, \quad \begin{array}{ll}
& v = v_0, \\
& v = v_{(j-1)} < v_{(j)}.
\end{array} \right\}$$

$$G_v, w, > 0 = \left\{ g = g_1 g_2 \cdots g_k \mid g_j = y_i(a_j) \text{ for } a_j \in R_{> 0}, \quad \begin{array}{ll}
& v = v_0, \\
& v = v_{(j-1)} < v_{(j)}.
\end{array} \right\}$$

Marsh and Rietsch have proved that the morphism $g \rightarrow g B^+$ maps $G_{v, w}$ into $R_{v, w}$ (see [MR, 5.2]) and $G_{v, w}, > 0$ bijectively onto $R_{v, w, > 0}$ (see [MR, 11.3]).

The following proposition is a technical tool needed in the proof of the main theorem.

**Proposition 3.7.** For any $g \in G_{v, w}, > 0$, we have

$$\bigcap_{u \in U_{w, > 0}} (\pi_{U_{J}}(u g))^{-1} \cdot U_{w, > 0, \geq 0} = \begin{cases}
U_{w, > 0}, & \text{if } v \in W^J; \\
\emptyset, & \text{otherwise}.
\end{cases}$$

The proof will be given in 3.13.

**Lemma 3.8.** Suppose $\alpha_{i_0}$ is a simple root such that $v_i^{-1} \alpha_{i_0} > 0$ for $v \leq v_i \leq w$. Then for all $g \in G_{v, w}, > 0$ and $a \in R$, we have $x_{i_0}(a) g = g t g'$ for some $t \in T_{> 0}$ and $g' \in \prod_{\alpha \in R(v)} U_\alpha \cdot (v^{-1} x_{i_0}(a) v)$, where $R(v) = \{ \alpha \in \Phi^+ \mid v \alpha \in -\Phi^+ \}$.

**Proof.** Marsh and Rietsch proved in [MR, 11.8] that $g$ is of the form

$$g = \left( \prod_{i \in \Delta_+} y_{(j)} t_j \right) v$$

and $v_{(j-1)} \alpha_i \neq \alpha_{i_0}$, for all $j = 1, 2, \ldots, n$. Thus $g = g_1 v$ for some

$$g_1 \in \prod_{\alpha \in \Phi^+-\{\alpha_{i_0}\}} U_{-\alpha}.$$

Set $T_1 = \{ t \in T \mid \alpha_{i_0}(t) = 1 \}$, then $T_1 \prod_{\alpha \in \Phi^+-\{\alpha_{i_0}\}} U_{-\alpha}$ is a normal subgroup of $\psi(P_{i_0})$. Now set $x = x_{i_0}(a)$, then $x g_1 x^{-1} \in B^-$. We may assume that $x g_1 x^{-1} = u_1 t_1$ for some $u_1 \in U^-$ and $t_1 \in T$. Now $x g_1 x^{-1} v = u_1 v_1 (v^{-1} t_1 v) (v^{-1} x v)$. Moreover, by [MR, 11.8], $x g \in g B^+$. Thus $x g = g_1 t_2 g_2 g_3 = g_1 (v t_2 g_2) (v^{-1}) t_2 g_3$, for some $t_2 \in T$, $g_2 \in \prod_{\alpha \in R(v)} U_\alpha$ and $g_3 \in \prod_{\alpha \in \Phi^+-R(v)} U_\alpha$. Note that $g_1 (v t_2 g_2) (v^{-1}) t_1 \in U^-$, $t_2, t_1 t_2 v \in T$ and $g_3, v^{-1} x v \in \prod_{\alpha \in \Phi^+-R(v)} U_\alpha$. Thus $g_1 (v t_2 g_2) (v^{-1}) t_1 = u_1$, $t_2 = v^{-1} t_1 v$ and $g_3 = v^{-1} x v$. Note that $g^{-1} x_0(b) g \in B^+$ for $b \in R$ (see [MR, 11.8]). We have that $\pi_T(g^{-1} x_0(b) g) \mid b \in R$ is connected and contains $\pi_T(g^{-1} x_0(0) g) = 1$. Hence $\pi_T(g^{-1} x_0(b) g) \in T_{> 0}$ for $b \in R$. 

In particular, $\pi_T(g^{-1}xg) = t_2 \in T_{>0}$. Therefore $xg = gt_2g'$ with $t_2 \in T_{>0}$ and $g' = \alpha g \beta = \prod_{\alpha \in R(v)} U_\alpha \cdot (\dot{w}^{-1}x\dot{v})$. 

\textbf{Remark.} In \cite{MR} 11.9, Marsh and Rietsch pointed out that for any $j \in J^{v}_{X}$, we have $\omega^{-1}\alpha_j > 0$ for all $v_{(j)}^{-1}v \leq u \leq w_{(j)}^{-1}w$.

3.9. Suppose that $J^{v}_{X_j} = \{j_1, j_2, \ldots, j_k\}$, where $j_1 < j_2 < \cdots < j_k$ and $g = g_1g_2 \cdots g_k$, where

$$g_j = \begin{cases} (y_j(a_j) \text{ for } a_j \in R_{>0}, & \text{if } j \in J^v_{X_j}; \\ s_j, & \text{if } j \in J^v_{X_j}. \end{cases}$$

For any $m = 1, \ldots, k$, define $v_m = v_{(j_m)}^{-1}v$, $g_m = g_{j_1+1}g_{j_2+2} \cdots g_m$ and $f_m(a) = g_{(m)}^{-1}x_{ij_m}(-a)g_{(m)} \in B^+$ (see \cite{MR} 11.8). Now I will prove the following lemma.

\textbf{Lemma 3.10.} Keep the notation in 3.9. Then

(a) For any $u \in U^+_{v_{i-1,j_0}}$, $ug = g'tu'$ for some $g' \in U^{-}_{w_{i,j_0}}$, $t \in T_{>0}$ and $u' \in U^+$.

(b) $\pi_T(U^+_{v_{i-1,j_0}}) = \{\pi_T(f_k(a_k)f_{k-1}(a_{k-1}) \cdots f_1(a_1)) | a_1, a_2, \ldots, a_k \in R_{>0}\}$.

\textbf{Proof.} I will prove the lemma by induction on $l(v)$. It is easy to see that the lemma holds when $v = 1$. Now assume that $v \neq 1$.

For any $u \in U^+_{v_{i-1,j_0}}$, since $B^+ \in \mathcal{R}_{v,w,>0}$, we have $ug \in \mathcal{R}_{v,w,>0}$. Thus $ug = g'tu'$ for some $g' \in U^{-}_{w_{i,j_0}}$, $t \in T$ and $u' \in U^+$. Set $y = g_1g_2 \cdots g_{j_{i-1}}$. Note that $y \in U_{>0}$, we have $uy = y'tu'$ for some $y' \in U^{-}$, $u' \in U^+_{v_{i-1,j_0}}$ and $t \in T_{>0}$. Hence $\pi_T(ug) = \pi_T(uy's_{j_i}g(1)) = \pi_T(y'tu's_{j_i}g(1)) \in T_{>0} \pi_T(u's_{j_i}g(1))$. To prove that $\pi_T(U^+_{v_{i-1,j_0}}) \subset T_{>0}$, it is enough to prove that $\pi_T(u's_{j_i}g(1)) \in T_{>0}$ for all $u \in U^+_{v_{i-1,j_0}}$.

For any $u \in U^+_{v_{i-1,j_0}}$, we have $u = u_1x_{ij_1}(a)$ for some $u_1 \in U^+_{v_{i-1,j_1}}, a \in R_{>0}$. It is easy to see that $x_{ij_1}(a)s_{j_1}g(1) = \alpha_{ij_1}(a)y_{ij_1}(a)x_{ij_1}(-a^{-1})g(1)$. Note that $\alpha_{ij_1}(a) \in T_{>0}$ and by 3.8, $g(1)^{-1}x_{ij_1}(a)x_{ij_1}(-a^{-1})g(1) \in T_{>0} U^+$. Hence by 1.7, we have

$$\pi_T(u's_{j_i}g(1)) = \pi_T(u_1\alpha_{ij_1}(a)y_{ij_1}(a)g(1)g(1)^{-1}x_{ij_1}(-a^{-1})g(1)) \in T_{>0} \pi_T(U^+_{v_{i-1,j_1}}, \rho_{ij_1,>0}y_{ij_1}(a)g(1))T_{>0}.$$

Set

$$w' = (1, w_{j_1-1}^{-1}w_{j_1}, \ldots, w_{j_1-1}^{-1}w_{n}),$$

$$v' = (1, s_{j_1}v_{j_1}, s_{j_1}v_{j_1+1}, \ldots, s_{j_1}v_{n}).$$

Then $w'$ is a reduced expression of $w_{(j_1-1)}^{-1}w_{(n)}$ and $v'$ is a positive subexpression of $w'$. For any $a \in R_{>0}$, $y_{ij_1}(a)g(1) \in G_{w',>,>0}$. Thus by induction hypothesis, for any $a \in R_{>0}$, $\pi_T(U^+_{v_{i-1,j_1},>0}y_{ij_1}(a)g(1)) \subset T_{>0}$. Therefore, $\pi_T(ug) \in T_{>0}$. Part (a) is proved.
Therefore, the referee pointed out to me that the assertion
exists also be proved using generalized minors.

Assume that there exist a \( t \in T_0 \) such that
\[ \lim_{n \to \infty} t^n u^n t^n = 1. \]
Thus
\[ \lim_{n \to \infty} t^n u^n t^n = u \in U_{\geq}^+. \]
Thus \( u = x_i(a) \) for some \( i \in I \) and \( a \in R_{\geq}. \)

\[ \text{Remark. The referee pointed out to me that the assertion } t \in T_0 \text{ of 3.10(a) could also be proved using generalized minors.} \]

**Lemma 3.11.** Assume that \( \alpha \) is a positive root and \( u \in U_\alpha \), \( u' \in U^+ \) such that \( u'^n u' \in U_{\geq}^+ \) for all \( n \in N \). Then \( u = x_i(a) \) for some \( i \in I \) and \( a \in R_{\geq}. \)

**Proof.** There exists \( t \in T_0 \), such that \( \alpha_i(t) = 2 \) for all \( i \in I \). Then \( t u^{-1} = u^m \) for some \( m \in N \). By assumption, \( t^n u^n u' \in U_{\geq}^+ \) for all \( n \in N \). Thus
\[ u(t^n u^n) = t^n (t^n u^n u') = \lim_{n \to \infty} t^n u^n t^n = 1. \]
Since \( U_{\geq}^+ \) is a closed subset of \( U^+ \), \( \lim_{n \to \infty} t^n u^n t^n = u \in U_{\geq}^+ \). Thus \( u = x_i(a) \) for some \( i \in I \) and \( a \in R_{\geq}. \)

**Lemma 3.12.** Assume that \( w \in W \) and \( i, j \in I \) such that \( w^{-1} \alpha_i = \alpha_j \). Then there exists \( c \in R_{\geq}, \) such that \( w^{-1} x_i(a) w = x_j(c) a \) for all \( a \in R \).

**Proof.** There exist \( c, c' \in R \setminus \{0\} \), such that \( y_i(a) w = wy_j(c') a \) and \( x_i(a) w = \dot{w} x_j(c) a \) for all \( a \in R \). Since \( \dot{w} B^- = B_{\geq} \), we have \( y_i(1) w B^+ = y_j(c') B^+ \in B_{\geq} \). By 3.6, \( c' > 0 \). Thus \( c' > 0 \). Moreover, since \( w a_j = \alpha_i > 0 \), we have \( w s_j w^{-1} = s_i \) and \( l(ws_j) = l(s_i w) = l(w) + 1 \). Hence, setting \( \dot{w}' = w s_j = s_i w, \) we have \( \dot{w}' = s_i \dot{w} = s_i \dot{w} = \dot{w} \). Therefore, \( c = c'^{-1} > 0 \).
3.13. Proof of Proposition 3.7. If \( v \in W^J \), then \( v \alpha > 0 \) for \( \alpha \in \Phi_+^J \). So 
\[ \pi_{U^+}^J(\prod_{\alpha \in R(v)} U_\alpha) = \{1\} \]. By 3.8, \( f_m(a) \in T(\prod_{\alpha \in R(v_m)} U_\alpha \cdot U_{v_m^{-1}\alpha^-}_m) \) for all \( m \in \{1, 2, \ldots, k\} \). Note that \( v_{\alpha > 0} \in -\Phi^+ \) for all \( \alpha \in R(v_m) \) and \( v_{v_m^{-1}\alpha^-}_m \in -\Phi^+ \). So \( f_m(a) \in T(\prod_{\alpha \in R(v)} U_\alpha) \pi_{k_1}(ak_1) \cdots \pi_{k_l}(ak_l) \) in 
\[ T(\prod_{\alpha \in R(v)} U_\alpha) \]. Hence by 3.10(b), \( \pi_{U^+}^J(ug) = 1 \) for all \( u \in U^+_{v^{-1}, > 0} \). Therefore 
\[ \bigcap_{u \in U^+_{v^{-1}, > 0}} (\pi_{U^+}^J(ug))^{-1} \cdot U^+_{w_d, > 0} = U^+_{w_d, > 0}. \]

If \( v \notin W^J \), then there exists \( \alpha \in \Phi_+^J \) such that \( v\alpha \in -\Phi^+ \), that is, \( v_{m^{-1}\alpha^-}_m \in \Phi_+^J \) for some \( m \in \{1, 2, \ldots, k\} \). Set \( k_0 = \max\{m \mid v_{m^{-1}\alpha^-}_m \in \Phi_+^J\} \). Then since 
\[ R(v_{k_0}) = \{v_{m^{-1}\alpha^-}_m \mid m > k_0\} \], we have that \( v_{k_0} \alpha > 0 \) for \( \alpha \in \Phi_+^J \). Hence by 3.8, 
\[ \pi_{U^+}^J(f_{k_0}(a)) = v_{k_0}^{-1}x_{i_{k_0}}(-1)v_{k_0}. \]
If \( u' \in \bigcap_{u \in U^+_{v^{-1}, > 0}} (\pi_{U^+}^J(ug))^{-1} \cdot U^+_{w_d, > 0} \), then 
\[ \pi_{U^+}^J(f_{k_0}(ak_1) \cdots f_{k_l}(ak_l))u' \in U^+_{w_d, > 0} \] for all \( a_1, a_2, \ldots, a_k \in \mathbb{R}_{> 0} \). Since 
\[ U^+_{w_d, > 0} \] is a closed subset of \( G \), \( \pi_{U^+}^J(f_{k_0}(ak_1) \cdots f_{k_l}(ak_l))u' \in U^+_{w_d, > 0} \) for all \( a \in \mathbb{R}_{> 0} \). Now take \( a_m = 0 \) for \( m \in \{1, 2, \ldots, k\} \). Then 
\[ \pi_{U^+}^J(f_{k_0}(a)u') \in U^+_{w_d, > 0} \] for all \( a \in \mathbb{R}_{> 0} \). Set \( u_1 = v_{k_0}^{-1}x_{i_{k_0}}(-1)v_{k_0} \). Then 
\[ u_0u' \in U^+_{w_d, > 0} \] for all \( u' \in U^+_{w_d, > 0} \). By 3.11, \( v_{k_0}^{-1}\alpha_{k_0} \) is \( \alpha_j' \) for some \( j \) in \( J \) and 
\( u_1 = x_{j'}(-c) \) for some \( c \in \mathbb{R}_{> 0} \). That is a contradiction. The proposition is proved.

Let me recall that \( L = P_J \cap Q_J \) (see 2.4). Now I will prove the main theorem.

**Theorem 3.14.** For any \( v, w, v', w' \in W^J \) such that \( v \leq w, v' \leq w' \), set 
\[ \tilde{Z}_{v, w, v', w'}^J = \{ (9^J P_J, \psi(g)^{-1} Q_J, gH_{P_J} \cdot U_{Q_J} \psi(g) \mid g \in G_{v^+, w_0} \rangle \}. \]

Then 
\[ Z_{v, w, v', w'}^J = \begin{cases} \tilde{Z}_{v, w, v', w'}^J, & \text{if } v, w, v', w' \in W^J, v \leq w, v' \leq w'; \\ \emptyset, & \text{otherwise}. \end{cases} \]

**Proof.** Note that \( \{(P, Q, \gamma) \in Z_J \mid P \in P_{\geq 0}^J, \psi(Q) \in P_{\geq 0}^J \} \) is a closed subset 
containing \( Z_{J, > 0} \). Hence it contains \( Z_{J, > 0} \). Now fix \( g \in G_{v^+, w_0} \rangle, g' \in G_{v^+, w_0} \rangle \) and 
\( l \in L \). By 3.10(a), for any \( u \in U_{v^{-1}, > 0} \rangle, \) 
\[ u \in \psi(u')^{-1} Q_J, aH_{P_J} \cdot U_{Q_J} \psi(g') \]. We have 
\[ \begin{align*} 
(u, \psi(u'))^{-1} \cdot z &= \left( a P_J, \psi(a')^{-1} Q_J, \psi(\pi_{U^+}(ug)H_{P_J} \cdot U_{Q_J} \psi(\pi_{U^+}(u'g')))t' \psi(a') \right) \\
&= \left( a P_J, \psi(a')^{-1} Q_J, aH_{P_J} \cdot \psi(\pi_{U^+}(ug))t' \psi(a') \right) \\
&= \left( a P_J, \psi(a')^{-1} Q_J, aH_{P_J} \cdot U_{Q_J} \psi(\pi_{U^+}(u'g')))t' \psi(a') \right). 
\end{align*} \]

Then \( (u, \psi(u')^{-1}) \cdot z \in Z_{J, > 0} \) if and only if 
\[ t \pi_{U^+}(ug) \psi(\pi_{U^+}(u'g')) \in L_{> 0} Z(L), \]
that is, 
\[ l \in \pi_{U^+} \pi_{U^+}^{-1} L_{> 0} Z(L) \psi(\pi_{U^+}(u'g'))^{-1} \]
\[ = (\pi_{U^+} \pi_{U^+}^{-1} U_{w_d, > 0}) T_{> 0} Z(L) \psi(\pi_{U^+}(u'g'))^{-1} U_{w_d, > 0}). \]
So by 3.5, \( z \in Z_{J,0} \) if and only if
\[
l \in \bigcap_{u \in U^+_{v,>0}} \left( \pi_{U^+} \left( u g \right) \right)^{-1} U^+_{u_0,>0} T_{>0} Z(L) \psi \left( \pi_{U^+} \left( u' g' \right) \right)^{-1} U^+_{u_0',>0}.
\]

By 3.7, \( z \in Z_{I,0} \) if and only if \( v, v' \in W^J \) and \( l \in L_{>0} Z(L) \). The theorem is proved. \( \square \)

3.15. It is known that \( G_{>0} = \bigcup_{w,w' \in W} U_{w,>0} U_{w',>0} \), where for any \( w, w' \in W \), \( U_{w,>0} U_{w',>0} \) is a semi-algebraic cell (see [11 2.11]) and is a connected component of \( B^w B^+ \cap B^{-w'} B^- \) (see [FZ]). Moreover, Rietsh proved in [R2 2.8] that \( \mathcal{R}_{v,w} = \bigcup_{v \in w} \mathcal{R}_{v,w,>0} \), where for any \( v, w \in W \) such that \( v \leq w \), \( \mathcal{R}_{v,w,>0} \) is a semi-algebraic cell and is a connected component of \( \mathcal{R}_{v,w} \).

The following result generalizes these facts.

Corollary 3.16. \( \overline{G_{>0}} = \bigcup_{J \subset I} \bigcup_{v,w,v',w' \in W^+} \bigcup_{y,y' \in W^J} Z_{J,v,w,v',w';y,y'} \). Moreover, for any \( v, w, v', w' \in W^J, y, y' \in W^J \) with \( v \leq w, v' \leq w' \), \( Z_{J,v,w,v',w';y,y'} \) is a connected component of \( Z_{J,v,w,v',w';y,y'} \) and is a semi-algebraic cell which is isomorphic to \( \mathcal{R}_{v,w}^J \), where \( d = l(w) + l(w') + 2l(w'_0) + 1 \).

Proof. \( \mathcal{P}_{v,w,>0} \) is a connected component of \( \mathcal{P}_{v,w} \) (resp. \( \mathcal{P}_{v',w'} \)) (see [L3]). Thus \( \{(P, Q, \gamma) \in Z_{J,v,w,v',w';y,y'}^J \mid P \in \mathcal{P}_{v,w,>0} \}, \psi(Q) \in \mathcal{P}_{v',w',>0} \) \} is open and closed in \( Z_{J,v,w,v',w';y,y'} \). To prove that \( \overline{Z_{J,v,w,v',w';y,y'}^J} \) is a connected component of \( Z_{J,v,w,v',w';y,y'} \), it is enough to prove that \( \overline{Z_{J,v,w,v',w';y,y'}^J} \) is a connected component of \( \{(P, Q, \gamma) \in Z_{J,v,w,v',w';y,y'}^J \mid P \in \mathcal{P}_{v,w,>0} \}, \psi(Q) \in \mathcal{P}_{v',w',>0} \} \).

Assume that \( g \in G_{v,w,>0} \) and \( l \in L \). We have that \( \left( \mathcal{P}_{J,B^+} \right) \) is the unique element in \( \mathcal{R}_{v,w} \) that is contained in \( \mathcal{P}_{J,B^+} \) (see 1.4). Therefore \( \left( \mathcal{P}_{J,B^+} \right) = \mathcal{P}_{J,B^+} \). Similarly, \( \mathcal{P}_{J,B^-} = \mathcal{P}_{J,B^-} \) and \( \mathcal{P}_{J,B^-} = \mathcal{P}_{J,B^-} \) and \( \mathcal{P}_{J,B^-} = \mathcal{P}_{J,B^-} \). Thus \( \mathcal{P}_{J,B^+} \) is a connected component of \( \mathcal{R}_{v,w}^J \).

Therefore we have that \( \mathcal{P}_{J,B^+} \) is a connected component of \( \mathcal{R}_{v,w}^J \) if and only if \( l \in B^+ \mathcal{P}_{J,B^+} \) and \( \mathcal{R}_{v,w}^J \) is a connected component of \( \mathcal{R}_{v,w}^J \).

Note that \( L \cap B^+ \subset \mathcal{P}_{J,B^+} \) and \( \mathcal{P}_{J,B^+} \) is a connected component of \( \mathcal{R}_{v,w}^J \).

Therefore, \( L \cap B^+ \mathcal{P}_{J,B^+} \) is a connected component of \( \mathcal{R}_{v,w}^J \).

Similarly, \( L \cap \mathcal{P}_{J,B^+} \mathcal{P}_{J,B^+} \) is a connected component of \( \mathcal{R}_{v,w}^J \).

Then \( \mathcal{P}_{J,B^+} \) is a connected component of \( \mathcal{R}_{v,w}^J \) and \( \mathcal{R}_{v,w}^J \) is a connected component of \( \mathcal{R}_{v,w}^J \).

Note that \( L \cap B^+ \subset \mathcal{P}_{J,B^+} \) and \( \mathcal{P}_{J,B^+} \) is a connected component of \( \mathcal{R}_{v,w}^J \).

Therefore, \( L \cap B^+ \mathcal{P}_{J,B^+} \) is a connected component of \( \mathcal{R}_{v,w}^J \).

Similarly, \( L \cap \mathcal{P}_{J,B^+} \mathcal{P}_{J,B^+} \) is a connected component of \( \mathcal{R}_{v,w}^J \).

Then \( \mathcal{P}_{J,B^+} \) is a connected component of \( \mathcal{R}_{v,w}^J \) and \( \mathcal{R}_{v,w}^J \) is a connected component of \( \mathcal{R}_{v,w}^J \).

Note that \( L \cap B^+ \subset \mathcal{P}_{J,B^+} \) and \( \mathcal{P}_{J,B^+} \) is a connected component of \( \mathcal{R}_{v,w}^J \).

Therefore, \( L \cap B^+ \mathcal{P}_{J,B^+} \) is a connected component of \( \mathcal{R}_{v,w}^J \).

Similarly, \( L \cap \mathcal{P}_{J,B^+} \mathcal{P}_{J,B^+} \) is a connected component of \( \mathcal{R}_{v,w}^J \).

Then \( \mathcal{P}_{J,B^+} \) is a connected component of \( \mathcal{R}_{v,w}^J \) and \( \mathcal{R}_{v,w}^J \) is a connected component of \( \mathcal{R}_{v,w}^J \).
that \((L \cap B^+) y \hat{w}_0^d (L \cap B^+) \cap (L \cap B^-) \hat{w}_0^d y' (L \cap B^-) \cap L \geq 0 = U_{y \hat{w}_0^d, >0}^- T_{>0} U_{y \hat{w}_0^d, >0}^+ y', >0\).

Therefore
\[
Z_{v_0, >0}^{v, w_0, w, y, y'} \cong G_{v, w_0, >0} \times G_{v', w', >0} \times U_{y \hat{w}_0^d, >0}^- T_{>0} U_{y \hat{w}_0^d, >0}^+ y', >0 / (Z(L) \cap T_{>0})
\]
\[
\cong R_{>0}^{l(w)+l(w')+2l(w_0')+l(v)-l(v')-l(y)-l(y')}.
\]

By 3.15, we have that \(U_{y \hat{w}_0^d, >0}^- T_{>0} U_{y \hat{w}_0^d, >0}^+ y', >0 / (Z(L) \cap T_{>0})\) is a connected component of \((L \cap B^+) y \hat{w}_0^d (L \cap B^+) \cap (L \cap B^-) \hat{w}_0^d y' (L \cap B^-) / Z(L)\). The corollary is proved. \( \square \)

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References


