

CHARACTER SHEAVES ON DISCONNECTED GROUPS, III

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ABSTRACT. In this paper we define and study generalized Green functions for possibly disconnected groups.

INTRODUCTION

Throughout this paper, G denotes a fixed, not necessarily connected, reductive algebraic group over an algebraically closed field \mathbf{k} . This paper is a part of a series [L9] which attempts to develop a theory of character sheaves on G .

Assume now that \mathbf{k} is an algebraic closure of a finite field \mathbf{F}_q and that G has a fixed \mathbf{F}_q -structure with Frobenius map F . Let $(L, S, \mathcal{E}, \phi_0)$ be a quadruple where L is an F -stable Levi of some parabolic of G^0 , \mathcal{E} is a local system on an isolated F -stable stratum S of $N_G L$ with certain properties and ϕ_0 is an isomorphism of \mathcal{E} with its inverse image under the Frobenius map. To (L, S, \mathcal{E}) we have associated in 5.6 an intersection cohomology complex $\mathfrak{K} = IC(\bar{Y}, \pi_1 \mathcal{E})$ on G . Moreover, ϕ_0 gives rise to an isomorphism ϕ between \mathfrak{K} and its inverse image under the Frobenius map. There is an associated characteristic function $\chi_{\mathfrak{K}, \phi}$ (see 15.12(a)) which is a function $G^F \rightarrow \mathbf{Q}_l$, constant on $(G^0)^F$ -conjugacy classes. The main result of this paper is Theorem 16.14 which shows that the computation of this function can be reduced to an analogous computation involving only unipotent elements in a smaller group (the centralizer of a semisimple element). (This is a generalization of [L3, II, Theorem 8.5]. However, even if G is assumed to be connected, as in [L3, II], our Theorem 16.14 is more general than that in [L3, II], since here we do not make the assumption that \mathcal{E} is cuspidal. Also, unlike the proof in [L3, II] the present proof does not rely on the classification of cuspidal local systems.) A main ingredient in Theorem 16.14 are the generalized Green functions (see 15.12(c)) which generalize those in [L3, II, 8.3.1] One of the key properties of the generalized Green functions is the invariance property 15.12(d). In the connected case, this property was stated in [L3, II, 8.3.2], but the proof given there was incomplete (as pointed out by E. Letellier). Most of Section 15 is devoted to establishing this invariance property.

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15. GENERALIZED GREEN FUNCTIONS

15.1. In this section we fix a pair (L, Σ) where L is a subgroup of G^0 and Σ is a non-empty subset of $N_G L$; we assume that there exists a parabolic of G^0 normalized by Σ , with Levi L , and that $\Sigma = \bigcup_{j \in J} S_j$ where S_j are distinct isolated strata of $N_G L$ with $\dim S_j$ independent of j . Several definitions in Sections 3 and 5 which concern the special case where Σ is a single stratum will now be extended to the present, more general case. Let $\bar{\Sigma}, \bar{S}_j$ be the closure of Σ, S_j in $N_G L$. Then $\bar{\Sigma} = \bigcup_{j \in J} \bar{S}_j$. Since \bar{S}_j is a union of isolated strata of G , the same holds for $\bar{\Sigma}$. Let \mathcal{E} be a local system on Σ such that $\mathcal{E}_j := \mathcal{E}|_{S_j} \in \mathcal{S}(S_j)$ for all $j \in J$. Since Σ is a smooth, open, dense subvariety of pure dimension of $\bar{\Sigma}$, $IC(\bar{\Sigma}, \mathcal{E}) \in \mathcal{D}(\bar{\Sigma})$ is well defined. Let

$$\Sigma^* = \{g \in \Sigma; Z_G(g_s)^0 \subset L\} = \bigcup_{j \in J} S_j^*$$

($S_j^* = \{g \in S_j; Z_G(g_s)^0 \subset L\}$ as in 3.11). Then Σ^* is an open dense subset of Σ (this follows from the fact that S_j^* is open dense in S_j for any j ; see 3.11). Let

$$\tilde{Y}_{L, \Sigma} = \{(g, xL) \in G \times G^0/L; x^{-1}gx \in \Sigma^*\} = \bigcup_{j \in J} \tilde{Y}_{L, S_j}$$

(a disjoint union),

$$Y_{L, \Sigma} = \bigcup_{x \in G^0} x\Sigma^*x^{-1} = \bigcup_{j \in J} Y_{L, S_j}.$$

(a not necessarily disjoint union). Thus $Y_{L, \Sigma}$ is a finite union of strata of equal dimension (see 3.13(a)) of G . Hence $Y_{L, \Sigma}$ is a locally closed smooth subvariety of pure dimension of G and any of its irreducible components is of the form Y_{L, S_j} for some j (which is not necessarily unique). Let $\tilde{Y}_{L, S_j}, \tilde{Y}_{L, \Sigma}$ be the closure of $Y_{L, S_j}, Y_{L, \Sigma}$ in G . We define a local system $\tilde{\mathcal{E}}$ on $\tilde{Y}_{L, \Sigma}$ by the requirement that $b^*\mathcal{E} = a^*\tilde{\mathcal{E}}$ where $a(g, x) = (g, xL), b(g, x) = x^{-1}gx$ in the diagram

$$\tilde{Y}_{L, \Sigma} \xleftarrow{a} \{(g, x) \in G \times G^0; x^{-1}gx \in \Sigma^*\} \xrightarrow{b} \Sigma.$$

(We use the fact that a is a principal L -bundle and $b^*\mathcal{E}$ is L -equivariant.)

Define $\pi : \tilde{Y}_{L, \Sigma} \rightarrow Y_{L, \Sigma}$ by $\pi(g, xL) = g$. Using 3.13(a), we see that π is a finite unramified covering (for any irreducible component U of $Y_{L, \Sigma}$, $\pi : \pi^{-1}(U) \rightarrow U$ may be identified with $\bigsqcup_{j \in J; Y_{L, S_j} = U} \tilde{Y}_{L, S_j} \rightarrow U$ given by $\pi_j : \tilde{Y}_{L, S_j} \rightarrow Y_{L, S_j}, (g, xL) \mapsto g$). It follows that $\pi_1\tilde{\mathcal{E}}$ is a local system on $Y_{L, \Sigma}$ such that for any irreducible component U of $Y_{L, \Sigma}$, we have

$$\pi_1\tilde{\mathcal{E}}|_U = \bigoplus_{j \in J; Y_{L, S_j} = U} \pi_{j!}\tilde{\mathcal{E}}_j$$

where $\tilde{\mathcal{E}}_j$ is the local system on \tilde{Y}_{L, S_j} defined in terms of \mathcal{E}_j as in 5.6. Then $IC(\tilde{Y}_{L, \Sigma}, \pi_1\tilde{\mathcal{E}}) \in \mathcal{D}(\tilde{Y}_{L, \Sigma})$ is well defined and

$$(a) \quad IC(\tilde{Y}_{L, \Sigma}, \pi_1\tilde{\mathcal{E}}) = \bigoplus_{j \in J} IC(\tilde{Y}_{L, S_j}, \pi_{j!}\tilde{\mathcal{E}}_j)$$

where $IC(\tilde{Y}_{L, S_j}, \pi_{j!}\tilde{\mathcal{E}}_j)$ is regarded as a complex on $\tilde{Y}_{L, \Sigma}$ which is zero on $\tilde{Y}_{L, \Sigma} - \tilde{Y}_{L, S_j}$.

Proposition 15.2. *Assume that $S_j^\omega = \{g \in S_j; g \text{ unip.}\} \neq \emptyset$ for any $j \in J$. Let $\Sigma^\omega = \bigcup_{j \in J} S_j^\omega$, let $p : \Sigma \rightarrow \Sigma$ be the morphism $p(g) = g_u$ and let $\mathcal{E}^1 = p^*\mathcal{E}$. Then $\mathcal{E}^1|_{S_j} \in \mathcal{S}(S_j)$ for all $j \in J$ and we can define $\tilde{\mathcal{E}}^1$ in terms of \mathcal{E}^1 just like $\tilde{\mathcal{E}}$ is defined in terms of \mathcal{E} . Let $\tilde{Y}_{L,\Sigma}^\omega = \{g \in \tilde{Y}_{L,\Sigma}; g \text{ unipotent}\}$. We have a canonical isomorphism*

$$(a) \quad IC(\tilde{Y}_{L,\Sigma}, \pi_1 \tilde{\mathcal{E}})|_{\tilde{Y}_{L,\Sigma}^\omega} \xrightarrow{\sim} IC(\tilde{Y}_{L,\Sigma}, \pi_1 \tilde{\mathcal{E}}^1)|_{\tilde{Y}_{L,\Sigma}^\omega}$$

in $\mathcal{D}(\tilde{Y}_{L,\Sigma}^\omega)$.

In other words, $IC(\tilde{Y}_{L,\Sigma}, \pi_1 \tilde{\mathcal{E}})|_{\tilde{Y}_{L,\Sigma}^\omega}$ depends only on $\mathcal{E}|_{\Sigma^\omega}$, not on \mathcal{E} on the whole of Σ . The proof will be given in 15.5–15.11.

15.3. We return to the setup in 15.1. Let P be a parabolic of G^0 with Levi L such that $\Sigma \subset N_G P$. By 3.14 we have $\tilde{Y}_{L,S_j} = \bigcup_{x \in G^0} x \tilde{S}_j U_P x^{-1}$ hence

$$\tilde{Y}_{L,\Sigma} = \bigcup_{j \in J} \tilde{Y}_{L,S_j} = \bigcup_{x \in G^0} x \tilde{\Sigma} U_P x^{-1}.$$

Let $X_j = \{(g, xP) \in G \times G^0/P; x^{-1}gx \in \tilde{S}_j U_P\}$. Let

$$X = \{(g, xP) \in G \times G^0/P; x^{-1}gx \in \tilde{\Sigma} U_P\} = \bigcup_{j \in J} X_j.$$

Define $\psi : X \rightarrow \tilde{Y}_{L,\Sigma}$ by $\psi(g, xP) = g$. This map is onto. We have the following generalization of Lemma 5.5:

$$(a) \quad (g, xL) \mapsto (g, xP) \text{ is an isomorphism } \tilde{Y}_{L,\Sigma} \xrightarrow{\sim} \psi^{-1}(Y_{L,\Sigma}).$$

We verify this only at the level of sets. The proof of injectivity is the same as that in Lemma 5.5. We prove surjectivity. Let $(g, xP) \in \psi^{-1}(Y_{L,\Sigma})$. We have $(g, xP) \in X_j$ for some $j \in J$ (hence $g \in \tilde{Y}_{L,S_j}$) and $g \in Y_{L,S_{j'}}$ for some $j' \in J$. Since $Y_{L,S_{j'}}$ is a stratum of G that meets \tilde{Y}_{L,S_j} (which is a union of strata, see 3.15) we see that $Y_{L,S_{j'}} \subset \tilde{Y}_{L,S_j}$. Since $\dim Y_{L,S_{j'}} = \dim Y_{L,S_j}$ (see 3.13(b)) and Y_{L,S_j} is the only stratum in \tilde{Y}_{L,S_j} of its dimension, we see that $Y_{L,S_{j'}} = Y_{L,S_j}$. Thus, $g \in Y_{L,S_j}$. Using the surjectivity of the map in Lemma 5.5 (for \tilde{S}_j instead of S) we see that there exists $(g, x'P) \in \tilde{Y}_{L,S_j}$ such that $(g, x'P) = (g, xP)$. This proves the surjectivity of our map.

15.4. We are still in the setup of 15.1. For any stratum S' of $N_G L$ that is contained in $\tilde{\Sigma}$ let $X_{S'} = \{(g, xP) \in G \times G^0/P; x^{-1}gx \in S' U_P\} \subset X$. We have $X = \bigsqcup_{S'} X_{S'}$ (a finite union over all S' as above). Let $X_\Sigma = \bigcup_{j \in J} X_{S_j}$. Then X_Σ is a smooth, open dense subvariety of pure dimension of X (since Σ is a smooth, open dense subvariety of pure dimension of $\tilde{\Sigma}$).

We define a local system $\bar{\mathcal{E}}$ on X_Σ by the requirement that $b'^*\mathcal{E} = a'^*\bar{\mathcal{E}}$ where $a'(g, x) = (g, xP), b'(g, x) = f(x^{-1}gx)$ (f as in 5.4) in the diagram

$$X_\Sigma \xleftarrow{a'} \{(g, x) \in G \times G^0; x^{-1}gx \in \Sigma U_P\} \xrightarrow{b'} \Sigma.$$

(We use the fact that a' is a principal P -bundle and $b'^*\mathcal{E}$ is P -equivariant.) Then $IC(X, \bar{\mathcal{E}}) \in \mathcal{D}(X)$ is well defined and we have $a''^*IC(X, \bar{\mathcal{E}}) = b''^*IC(\tilde{\Sigma}, \mathcal{E})$ where $a''(g, x) = (g, xP), b''(g, x) = f(x^{-1}gx)$ in the diagram

$$X \xleftarrow{a''} \{(g, x) \in G \times G^0; x^{-1}gx \in \tilde{\Sigma} U_P\} \xrightarrow{b''} \tilde{\Sigma}.$$

We have the following generalization of Lemma 5.7:

(a) $\psi_!(IC(X, \bar{\mathcal{E}}))$ is canonically isomorphic to $IC(\bar{Y}_{L,\Sigma}, \pi_! \bar{\mathcal{E}})$.

The proof is similar to that of Lemma 5.7. Let $K = IC(X, \bar{\mathcal{E}})$ and let $K^* = IC(X, \check{\mathcal{E}})$ where $\check{\mathcal{E}}$ is defined like $\bar{\mathcal{E}}$, replacing \mathcal{E} by the dual local system. Using 15.3(a), we see that $\psi_! K|_{Y_{L,\Sigma}} = \pi_! \check{\mathcal{E}}$. As in the proof of Lemma 5.7, it is enough to verify the following statement.

For any $i > 0$ we have $\dim \text{supp} \mathcal{H}^i(\psi_! K) < \dim \bar{Y}_{L,\Sigma}$ and $\dim \text{supp} \mathcal{H}^i(\psi_! K^*) < \dim \bar{Y}_{L,\Sigma} - i$.

We shall verify this only for K ; the corresponding statement for K^* is entirely similar. As in the proof of Lemma 5.7, it is enough to prove:

For any $i > 0$ and any stratum S' of $N_G L$ contained in $\bar{\Sigma}$ we have

$$\dim\{g \in \bar{Y}_{L,\Sigma}; H_c^i(\psi^{-1}(g) \cap X_{S'}, K) \neq 0\} < \dim \bar{Y}_{L,\Sigma} - i.$$

Assume first that S' is not one of the S_j . As in the proof of Lemma 5.7, we see that it is enough to prove that

$$\dim\{g \in \bar{Y}_{L,\Sigma}; \dim(\psi^{-1}(g) \cap X_{S'}) > \frac{i}{2} - \frac{1}{2}(\dim \Sigma - \dim S')\} < \dim \bar{Y}_{L,\Sigma} - i.$$

Since $\bar{Y}_{L,\Sigma} = \bigcup_{j \in J} \bar{Y}_{L,S_j}$ this follows from the analogous inequality in the proof of Lemma 5.7 where $\bar{Y}_{L,\Sigma}$ is replaced by \bar{Y}_{L,S_j} .

Next assume that $S' = S_j$ for some $j \in J$. As in the proof of Lemma 5.7, we see that it is enough to prove that

$$\dim\{g \in \bar{Y}_{L,\Sigma}; \dim(\psi^{-1}(g) \cap X_{S_j}) \geq \frac{i}{2}\} < \dim \bar{Y}_{L,\Sigma} - i.$$

If $g \in \bar{Y}_{L,\Sigma}$ satisfies $\dim(\psi^{-1}(g) \cap X_{S_j}) \geq \frac{i}{2}$, then $\psi^{-1}(g) \cap X_{S_j} \neq \emptyset$ and we have $x^{-1}gx \in S_j U_P$ for some $x \in G^0$. Hence $g \in \psi(X_j) = \bar{Y}_{L,S_j}$. Thus it is enough to prove that

$$\dim\{g \in \bar{Y}_{L,S_j}; \dim(\psi^{-1}(g) \cap X_{S_j}) \geq \frac{i}{2}\} < \dim \bar{Y}_{L,S_j} - i.$$

This is actually contained in the proof of Lemma 5.7. This completes the proof.

15.5. The remainder of this section (except 15.12) is concerned with the proof of Proposition 15.2. If the analogue of the isomorphism 15.2(a) is known when Σ is replaced by S_j then, taking direct sum over $j \in J$ and using the isomorphism 15.1(a) and its analogue for \mathcal{E}^1 instead of \mathcal{E} , we obtain an isomorphism as in 15.2(a) for Σ . Thus, to prove Proposition 15.2, we may assume that $\Sigma = S$ is a single isolated stratum of $N_G L$ with $(L, S) \in \mathbf{A}$. Then $\mathcal{E} \in \mathcal{S}(S)$.

Let D be the connected component of G that contains S . Let δ be the connected component of $N_G L$ that contains S . By the assumption of Proposition 15.2, the set S^ω of unipotent elements in S is a single L -conjugacy class. We have $S = {}^\delta \mathcal{Z}_L^0 S^\omega$. Let $S^*, Y = Y_{L,S}, \bar{Y} = \bar{Y}_{L,S}, \tilde{Y} = \tilde{Y}_{L,S}, \pi : \tilde{Y} \rightarrow Y$ be as in 15.1.

Let $\mu_\infty = \mu_\infty({}^\delta \mathcal{Z}_L^0)$ be as in 5.3. There is a canonical direct sum decomposition $\mathcal{E} = \bigoplus_\lambda \mathcal{E}^\lambda$ in $\mathcal{S}(S)$ where λ runs over the set of homomorphisms $\mu_\infty \rightarrow \mathbf{Q}_l^*$ that factor through some $\mu_n = \mu_n({}^\delta \mathcal{Z}_L^0)$ (see 5.3). Here \mathcal{E}^λ has the property that, for any $z \in {}^\delta \mathcal{Z}_L^0, h \in S^\omega$, the monodromy action of μ_∞ on the stalk \mathcal{E}_{zh}^λ of the local system $\mathcal{E}^\lambda|_{{}^\delta \mathcal{Z}_L^0 h}$ (equivariant for the transitive ${}^\delta \mathcal{Z}_L^0$ -action $z_1 : zh \mapsto z_1^n zh$ for some $n \in \mathbf{N}_k^*$) is through λ . We have canonically

$$IC(\bar{Y}, \pi_! \bar{\mathcal{E}}) = \bigoplus_\lambda IC(\bar{Y}, \pi_! \bar{\mathcal{E}}^\lambda), \quad IC(\bar{Y}, \pi_! \bar{\mathcal{E}}^1) = \bigoplus_\lambda IC(\bar{Y}, \pi_! \bar{\mathcal{E}}^{1\lambda}).$$

Hence if we can construct the isomorphism 15.2(a) for each \mathcal{E}^λ instead of \mathcal{E} , then by taking direct sums we get the isomorphism (a) for \mathcal{E} . Thus we may assume that $\mathcal{E} = \mathcal{E}^\lambda$ for some λ .

Let \mathcal{L} be a local system of rank 1 on ${}^\delta\mathcal{Z}_L^0$, equivariant for the transitive ${}^\delta\mathcal{Z}_L^0$ -action $z_1 : z \mapsto z_1^n z$ for some $n \in \mathbf{N}_k^*$, whose associate homomorphism $\mu_\infty \rightarrow \mathbf{Q}_l^*$ is λ^{-1} ; let v_0 be a basis vector of the stalk \mathcal{L}_1 . The pair (\mathcal{L}, v_0) is defined up to a unique isomorphism.

Let $b : S \rightarrow {}^\delta\mathcal{Z}_L^0$ be $g \mapsto g_s$. From the definitions, the restriction of $\mathcal{E} \otimes b^*\mathcal{L}$ to any fibre of $S \rightarrow S^\omega, g \mapsto g_u$ is isomorphic to $\bar{\mathbf{Q}}_l^e$ where e is the rank of \mathcal{E} . Also

$$(a) \quad (\mathcal{E} \otimes b^*\mathcal{L})|_{S^\omega} = \mathcal{E}^1|_{S^\omega} = \mathcal{E}|_{S^\omega}$$

canonically, using the identification $\mathcal{L}_1 = \bar{\mathbf{Q}}_l, v_0 \leftrightarrow 1$. It follows that $\mathcal{E} \otimes b^*\mathcal{L}$ is isomorphic to \mathcal{E}^1 and there is a unique isomorphisms between these two local systems which induces for the restrictions to S^ω the identification (a). Thus we have a canonical isomorphism

$$(b) \quad \mathcal{E} \otimes b^*\mathcal{L} \cong \mathcal{E}^1.$$

It is then enough to construct a canonical isomorphism

$$(c) \quad IC(\bar{Y}, \pi_1 \tilde{\mathcal{E}})|_{\bar{Y}^\omega} \xrightarrow{\sim} IC(\bar{Y}, \pi_1 \widetilde{\mathcal{E} \otimes b^*\mathcal{L}})|_{\bar{Y}^\omega}.$$

where $\bar{Y}^\omega = \{g \in \bar{Y}; g \text{ unipotent}\}$.

15.6. Let S^\dagger be the subset of S consisting of those $zh, z \in {}^\delta\mathcal{Z}_L^0, h \in S^\omega$ such that

$$n \in N_{G^0}\delta = \{x \in G^0; x\delta x^{-1} = \delta\}, nz = zn \implies n \in L$$

(that is, z has trivial stabilizer for the conjugation action of $N_{G^0}\delta/L$ on ${}^\delta\mathcal{Z}_L^0$). Now S^\dagger is open dense in S . (Since $N_{G^0}\delta/L$ is finite, it is enough to show that, for any $n \in N_{G^0}\delta - L$, the closed subset $\{zh; z \in {}^\delta\mathcal{Z}_L^0, h \in S^\omega, nz = zn\}$ of S is $\neq S$ or that the closed subset $\{z \in {}^\delta\mathcal{Z}_L^0; nz = zn\}$ of ${}^\delta\mathcal{Z}_L^0$ is $\neq {}^\delta\mathcal{Z}_L^0$. If it is equal to ${}^\delta\mathcal{Z}_L^0$, then $n \in Z_{G^0}({}^\delta\mathcal{Z}_L^0)$ hence using 1.10(a), $n \in L$, a contradiction.)

Since S^* is an open dense subset of S it follows that $S^{*\dagger} = S^* \cap S^\dagger$ is an open dense subset of S . Hence

$$\tilde{Y}^\dagger = \{(g, xL) \in G \times G^0; x^{-1}gx \in S^{*\dagger}\}$$

is an open dense subset of \tilde{Y} (we use that $S^{*\dagger}$ is stable under L -conjugacy). Also, \tilde{Y}^\dagger is stable under the free action of \mathcal{W}_S on \tilde{Y} (see 3.13); it follows that $Y^\dagger = \pi(\tilde{Y}^\dagger)$ is open dense in Y and $\tilde{Y}^\dagger = \pi^{-1}(Y^\dagger)$.

Let $\sigma : D \rightarrow D//G^0$ be as in 7.1. Let $\mathbf{a} = \sigma(\bar{Y})$. Let u be a unipotent, quasi-semisimple element of N_GL such that $u \in \delta$. As in the proof of Lemma 7.3(b) we see that $\mathbf{a} = \{\sigma(zu); z \in {}^\delta\mathcal{Z}_L^0\}$. Let

$$\Upsilon = \{(g, z) \in \bar{Y} \times {}^\delta\mathcal{Z}_L^0; \sigma(g) = \sigma(zu)\}.$$

Define $\kappa : \tilde{Y}^\dagger \rightarrow \Upsilon$ by $(g, xL) \mapsto (g, z)$ where $x^{-1}gx = zh \in S^{*\dagger}, z \in {}^\delta\mathcal{Z}_L^0, h \in S^\omega$. (This definition is correct since $\sigma(zh) = \sigma(zu)$ for $z \in {}^\delta\mathcal{Z}_L^0, h \in S^\omega$; see 7.3.) Let Υ' be the closure of $\kappa(\tilde{Y}^\dagger)$ in Υ . Define $\psi' : \Upsilon' \rightarrow \bar{Y}$ by $(g, z) \mapsto g$.

Lemma 15.7. (a) $\kappa(\tilde{Y}^\dagger)$ is open in Υ' .

(b) κ restricts to an isomorphism $\tilde{Y}^\dagger \xrightarrow{\sim} \kappa(\tilde{Y}^\dagger)$.

(c) ψ' is a finite surjective morphism.

We verify (b) at the level of sets. We must show that $\kappa : \tilde{Y}^\dagger \rightarrow \Upsilon$ is injective. Assume that $(g, xL), (g', x'L)$ in \tilde{Y}^\dagger have the same image under κ . Then $g = g'$ and

$$x^{-1}gx = zh, x'^{-1}gx' = zh', z \in {}^\delta Z_L^0, h \in S^\omega, h' \in S^\omega, zh \in S^{*\dagger}.$$

From 3.13(a) we see that $x' = xn^{-1}$ for some $n \in N_{G^0}L, nS^\omega n^{-1} = S^\omega$. Then $zh' = nzhn^{-1} = nzn^{-1}(nhn^{-1})$. Now z, z' are semisimple elements commuting with the unipotent elements h', nhn^{-1} . It follows that $z = nzn^{-1}$. Since $n \in N_{G^0}\delta$ and z has trivial stabilizer in $N_{G^0}\delta/L$, we see that $n \in L$. Thus $xL = x'L$ and κ is injective, as required.

We prove (a). Let P be a parabolic of G^0 with Levi L such that $S \subset N_G P$. Let $\psi : X \rightarrow \bar{Y}$ be as in 3.14. Define $f : X \rightarrow \Upsilon$ by $f(g, xP) = (g, z)$ where $x^{-1}gx \in z\bar{S}^\omega U_P, z \in {}^\delta Z_L^0$. (We show that z is uniquely determined by (g, xP) . It is enough to show that, if $zhv = p^{-1}z'h'v'p$ with $z, z' \in {}^\delta Z_L^0, h, h' \in \bar{S}^\omega, v, v' \in U_P, p \in P$, then $z = z'$. Writing $p \in lU_P$ with $l \in L$, we have $zh = l^{-1}z'h'l = z'l^{-1}h'l$. By the uniqueness of Jordan decomposition we have $z = z'$ as required. To show that f is well defined we must also show that, if $g \in \bar{Y}, x \in G^0, z \in {}^\delta Z_L^0, h \in \bar{S}^\omega, v \in U_P$ satisfy $x^{-1}gx = zhv$, then $\sigma(zu) = \sigma(g)$. Clearly, $\sigma(g) = \sigma(zhv)$ hence we must show that $\sigma(zu) = \sigma(zhv)$. This follows from the description of σ given in 7.1.) We have $\psi = pr_1 \circ f$ where $pr_1 : \Upsilon \rightarrow \bar{Y}$ is $(g, z) \mapsto z$. Since ψ is proper, it follows that f is proper and $f(X)$ is closed in Υ . Since Y^\dagger is open in Y and Y is open in \bar{Y} we see that Y^\dagger is open in \bar{Y} and $\psi^{-1}(\bar{Y} - Y^\dagger)$ is closed in X . Since f is proper it follows that $f(\psi^{-1}(\bar{Y} - Y^\dagger))$ is closed in $f(X)$. We have $f(\psi^{-1}(\bar{Y} - Y^\dagger)) \cap f(\psi^{-1}(Y^\dagger)) = \emptyset$. (Indeed if $(g, xP), (g', x'P) \in X$ have the same image under f and $g \in Y^\dagger$, then $g' \in Y^\dagger$.) Thus, $f(\psi^{-1}(Y^\dagger))$ is the complement of $f(\psi^{-1}(\bar{Y} - Y^\dagger))$ in $f(X)$ hence it is open in $f(X)$. From Lemma 5.5 we see that $\gamma_0 : \tilde{Y}^\dagger \rightarrow \psi^{-1}(Y^\dagger), (g, xL) \mapsto (g, xP)$ is an isomorphism. Since $f\gamma_0 = \kappa$, we see that $\kappa(\tilde{Y}^\dagger) = f(\psi^{-1}(Y^\dagger))$ hence $\kappa(\tilde{Y}^\dagger)$ is open in $f(X)$. Since Υ' is the closure of $\kappa(\tilde{Y}^\dagger)$ in Υ and $f(X)$ is a closed subset of Υ containing $\kappa(\tilde{Y}^\dagger)$, we see that $\Upsilon' \subset f(X)$. Since X, \bar{Y} are irreducible of the same dimension (see the proof of 3.14) and \tilde{Y}^\dagger is open dense in \bar{Y} , we see that $f(X), \kappa(\tilde{Y}^\dagger), \Upsilon'$ are irreducible and

$$\dim \bar{Y} = \dim \tilde{Y}^\dagger = \dim \kappa(\tilde{Y}^\dagger) \leq \dim \Upsilon' \leq \dim f(X) \leq \dim X = \dim \bar{Y}.$$

It follows that $\dim \Upsilon' = \dim f(X) = \dim X$ hence $\Upsilon' = f(X)$. Thus, $\kappa(\tilde{Y}^\dagger)$ is open in Υ' .

We prove (c). Let T_1 be a maximal torus of $Z_G(u)^0$ that contains ${}^\delta Z_L^0$. As in 7.1, σ induces a finite morphism $uT_1 \rightarrow D//G^0$. This restricts to a finite morphism $u{}^\delta Z_L^0 \rightarrow \mathbf{a}$. Since $pr_1 : \Upsilon \rightarrow \bar{Y}$ is obtained from this finite morphism by change of base, it follows that pr_1 is a finite morphism. Restricting to the closed subset Υ' of Υ , we deduce that $\psi' : \Upsilon' \rightarrow \bar{Y}$ is a finite morphism. To see that it is surjective, we note that $\psi : X \rightarrow \bar{Y}$ is surjective and ψ factorizes as $X \xrightarrow{f} \Upsilon' \xrightarrow{\psi'} \bar{Y}$. The lemma is proved.

15.8. Let $\underline{\mathcal{E}} = \kappa_!(\tilde{\mathcal{E}}|_{\tilde{Y}^\dagger})$, a local system on $\kappa(\tilde{Y}^\dagger)$. Since \tilde{Y} is smooth (see 3.17) and \tilde{Y}^\dagger is open in \tilde{Y} , we see that \tilde{Y}^\dagger is smooth. Using Lemma 15.7(a),(b) we see that $\kappa(\tilde{Y}^\dagger)$ is a smooth open dense subvariety of Υ' . Hence $IC(\Upsilon', \underline{\mathcal{E}}) \in \mathcal{D}(\Upsilon')$ is well defined.

Lemma 15.9. $\psi'_!IC(\Upsilon', \underline{\mathcal{E}})$ is canonically isomorphic to $IC(\bar{Y}, \pi_1^* \underline{\mathcal{E}})$.

Since Y^\dagger is open dense in Y , we have canonically $IC(\bar{Y}, \pi_1 \tilde{\mathcal{E}}) = IC(\bar{Y}, (\pi_1 \tilde{\mathcal{E}})_{Y^\dagger})$.

Let $K_1 = IC(\Upsilon', \underline{\mathcal{E}})$, $K_1^* = IC(\Upsilon', \check{\underline{\mathcal{E}}})$ where $\check{\underline{\mathcal{E}}}$ is defined like $\underline{\mathcal{E}}$ replacing \mathcal{E} by $\check{\mathcal{E}}$. Then K_1^* is the Verdier dual of K_1 with a suitable shift. Since ψ' is proper, it follows that $\psi'_!(K_1^*)$ is the Verdier dual of $\psi'_!K_1$ with a suitable shift. From the definitions it is clear that $(\psi'_!K_1)|_{Y^\dagger} = (\pi_1 \tilde{\mathcal{E}})|_{Y^\dagger}$. By the definition of an intersection cohomology complex we see that it is enough to verify the following statement.

For any $i > 0$ we have

$$\dim \operatorname{supp} \mathcal{H}^i(\psi'_!K_1) < \dim \bar{Y} - i \quad \text{and} \quad \dim \operatorname{supp} \mathcal{H}^i(\psi'_!(K_1^*)) < \dim \bar{Y} - i.$$

We shall only verify this for K_1 ; the corresponding statement for K_1^* is entirely analogous. For $g \in \bar{Y}$, $\psi'^{-1}(g)$ is finite hence

$$\mathcal{H}_g^i(\psi'_!K_1) = H_c^i(\psi'^{-1}(g), K_1) = \bigoplus_{y \in \psi'^{-1}(g)} \mathcal{H}_y^i K_1.$$

It is enough to check that $\dim \psi'(\{y \in \Upsilon'; \mathcal{H}_y^i K_1 \neq 0\}) < \dim \bar{Y} - i$. But

$$\dim \psi'(\{y \in \Upsilon'; \mathcal{H}_y^i K_1 \neq 0\}) \leq \dim \{y \in \Upsilon'; \mathcal{H}_y^i K_1 \neq 0\} < \dim \Upsilon' - i = \dim \bar{Y} - i$$

(since K_1 is an intersection cohomology complex on Υ'). The lemma is proved.

Lemma 15.10. *Let $\mathcal{L}' = pr_2^* \mathcal{L}$ where $pr_2 : \Upsilon' \rightarrow {}^\delta \mathcal{Z}_L^0$ is the second projection. The restriction of the local system \mathcal{L}' on Υ' to $\kappa(\tilde{Y}^\dagger)$ is denoted again by \mathcal{L}' . We have a canonical isomorphism*

$$(a) \quad IC(\Upsilon', \underline{\mathcal{E}}) \otimes \mathcal{L}' \xrightarrow{\sim} IC(\Upsilon', \underline{\mathcal{E}} \otimes \mathcal{L}').$$

The restrictions of the two sides of (a) to the open dense subset $\kappa(\tilde{Y}^\dagger)$ of Υ' are canonically isomorphic (they can both be identified with $\underline{\mathcal{E}} \otimes \mathcal{L}'$). From the properties of intersection cohomology complexes it follows that this extends to an isomorphism as in (a) provided we can show that the left-hand side of (a) is an intersection cohomology complex on Υ' . To do this we choose a parabolic P as in the proof of Lemma 15.7(a). Let $f : X \rightarrow \Upsilon'$ be as in that proof. Then f is proper, surjective. Let $X_S, \bar{\mathcal{E}}$ be as in 5.6 and let K, K^* be as in 5.7. We show that

$$(b) \quad f_! K = IC(\bar{Y}', \underline{\mathcal{E}}).$$

It is clear that $(f_! K)|_{\kappa(\tilde{Y}^\dagger)} = \underline{\mathcal{E}}$. As in the proof of Lemma 5.7, it is enough to verify the following statement.

For any $i > 0$ we have

$$\dim \operatorname{supp} \mathcal{H}^i(f_! K) < \dim \Upsilon' - i \quad \text{and} \quad \dim \operatorname{supp} \mathcal{H}^i(f_! K^*) < \dim \Upsilon' - i.$$

We shall only verify this for K ; the corresponding statement for K^* is entirely similar. Let $y \in \Upsilon'$. If $\mathcal{H}_y^i(f_! K) \neq 0$, then $\bigoplus_{y' \in \Upsilon'; \psi'(y') = \psi'(y)} \mathcal{H}_{y'}^i(f_! K) \neq 0$ hence $\mathcal{H}_{\psi'(y)}^i(\psi'_! f_! K) \neq 0$. (We use that $\psi'^{-1}(\psi'(y))$ is finite.) Thus, $y \in \operatorname{supp} \mathcal{H}^i(f_! K)$ implies $\psi'(y) \in \operatorname{supp} \mathcal{H}^i(\psi'_! K)$ that $y \in \psi'^{-1}(\operatorname{supp} \mathcal{H}^i(\psi'_! K))$. We see that

$$\operatorname{supp} \mathcal{H}^i(f_! K) \subset \psi'^{-1}(\operatorname{supp} \mathcal{H}^i(\psi'_! K)).$$

Hence

$$\dim \operatorname{supp} \mathcal{H}^i(f_! K) \leq \dim \psi'^{-1}(\operatorname{supp} \mathcal{H}^i(\psi'_! K)).$$

Since ψ' is finite, surjective, we have $\dim \psi'^{-1}(\operatorname{supp} \mathcal{H}^i(\psi'_! K)) \leq \dim \operatorname{supp} \mathcal{H}^i(\psi'_! K)$. It is then enough to show that $\dim \operatorname{supp} \mathcal{H}^i(f_! K) < \dim \Upsilon' - i$. But this follows from an estimate in the proof of Lemma 5.7 since $\dim \Upsilon' = \dim Y$. This proves (b).

We show that

(c) $K \otimes f^* \mathcal{L}' = IC(X, \bar{\mathcal{E}}^1)$

where $\bar{\mathcal{E}}^1$ is the local system on X_S defined in terms of \mathcal{E}^1 in the same way as $\bar{\mathcal{E}}$ is defined in terms of \mathcal{E} (see 5.6). With the notation in 5.6(a), it is enough to show that $a''^*(K \otimes f^* \mathcal{L}') = a''^* IC(X, \bar{\mathcal{E}}^1)$ or that $b''^* IC(\bar{S}, \mathcal{E}) \otimes b''^* \bar{b}^* \mathcal{L} = b''^* IC(\bar{S}, \mathcal{E}^1)$ where $\bar{b} : \bar{S} \rightarrow {}^\delta \mathcal{Z}_L^0$ is $g \mapsto g_s$. It is enough to show that $IC(\bar{S}, \mathcal{E}) \otimes \bar{b}^* \mathcal{L} = IC(\bar{S}, \mathcal{E}^1)$. This follows immediately from the definitions. Thus (c) is proved.

Using (b),(c), we have

$$IC(\Upsilon', \underline{\mathcal{E}}) \otimes \mathcal{L}' = f_! K \otimes \mathcal{L}' = f_!(K \otimes f^* \mathcal{L}') = f_!(IC(X, \bar{\mathcal{E}}^1)) = IC(\Upsilon', \underline{\mathcal{E}}^1),$$

the last step being (b) applied to \mathcal{E}^1 instead of \mathcal{E} . (We define $\underline{\mathcal{E}}^1$ in terms of \mathcal{E}^1 in the same way as $\underline{\mathcal{E}}$ is defined in terms of \mathcal{E} .) This completes the proof.

It is clear that, although P is used in the proof above, the isomorphism (a) that we construct does not depend on the choice of P hence it is truly canonical.

15.11. Let $\Upsilon^\omega = \bar{Y}^\omega \times \{1\} \subset \bar{Y} \times {}^\delta \mathcal{Z}_L^0$. Then Υ^ω is a closed subset of Υ . More precisely, we have $\Upsilon^\omega \subset \Upsilon'$. (With notation in the proof of Lemma 15.7, it is enough to show that $\Upsilon^\omega \subset f(X)$. Let $g \in \bar{Y}^\omega$. Since $\psi : X \rightarrow \bar{Y}$ is surjective, we can find $xP \in G^0/P$ such that $x^{-1}gx \in z\bar{S}^\omega U_P$ where $z \in {}^\delta \mathcal{Z}_L^0$. Since $x^{-1}gx$ is unipotent, we must have $z = 1$. Hence $f(g, xP) = (g, 1)$. Thus $(g, 1) \in f(X)$ as required.) Thus we have $\Upsilon^\omega \subset \psi'^{-1}(\bar{Y}^\omega)$. This is in fact an equality. (If $(g, z) \in \Upsilon'$ and g is unipotent, then from $\sigma(zu) = \sigma(g) = \omega$ we see that zu is unipotent hence $z = 1$.) Since ψ' restricts to an isomorphism $\Upsilon^\omega \xrightarrow{\sim} \bar{Y}^\omega$, it also restricts to an isomorphism $\psi'_0 : \psi'^{-1}(\bar{Y}^\omega) \xrightarrow{\sim} \bar{Y}^\omega$. Via ψ'_0 we may identify \bar{Y}^ω with $\Upsilon^\omega = \psi'^{-1}(\bar{Y}^\omega)$. By change of base, we have

$$IC(\bar{Y}, \pi_1 \tilde{\mathcal{E}})|_{\bar{Y}^\omega} = (\psi'_! IC(\Upsilon', \underline{\mathcal{E}}))|_{\bar{Y}^\omega} = \psi'_{0!}(IC(\Upsilon', \underline{\mathcal{E}})|_{\psi'^{-1}(\bar{Y}^\omega)}).$$

Thus, via ψ'_0 , we may identify $IC(\bar{Y}, \pi_1 \tilde{\mathcal{E}})|_{\bar{Y}^\omega} = IC(\Upsilon', \underline{\mathcal{E}})|_{\Upsilon^\omega}$.

Similarly, we may identify $IC(\bar{Y}, \pi_1 \mathcal{E} \otimes b^* \mathcal{L})|_{\bar{Y}^\omega} = IC(\Upsilon', \underline{\mathcal{E}} \otimes b^* \mathcal{L})|_{\Upsilon^\omega}$. Hence, in order to construct the isomorphism 15.5(c), it is enough to construct a canonical isomorphism

$$IC(\Upsilon', \underline{\mathcal{E}})|_{\Upsilon^\omega} \xrightarrow{\sim} IC(\Upsilon', \underline{\mathcal{E}} \otimes b^* \mathcal{L})|_{\Upsilon^\omega}.$$

From the definitions we have $\underline{\mathcal{E}} \otimes b^* \mathcal{L} = \underline{\mathcal{E}} \otimes \mathcal{L}'$. Hence it is enough to construct a canonical isomorphism

$$IC(\Upsilon', \underline{\mathcal{E}})|_{\Upsilon^\omega} \xrightarrow{\sim} IC(\Upsilon', \underline{\mathcal{E}} \otimes \mathcal{L}')|_{\Upsilon^\omega}.$$

We have canonically $\mathcal{L}'_{\Upsilon^\omega} = \bar{\mathbf{Q}}_l$ (we identify $\mathcal{L}_1 = \bar{\mathbf{Q}}_l$ by $v_0 \leftrightarrow 1$). Hence $(IC(\Upsilon', \underline{\mathcal{E}}) \otimes \mathcal{L}')|_{\Upsilon^\omega} = IC(\Upsilon', \underline{\mathcal{E}})|_{\Upsilon^\omega}$. Thus it is enough to construct a canonical isomorphism

$$(IC(\Upsilon', \underline{\mathcal{E}}) \otimes \mathcal{L}')|_{\Upsilon^\omega} \xrightarrow{\sim} IC(\Upsilon', \underline{\mathcal{E}} \otimes \mathcal{L}')|_{\Upsilon^\omega}.$$

This is obtained by restricting to Υ^ω the isomorphism 15.10(a). This completes the proof of Proposition 15.2.

15.12. In the remainder of this section we assume that \mathbf{k} is an algebraic closure of a finite field \mathbf{F}_q and that G has a fixed \mathbf{F}_q -structure with Frobenius map $F : G \rightarrow G$.

For any algebraic variety Z defined over \mathbf{F}_q with Frobenius map $F : Z \rightarrow Z$, an object $A \in \mathcal{D}(Z)$ and an isomorphism $\phi : F^* A \xrightarrow{\sim} A$ in $\mathcal{D}(Z)$ we define the *characteristic function* $\chi_{A, \phi} : Z^F \rightarrow \bar{\mathbf{Q}}_l$ by

(a)
$$\chi_{A, \phi}(z) = \sum_i (-1)^i \text{tr}(\phi, \mathcal{H}_z^i A), \quad (z \in Z^F).$$

(The map induced by ϕ on the stalk $\mathcal{H}_z^i A$ is denoted again by ϕ .)

Consider a quadruple $(L, \Sigma^\omega, \mathcal{F}, \phi_1)$ where

L is an F -stable Levi of some (not necessarily F -stable) parabolic of G^0 ;

Σ^ω is the set of unipotent elements in a subset Σ of $N_G L$ as in 15.1, 15.2 such that $F(\Sigma^\omega) = \Sigma^\omega$ (or equivalently $F(\Sigma) = \Sigma$); note that Σ^ω, Σ determine each other;

\mathcal{F} is an L -equivariant local system on Σ^ω ;

$\phi_1 : F^* \mathcal{F} \xrightarrow{\sim} \mathcal{F}$ is an isomorphism of local systems on Σ^ω .

Consider the complex $\mathfrak{K} = IC(\bar{Y}_{L,\Sigma}, \pi_1 \tilde{\mathcal{E}}) \in \mathcal{D}(\bar{Y}_{L,\Sigma})$ where $\bar{Y}_{L,\Sigma}, \pi : \tilde{Y}_{L,\Sigma} \rightarrow Y_{L,\Sigma}$ are defined as in 15.1 and $\tilde{\mathcal{E}}$ is defined as in 15.1 in terms of a local system \mathcal{E} on Σ (as in 15.1) such that \mathcal{F} is the inverse image of \mathcal{E} under the inclusion $\Sigma^\omega \rightarrow \Sigma$.

We assume that we are given an isomorphism $\phi'_1 : F^* \mathcal{E} \xrightarrow{\sim} \mathcal{E}$ of local systems on Σ extending ϕ_1 . (Note that we can always find \mathcal{E}, ϕ'_1 as above: for example, we have the “trivial choice” where \mathcal{E} is the inverse image of \mathcal{F} under $\Sigma \rightarrow \Sigma^\omega, g \mapsto g_u$ and ϕ'_1 is induced by ϕ_1 . However, for future applications, it is necessary to allow other choices of \mathcal{E}, ϕ'_1 .) Now $\bar{Y}_{L,\Sigma}, \tilde{Y}_{L,\Sigma}, Y_{L,\Sigma}$ have natural \mathbf{F}_q -structures with Frobenius maps F and ϕ'_1 induces an isomorphism $F^* \tilde{\mathcal{E}} \xrightarrow{\sim} \tilde{\mathcal{E}}$ of local systems on $\tilde{Y}_{L,\Sigma}$, an isomorphism $F^* \pi_1 \tilde{\mathcal{E}} \xrightarrow{\sim} \pi_1 \tilde{\mathcal{E}}$ of local systems on $Y_{L,\Sigma}$ and an isomorphism $\phi : F^* \mathfrak{K} \xrightarrow{\sim} \mathfrak{K}$ in $\mathcal{D}(\bar{Y}_{L,\Sigma})$. We define a function

$$(b) \quad Q_{L,G,\Sigma^\omega,\mathcal{F},\phi_1} : \{\text{unipotent elements in } G^F\} \rightarrow \bar{\mathbf{Q}}_l$$

by

$$(c) \quad Q_{L,G,\Sigma^\omega,\mathcal{F},\phi_1}(u) = \chi_{\mathfrak{K},\phi}(u)$$

(see (a)) if $u \in \bar{Y}_{L,\Sigma}$ and $Q_{L,G,\Sigma^\omega,\mathcal{F},\phi_1}(u) = 0$ if $u \notin \bar{Y}_{L,\Sigma}$. The function (c) is called a *generalized Green function*. It extends (up to a sign) a definition given in [L3, II, 8.3.1] (in the case where $G = G^0$ and Σ^ω is a single unipotent class). From Proposition 15.2 we see that

(d) $Q_{L,G,\Sigma^\omega,\mathcal{F},\phi_1}(u)$ is independent of the choice of \mathcal{E}, ϕ'_1 , namely it is the same for a general \mathcal{E}, ϕ'_1 as for the “trivial choice”. (The isomorphism in Proposition 15.2 is compatible with the Frobenius maps.)

16. THE CHARACTERISTIC FUNCTION $\chi_{\mathfrak{K},\phi}$

16.1. In this section we fix $(L, S) \in \mathbf{A}$ and $\mathcal{E} \in \mathcal{S}(S)$. Let δ be the connected component of $N_G L$ that contains S . Recall (cf. 1.22) that

$$(a) \quad S_s = \{g_s; g \in S\} \text{ is a single } \delta \mathcal{Z}_L^0 \times L\text{-orbit on } N_G L \text{ for the action } (z, x) : y \mapsto xzyx^{-1}.$$

Lemma 16.2. (a) Let $s' \in S_s$. Let $\mathbf{c}_{(s')} = \{v \in Z_G(s'); v \text{ unipotent}, s'v \in S\}$. Then $\mathbf{c}_{(s')} = \bigsqcup_{j \in J} \mathbf{c}_j$ where J is finite and \mathbf{c}_j are (unipotent) $Z_L(s')^0$ -conjugacy classes of dimension independent of j .

(b) For any $j \in J$, the stratum S_j of $Z_{N_G L}(s')$ that contains \mathbf{c}_j is $\delta \mathcal{Z}_L^0 \mathbf{c}_j$. In particular, $\dim S_j$ is independent of j .

First we note that

$$(c) \text{ any } G^0\text{-conjugacy class in } G \text{ has finite intersection with } \delta \mathcal{Z}_L^0 s'.$$

This follows from 1.14(a),(d) applied to $g = s'$ and to a maximal torus of $Z_G(g)^0$ that contains $\delta \mathcal{Z}_L^0$.

In particular, any L -conjugacy class in $N_G L$ has finite intersection with ${}^\delta \mathcal{Z}_L^0 s'$. Hence the group $\tilde{Z} = \{y \in L; y s' y^{-1} \in {}^\delta \mathcal{Z}_L^0 s'\}$ contains $Z_L(s')$ as a subgroup of finite index. Thus, $\tilde{Z}^0 = Z_L(s')^0$. From 16.1(a) we see that \tilde{Z} acts transitively (by conjugation) on $\mathfrak{c}_{(s')}$. Since $Z_L(s')^0$ is normal in \tilde{Z} , it follows that \tilde{Z} permutes transitively the $Z_L(s')^0$ -orbits in $\mathfrak{c}_{(s')}$; hence all these orbits have the same dimension. This proves (a).

We prove (b). We have $S_j = \{z \in \mathcal{Z}_{Z_L(s')^0}^0; z v = v z\}^0 \mathfrak{c}_j = T_{N_G L}(s'v) \mathfrak{c}_j$ where v is any element of \mathfrak{c}_j . Since $s'v \in S$, $s'v$ is isolated in $N_G L$ hence $T_{N_G L}(s'v) = {}^\delta \mathcal{Z}_L^0$; see 2.2. This proves (b).

16.3. We fix a semisimple element $s \in G$ and a unipotent element $u \in Z_G(s)$ such that $su \in \bar{Y}$, the closure of $Y_{L,S}$ in G . Let P be a parabolic of G^0 with Levi L such that $S \subset N_G P$. Let

$$M = \{x \in G^0; x^{-1} s x \in S_s\}, \hat{M} = \{x \in G^0; x^{-1} s x \in S_s U_P\}.$$

Let Γ be the set of orbits for the $Z_G(s)^0 \times L$ action $(h, l) : x \mapsto h x l^{-1}$ on M . We show that

(a) Γ is finite.

We may assume that $M \neq \emptyset$. Let $x_0 \in M$. Using 16.1(a), we see that it is enough to show that $\{x \in G^0; x^{-1} s x \in {}^\delta \mathcal{Z}_L^0 x_0^{-1} s x_0\}$ is a union of finitely many orbits under left translation by $Z_G(s)^0$, or equivalently, a union of finitely many orbits under left translation by $Z_G(s)$, which contains $Z_G(s)^0$ with finite index. It is enough to note that any G^0 -conjugacy class in G has finite intersection with ${}^\delta \mathcal{Z}_L^0 x_0^{-1} s x_0$; see 16.2(c).

The group $\{n \in N_{G^0} L; n S n^{-1} = S\}$ acts on Γ by $n : \eta \mapsto \eta n^{-1}$; this induces an action of \mathcal{W}_S (see 3.13) on Γ .

Let $\hat{\Gamma}$ be the set of orbits for the $Z_G(s)^0 \times P$ action $(h, p) : x \mapsto h x p^{-1}$ on \hat{M} . Any orbit η in Γ is contained in a unique orbit $\hat{\eta}$ in $\hat{\Gamma}$.

(b) The map $\Gamma \rightarrow \hat{\Gamma}, \eta \mapsto \hat{\eta}$, is a bijection.

We show that our map is injective. Let $x, x' \in M$ be such that $x' = h x p^{-1}$ for some $(h, p) \in Z_G(s)^0 \times P$. We must show that x, x' are in the same $Z_G(s)^0 \times L$ orbit. Replacing x by an element in the same $Z_G(s)^0 \times L$ orbit we may assume that $x' = x u^{-1}$ for some $u \in U_P$. Let $s'' = x^{-1} s x, s' = u x^{-1} s x u^{-1}$. Then s', s'' belong to S_s hence to $N_G L \cap N_G P$ and

$$s'' s'^{-1} = u^{-1} (s' u s'^{-1}) \in (N_G L \cap N_G P) \cap U_P = \{1\}.$$

Thus, $u^{-1} s' u s'^{-1} = 1$ that is $u \in U_P \cap Z_G(s') = U_P \cap Z_G(s')^0$. (We use 1.11.) Then $\zeta := x u x^{-1} = x u^{-1} u u x^{-1} \in Z_G(x u^{-1} s' u x^{-1})^0 = Z_G(s)^0$ and $x' = x u^{-1} = \zeta^{-1} x$. Since $\zeta \in Z_G(s)^0$, we see that x, x' are in the same $Z_G(s)^0 \times L$ orbit, as required.

We show that our map is surjective. Let $x \in \hat{M}$. It is enough to show that for some $v \in U_P$ we have $x v \in M$. Now $x^{-1} s x \in N_G P$ is semisimple. Hence, using 1.4(a), $x^{-1} s x$ normalizes $v L v^{-1}$ for some $v \in U_P$. Replacing x by $x v$ we may assume that $x^{-1} s x \in N_G P \cap N_G L$. We have $x^{-1} s x = g' g''$ where $g' \in S_s, g'' \in U_P$. Since $S_s \subset N_G P \cap N_G L$, we have $g'^{-1} x^{-1} s x \in (N_G P \cap N_G L) \cap U_P = \{1\}$; see 1.26. Thus, $x^{-1} s x = g' \in S_s$ and $x \in M$. This completes the proof of (b).

Since Γ is finite, it follows that $\hat{\Gamma}$ is finite.

The orbits of $Z_G(s)^0$ acting by left translation on $\{x P \in G^0/P; x^{-1} s x \in S_s U_P\}$ are complete varieties. (Indeed, such an orbit is of the form $Z_G(s)^0 / (Z_G(s)^0 \cap x P x^{-1})$ where $x \in G^0, x^{-1} s x \in S_s U_P$ and it is enough to show that $Z_G(s)^0 \cap x P x^{-1}$

is a parabolic of $Z_G(s)^0$, or equivalently that $Z_G(x^{-1}sx)^0 \cap P$ is a parabolic of $Z_G(x^{-1}sx)^0$. This follows from 1.12(a) since $x^{-1}sx \in N_G P$.) Hence these orbits are closed. Since there are only finitely many such orbits (their number is $|\hat{\Gamma}| = |\Gamma|$), these orbits are also open.

Let δ_1 be the connected component of $Z_G(s)$ that contains u .

Lemma 16.4. *There exists an open subset \mathcal{U} of δ_1 such that*

- (i) \mathcal{U} contains any unipotent element in δ_1 ;
- (ii) $g\mathcal{U}g^{-1} = \mathcal{U}$ for all $g \in Z_G(s)^0$;
- (iii) for any P as in 16.3 we have $h \in \mathcal{U}, x \in G^0, x^{-1}shx \in \bar{S}U_P \implies x^{-1}h_sx \in {}^\delta Z_L^0 U_P, x^{-1}sx \in S_s U_P$.

Let D be a connected component of G . A subset of D is said to be *stable* if it is a union of fibres of the map $\sigma : D \rightarrow D//G^0$ in 7.1. Let $g \in D$ be quasi-semisimple and let T_1 be a maximal torus of $Z_G(g)^0$. From 7.1(a) we deduce:

- (a) a stable subset \mathcal{R} of D is closed in D if and only if $\mathcal{R} \cap gT_1$ is closed in gT_1 .

Next we show that

- (b) Assume that D contains some unipotent elements. Let \mathcal{R}_0 be a subset of G^0 which is a union of G^0 -conjugacy classes such that the intersection of \mathcal{R}_0 with some/any maximal torus in G^0 is closed in that torus. Then $\mathcal{R} := \{g \in D; g_s \in \mathcal{R}_0\}$ is a closed and stable subset of D .

We show that \mathcal{R} is stable. Let $y \in \mathcal{R}$ and let $y' \in D$ be such that $\sigma(y) = \sigma(y')$. We must show that $y' \in \mathcal{R}$. Let $v \in y_u Z_G(y_s)^0$ be unipotent, quasi-semisimple in $Z_G(y_s)$ and let $v' \in y'_u Z_G(y'_s)^0$ be unipotent, quasi-semisimple in $Z_G(y'_s)$. Then $y'_s v' = z y_s v z^{-1}$ for some $z \in G^0$ (see 7.1). It follows that $y'_s = z y_s z^{-1}$. Since $y_s \in \mathcal{R}_0$ we see that $y'_s \in \mathcal{R}_0$ hence $y' \in \mathcal{R}$. We show that \mathcal{R} is closed in D . Let g, T_1 be as above. Let T'_1 be a maximal torus of G^0 that contains T_1 . Since g is unipotent, we have

$$\mathcal{R} \cap gT_1 = \{gt; t \in T_1 \cap \mathcal{R}_0\} = g(T'_1 \cap \mathcal{R}_0) \cap gT_1.$$

This is closed in gT_1 since $T'_1 \cap \mathcal{R}_0$ is closed in G . This proves (b).

- (c) Let $s' \in G$ be semisimple and let $g' \in Z_G(s')$ be such that $g'_s \in Z_G(s')^0$ and $Z_G(s'g'_s) \subset Z_G(s')$. If $s'g'$ is isolated in G , then $s'g'_u = g'_u s'$ is isolated in G .

Let T_1 be a maximal torus of $Z_G(s')^0$ such that $g'_s \in T_1$. Then T_1 is also a maximal torus of $Z_G(s'g'_s)^0$. With the notation of 1.5 we have

$$\text{Lie } Z_G(s'g'_s) = \mathfrak{t} \oplus \bigoplus_{\alpha \in R'} \mathfrak{g}_\alpha, \quad \text{Lie } Z_G(s') = \mathfrak{t} \oplus \bigoplus_{\alpha \in R''} \mathfrak{g}_\alpha$$

where $R' \subset R''$ are subsets of R and $\mathfrak{t} = \text{Lie } T_1$. The centre of $Z_G(s'g'_s)^0$ is $\{t \in T_1; \alpha(t) = 1 \forall \alpha \in R'\}$ and the centre of $Z_G(s')^0$ is $\{t \in T_1; \alpha(t) = 1 \forall \alpha \in R''\}$. Since $R' \subset R''$, the centre of $Z_G(s'g'_s)^0$ contains the centre of $Z_G(s')^0$. Hence

$$\mathcal{Z}_{Z_G(s')^0}^0 \subset \mathcal{Z}_{Z_G(s'g'_s)^0}^0 \quad \text{and} \quad \mathcal{Z}_{Z_G(s')^0}^0 \cap Z_G(g_u) \subset \mathcal{Z}_{Z_G(s'g'_s)^0}^0 \cap Z_G(g'_u).$$

It follows that $T(s'g'_u) \subset T(s'g'_s g'_u)$ (see 2.1). Since $s'g'_s g'_u$ is isolated in G , we have $T(s'g'_s g'_u) \subset \mathcal{Z}_{G^0}$. It follows that $T(s'g'_u) \subset \mathcal{Z}_{G^0}$, hence $s'g'_u$ is isolated in G . This proves (c).

- (d) Let $s' \in G$ be semisimple and let $g \in Z_G(s')$ be such that $g_s \in Z_G(s')^0$, $s'g \in \bar{S}U_P$ (P as in 16.3), $s'g_s \in S_s$ and $Z_G(s'g_s) \subset Z_G(s')$. Then there exists $a \in N_G L$ such that a is unipotent, $as' = s'a \in \delta$ and $s'a$ is isolated in $N_G L$.

We have ${}^\delta Z_L^0 \subset Z_G(s'g_s)$ hence ${}^\delta Z_L^0 \subset Z_G(s')$. Thus, $s \in Z_G({}^\delta Z_L^0)$. Since $Z_{G^0}({}^\delta Z_L^0) = L$ (see 1.10) and $Z_{G^0}({}^\delta Z_L^0)$ is normal in $Z_G({}^\delta Z_L^0)$, we have $s' \in N_G L$. From $s'g_s \in S_s$ we see that $g_s \in N_G L$. Since $g \in Z_G(s')$, we have $g_u \in Z_G(s')$. Now $Z_G(s')^0$ contains g_s and ${}^\delta Z_L^0$; moreover, g_s commutes with any element of ${}^\delta Z_L^0$ (since $s'g_s$ and s' do). Hence we can find a maximal torus T_1 of $Z_G(s')^0$ such that $g_s \in T_1$, ${}^\delta Z_L^0 \subset T_1$. Since $L = Z_{G^0}({}^\delta Z_L^0)$ we have $T_1 \subset L$. Thus, T_1 is a torus in $Z_L(s')^0$. Since $g_s \in T_1$, we have $g_s \in Z_L(s')^0$. Since $s'g \in \bar{S}U_P \subset N_G P$ we have $g_u = (s'g)_u \in N_G P$. Hence $g_u = ab$ where $a \in N_G L \cap N_G P, b \in U_P$ are uniquely determined and a is unipotent. Now $s'g_s$ commutes with g_u hence $(s'g_s a g_s^{-1} s'^{-1})(s'g_s b g_s^{-1} s'^{-1}) = ab$. Since $s'g_s \in S_s \subset N_G L \cap N_G P$, we have $a^{-1}(s'g_s a g_s^{-1} s'^{-1}) = b(s'g_s b g_s^{-1} s'^{-1})^{-1} \in N_G L \cap U_P = \{1\}$. Hence $a \in Z_G(s'g_s)$. Since $Z_G(s'g_s) \subset Z_G(s')$ we have $a \in Z_G(s')$ and $a \in Z_G(g_s)$. We have $s'g \in \bar{S}U_P$ hence $s'g_s a b \in \bar{S}U_P$. Since $s'g_s a \in N_G L \cap N_G P, b \in U_P$, it follows that $s'g_s a \in \bar{S}$. Let $g' = g_s a$. Then $g' \in N_G L, s'g' \in \bar{S}$. We have $g' \in Z_{N_G L}(s')$. Since a is unipotent we have $g'_s = g_s \in Z_L(s')^0$. Also, $Z_{N_G L}(s'g'_s) \subset Z_{N_G L}(s')$. Since $s'g' \in \bar{S}$, we see that $s'g'$ is isolated in $N_G L$. Applying (c) to $N_G L$ instead of G we see that $s'a = as'$ is isolated in $N_G L$. Since $s'g_s a \in \bar{S}$ we have $s'g_s a \in \delta$. Since $g_s \in L$ we have $s'a \in \delta$. This proves (d).

- (e) Let F be the image of $\{y \in \delta; y \text{ isolated in } N_G L\}$ under $y \mapsto y_s$. Let E be the set of all $g \in \delta_1$ such that there exists $x \in G^0$ with $x^{-1}sg_s x \in S_s, x^{-1}sx \in F, x^{-1}g_s x \notin {}^\delta Z_L^0$. Then E is a closed stable subset of δ_1 .

Let δ' be the connected component of $N_G L$ such that $h \in \delta \implies h_s \in \delta'$. From 2.7 and 1.22 we see that there exist finitely many semisimple L -conjugacy classes C_0, C_1, \dots, C_m in δ' such that $F - S_s = \bigcup_{j=1}^m {}^\delta Z_L^0 C_j$ and $S_s = {}^\delta Z_L^0 C_0$. Applying an argument in 16.3(a) to ${}^\delta Z_L^0 C_j, (j \in [0, m])$ instead of S_s we see that there are only finitely many orbits for the action $(h, l) : x \mapsto hxl^{-1}$ of $Z_G(s)^0 \times L$ on $\{x \in G^0; x^{-1}sx \in {}^\delta Z_L^0 C_j\}$. Hence for $j \in [0, m]$ we can find elements $x_{ij} \in G^0, i \in [1, p_j], p_j < \infty$ such that $x_{ij}^{-1}sx_{ij} \in {}^\delta Z_L^0 C_j$ and $E = \bigcup_{j \in [0, m], i \in [1, p_j]} E_{ij}$ where

$$E_{ij} = \bigcup_{z \in Z_G(s)^0} \{g \in \delta_1; z^{-1}sg_s z \in x_{ij} {}^\delta Z_L^0 C_0 x_{ij}^{-1}, z^{-1}g_s z \notin x_{ij} {}^\delta Z_L^0 x_{ij}^{-1}\}.$$

It is then enough to show that, for any j, i as above, E_{ij} is a closed stable subset of δ_1 . We set $a = x_{ij}$. We have $a^{-1}sa \in \delta'$ hence $a^{-1}sa \in N_G L$. Applying 1.27(a) with $N_G L, Z_{N_G L}(a^{-1}sa), C_0$ instead of H', H, \mathbf{c} we see that $C_0 \cap Z_{N_G L}(a^{-1}sa) = \bigcup_{r=1}^{m'} C'_r$ where $m' < \infty$ and C'_r are semisimple $Z_L(a^{-1}sa)^0$ -conjugacy classes in $Z_{N_G L}(a^{-1}sa)$. In the definition of E_{ij} we have automatically

$$z^{-1}sg_s z \in a({}^\delta Z_L^0)(c_0 \cap Z_{N_G L}(a^{-1}sa))a^{-1}$$

(we use that ${}^\delta Z_L^0 \subset Z_G(a^{-1}sa)$ since $a^{-1}sa \in \delta'$). Hence $E_{ij} = \bigcup_{r=1}^{m'} E_{ijr}$ where

$$E_{ijr} = \bigcup_{z \in Z_G(s)^0} \{g \in \delta_1; z^{-1}sg_s z \in \tau a C'_r a^{-1}, z^{-1}g_s z \notin \tau\}$$

and $\tau := a({}^\delta Z_L^0)a^{-1}$. It is then enough to show that, for any $r \in [1, m']$ as above, E_{ijr} is a closed stable subset of δ_1 . Let $f \in C'_r$. Since $aZ_L(a^{-1}sa)^0a^{-1} \subset Z_G(s)^0$, we have

$$E_{ijr} = \bigcup_{z \in Z_G(s)^0} \{g \in \delta_1; z^{-1}sg_s z \in \tau afa^{-1}, z^{-1}g_s z \notin \tau\}.$$

We may assume that E_{ijr} is non-empty. Then $afa^{-1} = sf_0$ where f_0 is a semisimple element of $Z_G(s)^0$. Now f_0 centralizes τ , a torus in $Z_G(s)^0$. Hence there exists a maximal torus T_1 of $Z_G(s)^0$ such that T_1 contains τ and f_0 . Applying (b) to $Z(s), \delta_1, E_{ijr}$ instead of G, D, \mathcal{R} , we see that it is enough to show that

$$\{h \in T_1; z^{-1}shz \in sf_0\tau, z^{-1}hz \notin \tau \text{ for some } z \in Z_G(s)^0\}$$

is closed in T_1 . Since this is non-empty, we have $f_0 \notin \tau$ and the last variety becomes

$$\{h \in T_1; z^{-1}hz \in f_0\tau \text{ for some } z \in Z_G(s)^0\},$$

that is, $\bigcup_{w \in W_1} w\tau f_0 w^{-1}$ where $W_1 = N_{Z_G(s)^0}T_1/T_1$. This is closed in T_1 since W_1 is finite and τf_0 is closed in T_1 . This proves (e).

(f) *there exists an open stable subset \mathcal{U}_1 of δ_1 such that \mathcal{U}_1 contains any unipotent element in δ_1 and such that $g \in \mathcal{U}_1 \implies Z_G(sg_s) \subset Z_G(s)$.*

We imbed G into $\hat{G} = GL_n(\mathbf{k})$ as a closed subgroup. Let $\mathcal{U}'_1 = \{g \in \hat{G}; Z_{\hat{G}}(g_s) \subset Z_{\hat{G}}(s)\}$. Let $\mathcal{U}_1 = s^{-1}\mathcal{U}'_1 \cap \delta_1$. Clearly, \mathcal{U}_1 has the required properties.

We can now prove the lemma. Let $\mathcal{U} = \{g \in \mathcal{U}_1; g \notin E\}$ with \mathcal{U}_1 as in (f), E as in (e). From (e),(f) we see that \mathcal{U} is an open stable subset of δ_1 . If $g \in \delta_1$ is unipotent, then $g \in \mathcal{U}_1$ by (f) and $g \notin E$ (if we had $g \in E$, then there would exist $x \in G^0$ such that $1 = x^{-1}1x \notin {}^\delta Z_L^0$, absurd). Thus \mathcal{U} contains any unipotent element in δ_1 . Assume now that $g \in \mathcal{U}, x \in G^0, x^{-1}sg_s x \in \bar{S}U_P$ (with P as in 16.3). We must show that $x^{-1}sx \in S_s U_P$ and $x^{-1}g_s x \in {}^\delta Z_L^0 U_P$. Now any element in $\bar{S}U_P$ is U_P -conjugate to an element whose semisimple part is in S_s . (See the proof of 3.15 and 1.22(b).) Hence, replacing x by xv for some $v \in U_P$ we may assume that we have, in addition, $x^{-1}sg_s x \in S_s$. Since $g \in \mathcal{U}_1$, we have $Z_G(sg_s) \subset Z_G(s)$ hence $Z_G(xsg_s x^{-1}) \subset Z_G(xsx^{-1})$. We apply (d) with xsx^{-1}, xgx^{-1} instead of s', g . (We have $g \in Z_G(s)^0$ since $g \in \delta_1$ and δ_1 contains unipotent elements. Hence $xgx^{-1} \in Z_G(xsx^{-1})^0$.) We see that there exists $a \in N_G L$ such that a is unipotent, $axsx^{-1} = xsx^{-1}a \in \delta$ and $xsx^{-1}a$ is isolated in $N_G L$. We then have $xsx^{-1} \in F$ (F as in (e)). Since $x^{-1}sg_s x \in S_s, x^{-1}sx \in F$ and $g \notin E$ we must have $x^{-1}g_s x \in {}^\delta Z_L^0$, by the definition of E . We have $x^{-1}sx = (x^{-1}sg_s x)(x^{-1}g_s x)^{-1} \in S_s {}^\delta Z_L^0 = S_s$. This completes the proof of the lemma.

16.5. Let \mathcal{U} be as in Lemma 16.4. Let P be as in 16.3. Let $\psi : X \rightarrow \bar{Y}$ be as in 3.14. We show that

(a) *the sets $X_{\mathcal{U}, \eta} = \{(g, xP) \in X; g \in s\mathcal{U}, x \in \hat{\eta}\}$ ($\eta \in \Gamma$) form a finite partition of $X_{\mathcal{U}} = \{(g, xP) \in X; g \in s\mathcal{U}\}$ into open and closed subsets.*

From Lemma 16.4(iii) we see that the second projection defines a morphism $pr_2 : X_{\mathcal{U}} \rightarrow \{xP \in G^0/P; x^{-1}sx \in S_s U_P\}$. We have $X_{\mathcal{U}, \eta} = pr_2^{-1}(\hat{\eta}/P)$ and it remains to use the fact that the subsets $\hat{\eta}/P$ form a finite partition of $\{xP \in G^0/P; x^{-1}sx \in S_s U_P\}$ into open and closed subsets.

For any $x \in M, P_x := xPx^{-1} \cap Z_G(s)^0$ is a parabolic of $Z_G(s)^0$; see 16.3. Moreover, $L_x := xLx^{-1} \cap Z_G(s)^0$ is a Levi of P_x , since $x^{-1}sx \in N_G L \cap N_G P$; see

1.12(a). Let

$$\begin{aligned} \mathbf{c}_x &= \{v \in Z_G(s); v \text{ unipotent, } x^{-1}svx \in S\}, \\ \Sigma_x &= x({}^\delta Z_L^0)x^{-1}\mathbf{c}_x. \end{aligned}$$

Let \mathcal{E}_x be the local system on Σ_x obtained as the inverse image of \mathcal{E} under $\Sigma_x \rightarrow S, g \mapsto x^{-1}sgx$. The results of 15.1, 15.3, 15.4 are applicable to $Z_G(s), P_x, L_x, \Sigma_x, \mathcal{E}_x$ instead of $G, P, L, \Sigma, \mathcal{E}$ (see 16.2). Let

$$\pi_x : \tilde{Y}'_x \rightarrow Y'_x, \tilde{\mathcal{E}}_x, \psi_x : X'_x \rightarrow \bar{Y}'_x, K_x$$

be obtained from $\pi : \tilde{Y}_{L,\Sigma} \rightarrow Y_{L,\Sigma}, \tilde{\mathcal{E}}, \psi : X \rightarrow \bar{Y}_{L,\Sigma}, K$ in 15.1, 15.3, 15.4, replacing $G, P, L, \Sigma, \mathcal{E}$ by $Z_G(s), P_x, L_x, \Sigma_x, \mathcal{E}_x$.

For any $\eta \in \Gamma$ we choose a base point $x_\eta \in \eta$. We set

$$\begin{aligned} P_\eta &= P_{x_\eta}, L_\eta = L_{x_\eta}, \mathbf{c}_\eta = \mathbf{c}_{x_\eta}, \Sigma_\eta = \Sigma_{x_\eta}, \mathcal{E}_\eta = \mathcal{E}_{x_\eta}, \pi_\eta = \pi_{x_\eta}, \\ \tilde{Y}'_\eta &= \tilde{Y}'_{x_\eta}, Y'_\eta = Y'_{x_\eta}, \tilde{\mathcal{E}}_\eta = \tilde{\mathcal{E}}_{x_\eta}, \psi_\eta = \psi_{x_\eta}, X'_\eta = X'_{x_\eta}, \bar{Y}'_\eta = \bar{Y}'_{x_\eta}, K_\eta = K_{x_\eta}, \\ X'_{\mathcal{U},\eta} &= \{(h, zP_\eta) \in \mathcal{U} \times Z_G(s)^0/P_\eta; z^{-1}hz \in \bar{\Sigma}_\eta U_{P_\eta}\} \subset X'_\eta. \end{aligned}$$

Lemma 16.6. *We have an isomorphism $X'_{\mathcal{U},\eta} \xrightarrow{\sim} X_{\mathcal{U},\eta}, (h, zP_\eta) \mapsto (sh, zx_\eta P)$.*

We prove this only at the level of sets. First we show that our map is well defined. Assume that $(h, zP_\eta) \in X'_{\mathcal{U},\eta}$. Then $z^{-1}hz = \alpha\beta\chi$ with $\alpha \in x_\eta {}^\delta Z_L^0 x_\eta^{-1}, \beta \in \bar{\mathbf{c}}_\eta, \chi \in U_{P_\eta}$ and

$$\begin{aligned} (zx_\eta)^{-1}shzx_\eta &= x_\eta^{-1}sz^{-1}hzx_\eta = x_\eta^{-1}s\alpha\beta\chi x_\eta \\ &= (x_\eta^{-1}\alpha x_\eta)(x_\eta^{-1}s\beta x_\eta)(x_\eta^{-1}\chi x_\eta) \in {}^\delta Z_L^0 \bar{S}U_P = \bar{S}U_P, \end{aligned}$$

$sh \in s\mathcal{U}, zx_\eta \in \eta$, hence $(sh, zx_\eta P) \in X_{\mathcal{U},\eta}$. (We use that $x_\eta^{-1}s\bar{\mathbf{c}}_\eta x_\eta \subset \bar{S}, x_\eta^{-1}U_{P_\eta}x_\eta \subset U_P$.)

We show that our map is injective. Assume that $(h, zP_\eta), (h', z'P_\eta) \in \mathcal{U} \times Z_G(s)^0/P_\eta$ satisfy $(sh, zx_\eta P) = (sh', z'x_\eta P)$. Then clearly $h = h'$ and $z^{-1}z' \in x_\eta P x_\eta^{-1} \cap Z_G(s)^0 = P_\eta$ hence $zP_\eta = z'P_\eta$.

We show that our map is surjective. Let $(g, xP) \in X_{\mathcal{U},\eta}$. We have $g = sh, xP = zx_\eta P$ where

$$h \in \mathcal{U}, z \in Z_G(s)^0, x_\eta^{-1}z^{-1}shzx_\eta \in \bar{S}U_P.$$

We have $z^{-1}hz = a'c$ where $a' \in s^{-1}x_\eta \bar{S}x_\eta^{-1}, c \in U_{x_\eta P x_\eta^{-1}}$. Since $z^{-1}hz \in Z_G(s)$, we must have $a' \in Z_G(s), c \in Z_G(s)$. Now

$$c \in U_{x_\eta P x_\eta^{-1}} \cap Z_G(s) = U_{x_\eta P x_\eta^{-1}} \cap Z_G(s)^0 = U_{P_\eta},$$

by 1.11. Thus $z^{-1}hz \in a'U_{x_\eta P x_\eta^{-1}}$ where $a' \in s^{-1}x_\eta \bar{S}x_\eta^{-1} \subset N_G(x_\eta Lx_\eta^{-1})$. Hence

$$(z^{-1}hz)_s \in a'_s U_{P_\eta} \text{ with } a'_s \in N_G(x_\eta Lx_\eta^{-1}).$$

Using 16.4(iii) we have $x_\eta^{-1}z^{-1}h_s z x_\eta \in {}^\delta Z_L^0 U_P$. Thus, $(z^{-1}hz)_s = z^{-1}h_s z \in a(x_\eta U_P x_\eta^{-1})$ with $a \in x_\eta {}^\delta Z_L^0 x_\eta^{-1}$. Since

$$a^{-1}a'_s \in N_G(x_\eta Lx_\eta^{-1}) \cap U_{x_\eta P x_\eta^{-1}} = \{1\}$$

we have $a'_s = a \in x_\eta {}^\delta Z_L^0 x_\eta^{-1}$. Let $b = a'_u$. It remains to show that $b \in \bar{\mathbf{c}}_\eta$. Since $a' \in Z_G(s)$, we have $b \in Z_G(s)$. From $sa' \in x_\eta \bar{S}x_\eta^{-1}$ we deduce

$$sa'_u \in x_\eta \bar{S}x_\eta^{-1} a'_s{}^{-1} \subset x_\eta \bar{S}x_\eta^{-1} x_\eta {}^\delta Z_L^0 x_\eta^{-1} = x_\eta \bar{S}x_\eta^{-1}.$$

It follows that $a'_u \in \bar{c}_\eta$, as required. The lemma is proved.

16.7. Let $\tilde{Y} = \tilde{Y}_{L,S}, Y = Y_{L,S}$ and let $\pi : \tilde{Y} \rightarrow Y$ be as in 3.13. Let $\tilde{Y}_U = \{(g, xL) \in \tilde{Y}; g \in sU\}$. For any $\eta \in \Gamma$ we set

$$\begin{aligned} \tilde{Y}_{U,\eta} &= \{(g, xL) \in \tilde{Y}; g \in sU, x \in \eta\}, & Y_{U,\eta} &= \pi(\tilde{Y}_{U,\eta}), \\ \tilde{Y}'_{U,\eta} &= \{(h, zL_\eta) \in \tilde{Y}'_\eta; h \in U\}. \end{aligned}$$

Lemma 16.8. (a) *The map $(g, xL) \mapsto (g, xP)$ is an isomorphism of \tilde{Y}_U onto the open subset $\psi^{-1}(Y \cap sU)$ of X_U .*

(b) *The subsets $\tilde{Y}_{U,\eta}, (\eta \in \Gamma)$ form a finite partition of \tilde{Y}_U into open and closed subsets.*

(c) *The map $\psi : X_U \rightarrow \bar{Y} \cap sU$ is proper, surjective, and $Y \cap sU$ is open in $\bar{Y} \cap sU$.*

(d) *The map $(h, zL_\eta) \mapsto (h, zP_\eta)$ is an isomorphism of $\tilde{Y}'_{U,\eta}$ onto the open subset $\psi_\eta^{-1}(Y'_\eta \cap U)$ of $X'_{U,\eta}$. The map ψ_η restricts to a proper map of $X'_{U,\eta}$ onto $\bar{Y}'_\eta \cap U$ and $Y'_\eta \cap U$ is open, dense in $\bar{Y}'_\eta \cap U$.*

We prove (a). By 5.5, the same formula gives an isomorphism $\tilde{Y} \xrightarrow{\sim} \psi^{-1}(Y)$. Hence the map in (a) is an isomorphism onto $\psi^{-1}(Y \cap sU)$. It remains to show that $\psi^{-1}(Y \cap sU)$ is open in X_U . Since $\psi^{-1}(Y \cap sU) = \psi^{-1}(Y) \cap X_U$ it is enough to show that $\psi^{-1}(Y)$ is open in X . This follows from the fact that Y is open in $\bar{Y} = \psi(X)$.

We prove (b). The map in (a) identifies \tilde{Y}_U with an open subset of X_U and $\tilde{Y}_{U,\eta}$ with $\tilde{Y}_U \cap X_{U,\eta}$. This together with 16.5(a) yields (b).

We prove (c). We have $X_U = \psi^{-1}(\bar{Y} \cap sU)$ hence the first assertion of (b) follows by change of base from the fact that ψ is proper. Since Y is open in \bar{Y} , we see that $Y \cap (\bar{Y} \cap sU)$ is open in $\bar{Y} \cap sU$. Hence $Y \cap sU$ is open in $\bar{Y} \cap sU$.

We prove (d). From 15.3(a) we see that $(h, zL_\eta) \mapsto (h, zP_\eta)$ gives an isomorphism of \tilde{Y}'_η onto the open subset $\psi_\eta^{-1}(Y'_\eta)$ of X'_η . Hence the same formula gives an isomorphism of $\tilde{Y}'_{U,\eta}$ onto the open subset $\psi_\eta^{-1}(Y'_\eta \cap U)$ of $X'_{U,\eta}$. The map $\psi_\eta : X'_\eta \rightarrow \bar{Y}'_\eta$ is proper, surjective. Since $\psi_\eta : X'_{U,\eta} \rightarrow \bar{Y}'_\eta \cap U$ is obtained from the previous map by change of base, it is also proper, surjective. Since Y'_η is open in \bar{Y}'_η , $Y'_\eta \cap U$ is open in $\bar{Y}'_\eta \cap U$. We show it is dense. We have $\bar{Y}'_\eta = \cup_F (F \cap U)$ where F runs over the irreducible components of \bar{Y}'_η . It is enough to show that $F \cap Y'_\eta \cap U$ is dense in $F \cap U$ for any F . We may assume that $F \cap U \neq \emptyset$. Since $F \cap U$ is open, non-empty in the irreducible variety F , it is also dense in F . Since Y'_η is open dense in \bar{Y}'_η (see 15.1), we see that $F \cap Y'_\eta$ is open dense in F . Since $F \cap U, F \cap Y'_\eta$ are open dense in F , their intersection $F \cap Y'_\eta \cap U$ is open dense in $F \cap U$. The lemma is proved.

Lemma 16.9. (a) *The map $\psi : X_{U,\eta} \rightarrow \bar{Y} \cap sU$ is proper, with image equal to $s(\bar{Y}'_\eta \cap U)$.*

(b) *Let us identify $\tilde{Y}_{U,\eta}$ with a subset of X via the imbedding $\tilde{Y} \subset X$ in 5.5. Then $\tilde{Y}_{U,\eta} = \psi^{-1}(Y_{U,\eta}) \cap X_{U,\eta}$.*

(c) *$Y_{U,\eta}$ is open in $s(\bar{Y}'_\eta \cap U)$.*

(d) *$Y_{U,\eta}$ is open and closed in $Y \cap sU$. We have $\bigcup_{\eta \in \Gamma} Y_{U,\eta} = Y \cap sU$. For $\eta, \eta' \in \Gamma, Y_{U,\eta}, Y_{U,\eta'}$ coincide if η, η' are in the same \mathcal{W}_S -orbit in Γ and are disjoint, otherwise.*

- (e) For any $\eta \in \Gamma$, $\tilde{Y}_{\mathcal{U},\eta}$ is a dense subset of $X_{\mathcal{U},\eta}$ (see (b)).
- (f) For any $\eta \in \Gamma$, $Y_{\mathcal{U},\eta}$ is an open dense subset of $s(\tilde{Y}'_{\eta} \cap \mathcal{U})$.

We prove (a). The fact that $\psi : X_{\mathcal{U},\eta} \rightarrow \bar{Y} \cap s\mathcal{U}$ is proper follows from Lemma 16.8(c) since $X_{\mathcal{U},\eta}$ is closed in $X_{\mathcal{U}}$ (see 16.5(a)). The statement about its image can be reduced using the isomorphism in Lemma 16.6 to a statement in Lemma 16.8(d).

We prove (b). We must show that $\tilde{Y}_{\mathcal{U},\eta}$ is a union of fibres of $\psi : X_{\mathcal{U},\eta} \rightarrow s(\tilde{Y}'_{\eta} \cap \mathcal{U})$. This is clear from the definitions.

We prove (c). From the proof of (b) we see that $X_{\mathcal{U},\eta} - \tilde{Y}_{\mathcal{U},\eta}$ is also a union of fibres of $\psi : X_{\mathcal{U},\eta} \rightarrow s(\tilde{Y}'_{\eta} \cap \mathcal{U})$. Hence its image under the proper surjective map $\psi : X_{\mathcal{U},\eta} \rightarrow s(\tilde{Y}'_{\eta} \cap \mathcal{U})$ is a closed subset of $s(\tilde{Y}'_{\eta} \cap \mathcal{U})$ complementary to the image $Y_{\mathcal{U},\eta}$ of $\tilde{Y}_{\mathcal{U},\eta}$. This proves (c).

We prove (d). The map $\pi : \tilde{Y}_{\mathcal{U}} \rightarrow Y \cap s\mathcal{U}$ is proper, surjective, since it is obtained by change of base from the proper surjective map $\pi : \tilde{Y} \rightarrow Y$. Since $\tilde{Y}_{\mathcal{U},\eta}$ is closed in $\tilde{Y}_{\mathcal{U}}$ (see Lemma 16.8(b)), it follows that $Y_{\mathcal{U},\eta} = \pi(\tilde{Y}_{\mathcal{U},\eta})$ is closed in $Y \cap s\mathcal{U}$. Since $\bigcup_{\eta} \tilde{Y}_{\mathcal{U},\eta} = \tilde{Y}_{\mathcal{U}}$ (see Lemma 16.8(b)), it follows that $\bigcup_{\eta} Y_{\mathcal{U},\eta} = Y \cap s\mathcal{U}$.

Assume that $Y_{\mathcal{U},\eta}, Y_{\mathcal{U},\eta'}$ are not disjoint; let $g \in Y_{\mathcal{U},\eta} \cap Y_{\mathcal{U},\eta'}$. Then there exist $x \in \eta, x' \in \eta'$ such that $(g, xL) \in \tilde{Y}, (g, x'L) \in \tilde{Y}$. Using 3.13(a), we see that there exists $n \in N_{G^0}L$ such that $nSn^{-1} = S$ and $x' = xn^{-1}$. Then $(g_1, x_1L) \mapsto (g_1, x_1n^{-1}L)$ is a bijection $\tilde{Y}_{\mathcal{U},\eta} \rightarrow \tilde{Y}_{\mathcal{U},\eta'}$ hence $Y_{\mathcal{U},\eta} = Y_{\mathcal{U},\eta'}$. The same argument shows that, if η, η' are in the same \mathcal{W}_S -orbit, then $Y_{\mathcal{U},\eta} = Y_{\mathcal{U},\eta'}$. We see that the complement of $Y_{\mathcal{U},\eta}$ in $Y \cap s\mathcal{U}$ is the union of the closed subsets $Y_{\mathcal{U},\eta'}$ (with η' not in the \mathcal{W}_S -orbit of η); hence $Y_{\mathcal{U},\eta}$ is open in $Y \cap s\mathcal{U}$. This proves (d).

We prove (e). We have $\mathbf{c}_{\eta} = \bigsqcup_{j \in J} \gamma_j$ where J is finite and γ_j are (unipotent) L_{η} -conjugacy classes in $N_{Z_G(s)}L_{\eta}$. We have $X'_{\eta} = \bigcup_{j \in J} X'_j, \tilde{Y}'_{\eta} = \bigsqcup_{j \in J} \tilde{Y}'_j$ where

$$\begin{aligned} X'_j &= \{(h, zP_{\eta}) \in Z_G(s) \times Z_G(s)^0; z^{-1}hz \in x_{\eta}^{\delta} \mathcal{Z}_L^0 x_{\eta}^{-1} \tilde{\gamma}_j U_{P_{\eta}}\}, \\ \tilde{Y}'_j &= \{(h, zL_{\eta}) \in Z_G(s) \times Z_G(s)^0; \\ &\quad z^{-1}hz = ab, a \in x_{\eta}^{\delta} \mathcal{Z}_L^0 x_{\eta}^{-1}, b \in \gamma_j, Z_{Z_G(s)}(a)^0 \subset L_{\eta}\}. \end{aligned}$$

Let $X'_{\mathcal{U},j} = \{(h, zP_{\eta}) \in X'_j; h \in \mathcal{U}\}, \tilde{Y}'_{\mathcal{U},j} = \{(h, zL_{\eta}) \in \tilde{Y}'_j; h \in \mathcal{U}\}$.

We have $X'_{\mathcal{U},\eta} = \bigcup_{j \in J} X'_{\mathcal{U},j}, \tilde{Y}'_{\mathcal{U},\eta} = \bigcup_{j \in J} \tilde{Y}'_{\mathcal{U},j}$.

Since X'_j is irreducible and \mathcal{U} is open in $Z_G(s)$ we see that $X'_{\mathcal{U},j}$ is open in X'_j , so that $X'_{\mathcal{U},j}$ is either empty or irreducible. If $X'_{\mathcal{U},j} \neq \emptyset$, then we can find $a \in x_{\eta}^{\delta} \mathcal{Z}_L^0 x_{\eta}^{-1}, b \in \tilde{\gamma}_j, c \in U_{P_{\eta}}$ with $abc \in \mathcal{U}$. Since a and c are contained in $Z_G(s)^0$, we see that b is contained in δ_1 , the connected component of $Z_G(s)$ that contains \mathcal{U} . Hence $\gamma_j \subset \delta_1$. Let us identify $X'_{\mathcal{U},j}$ with a subset $X_{\mathcal{U},j}$ of $X_{\mathcal{U},\eta}$ via the isomorphism $X'_{\mathcal{U},\eta} \rightarrow X_{\mathcal{U},\eta}$ in Lemma 16.6. Since $X_{\mathcal{U},\eta} = \bigcup_{j \in J; \gamma_j \subset \delta_1} X_{\mathcal{U},j}$, we see that it is enough to show that, if $\gamma_j \subset \delta_1$, then

- (g) $\tilde{Y}_{\mathcal{U},\eta} \cap \tilde{Y}'_{\mathcal{U},j}$ is dense in $\tilde{Y}'_{\mathcal{U},j}$,
- (h) $\tilde{Y}'_{\mathcal{U},j}$ is dense in $X'_{\mathcal{U},j} = X_{\mathcal{U},j}$.

(We regard $\tilde{Y}'_{\mathcal{U},j}$ as a subspace of $X'_{\mathcal{U},j}$ via the imbedding $\tilde{Y}'_{\mathcal{U},\eta} \rightarrow X'_{\mathcal{U},\eta}$ in 16.8(d) hence as a subspace of $X_{\mathcal{U},j}$.)

We prove (g). This is equivalent to the following statement:

$$\{(h, zL_\eta); h \in \mathcal{U}, z \in Z_G(s)^0, z^{-1}hz = ab, a \in x_\eta^\delta \mathcal{Z}_L^0 x_\eta^{-1}, \\ b \in \gamma_j, (Z(s) \cap Z(a))^0 \subset L_\eta, Z_G(sa)^0 \subset x_\eta L x_\eta^{-1}\}$$

is dense in

$$\{(h, zL_\eta); h \in \mathcal{U}, z \in Z_G(s)^0, z^{-1}hz = ab, a \in x_\eta^\delta \mathcal{Z}_L^0 x_\eta^{-1}, \\ b \in \gamma_j, (Z(s) \cap Z(a))^0 \subset L_\eta\}.$$

Since the condition $Z_G(sa)^0 \subset x_\eta L x_\eta^{-1}$ implies that $(Z(s) \cap Z(a))^0 \subset L_\eta$, we see that it is enough to show that for any $b \in \gamma_j$,

$$\{a \in x_\eta^\delta \mathcal{Z}_L^0 x_\eta^{-1}; Z_G(sa)^0 \subset x_\eta L x_\eta^{-1}, ab \in \mathcal{U}\} \text{ is dense in } x_\eta^\delta \mathcal{Z}_L^0 x_\eta^{-1}$$

or that

$$\{a \in x_\eta^\delta \mathcal{Z}_L^0 x_\eta^{-1}; Z_G(sa)^0 \subset x_\eta L x_\eta^{-1}\} \cap (x_\eta^\delta \mathcal{Z}_L^0 x_\eta^{-1} \cap \mathcal{U}b^{-1}) \text{ is dense in } x_\eta^\delta \mathcal{Z}_L^0 x_\eta^{-1}.$$

Since $b \in \mathcal{U}$, $x_\eta^\delta \mathcal{Z}_L^0 x_\eta^{-1} \cap \mathcal{U}b^{-1}$ is an open subset of the torus $x_\eta^\delta \mathcal{Z}_L^0 x_\eta^{-1}$ containing the unit element; hence it is an open dense subset of $x_\eta^\delta \mathcal{Z}_L^0 x_\eta^{-1}$. On the other hand, $\{a \in x_\eta^\delta \mathcal{Z}_L^0 x_\eta^{-1}; Z_G(sa)^0 \subset x_\eta L x_\eta^{-1}\}$ is an open dense subset of $x_\eta^\delta \mathcal{Z}_L^0 x_\eta^{-1}$ by 3.10(a). Since the intersection of two open dense subsets of a torus is dense in that torus, (g) is proved.

We prove (h). Since $\tilde{Y}'_{\mathcal{U},j}$ is open in \tilde{Y}'_j which is open in X'_j , we see that $\tilde{Y}'_{\mathcal{U},j}$ is open in X'_j . Since X'_j is irreducible, to prove (h) it suffices to show that $\tilde{Y}'_{\mathcal{U},j} \neq \emptyset$. But this is contained in the proof of (g). This proves (h) hence (e).

We prove (f). The openness follows from (c). From (e) we see that $\psi(\tilde{Y}_{\mathcal{U},\eta})$ is a dense subset of $\psi(X_{\mathcal{U},\eta})$ hence $Y_{\mathcal{U},\eta}$ is a dense subset of $s(\tilde{Y}'_\eta \cap \mathcal{U})$ (see 16.9(a)). The lemma is proved.

16.10. For a \mathcal{W}_S -orbit Z in Γ we set $Y_{\mathcal{U},Z} = Y_{\mathcal{U},\eta}$, $\bar{Y}'_{Z,\mathcal{U}} = \bar{Y}'_\eta \cap \mathcal{U}$ where $\eta \in \Gamma$. This is well defined, by 16.9(d),(f). For $\eta \in \Gamma$,

(a) $s(Y'_\eta \cap \mathcal{U}) \cap Y_{\mathcal{U},\eta}$ is open in $Y_{\mathcal{U},\eta}$.

(Since $Y_{\mathcal{U},\eta} \subset s(\bar{Y}'_\eta \cap \mathcal{U})$, it suffices to show that $s(Y'_\eta \cap \mathcal{U})$ is open in $s(\bar{Y}'_\eta \cap \mathcal{U})$. This follows from Lemma 16.8(d).) Hence, for any \mathcal{W}_S -orbit Z in η ,

$$\mathcal{V}_Z := \bigcap_{\eta \in Z} (s(Y'_\eta \cap \mathcal{U}) \cap Y_{\mathcal{U},Z})$$

is an open subset of $Y_{\mathcal{U},Z}$. It also follows that \mathcal{V}_Z is open in $Y \cap s\mathcal{U}$. Let $\mathcal{V} = \bigcup_Z \mathcal{V}_Z$ where Z runs over the \mathcal{W}_S -orbits in Γ . We show that

(b) \mathcal{V} is an open smooth subset of $Y \cap s\mathcal{U}$ of pure dimension $\dim Z_G(s)^0 - \dim L + \dim S$ and the \mathcal{V}_Z form a finite partition of \mathcal{V} into open and closed subsets.

Since \mathcal{V}_Z is open in $Y \cap s\mathcal{U}$, the union $\mathcal{V} = \bigcup_Z \mathcal{V}_Z$ is open in $Y \cap s\mathcal{U}$ and \mathcal{V}_Z is open in \mathcal{V} for all Z . For $Z \neq Z'$, the sets $\mathcal{V}_Z, \mathcal{V}_{Z'}$ are disjoint since they are contained in $Y_{\mathcal{U},Z}, Y_{\mathcal{U},Z'}$ which are disjoint (see Lemma 16.9(d)). Hence the sets \mathcal{V}_Z are also closed in \mathcal{V} . For any Z and any $\eta \in Z$, \mathcal{V}_Z is open in $s(Y'_\eta \cap \mathcal{U})$. (It is enough to show that $Y_{\mathcal{U},Z} \cap \bigcap_{\eta' \in Z} s(Y'_{\eta'} \cap \mathcal{U})$ is open in $s(\bar{Y}'_\eta \cap \mathcal{U})$. This follows from the fact that $Y_{\mathcal{U},\eta}$ is open in $s(\bar{Y}'_\eta \cap \mathcal{U})$ (see Lemma 16.9(c)), and that $Y_{\mathcal{U},Z} \cap \bigcap_{\eta' \in Z} s(Y'_{\eta'} \cap \mathcal{U})$ is open in $Y_{\mathcal{U},Z}$; see (a).) Since $s(Y'_\eta \cap \mathcal{U})$ is isomorphic to an open set in Y'_η , it follows that \mathcal{V}_Z is isomorphic to an open set in Y'_η . Since Y'_η is smooth of pure dimension

$\dim Z_G(s)^0 - \dim L_\eta + \dim {}^\delta \mathcal{Z}_L^0 + \dim \mathfrak{c}_\eta$ (see 15.1 and 3.13(b)), it follows that \mathcal{V}_Z is smooth of pure dimension $\dim Z_G(s)^0 - \dim L_\eta + \dim {}^\delta \mathcal{Z}_L^0 + \dim \mathfrak{c}_\eta$ where $\eta \in Z$. Now L_η is isomorphic to the connected centralizer in L of an element in S_s hence $\dim L_\eta = \dim L - (\dim S_s - \dim {}^\delta \mathcal{Z}_L^0)$; moreover, $\dim \mathfrak{c}_\eta = \dim S - \dim S_s$. Hence \mathcal{V}_Z is smooth of pure dimension $\dim Z_G(s)^0 - \dim L + \dim S$ which is independent of Z . The same is then true for \mathcal{V} . This proves (b).

We show that

(c) \mathcal{V} is open dense in $\bar{Y} \cap s\mathcal{U}$.

By (b), \mathcal{V} is open in $Y \cap s\mathcal{U}$ and by Lemma 16.8(c), $Y \cap s\mathcal{U}$ is open in $\bar{Y} \cap s\mathcal{U}$. Hence \mathcal{V} is open in $\bar{Y} \cap s\mathcal{U}$. We prove that it is also dense. We have

$$\bar{Y} \cap s\mathcal{U} = \psi(X_{\mathcal{U}}) = \bigcup_{\eta} \psi(X_{\mathcal{U},\eta}) = s \cup_{\eta} \psi_{\eta}(X'_{\mathcal{U},\eta}) = s \cup_{\eta} (\bar{Y}'_{\eta} \cap \mathcal{U}) = \bigcup_Z s\bar{Y}'_{Z,\mathcal{U}}$$

where Z runs over the \mathcal{W}_S -orbits in Γ and $\bar{Y}'_{Z,\mathcal{U}}$ is as in 16.10. (We have used 16.8(c), 16.5(a), 16.6, 16.8(d).) Since $\mathcal{V} = \bigcup_Z \mathcal{V}_Z$, it is enough to show that, for any Z , \mathcal{V}_Z is dense in $s\bar{Y}'_{Z,\mathcal{U}}$. By Lemmas 16.8(d) and 16.9(f), for any $\eta \in Z$, $s(Y'_\eta \cap \mathcal{U})$ is open dense in $s\bar{Y}'_{Z,\mathcal{U}}$ and $Y_{\mathcal{U},\eta}$ is open dense in $s\bar{Y}'_{Z,\mathcal{U}}$. Hence $s(Y'_\eta \cap \mathcal{U}) \cap Y_{\mathcal{U},\eta}$ is open dense in $s\bar{Y}'_{Z,\mathcal{U}}$. Hence $\mathcal{V}_Z = \bigcap_{\eta \in Z} (s(Y'_\eta \cap \mathcal{U}) \cap Y_{\mathcal{U},\eta})$ is open dense in $s\bar{Y}'_{Z,\mathcal{U}}$. This proves (c).

Lemma 16.11. *Let ${}^0\tilde{Y} = \{(g, xL) \in \tilde{Y}; g \in \mathcal{V}\}$. For $\eta \in \Gamma$ let*

$${}^0\tilde{Y}'_{\eta} = \{(h, zL_{\eta}) \in \tilde{Y}'_{\eta}; h \in s^{-1}\mathcal{V}_Z\}$$

where $\eta \in Z$. We have a well defined isomorphism $\alpha : \bigsqcup_{\eta \in \Gamma} {}^0\tilde{Y}'_{\eta} \xrightarrow{\sim} {}^0\tilde{Y}, (h, zL_{\eta}) \mapsto (sg, zx_{\eta}L)$.

For $\eta \in \Gamma$ let ${}^0\tilde{Y}_{\eta} = \{(g, xL) \in \tilde{Y}; g \in \mathcal{V}, x \in \eta\}$. Since $\mathcal{V} \subset s\mathcal{U}$, the subsets ${}^0\tilde{Y}_{\eta}$ form a partition of ${}^0\tilde{Y}$ into open and closed subsets (see Lemma 16.8(b)). It is enough to show that for any η , we have a well-defined isomorphism ${}^0\tilde{Y}'_{\eta} \xrightarrow{\sim} {}^0\tilde{Y}_{\eta}, (h, zL_{\eta}) \mapsto (sg, zx_{\eta}L)$. The imbedding $\tilde{Y}_{\mathcal{U}} \rightarrow X_{\mathcal{U}}$ (see 16.8(a)) identifies ${}^0\tilde{Y}_{\eta}$ with an open subset of $X_{\mathcal{U},\eta}$; the imbedding $\tilde{Y}'_{\mathcal{U},\eta} \rightarrow X'_{\mathcal{U},\eta}$ identifies ${}^0\tilde{Y}'_{\eta}$ with an open subset of $X'_{\mathcal{U},\eta}$. It is enough to show that the isomorphism $X'_{\mathcal{U},\eta} \xrightarrow{\sim} X_{\mathcal{U},\eta}$ in Lemma 16.6 carries the subspace $\tilde{Y}'_{\mathcal{U},\eta}$ onto the subspace ${}^0\tilde{Y}_{\eta}$. Thus, it is enough to show that, for $(h, zP_{\eta}) \in X'_{\eta}$, the following two conditions are equivalent:

- (i) $h \in s^{-1}\mathcal{V}_Z, h \in Y'_{\eta}$;
- (ii) $sh \in \mathcal{V}, sh \in Y_{\mathcal{U},\eta}$.

Both (i) and (ii) are equivalent to the condition $sh \in \mathcal{V}_Z$. The lemma is proved.

16.12. By Lemma 16.11 we have a commutative diagram

$$\begin{array}{ccc} \bigsqcup_{\eta \in \Gamma} {}^0\tilde{Y}'_{\eta} & \xrightarrow{\alpha} & {}^0\tilde{Y} \\ \downarrow & & \downarrow \\ s^{-1}\mathcal{V} & \xrightarrow{\epsilon} & \mathcal{V} \end{array}$$

where $\epsilon(h) = sh$ and the vertical maps are given by the first projection. Hence we have a canonical isomorphism

$$\bigoplus_{\eta \in \Gamma} (\pi_{\eta!} \tilde{\mathcal{E}}_{\eta})|_{s^{-1}\mathcal{V}} \xrightarrow{\sim} \epsilon^*((\pi_1 \tilde{\mathcal{E}})|_{\mathcal{V}})$$

of local systems on $s^{-1}\mathcal{V}$. (Here $(\pi_{\eta!}\tilde{\mathcal{E}}_{\eta})|_{s^{-1}\mathcal{V}}$ is by definition the restriction of $\pi_{\eta!}\tilde{\mathcal{E}}_{\eta}$ to $s^{-1}\mathcal{V}_Z$ where $\eta \in Z$ and is zero on $s^{-1}\mathcal{V}_{Z'}$ for $Z' \neq Z$.) This can be also regarded as an isomorphism

$$(a) \quad \bigoplus_{\eta \in \Gamma} IC(\bar{Y}'_{\eta}, \pi_{\eta!}\tilde{\mathcal{E}}_{\eta})|_{s^{-1}\mathcal{V}} \xrightarrow{\sim} \epsilon^*(IC(\bar{Y}, \pi_!\tilde{\mathcal{E}})|_{\mathcal{V}}).$$

Assume that we can show that the isomorphism (a) is the restriction to $s^{-1}\mathcal{V}$ of an isomorphism

$$(b) \quad \bigoplus_{\eta \in \Gamma} IC(\bar{Y}'_{\eta}, \pi_{\eta!}\tilde{\mathcal{E}}_{\eta})|_{\bar{Y}'_{\eta} \cap \mathcal{U}} \xrightarrow{\sim} \epsilon^*(IC(\bar{Y}, \pi_!\tilde{\mathcal{E}})|_{\bar{Y} \cap s\mathcal{U}}).$$

(Here ϵ is regarded as an isomorphism $s^{-1}\bar{Y} \cap \mathcal{U} \rightarrow \bar{Y} \cap s\mathcal{U}, g \mapsto sg$; moreover, $IC(\bar{Y}'_{\eta}, \pi_{\eta!}\tilde{\mathcal{E}}_{\eta})|_{\bar{Y}'_{\eta} \cap \mathcal{U}}$ is regarded as a complex on $s^{-1}\bar{Y} \cap \mathcal{U}$, zero outside $\bar{Y}'_{\eta} \cap \mathcal{U}$). An isomorphism (b) extending (a) is unique, if it exists. (This follows from the fact that the left-hand side of (b) is the intersection cohomology complex of $s^{-1}\bar{Y} \cap \mathcal{U}$ with coefficients in a local system on the open dense smooth subvariety $s^{-1}\mathcal{V}$ of pure dimension, namely $\bigoplus_{\eta \in \Gamma} (\pi_{\eta!}\tilde{\mathcal{E}}_{\eta})|_{s^{-1}\mathcal{V}}$.) The isomorphism (b) gives rise for any i to an isomorphism of stalks

$$(c) \quad \bigoplus_{\eta \in \Gamma} \mathcal{H}_u^i IC(\bar{Y}'_{\eta}, \pi_{\eta!}\tilde{\mathcal{E}}_{\eta}) \xrightarrow{\sim} \mathcal{H}_{su}^i IC(\bar{Y}, \pi_!\tilde{\mathcal{E}}).$$

We have $u \in s^{-1}\bar{Y} \cap \mathcal{U}$. Indeed, $su \in \bar{Y}$ and $u \in \mathcal{U}$ since $u \in \delta_1$ and \mathcal{U} contains any unipotent element in δ_1 .

We now show the existence of the isomorphism (b). To do this we will use P in 16.3. (However, the isomorphism we construct will be independent of the choice of P in view of its uniqueness.) Using Lemma 5.7 and 15.4(a) we find isomorphisms

$$(d) \quad IC(\bar{Y}, \pi_!\tilde{\mathcal{E}})|_{\bar{Y} \cap s\mathcal{U}} \xrightarrow{\sim} \psi_!(K|_{X_{\mathcal{U}}}) \text{ in } \mathcal{D}(\bar{Y} \cap s\mathcal{U}),$$

$$(e) \quad IC(\bar{Y}'_{\eta}, \pi_{\eta!}\tilde{\mathcal{E}}_{\eta})|_{\bar{Y}'_{\eta} \cap \mathcal{U}} \xrightarrow{\sim} \psi_{\eta!}(K_{\eta}|_{X'_{\mathcal{U}, \eta}}) \text{ in } \mathcal{D}(\bar{Y}'_{\eta} \cap \mathcal{U}),$$

(K as in 5.7.) From 16.5(a) and Lemma 16.6 we get an isomorphism

$$(f) \quad \bigoplus_{\eta \in \Gamma} \psi_{\eta!}(K_{\eta}|_{X'_{\mathcal{U}, \eta}}) \xrightarrow{\sim} \epsilon^*(\psi_!(K|_{X_{\mathcal{U}}})) \text{ in } \mathcal{D}(s^{-1}\bar{Y} \cap \mathcal{U}).$$

(We regard $\psi_{\eta!}(K_{\eta}|_{X'_{\mathcal{U}, \eta}})$ as a complex on $s^{-1}\bar{Y} \cap \mathcal{U}$ equal to 0 outside $\bar{Y}'_{\eta} \cap \mathcal{U}$.) Combining the isomorphisms (d),(e),(f) we obtain an isomorphism as in (b).

16.13. In the remainder of this section we assume that \mathbf{k} is an algebraic closure of a finite field \mathbf{F}_q and that G has a fixed \mathbf{F}_q -structure with Frobenius map $F : G \rightarrow G$. Assume that $F(L) = L, F(S) = S, \mathcal{E} \in \mathcal{S}(S)$ and that we are given an isomorphism $\phi_0 : F^*\mathcal{E} \xrightarrow{\sim} \mathcal{E}$ of local systems on S . Then $F(\bar{Y}) = \bar{Y}$ and ϕ_0 induces an isomorphism $\phi : F^*\mathfrak{K} \xrightarrow{\sim} \mathfrak{K}$ where $\mathfrak{K} = IC(\bar{Y}, \pi_!\tilde{\mathcal{E}})$. Assume that s, u in 16.3 satisfy $F(s) = s, F(u) = u$. For any $x \in (G^0)^F$ such that $x^{-1}sx \in S_s$ let $L_x, \mathbf{c}_x, \Sigma_x, \mathcal{E}_x$ be as in 16.5. Let $\mathcal{F}_x = \mathcal{E}_x|_{\mathbf{c}_x}$, a local system on \mathbf{c}_x . Now $Z_G(s), L_x, \mathbf{c}_x, \Sigma_x$ are defined over \mathbf{F}_q and $\phi_0 : F^*\mathcal{E} \xrightarrow{\sim} \mathcal{E}$ induces an isomorphism $\phi'_x : F^*\mathcal{E}_x \xrightarrow{\sim} \mathcal{E}_x$ and an isomorphism $\phi_x : F^*\mathcal{F}_x \xrightarrow{\sim} \mathcal{F}_x$. Then $L_x, Z_G(s), \mathbf{c}_x, \Sigma_x, \mathcal{F}_x, \phi_x$ are like $L, G, \Sigma^\omega, \Sigma, \mathcal{F}, \phi_1$ in 15.12 hence the generalized Green function

$$Q_{L_x, Z_G(s), \mathbf{c}_x, \mathcal{F}_x, \phi_x} : \{\text{unipotent elements in } Z_G(s)^F\} \rightarrow \bar{\mathbf{Q}}_l$$

is defined as in 15.12(c). We have the following result.

Theorem 16.14.

$$\chi_{\mathfrak{R},\phi}(su) = \sum_{x \in (G^0)^F; x^{-1}sx \in S_s} \frac{|L_x^F|}{|Z_G^0(s)^F| |L^F|} Q_{L_x, Z_G(s), \mathbf{c}_x, \mathcal{F}_x, \phi_x}(u).$$

We can choose the base points x_η in η (see 16.5) in such a way that $F(x_\eta) = x_{F(\eta)}$ for any $\eta \in \Gamma$. (We use the fact that, if η is F^n -stable, then $\eta^{F^n} \neq \emptyset$. This follows from the fact that η is a homogeneous space under a connected group.) Now the sum over x in the theorem can be broken into sums over $x \in \eta^F$ for various $\eta \in \Gamma$ with $F(\eta) = \eta$. The sum over $x \in \eta^F$ is equal to $Q_{L_\eta, Z_G(s), \mathbf{c}_\eta, \mathcal{F}_\eta, \phi_\eta}(u)$ (notation of 16.5) since all terms of the sum are equal and the number of terms in the sum is $|\eta^F| = \frac{|Z_G^0(s)^F| |L^F|}{|L_\eta^F|}$. Thus the right-hand side of the equality in the theorem is

$$\sum_{\eta \in \Gamma; F(\eta) = \eta} Q_{L_\eta, Z_G(s), \mathbf{c}_\eta, \mathcal{F}_\eta, \phi_\eta}(u)$$

or equivalently,

$$\sum_{\eta \in \Gamma; F(\eta) = \eta} \chi_{\mathfrak{R}_\eta, \tilde{\phi}}(u)$$

where \mathfrak{R}_η is $IC(\bar{Y}'_\eta, \pi_\eta, \tilde{\mathcal{E}}_\eta)$ (notation of 16.5) extended by 0 on $\bar{Y} - \bar{Y}'_\eta$ and $\tilde{\phi} : F^* \mathfrak{R}_\eta \xrightarrow{\sim} \mathfrak{R}_\eta$ is induced by $\phi'_x : F^* \mathcal{E}_x \xrightarrow{\sim} \mathcal{E}_x$. (At this point, it is important that $Q_{L_\eta, Z_G(s), \mathbf{c}_\eta, \mathcal{F}_\eta, \phi_\eta}(u)$ can be computed in terms of a not necessarily “trivial choice”; see 15.12(d).) We see that it is enough to prove that

$$(a) \chi_{\mathfrak{R},\phi}(su) = \sum_{\eta \in \Gamma; F(\eta) = \eta} \chi_{\mathfrak{R}_\eta, \tilde{\phi}}(u).$$

We shall now make the choice of \mathcal{U} in Lemma 16.4 more precise. Namely we will choose it so that, in addition, it satisfies $F(\mathcal{U}) = \mathcal{U}$. (In the proof of 16.4(f) we choose the imbedding $G \subset GL_n(\mathbf{k})$ so that it is defined over \mathbf{F}_q . Then \mathcal{U}_1 defined in that proof is automatically F -stable hence $\mathcal{U} = \{g \in \mathcal{U}_1; g \notin E\}$ (see the proof of Lemma 16.4) is again F -stable.) With this choice of \mathcal{U} , the isomorphism 16.12(c) commutes with the natural Frobenius maps on its two sides. This gives rise to the equality of (alternating sums of) traces (a). The theorem is proved.

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