A NORMAL FORM FOR ADMISSIBLE CHARACTERS IN THE SENSE OF LYNCH

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ABSTRACT. Parabolic subalgebras $\mathfrak{p}$ of semisimple Lie algebras define a $\mathbb{Z}$-grading of the Lie algebra. If there exists a nilpotent element in the first graded part of $\mathfrak{g}$ on which the adjoint group of $\mathfrak{p}$ acts with a dense orbit, the parabolic subalgebra is said to be nice. The corresponding nilpotent element is also called admissible. Nice parabolic subalgebras of simple Lie algebras have been classified. In the case of Borel subalgebras a Richardson element of $\mathfrak{g}_1$ is exactly one that involves all simple root spaces. It is, however, difficult to write down such nilpotent elements for general parabolic subalgebras. In this paper we give an explicit construction of admissible elements in $\mathfrak{g}_1$ that uses as few root spaces as possible.

INTRODUCTION

Let $\mathfrak{g}$ be a semisimple Lie algebra over $\mathbb{C}$, $\mathfrak{p} \subset \mathfrak{g}$ a parabolic subalgebra. There is a $\mathbb{Z}$-grade of $\mathfrak{g}$, $\mathfrak{g} = \sum_j \mathfrak{g}_j$ such that $\mathfrak{p} = \sum_{j \geq 0} \mathfrak{g}_j$ and $\mathfrak{n} := \sum_{j > 0} \mathfrak{g}_j$ is the nilradical of $\mathfrak{p}$. By a theorem of Richardson ([R]) there is always a Richardson element in $\mathfrak{n}$, i.e. an element $X \in \mathfrak{n}$ satisfying $[\mathfrak{p}, X] = \mathfrak{n}$. We say that $\mathfrak{p}$ is nice if there is a Richardson element in the first graded part $\mathfrak{g}_1$. Nice parabolic subalgebras have been classified in [BW]. If $\mathfrak{p}$ is a Borel subalgebra, then a Richardson element of $\mathfrak{g}_1$ involves all simple root spaces. For arbitrary nice parabolic subalgebras, the support of a Richardson element in the first graded part may consist of all roots of $\mathfrak{g}_1$. In this sense it is far from being a simple representative of a Richardson element. The goal of this paper is to give a normal form of Richardson elements for nice parabolic subalgebras in the classical case. The construction uses as few root spaces of the nilradical as possible. It turns out that in many cases, the support of this normal form spans a simple system of roots. Since Richardson elements correspond to admissible characters $\nu : \overline{\mathfrak{n}} \to \mathbb{C}$, the normal form describes how admissible characters of the (opposite) nilradical actually appear.

In his thesis [L], Lynch studied Whittaker modules for which there is an admissible homomorphism $\nu : \mathfrak{n} \to \mathbb{C}$: Let $\mathcal{U}$ be the universal enveloping algebra of $\mathfrak{g}$ and let V be a $\mathcal{U}$-module. A vector $v \in V$ is called a Whittaker vector if there exists a nonsingular character $\nu : \mathfrak{n} \to \mathbb{C}$ such that $xv = \nu(x)v$ for all $x \in \mathfrak{n}$. The module V is a Whittaker module if V is cyclically generated by a Whittaker vector. An element $x \in \mathfrak{g}$ is called admissible if $x \in \mathfrak{g}_{-1}$ and $\mathfrak{n}^x = \{0\}$. In particular, x is a

nilpotent element of $\mathfrak{g}$. There is a natural bijection between $\mathfrak{g}_{-1}$ and the characters of $\mathfrak{n}$. That is, if x is an element of $\mathfrak{g}_{-1}$, x corresponds to the character $\nu : \mathfrak{n} \to \mathbb{C}$ if $B(x, y) = \nu(y)$ for all $y \in \mathfrak{n}$. Here, $B(\cdot, \cdot)$ is the Killing form on $\mathfrak{g}$. The character ν is called admissible if the corresponding element x is admissible.

Lynch has studied real parabolic subalgebras whose nilradicals support admissible Lie algebra homomorphisms to $i\mathbb{R}$. He calls a parabolic subalgebra $\mathfrak{p}$ of $\mathfrak{g}$ admissible if there exists such an admissible homomorphism. Nice parabolic subalgebras are exactly the complexifications of admissible parabolic subalgebras in the sense of Lynch. So the classification of nice parabolic subalgebras tells us when admissible characters exist. But it is not easy to write one down. This paper gives a normal form for admissible elements of nice parabolic subalgebras in the classical Lie algebras. Via the correspondence admissible elements—admissible characters given above, the normal forms give a description of admissible characters.

The basic idea of the construction is to spread out small blocks of skew-diagonal matrices at the appropriate places of the entries of the first graded part $\mathfrak{g}_1$ of $\mathfrak{p}$. In the fifth section we show that in many cases the normal form of a generic element involves only roots that form a simple system of roots. In particular, in all these cases we can construct a Richardson element using at most $n = \operatorname{rk} \mathfrak{g}$ root spaces.

The normal forms are given in section one. In section two we recall the properties of nice parabolic subalgebras. Sections three and four describe the constructions and give a modification for the orthogonal Lie algebras. In section five we discuss representation theoretic aspects of the results. We explain in which cases the roots involved in the normal form, form a simple system of roots and describe the factors of the obtained root system. In the last section we turn our attention to the exceptional Lie algebras. Roughly half of the nice parabolic subalgebras of the exceptional Lie algebras have a Richardson element whose support forms a simple system of roots. We list all of these to complete the picture.

Finally, I thank the referee for the suggestion concerning the exceptional Lie algebras.

1. RESULTS

In what follows, $\mathfrak{g}$ will be a classical Lie algebra over the complex numbers. As usual, $\mathfrak{g}$ will be denoted by A_n, B_n, C_n, D_n respectively. These Lie algebras are realized as subalgebras of the appropriate $\mathfrak{gl}_N$ ($N = n + 1, 2n + 1, 2n$ respectively $2n$). We follow the usage of [GW]: A_n are the matrices of trace zero, B_n, D_n the orthogonal Lie algebra of the symmetric form given by the matrix with entries 1 on the skew-diagonal and otherwise 0 (we will denote such a skew-diagonal $N \times N$-matrix by J_N). And C_n is the symplectic Lie algebra for the symplectic form given by the matrix with entries 0 outside the skew-diagonal and entry 1 for the first n entries on this skew-diagonal, entry -1 on the last n entries of it (i.e. the form is given by the matrix $\begin{bmatrix} 0 & J_n \\ -J_n & 0 \end{bmatrix}$). Thus B_n and D_n are the matrices that are skew-symmetric around the skew-diagonal: If $A = (a_{ij}) \in \mathfrak{so}_N$, we have $a_{N-j+1,N-i+1} = -a_{ij}$. The elements of C_n are the $2n \times 2n$-matrices $A = (a_{ij})$ with $a_{2n-j+1,2n-i+1} = -a_{ij}$ if $1 \leq i, j \leq n$ and $a_{2n-j+1,2n-i+1} = a_{ij}$ if $1 \leq i \leq n$, $n < j$ or $n < i$, $1 \leq j \leq n$.

With this realization the Borel subalgebra we choose is the intersection of the corresponding Lie algebra with the upper triangular matrices in $\mathfrak{gl}_N$, the Cartan subalgebra is the set of diagonal matrices in the corresponding Lie algebra. We call a parabolic subalgebra containing this Borel subalgebra standard. Similarly, a Levi

factor of $\mathfrak{p}$ that contains the diagonal matrices is called a standard Levi factor of $\mathfrak{p}$. If $\sum_j \mathfrak{g}_j$ is the $\mathbb{Z}$-grade associated to $\mathfrak{p}$, then $\mathfrak{g}_0$ is the standard Levi factor of $\mathfrak{p}$.

The Levi factor $\mathfrak{g}_0$ of a parabolic subalgebra consists of a sequence of square matrices on the diagonal. Caveat: in the case of $\mathfrak{so}_N$, $\mathfrak{sp}_{2n}$, the central block is skew-symmetric around the skew-diagonal or symmetric around the skew-diagonal, respectively. Thus in the case of D_n there is an ambiguity in describing the parabolic subalgebras: the intersection of the standard parabolic subalgebra of $\mathfrak{gl}_{2n}$ with block lengths $(a_1, \ldots, a_{r-1}, 2, a_{r-1}, \ldots, a_1)$ with the $\mathfrak{so}_{2n}$ is the same as the intersection of the parabolic subalgebra of $\mathfrak{gl}_{2n}$ described by $(a_1, \ldots, a_{r-1}, 1, 1, a_{r-1}, \ldots, a_1)$ with $\mathfrak{so}_{2n}$. In what follows we will always use the first version of this parabolic subalgebra of $\mathfrak{so}_{2n}$.

That said we can think of $\mathfrak{p}$ to be given by the sequence of block lengths of the standard Levi factor. The space $\mathfrak{g}_1$ is then a sequence of rectangles $R_{i,i+1}$ on the first super-diagonal. The sides of the rectangles are given by the lengths of the square matrices in $\mathfrak{g}_0$. For example, if $\mathfrak{p}$ is given by $(a_1, \ldots, a_r)$, then the $r-1$ rectangles of $\mathfrak{g}_1$ have size $a_1 \times a_2, a_2 \times a_3, \ldots, a_{r-1} \times a_r$.

Recipe 1.1. Let $\mathfrak{g}$ be of type A_n, let the nice parabolic subalgebra $\mathfrak{p} \subset \mathfrak{g}$ be given by the unimodal sequence $(a_1, \ldots, a_{r+1})$. Let

$$R_{i,i+1} \text{ have the form } \begin{cases} \begin{bmatrix} J_{a_i} & 0 \end{bmatrix} & \text{if } i \text{ is odd,} \\ \begin{bmatrix} 0 & J_{a_i} \end{bmatrix} & \text{if } i \text{ is even.} \end{cases}$$

Then we define $X_R \in \mathfrak{g}_1$ to be the matrix formed by these rectangles.

In the case of the symplectic and orthogonal Lie algebras with an even number of blocks in the standard Levi factor, X_R is constructed similarly. However, if the standard Levi factor has an odd number of blocks, we have to spread out the entries of the rectangles and choose small skew-diagonal matrices in both corners of each rectangle, resp., in the central part of such a rectangle as we will see. To do so, set $B_i := \lceil \frac{a_i}{2} \rceil$ and $b_i := \lfloor \frac{a_i}{2} \rfloor$.

If the standard Levi factor has an even number of blocks, we talk about case (A), i.e. case (A) is the situation where the lengths in the standard Levi factor are given by $(a_1, \ldots, a_r, a_r, \ldots, a_1)$. If the lengths of the blocks in the standard Levi factor are $(a_1, \ldots, a_r, a_{r+1}, a_r, \ldots, a_1)$, then we talk about case (B) (i.e. an odd number of blocks in the standard Levi factor).

Recipe 1.2. Let $\mathfrak{g}$ be of type C, $\mathfrak{p} \subset \mathfrak{g}$ a nice parabolic subalgebra, given by an unimodal sequence.

Case (A). Let $R_r := J_{a_r}$ and for $i \leq r - 1$ let

$$R_{i,i+1} \text{ have the form } \begin{cases} \begin{bmatrix} J_{a_i} & 0 \end{bmatrix} & \text{if } i \text{ is odd,} \\ \begin{bmatrix} 0 & J_{a_i} \end{bmatrix} & \text{if } i \text{ is even.} \end{cases}$$

Case (B) For $i = 1, \ldots, r$ we let

$$R_{i,i+1} \text{ be of the form } \begin{cases} \begin{bmatrix} J_{B_i} & J_{b_i} \\ & \end{bmatrix} & \text{if } i \text{ is odd,} \\ \\ \begin{bmatrix} & J_{B_i} \\ J_{b_i} & \end{bmatrix} & \text{if } i \text{ is even.} \end{cases}$$

Then we define $X_R \in \mathfrak{g}_1$ to be the matrix that has the rectangles $R_{i,i+1}$ in the upper left half, that has the corresponding entries ± 1 in the lower right half and—in

case (A) of an even number of parts—has rectangle R_r in the central position of the super-diagonal.

In the cases of the orthogonal Lie algebras we allow the sequence of block lengths to be unimodal up to some smaller blocks in the middle. If $a_i > a_{i+1}$, the prescription is modified as follows. In case (A) we let $R_{i,i+1}$ have the form $\begin{bmatrix} J_{a_{i+1}} \\ 0 \end{bmatrix}$ for odd i, resp., $\begin{bmatrix} 0 \\ J_{a_{i+1}} \end{bmatrix}$ for even i. In case (B) we set $R_{i,i+1} = \begin{bmatrix} & J_{B_i} \\ J_{B_i} & \end{bmatrix}$. Note that the entries of the two J_{B_i} overlap if a_i is odd, that is, the B_ith column has two nonzero entries.

Recipe 1.3. Let $\mathfrak{g}$ of type B, D, $\mathfrak{p} \subset \mathfrak{g}$ be a nice parabolic subalgebra, given by a unimodal sequence or let $a_1 \leq \cdots \leq a_l > a_{l+1} = \cdots = a_r$ (resp. $= a_{r+1}$) with $a_{l+1} = a_l + 1$.
    Case (A). For $i \leq r - 1$ with $a_i \leq a_{i+1}$ let

$$R_{i,i+1} \text{ have the form } \begin{cases} \begin{bmatrix} J_{a_i} & 0 \end{bmatrix} & \text{if } i \text{ is odd,} \\ \begin{bmatrix} 0 & J_{a_i} \end{bmatrix} & \text{if } i \text{ is even.} \end{cases}$$

If $a_i > a_{i+1}$ let $R_{i,i+1}$ have the form $\begin{bmatrix} 0 \\ J_{a_{i+1}} \end{bmatrix}$ for odd i resp. $\begin{bmatrix} J_{a_{i+1}} \\ 0 \end{bmatrix}$ for even i.

Furthermore, we let R_r be the matrix that has b_r two-by-two blocks $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ starting from the lower left corner if r is odd resp. from the upper right corner if r is even.
    Case (B). For $i \leq r$ let $R_{i,i+1}$ have the form

$$\begin{cases} \begin{bmatrix} 0 & J_{B_i} & 0 \end{bmatrix} & \text{if } a_i, a_{i+1} \text{ have the same parity,} \\[2ex] \begin{bmatrix} & J_{B_i - 1} \\ J_{B_i} & \end{bmatrix} & \text{if } a_i = a_{i+1} + 1 \text{ and } a_i \text{ is even, } i \text{ odd,} \\[2ex] \begin{bmatrix} & J_{B_i} \\ J_{B_i - 1} & \end{bmatrix} & \text{if } a_i = a_{i+1} + 1 \text{ and } a_i \text{ is even, } i \text{ even,} \\[2ex] \begin{bmatrix} & J_{B_i} \\ J_{B_i} & \end{bmatrix} & \text{else (including } a_i = a_{i+1} + 1 \text{ with } a_i \text{ odd).} \end{cases}$$

Let $X_R \in \mathfrak{g}_1$ be the matrix that has the rectangles $R_{i,i+1}$ in its upper left half and the corresponding entries -1 in the lower right half and that has the square matrix R_r as the rst rectangle in the super-diagonal in case (A).

Note that in Recipe 1.3 part (B), if $a_l = a_{l+1} + 1$, with a_l odd (i.e. the Lie algebra is an even orthogonal one) the recipe picks $a_l + 1$ entries for the rectangle $R_{l,l+1}$. The rank of $R_{l,l+1}$ is $a_l - 1$.

Theorem 1.4. *Let $\mathfrak{p} \subset \mathfrak{g}$ be a nice parabolic subalgebra of a classical Lie algebra. Let X_R be constructed as above (Recipes 1.1, 1.2, 1.3).*
    Then $[\mathfrak{p}, X_R] = \mathfrak{n}$.

Remark 1.5. In their recent article [GR], S. Goodwin and G. Röhrle use an alternative description of Richardson elements in $\mathfrak{g}_1$ for parabolic subalgebras of $\mathfrak{gl}_n$. They follow a construction of [BHRR]. The Richardson element obtained that way uses the identity matrix of size $\min\{a_i, a_{i+1}\}$ in the rectangle $R_{i,i+1}$, starting in the

upper left corner of the rectangle. This choice ensures that the matrix and all its powers have maximal rank.

For our purposes the construction we present in this paper fits better since we are interested in the representation theoretic meaning of the root spaces involved. This will be discussed in Section 5.

2. Background on nice parabolic subalgebras

In this section we recall the properties of nice parabolic subalgebras that we will need later. The proofs of these statements can be found in [BW] and in [W]. The following result, Theorem 2.5 in [BW], determines whether an element of $\mathfrak{g}_1$ is a Richardson element or not.

Theorem 2.1. *Let $\mathfrak{p} \subset \mathfrak{g}$ be a parabolic subalgebra, $\mathfrak{p} = \mathfrak{m} \oplus \mathfrak{u}$ where $\mathfrak{m}$ is a Levi factor of $\mathfrak{p}$ and $\mathfrak{u}$ the corresponding unipotent radical. Let $\mathfrak{g} = \sum \mathfrak{g}_i$ be the corresponding grade. Then $x \in \mathfrak{g}_1$ is a Richardson element of $\mathfrak{p}$ if and only if $\dim \mathfrak{g}^x = \dim \mathfrak{m}$.*

This dimension criterion is essential both in the classical and in the exceptional case. In the classical case, the method used in [BW] is the following: one calculates the Jordan normal form of a generic nilpotent element of $\mathfrak{g}_1$. Then from the Jordan normal form, the dimension of a generic centralizer is computed. By Theorem 2.1, the parabolic subalgebra is nice if and only if this dimension is the same as the dimension of a Levi factor.

Next we recall the obtained characterization of nice parabolic subalgebras of the classical Lie algebras as it is given in the first section of [BW]:

Theorem 2.2. *A parabolic subalgebra in a Lie algebra of a type $\mathrm{A, C}$ is nice if and only if the corresponding sequence of block lengths in the standard Levi factor is unimodal.*

Theorem 2.3. *A parabolic subalgebra in a Lie algebra of type B_n is nice if and only if the corresponding sequence of block lengths in the standard Levi factor is unimodal or satisfies*

$$a_1 \leq \cdots < a_l > b_1 = \cdots = b_s < a_l \geq \cdots \geq a_1$$

with $b_1 = a_l - 1$, $l \geq 1$.

Theorem 2.4. *A parabolic subalgebra in a Lie algebra of type B_n is nice if and only if the corresponding sequence of block lengths in the standard Levi factor has one of the following forms:*
1) It is unimodal with an odd number of blocks.
2) It is unimodal with an even number of blocks and odd block lengths occur exactly twice.
3) The block lengths satisfy

$$a_1 \leq \cdots \leq a_l > b_1 = \cdots = b_s < a_l \geq \cdots \geq a_1$$

with $b_1 = a_l - 1$, a_l is odd and if s is even, the odd block lengths occur exactly twice.

To be able to compute the dimensions of the centralizer of a nilpotent element, we use the following well-known result (see [CM]):

Theorem 2.5. *Let $x \in \mathfrak{g}$ be a nilpotent element, let $n_1 \geq \cdots \geq n_r > 0$ be its Jordan normal form and $m_1 \geq \cdots \geq m_s > 0$ the dual partition. Then the dimension of the centralizer of x in $\mathfrak{g}$ is*

$$\begin{cases} \sum_i m_i^2 & \text{if } \mathfrak{g} = \mathfrak{sl}_{n+1}, \\ \sum_i m_i^2 + \frac{1}{2}\{i \mid n_i \text{ odd}\} & \text{if } \mathfrak{g} = \mathfrak{sp}_{2n}, \\ \sum_i m_i^2 - \frac{1}{2}\{i \mid n_i \text{ odd}\} & \text{if } \mathfrak{g} = \mathfrak{so}_n. \end{cases}$$

3. Construction, Proofs

Let $\mathfrak{p} \subset \mathfrak{p}$ be a nice parabolic subalgebra. Let the corresponding sequence of the block lengths of $\mathfrak{g}_0$ be $(a_1, \ldots, a_{r+1})$. Denote the entries of X_R in the rectangles $R_{i,i+1}$ by X_i, $i = 1, \ldots, r$. In the other cases, we have (skew-)symmetries around the skew-diagonal. As before, case (A) is the case when the standard Levi factor has an odd number of blocks. In that case, $\mathfrak{p}$ is given by $(a_1, \ldots, a_r, a_r, \ldots, a_1)$, the elements of $\mathfrak{g}_1$ have block form, say

$$X_R = \begin{bmatrix} 0 & X_1 & 0 & 0 & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & X_{r-1} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & X_r & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & Y_{r-1} & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & Y_1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

If there is an even number of blocks, case (B), $\mathfrak{p}$ is given by $(a_1, \ldots, a_r, a_{r+1}, a_r, \ldots, a_1)$. The elements of $\mathfrak{g}_1$ have block form, say

$$X_R = \begin{bmatrix} 0 & X_1 & 0 & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & X_r & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & Y_r & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & Y_1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Note that in the (A)-case X_r is symmetric if the Lie algebra is symplectic, it is skew-symmetric if the Lie algebra is orthogonal. In the (A)-case, each Y_i is skew-symmetric to X_i.

In the (B)-case, if the Lie algebra is symplectic, then the first half of the rows of Y_r are symmetric to the last half of the columns of X_r, and the last half of the rows of Y_r are skew-symmetric to the first half of columns of X_r. For $i < r$, Y_i is skew-symmetric to X_i. In the orthogonal case, each X_i is skew-symmetric to X_i.

Remark 3.1. Consider the powers X_R^j. The entries of X_R^j are in the blocks in the j-th diagonal, i.e. in the diagonal that consists of the rectangles $R_{i,i+j}$.

(1) If $j = 2$ with $a_i \leq a_{i+1} \leq a_{i+2}$, the products are described easily as of the form

$$\begin{bmatrix} J_{a_i} & 0 \end{bmatrix} \begin{bmatrix} 0 & J_{a_{i+1}} \end{bmatrix} = \begin{bmatrix} 0 & I_{a_i} \end{bmatrix}$$

or as

$$\begin{bmatrix} & J_{B_i} \\ J_{b_i} & \end{bmatrix} \begin{bmatrix} & J_{b_{i+1}} \\ J_{B_{i+1}} & \end{bmatrix} = \begin{bmatrix} I_{B_i} & \\ & I_{b_i} \end{bmatrix}$$

or as

$$\begin{bmatrix} & J_{B_i} \\ J_{B_i} & \end{bmatrix} \begin{bmatrix} & J_{B_{i+1}} \\ J_{B_{i+1}} & \end{bmatrix} = \begin{bmatrix} I_{B_i} & \\ & I_{B_i} \end{bmatrix}$$

depending on the case (where $b_i = \lfloor \frac{a_i}{2} \rfloor$, $B_i = \lceil \frac{a_i}{2} \rceil$).

(2) More generally for $j \geq 1$ there are of three possible types:
   1-A. $X_i X_{i+1} \cdots X_{i+j-1}$ if $i \geq 1$ and $j + i \leq r + 1$ (the i-th entry).
   2-A. $X_i X_{i+1} \cdots X_{r+1} Y_r \cdots Y_{2r-i-j+1}$ if $i \geq 1$, $2r > i + j > r + 1$ and $i \leq r$ (the i-th entry).
   3-A. $Y_{r-i} Y_{r-i-1} \cdots Y_{r-i-j+1}$ if $i \geq 0$, $i + j \leq r$ the ($r + i$-th entry).
      In case (B) there are also three possibilities:
   1-B. $X_i X_{i+1} \cdots X_{i+j-1}$ if $i \geq 1$ and $j + i \leq r + 1$ (the i-th entry).
   2-B. $X_i X_{i+1} \cdots X_r Y_r \cdots Y_{2r-i-j+1}$ if $i \geq 1$ and $2r > i + j > r + 1$ and $i \leq r$ (the i-th entry).
   3-B. $Y_{r-i} Y_{r-i-1} \cdots Y_{r-i-j+1}$ if $i \geq 0$ and $i + j \leq r$ the ($r + i + 1$-th entry).

Proof of Theorem 1.4. Let $\mathfrak{p} \subset \mathfrak{g}$ be a nice parabolic subalgebra, let $X_R \in \mathfrak{g}_1$ be the corresponding nilpotent element.

Since $\mathfrak{p}$ is nice, there is a Richardson element X in $\mathfrak{g}_1$. By Theorem 2.1, $\dim \mathfrak{g}^X = \dim \mathfrak{g}_0$. Recall that the dimension of the centralizer of a nilpotent element is calculated from its Jordan normal form (cf. Theorem 2.5). The latter only depends on the ranks $\operatorname{rk} X^j$. So if we show that X_R is generic (i.e. $\operatorname{rk} X^j = \operatorname{rk} X_R^j$ for all j), we obtain $\dim \mathfrak{g}^X = \dim \mathfrak{g}^{X_R}$. In particular, X_R is a Richardson element.

To see that X_R is generic, we use the description of the blocks X_R^j in Remark 3.1. It is easy to check that in the case of A, C, the rank of X_R^j is maximal for each j.

In the B, D-case one has to be careful with the products of type 2-A and 2-B as described in Remark 3.1. We leave the details to the reader and refer to [BW], Section 3 where the generic ranks are computed explicitly in terms of the block lengths (Lemma 3.2 for A_n, Propositions 3.4 and 3.5 for the symplectic and the orthogonal Lie algebras). $\qquad\square$

4. MORE ON THE ORTHOGONAL LIE ALGEBRAS

We consider the case of a parabolic subalgebra $\mathfrak{p} \subset \mathfrak{g}$ of an orthogonal Lie algebra. The construction of the Richardson element X_R given in Recipe 1.3 seems unsatisfactory in the (B)-case, i.e. if the standard Levi factor has an odd number of blocks. In this section we will study this case in detail. The goal is to optimize the choice of S_1, the subset of the roots of $\mathfrak{g}_1$ whenever this is possible. We will see that this is only possible if the sequence of the blocks in the standard Levi factor is not unimodal.

Let $\mathfrak{p}$ be given by the sequence $(a_1, \ldots, a_r, a_{r+1}, a_r, \ldots, a_1)$. Let $B_i := \lceil a_i/2 \rceil$ and $b_i = \lfloor a_i/2 \rfloor$. Recall that the element X_R has block form

$$
X_R = \begin{bmatrix}
0 & X_1 & 0 & 0 & 0 & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & X_r & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & Y_r & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & 0 & 0 & 0 & Y_1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
$$

with

$$X_i = \begin{cases} (0 \quad J_{B_i} \quad 0) & \text{if } a_i \text{ and } a_{i+1} \text{ have the same parity,} \\[2mm] \begin{bmatrix} & J_{B_i} \\ J_{b_i} & \end{bmatrix} \text{ or } \begin{bmatrix} & J_{b_i} \\ J_{B_i} & \end{bmatrix} & \text{if } a_i > a_{i+1}, a_i \text{ even, } i \text{ odd resp. } i \text{ even,} \\[4mm] \begin{bmatrix} & J_{B_i} \\ J_{B_i} & \end{bmatrix} & \text{else} \end{cases}$$

and where Y_i is the negative of the skew-transpose of X_i.

In particular, if the parity of the a_i changes from odd to even, the recipe chooses two roots of $R_{i,i+1}$ that subtract from each other. The next result shows that the extra root chosen is actually necessary in many cases.

Lemma 4.1. *Let the sequence $(a_1, \ldots, a_{r+1}, \ldots, a_1)$ be unimodal, such that the corresponding parabolic subalgebra $\mathfrak{p} \subset \mathfrak{so}_n$ is nice. Assume that there is an $i \leq r$ such that a_i is odd and a_{i+1} is even. Let $\hat{X}_i$ be a $a_i \times a_{i+1}$-matrix of rank a_i with exactly a_i nonzero entries, let $\hat{Y}_i$ be the negative of its skew-transpose. Define $\hat{X}_R$ to be the matrix obtained from X_R by replacing X_i by $\hat{X}_i$ and Y_i by $\hat{Y}_i$.*

Then $\hat{X}_R$ is not a Richardson element for $\mathfrak{p}$.

Proof. We have to show that $\hat{X}_R$ is not generic, in the sense that the ranks of $\hat{X}_R^j$ are not all maximal. By the construction of X_R, for each $j > i$ the matrix $Z_j := X_j \cdots X_r \cdot Y_r \cdots Y_j$ is a $a_j \times a_j$ diagonal matrix of rank a_j that is symmetric around the skew-diagonal.

We can assume that $a_i = a_{i+1} - 1$, the general statement follows with the same arguments. By assumption, $\hat{X}_i$ has a column of zeroes, say the kth. So the (a_i+2-k)th row of $\hat{Y}_i$ is zero. The product $\hat{X}_i Z_{i+1} \hat{Y}_i$ is a $a_i \times a_i$-matrix. Its entries are the entries of the diagonal matrix Z_{i+1} except for the kth diagonal element of Z_{i+1} and the k to the last entry of Z_{i+1}. Since a_{i+1} is even, these two entries of Z_{i+1} are at different positions. Hence the product $\hat{X}_i Z_{i+1} \hat{Y}_i$ only has $a_{i+1} - 2 = a_i - 1$ nonzero entries and is not of maximal rank. $\square$

The next result treats the non-unimodal cases. Here we define $\hat{X}_i$ to be the matrix $\begin{bmatrix} & J_{b_i} \\ J_{B_i} & \end{bmatrix}$ if i is odd and $\begin{bmatrix} & J_{B_i} \\ J_{b_i} & \end{bmatrix}$ if i is even.

Lemma 4.2. *Let the sequence $(a_1, \ldots, a_{r+1}, \ldots, a_1)$ describe the nice parabolic subalgebra $\mathfrak{p} \subset \mathfrak{so}_{2n+1}$. Assume that there is an l such that $a_l = a_{l+1} + 1$. Let a_{l-1} be odd.*

Then the matrix $\hat{X}_R$ obtained by replacing X_{l-1} (Y_{l-1}) by $\hat{X}_{l-1}$ (by $\hat{Y}_{l-1}$) is a Richardson element for $\mathfrak{p}$ if and only if $a_{l-1} = a_{l+1}$.

Proof. As before we study the rank of the powers $\hat{X}_R^j$. Note that the matrices $X_{l+1}, \ldots, X_r, Y_r, \ldots, Y_{r+1}$ all are square matrices with ones on the skew-diagonal. In particular, products of these are either equal to I_{a_r} or to J_{a_r}. Hence we can assume w.l.o.g. that $l = r - 1$, i.e. that the sequence satisfies

$$a_1 \leq \cdots \leq a_r > a_{r+1} < a_r \geq \cdots \geq a_1.$$

Let $a_{r-1} = a_{r+1}$, say $r - 1$ is odd. Then

$$\hat{X}_{r-1} X_r = \begin{bmatrix} & J_{b_{r-1}} \\ J_{B_{r-1}} & \end{bmatrix} \begin{bmatrix} & J_{B_{r+1}} \\ J_{b_{r+1}} & \end{bmatrix} = \begin{bmatrix} I_{b_{r+1}} & \\ & I_{B_{r+1}} \end{bmatrix} = I_{a_{r+1}}$$

which is equal to $-Y_r \hat{Y}_{r-1}$ and so $\hat{X}_{r-1} X_r Y_r \hat{Y}_{r-1} = -I_{a_{r+1}}$.

On the other hand, each product $X_k \cdots \hat{X}_i$ has (maximal) rank a_k and the same holds for $\hat{Y}_i \cdots Y_k$. So, in particular, all products of the blocks of $\hat{X}_R$ that involve $\hat{X}_i$ or $\hat{Y}_i$ have the same rank as the corresponding products for X_R. Therefore, $\hat{X}_R$ is a Richardson element for $\mathfrak{p}$.

If $a_{r-1} \leq a_{r_1} - 2$, we obtain as

$$\hat{X}_{r-1}X_r = \begin{bmatrix} & J_{b_{r-1}} \\ J_{B_{r-1}} & \end{bmatrix} \begin{bmatrix} & J_{B_{r+1}} \\ J_{b_{r+1}} & \end{bmatrix} = \begin{bmatrix} I_{b_{r-1}} & \\ & I_{B_{r-1}} \end{bmatrix},$$

which is a matrix with a_{r-1} rows and a_{r+1} columns. So $Y_r\hat{Y}_{r-1} = -\begin{bmatrix} I_{B_{r-1}} & \\ & I_{b_{r-1}} \end{bmatrix}$

and $\hat{X}_{r-1}X_rY_r\hat{Y}_{r-1} = \begin{bmatrix} I_{b_{r-1}} & \\ & I_{b_{r-1}} \end{bmatrix}$. Since a_{r-1} is odd, the rank of this product is not maximal and therefore $\hat{X}_R$ is not a Richardson element of $\mathfrak{p}$. $\qquad\square$

We summarize these results and obtain one case where the Richardson element X_R can be replaced by one with fewer root spaces involved.

Proposition 4.3. *Let $\mathfrak{p} \subset \mathfrak{so}_N$ be a nice parabolic subalgebra with an odd number of blocks in the standard Levi factor. If a_i is odd and a_{i+1} is even then the element $\hat{X}_R$ obtained from X_R by replacing X_i and Y_i by $\hat{X}_i$ and $\hat{Y}_i$ is a Richardson element for $\mathfrak{p}$ exactly in the following case:*
$\mathfrak{g} = \mathfrak{so}_{2n+1}$, $i = l - 1$ *with* $a_{l-1} = a_{l+1}$ *and* $a_l > a_{l+1}$.

From now on we will use the element $\hat{X}_R$ for nonunimodal block lengths of $\mathfrak{g}_0$ in $\mathfrak{so}_{2n+1}$ whenever applicable. Let S_1 be the set of roots that are involved in the construction of the Richardson element.

Corollary 4.4. *Let $\mathfrak{p} \subset \mathfrak{so}_N$ be a nice parabolic subalgebra. Then there exist α, β in S_1 such that $\alpha - \beta$ is a root of $\mathfrak{g}$ whenever:*
  (1) *The sequence of the block lengths is unimodal, a_i odd, a_{i+1} even for some $i < r$.*
  (2) *If $a_l > a_{l+1}$ there is $i < l$ with a_i odd, a_{i+1} even and if $i = l - 1$ then $a_{r-1} < a_{r+1}$.*
  (3) *If $a_l > a_{l+1}$, a_l is odd.*

Proof. In all these cases the recipes for X_R resp. for $\hat{X}_R$ chooses more elements of $R_{i,i+1}$ than the "rank" of this rectangle, i.e. more than $\min\{a_i, a_{i+1}\}$. $\qquad\square$

Definition 4.5. Let $\mathfrak{p} \subset \mathfrak{so}_N$ be a nice parabolic subalgebra as in (1), (2) or (3) of Corollary 4.4 above. Then we say that $\mathfrak{p}$ is non-simple.

In other words, the parabolic subalgebras of non-simple type are exactly those where there are at least two roots in S_1 that subtract from each other. In Section 5 we will examine the structure of the roots of S_1 in detail.

5. ROOT STRUCTURE OF X_R

In this section we first translate the entries of X_R to the corresponding roots. The constructed Richardson elements were chosen in a way that they involve as few root spaces as possible. We will see that very often the roots corresponding to X_i do not subtract from each other. That means that they form a simple system of

roots. We will discuss the different cases and show in which cases all roots of X_R form a simple system. Also, we will explain what the factors of such a system are.

Let $\mathfrak{p} \subset \mathfrak{g}$ be a parabolic subalgebra, let $\mathfrak{m} = \mathfrak{g}_0$ be its standard Levi factor. We denote the adjoint group of $\mathfrak{m}$ by M. It acts on $\mathfrak{g}_1$ by conjugation. Under this action, the rectangles $R_{i,i+1}$ (and there counterparts below the skew-diagonal in the case of B, C, D) are the irreducible components of $\mathfrak{g}_1$.

Note that M is a product of the form $\mathrm{GL}_{a_1} \times \cdots \times \mathrm{GL}_{a_{r+1}}$ in the case of A_n. If $(h_1, \ldots, h_{r+1})$ is an element of M, it sends $(X_1, \ldots, X_r) \in \mathfrak{g}_1$ to the element $(h_1^{-1} X_1 h_2, h_2^{-1} X_2 h_3, \ldots, h_r^{-1} X_r h_{r+1})$.

In the case of the other classical Lie algebras, (A)-cases, M is of the form $\mathrm{GL}_{a_1} \times \cdots \times \mathrm{GL}_{a_r}$ and acts as $(h_1^{-1} X_1 h_2, \ldots, h_{r-1}^{-1} X_{r-1} h_r, h_r^{-1} X_r h_r)$.

In the (B)-cases, M is the product $\mathrm{GL}_{a_1} \times \cdots \times \mathrm{H}_{a_r}$ with $H \subset \mathrm{GL}_{a_{r+1}}$ the group $\mathrm{SO}_{a_{r+1}}$ resp. $\mathrm{SP}_{a_{r+1}}$. It acts on $\mathfrak{g}_1$ as $(h_1^{-1} X_1 h_2, \ldots, h_{r-1}^{-1} X_{r-1} h_r, h_r^{-1} X_r h_{r+1})$.

Under the action of the adjoint group M the vector space $\mathfrak{g}_1$ decomposes into r irreducible components. The irreducible components are the vector spaces corresponding to the rectangles $R_{i,i+1}$ for $i \leq r - 1$ ($i \leq r$ for A_n) together with R_i in the (A)-cases (even number of blocks in the standard Levi factor). We thus obtain r subsets of the positive roots of $\mathfrak{g}$, in fact, of the roots of $\mathfrak{g}_1$. Each of these subsets fills out the corresponding rectangle $R_{i,i+1}$. The way we set up the construction of Richardson elements translates to the following:

In the case of $\mathfrak{sl}_{n+1}$ or in the (A)-cases (an even number of blocks in the standard Levi factor of the symplectic or of the orthogonal Lie algebra) we choose a_i entries starting alternatingly from the lower left corner resp. from the upper right corner. In terms of the roots of the rectangles, we choose the lowest resp. the highest root together with the next $a_i - 1$ roots on the skew-diagonal. We describe the former case: In rectangle $R_{1,2}$ the roots are α_{a_1}, α_{a_1-1,a_1,a_1+1}, $\alpha_{a_1-2,\ldots,a_1+2}$, up to $\alpha_{1,\ldots,a_1,\ldots,2a_1-1)}$ where $\alpha_{i,i+1,\ldots,i+k}$ stands for the sum $\sum_{l=0}^{k} \alpha_{i+l}$.

In R_r for $\mathfrak{sp}_{2n}$ we choose all skew-diagonal entries. They correspond to the long roots of C_n whose root spaces lie in $\mathfrak{g}_1$.

If the Lie algebra is $\mathfrak{so}_{2n}$, the last rectangle R_r is skew-symmetric around the skew-diagonal, so we start in the lower left or upper right corner just above the skew-diagonal: We choose the entries above (i.e. around) the skew-diagonal, leaving out every second. Say $r + 1$ is odd. Then our choice corresponds to the roots α_{n-1}, $\alpha_{n-3} + 2\alpha_{n-2} + \alpha n - 1 + \alpha_n$, $\alpha_{n-5} + 2\alpha_{n-4} + \cdots + 2\alpha_{n-2} + \alpha_{n-1} + \alpha_n$, etc.

(B)-cases. If there is an odd number of blocks in $\mathfrak{g}_0$ we refined our choice of elements in the rectangles and started both from the upper right and the lower left corners. The pattern is similar as in case (A). The only difference is if for some i a_i is odd and a_{i+1} is even, $\mathfrak{g} = \mathfrak{so}_N$. In that case we choose $a_i + 1$ entries in $R_{i,i+1}$.

To examine these roots better, let us denote the positive roots of $\mathfrak{g}$ by R^+. Let the roots of the ith irreducible component of $\mathfrak{g}_1$ under the adjoint group of the Levi factor be R_1^i. Let S_1^i be the subsets of the roots whose root spaces contribute to X_R.

On the matrix level, the roots in S_1^i are the entries of X_i, Y_i for $i < r$ and of X_r in the (A)-case, they are the entries of X_i, Y_i for $i \leq r$ in the (B)-case and they are the entries of X_i in the case of $\mathfrak{sl}_{n+1}$.

Lemma 5.1. *Let α, β be elements of S_1^i, $\gamma \in S_1^j$. Then*

(1) $\alpha - \beta$ is not a root of $\mathfrak{g}$ except if $\mathfrak{g}$ is an orthogonal Lie algebra and the parabolic subalgebra is of non-simple type.

(2) $\alpha - \gamma$ is not a root of $\mathfrak{g}$.

For the definition of the parabolic subalgebras of non-simple type see Definition 4.5.

Proof. (1) Except for the non-simple cases, the recipes only choose roots that are in different rows/columns of $R_{i,i+1}$.

(2) The roots of two different rectangles lie always in different rows. So in particular, their difference is not a root of $\mathfrak{g}$. $\square$

Lemma 5.2. *Let $\mathfrak{p} \subset \mathfrak{g}$ be a nice parabolic subalgebra with unimodal sequence of block lengths in $\mathfrak{g}_0$. The roots of S_1^i commute exactly in the following cases:*

(i) $\mathfrak{g}$ is of type A or C case (A), $1 \leq i \leq r$,
(ii) $\mathfrak{g}$ is of type C case (B) and $i < r$,
(iii) $\mathfrak{g}$ is of type $\mathfrak{so}_{2n}$ case (A), $1 \leq i \leq r$,
(iv) $\mathfrak{g} = \mathfrak{so}_N$ case (B), $i < r$ and if a_i is odd there is no $k > i$ with a_k even.

Proof. In all these cases, the elements of X_i (and of Y_i) are either diagonal neighbors (from lower left to upper right) or further apart from each other. If $\mathfrak{g}$ is $\mathfrak{sl}_{n+1}$ or if $i < r$, the roots of X_i (and of Y_i) are the roots of $\mathfrak{gl}_N$. Say we are dealing with entries (1,1) and (2,2) of the rectangle $R_{i,i+1}$. They correspond to the roots α_l and $\alpha_{l-1} + \alpha_l + \alpha_{l+1}$ for some l. Their sum is not a root of $\mathfrak{g}$. In particular, $(\alpha_l, \alpha_{l-1} + \alpha_l + \alpha_{l+1}) = 0$.

It remains to consider $i = r$, $\mathfrak{g} = C_n$ or $= D_n$ with an even number of blocks in the standard Levi factor. In the former case, the recipe picks all long roots of $\mathfrak{g}_1$. They do commute. In the latter case, the chosen entries of $R_{r,r+1}$ all contain α_n. In particular, no two of them can add up to a root of $\mathfrak{g}$. $\square$

The following result describes the structure of the roots of S_1^r in the symplectic and orthogonal Lie algebras if the standard Levi factor of $\mathfrak{p}$ has an odd number of blocks.

Lemma 5.3. *Let $\mathfrak{p}$ be a nice parabolic subalgebra of $\mathfrak{sp}_{2n}$ or of $\mathfrak{so}_N$, given by the unimodal sequence $(a_1, \ldots, a_r, a_{r+1}, a_r, \ldots, a_1)$.*

Then the roots in S_1^r span

$$\begin{cases} \frac{a_r}{2} \text{ factors } A_2 & \text{if } a_r \text{ is even,} \\ \frac{a_r-1}{2} \text{ factors } A_2 \text{ and one factor } A_1 & \text{if } a_r \text{ is odd and } \mathfrak{g} = \mathfrak{sp}_{2n}. \end{cases}$$

Proof. Recall that the roots chosen by the recipes form a rectangle of the form $\begin{bmatrix} & {}^{J_{B_r}} \\ {}_{J_{b_r}} & \end{bmatrix}$ where $b_r = \lfloor a_r/2 \rfloor$ and $B_r = \lceil a_r/2 \rceil$. We consider the case of C_n, with sequence (a_1, a_2, a_1). The case of the orthogonal Lie algebras is similar and thus is left to the reader.

Let $1 \leq l \leq a_1$. Then the lth entry from the lower left corner corresponds to the root $\alpha_{a_1-l+1} + \cdots + \alpha_{a_1} + \cdots + \alpha_{a_1+l-1}$. And the lth entry from the upper right corner corresponds to the root $\alpha_l + \cdots + \alpha_{a_1} + \cdots + \alpha_{a_1+l-1} + 2\alpha_{a_1+l} + \cdots + 2\alpha_{n-l} + \alpha_{n-l+1}$. Adding these two gives a root of $\mathfrak{g}_2$. Then one checks that the lth entry from the lower left corner and the kth entry from the upper right corner do not add up to a root of $\mathfrak{g}$ if $k \neq l$. $\square$

Now we have everything needed to understand the structure of $S_1 := \bigcup_i S_1^i$.

Proposition 5.4. *Let $\mathfrak{p} \subset \mathfrak{g}$ be a nice parabolic subalgebra, let S_1 be the roots that are involved in the construction of X_R. The elements of S_1 form a simple system of roots exactly in the following cases:*

$$\begin{cases} \mathfrak{g} \text{ is of type } A_n \text{ or } C_n, \\ \mathfrak{g} \text{ is orthogonal and } \mathfrak{p} \text{ is not of non-simple type.} \end{cases}$$

Proof. Lemma 5.1 shows that no two roots of the set S_1 subtract from each other.
□

Corollary 5.5. *Let $\mathfrak{p} \subset \mathfrak{g}$ be a nice parabolic subalgebra, let $\mathfrak{g}$ have rank n. If $\mathfrak{g}$ is a special or a symplectic Lie algebra, or if $\mathfrak{p} \subset \mathfrak{g}$ is not of non-simple type, then S_1 consists of at most n roots.*

6. THE EXCEPTIONAL LIE ALGEBRAS

If $\mathfrak{g}$ is one of the exceptional Lie algebras, we may also look for representatives of a Richardson element that involve as few root spaces as possible. It turns out that in many cases, the support of such an element does not form a simple system of roots. We list all parabolic subalgebras of the exceptional Lie algebras where there actually exists a Richardson element whose support forms a simple system of roots. In each of these cases we give an explicit representative. Thus we complete the construction of Richardson elements with simple support for the simple Lie algebras.

We sketch the way to find the parabolic subalgebras with such a Richardson element and how to obtain a representative with simple support: By Theorem 2.1 we know that the dimension of a Richardson orbit $\mathcal{O}_X$ for $\mathfrak{p}$ is equal to $\dim \mathfrak{g} - \dim \mathfrak{m}$. In [BW], all nice parabolic subalgebras of the exceptional Lie algebras are listed. For each of these, we compute the dimension of the Levi factor and thus obtain the dimension of the Richardson orbit. Using the lists of nilpotent orbits of [CM], one finds the Bala-Carter labels for nilpotent orbits of the given dimension. The Bala-Carter label essentially gives the group for which the Richardson element is a regular nilpotent element. Note that in some cases there is ambiguity (i.e. there exist two or three orbits of the same dimension). An analysis of the roots of the irreducible components of $\mathfrak{g}_1$ then helps to determine the correct label. Once the label is obtained, we have to look for such a simple system of roots among the roots of $\mathfrak{g}_1$.

We use the Bourbaki ordering of the simple roots (so in the case of E_n, α_2 is the root at the top, attached to the third simple root of the horizontal string). For simplicity, we denote the positive roots of the exceptional Lie algebras by writing α_I where the index I lists the simple roots involved. For example, the root $\alpha_1 + \alpha_2 + \alpha_3 + 2\alpha_4 + \alpha_5$ of E_6 will be written as α_{1234^25}. The parabolic subalgebra will be described by an n-tuple $(u_1, \ldots, u_n) \in \{0,1\}^n$: an entry 0 at position i means that the ith simple root is a root of the (standard) Levi factor, an entry 1 at position j means that the jth simple root is a root of the nilradical of $\mathfrak{p}$ (as in [BW]).

Note that there are two trivial cases: if all simple roots are roots of the Levi factor of $\mathfrak{p}$ (i.e. $\mathfrak{p}$ is given by the tuple $(0, \ldots, 0)$), then the Richardson orbit is the zero orbit. If the Levi factor of $\mathfrak{p}$ is the Cartan subalgebra (i.e. $\mathfrak{p}$ is given by the tuple $(1, \ldots, 1)$), then the nilpotent element $X = \sum_{\alpha \in \Delta} X_\alpha$ is a Richardson element (where Δ is the set of simple roots of $\mathfrak{g}$). In G_2 the only parabolic subalgebras

where the (minimal) support of a Richardson element forms a simple system of roots are the two trivial cases. So the first interesting case is F_4. In type F_4, we have several parabolic subalgebra where there exists a Richardson element with a simple support. In the following tables we list all nice parabolic subalgebras of the simple exceptional Lie algebras that have a Richardson element whose support is a simple system of roots. For each of these we give the support of such a Richardson element X_0 in the second column.

$$\mathfrak{p} \subset \mathfrak{f}_4 \qquad \text{Support of } X_0$$

$$(1,0,0,0) \qquad \alpha_1 \longrightarrow \alpha_{1 2^3 3^4 4^2}$$

$$(0,0,0,1) \qquad \alpha_{1234} \longrightarrow \alpha_{2 3^2 4}$$

$$(1,1,0,0) \qquad \alpha_{234} \longrightarrow \alpha_1 \Longrightarrow \alpha_{2 3^2}$$

For E_6 we list the parabolic subalgebras up to the symmetry of the Dynkin diagram given by interchanging the vertices $1 \leftrightarrow 6$ and $3 \leftrightarrow 5$.

$$\mathfrak{p} \subset \mathfrak{e}_6 \qquad \qquad \text{Support of } X_0$$

$(1,1,1,0,1,0)$
$$\begin{array}{c} \alpha_{56} \\ | \\ \alpha_{45} - \alpha_2 - \alpha_{34} - \alpha_1 \end{array}$$

$(1,1,0,1,0,1)$
$$\begin{array}{c} \alpha_{56} \\ | \\ \alpha_{345} - \alpha_2 - \alpha_4 - \alpha_{13} \end{array}$$

$(1,1,0,0,0,1) \qquad \alpha_{234} - \alpha_{456} - \alpha_{13} - \alpha_{245}$

$(1,1,0,0,0,0) \qquad \alpha_{2345} - \alpha_{134} - \alpha_{245}$

$(1,0,1,1,0,1)$
$$\begin{array}{c} \alpha_{24} \\ | \\ \alpha_6 - \alpha_{45} - \alpha_3 - \alpha_1 \end{array}$$

$(1,0,1,0,0,1) \qquad \alpha_{345} - \alpha_1 - \alpha_{234} - \alpha_{56}$

$(1,0,1,0,0,0) \qquad \alpha_{234} - \alpha_1 - \alpha_{345}$

$(1,0,0,1,0,0) \qquad \alpha_{345} - \alpha_1 - \alpha_{234} - \alpha_{456} \qquad \alpha_{245}$

$(1,0,0,0,0,1) \qquad \alpha_{1234} - \alpha_{3456} \qquad \alpha_{1345} - \alpha_{2456}$

$(1,0,0,0,0,0) \qquad \alpha_{1234} \qquad \alpha_{1345}$

$(0,1,0,1,0,0)$
$$\begin{array}{c} \alpha_{345} \\ | \\ \alpha_{134} - \alpha_2 - \alpha_{456} \end{array}$$

$(0,1,0,0,0,0) \qquad \alpha_{1 2^3 2^4 3^5 2^6} - \alpha_2$

$(0,0,1,0,0,0) \qquad \alpha_{1234} - \alpha_{3456} \qquad \alpha_{1345} \qquad \alpha_{2345}$

$\mathfrak{p} \subset \mathfrak{e}_7$	Support of X_0

$(1,0,1,1,0,1,0)$

$$\alpha_1$$
$$|$$
$$\alpha_{56} - \alpha_4 - \alpha_3 - \alpha_{245} - \alpha_{67}$$

$(1,0,1,0,0,1,0)$

$$\alpha_{456}$$
$$|$$
$$\alpha_{345} - \alpha_1 - \alpha_{234} - \alpha_{567}$$

$(1,0,1,0,0,0,0)$

$$\alpha_{23456}$$
$$|$$
$$\alpha_{234^25} - \alpha_1 - \alpha_{34567}$$

$(1,0,0,0,0,1,1)$

$$\alpha_{1234} - \alpha_{3456} - \alpha_7 - \alpha_{2456} - \alpha_{1345}$$

$(1,0,0,0,0,1,0)$

$$\alpha_{1234} - \alpha_{34567} - \alpha_{2456} - \alpha_{1345}$$

$(1,0,0,0,0,0,1)$

$$\alpha_{567} - \alpha_{1234} - \alpha_{234^25^267} \qquad \alpha_{1345}$$

$(1,0,0,0,0,0,0)$

$$\alpha_1 - \alpha_{12^23^34^45^36^27}$$

$(0,1,0,0,0,0,0)$

$$\alpha_{2456} - \alpha_{123^24^25^267} \qquad \alpha_{1234} \qquad \alpha_{2345} \qquad \alpha_{123^24^25^26^27}$$

$(0,0,1,0,0,0,1)$

$$\alpha_{2345} - \alpha_{4567} - \alpha_{1234} - \alpha_{3456} \qquad \alpha_{1345}$$

$(0,0,0,1,0,1,0)$

$$\alpha_{345} - \alpha_{67} - \alpha_{245} - \alpha_{134} - \alpha_{56} - \alpha_{234}$$

$(0,0,0,1,0,0,0)$

$$\alpha_{3456} - \alpha_{1234} - \alpha_{4567} - \alpha_{2345} \qquad \alpha_{1345} - \alpha_{2456}$$

$(0,0,0,0,1,0,1)$

$$\alpha_{1345} - \alpha_{2456} - \alpha_7 - \alpha_{3456} \qquad \alpha_{2345}$$

$(0,0,0,0,1,0,0)$

$$\alpha_{123^24^2567} - \alpha_{2456} - \alpha_{1345} \qquad \alpha_{2345} - \alpha_{4567} \qquad \alpha_{3456}$$

$(0,0,0,0,0,1,0)$

$$\alpha_{56} - \alpha_{123^24^35^267} \qquad \alpha_{67} - \alpha_{12^23^24^35^26}$$

$(0,0,0,0,0,0,1)$

$$\alpha_7 \qquad \alpha_{234^25^26^27} \qquad \alpha_{1^22^23^34^45^36^27}$$

$\mathfrak{p} \subset \mathfrak{e}_8$ Support of X_0

$(1,0,0,0,0,1,1,1)$

$$
\begin{array}{c}
\alpha_8 \\
| \\
\alpha_{1234} - \alpha_{3456} - \alpha_7 - \alpha_{2456} - \alpha_{1345}
\end{array}
$$

$(1,0,0,0,0,1,0,0)$

$$\alpha_{34567} - \alpha_{1234} - \alpha_{45678} - \alpha_{23456} - \alpha_{1345} - \alpha_{24567}$$

$(1,0,0,0,0,0,1,1)$

$$
\begin{array}{c}
\alpha_{1234^2 5} \\
| \\
\alpha_{24567} - \alpha_8 - \alpha_{34567} - \alpha_{123456}
\end{array}
$$

$(1,0,0,0,0,0,0,1)$

$$\alpha_{1234} - \alpha_{345678} - \alpha_{1345} - \alpha_{245678}$$

$(1,0,0,0,0,0,0,0)$

$$\alpha_{1234} - \alpha_{12^2 3^2 4^4 5^4 6^3 7^2 8} \qquad \alpha_{1345} - \alpha_{12^2 3^3 4^4 5^3 6^3 7^2 8}$$

$(0,1,0,0,0,0,0,1)$

$$
\begin{array}{c}
\alpha_{2345} \\
| \\
\alpha_{1234} - \alpha_{45678} - \alpha_{123^2 4^2 5^2 6^2 7} \qquad \alpha_{2456} - \alpha_{123^2 4^3 5^2 6^7}
\end{array}
$$

$(0,0,1,0,0,0,1,0)$

$$
\begin{array}{c}
\alpha_{567} \\
| \\
\alpha_{1345} - \alpha_{678} - \alpha_{2345} - \alpha_{1234} - \alpha_{3456}
\end{array}
$$

$(0,0,0,0,1,0,0,1)$

$$
\begin{array}{c}
\alpha_{1345} \\
| \\
\alpha_{3456} - \alpha_{78} - \alpha_{2456} - \alpha_{123^2 4^2 567} \qquad \alpha_{2345} - \alpha_{4567}
\end{array}
$$

$(0,0,0,0,0,1,0,0)$

$$\alpha_{24567} - \alpha_{13456} - \alpha_{12^2 3^2 4^3 5^2 678} - \alpha_{34567} \qquad \alpha_{23456} - \alpha_{45678}$$

$(0,0,0,0,0,0,1,1)$

$$
\begin{array}{c}
\alpha_{123^2 4^2 567} \\
| \\
\alpha_{234^2 5^2 6^2 7} \text{———} \alpha_8 \text{———} \alpha_{1234^2 5^2 67}
\end{array}
$$

$(0,0,0,0,0,0,0,1)$

$$\alpha_8 - \alpha_{1^2 2^3 3^4 4^6 5^5 6^4 7^3 8}$$

References

[BHRR] T. Brüstle, L. Hille, C. M. Ringel, G. Röhrle, *The Δ-filtered modules without self-extensions for the Auslander algebra of $k[T]/\langle T^n \rangle$*, Algebr. Represent. Theory 2 (1999), no. 3, 295–312. MR1715751 (2000j:16022)

[BW] K. Baur, N. R. Wallach, *Nice parabolic subalgebras of reductive Lie algebras*, Represent. Theory 9 (electronic), Amer. Math. Soc. (2005), 1–29.

[CM] D.H. Collingwood, W.M. McGovern, *Nilpotent orbits in semisimple Lie algebras*, Van Nostrand Reinhold Mathematics Series. Van Nostrand Reinhold Co., New York, 1993. MR1251060 (94j:17001)

[GR] S. Goodwin, G. Röhrle, *Prehomogeneous spaces for parabolic group actions in classical groups*, Journal of Algebra 276 (2004), 383–398. MR2054402

[GW] R. Goodman, N.R. Wallach, *Representations and invariants of the classical groups*, Cambrigde University Press, Cambridge 1998. MR1606831 (99b:20073)

[K] B. Kostant, *On Whittaker vectors and representation theory*, Invent. Math. 48 (1978), no. 2, 101–184. MR0507800 (80b:22020)

[L] T.E. Lynch, *Generalized Whittaker vectors and representation theory*, Ph.D. Thesis, 1979, MIT Libraries (http://theses.mit.edu)

[R]   R.W. RICHARDSON, *Conjugacy classes in parabolic subgroups of semisimple algebraic groups*, Bulletin London Math. Society 6 (1974), 21–24. MR0330311 (48 #8648)
[W]   N.R. WALLACH, *Holomorphic continuation of generalized Jacquet integrals for degenerate principal series*, in preparation.

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