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THE BURGER-SARNAK METHOD AND OPERATIONS ON THE UNITARY DUAL OF GL(n)

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Abstract. We study the effect of restriction to Levi subgroups, induction from Levi subgroups, and tensor product, on unitary representations of GL(n) over a local field k. These results give partial answers to questions raised by Clozel.

1. Introduction

In a recent paper [4], Clozel observed that Arthur's conjectures had some striking consequences for local harmonic analysis. Roughly speaking, the Arthur parameterization partitions (some of) the unitary dual of a reductive group over a local field. Clozel observed that this partition should behave in a stable fashion under restriction to or induction from proper subgroups.

The purpose of the present note is to clarify these consequences for GL(n): we compute the correspondences of unitary representations suggested by Clozel in a weakened sense, providing justification for the conjecture of the author that was stated in [4]. Clozel's motivation stemmed in part from the work of Burger, Li and Sarnak [2], which showed that the "automorphic spectrum" is stable under certain operations arising from restriction or induction; here we explicate quite precisely what these operations are in the case of GL(n).

The argument is very naive and only uses Mackey theory in the crudest way to compute with representations with nice models, and a global argument (essentially due to Clozel, which we have explicated) to deal with the general case. We are unable to rule out, in general, the presence of complementary series when one restricts or inducts a representation of "Arthur type" (see Section 2 for definition).

More precisely, our primary goal is to understand, for GL(n), the effect on the unitary dual of the following operations: *restriction to a Levi subgroup, induction from a Levi subgroups* and *tensor product*. There are other operations corresponding to other embeddings of (products of) groups of type A into other such groups, but (see remarks a little later), the three cases discussed seem the most interesting. In some special cases these have been considered: for example, the spectral decomposition of $L^2(\mathrm{SL}(a,\mathbb{R}) \times \mathrm{SL}(b,\mathbb{R}) \backslash \mathrm{SL}(a+b,\mathbb{R}))$ has been computed as part of the (general) Plancherel formula for symmetric spaces; of course the Plancherel formula gives much more precise information than the discussion here.

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The situation for other groups is quite unclear to the author. In general, given an embedding $H \hookrightarrow G$ of algebraic groups, the relation between the *dual groups* $\hat{H}$ and $\hat{G}$ seems unclear. However, for the situations considered here, where H is either a Levi subgroup of G or a diagonal copy of G inside $G \times G$, the relationship between the dual groups is much clearer. It may be, then, that in this restricted setting one can formulate a general principle.

We note that if H is "small" compared to G, restriction of unitary representations from G to H very quickly "contracts" the entire spectrum to temperedness. (Thus, in the case of $\mathrm{GL}(n)$, the cases under consideration are perhaps the most interesting; other subgroups $H \subset \mathrm{GL}(n)$ tend to be much smaller.) One can check temperedness in any given case by using the temperedness criterion due to Cowling, Haagerup and Howe [6, Thm 1]. A similar situation occurs with induction: if $H \subset G$ is very small the induction of any H-representation to G tends to be tempered. However, in the induction case, (although one can verify temperedness in many cases by *ad hoc* methods) the author does not know a general technique. For a very simple result of this nature see Corollary 1.

Acknowledgments. This paper arose out of an attempt to understand the ideas explained by L. Clozel in his 2002 Park City lectures; to appear [3]. I have benefited greatly from a visit to Orsay and discussions with Professor Clozel; I thank him for his generosity in sharing his ideas.

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Also E. Lapid and J. Rogawski independently have proved most of these results (their method seems somewhat sharper than our rather clumsy method in Section 3).

2. Enunciation of conjectures

2.1. **Preliminaries.** Throughout this paper, we shall use "representation" to always mean *unitary* representation, and by "induction" we shall always mean *unitary* induction, as in [12, Sec. 3.2].

Let $G = \mathrm{GL}(n)$, and let k be a local field of characteristic 0. As usual $\widehat{G(k)} = \widehat{\mathrm{GL}(n,k)}$ will denote the unitary dual of the topological group $G(k)$; it is endowed with the Fell topology (see [8] or [22, Section 1]; we give a very brief discussion of weak containment in this section.)

We also use several (deep) results from representation theory and automorphic forms without explicit comment: we shall use the results of Tadić [18] on the classification of the unitary dual of $\mathrm{GL}(n)$ over a nonarchimedean field and on its topology [19], and the corresponding results of Vogan [21] in the archimedean case.

To be precise, we only need the fact that the Fell topology coincides with the natural topology on the space of Langlands parameters if one stays away from the endpoints of complementary series. This enters into our paper only during Proposition 3. The required assertion follows from [19] in the nonarchimedean case, and in the archimedean case it should follow from the results of Miličić; in any case, enough of it for our purposes can be deduced from the arguments contained in [20], proof of Theorem A.3.

Finally, we shall use in the final section, Section 3.6, the results of Luo, Rudnick and Sarnak from [11]. Although originally written only for unramified places, these are valid even at ramified places; see [15, Prop 3.3] for a statement and proof in the general setting. In our context, they will guarantee that the automorphic spectrum for GL_n (see definition prior to Remark 1) stays away from the endpoints of complementary series, avoiding "pathologies" of the Fell topology.

We now review, very briefly, some notions about unitary representations that will be used throughout the text, giving references to more detailed accounts. First, the notion of *weak containment*: if G_1 is a separable locally compact topological group, a representation σ is *weakly contained* in a representation τ if every diagonal matrix coefficient of σ can be approximated, uniformly on compacta, by a convex combination of diagonal matrix coefficients of τ. That is, given $\varepsilon > 0$, a compact subset $K \subset G_1$, and any vector u in the space of σ, there exist vectors $v_1, \dots, v_n$ in the space of τ so that

$$\sup_{g \in K} \left| \langle \sigma(g)u, u\rangle - \sum_{i=1}^{n} \langle \tau(g)v_i, v_i\rangle \right| < \varepsilon.$$

We refer to [17, Sec. 2] for a discussion of the other notion of weak containment that occurs in the literature, and its relation with the definition just given.

Now suppose that G_1 is the group of k-points of a reductive algebraic group, or a product of such groups. Then any unitary representation τ may be disintegrated into irreducibles as

$$\tau = \int_{x \in \widehat{G_1}} m_x \pi_x \, d\mu(x), \tag{1}$$

for certain multiplicities $m_x \in \mathbb{N} \cup \{\infty\}$ and an appropriate measure μ on $\widehat{G_1}$. We refer to [12, Sec. 2.4] or [7, Chapter 8] for a more detailed treatment of the notions of Hilbert direct integral and disintegration; in any case we will not need to work with such disintegrations during our main argument.

This decomposition of (1) is essentially unique ([1], [7, Thm 15.5.2], [9, Thm 8.1]), in the following sense: the measure class of μ is uniquely determined, and the function $x \mapsto m_x$ is determined up to modification on a μ-null set. In particular, one can unambiguously refer to the *support* of μ, a certain closed subset of $\widehat{G_1}$.

Abusing notation slightly, we shall refer to support of the measure μ as the "support of τ." It is a certain subset of the unitary dual that is associated to the unitary representation τ. Intuitively speaking, it should be thought of as the "representations that occur in τ."

There is an alternate way of defining the support of τ, which adds some rigour to the phrase "representations that occur in τ." Namely, one knows that an irreducible representation σ is weakly contained in τ iff σ belongs to the support of τ: [7, Prop. 8.6.8]. Thus the support of τ consists precisely of those irreducibles $\sigma \in \widehat{G_1}$ that *weakly occur* in τ. From this one deduces from the definitions the following slight strengthening: a (not necessarily irreducible) representation σ is weakly contained in τ precisely when the support of σ is contained in the support of τ.

The notion of "automorphic dual" is used in [2] for semisimple groups; we will extend its definition to reductive groups in the following way. Let F be a number field and S a finite subset of places. Set $F_S = \prod_{v \in S} F_v$. Let G_1 be a reductive algebraic group over F. We let $\widehat{G_1(F_S)}$ denote the unitary dual of $G_1(F_S)$, and

$\widehat{G_1(F_S)}_{Aut} \subset \widehat{G_1(F_S)}$ the "automorphic dual," i.e. the support (in $\widehat{G_1(F)}_S$) of $L^2(G_1(F)\backslash G_1(\mathbb{A}_F))$, considered as a $G_1(F_S)$-representation.

The following remark allows us to reduce questions about the automorphic dual of GL_n to questions about the automorphic dual of the semisimple group SL_n.

Remark 1. An irreducible representation $\pi \in \widehat{G(F_S)}$ belongs to the automorphic dual if and only if all (equivalently one of) its irreducible constituents, when restricted to $\mathrm{SL}(n, F_S)$, belong to the automorphic dual $\widehat{SL(n, F_S)}_{Aut}$.

Proof (Sketch). First, the "if" direction. The idea is entirely due to Burger and Sarnak. Suppose that $\pi \in \widehat{G(F_S)}$, and let $\sigma \in \widehat{\mathrm{SL}_n(F_S)}_{Aut}$ be an irreducible constituent of the restriction of π to $\mathrm{SL}_n(F_S)$. It is easy to check that π is contained weakly in the induction from $\mathrm{SL}_n(F_S)$ to $\mathrm{GL}_n(F_S)$ of σ. This implies that π occurs weakly in $L^2(\Gamma_S\backslash \mathrm{GL}_n(F_S))$, where $\Gamma_S \subset \mathrm{SL}_n(F_S)$ is an S-arithmetic subgroup. Without loss we may assume $\Gamma_S = \{g \in \mathrm{SL}_n(\mathcal{O}_{F,S}) | g \equiv 1 \bmod \mathfrak{a}\}$ where $\mathfrak{a}$ is an ideal of $\mathcal{O}_{F,S}$, the ring of S-integers in F. Let $\mathfrak{p}$ be any ideal of $\mathcal{O}_{F,S}$ and, for $j \geq 1$ integral, set $\Gamma'_{j,S} = \{g \in \mathrm{GL}_n(\mathcal{O}_{F,S}) | g \equiv 1 \bmod \mathfrak{a}, \det(g) \equiv 1 \mod \mathfrak{p}^j\}$. Then $\Gamma'_{j,S}$ is an S-arithmetic subgroup of $\mathrm{GL}_n(F_S)$ and $\bigcap_{j\geq 1} \Gamma'_{j,S} = \Gamma_S$. Then one sees that matrix coefficients of the $\mathrm{GL}_n(F_S)$-representation $L^2(\Gamma_S\backslash \mathrm{GL}_n(F_S))$ may be approximated by those of $\bigoplus_{j\geq 1} L^2(\Gamma'_{j,S}\backslash \mathrm{GL}_n(F_S))$. It follows that $\pi \in \widehat{\mathrm{GL}_n(F_S)}_{Aut}$.

Next, the "only if" direction. Let Z_S be the center of $\mathrm{GL}_n(F_S)$, and let Γ be a congruence subgroup of $\mathrm{GL}_n(\mathcal{O}_{F,S})$. Let $\det(\Gamma) \subset F_S^*$ be the subgroup of determinants of elements in Γ, let $\{z_1, \ldots, z_n\}$ be a set of representatives in F^* for $F_S^*/((F_S^*)^n \det(\Gamma))$, and choose $\delta_i \in \mathrm{GL}_n(F)$ with $\det(\delta_i) = z_i$. Then $\mathrm{GL}_n(F_S)$ is the disjoint union of $\Gamma \cdot \delta_i \cdot Z_S \mathrm{SL}_n(F_S)$. The L^2 space of functions on $\Gamma\backslash \mathrm{GL}_n(F_S)$ that are supported in $\Gamma \cdot \delta_i \cdot Z_S \mathrm{SL}_n(F_S)$ is isomorphic, as an $\mathrm{SL}_n(F_S)$-representation, to a countable direct sum of $L^2(\delta_i^{-1}\Gamma\delta_i \cap \mathrm{SL}_n(F_S)\backslash \mathrm{SL}_n(F_S))$. It follows that the restriction of any $\pi \in \widehat{\mathrm{GL}_n(F_S)}_{Aut}$ belongs to $\widehat{\mathrm{SL}_n(F_S)}_{Aut}$. □

We recall the Burger-Sarnak principle [2, Theorem 1]. We note that a careful treatment has been given in the S-arithmetic context by Clozel and Ullmo [5].

Proposition 1. *Let $G_1 \subset G_2$ be semisimple algebraic groups defined over F. Suppose that $\pi \in \widehat{G_2(F_S)}_{Aut}$ and $\sigma \in \widehat{G_1(F_S)}_{Aut}$. Then any irreducible representation of $G_2(F_S)$ weakly contained in the induction of σ to $G_2(F_S)$ is contained in $\widehat{G_2(F_S)}_{Aut}$. Any irreducible representation of $G_1(F_S)$ weakly contained in the restriction of π to $G_1(F_S)$ is contained in $\widehat{G_1(F_S)}_{Aut}$.*

Note first that Proposition 1 gives a result for tensor products, by application to the situation where G_1 is diagonally embedded in $G_2 = G_1 \times G_1$. Note also that, although the proposition is stated for semisimple groups, by combining Proposition 1 with Remark 1 (and slight variants thereof) we may extend the applicability of the proposition to the situation involving $\mathrm{GL}(n)$ or products of $\mathrm{GL}(n)$. In particular, it follows immediately from combining Proposition 1 and Remark 1 that restriction to a Levi subgroup, induction from a Levi subgroup, and tensor product preserve the automorphic dual for $\mathrm{GL}(n)$. (The first and last assertions are almost immediate; for induction, one uses the ideas involved in the proof of Remark 1.)

We shall make free use of the word "tempered" for reductive groups; a representation π is tempered if its support consists only of tempered representations. (For definitions, a discussion, and a careful treatment of the basic properties of this notion for reductive groups over a local field, we refer to [16, Sec 2.4].)

2.2. **Type.** We associate to each $\pi \in \widehat{G(k)}$ a partition of n, the *$SL(2)$-type* or just *type* of π. We caution that the type (as we define it) is defined for *all* $\pi \in \widehat{G(k)}$, not merely those attached to Arthur parameters; this extension is convenient.

We recall the classification of the unitary dual. We follow notation from the paper of Tadić, [18]; specifically, his Theorem D, which is (as noted in [18]) also valid at archimedean places. Let m, j be positive integers. Let δ be a discrete series representation of $\mathrm{GL}(m, k)$, i.e. a representation whose matrix coefficients are square integrable modulo the center. The representation of $\mathrm{GL}(mj, k)$ parabolically induced from $(\delta|\det|^{(j-1)/2} \otimes \delta|\det|^{(j-2)/2} \otimes \cdots \otimes \delta|\det|^{(1-j)/2})$ has a unique irreducible quotient; we denote it by $u(\delta, j)$.

Suppose $0 < \alpha < 1/2$. The representation of $GL(2mj, k)$ parabolically induced from $u(\delta, j)|\det|^{\alpha} \times u(\delta, j)|\det|^{-\alpha}$ is unitarizable; we denote it by $u(\delta, j)[\alpha, -\alpha]$.

Any unitary representation of $\mathrm{GL}(n, k)$ is unitarily induced from representations of type $u(\delta, j)$ or $u(\delta, j)[\alpha, -\alpha]$, and this expression is unique up to permutation ([18] and [21]).

We define $\widehat{\mathrm{GL}(n, k)}_{Ar} \subset \widehat{\mathrm{GL}(n, k)}$ (the part of the spectrum associated to Arthur packets) to consist of those π which do not involve any $u(\delta, j)[\alpha, -\alpha]$; we say an element belonging to this subset is *of Arthur type.*

We shall use angle brackets $\langle ? \rangle$ to denote partitions (and, more generally, unordered sequences of integers).

Definition 1. The *type* of an irreducible representation $\pi \in \widehat{\mathrm{GL}(n, k)}$ is the partition of n specified by the following two conditions:

(1) If π is unitarily induced from $\pi_1 \in \widehat{\mathrm{GL}(a, k)}$ and $\pi_2 \in \widehat{\mathrm{GL}(b, k)}$, with $a + b = n$, then the type of π is obtained by concatenating the types of π_1 and π_2.
(2) Suppose δ is a discrete series representation of $\mathrm{GL}(m, k)$. The type of $u(\delta, j)$ is $\langle j, j, \ldots, j \rangle$ (with m js).
(3) The type of $u(\delta, j)[\alpha, -\alpha]$ is $\langle j, j, j, \ldots, j \rangle$ (with $2m$ js).

Finally, we say a (not necessarily irreducible) representation π has type τ if its support consists only of irreducible representations with type τ.

Note that the trivial representation has type $\langle n \rangle$ and any tempered representation has type $\langle 1, 1, 1, \ldots, 1 \rangle$. The converse is true for representations of Arthur type: any $\pi \in \widehat{\mathrm{GL}(n, k)}_{Ar}$ with type $\langle 1, \ldots, 1 \rangle$ is tempered. On the other hand, a complementary series for $\mathrm{GL}(2)$ is also assigned the type $\langle 1, 1 \rangle$.

The notion of type extends to products of $\mathrm{GL}(n)$s in a natural way: if π is a representation of $\mathrm{GL}(n_1, k) \times \cdots \times \mathrm{GL}(n_r, k)$, and τ_i is a partition of n_i for each $1 \le i \le r$, then we say that π has type $(\tau_1, \ldots, \tau_r)$ if its restriction to $\mathrm{GL}(n_i, k)$ has type τ_i, for each $1 \le i \le r$.

2.3. **Results and Conjectures.** In this assertion, we state our main result about operations on SL(2)-types (Proposition 2). Part (3) of this proposition appeared as a conjecture in Clozel's article; [4, Conj. 4.1].

If $\sigma = (n_1, n_2, \dots, n_r)$ is a sequence of integers with $\sum_{i=1}^r n_i = n$, we denote by P_σ the corresponding parabolic and M_σ the corresponding Levi subgroup. (Thus P_σ is the stabilizer of a flag of type $(n_1, n_1+n_2, n_1+n_2+n_3, \dots)$, and M_σ consists of block-diagonal matrices, with blocks of size $n_1, n_2, \dots, n_r$.)

Note that the conjugacy class of a parabolic is determined by an ordered sequence of integers whereas its associate class (i.e. conjugacy class of the Levi) does not depend on the order. We shall generally use $(\dots)$ to denote ordered sequences. As previously noted, we use $\langle \dots \rangle$ to denote unordered sequences (i.e. multisets). (Recall that a multiset on the set X is a function from X to the nonnegative integers; we shall identify them with sequences of elements of X where order is unimportant.)

Let $\sigma = \langle a_j \rangle$ or $\sigma = (a_j)$ be either a multiset of integers or a sequence of integers, satisfying $\sum_j \max(0, a_j) \le n$. We denote by $\langle \sigma \rangle_n$ the partition of n consisting of all positive a_js together with enough 1s to ensure that the resulting sequence sums to n. For example, if $\sigma = \langle 2, 2, -3 \rangle$, then $\langle \sigma \rangle_7 = \langle 2, 2, 1, 1, 1 \rangle$ and $\langle \sigma \rangle_8 = \langle 2, 2, 1, 1, 1, 1 \rangle$.

By the *standard embedding* of $\mathrm{GL}(m)$ into $\mathrm{GL}(n)$ (for $m < n$) we mean the embedding into the "top left-hand corner," equivalently the embedding corresponding to the n-dimensional representation of $\mathrm{GL}(m)$ given by the sum of a standard representation and $n - m$ copies of the trivial representation.

We first enunciate a weak version of the conjecture. We shall prove this in the following section; thus it is titled as a proposition.

Proposition 2. *"Weak conjecture."*

(1) *Restriction to a Levi subgroup: Suppose that $\sigma = \langle n_i \rangle_{1 \le i \le I}$ is a partition of n, and $\mathrm{GL}(m)$ is embedded into $\mathrm{GL}(n)$ via the standard embedding. Suppose $\pi \in \widehat{\mathrm{GL}(n,k)}_{Ar}$ has type σ.*

   Then $Res^G_{\mathrm{GL}(m)}\pi$ has type $\langle (n_i + m - n)_{1 \le i \le I} \rangle_m$. (Here $\mathrm{Res}^G_{\mathrm{GL}(m)}$ denotes restriction from $G(k) = \mathrm{GL}(n,k)$ to $\mathrm{GL}(m,k)$.)

(2) *Induction from a Levi subgroup: suppose σ is as above, M_σ is the associated Levi subgroup, and for each $1 \le i \le I$ we are given a partition $\tau_i = \langle m_{i,k} \rangle_{1 \le k \le K_i}$ of n_i. Suppose $\pi_i \in \widehat{\mathrm{GL}(n_i,k)}_{Ar}$ has type τ_i.*

   Then $I^G_{M_\sigma}(\bigotimes_{1 \le i \le I} \pi_i)$ has type $\langle (m_{i,k} + n_i - n, 1 \le i \le I, 1 \le k \le K_i) \rangle_n$. (Here $I^G_{M_\sigma}$ denotes induction from the Levi subgroup $M_\sigma(k)$ to $G(k)$.)

(3) *Tensor product: suppose that $\sigma_1 = \langle n_i \rangle, \sigma_2 = \langle m_j \rangle$ are partitions of n, and set $\tau = \langle n_i + m_j - n \rangle_n$. Suppose $\pi_1, \pi_2 \in \widehat{\mathrm{GL}(n,k)}_{Ar}$ have types σ_1, σ_2 respectively. Then $\pi_1 \otimes \pi_2$ has type τ.*

Note that, "type" for products of $\mathrm{GL}(n)$ being defined as at the end of Sec. 2.2, then assertion (1) above is sufficient to determine the type of the restriction of π to a Levi subgroup (even though the assertion is only about the restriction to one component $\mathrm{GL}(m)$ of the Levi factor).

We give an example to help clarify the meaning of the proposition.

Example 1. Let π be a representation of $\mathrm{GL}(20, \mathbb{R})$ of Arthur type with type $(12, 6, 2)$. Then $\pi \otimes \pi$ has type $(4, 1, 1, 1, \dots, 1)$ (with 16 ones). The restriction of π to the Levi subgroup $\mathrm{GL}(15, \mathbb{R}) \times \mathrm{GL}(5, \mathbb{R})$ has type $(1, 1, 1, 1, 1)$ on the $\mathrm{GL}(5, \mathbb{R})$

factor and type $(7, 1, 1, 1, \ldots, 1)$ (with 8 ones) on the $\mathrm{GL}(15, \mathbb{R})$ factor. Finally, consider $\mathrm{GL}(20, \mathbb{R}) \times \mathrm{GL}(3, \mathbb{R})$ as a Levi subgroup of $\mathrm{GL}(23, \mathbb{R})$, and consider the induction of $\pi \otimes 1$. The resulting representation of $\mathrm{GL}(23, \mathbb{R})$ has type $(9, 3, 1, 1, 1, \ldots, 1)$ (with 11 ones).

One verifies easily that the assertions of this proposition are well defined and satisfy the required compatibilities: associativity of the tensor product, projection formula, and compatibility of the formula with "iterated" induction and restriction.

This proposition is termed the *weak* conjecture, since it does not rule out the possibility that one could obtain a complementary series representation occurring weakly in (e.g.) the tensor product of two representations of Arthur type. That this should not occur is the content of the following:

Conjecture 1 (Strong conjecture)**.** *The operations described in Proposition* 2 *preserve the part of the spectrum associated to Arthur packets.*

This conjecture would follow from the Ramanujan conjecture; cf. argument in Section 3.6. (Here we understand "Ramanujan conjectures" to mean "temperedness of cusp forms on $\mathrm{GL}(n)$.") Of course this does not seem a satisfactory way to approach it!

3. Proof of Proposition 2

3.1. **Preliminaries.** In this section we shall prove Proposition 2. The author does not know how to prove Conjecture 1 in full generality.

The strategy of proof is as follows. In Section 3.1 we recall some basic definitions and results. In Section 3.2 we enunciate Lemma 3, a weakened version of Proposition 2; this lemma is proved in Sections 3.3–3.5. We deduce Proposition 2 from Lemma 3 via a global argument, given in Section 3.6.

This argument is rather unsatisfactory; in particular, it would be desirable to deal directly with representations that do not have nice models. Of course the considerations of this section are very elementary.

In what follows, if $G_1 \subset G_2$ are locally compact separable topological groups, $I_{G_1}^{G_2}$ (respectively $\mathrm{Res}_{G_1}^{G_2}$) denotes induction (respectively restriction) of unitary representations. (Recall that *induction* always refers to induction of *unitary* representations, in the sense of Mackey [12, Sec. 3.2].)

We also sometimes write for brevity $\pi|_{G_1}$, instead of $\mathrm{Res}_{G_1}^{G_2}\pi$, for the restriction of the representation π to G_1.

We will make a number of arguments about induced representations which we will not give detailed justification for; the details all follow from the following facts. All may be verified directly from the definitions.

(1) Induction in stages: if $G_1 \subset G_2 \subset G_3$ and U is a unitary representation of G_1, then $I_{G_2}^{G_3} I_{G_1}^{G_2} U = I_{G_1}^{G_3} U$.
(2) Decomposition: If G_1, G_2 are subgroups of G, we will often analyze the composition $\mathrm{Res}_{G_1}^{G} I_{G_2}^{G}$ in the "naive" way, viz. by decomposing G into (G_1, G_2) double cosets. This is justified if G_1 and G_2 are *regularly related* in G, in the sense of Mackey (see [12, Sec. 3.4]).

    This is, in particular, true if there is a set $Z \subset G/G_2$ with measure 0 so that G_1 acts transitively on $G/G_2 - Z$. This is the only case in which we will use it.

(3) Continuity of induction: say $H \subset G$, and σ, τ are representations of H such that σ is weakly contained in τ; then $I_H^G \sigma$ is weakly contained in $I_H^G \tau$.
(4) Continuity of restriction: if π_1, π_2 are unitary representations of G so that π_1 is weakly contained in π_2, then $\mathrm{Res}_H^G \pi_1$ is weakly contained in $\mathrm{Res}_H^G \pi_2$.
(5) Projection formula: if σ is a representation of G_1, and τ a representation of $G_2 \subset G_1$, then $I_{G_2}^{G_1}(\sigma|_{G_2} \otimes \tau)$ is isomorphic to $I_{G_2}^{G_1}(\tau) \otimes \sigma$.

The following lemma is a consequence of Harish-Chandra's Plancherel formula. However, an elementary proof is given in [16], following the ideas of [6].

Lemma 1. *Suppose G is a reductive group over a local field k. Then any tempered representation of $G(k)$ is weakly contained in $L^2(G(k))$. Conversely, any irreducible representation of $G(k)$ weakly contained in $L^2(G(k))$ is tempered; indeed, if $G(k) \times G(k)$ acts on $L^2(G(k))$ in the natural way, any irreducible $G(k) \times G(k)$ representation weakly occurring in $L^2(G(k))$ is tempered.*

Proof. See [16, Thm, Sec. 2.4] for the first two assertions. The final statement (concerning $G(k) \times G(k)$) is a consequence of the previous ones, by considering the restriction to each $G(k)$ factor separately. $\square$

Lemma 2. *Suppose G is a reductive group over a local field k and H an algebraic subgroup so that $I_{H(k)}^{G(k)}(1)$ is tempered. Then, for any unitary $H(k)$-representation σ, the representation $I_{H(k)}^{G(k)}(\sigma)$ is tempered.*

Proof. Every matrix coefficient of $I_{H(k)}^{G(k)}(\sigma)$ is pointwise dominated by a matrix coefficient of $I_{H(k)}^{G(k)}(1)$.

Indeed, this remark is valid if one replaces $H(k) \subset G(k)$ by an inclusion of locally compact groups $G_1 \subset G_2$. Assume for simplicity that $G_1 \backslash G_2$ carries an invariant measure $d\mu$, and suppose that σ is a representation of G_1 realized on the Hilbert space V. Let $\langle \cdot, \cdot \rangle_V$ be the inner product on V. Then $I_{G_1}^{G_2}\sigma$ is realized on the space of functions $f : G_2 \to V$ satisfying $f(g_1 g) = \sigma(g_1) f(g)\,(g \in G_2)$, the inner product being $\langle f_1, f_2 \rangle = \int_{G_1 \backslash G_2} \langle f_1(g), f_2(g) \rangle_V \, d\mu(g)$. For such f_1, f_2 the functions $h_j : g \mapsto \sqrt{\langle f_j(g), f_j(g) \rangle_V}$ define elements of $L^2(G_1 \backslash G_2) = I_{G_1}^{G_2} 1$, and

$$|\langle \pi(g) f_1, f_2 \rangle_{I_{G_1}^{G_2}\sigma}| \leq \langle \pi(g) h_1, h_2 \rangle_{I_{G_1}^{G_2} 1}.$$

If G is semisimple, this is enough, since in that case a representation is tempered if and only if all matrix coefficients belong to $L^{2+\epsilon}$ [6, Thm. 1].

In the general case, let Z_G be the center of G. Then there exists a semisimple subgroup $G^{(0)} \subset G$ so that the map $Z_G(k) \cdot G^{(0)}(k) \to G(k)$ has finite kernel and cokernel. For such a subgroup $G^{(0)}(k)$, a representation of $G(k)$ is tempered if and only if its restriction to $G^{(0)}$ is tempered.

Indeed, to check that the restriction of a tempered representation of $G(k)$ is tempered, it suffices to check this for irreducible representations, by [16, Thm, Sec. 2.4]. The statement for irreducible representations follows by [16, Thm, assertion (1), Sec. 2.4]. For the converse direction, note first that $G(k)/G^{(0)}(k)$ is *amenable*, that is, the regular representation of $G(k)/G^{(0)}(k)$ on $L^2(G(k)/G^{(0)}(k))$ weakly contains the trivial representation. This can be checked by hand, since the map $Z_G(k) \to G(k)/G^{(0)}(k)$ has finite kernel and cokernel. Now, it follows from this fact and the projection formula (see property 5 above) that π is weakly contained in

$I_{G^{(0)}(k)}^{G(k)} \pi|_{G^{(0)}(k)}$. Since $\pi|_{G^{(0)}(k)}$ is tempered, it is weakly contained in $L^2(G^{(0)}(k))$, by Lemma 1 above. We deduce that π is weakly contained in $L^2(G(k))$ and so is also tempered (Lemma 1 again).

Since every matrix coefficient of $I_{H(k)}^{G(k)}(\sigma)$ is pointwise dominated by a matrix coefficient of $I_{H(k)}^{G(k)}(1)$, it follows (from the matrix coefficient criterion for temperedness; see again [16, Thm, Sec. 2.4]) that $I_{H(k)}^{G(k)} \sigma$ is tempered as a $G^{(0)}(k)$-representation, and thus also as a $G(k)$-representation. □

Corollary 1 (of Lemma 2 and Lemma 1)**.** *Let notation be as in Lemma* 1 *and let σ be an irreducible unitary representation of $G(k)$, and regard $G(k)$ as being diagonally embedded in $G(k) \times G(k)$. Then $I_{G(k)}^{G(k)\times G(k)} \sigma$ is tempered.*

Proof. Apply Lemma 2 to H the diagonal subgroup of $G \times G$. The hypothesis of Lemma 2 is precisely the final statement of Lemma 1; the assertion of Lemma 2 now yields the Corollary. □

3.2. **A weakened version of Proposition 2.** From Section 3.2 to Section 3.5 we shall work with the local field $k = \mathbb{C}$.

The weakened version of Proposition 2 will be stated as Lemma 3 and will be proven by induction in the sections that follow. In Section 3.6, we will deduce the full Proposition 2 from Lemma 3 by global methods.

Roughly, this weakened version is a statement only over the local field $\mathbb{C}$, and states that (e.g.) the tensor product of two $\mathrm{GL}(n, \mathbb{C})$-representations (both of Arthur type, as in Proposition 2) weakly contains *at least one* $\mathrm{GL}(n, \mathbb{C})$ representation of the "predicted" $\mathrm{SL}(2)$-type. The necessity for this in our procedure is as follows: in one part of our argument—the approximation argument (see (4))—we use a technique which produces a weak constituent of a certain representation, but we lose information about all other possible constituents.

In the case of $\mathrm{GL}(n, \mathbb{C})$, the Arthur spectrum $\widehat{\mathrm{GL}(n, \mathbb{C})}_{Ar}$ can be described in an especially nice way:

Say that an irreducible representation π of $\mathrm{GL}(n, \mathbb{C})$ is *principal* if it is unitarily induced from a unitary one-dimensional character of a parabolic subgroup. In other words, it is a unitary principal series, or otherwise a degenerate unitary principal series. Such a representation is automatically irreducible; The same notion applies to a representation of a product $\mathrm{GL}(n_1, \mathbb{C}) \times \mathrm{GL}(n_2, \mathbb{C}) \times \dots$; in particular, it applies to representations of Levi subgroups of $\mathrm{GL}(n, \mathbb{C})$. Clearly, if one parabolically induces a principal representation of a Levi subgroup, one obtains a principal representation.

Note that in fact, the principal representations of $\mathrm{GL}(n, \mathbb{C})$ are precisely those of Arthur type in the sense of Section 2.2; for instance, a complementary series for $\mathrm{GL}(2, \mathbb{C})$ is not principal. However, we have used the adjective "principal" in this and the subsequent sections since it is really the property of being principal that is used repeatedly in the proofs in Section 3.3–Section 3.5. We will, however, need the fact that any tempered representation of $\mathrm{GL}(n, \mathbb{C})$ is principal, see below.

Remark that any one-dimensional character of $\mathrm{GL}(n, \mathbb{C})$ factors through the determinant (since $\mathrm{SL}(n, \mathbb{C})$, as an abstract group, is generated by commutators $g_1 g_2 g_1^{-1} g_2^{-1}$ with $g_1, g_2 \in \mathrm{SL}(n, \mathbb{C})$). In particular, any one-dimensional character of

$\mathrm{GL}(n, \mathbb{C})$ is trivial when restricted to any subgroup consisting of unipotent elements. We will use this simple fact several times.

Any tempered irreducible representation of $\mathrm{GL}(n, \mathbb{C})$ is principal, by the classification of irreducible representations of $\mathrm{GL}(n, \mathbb{C})$; in particular, any tempered (not necessarily irreducible) representation of $\mathrm{GL}(n, \mathbb{C})$ weakly contains a principal representation. This is why we work over $\mathbb{C}$, and we shall use it repeatedly without further comment.

Lemma 3. *Let π be a principal representation of $\mathrm{GL}(n, \mathbb{C})$. Then its restriction to any Levi subgroup weakly contains a principal representation whose $\mathrm{SL}(2)$-type is given by Prop. 2, (1).*

More precisely: suppose the principal representation π has type $\sigma = \langle n_i \rangle_{1 \leq i \leq I}$ and M is the Levi subgroup $\mathrm{GL}(r_1, \mathbb{C}) \times \cdots \times \mathrm{GL}(r_j, \mathbb{C})$ with $\sum_{i=1}^{I} n_i = \sum_{k=1}^{j} r_k = n$. Then the restriction of π to M contains a principal representation of type $(\omega_1, \ldots, \omega_j)$, where for each $1 \leq k \leq j$, ω_k is the partition of r_k given by $\langle (n_i + r_k - n)_{1 \leq i \leq I} \rangle_{r_k}$.

Similar assertions holds for induction and tensor product. If π is a principal representation of a Levi subgroup, its induction to $\mathrm{GL}(n, \mathbb{C})$ weakly contains a principal representation whose type is given by Proposition 2, (2). If π, σ are principal representations of $\mathrm{GL}(n, \mathbb{C})$, then $\pi \otimes \sigma$ weakly contains a principal representation of $\mathrm{GL}(n, \mathbb{C})$ whose type is that specified by Proposition 2, (3).

We illustrate the content of the lemma and its relation to Proposition 2 with an example. Take the principal representation π of $\mathrm{GL}(20, \mathbb{C})$ that is unitarily induced from a one-dimensional character of a parabolic of type $(15, 5)$. Then π has type $(15, 5)$. The (third assertion of the) lemma states that $\pi \otimes \pi$ weakly contains a principal representation of $\mathrm{GL}(20, \mathbb{C})$ with type $(10, 1, 1, \ldots, 1)$ (with 10 ones). Contrast this with Proposition 2, which asserts that $\pi \otimes \pi$ has type $(15, 5)$. The lemma therefore only produces information about *one single constitutent* of $\pi \otimes \pi$; unlike Proposition 2, the lemma does not rule out the possibility that $\pi \otimes \pi$ might weakly contain constituents of other types.

We will prove this by induction: assuming it true for all $\mathrm{GL}(m, \mathbb{C})$ with $m < n$, we shall prove it for $\mathrm{GL}(n, \mathbb{C})$. We deduce Proposition 2 from this lemma in Section 3.6 by a global argument, due to Clozel. This allows us to replace $\mathbb{C}$ by a general local field in Lemma 3, and also to get information about all weak constituents, not just one.

The proof of Lemma 3, as we have remarked, is standard Mackey theory; the only twist is in Section 3.3 where we use an approximation argument (see equation (4)). The assertions of the lemma pertaining to induction, restriction and tensor product will be proven in Sections 3.3, 3.4 and 3.5 respectively.

We finally note that it suffices to show Lemma 3 in the cases of restriction to, or induction from, *maximal* Levi subgroups; the general case can be expressed as a sequence of such operations, and as remarked before the assertions of Proposition 2 are compatible with iterated induction and restriction. Also, since we are always considering principal π, it suffices to prove the assertions of Lemma 3 in the case where π is parabolically induced from a representation $\rho_a \otimes \rho_b$ of $\mathrm{GL}(a, \mathbb{C}) \times \mathrm{GL}(b, \mathbb{C})$, where $a + b = n$, ρ_a, ρ_b are unitary, and ρ_a is one-dimensional. We will make similar easy reductions in the proofs of all three parts of Lemma 3.

3.3. **Proof of Lemma 3 for Induction.** Set $G = \mathrm{GL}(n, \mathbb{C})$ from this Section through Section 3.5.

To prove the assertion of Lemma 3 concerning induction from Levi subgroups, it suffices to consider the case of induction from a *maximal* Levi subgroup.

Let $\sigma = (a, b)$, where $a + b = n$, and let ρ_a, ρ_b be principal representations of $\mathrm{GL}(a, \mathbb{C})$ and $\mathrm{GL}(b, \mathbb{C})$ respectively. We may assume $a \leq b$. Notation being as in Section 2.3 and the previous section, we set P_σ to be the parabolic of type σ, that is, the stabilizer of an a-dimensional subspace. Then we have a Levi decomposition $P_\sigma = M_\sigma N_\sigma$, with N_σ abelian. Let $\Sigma = \rho_a \otimes \rho_b$ as a representation of M_σ. We shall analyze $I = I_{M_\sigma}^G \Sigma$ by induction in stages and decomposing into characters of N. One has

$$I = I_{M_\sigma N_\sigma}^G I_{M_\sigma}^{M_\sigma N_\sigma} \Sigma. \tag{2}$$

We may identify N_σ with $\mathrm{Hom}_{\mathbb{C}}(\mathbb{C}^b, \mathbb{C}^a)$ (maps of $\mathbb{C}$-vector spaces, considered as an additive group). M_σ acts by conjugation on the character group of N_σ; this action has a unique open orbit, and we say a character belonging to this open orbit is *generic*. Let ψ be a generic character of N_σ and M_ψ its stabilizer in M. Fourier analysis on N_σ yields an isomorphism:

$$I_{M_\sigma}^{M_\sigma N_\sigma} \Sigma = I_{M_\psi N_\sigma}^{M_\sigma N_\sigma} (\mathrm{Res}_{M_\psi}^M \Sigma \cdot \psi). \tag{3}$$

(Here $\mathrm{Res}_{M_\psi}^M \Sigma \cdot \psi$ denotes the representation $\mathrm{Res}_{M_\psi}^M \Sigma$ on M_ψ extended by ψ on N_σ.)

We regard N_σ in the obvious fashion as a $\mathbb{C}$-vector space; in particular, $\mathbb{C}^\times$ acts on N_σ. For $z \in \mathbb{C}^\times$, set $\psi_z(n) = \psi(zn)$. Note that $M_{\psi_z} = M_\psi$, thus (3) holds with ψ replaced by ψ_z. Now take $z \to 0$ so that ψ_z approaches (weakly) the trivial character of N. Utilizing the continuity of induction w.r.t. the Fell topology, we obtain

$$\begin{aligned} I_{M_\sigma}^{M_\sigma N_\sigma} \Sigma \text{ weakly contains } I_{M_\psi N_\sigma}^{M_\sigma N_\sigma} (\mathrm{Res}_{M_\psi}^{M_\sigma} \Sigma \cdot 1) &= (I_{M_\psi}^{M_\sigma} \mathrm{Res}_{M_\psi}^{M_\sigma} \Sigma) \cdot 1 \\ &= (I_{M_\psi}^{M_\sigma}(1) \otimes \Sigma) \cdot 1 \end{aligned} \tag{4}$$

where we have used the projection formula (see Section 3.1, property 5), and the final expression denotes the representation $I_{M_\psi}^{M_\sigma}(1) \otimes \Sigma$ on M_σ, extended by 1 on N_σ.

After choosing a fixed nontrivial additive character of $\mathbb{C}$, one may identify the character group of N_σ with $\mathrm{Hom}_{\mathbb{C}}(\mathbb{C}^a, \mathbb{C}^b)$; a generic character then corresponds to an injection of $\mathbb{C}^a$ into $\mathbb{C}^b$.

From this, one may deduce the description of M_ψ as follows. $M_\sigma = \mathrm{GL}(b, \mathbb{C}) \times \mathrm{GL}(a, \mathbb{C})$ has a Levi subgroup of the form $M_1 = (\mathrm{GL}(b - a, \mathbb{C}) \times \mathrm{GL}(a, \mathbb{C})) \times \mathrm{GL}(a, \mathbb{C})$ with unipotent radical N_1 isomorphic to $\mathbb{C}^{a(b-a)}$. Let $P_1 = M_1 N_1$ be the corresponding parabolic subgroup of M_σ.

Then, for appropriate choice of ψ, $M_\psi \subset M_1$ is isomorphic to $(\mathrm{GL}(b - a, \mathbb{C}) \times \mathrm{GL}(a, \mathbb{C})).N_1$, with the $\mathrm{GL}(a, \mathbb{C})$ factor embedded diagonally inside M_1.

A computation (using Lemma 1 and induction in stages: induce first to P_1, then to M_σ) shows that $I_{M_\psi}^{M_\sigma}(1)$ weakly contains a principal representation of the form $\gamma_{temp} \otimes \gamma_b$, where γ_{temp} is a tempered representation of $\mathrm{GL}(a, \mathbb{C})$ and γ_b is a representation of $\mathrm{GL}(b, \mathbb{C})$ of type $(b - a, 1, \ldots, 1)$.

We may then apply the inductive hypothesis (for tensor product) to $I_{M_\psi}^{M_\sigma}(1) \otimes \Sigma$. More precisely, we apply the assertion of Lemma 3 for tensor products on $\mathrm{GL}(a, \mathbb{C})$ and $\mathrm{GL}(b, \mathbb{C})$.

Using (2) and (4), one deduces the validity of Lemma 3 in the case of induction to $\mathrm{GL}(n, \mathbb{C})$.

3.4. **Proof of Lemma 3 for Restriction.** In this section the subgroup computations required for Mackey theory are more involved. We therefore discuss them "geometrically" in an attempt to make these manipulations more transparent.

Again, to prove the assertion of Lemma 3 concerning restriction to a Levi subgroup, it suffices to treat the case of restriction to a *maximal* Levi subgroup. Suppose we are analyzing the restriction of a principal representation π of $\mathrm{GL}(n, \mathbb{C})$ of type $(n_1, \ldots, n_r)$ to a Levi subgroup of type (a', b'). Without loss of generality, $a' \leq b'$. A computation with matrix coefficients (using [16, Thm, Sec. 2.4]) shows that, if $\max_{1 \leq i \leq r} n_i \leq a' + 1$, then the restriction is tempered. (This verification proceeds along the same lines as the computations performed in [4, Section 4]; note that [6, page 108] reduces the question to a finite combinatorial problem.) In that case the assertion of Lemma 3 is true. So it suffices to treat the case when $a' < \max_{1 \leq i \leq r} n_i$. In this case, put $b = \max_{1 \leq i \leq r} n_i$; then we can write π as being induced from a parabolic of type $(n - b, b)$ with a one-dimensional representation on the $\mathrm{GL}(b, \mathbb{C})$ factor; moreover, $a' < b$ (and so also $n - b < b'$).

Let notations $\sigma, \Sigma = \rho_a \otimes \rho_b, P_\sigma = M_\sigma N_\sigma$ be as in the previous section. Further, let $\tau = (a', b')$ and P_τ, M_τ the associated parabolic subgroup and Levi. We shall compute the restriction of $I_{P_\sigma}^G \Sigma$ to $M_\tau = \mathrm{GL}(a', \mathbb{C}) \times \mathrm{GL}(b', \mathbb{C})$.

In view of the previous remarks, it suffices to consider only the case where $a' \leq b'$, ρ_b is 1-dimensional, and $b > a'$ (so also $a < b'$). We divide into two cases according to whether $a \leq a'$ or $a > a'$.

Case 1: $a \leq a' \leq b'$. We may identify G/P_σ with the Grassmannian of a-dimensional subspaces of an n-dimensional vector space V over $\mathbb{C}$; on the other hand, M_τ is the stabilizer of a splitting $V = S \oplus T$, with $\dim(S) = a', \dim(T) = b'$.

Given such a splitting $V = S \oplus T$, a generic a-dimensional subspace $W \subset V$ is specified by giving an a-dimensional subspace of S, an a-dimensional subspace of T, and an isomorphism between them. (These are, respectively, the projection of W onto S, the projection of W onto T, and the map between these projections whose graph is given by W.) It follows from this description that M_τ acts with an open orbit on G/P_σ.

Let us describe the stabilizer in M_τ of a generic point in G/P_σ. Fix an a-dimensional subspace W_S of S, an a-dimensional subspace W_T of T, an isomorphism $\varphi : W_S \xrightarrow{\sim} W_T$. Choose complements S' to W_S in S and T' to W_T in T, so that we have splittings $S = W_S \oplus S'$, $T = W_T \oplus T'$. (So $\dim(W_S) = \dim(W_T) = a$, and $\dim(S') = a' - a, \dim(T') = b' - a$). By the remark above, W_S, W_T, φ determine an a-dimensional subspace $\mathrm{graph}(\varphi) \subset V$. We can and will assume that P_σ is the stabilizer of the subspace $\mathrm{graph}(\varphi)$. With this choice, the identity coset in G/P_σ belongs to the open M_τ-orbit on G/P_σ.

Let P_v be the subgroup of $\mathrm{GL}(S) \times \mathrm{GL}(T)$ stabilizing W_S and W_T. Let M_v be the subgroup of $\mathrm{GL}(S) \times \mathrm{GL}(T)$ stabilizing W_S, W_T, S' and T'. Let N_v be the subgroup of P_v consisting of elements that induce the identity endomorphism on each of $W_S, W_T, S/W_S$ and T/W_T. Then P_v is a parabolic subgroup of $\mathrm{GL}(S) \times \mathrm{GL}(T)$, and $P_v = M_v N_v$ is a Levi decomposition.

Let $M_H \subset M_\upsilon$ be the subgroup of M_υ consisting of elements preserving the isomorphism $\varphi : W_S \to W_T$, i.e. $M_H = \{(g_1, g_2) \in GL(S) \times \mathrm{GL}(T) : (g_1, g_2) \in M_\upsilon, \varphi g_1|_{W_S} = g_2\varphi\}$. One can regard M_H as the subgroup

$$
\begin{aligned}
(5) \qquad M_H &= \mathrm{GL}(S') \times \mathrm{GL}(T') \times \mathrm{GL}(W_S) \\
&\subset M_\upsilon = \mathrm{GL}(S') \times \mathrm{GL}(T') \times \mathrm{GL}(W_S) \times GL(W_T)
\end{aligned}
$$

where $\mathrm{GL}(W_S)$ is "diagonally" embedded into $\mathrm{GL}(W_S) \times \mathrm{GL}(W_T)$. Here "diagonally embedded" implicitly makes use of the identification of $\mathrm{GL}(W_S)$ and $\mathrm{GL}(W_T)$ induced by φ. With these notations, we see that $P_\sigma \cap M_\tau = M_H N_\upsilon$, and so $M_H N_\upsilon$ is the stabilizer in M_τ of a point in the open M_τ-orbit on G/P_σ.

One sees by Mackey theory (Section 3.1, property 2) that

$$
(6) \qquad \mathrm{Res}^G_{M_\tau} I^G_{P_\sigma} \Sigma \text{ is isomorphic to } I^{M_\tau}_{M_H N_\upsilon} \mathrm{Res}^{P_\sigma}_{M_H N_\upsilon} \Sigma.
$$

One may compute $\mathrm{Res}^{P_\sigma}_{M_H N_\upsilon} \Sigma$ as follows: First, the restriction to N_υ of Σ is trivial. Indeed, in terms of the "geometric" description given above, N_υ induces the identity endomorphism on graph(φ); this shows that N_υ projects trivially to the $\mathrm{GL}(a, \mathbb{C})$ factor in P_σ, and so N_υ acts trivially in Σ by the one-dimensionality of ρ_b (see remark on one-dimensional characters in Section 3.2). The $\mathrm{GL}(S') \times \mathrm{GL}(T')$ factor of M_H also induces the identity endomorphism on graph(φ). The one-dimensionality of ρ_b now implies that $\mathrm{GL}(S') \times \mathrm{GL}(T')$ acts by a character on Σ. Finally, it is easy to see that the restriction of Σ to $\mathrm{GL}(W_S) \subset M_H$ (cf. (5)) is just a twist of the representation ρ_a by some one-dimensional character.

Using the triviality of $\Sigma|_{N_\upsilon}$ and induction in stages, we obtain from (6)

$$
(7) \qquad \mathrm{Res}^G_{M_\tau} I^G_{P_\sigma} \Sigma \text{ is isomorphic to } I^{M_\tau}_{P_\upsilon} \left(I^{M_\upsilon}_{M_H} \mathrm{Res}^{P_\sigma}_{M_H} \Sigma \cdot 1 \right).
$$

Again, the notation $\cdot 1$ in (7) denotes extending a representation trivially on N_υ.

Now we have seen that $\mathrm{Res}^{P_\sigma}_{M_H} \Sigma$ is a representation of the form $(m_1, m_2, m_3) \in \mathrm{GL}(S') \times \mathrm{GL}(T') \times \mathrm{GL}(W_S) \mapsto \gamma_{a'-a}(m_1) \otimes \gamma_{b'-a}(m_2) \otimes \gamma_a(m_3)\rho_a(m_3)$ where $\gamma_{a'-a}, \gamma_{b'-a}, \gamma_a$ are 1-dimensional characters of $\mathrm{GL}(S')$, $\mathrm{GL}(T')$ and $\mathrm{GL}(W_S)$ respectively. In view of Corollary 1, the restriction of $I^{M_\upsilon}_{M_H} \mathrm{Res}^{P_\sigma}_{M_H} \Sigma$ to $\mathrm{GL}(W_S) \times \mathrm{GL}(W_T) \subset M_\upsilon$ is tempered.

It follows that the representation $I^{M_\upsilon}_{M_H} \mathrm{Res}^{P_\sigma}_{M_H} \Sigma$ of $M_\upsilon \cong \mathrm{GL}(a' - a, \mathbb{C}) \times \mathrm{GL}(b' - a, \mathbb{C}) \times \mathrm{GL}(a, \mathbb{C}) \times \mathrm{GL}(a, \mathbb{C})$ weakly contains an irreducible representation of the form

$$
\gamma_{a'-a} \otimes \gamma_{b'-a} \otimes \gamma_{temp} \otimes \gamma'_{temp}
$$

where $\gamma_{temp}, \gamma'_{temp}$ are tempered (thus principal) representations of $\mathrm{GL}(a, \mathbb{C})$, and $(\gamma_{a'-a} \otimes \gamma_{b'-a})$ is a one-dimensional representation of $\mathrm{GL}(a' - a, \mathbb{C}) \times \mathrm{GL}(b' - a, \mathbb{C})$.

The assertion of Lemma 3 in this case now follows from (7).

Case 2: $a' < a \leq b'$. As before suppose M_τ is the stabilizer of a splitting $V = S \oplus T$ with $\dim(S) = a', \dim(T) = b'$. Again, G/P_σ can be identified with the Grassmannian of a-dimensional subspaces in an n-dimensional vector space; in this case, a generic a-dimensional subspace is specified by giving an a-dimensional subspace of T together with a surjection of this subspace onto S. It is clear from this description that M_τ acts with an open orbit on G/P_σ.

Let us describe the stabilizer in $\mathrm{GL}(T) \times \mathrm{GL}(S)$ of a point in the open $\mathrm{GL}(T) \times \mathrm{GL}(S)$-orbit on G/P_σ. Let W_T be an a-dimensional subspace of T, let $K_T \subset W_T$ be an $a - a'$-dimensional subspace of W_T, and let $\varphi : W_T/K_T \xrightarrow{\sim} S$ be an isomorphism.

Then φ determines an a-dimensional subspace of V, viz. graph(φ). We can and will assume that P_σ is the stabilizer of graph(φ). Then the identity coset in G/P_σ belongs to the open $\mathrm{GL}(T)\times\mathrm{GL}(S)$-orbit on G/P_σ.

Fix a complement T' to W_T inside T, so that $T'\oplus W_T = T$. (Thus $\dim(W_T)=a$, $\dim(K_T)=a-a'$, $\dim(T')=b'-a$.)

Let P_1 be the stabilizer in $\mathrm{GL}(T)\times\mathrm{GL}(S)$ of W_T, i.e. those elements $(g_1,g_2)\in\mathrm{GL}(T)\times\mathrm{GL}(S)$ s.t. g_1 preserves W_T. Let $N_1\subset P_1$ be the subgroup inducing the identity endomorphism on W_T and T/W_T, and let $M_1=\mathrm{GL}(T')\times\mathrm{GL}(W_T)\times\mathrm{GL}(S)$. Then P_1 is a parabolic subgroup of $\mathrm{GL}(T)\times\mathrm{GL}(S)$, and $P_1=M_1.N_1$ is a Levi decomposition.

Let $H\subset\mathrm{GL}(W_T)\times\mathrm{GL}(S)$ be the subgroup consisting of elements that preserve K_T and induce compatible endomorphisms of W_T/K_T and S, i.e. $H=\{(g_1,g_2)\in\mathrm{GL}(W_T)\times\mathrm{GL}(S):\varphi g_1=g_2\varphi\}$. The stabilizer in $\mathrm{GL}(T)\times\mathrm{GL}(S)$ of graph(φ), i.e. $(\mathrm{GL}(T)\times\mathrm{GL}(S))\cap P_\sigma$, is then seen to be $(\mathrm{GL}(T')\times H).N_1$.

One now sees by Mackey theory that

$$\text{(8)}\qquad \mathrm{Res}^G_{\mathrm{GL}(T)\times\mathrm{GL}(S)} I^G_{P_\sigma}\Sigma \text{ is isomorphic to } I^{\mathrm{GL}(T)\times\mathrm{GL}(S)}_{(\mathrm{GL}(T')\times H).N_1}\mathrm{Res}^{P_\sigma}_{(\mathrm{GL}(T')\times H).N_1}\Sigma.$$

We now compute $\mathrm{Res}^{P_\sigma}_{(\mathrm{GL}(T')\times H).N_1}\Sigma$. First, elements of N_1 induce the identity automorphism on W_T, so they also induce the identity automorphism on graph(φ). Using the one-dimensionality of ρ_b, one sees this implies that N_1 acts trivially in Σ. The same reasoning shows that $\mathrm{GL}(T')$ acts by scalars on Σ.

Finally, we compute the H-action on Σ. We can restrict the one-dimensional character ρ_b of $\mathrm{GL}(b,\mathbb{C})$ to $\mathrm{GL}(S)$ (embed $\mathrm{GL}(S)\cong\mathrm{GL}(a',\mathbb{C})$ into $\mathrm{GL}(b,\mathbb{C})$ via the standard embedding). Let $\tilde\rho_a$ be the representation $\rho_a\otimes\rho_b$ of $\mathrm{GL}(W_T)\times\mathrm{GL}(S)$. With this convention, then the restriction of Σ to H is isomorphic to $\tilde\rho_a|_H$.

Let $\gamma_{b'-a}$ be the character of $\mathrm{GL}(T')$ defined by Σ. Then, applying (8) and inducing in stages through P_1 we get

$$\begin{aligned}
\text{(9)}\quad & \mathrm{Res}^G_{\mathrm{GL}(T)\times\mathrm{GL}(S)} I^G_{P_\sigma}\Sigma\\
&= I^{\mathrm{GL}(T)\times\mathrm{GL}(S)}_{P_1} I^{P_1}_{(\mathrm{GL}(T')\times H).N_1}\mathrm{Res}^{P_\sigma}_{(GL(T')\times H).N_1}\Sigma\\
&= I^{\mathrm{GL}(T)\times\mathrm{GL}(S)}_{P_1} I^{P_1}_{(\mathrm{GL}(T')\times H).N_1}((\gamma_{b'-a}\otimes\tilde\rho_a|_H).1)\\
&= I^{\mathrm{GL}(T)\times\mathrm{GL}(S)}_{P_1}((\gamma_{b'-a}\otimes I^{\mathrm{GL}(W_T)\times\mathrm{GL}(S)}_H\tilde\rho_a|_H)\cdot 1).
\end{aligned}$$

Here, in the middle line, $(\gamma_{b'-a}\otimes\rho_a|_H).1$ denotes the representation $\gamma_{b'-a}\otimes\rho_a|_H$ on $GL(T')\times H$, extended trivially on N_1. One interprets the final line similarly.

One sees from (9) that $\mathrm{Res}^G_{\mathrm{GL}(T)\times\mathrm{GL}(S)} I^G_{P_\sigma}\Sigma$ is the representation of $\mathrm{GL}(T)\times\mathrm{GL}(S)\cong\mathrm{GL}(b',\mathbb{C})\times\mathrm{GL}(a',\mathbb{C})$ induced from the parabolic P_1 with a one-dimensional representation on the $\mathrm{GL}(b'-a,\mathbb{C})$ factor and the following representation on the $\mathrm{GL}(W_T)\times\mathrm{GL}(S)\cong\mathrm{GL}(a,\mathbb{C})\times\mathrm{GL}(a',\mathbb{C})$ factor:

$$\text{(10)}\qquad I^{\mathrm{GL}(W_T)\times\mathrm{GL}(S)}_H\tilde\rho_a|_H=\tilde\rho_a\otimes L^2(H\backslash\mathrm{GL}(W_T)\times\mathrm{GL}(S)).$$

Here we have again used the projection formula (Section 3.1, property 5). However, $L^2(H\backslash\mathrm{GL}(W_T)\times\mathrm{GL}(S))$ may be analyzed by induction in stages: if one sets P_2 to be the parabolic subgroup of $\mathrm{GL}(W_T)\times\mathrm{GL}(S)$ consisting of (g_1,g_2) such that g_1 preserves K_T, then $H\subset P_2$, and one computes using $L^2(H\backslash\mathrm{GL}(W_T)\times\mathrm{GL}(S))=I^{\mathrm{GL}(S)\times\mathrm{GL}(T)}_{P_2}I^{P_2}_H 1$.

One verifies thereby that $L^2(H\backslash \mathrm{GL}(W_T) \times \mathrm{GL}(S))$—considered as a representation of $\mathrm{GL}(W_T) \times \mathrm{GL}(S) \cong \mathrm{GL}(a, \mathbb{C}) \times \mathrm{GL}(a', \mathbb{C})$—weakly contains a principal representation $\gamma_a \otimes \gamma_{temp}$, where γ_a is a representation of $\mathrm{GL}(a, \mathbb{C})$ of type $(a - a', 1, 1, \ldots, 1)$, and γ_{temp} is a tempered representation of $\mathrm{GL}(a', \mathbb{C})$.

Then using the inductive hypothesis (for tensor product) we deduce from (10) that $I_H^{\mathrm{GL}(W_T)\times \mathrm{GL}(S)} \tilde{\rho}_a|_H$ weakly contains a principal representation of a type specified by Proposition 2. (More precisely, we use the assertion of Lemma 3 for tensor products on $\mathrm{GL}(W_T) \cong \mathrm{GL}(a, \mathbb{C})$ and $\mathrm{GL}(S) \cong \mathrm{GL}(a', \mathbb{C})$.)

One now concludes from (9) that Lemma 3 holds in this case.

3.5. **Proof of Lemma 3 for tensor product.** We now wish to analyze the tensor product of two principal representations. Suppose for a moment that they are of type $\langle a_1, \ldots, a_r\rangle$ and $\langle b_1, \ldots, b_r\rangle$. Note that if every sum $a_i + b_j$ is less than n, then an argument with matrix coefficients shows the tensor product to be tempered and the conclusion of Lemma 3 holds. (Again, this verification proceeds along the same lines as the computations performed in [4, Section 4]; the point is, again, that [6, page 108] reduces the question to a finite combinatorial problem.)

We may therefore assume that there is an inequality $a_i + b_j > n$ (some i, j).

Let $\sigma = (a, b)$ and $\tau = (a', b')$. We will assume we are given representations $\rho_a, \rho_b, \eta_{a'}, \eta_{b'}$ of $\mathrm{GL}(a, \mathbb{C}), \mathrm{GL}(b, \mathbb{C}), \mathrm{GL}(a', \mathbb{C}), \mathrm{GL}(b', \mathbb{C})$ respectively; let $\Sigma = \rho_a \otimes \rho_b$ and $\Upsilon = \eta_{a'} \otimes \eta_{b'}$ be the corresponding representations of M_σ and M_τ (thus P_σ and P_τ, by extending trivially on the unipotent radicals).

We shall analyze $I_{P_\sigma}^G \Sigma \otimes I_{P_\tau}^G \Upsilon$. In view of the above remarks we may assume that $b + b' > n$ (equivalently $a + a' < n$) and that ρ_b and $\eta_{b'}$ are 1-dimensional. By the Bruhat decomposition G acts with a single open orbit on $P_\sigma\backslash G \times P_\tau\backslash G$.

The stabilizer H of a point in this open orbit may be described as the stabilizer of a configuration $V_1 \oplus V_2 \subset V$, where V_1, V_2 are (respectively) a, a'-dimensional $\mathbb{C}$-subvector spaces of the n-dimensional $\mathbb{C}$-vector space V.

Let $\upsilon = (a + a', n - a - a')$; consider the corresponding parabolic P_υ, Levi subgroup $M_\upsilon = \mathrm{GL}(a + a', \mathbb{C}) \times \mathrm{GL}(n - a - a', \mathbb{C})$ and unipotent radical N_υ. Now $\mathrm{GL}(a, \mathbb{C}) \times \mathrm{GL}(a', \mathbb{C})$ may be embedded as a Levi subgroup of $\mathrm{GL}(a + a', \mathbb{C})$; in particular, $M_H = \mathrm{GL}(a, \mathbb{C}) \times \mathrm{GL}(a', \mathbb{C}) \times \mathrm{GL}(n - a - a', \mathbb{C})$ is a subgroup of M_υ. Set $H = M_H \cdot N_\upsilon$; then H is the stabilizer in G of an appropriate point in the open G-orbit on $P_\sigma\backslash G \times P_\tau\backslash G$. We can and will assume that this point is the identity coset, so that $H \subset P_\sigma \times P_\tau$. (More precisely: the choice of M_H defines a splitting $\mathbb{C}^n = \mathbb{C}^a \oplus \mathbb{C}^{a'} \oplus \mathbb{C}^{n-a-a'}$, and we take P_σ and P_τ to be the stabilizers of the $\mathbb{C}^a$ and $\mathbb{C}^{a'}$ respectively.)

We then have

$$\begin{aligned} I_{P_\sigma}^G \Sigma \otimes I_{P_\tau}^G \Upsilon &= I_H^G(\mathrm{Res}_H^{P_\sigma \times P_\tau} \Sigma \otimes \Upsilon) \\ &= I_{M_\upsilon N_\upsilon}^G I_{M_H N_\upsilon}^{M_\upsilon N_\upsilon}(\mathrm{Res}_H^{P_\sigma \times P_\tau} \Sigma \otimes \Upsilon). \end{aligned} \tag{11}$$

Reasoning similar to that of Section 3.4 (using the one-dimensionality of $\rho_b, \eta_{b'}$) shows that $\mathrm{Res}_H^{P_\sigma \times P_\tau} \Sigma \otimes \Upsilon$ is trivial on N_υ and given on M_H via

$$\begin{aligned} (m_1, m_2, m_3) &\in \mathrm{GL}(a, \mathbb{C}) \times \mathrm{GL}(a', \mathbb{C}) \times \mathrm{GL}(n - a - a', \mathbb{C}) \\ &\mapsto \rho_a(m_1) \otimes \eta_{a'}(m_2) \otimes \rho_b((m_2, m_3)) \otimes \eta_{b'}((m_1, m_3)) \end{aligned} \tag{12}$$

where, for example, (m_1, m_3) is considered as an element of $\mathrm{GL}(b', \mathbb{C})$ via the Levi embedding $\mathrm{GL}(a, \mathbb{C}) \times \mathrm{GL}(n - a - a', \mathbb{C}) \to \mathrm{GL}(n - a', \mathbb{C}) = \mathrm{GL}(b', \mathbb{C})$. Since ρ_b

and $\eta_{b'}$ are 1-dimensional and ρ_a and $\eta_{a'}$ are principal, it follows that (12) defines a principal representation of M_H.

We may now apply the inductive hypothesis (for induction) to show that the representation $I_{M_H}^{M_v} \mathrm{Res}_H^{P_\sigma \times P_\tau}(\Sigma \otimes \Upsilon)$ weakly contains a principal representation of the type specified by Proposition 2, (3). (More precisely, we use Lemma 3 for induction from a Levi subgroup $\mathrm{GL}(a, \mathbb{C}) \times \mathrm{GL}(a', \mathbb{C})$ to $\mathrm{GL}(a + a', \mathbb{C})$.)

Combining this with (11), one verifies that $I_{P_\sigma}^G \Sigma \otimes I_{P_\tau}^G \Upsilon$ weakly contains a principal representation of the type predicted by Proposition 2, (3).

This proves Lemma 3 in the tensor product case.

3.6. **Global argument: from Lemma 3 to Proposition 2.** We now conclude the proof of Proposition 2 with a global argument; we carry it out for the case of the tensor product, but it is clear the method applies to the other cases. It should be noted that the ideas here are already in Clozel's recent note [4], and they are explicated there in the case of a function field. We treat the case of a global field of characteristic zero only for completeness (cf. last paragraph of Section 3 in [4, page 516]); more details in the characteristic 0 case are given in Clozel's Park City lecture notes [3].

We continue with the notation introduced at the start of Section 2; in particular, $G = \mathrm{GL}(n)$.

Let F be a number field with at least one complex place, and $\mathbb{A}_F$ its adèle ring. Let S be a finite subset of places of F containing at least one complex place, set $F_S = \prod_{v \in S} F_v$ and let $\widehat{G(F_S)}_{Aut} \subset \widehat{G(F_S)}$ be defined as in Section 2.1.

Proposition 3. *Suppose $\pi_S \in \widehat{G(F_S)}_{Aut}$, and write $\pi_S = \bigotimes_{v \in S} \pi_v$, with each $\pi_v \in \widehat{G(F_v)}$. Then the type of π_v is the same for each $v \in S$.*

Proof (Sketch). The theory of Eisenstein series ([10], [14]) gives an explicit spectral decomposition of $L^2(G(F) \backslash G(\mathbb{A}_F))$ in terms of the discrete spectrum occurring in $L^2(M(F) \backslash M(\mathbb{A}_F))$, where M varies through Levi subgroups of G. In the case of $\mathrm{GL}(n)$ the discrete spectrum of Levi subgroups is known, owing to the work of Moeglin and Waldspurger [13]. The assertion follows from these results; note at this point we are implicitly and crucially using the result of Luo, Rudnick and Sarnak (see [15, Prop 3.3]) which ensures that the automorphic spectrum stays away from the endpoints of complementary series. $\square$

On the other hand, set $\widehat{G(F_S)}_{Ar} \subset \widehat{G(F_S)}$ to consist of $\pi = \bigotimes \pi_v$ so that each $\pi_v \in \widehat{G(F_v)}_{Ar}$ *and* each π_v (for $v \in S$) has the same type.

Lemma 4.

$$\widehat{G(F_S)}_{Ar} \subset \widehat{G(F_S)}_{Aut}.$$

Of course, one expects equality; this is the Ramanujan conjecture for $\mathrm{GL}(n)$!

Proof (Sketch). It suffices to see that every $\pi_{1,S} \in \widehat{G(F_S)}_{Ar}$ can be arbitrarily well approximated by local constituents of (unitary) automorphic representations. If $\pi_{1,S}$ is *tempered*, this is Lemma 5 below. In the general case, one may express $\pi_{1,S}$ as a Langlands quotient of an induced-from-tempered representation, and use the results of Moeglin and Waldspurger (see [13], especially Section I.11 and results cited there), together with the theory of Eisenstein series, to conclude the result. $\square$

Lemma 5. *Any tempered representation of $G(F_S)$ may be arbitrarily well approximated (in the Fell topology) by constituents of cuspidal automorphic representations of $G(F_{\mathbb{A}})$.*

Proof. In the spirit of the paper, we present a proof using [2, Thm 1] (roughly equivalent to an elementary proof using Poincaré series). We will use the subscript *temp* to denote the tempered part of the unitary dual for a reductive group over a product of local fields.

Suppose $\rho_S \in \widehat{G(F_S)}_{temp}$. Let w be an auxiliary finite place of F not belonging to S. By the Burger-Sarnak principle (Proposition 1) applied to induction from the trivial group, together with Lemma 1, one gets

$$G(\widehat{F_{\{S,w\}}})_{temp} \subset G(\widehat{F_{\{S,w\}}})_{Aut}. \tag{13}$$

On the other hand, if ρ_w is a supercuspidal representation of $G(F_w)$, the representation $\rho_S \otimes \rho_w$ belongs to $G(\widehat{F_{\{S,w\}}})_{temp}$, and so, by (13), to $G(\widehat{F_{\{S,w\}}})_{Aut}$; as such, it can be approximated in the Fell topology by local constituents of automorphic forms; but these automorphic forms may be taken to be cuspidal (for any supercuspidal irreducible is isolated from non-supercuspidal representations in the unitary dual of $G(F_w)$). □

Conclusion of proof of Proposition 2. Now take S to consist of two places: an arbitrary v, together with a complex place of F. Suppose given $\pi_{1,v}, \pi_{2,v} \in \widehat{G(F_v)}_{Ar}$ with SL(2)-types σ_1, σ_2 respectively. Suppose that $\pi_{1,\mathbb{C}} \in \widehat{G(\mathbb{C})}_{Ar}$ and $\pi_{2,\mathbb{C}} \in \widehat{G(\mathbb{C})}_{Ar}$ also have SL(2)-types σ_1 and σ_2, respectively. By Lemma 4, one has $\pi_{j,\mathbb{C}} \otimes \pi_{j,v} \in \widehat{G(F_S)}_{Aut}$ for $j = 1, 2$. Then, by Burger and Sarnak (Proposition 1) any unitary representation of $G(F_S)$ weakly occurring in $(\pi_{1,\mathbb{C}} \otimes \pi_{2,\mathbb{C}}) \otimes (\pi_{1,v} \otimes \pi_{2,v})$ also belongs to $\widehat{G(F_S)}_{Aut}$.

In particular, by Proposition 3, any representation of $G(F_v)$ occurring weakly in $\pi_{1,v} \otimes \pi_{2,v}$ is associated to the *same* partition τ of n as any representation of $G(\mathbb{C})$ occurring weakly in $\pi_{1,\mathbb{C}} \otimes \pi_{2,\mathbb{C}}$.

Therefore Proposition 2 follows from Lemma 3 by choosing a global field F with a prescribed local completion F_v. (More precisely, if k is a local field of characteristic 0, it is easily checked that there is a number field F and two distinct places v, w of F so that $F_w = \mathbb{C}$ and $F_v = k$). □

3.7. **Twisted forms.** The reasoning above is also valid in twisted cases. We give some simple examples. Let E/F be a quadratic extension of local fields; we will give some instances which come from maps between F-groups of the form GL_n and $\mathrm{Res}_{E/F}\mathrm{GL}_n$.

Let π be a unitary irreducible representation of $\mathrm{GL}(n, E)$ of Arthur type and SL(2)-type $\langle n_1, \ldots, n_r \rangle$. Then the restriction of π to $\mathrm{GL}(n, F)$ has SL(2)-type $\langle 2n_1 - n, 2n_2 - n, \ldots, 2n_r - n \rangle_n$, as follows from our description of the tensor product.

Similarly, the type of all representations occurring in $L^2(\mathrm{GL}(n, F)\backslash \mathrm{GL}(n, E))$ is the same as those occurring in $L^2(\mathrm{GL}(n, F)\backslash \mathrm{GL}(n, F) \times \mathrm{GL}(n, F))$, and thus all these representations have type $\langle 1, \ldots, 1 \rangle$. In particular, one expects all these representations to be tempered (and the validity of this would follow from the Ramanujan conjecture!).

Finally, there is a natural embedding $\mathrm{GL}(n, E) \to \mathrm{GL}(2n, F)$. Then if π is a unitary irreducible representation of $\mathrm{GL}(2n, F)$ of Arthur type and with SL(2)-type $(n_1, \ldots n_r)$, then the restriction of π to $\mathrm{GL}(n, E)$ has SL(2)-type $\langle n_1 - n, n_2 - n, \ldots, n_r - n\rangle_n$.

Of course, each of these could be "guessed" by Mackey-theory in the same way that we have done earlier; also, in some cases far more precise results are known in the context of the Plancherel formula for symmetric spaces. The striking point is that, at the crude level where one is only interested in the behavior of SL(2)-types, the qualitative results depend on the corresponding map of groups over the algebraic closure.

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