ON THE CENTRALIZER OF A REGULAR, SEMI-SIMPLE, STABLE CONJUGACY CLASS

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Abstract. We describe the isomorphism class of the torus centralizing a regular, semi-simple, stable conjugacy class in a simply-connected, semi-simple group.

Let \( k \) be a field, and let \( G \) be a semi-simple, simply-connected algebraic group, which is quasi-split over \( k \). The theory of semi-simple conjugacy classes in \( G \) is well understood, from work of Steinberg [S] and Kottwitz [K]. Any semi-simple conjugacy class \( s \) which is defined over \( k \) is represented by a semi-simple element \( \gamma \) in \( G(k) \). The centralizer \( G_\gamma \) of \( \gamma \) in \( G \) is connected and reductive. It is determined by the stable class \( s \) up to inner twisting, and one can choose a representative \( \gamma \) so that \( G_\gamma \) is quasi-split over \( k \).

In this paper, we will only consider the case when the semi-simple stable class \( s \) is regular. Then \( G_\gamma \) is a maximal torus in \( G \), whose \( k \)-isomorphism class depends only on the class \( s \). Our aim is to determine the isomorphism class of this torus, which we denote \( T_s \) over \( k \), from the data specifying \( s \) in the variety of semi-simple stable conjugacy classes.

We will first give an abstract description of the character group \( X(T_s) \), as an integral representation of the Galois group of \( k \). We will then describe \( T_s \) concretely, in some special cases. In particular, for a simple, split group \( G \) which is not simply-laced, we use a semi-direct product decomposition of the Weyl group to reduce the problem to a semi-simple, quasi-split subgroup \( H_s \) containing \( T_s \) and the long root subgroups of \( G \).

The concrete description of \( T_s \) allows one to compute the terms corresponding to regular classes \( s \) in the stable trace formula (cf. [G-P]). For the general semi-simple class, one would like to have a description of the motive \( M(G_\gamma) \) of the centralizer.

Table of Contents

1. The extended Weyl group
2. The variety of semi-simple classes
3. The discriminant locus
4. The character group \( X(T_s) \)
5. Long and short roots
6. Linear and unitary groups

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1. The extended Weyl group

We recall that $G$ is assumed quasi-split over $k$. Let $B$ be a Borel subgroup, and let $T$ be a Levi factor of $B$ — which is a maximal torus in $G$.

Let $k^s$ denote a separable closure of $k$, and let $\Gamma = \text{Gal}(k^s/k)$. Let $X(T) = \text{Hom}_{k^s}(T, \mathbb{G}_m)$ be the character group of $T$ over $k^s$, which is an integral representation of $\Gamma$. Let $E$ be the fixed field of the kernel of this action, so the quotient $\Gamma_E = \text{Gal}(E/k)$ acts faithfully on $X(T)$. Both the torus $T$ and the group $G$ are split by the finite Galois extension $E$ of $k$.

Let $\mathbf{W} = N_G(T)/T$ be the Weyl group of $T$ in $G$. This is a finite, étale group scheme over $k$, which is pointwise rational over $E$. We put $W = \mathbf{W}(E)$. The Galois group $\Gamma_E$ acts on $W$, and the semi-direct product $W \Gamma_E$ acts on $X(T)$ via the reflection representation

$$r : W \Gamma_E \rightarrow GL(X(T)).$$

We call $W \Gamma_E$ the extended Weyl group.

The roots of $\alpha$ of $T$ are the non-zero elements of $X(T) = \text{Hom}_{E}(T, \mathbb{G}_m)$ which occur in the action of $T$ on $\text{Lie}(G)$ over $E$. They are permuted under the reflection action of the extended Weyl group $W \Gamma_E$. Associated to each root $\alpha$ is a co-root $\alpha^\vee$ in $\text{Hom}_{E}(\mathbb{G}_m, T)$, as well as a reflection $r_\alpha$ in $W$, whose action on $X(T)$ is given by $r_\alpha(x) = x - (x, \alpha^\vee) \cdot \alpha$.

Let $\{\alpha_1, \ldots, \alpha_n\}$ be the simple roots of $T$ determined by $B$. These simple roots are permuted by the action of $\Gamma_E$ on $X(T)$, and the simple reflections $r_\alpha$, generate $W$.

2. The variety of semi-simple classes

The simple co-roots $\{\alpha_1^\vee, \ldots, \alpha_n^\vee\}$ form a basis of $\text{Hom}_{E}(\mathbb{G}_m, T)$, as $G$ is simply-connected. Let $\{\omega_1, \ldots, \omega_n\}$ be the dual basis of $X(T)$. This basis is permuted by the action of $\Gamma_E$. If $\sigma \in \Gamma_E$, we write $\sigma$ for the associated permutation of the $\omega_i$.

Let $V_i$ denote the irreducible representation of $G$ over $E$ with highest weight $\omega_i$ for $B$. Then, for all $\sigma$ in $\Gamma_E$, we have

$$V_i^\sigma \simeq V_{\omega_i(\sigma)}.$$

In particular, if $\gamma$ is any element in $G(k)$, we have

$$\text{Tr}(\gamma|V_i)^\sigma = \text{Tr}(\gamma|V_{\omega_i(\sigma)}) \quad \text{in} \quad E.$$

Let $S$ be the twisted form of affine $n$-space over $k$, given by the permutation representation $\sigma$ of $\Gamma_E$ on the coordinates:

$$S(k) = \{(x_1, \ldots, x_n) \in E^n : x_i^\sigma = x_{\omega_i(\sigma)}\}.$$

If $\gamma$ is an element in $G(k)$, then

$$x(\gamma) = (\text{Tr}(\gamma|V_1), \ldots, \text{Tr}(\gamma|V_n))$$

is a point of $S(k)$, which depends only on the stable conjugacy class of $\gamma$ in $G(k^s)$.

The following fundamental result is due to Steinberg [S].
Proposition. If $s$ is any point in $S(k)$, there is a semi-simple element $\gamma$ in $G(k)$ with $x(\gamma) = s$. The element $\gamma$ is well defined up to conjugacy in $G(k^s)$. The map that assigns to each semi-simple element $\gamma$ the point $x(\gamma)$ identifies $S$ with the variety of semi-simple stable conjugacy classes in $G$.

3. The discriminant locus

Steinberg constructs the variety $S$ as a quotient:

$$T \rightarrow T/W = S.$$  

This covering is étale over the complement of a divisor $D \subset S$.

Over the extension $E$, $S = \mathbb{A}^n = T/W$ and the divisor $D$ is given by the zero locus of a polynomial $D(x_1, \ldots, x_n)$. As a $W$-invariant function on $T$, $D(t)$ can be given by the formula

$$D(t) = (-1)^N \cdot \prod_\alpha (t^\alpha - 1).$$

Here the product is taken over all the roots, and $N$ is the number of positive roots.

For example, when $G = \text{SL}_2$, we have $x = t + t^{-1}$, and

$$D(t) = (-1)(t^2 - 1)(t^{-2} - 1) = x^2 - 4.$$  

The square-root $\Delta$ of $D$ is the usual denominator in the Weyl character formula:

$$\Delta(t) = \prod_{\alpha > 0} \left( t^{\alpha/2} - t^{-\alpha/2} \right).$$

This function on $T$ satisfies $\Delta(wt) = \text{sign}(w)\Delta(t)$.

Since $D(t)$ is also invariant under the action of $\Gamma_E$, it defines a divisor $D$ on $S$ over $k$. The complement $S' = S - D$ defines the variety of regular, semi-simple, stable conjugacy classes in $G$. If $s$ is a point of $S'(k)$, there is a regular, semi-simple conjugacy class $\gamma$ in $G(k)$ with $x(\gamma) = s$. The centralizer $G_\gamma$ of $\gamma$ in $G$ is a maximal torus, whose isomorphism class $T_s$ over $k$ depends only on $s$.

4. The character group $X(T_s)$

Since the covering $T \rightarrow S$ is Galois over $S' = S - D$, it gives rise to a homomorphism of the fundamental group

$$\rho : \pi_1(S') \rightarrow W.\Gamma_E$$

well defined up to conjugacy by $W$. The subgroup $\pi_1^{\text{geom}}(S')$ maps to $W$, and the resulting homomorphism from the quotient $\Gamma = \pi_1^{\text{geom}}/\pi_1^{\text{geom}}$ to $\Gamma_E$ is the standard projection.

Specializing $\rho$ to the point $s$ in $S'(k)$, we obtain a homomorphism

$$\rho_s : \Gamma \rightarrow W.\Gamma_E,$$

well defined up to conjugation by $W$, such that the resulting map $\Gamma \rightarrow \Gamma_E$ is the standard projection. In particular, the normal subgroup $\text{Gal}(k^s/E)$ maps into $W$.

The following result gives an abstract determination of the torus $T_s$ over $k$, via a description of its character group $X(T_s)$. 
Proposition. The character group $X(T_s)$ is isomorphic to the free $\mathbb{Z}$-module $X(T)$, with Galois action given by the composite homomorphism

$$\Gamma \xrightarrow{\rho^*} W.\Gamma_E \xrightarrow{r} \text{GL}(X(T)),$$

where $r$ is the reflection representation.

Proof. We give the argument in the split case, when $E = k$. Let $\gamma$ be a regular, semi-simple class in $G(k)$ which maps to $s$ in $S'(k)$, and let $t$ be an element in $T(k^s)$ which lies above $s$ in the covering $T \to S = T/W$.

Since $\gamma$ and $t$ have the same image in $S(k)$, they are conjugate in $G(k^s) : g\gamma g^{-1} = t$. Conjugation by $g$ gives an isomorphism of their centralizers, which is defined over $k^s$:

$$\varphi : G_\gamma \to T.$$

The fiber over $s$ in the covering $T \to S$ can be identified with the orbit $Wt$ in $T(k^s)$. In particular, since $s$ is defined over $k$, $t^\gamma = w_\alpha(t)$ for every $\gamma \in \Gamma$. The map $\sigma \mapsto w_\sigma$ is the homomorphism $\rho_s : \Gamma \to W$. In particular, the isomorphism $\varphi^\sigma$ (which is conjugation by $g^\sigma$) is equal to $\rho_s(\sigma) \circ \varphi$. Hence the 1-cocycle $\sigma \mapsto \varphi^\sigma$ defines $G_\gamma$ as a twist of $T$ over $k$ is given by the homomorphism $\rho_s : \Gamma \to W \subset \text{Aut}_k(T)$. It follows that the action of $\Gamma$, on $X(T_s) = X(G_\gamma)$ is given by the composition of $\rho_s$ with the reflection representation.

The quasi-split case is similar, but the 1-cocycle $\rho_s$ is not a homomorphism, as $\Gamma_E$ acts nontrivially on $W$. This can be converted to a homomorphism $w_\sigma : \Gamma \to W \subset \text{Aut}_k(T)$ from $\Gamma$ to the extended Weyl group $W.\Gamma_E$ [S3 p. 43]. The rest of the argument is similar. \qed

Note. The above argument shows that the class of the 1-cocycle $\rho_s$ in $H^1(k,W)$ is in the image of $\ker : (H^1(k,N(T)) \to H^1(k,G))$. Any cocycle with this property (or the equivalent homomorphism from $\Gamma$ to the extended Weyl group $W.\Gamma_E$) arises from a stable, regular, semi-simple class in $G$.

5. Long and Short Roots

In this section, we assume that $G$ is quasi-simple and split, and that $W$ has two orbits on the set of roots in $X(T)$. These are the long and short roots; the long roots are in the orbit of the highest root (the highest weight of $B$ on $\text{Lie}(G)$).

Let $W_\ell$ denote the normal subgroup of $W$ generated by the reflections in the long roots. Let $W_{ss}$ denote the subgroup of $W$ generated by the reflections in the short simple roots (relative to $B$).

Proposition. $W$ is isomorphic to the semi-direct product

$$W = W_\ell.W_{ss}.$$

The group $W_\ell$ is the Weyl group of the sub-root system (of the same rank) of long roots. The group $W_{ss}$ isomorphic to the symmetric group $S_m$, where $(m-1)$ is the number of short simple roots.

Proof. The following argument was shown to me by Mark Reeder. Let $P$ be the set of positive roots for $G$, relative to $B$, and let $P_\ell$ be the set of long positive roots. Then $P_\ell$ is a positive system for the sub-root system of long roots.

If $\alpha$ is a simple root, then $r_\alpha(P) = P - \{\alpha\} \cup \{-\alpha\}$. Hence every element in $W_{ss}$ stabilizes $P_\ell$. Since $W_\ell$ acts simply-transitively on the positive systems of long roots, $W_\ell \cap W_{ss} = 1$. 

To show $W = W_\ell, W_{ss}$, let $w$ be an arbitrary element of $W$. Then $w(P) = P'$ is another system of positive roots, and $w(P_\ell) = P'_\ell$ is another system of positive long roots. Hence there is a unique element $w_\ell$ in $W_\ell$ with $P'_\ell = w_\ell(P_\ell)$. The element $v = w^{-1} \cdot w$ then stabilizes $P_\ell$.

To show that $v$ is in $W_{ss}$, we use an argument familiar in Lie theory. Let $J$ be the subset of short simple roots. In the coset $v \cdot W_{ss}$, choose an element $u$ of shortest length. (In fact, the element $u$ is unique.) Then $u(J) \subset P$, for if $u(\alpha) < 0$ for any $\alpha \in J$, $u$ would have a reduced expression ending in $r_\alpha$, contradicting the minimality of its length. Since $u(P_\ell) = P_\ell$ and $u(J) \subset P$, the element $u$ of $W$ stabilizes $P$. Hence $u = 1$, so $v$ is in $W_{ss}$, and $w = w_\ell \cdot v = w_\ell \cdot w_{ss}$ as claimed. □

Here is a table of the cases:

<table>
<thead>
<tr>
<th>$W$ of type</th>
<th>$W_\ell$ of type</th>
<th>$W_{ss}$ isomorphic to</th>
</tr>
</thead>
<tbody>
<tr>
<td>$B_n$</td>
<td>$D_n$</td>
<td>$S_2$</td>
</tr>
<tr>
<td>$C_n$</td>
<td>$(A_1)^n$</td>
<td>$S_n$</td>
</tr>
<tr>
<td>$G_2$</td>
<td>$A_2$</td>
<td>$S_2$</td>
</tr>
<tr>
<td>$F_4$</td>
<td>$D_4$</td>
<td>$S_3$</td>
</tr>
</tbody>
</table>

In this case, the discriminant divisor $D \subset S$ is reducible, as $D(x) = D_\ell(x)D_s(x)$ with

$$D_\ell(t) = (-1)^{N_\ell} \prod_{\alpha \text{ long}} (t^\alpha - 1),$$

$$D_s(t) = (-1)^{N_s} \prod_{\alpha \text{ short}} (t^\alpha - 1).$$

Here $N_\ell$ and $N_s$ are half the number of long and short roots, respectively.

Now let $s$ be a regular, semi-simple, stable class in $G$, and fix an isomorphism $W_{ss} \simeq S_m$. (When $m \neq 6$, this is unique up to inner automorphism. When $m = 6$ it can be fixed by having $W_{ss}$ act on the ± weight spaces in the standard representation of $G = Sp_{12}$.) The composite homomorphism

$$\delta: \Gamma \longrightarrow W \longrightarrow W_{ss} \simeq S_m$$

(up to conjugacy) defines an étale $k$-algebra $K$ of rank $m$: the algebra $K$ is the twist of $k^m$ by the 1-cocycle $\delta$ [Se2 p. 652]. Let $E$ be the finite Galois extension of $k$, fixed by the kernel of $\delta$. Then $E$ is the “Galois closure” of $K$, and $\Gamma_E \subset S_m$ is the image of $\delta$. The automorphism group of the algebra $K$ over $k$ is the centralizer of the subgroup $\Gamma_E$ in $S_m$.

We may view $\rho_s$ as a homomorphism

$$\rho_s: \Gamma \rightarrow W_\ell, \Gamma_E$$

corresponding to a stable class in the quasi-split subgroup $H_s \subset G$ with root system the long roots and Weyl group $W_\ell$. This subgroup is split by $E$, and $T_s \subset H_s \subset G$. This approach often simplifies the computation of $T_s$, as we will see in §7 and §8.

One caveat—several distinct stable classes $s'$ in $H_s$ may become fused (i.e., conjugate) with $s$ in $G$. Indeed, if we conjugate $\rho_s$ above with any element of $W_{ss} \simeq S_m$ which centralizes $\Gamma_E$, we get a homomorphism $\rho_{s'}$ corresponding to a different stable class in $H_s$ which is stably conjugate to $s$ in $G$. This corresponds to the fact that the finite group $\text{Aut}_k(K)$ normalizes the subgroup $H_s$ in $G$. 

CENTRALIZER OF A REGULAR, SEMI-SIMPLE, STABLE CONJUGACY CLASS 291
6. Linear and Unitary Groups

The description of $T_s$ in §4 is fairly abstract. For some classical groups $G$, we can give a more concrete realization of $T_s$, using the characteristic polynomial of the standard representation (cf. [S-S] and [G-Mc, Appendix]). We will describe the group of points $T_s(k)$, and when $k$ is local or global the Artin $L$-function of the Galois representation $X(T_s)$.

Consider the split group $G = SL(V)$ with $n = \dim(V) \geq 2$. The fundamental representations $V_i$ are given by the exterior powers $\Lambda^i V$, for $i = 1, 2, \ldots, n-1$. Giving the point $s = (x_1, \ldots, x_n)$ in $S$ with $x_i = \Tr(\gamma^i \Lambda V)$ is equivalent to specifying the characteristic polynomial of $\gamma$ on $V$:

\[ f(z) = \det(z \cdot 1 - \gamma | V) \]
\[ = z^n - x_1 z^{n-1} + x_2 z^{n-2} - \cdots + (-1)^n. \]

The discriminant $D(s)$ is equal to $\text{disc}(f(z))$, so $s$ lies in $S'$ if and only if the characteristic polynomial $f(z)$ is separable (by which we mean that $f(z)$ has distinct roots in an algebraic closure of $k$).

Assume $s$ lies in $S'(k)$. The $k$-algebra $K = k[z]/(f(z))$ is then étale of rank $n$. The permutation action of $\Gamma$ on the finite set $\text{Hom}(K, k^\times)$ gives a homomorphism $\delta : \Gamma \to S_n$. If we identify $W$ with $S_n$, by having it permute the weight spaces for $T$ on $V$, then $\delta$ is conjugate to $\rho_s$. The torus $T_s$ has points

\[ T_s(k) = \{ t \in K^* : Nt = 1 \} \subset G(k) = SL(K). \]

The $L$-function of the character group is given by $\zeta_K(s)/\zeta_k(s)$.

Now consider the quasi-split unitary group $G = SU(V)$, associated to a (quasi-split) Hermitian space $V$ with $n = \dim(V) \geq 3$ over the separable quadratic field extension $E$. Let $\beta \mapsto \overline{\beta}$ denote the nontrivial automorphism of $E$ over $k$.

Again, the fundamental representations of $G$ are the $\Lambda^i V$. These are defined over $E$, and for $\gamma \in G(k)$, $x_i = \Tr(\gamma^i \Lambda V)$ is conjugate to $x_{n-i}$. Furthermore, if $n = 2m$, $x_m$ lies in $k$. Giving the point $s = (x_1, \ldots, x_{n-1})$ in $S(k)$ is equivalent to specifying the characteristic polynomial of $\gamma$ on $V$:

\[ f(z) = z^n - x_1 z^{n-1} + x_2 z^{n-2} - \cdots + (-1)^n. \]

Again we have $D(s) = \text{disc}(f)$, so $s$ lies in $S'(k)$ precisely when $L = E[z]/(f(z))$ is an étale $E$-algebra of rank $n$.

Since $f(z) = (-z)^n f(1/z)$, the involution $\beta \mapsto \overline{\beta}$ of $E$ extends to an involution $z^\tau = 1/z$ of $L$. Let $M$ be the fixed algebra of $\tau$. If

\[ f(z) \cdot \overline{f(z)} = z^n g(z + 1/z), \]

then $g(y)$ is separable of degree $n$, and $M \simeq k[y]/(g(y))$. Here is a diagram of étale $k$-algebras:
The Hermitian form $\varphi(x,y) = \text{Tr}_{L/E}(cxy^\tau)$ on $L/E$ is nondegenerate provided $c \in M^\ast$. For some choice of $c$, this space is quasi-split. The torus $T_s$ has points
\begin{align*}
T_s(k) &= \{ t \in L^\ast : N_M t = N_E t = 1 \} \\
G(k) &= SU(L, \varphi)
\end{align*}

When $k$ is local or global, the $L$-function of the character group is
\[ \zeta_L(s) \zeta_k(s)/\zeta_M(s) \zeta_E(s). \]

7. Symplectic groups

In this section, we consider the split group $G = \text{Sp}(V)$, where $V$ is a non-degenerate symplectic space over $k$ of dimension $2n$. We use the method of §5 to reduce to the quasi-split subgroup $H_s = \text{Res}_{K/k} \text{SL}_2$, where $K$ is an étale $k$-algebra of rank $n$.

The fundamental representations of $G$ are the virtual modules $\Lambda V - i \Lambda V$, for $1 \leq i \leq n$, when $k$ has characteristic zero. In general, they are always a virtual sum of the $\Lambda V$. Hence the point $s = (x_1, \ldots, x_n)$ in $S$, with $x_i = \text{Tr}(\gamma|V_i)$ determines, and is determined by, the characteristic polynomial of $\gamma$ on $V$:
\[ f(z) = \det(z \cdot 1 - \gamma|V) \]
\[ = z^{2n} - x_1 z^{2n-1} + \cdots - x_1 z + 1. \]

This polynomial is palindromic:
\[ f(z) = z^{2n} f(1/z) \]
as $\Lambda V$ is isomorphic to $2n \Lambda V$. Hence
\[ f(z) = z^n g(z + 1/z) \]
with $g(y) = y^n - x_1 y + \cdots$ of degree $n$.

We have the formulae:
\begin{align*}
D &= D_\ell \cdot D_s, \\
D_\ell(s) &= (-1)^n f(1)f(-1), \\
D_s(s) &= \text{disc}(g) \\
D_\ell \cdot D_s^2 &= \text{disc}(f).
\end{align*}

Hence $\gamma$ is regular if and only if $f(z)$ is separable. If $s \in S'(k)$, we let $K$ be the étale $k$-algebra of rank $n$ defined by $K = k[y]/(g(y))$, and $L$ the étale $k$-algebra of rank $2n$ defined by $L = k[z]/(f(z)) = K[z]/(z^n - yz + 1)$. Here is an algebra diagram:

```
L
  2
  \_\_\_
  K
  \_\_\_
  n
  \_\_\_
  k
```
Let \( \tau \) be the nontrivial involution of \( L \) over \( K \), defined by \( z^\tau = 1/z \).

The homomorphism \( \rho_s : \Gamma \to W = \langle \pm 1 \rangle^n \cdot S_n = W_\ell \cdot W_{ss} \) is given by the action of \( \Gamma \) on the covering of finite sets \( \text{Hom}(L, k^s) \to \text{Hom}(K, k^s) \).

The projection \( \delta : \Gamma \to W \to W_{ss} = S_n \) is given by the \( \acute{e} \text{tale} \) algebra \( K \), with Galois closure \( E \). The subgroup \( H_s = SL_2(K) \) is defined by the long roots, and \( T_s \) is the maximal torus in \( H_s \) defined by the quadratic extension \( L \) of \( K \):

\[
T_s(k) = \{ t \in L^* : t^{1+\tau} = 1 \} \subset SL_2(K).
\]

When \( k \) is local or global, the \( L \)-function of \( X(T_s) \) is equal to \( \zeta_L(s)/\zeta_K(s) \).

8. The Group \( G_2 \)

We now use the method of \( \S 5 \) to treat the split group of type \( G_2 \), the automorphisms of a split octonion algebra over \( k \). Let \( V_1 \) denote the 7-dimensional representation of \( G \) on the octonions of trace 0; this is irreducible and fundamental provided that \( \text{char}(k) \neq 2 \). Let \( V_2 \) denote the 14-dimensional adjoint representation; this is irreducible and fundamental provided that \( \text{char}(k) \neq 3 \).

In general, let \( x_1 = \text{Tr}(\gamma|V_1) \) and \( x_2 = \text{Tr}(\gamma|V_2) \). These elements of \( k \) determine the stable conjugacy class of a semi-simple element \( \gamma \). The characteristic polynomial of \( \gamma \) on the 7-dimensional representation \( V_1 \) has the form \( (z - 1)f(z) \), with

\[
f(z) = z^6 - A z^5 + B z^4 - C z^3 + B z^2 - A z + 1,
\]

\[
A = x_1 - 1,
\]

\[
B = x_2 + 1,
\]

\[
C = x_1^2 - 2 x_2 + 1 = A^2 + 2 A - 2 B + 2.
\]

Furthermore, we have

\[
D_\ell = -4 x_1^3 + x_2^2 + 10 x_1 x_2 + x_1^2 + 2 x_1 + 10 x_2 - 7,
\]

\[
D_s = x_1^2 + 2 x_1 - 4 x_2 - 7.
\]

Assume that the stable class defined by \( \gamma \) is regular. Let

\[
h(\beta) = \beta^2 - A \beta + (B - A) = \beta^2 - (x_1 - 1) \beta + (x_2 - x_1 + 2).
\]

Since

\[
\text{disc}(h) = D_s,
\]

the quadratic algebra \( K = k[\beta]/(h(\beta)) \) is \( \acute{e} \text{tale} \). This is the \( \acute{e} \text{tale} \) algebra defined by the projection \( \rho_s : \Gamma \to W = W_\ell \cdot W_{ss} \to W_{ss} = S_2 \). Its Galois closure \( E \) is either \( K \) (if \( K \) is a field) or \( k \) (if \( K \simeq k + k \)). In both cases, \( \text{Aut}_k(K) = S_2 \).

Over the algebra \( K \), we have the factorization

\[
f(z) = (z^3 - \beta z^2 + \overline{\beta} z - 1)(z^3 - \overline{\beta} z^2 + \beta z - 1),
\]
where \( \alpha \mapsto \overline{\alpha} \) denotes the nontrivial automorphism of \( K \). The quasi-split group \( H_s \) defined by the long roots is \( SU_3(K) \) (which is isomorphic to the split group \( SL_3 \) when \( K = k + k \)), and \( s \) is the stable class in \( H_s \) with separable characteristic polynomial \( z^3 - \beta z^2 + \overline{\beta} z - 1 \). The class \( s' \) with polynomial \( z^3 - \beta z^2 + \beta z - 1 \) is fused with \( s \) in \( G = G_2 \).

Let \( L = K[z]/(z^3 - \beta z^2 + \overline{\beta} z - 1) \) be the fixed algebra. Then 
\[
g(y) = y^3 - (x_1 - 1)y^2 + (x_2 - 2)y - (x_1^2 - 2x_2 - 2x_1 + 1),
\]
\[
f(z) = z^3 (z + 1/z),
\]
\[
disc(g) = D = D_L \cdot D_s.
\]

Here is a diagram of the étale \( k \)-algebras in question:

The torus \( T_s \) has points
\[
T_s(k) = \{ t \in L^* : \mathbb{N}_K t = \mathbb{N}_M t = 1 \}.
\]

If \( k \) is local or global, the \( L \)-function of \( X(T) \) is \( \zeta_L(s) \zeta_k(s)/\zeta_K(s) \zeta_M(s) \). If \( K = k + k \), the \( L \)-function is simply \( \zeta_M(s)/\zeta_k(s) \), and the torus \( T_s \) has points \( T_s(k) = \{ t \in M^* : \mathbb{N}_k t = 1 \} \).

A similar method works for the stable regular semi-simple classes \( s \) in the group \( G = F_4 \). Here the projection of \( \rho_s \) to \( W_{ss} \cong S_3 \) determines an étale cubic algebra \( K \), and the characteristic polynomial of \( s \) on the 26-dimensional representation of \( G \) factors over \( K \) as \( (z - 1)^2 h_8(z) g_{16}(z) \). This allows one to reduce the calculation of \( T_s \) to tori in the quasi-split long root subgroup \( H_s = \text{Spin}_8^K \).

REFERENCES


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