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INTEGRAL STRUCTURES IN THE p-ADIC HOLOMORPHIC DISCRETE SERIES

ELMAR GROSSE-KLÖNNE

ABSTRACT. For a local non-Archimedean field K we construct $\mathrm{GL}_{d+1}(K)$-equivariant coherent sheaves $\mathcal{V}_{\mathcal{O}_K}$ on the formal $\mathcal{O}_K$-scheme $\mathfrak{X}$ underlying the symmetric space X over K of dimension d. These $\mathcal{V}_{\mathcal{O}_K}$ are $\mathcal{O}_K$-lattices in (the sheaf version of) the holomorphic discrete series representations (in K-vector spaces) of $\mathrm{GL}_{d+1}(K)$ as defined by P. Schneider. We prove that the cohomology $H^t(\mathfrak{X},\mathcal{V}_{\mathcal{O}_K})$ vanishes for $t>0$, for $\mathcal{V}_{\mathcal{O}_K}$ in a certain subclass. The proof is related to the other main topic of this paper: over a finite field k, the study of the cohomology of vector bundles on the natural normal crossings compactification Y of the Deligne-Lusztig variety Y^0 for GL_{d+1}/k (so Y^0 is the open subscheme of $\mathbb{P}^d_k$ obtained by deleting all its k-rational hyperplanes).

INTRODUCTION

Let K be a non-Archimedean locally compact field with residue field k of characteristic p. Let $d\in\mathbb{N}$ and let X be the Drinfel'd symmetric space over K of dimension d: the K-rigid space which is the complement in $\mathbb{P}^d_K$ of all K-rational hyperplanes. The group $G=\mathrm{GL}_{d+1}(K)$ acts on X, and it is expected that the cohomology of G-equivariant sheaves on X affords wide classes of interesting G-representations in infinite dimensional vector spaces. For example, by now we know that the ℓ-adic cohomology ($\ell\neq p$) of certain étale coverings of X contains all the *smooth* discrete series representations of G (in characteristic zero). A very different class of G-representations in infinite dimensional K-vector spaces is obtained by taking the global sections of G-equivariant vector bundles on X. The study of these has been initiated by Morita in the case $d=1$. In that case the relevant vector bundles are automorphic line bundles on X which are classified by their weight (an integer, or equivalently: an irredicible K-rational representation of GL_1). Generalizing to any d, Schneider [14] assigns to an irredicible K-rational representation V of GL_d a G-equivariant vector bundle $\mathcal{V}$ on X (in [14] he in fact only considers the action by $\mathrm{SL}_{d+1}(K)$; here we will consider a suitable extension to G which however depends on the choice of $\widehat{\pi}\in\widehat{K}$, see the text). The resulting G-representations $\mathcal{V}(X)$, which he called the "holomorphic discrete series representations", are at present very poorly understood, at least if $d>1$. Already the seemingly innocent case $\mathcal{V}=\omega_X$, the line bundle of d-forms on X, turned out to be fairly intricate and required a host of original techniques (Schneider and Teitelbaum [16]).

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It is natural to ask for integral structures inside $\mathcal{V}$. Let $\mathfrak{X}$ be the natural G-equivariant strictly semistable formal $\mathcal{O}_K$-scheme with generic fibre X constructed in [13] and consider $\mathcal{V}$ as a G-equivariant sheaf on $\mathfrak{X}$ via the specialization map $X \to \mathfrak{X}$. Let $\widehat{K}/K$ be a totally ramified extension of degree $d+1$. In the case $d=1$ we constructed in [5] a G-stable $\mathcal{O}_{\mathfrak{X}} \otimes_{\mathcal{O}_K} \mathcal{O}_{\widehat{K}}$-coherent subsheaf $\mathcal{V}_{\mathcal{O}_{\widehat{K}}}$ inside $\mathcal{V} \otimes_K \widehat{K}$, generalizing a previous construction of Teitelbaum from the case of even weight [17]. We completely determined the cohomology $H^*(\mathfrak{X}, \mathcal{V}_{\mathcal{O}_{\widehat{K}}})$ and obtained (if $\mathrm{char}(K)=0$) applications to the Γ-group cohomology $H^*(\Gamma, \mathcal{V}(X))$ of the above G-representations $\mathcal{V}(X)$ over K, for cocompact discrete subgroups $\Gamma \subset \mathrm{SL}_2(K)$. Let us also mention that Breuil uses (among many other tools) these $\mathcal{V}_{\mathcal{O}_{\widehat{K}}}$ to construct certain Banach space representations of $\mathrm{GL}_2(\mathbb{Q}_p)$ which he expects to occur in a hoped for p-adic continuous Langlands correspondence.

In the present paper we construct integral structures $\mathcal{V}_{\mathcal{O}_{\widehat{K}}}$ inside $\mathcal{V} \otimes_K \widehat{K}$ for arbitrary d. Roughly we proceed as follows. We have $\mathcal{V} \otimes_K \widehat{K} = V \otimes_{\mathcal{O}_K} \mathcal{O}_{\mathfrak{X}} \otimes_{\mathcal{O}_K} \mathcal{O}_{\widehat{K}}$ as an $\mathcal{O}_{\mathfrak{X}} \otimes_{\mathcal{O}_K} \widehat{K}$-module. The definition of the action of G is based on an embedding $\mathrm{GL}_d \to \mathrm{GL}_{d+1}$ which restricts to an embedding $T_1 \to T$ of the respective tori of diagonal matrices. We fix a $\mathrm{GL}_d(\mathcal{O}_K)$-stable $\mathcal{O}_K$-lattice V_0 inside V. It decomposes as $V_0 = \bigoplus_\mu V_{0,\mu}$ with the sum running over the weights μ of V with respect to T_1. We choose $\mathfrak{Y}$, an open formal subscheme of $\mathfrak{X}$ such that the set of irreducible components of the reduction $\mathfrak{Y} \otimes k$ is an orbit for the action of T on the set of irreducible components of $\mathfrak{X} \otimes k$. Restricted to $\mathfrak{Y}$ we define $\mathcal{V}_{\mathcal{O}_{\widehat{K}}}|_{\mathfrak{Y}} = \bigoplus_\mu (\mathcal{V}_{\mathcal{O}_{\widehat{K}}}|_{\mathfrak{Y}})_\mu$ where $(\mathcal{V}_{\mathcal{O}_{\widehat{K}}}|_{\mathfrak{Y}})_\mu$ is a $\mathcal{O}_{\mathfrak{X}} \otimes_{\mathcal{O}_K} \mathcal{O}_{\widehat{K}}$-coherent lattice inside $V_{0,\mu} \otimes_{\mathcal{O}_K} \mathcal{O}_{\mathfrak{X}} \otimes_{\mathcal{O}_K} \widehat{K}$ whose position relative to the constant $\mathcal{O}_{\mathfrak{X}} \otimes_{\mathcal{O}_K} \mathcal{O}_{\widehat{K}}$-lattice $V_{0,\mu} \otimes_{\mathcal{O}_K} \mathcal{O}_{\mathfrak{X}} \otimes_{\mathcal{O}_K} \mathcal{O}_{\widehat{K}}$ is given by μ. Then we prove that there exists a unique G-stable extension $\mathcal{V}_{\mathcal{O}_{\widehat{K}}}$ of $\mathcal{V}_{\mathcal{O}_{\widehat{K}}}|_{\mathfrak{Y}}$ to all of $\mathfrak{X}$.

We begin to compute the cohomology $H^*(\mathfrak{X}, \mathcal{V}_{\mathcal{O}_{\widehat{K}}})$, but the results we obtain here are by no means complete. However, we obtain a clean result for V which are "strongly dominant": if we identify as usual the weights μ of V with vectors $(a_1, \ldots, a_d) \in \mathbb{Z}^d$, then we require $\sum_{i \neq j} a_i \leq d a_j$ for all $1 \leq j \leq d$, for all weights. In particular, V is dominant in the sense that $0 \leq a_i$ for all $1 \leq i \leq d$, for all weights.

Theorem 4.5. *Suppose that V is strongly dominant. Then*

(i) $$H^t(\mathfrak{X}, \mathcal{V}_{\mathcal{O}_{\widehat{K}}}) = 0 \qquad (t > 0),$$

(ii) $$H^t(\mathfrak{X}, \mathcal{V}_{\mathcal{O}_{\widehat{K}}} \otimes_{\mathcal{O}_{\widehat{K}}} k) = 0 \qquad (t > 0),$$

(iii) $$H^0(\mathfrak{X}, \mathcal{V}_{\mathcal{O}_{\widehat{K}}}) \otimes_{\mathcal{O}_{\widehat{K}}} k = H^0(\mathfrak{X}, \mathcal{V}_{\mathcal{O}_{\widehat{K}}} \otimes_{\mathcal{O}_{\widehat{K}}} k).$$

If $d=1$, strong dominance is equivalent with (usual) dominance and our results in [5] showed that for nontrivial V, dominance is equivalent with the validity of the vanishing assertions from Theorem 4.5. If however $d > 1$, (usual) dominance is not enough to guarantee the vanishing assertions from Theorem 4.5; a first counterexample is the case where $d=2$ and V has highest weight corresponding to the vector $(8,3)$.

Examples of $\mathcal{V}_{\mathcal{O}_{\widehat{K}}}$'s arising from strongly dominant representations V are the terms $\Omega^s_{\mathfrak{X}}$ of the relative logarithmic de Rham complex of $\mathfrak{X} \to \mathrm{Spf}(\mathcal{O}_K)$ (with respect to canonical log structures). Of course these are defined even over $\mathcal{O}_K$, for them the extension to $\mathcal{O}_{\widehat{K}}$ is unnecessary. Thus Theorem 4.5 implies $H^t(\mathfrak{X}, \Omega^s_{\mathfrak{X}}) = 0$

for all $t > 0$ and $H^0(\mathfrak{X}, \Omega^s_{\mathfrak{X}}) \otimes k = H^0(\mathfrak{X}, \Omega^i_{\mathfrak{X}} \otimes k)$ (all s). Statements (i) and (iii) of Theorem 4.5 follow from statement (ii). The proof of (ii) is reduced (using the main result from [7]) to the vanishing of $H^t(Z, \mathcal{V}_{\mathcal{O}_{\widehat{K}}} \otimes_{\mathcal{O}_{\widehat{\mathfrak{X}}}} \mathcal{O}_Z)$ for $t > 0$, for all irreducible components Z of $\mathfrak{X} \otimes k$, where we write $\mathcal{O}_{\widehat{\mathfrak{X}}} = \mathcal{O}_{\mathfrak{X}} \otimes_{\mathcal{O}_K} \mathcal{O}_{\widehat{K}}$.

A typical irreducible component $Y = Z$ of $\mathfrak{X} \otimes k$ is isomorphic to the natural compactification, with normal crossings divisors at infinity, of the complement Y^0 in $\mathbb{P}^d_k$ of all k-rational hyperplanes; explicitly, Y is the successive blowing up of $\mathbb{P}^d_k$ in all its k-rational linear subvarieties. $\mathrm{GL}_{d+1}(k)$ acts on Y, and the study of the cohomology of $\mathrm{GL}_{d+1}(k)$-equivariant vector bundles on Y (like $\mathcal{V}_{\mathcal{O}_{\widehat{K}}} \otimes_{\mathcal{O}_{\widehat{\mathfrak{X}}}} \mathcal{O}_Y$) is the analog over k of the program over K described above. Of course this study should be of interest in its own right.

Here we begin (section 1) by establishing vanishing theorems for the cohomology of certain line bundles associated with divisors on Y which are stable under the subgroup of unipotent upper triangular matrices in $\mathrm{GL}_{d+1}(k)$. Given the cohomology of line bundles on projective space this turned out to be an essentially combinatorical matter (the underlying object is the building of PGL_{d+1}/k); in fact, we enjoyed this computation. We obtain $H^t(Y, \mathcal{V}_{\mathcal{O}_{\widehat{K}}} \otimes_{\mathcal{O}_{\widehat{\mathfrak{X}}}} \mathcal{O}_Y) = 0$ for $t > 0$ as desired (for strongly dominant V).

In principle our vanishing theorems for first cohomology groups allow the determination of the $\mathrm{GL}_{d+1}(k)$-representation on the finite dimensional k-vector space $H^0(Y, \mathcal{V}_{\mathcal{O}_{\widehat{K}}} \otimes_{\mathcal{O}_{\widehat{\mathfrak{X}}}} \mathcal{O}_Y)$, and hence of the G-representation on the infinite dimensional k-vector space

$$H^0(\mathfrak{X}, \mathcal{V}_{\mathcal{O}_{\widehat{K}}}) \otimes_{\mathcal{O}_{\widehat{K}}} k.$$

Note that $H^0(\mathfrak{X}, \mathcal{V}_{\mathcal{O}_{\widehat{K}}})$ is a G-stable $\mathcal{O}_{\widehat{K}}$-submodule of the holomorphic discrete series representation $\mathcal{V}(X) \otimes_K \widehat{K}$. We start this discussion in section 2 by analyzing the $\mathrm{GL}_{d+1}(k)$-representations $H^0(Y, \mathcal{V}_{\mathcal{O}_{\widehat{K}}} \otimes_{\mathcal{O}_{\widehat{\mathfrak{X}}}} \mathcal{O}_Y)$ with $\mathcal{V}_{\mathcal{O}_{\widehat{K}}} = \Omega^s_{\mathfrak{X}} \otimes_{\mathcal{O}_K} \mathcal{O}_{\widehat{K}}$ for some s. If $\Omega^\bullet_Y$ denotes the de Rham complex on Y with allowed logarithmic poles along $Y - Y^0$, then $\Omega^\bullet_Y \cong (\Omega^s_{\mathfrak{X}} \otimes_{\mathcal{O}_K} \mathcal{O}_{\widehat{K}}) \otimes_{\mathcal{O}_{\widehat{\mathfrak{X}}}} \mathcal{O}_Y = \Omega^s_{\mathfrak{X}} \otimes_{\mathcal{O}_{\mathfrak{X}}} \mathcal{O}_Y$, thus we are studying the $\mathrm{GL}_{d+1}(k)$-representations $H^0(Y, \Omega^s_Y)$. We describe an explicit k-basis of $H^0(Y, \Omega^s_Y)$ consisting of logarithmic differential forms and show that, as a $\mathrm{GL}_{d+1}(k)$-representation, it is a generalized Steinberg representation. As a corollary of our explicit computations we obtain the irreducibility of these generalized Steinberg representations. Moreover, we derive that the log crystalline cohomology $H^s_{crys}(Y/W(k))$ is a representation of $\mathrm{GL}_{d+1}(k)$ on a *free* finite $W(k)$-module, and that $H^0(Y, \Omega^s_Y)$ is its reduction modulo p. Note that $H^s_{crys}(Y/W(k)) \otimes \mathbb{Q} = H^s_{rig}(Y^0)$, the rigid cohomology with constant coefficients of the Deligne-Lusztig variety Y^0 for GL_{d+1}/k.

Another application of our vanishing theorems can be found in [8]; they enable us to compute the cohomology of sheaves of bounded logarithmic differential forms on $\mathfrak{X}$, with coefficients in certain algebraic $\mathrm{GL}_{d+1}(K)$-representations. This leads to the proof of certain (previously unknown) cases of a conjecture of Schneider, formulated in [14], concerning Hodge decompositions of the de Rham cohomology (with coefficients) of projective K-varieties uniformized by X.

Notations. K denotes a non-Archimedean locally compact field and K_a its algebraic closure, $\mathcal{O}_K$ its ring of integers, $\pi \in \mathcal{O}_K$ a fixed prime element and k the residue field with q elements, $q \in p^{\mathbb{N}}$. We denote by $\omega : K_a^\times \to \mathbb{Q}$ the extension of the discrete valuation $\omega : K^\times \to \mathbb{Z}$ normalized by $\omega(\pi) = 1$. We fix $\widehat{\pi} \in K_a$ with $\widehat{\pi}^{d+1} = \pi$ and set $\widehat{K} = K(\widehat{\pi})$.

We fix $d \in \mathbb{N}$ and enumerate the rows and columns of GL_{d+1}-elements by $0, \ldots, d$. We let $U \subset \mathrm{GL}_{d+1}$ denote the subgroup of unipotent upper triangular matrices,

$$U = \{(a_{ij})_{0 \le i,j \le d} \in \mathrm{GL}_{d+1} \quad | \quad a_{ii} = 1 \text{ for all } i, \quad a_{ij} = 0 \text{ if } i > j\}.$$

We set $G = \mathrm{GL}_{d+1}(K)$. For $r \in \mathbb{R}$ we define $\lfloor r \rfloor, \lceil r \rceil \in \mathbb{Z}$ by requiring $\lfloor r \rfloor \le r < \lfloor r \rfloor + 1$ and $\lceil r \rceil - 1 < r \le \lceil r \rceil$. For a divisor D on a smooth connected k-scheme S we denote by $\mathcal{L}_S(D)$ the associated line bundle on S, endowed with its canonical embedding into the constant sheaf generated by the function field of S.

1. Line bundle cohomology of rational varieties

The (right) action of $\mathrm{GL}_{d+1}(k) = \mathrm{GL}(k^{d+1})$ on $(k^{d+1})^* = \mathrm{Hom}_k(k^{d+1}, k)$ defines a (left) action of $\mathrm{GL}_{d+1}(k)$ on the affine k-scheme associated with $(k^{d+1})^*$, and this action passes to a (left) action of $\mathrm{GL}_{d+1}(k)$ on the projective space

$$Y_0 = \mathbb{P}((k^{d+1})^*) \cong \mathbb{P}_k^d.$$

For $0 \le j \le d-1$ let $\mathcal{V}_0^j$ be the set of all k-rational linear subvarieties Z of Y_0 with $\dim(Z) = j$, and let $\mathcal{V}_0 = \bigcup_{j=0}^{d-1} \mathcal{V}_0^j$. The sequence of projective k-varieties

$$Y = Y_{d-1} \longrightarrow Y_{d-2} \longrightarrow \ldots \longrightarrow Y_0$$

is defined inductively by letting $Y_{j+1} \to Y_j$ be the blowing up of Y_j in the strict transforms (in Y_j) of all $Z \in \mathcal{V}_0^j$. The action of $\mathrm{GL}_{d+1}(k)$ on Y_0 naturally lifts to an action of $\mathrm{GL}_{d+1}(k)$ on Y. Let $\Xi_0, \ldots, \Xi_d$ be the standard projective coordinate functions on Y_0 and hence on Y corresponding to the canonical basis of $(k^{d+1})^*$. Hence $Y_0 = \mathrm{Proj}(k[\Xi_i;\, i \in \Upsilon])$ with

$$\Upsilon = \{0, \ldots, d\}.$$

For $\emptyset \neq \tau \subset \Upsilon$ let $V_{\tau,0}$ be the reduced closed subscheme of Y_0 which is the common zero set of $\{\Xi_i\}_{i \in \tau}$. Let V_τ be the closed subscheme of Y which is the strict transform of $V_{\tau,0}$ under $Y \to Y_0$. These V_τ are particular elements of the following set of divisors on Y:

$$\mathcal{V} = \text{the set of all strict transforms in } Y \text{ of elements of } \mathcal{V}_0.$$

For $\tau \subset \Upsilon$ let

$$\tau^c = \Upsilon - \tau,$$

$$U_\tau = \{(a_{ij})_{0 \le i,j \le d} \in U(k) \quad | \quad a_{ij} = 0 \text{ if } i \neq j \text{ and } [j \in \tau^c \text{ or } \{i,j\} \subset \tau]\}.$$

Let

$$\mathcal{Y} = \{\tau \quad | \quad \emptyset \neq \tau \subsetneq \Upsilon\},$$
$$\mathcal{N} = \{(\tau, u) \quad | \quad \tau \in \mathcal{Y}, u \in U_\tau\}.$$

We have bijections

$$\mathcal{N} \cong \mathcal{V}, \qquad (\tau, u) \mapsto u.V_\tau,$$
$$\mathcal{Y} \cong (\text{the set of orbits of } U(k) \text{ acting on } \mathcal{V}), \quad \tau \mapsto \{u.V_\tau |\, u \in U_\tau\}.$$

Remark. We also have a bijection between $\mathcal{N}$ and the set of vertices of the building associated to PGL_{d+1}/k. The subset $\{(\tau, 1) |\, \tau \in \mathcal{Y}\}$ of $\mathcal{N}$ is then the set of vertices in a standard apartment.

For an element $\overline{a} = (\overline{a}_1, \ldots, \overline{a}_d)$ of $\mathbb{Z}^d$ let $\overline{a}_0 = -\sum_{j=1}^d \overline{a}_j$ and for $\sigma \in \mathcal{Y}$ let

$$b_\sigma(\overline{a}) = -\sum_{j\in\sigma} \overline{a}_j.$$

Given two more elements $n = (n_1, \ldots, n_d)$ and $m = (m_1, \ldots, m_d)$ of $\mathbb{Z}^d$ we define the divisor

$$D(\overline{a}, n, m) = \sum_{\substack{\sigma\in\mathcal{Y}\\ 0\notin\sigma}} (m_{|\sigma|} + b_\sigma(\overline{a})) \sum_{u\in U_\sigma} u.V_\sigma + \sum_{0\in\sigma\in\mathcal{Y}} (n_{|\sigma|} + b_\sigma(\overline{a})) \sum_{u\in U_\sigma} u.V_\sigma \tag{1}$$

on Y. The purpose of this section is to prove vanishing theorems for the cohomology of line bundles on Y of the type $\mathcal{L}_Y(D(\overline{a}, n, m))$ for suitable $\overline{a}, n, m$.

We need relative (or restricted) analogs of the above definitions. For a subset $\sigma \subset \Upsilon$ we define the sequence of projective k-varieties

$$Y^\sigma = Y^\sigma_{|\sigma|-2} \longrightarrow Y^\sigma_{|\sigma|-3} \longrightarrow \ldots \longrightarrow Y^\sigma_0$$

as follows: $Y^\sigma_0 = \mathrm{Proj}(k[\Xi_i]_{i\in\sigma}) \cong \mathbb{P}^{|\sigma|-1}_k$ and $Y^\sigma_{j+1} \to Y^\sigma_j$ is the blowing up of Y^σ_j in the strict transforms (under $Y^\sigma_j \to Y^\sigma_0 = \mathrm{Proj}(k[\Xi_i]_{i\in\sigma})$) of all j-dimensional k-rational linear subvarieties of $\mathrm{Proj}(k[\Xi_i]_{i\in\sigma})$. We write $Y^\sigma_{-1} = \mathrm{Spec}(k)$. For $\emptyset \neq \sigma \subset \Upsilon$ let

$$H_\sigma = \{(a_{ij})_{0\le i,j\le d} \in \mathrm{GL}_{d+1}(k) \quad | \quad \begin{array}{c} a_{ii} = 1 \text{ for all } i \in \sigma^c \\ a_{ij} = 0 \text{ if } i \neq j \text{ and } \{i,j\} \cap \sigma^c \neq \emptyset \end{array} \}.$$

Then H_σ-acts on Y^σ (by forgetting the a_{ij} with $\{i,j\} \cap \sigma^c \neq \emptyset$). In several subsequent proofs we will induct on d; for that purpose we note that Y^σ with its action by $H_\sigma \cong \mathrm{GL}_{|\sigma|}(k)$ and its fixed ordered set of projective coordinate functions $(\Xi_i)_{i\in\sigma}$ (as ordering on σ we take the one induced by its inclusion into the ordered set $\Upsilon = \{0, \ldots, d\}$) is just like the data

$(Y$ with its $\mathrm{GL}_{d+1}(k)$ action, $(\Xi_i)_{0\le i\le d}$ with the natural ordering on $\{0, \ldots, d\})$,

but of dimension $|\sigma| - 1$ instead of d. For $\tau \subset \sigma$ in $\mathcal{Y}$ let

$$U^\sigma_\tau = \{(a_{ij})_{0\le i,j\le d} \in U_\tau \quad | \quad a_{ij} = 0 \text{ if } i \neq j \text{ and } i \in \sigma^c\}.$$

This is a subgroup of H_σ. For $b \in \tau \in \mathcal{Y}$ it is sometimes convenient to write

$$U^{\{b\}}_\tau := U^{\Upsilon-\{b\}}_{\tau-\{b\}} = U^{\{b\}^c}_{\tau-\{b\}}.$$

For $\sigma \in \mathcal{Y}$ let

$$\mathcal{N}^\sigma = \{(\tau, u) \quad | \quad \emptyset \neq \tau \subsetneq \sigma \text{ and } u \in U^\sigma_\tau\},$$
$$\mathcal{V}^\sigma = \{V \in \mathcal{V} \quad | \quad V_\sigma \neq V \cap V_\sigma \neq \emptyset\}.$$

Then we have a bijection

$$\mathcal{N}^\sigma \coprod \mathcal{N}^{\sigma^c} \cong \mathcal{V}^\sigma \tag{2}$$

where the map $\mathcal{N}^\sigma \to \mathcal{V}^\sigma$ is given by $(\tau, u) \mapsto u.V_\tau$, and where the map $\mathcal{N}^{\sigma^c} \to \mathcal{V}^\sigma$ is given by $(\tau, u) \mapsto u.V_{\tau\cup\sigma}$.

Lemma 1.1. *For $\sigma \in \mathcal{Y}$ there is a canonical isomorphism*

$$V_\sigma \cong Y^\sigma \times Y^{\sigma^c},$$

equivariant for the respective actions of H_σ and H_{σ^c} on both sides.

Proof (cf. also [11], sect. 4). For $0 \leq j \leq |\sigma|-2$ let $V_{\sigma,j}$ be the closed subscheme of Y_j which is the strict transform of $V_{\sigma,0}$ under $Y_j \to Y_0$. We can naturally identify Y^{σ^c} with $V_{\sigma,d-|\sigma|}$. On the other hand, $V_{\sigma,d-|\sigma|+1} = \underline{\mathrm{Proj}}(\mathrm{Sym}_{\mathcal{O}_{V_{\sigma,d-|\sigma|}}}(\mathcal{J}/\mathcal{J}^2))$ according to [9] II, 8.24, where $\mathcal{J}$ denotes the ideal sheaf of $V_{\sigma,d-|\sigma|}$ in $Y_{d-|\sigma|}$. This is the pullback of the ideal sheaf of $V_{\sigma,0}$ in $Y_0 = \mathrm{Proj}(k[\Xi_0,\ldots,\Xi_d])$, i.e. the one corresponding to the homogeneous ideal $(\Xi_i)_{i\in\sigma} \subset k[\Xi_0,\ldots,\Xi_d]$. Hence $V_{\sigma,d-|\sigma|+1} = \mathrm{Proj}(k[\Xi_i]_{i\in\sigma}) \times V_{\sigma,d-|\sigma|}$. Under this isomorphism the successive blowing up of the first factor $\mathrm{Proj}(k[\Xi_i]_{i\in\sigma})$ in its k-rational linear subvarieties of dimension $\leq |\sigma|-3$ corresponds to taking the strict transform of $V_{\sigma,d-|\sigma|+1}$ in Y. □

Let $\sigma \in \mathcal{Y}$. Viewing Y^σ as the version of Y of dimension $|\sigma|-1$ instead of d, definition (1) provides us with particular divisors on Y^σ. Explicitly we name the divisors

$$D(0,\mathbf{1},0)^\sigma = \sum_{s\in\tau\subsetneq\sigma}\sum_{u\in U_\tau^\sigma} u.V_\tau^\sigma,$$
$$D(0,0,\mathbf{1})^\sigma = \sum_{\emptyset\neq\tau\subset\sigma-\{s\}}\sum_{u\in U_\tau^\sigma} u.V_\tau^\sigma$$

on Y^σ, where $s \in \sigma$ is the minimal element and where the prime divisor V_τ^σ on Y^σ is the strict transform under $Y^\sigma \to Y_0^\sigma$ of the common zero set of $\{\Xi_i\}_{i\in\tau}$, and where $\mathbf{1} = (1,\ldots,1) \in \mathbb{Z}^{|\sigma|-1}$.

Proposition 1.2. *Let $\sigma \in \mathcal{Y}$. With the divisor*

$$E = D(0,\mathbf{1},0)^\sigma \times Y^{\sigma^c} + Y^\sigma \times D(0,0,\mathbf{1})^{\sigma^c}$$

on $V_\sigma = Y^\sigma \times Y^{\sigma^c}$ we have the following isomorphism of line bundles on V_σ:

$$\mathcal{L}_Y(-V_\sigma) \otimes_{\mathcal{O}_Y} \mathcal{O}_{V_\sigma} \cong \mathcal{L}_{V_\sigma}(E).$$

Proof. For any $b \in \Upsilon$ the pullback to Y of the divisor $V_{\{b\},0}$ on Y_0 is the divisor $\sum_{b\in\tau\in\mathcal{Y}}\sum_{u\in U_\tau^{\{b\}}} u.V_\tau$. Let s, resp. t, be the minimal element of σ, resp. of σ^c. The equivalence of divisors $V_{\{s\},0} \sim V_{\{t\},0}$ on Y_0 gives rise to the equivalence

$$\sum_{s\in\tau\in\mathcal{Y}}\sum_{u\in U_\tau^{\{s\}}} u.V_\tau \quad\sim\quad \sum_{t\in\tau\in\mathcal{Y}}\sum_{u\in U_\tau^{\{t\}}} u.V_\tau$$

on Y. Thus

$$\sum_{s\in\tau\in\mathcal{Y}}\sum_{\substack{u\in U_\tau^{\{s\}}\\ u\notin U_\tau^{\{t\}}\text{ if } t\in\tau}} u.V_\tau \quad\sim\quad \sum_{t\in\tau\in\mathcal{Y}}\sum_{\substack{u\in U_\tau^{\{t\}}\\ u\notin U_\tau^{\{s\}}\text{ if } s\in\tau}} u.V_\tau$$

or equivalently

$$-V_\sigma \quad\sim\quad -(\sum_{t\in\tau\in\mathcal{Y}}\sum_{\substack{u\in U_\tau^{\{t\}}\\ u\notin U_\tau^{\{s\}}\text{ if } s\in\tau}} u.V_\tau) + \sum_{s\in\tau\in\mathcal{Y}}\sum_{\substack{u\in U_\tau^{\{s\}}\\ u\notin U_\tau^{\{t\}}\text{ if } t\in\tau\\ u\neq 1\text{ if } \tau=\sigma}} u.V_\tau.$$

Now we are interested only in the summands which belong to the set $\mathcal{V}^\sigma$ which we determined in (2). For example, all the summands in the bracketed term on the right-hand side do not belong to $\mathcal{V}^\sigma$ (the condition $t \in \tau$ excludes contributions from $\mathcal{N}^{\sigma}$, the condition $u \notin U_{\tau}^{\{s\}}$ if $s \in \tau$ and the fact $U_{\tau-\sigma}^{\sigma^c} \subset U_{\tau}^{\{s\}}$ for $\sigma \subset \tau$ exclude contributions from $\mathcal{N}^{\sigma^c}$). We get

$$\mathcal{L}_Y(-V_{\sigma}) \otimes_{\mathcal{O}_Y} \mathcal{O}_{V_{\sigma}} \cong \mathcal{L}_{V_{\sigma}}(E)$$

with

$$E = V_{\sigma} \cap (\sum_{s\in\tau\subsetneq\sigma} \sum_{u\in U_{\tau}^{\{s\}}\cap U_{\tau}^{\sigma}} u.V_{\tau} + \sum_{\sigma\subsetneq\tau\subsetneq\Upsilon} \sum_{\substack{u\in U_{\tau-\sigma}^{\sigma^c} \\ u\notin U_{\tau}^{\{t\}} \text{ if } t\in\tau}} u.V_{\tau}).$$

Now since s is minimal in σ we see that $U_{\tau}^{\sigma} \cap U_{\tau}^{\{s\}} = U_{\tau}^{\sigma}$ for all $\tau \subset \sigma$. On the other hand, since t is minimal in σ^c we have $U_{\tau}^{\{t\}} = U_{\tau}$ for all $\sigma \subset \tau$ with $t \in \tau$. We obtain

$$E = V_{\sigma} \cap (\sum_{s\in\tau\subsetneq\sigma} \sum_{u\in U_{\tau}^{\sigma}} u.V_{\tau} + \sum_{\sigma\subsetneq\tau\subset\Upsilon-\{t\}} \sum_{u\in U_{\tau-\sigma}^{\sigma^c}} u.V_{\tau}).$$

That this is the divisor E as stated follows from the construction of the isomorphism $V_{\sigma} = Y^{\sigma} \times Y^{\sigma^c}$. □

Proposition 1.3. *Assume $-d \le n_1 \le \ldots \le n_d$ and $0 \le m_1 \le \ldots \le m_d$; furthermore, $n_i - n_1 \le i-1$ and $m_i - m_1 \le i-1$ for all $1 \le i \le d$.*

(a) *We have $H^t(Y, \mathcal{L}_Y(D(0,n,m))) = 0$ for all $t \in \mathbb{Z}_{>0}$.*

(b) *If $n_d \le -1$ and $m_1 = 0$, we have $H^t(Y, \mathcal{L}_Y(D(0,n,m))) = 0$ for all $t \in \mathbb{Z}$.*

Proof. (b) Outer induction on d, inner induction on

$$s(n,m) = \sum_{i=1}^{d}(m_i + n_i - n_1).$$

Our assumptions imply $s(n,m) \ge 0$. If $s(n,m) = 0$ or if $d = 1$, we have $n_i = n_1$ and $m_i = 0$ for all $1 \le i \le d$. Then $D(0,n,m)$ is the pullback of $\mathcal{O}_{\mathbb{P}^d}(n_1)$ under the successive blowing up $Y \to \mathbb{P}^d$ and the claim follows from $H^t(\mathbb{P}^d, \mathcal{O}_{\mathbb{P}^d}(n_1)) = 0$ for all $t \in \mathbb{Z}$. Now let $s(n,m) > 0$. First suppose that there exists a $2 \le i \le d$ with $n_i \ne n_1$. Then let i_0 be minimal with this property. Let $n'_{i_0} = n_{i_0} - 1$ and $n'_i = n_i$ for all $i \ne i_0$. Then also $n' = (n'_1, \ldots, n'_d)$ satisfies our hypothesis, and $s(n',m) < s(n,m)$. We have an exact sequence

$$0 \longrightarrow \mathcal{L}_Y(D(0,n',m)) \longrightarrow \mathcal{L}_Y(D(0,n,m)) \longrightarrow \mathcal{C} \longrightarrow 0$$

and in view of the induction hypothesis it suffices to show $H^t(Y, \mathcal{C}) = 0$ for all $t \in \mathbb{Z}$. We may view $\mathcal{C}$ as living on the closed subscheme $\coprod_{(u,\sigma)} u.V_{\sigma}$ with σ running through all elements of $\mathcal{Y}$ with $0 \in \sigma$ and $|\sigma| = i_0$, and with $u \in U_{\sigma}$. We deal with every such (u,σ) separately. By equivariance we may assume $u = 1$. The restriction $\mathcal{C}|_{V_{\sigma}}$ of $\mathcal{C}$ to $V_{\sigma} = Y^{\sigma} \times Y^{\sigma^c}$ is isomorphic to

$$\mathcal{L}_{Y^{\sigma}}(D(0,\widetilde{n},\widetilde{m})) \otimes_{\mathcal{O}_{Y^{\sigma}}} \mathcal{O}_{V_{\sigma}} \otimes_{\mathcal{O}_{Y^{\sigma^c}}} \mathcal{E}$$

with a line bundle $\mathcal{E}$ on Y^{σ^c} and with $\widetilde{n} = (\widetilde{n}_1, \ldots, \widetilde{n}_{i_0-1})$ and $\widetilde{m} = (\widetilde{m}_1, \ldots, \widetilde{m}_{i_0-1})$ defined by $\widetilde{n}_i = n_i - n_{i_0}$ and $\widetilde{m}_i = m_i$ (for $1 \le i \le i_0 - 1$). This follows from Proposition 1.2. By induction hypothesis we have $H^t(Y^{\sigma}, \mathcal{L}_{Y^{\sigma}}(D(0,\widetilde{n},\widetilde{m}))) = 0$ for all $t \in \mathbb{Z}$. Thus $H^t(Y, \mathcal{C}|_{V_{\sigma}}) = 0$ follows from the Künneth formula.

If there is no $2 \le i \le d$ with $n_i \ne n_1$, then $s(n,m) > 0$ implies that there is a $2 \le i \le d$ with $m_i \ne 0$. Let i_0 be minimal with this property. Let $m'_{i_0} = m_{i_0} - 1$

and $m'_i = m_i$ for all $i \neq i_0$. Then also $m' = (m'_1, \dots, m'_d)$ satisfies our hypothesis, and $s(n, m') < s(n, m)$. We have an exact sequence

$$0 \longrightarrow \mathcal{L}_Y(D(0, n, m')) \longrightarrow \mathcal{L}_Y(D(0, n, m)) \longrightarrow \mathcal{C} \longrightarrow 0$$

and in view of the induction hypothesis it suffices to show $H^t(Y, \mathcal{C}) = 0$ for all $t \in \mathbb{Z}$. We may view $\mathcal{C}$ as living on the closed subscheme $\coprod_{(u,\sigma)} u.V_\sigma$ with σ running through all elements of $\mathcal{Y}$ with $0 \notin \sigma$ and $|\sigma| = i_0$, and with $u \in U_\sigma$. We deal with every such (u, σ) separately. By equivariance we may assume $u = 1$. The restriction $\mathcal{C}|_{V_\sigma}$ of $\mathcal{C}$ to $V_\sigma = Y^\sigma \times Y^{\sigma^c}$ is isomorphic to

$$\mathcal{L}_{Y^\sigma}(D(0, \widetilde{n}, \widetilde{m})) \otimes_{\mathcal{O}_{Y^\sigma}} \mathcal{O}_{V_\sigma} \otimes_{\mathcal{O}_{Y^{\sigma^c}}} \mathcal{E}$$

with a line bundle $\mathcal{E}$ on Y^{σ^c} and with $\widetilde{n} = (\widetilde{n}_1, \dots, \widetilde{n}_{i_0-1})$ and $\widetilde{m} = (\widetilde{m}_1, \dots, \widetilde{m}_{i_0-1})$ defined by $\widetilde{n}_i = m_i - m_{i_0}$ and $\widetilde{m}_i = m_i$ (for $1 \leq i \leq i_0 - 1$). This follows from Proposition 1.2. By induction hypothesis we have $H^t(Y^\sigma, \mathcal{L}_{Y^\sigma}(D(0, \widetilde{n}, \widetilde{m}))) = 0$ for all $t \in \mathbb{Z}$. Thus $H^t(Y, \mathcal{C}|_{V_\sigma}) = 0$ follows from the Künneth formula.

(a) Outer induction on d. If $d = 1$ our statement is the well-known fact $H^t(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(k)) = 0$ for all $k \in \mathbb{Z}_{\geq -1}$, all $t \in \mathbb{Z}_{>0}$. Inner induction on

$$r(n, m) = d^2 + \sum_{i=1}^{d} (m_i + n_i)$$

which is ≥ 0 as follows from our assumptions. The case $r(n, m) = 0$ corresponds to $n_i = -d$ and $m_i = 0$ for all i, hence was settled in (b). Now suppose $r(n, m) > 0$. Then at least one of the following cases occurs:

(i) There is a $1 \leq i_0 \leq d$ such that if we set $m'_i = m_i$ for $i \neq i_0$ and $m'_{i_0} = m_{i_0} - 1$, then also $m' = (m'_1, \dots, m'_d)$ satisfies our hypothesis.

(ii) There is a $1 \leq i_0 \leq d$ such that if we set $n'_i = n_i$ for $i \neq i_0$ and $n'_{i_0} = n_{i_0} - 1$, then also $n' = (n'_1, \dots, n'_d)$ satisfies our hypothesis.

Fix one of these two cases which holds true and let i_0 be the minimal element satisfying its condition. In case (i) let $n' = n$ and in case (ii) let $m' = m$. Since by induction hypothesis we have $H^t(Y, \mathcal{L}_Y(D(0, n', m'))) = 0$ for all $t \in \mathbb{Z}_{>0}$, the exact sequence

$$0 \longrightarrow \mathcal{L}_Y(D(0, n', m')) \longrightarrow \mathcal{L}_Y(D(0, n, m)) \longrightarrow \mathcal{C} \longrightarrow 0$$

shows that it suffices to show $H^t(Y, \mathcal{C}) = 0$ for all $t \in \mathbb{Z}_{>0}$. We may view $\mathcal{C}$ as living on the closed subscheme $\coprod_{(u,\sigma)} u.V_\sigma$ with $u \in U_\sigma$ and with σ running through all elements of $\mathcal{Y}$ with $|\sigma| = i_0$ and, in addition, with $0 \in \sigma$ in case (ii), resp. with $0 \notin \sigma$ in case (i). We deal with every such (u, σ) separately. By equivariance we may assume $u = 1$. The restriction $\mathcal{C}|_{V_\sigma}$ of $\mathcal{C}$ to $V_\sigma = Y^\sigma \times Y^{\sigma^c}$ is isomorphic (use Proposition 1.2) to

$$\mathcal{L}_{Y^\sigma}(D(0, \widetilde{n}, \widetilde{m})) \otimes_{\mathcal{O}_{Y^\sigma}} \mathcal{O}_{V_\sigma} \otimes_{\mathcal{O}_{Y^{\sigma^c}}} \mathcal{L}_{Y^{\sigma^c}}(D(0, \widehat{n}, \widehat{m}))$$

with $\widetilde{m} = (\widetilde{m}_1, \dots, \widetilde{m}_{i_0-1})$, $\widetilde{n} = (\widetilde{n}_1, \dots, \widetilde{n}_{i_0-1})$, $\widehat{m} = (\widehat{m}_1, \dots, \widehat{m}_{d-i_0})$ and $\widehat{n} = (\widehat{n}_1, \dots, \widehat{n}_{d-i_0})$ defined as follows: $\widehat{m}_i = m_{i+i_0} - m_{i_0}$ in case (i), and $\widehat{m}_i = n_{i+i_0} - n_{i_0}$ in case (ii). Moreover, $\widetilde{n}_i = m_i - m_{i_0}$ in case (i), and $\widetilde{n}_i = n_i - n_{i_0}$ in case (ii). Finally, $\widetilde{m}_i = m_i$ and $\widehat{n}_i = n_{i_0+i}$ in all cases. By induction hypothesis $H^t(Y^\sigma, \mathcal{L}_{Y^\sigma}(D(0, \widetilde{n}, \widetilde{m}))) = 0$ and $H^t(Y^{\sigma^c}, \mathcal{L}_{Y^{\sigma^c}}(D(0, \widehat{n}, \widehat{m}))) = 0$ for all $t \in \mathbb{Z}_{>0}$, so we conclude by the Künneth formula. □

Theorem 1.4. *Let m and n satisfy the hypothesis of Proposition* 1.3. *Let*

$$e=\min\{0\leq i\leq d \quad | \quad \overline{a}_t\leq 0 \text{ for all } t>i\}.$$

Then $H^t(Y,\mathcal{L}_Y(D(\overline{a},n,m)))=0$ for all $t>e$.

Proof. Let us introduce some more notation. For an element $\sigma\in\mathcal{Y}$ we define

$$\overline{a}|_\sigma=(\overline{a}|_{\sigma,1},\ldots,\overline{a}|_{\sigma,|\sigma|-1})\in\mathbb{Z}^{|\sigma|-1}$$

as follows. Let

$$\iota_\sigma:\{0,\ldots,|\sigma|-1\}\to\sigma$$

be the order preserving bijection. Then

$$\overline{a}|_{\sigma,i}=\overline{a}_{\iota_\sigma(i)}\qquad(1\leq i\leq|\sigma|-1).$$

Thus, $\overline{a}|_\sigma$ enumerates those components of the $(d+1)$-tupel $(\overline{a}_0,\ldots,\overline{a}_d)$ which have index in σ, but omits the first one of these. For an s-tuple $\overline{b}=(\overline{b}_1,\ldots,\overline{b}_s)\in\mathbb{Z}^s$ (some $s\in\mathbb{N}$) and an element $i\in\{1,\ldots,s\}$ we define

$$\overline{b}^{[i]}=(\overline{b}_1^{[i]},\ldots,\overline{b}_s^{[i]})=\overline{b}+(0,\ldots,0,1,0,\ldots,0)\in\mathbb{Z}^s$$

by setting $\overline{b}_j^{[i]}=\overline{b}_j$ for $j\neq i$, and $\overline{b}_i^{[i]}=\overline{b}_i+1$.

Now we begin. Outer induction on d. If $d=1$ our statement is the well-known fact $H^t(\mathbb{P}^1,\mathcal{O}_{\mathbb{P}^1}(k))=0$ for all $k\in\mathbb{Z}_{\geq-1}$, all $t\in\mathbb{Z}_{>0}$. For $d>1$ we proceed in two steps.

First Step: The case $\overline{a}_i\geq 0$ for all $1\leq i\leq d$.

Induction on $-\overline{a}_0=\sum_{i=1}^d\overline{a}_i$. The case $-\overline{a}_0=0$ is settled in Proposition 1.3. (If $e=0$ — the case relevant for our later Theorems 1.5 and 4.5 — this is the end of the present first step.) For the induction step suppose we are given a d-tupel $\overline{a}=(\overline{a}_1,\ldots,\overline{a}_d)$ with $\overline{a}_t\leq 0$ for all $t>e$ and an integer $1\leq j\leq e$ with $\overline{a}_j\geq 0$. Suppose we know $H^t(Y,\mathcal{L}_Y(D(\overline{a},n,m)))=0$ for all $t>e$ (induction hypothesis). Defining $\overline{a}'=(\overline{a}'_1,\ldots,\overline{a}'_d)$ by $\overline{a}'_i=\overline{a}_i$ for $i\neq j$, and $\overline{a}'_j=\overline{a}_j+1$, we need to show $H^t(Y,\mathcal{L}_Y(D(\overline{a}',n,m)))=0$ for all $t>e$. For $0\leq k\leq d$ let

$$D_k(\overline{a},n,m)=D(\overline{a},n,m)-\sum_{\substack{j\in\tau\in\mathcal{Y}\\|\tau|\leq k}}\ \sum_{u\in U_\tau-U_\tau^{\{j\}}}u.V_\tau. \tag{3}$$

Thus $D(\overline{a},n,m)=D_0(\overline{a},n,m)$. On the other hand, $D_d(\overline{a},n,m)\sim D(\overline{a}',n,m)$. Indeed,

$$D(\overline{a}',n,m)=D(\overline{a},n,m)+\sum_{0\in\tau\in\mathcal{Y}}\sum_{u\in U_\tau}u.V_\tau-\sum_{j\in\tau\in\mathcal{Y}}\sum_{u\in U_\tau}u.V_\tau$$

and we have

$$\sum_{0\in\tau\in\mathcal{Y}}\sum_{u\in U_\tau}u.V_\tau\sim\sum_{j\in\tau\in\mathcal{Y}}\ \sum_{u\in U_\tau^{\{j\}}}u.V_\tau$$

on Y because $\sum_{0\in\tau\in\mathcal{Y}}\sum_{u\in U_\tau}u.V_\tau-\sum_{j\in\tau\in\mathcal{Y}}\sum_{u\in U_\tau^{\{j\}}}u.V_\tau$ is the (principal) divisor of the rational function Ξ_0/Ξ_j on Y (note that $U_\tau=U_\tau^{\{0\}}$ if $0\in\tau$). Therefore, in view of our induction hypothesis and of the exact sequence

$$0\longrightarrow\mathcal{L}_Y(D_d(\overline{a},n,m))\longrightarrow\mathcal{L}_Y(D_0(\overline{a},n,m))\longrightarrow\frac{\mathcal{L}_Y(D_0(\overline{a},n,m))}{\mathcal{L}_Y(D_d(\overline{a},n,m))}\longrightarrow 0$$

we only need to show

$$H^t(Y, \frac{\mathcal{L}_Y(D_0(\overline{a},n,m))}{\mathcal{L}_Y(D_d(\overline{a},n,m))}) = 0$$

for all $t \geq e$. By the obvious induction we reduce to proving

$$H^t(Y, \frac{\mathcal{L}_Y(D_{k-1}(\overline{a},n,m))}{\mathcal{L}_Y(D_k(\overline{a},n,m))}) = 0$$

for all $t \geq e$, all $1 \leq k \leq d$. We have

$$\frac{\mathcal{L}_Y(D_{k-1}(\overline{a},n,m))}{\mathcal{L}_Y(D_k(\overline{a},n,m))} = \bigoplus_{\substack{j\in\tau\in\mathcal{Y}\\ |\tau|=k}} \bigoplus_{u\in U_\tau - U_\tau^{\{j\}}} \mathcal{L}_Y(D_{k-1}(\overline{a},n,m)) \otimes_{\mathcal{O}_Y} \mathcal{O}_{u.V_\tau},$$

so our task is to prove

$$H^t(Y, \mathcal{L}_Y(D_{k-1}(\overline{a},n,m)) \otimes_{\mathcal{O}_Y} \mathcal{O}_{u.V_\tau}) = 0$$

for any $t \geq e$, any τ with $j \in \tau$ and $|\tau| = k$, and any $u \in U_\tau - U_\tau^{\{j\}}$. Setting

$$\widehat{D}_{k-1}(\overline{a},n,m) = D(\overline{a},n,m) - \sum_{\substack{j\in\sigma\in\mathcal{Y}\\ |\sigma|\leq k-1}} \sum_{u\in U_\sigma} u.V_\sigma \tag{4}$$

we claim

$$\mathcal{L}_Y(D_{k-1}(\overline{a},n,m)) \otimes_{\mathcal{O}_Y} \mathcal{O}_{u.V_\tau} = \mathcal{L}_Y(\widehat{D}_{k-1}(\overline{a},n,m)) \otimes_{\mathcal{O}_Y} \mathcal{O}_{u.V_\tau}.$$

Indeed, to show this we need to show that for all τ' with $j \in \tau' \subsetneq \tau$, all $u' \in U_{\tau'}^{\{j\}}$ we have $u'.V_{\tau'} \cap u.V_\tau = \emptyset$. Assume that this is false. Then for the images $f(V_{\tau'}) = \mathbb{V}(\Xi_i)_{i\in\tau'}$ and $f(V_\tau) = \mathbb{V}(\Xi_i)_{i\in\tau}$ under $f : Y \to Y_0 = \mathrm{Proj}(k[\Xi_0,\ldots,\Xi_d])$ (we use the symbol $\mathbb{V}(.)$ to denote the vanishing locus of a set of projective coordinate functions) we have

$$u.\mathbb{V}(\Xi_i)_{i\in\tau} \subset u.\mathbb{V}(\Xi_i)_{i\in\tau'}$$

or equivalently

$$(u')^{-1}u.\mathbb{V}(\Xi_i)_{i\in\tau} \subset \mathbb{V}(\Xi_i)_{i\in\tau'}.$$

Write $u^{-1} = (c_{qr})_{qr}$ and $u' = (d_{qr})_{qr}$ and $u^{-1}u' = (e_{qr})_{qr}$. Because of $u^{-1} \in U_\tau - U_\tau^{\{j\}}$ there is a $s < j$, $s \notin \tau$, such that $c_{sj} \neq 0$. On the other hand, $d_{qj} = 0$ for all $q \neq j$. Thus $e_{sj} = c_{sj} \neq 0$. But then if $P \in \mathbb{V}(\Xi_i)_{i\in\tau}$ denotes the point with homogeneous coordinates $= 0$ for all indices $\neq s$ we have $((u')^{-1}u)P \notin \mathbb{V}(\Xi_j)$, in particular, $((u')^{-1}u)P \notin \mathbb{V}(\Xi_i)_{i\in\tau'}$. Thus the assumption was false and the claim is established. Now notice that $\widehat{D}_{k-1}(\overline{a},n,m)$ is invariant under U, implying that under the isomorphism $u : V_\tau \to u.V_\tau$ we have

$$\mathcal{L}_Y(\widehat{D}_{k-1}(\overline{a},n,m)) \otimes_{\mathcal{O}_Y} \mathcal{O}_{u.V_\tau} \cong \mathcal{L}_Y(\widehat{D}_{k-1}(\overline{a},n,m)) \otimes_{\mathcal{O}_Y} \mathcal{O}_{V_\tau}.$$

In this way we have transformed our task into that of proving

$$H^t(Y, \mathcal{L}_Y(\widehat{D}_{k-1}(\overline{a},n,m)) \otimes_{\mathcal{O}_Y} \mathcal{O}_{V_\tau}) = 0$$

for any $t \geq e$ and any τ with $j \in \tau$ and $|\tau| = k$ and $U_\tau - U_\tau^{\{j\}} \neq \emptyset$. We use Proposition 1.2 to show that on $V_\tau = Y^\tau \times Y^{\tau^c}$ we have

$$\begin{aligned} &\mathcal{L}_Y(\widehat{D}_{k-1}(\overline{a},n,m)) \otimes_{\mathcal{O}_Y} \mathcal{O}_{V_\tau} \\ &\cong \mathcal{L}_{Y^\tau}(D((\overline{a}|_\tau)^{[\iota_\tau^{-1}(j)]}, n', m')) \otimes_{\mathcal{O}_{Y^\tau}} \mathcal{O}_{V_\tau} \otimes_{\mathcal{O}_{Y^{\tau^c}}} \mathcal{L}_{Y^{\tau^c}}(D(\overline{a}|_{\tau^c}, n'', m'')) \end{aligned} \tag{5}$$

with $n'=(n'_1,\ldots,n'_{k-1})$, $m'=(m'_1,\ldots,m'_{k-1})$, $n''=(n''_1,\ldots,n''_{d-k})$ and $m''=(m''_1,\ldots,m''_{d-k})$ defined as follows:

$$m'_i=m_i \text{ for all } 1\le i\le k-1 \qquad n''_i=n_{k+i} \text{ for all } 1\le i\le d-k,$$

$$n'_i=\begin{cases} m_i-m_k-1 & : \ 0\notin\tau, \\ n_i-n_k-1 & : \ 0\in\tau, \end{cases} \qquad m''_i=\begin{cases} m_{k+i}-m_k & : \ 0\notin\tau, \\ n_{k+i}-n_k & : \ 0\in\tau. \end{cases}$$

By induction hypothesis we have

$$H^t(Y^\tau,\mathcal{L}_{Y^\tau}(D((\overline{a}|_\tau)^{[\iota_\tau^{-1}(j)]},n',m')))=0$$

for all $t>e'$, and

$$H^t(Y^{\tau^c},\mathcal{L}_{Y^{\tau^c}}(D(\overline{a}|_{\tau^c},n'',m'')))=0$$

for all $t>e''$. Here e' and e'' are defined as follows. For a nonempty subset σ of $\{0,\ldots,d\}$ denote by $\widehat{\sigma}$ the subset of σ obtained by deleting its smallest element, and all its elements $>e$. Then $e'=|\widehat{\tau}|$ and $e''=|\widehat{\tau^c}|$. Now observe $e'+e''<e$. Indeed, otherwise we would either have $\{0,\ldots,e\}\subset\tau^c$ — contradicting $j\in\{0,\ldots,e\}\cap\tau$ — or we would have $\{0,\ldots,e\}\subset\tau$ — contradicting $j\in\{0,\ldots,e\}$ and $U_\tau-U_\tau^{\{j\}}\ne\emptyset$. We conclude this step by the Künneth formula.

Second Step: The general case.

Induction on $-\sum_{\overline{a}_j<0}\overline{a}_j$. The case where this term is zero was settled in the first step. Now let $-\sum_{\overline{a}_j<0}\overline{a}_j>0$. Choose and fix a j with $\overline{a}_j<0$. Define $\overline{a}'=(\overline{a}'_1,\ldots,\overline{a}'_d)$ by $\overline{a}'_i=\overline{a}_i$ for $i\ne j$, and $\overline{a}'_j=\overline{a}_j+1$. For $0\le k\le d$ define $D_k(\overline{a},n,m)$ as in the first step through formula (3). Then we have again $D_d(\overline{a},n,m)\sim D(\overline{a}',n,m)$ and similarly to the first step we use the induction hypothesis to reduce to proving

$$H^t(Y,\frac{\mathcal{L}_Y(D_{k-1}(\overline{a},n,m))}{\mathcal{L}_Y(D_k(\overline{a},n,m))})=0$$

for all $t>e$, all $1\le k\le d$ (note that now we write $t>e$ rather than $t\ge e$). Similarly to the first step we then reduce to proving

$$H^t(Y,\mathcal{L}_Y(D_{k-1}(\overline{a},n,m))\otimes_{\mathcal{O}_Y}\mathcal{O}_{u.V_\tau})=0$$

for any $t>e$, any τ with $j\in\tau$ and $|\tau|=k$, and any $u\in U_\tau-U_\tau^{\{j\}}$. Again we define $\widehat{D}_{k-1}(\overline{a},n,m)$ through formula (4) and similarly as in the first case we reduce to proving

$$H^t(Y,\mathcal{L}_Y(\widehat{D}_{k-1}(\overline{a},n,m))\otimes_{\mathcal{O}_Y}\mathcal{O}_{V_\tau})=0$$

for any $t>e$ and any τ with $j\in\tau$ and $|\tau|=k$ and $U_\tau-U_\tau^{\{j\}}\ne\emptyset$. Again we have the formula (5) and by induction hypothesis we have

$$H^t(Y^\tau,\mathcal{L}_{Y^\tau}(D((\overline{a}|_\tau)^{[\iota_\tau^{-1}(j)]},n',m')))=0$$

for all $t>e'$, and

$$H^t(Y^{\tau^c},\mathcal{L}_{Y^{\tau^c}}(D(\overline{a}|_{\tau^c},n'',m'')))=0$$

for all $t>e''$, where e' and e'' are defined as in the first step. Now we clearly have $e'+e''\le e$ (but not necessarily $e'+e''<e$ as in the first case) and we conclude by the Künneth formula. □

For an element of $\mathcal{V}$ let us denote by a subscript 0 its image in $Y_0\cong\mathbb{P}^d$, i.e. the k-linear subspace of Y_0 whose strict transform is the given element of $\mathcal{V}$. Let $V,V'\in\mathcal{V}$ and suppose that $V_0\cup V'_0$ is contained in a *proper* k-linear subspace of Y_0; then denote by $[V,V']$ the element of $\mathcal{V}$ which is the strict transform of the minimal such subspace of Y_0. If $V_0 \cup V_0'$ is not contained in a proper k-linear subspace of Y_0, then $[V,V']$ is undefined. We say that a subset S of $\mathcal{V}$ is *stable* if for any two $V,V' \in S$ the element $[V,V']$ is defined and lies in S. For example, the empty set and all one-element subsets of $\mathcal{V}$ are stable.

For any subset S of $\mathcal{V}$ we define the divisor

$$D_S = \sum_{V \in S} V$$

on Y.

Theorem 1.5. *Suppose that S is stable and that $\overline{a} = (\overline{a}_1, \dots, \overline{a}_d) \in \mathbb{Z}^d$ satisfies $\overline{a}_i \leq 0$ for all $1 \leq i \leq d$. Then*

$$H^t(Y, \mathcal{L}_Y(D(\overline{a},0,0) - D_S)) = 0$$

for all $t \in \mathbb{Z}_{>0}$.

Proof. Outer induction on d. If $d = 1$ our statement is the well-known fact $H^t(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(k)) = 0$ for all $k \in \mathbb{Z}_{\geq -1}$, all $t \in \mathbb{Z}_{>0}$. Let now $d > 1$. For an element $V \in \mathcal{V}$ let

$$T(V) = \{V' \in \mathcal{V} | \quad V_0' \subset V_0\}.$$

If $S = \emptyset$, we cite Theorem 1.4. Otherwise there is a $V \in S$ with $S \subset T(V)$. It follows that there is a $W \in \mathcal{V}$ (not necessarily $W \in S$) with $\dim(W) = d-1$ and $S \subset T(W)$. For $1 \leq i \leq d$ set

$$Q_i = S \cup \{V \in T(W) | \ \dim(V_0) \geq i\}.$$

By induction on i we show

$$H^t(Y, \mathcal{L}_Y(D(\overline{a},0,0) - D_{Q_i})) = 0$$

for all $t \in \mathbb{Z}_{>0}$; for $i = d$ we then get our claim. For $i = 0$ we have $Q_0 = T(W)$ so we see that the divisor $D(\overline{a},0,0) - D_{Q_0}$ is linearly equivalent to $D(\overline{a},-\mathbf{1},0)$ with $-\mathbf{1} = (-1,\dots,-1)$. Thus Theorem 1.4 settles this case. For $i > 0$ we have an exact sequence

$$0 \longrightarrow \mathcal{L}_Y(D(\overline{a},0,0) - D_{Q_{i-1}}) \longrightarrow \mathcal{L}_Y(D(\overline{a},0,0) - D_{Q_i}) \longrightarrow \mathcal{C} \longrightarrow 0$$

with

$$\mathcal{C} = \bigoplus_{\substack{V \in T(W)-S \\ \dim(V_0)=i-1}} \mathcal{L}_Y(D(\overline{a},0,0) - D_{Q_i}) \otimes_{\mathcal{O}_Y} \mathcal{O}_V.$$

In view of the induction hypothesis it remains to show

$$H^t(Y, \mathcal{L}_Y(D(\overline{a},0,0) - D_{Q_i}) \otimes_{\mathcal{O}_Y} \mathcal{O}_V) = 0$$

for all $t \in \mathbb{Z}_{>0}$, all $V \in T(W) - S$ with $\dim(V_0) = i-1$. By equivariance we may assume $V = V_\tau$ with $|\tau| = d+1-i$. Recall the bijection (2) which we now view as an identification as follows:

$$\mathcal{N}^{\tau^c} = \{V \in \mathcal{V} \mid V_0 \subsetneq V_{\tau,0}\},$$

$$\mathcal{N}^{\tau} = \{V \in \mathcal{V} \mid V_{\tau,0} \subsetneq V_0\}.$$

On $V_\tau = Y^\tau \times Y^{\tau^c}$ we find (using Proposition 1.2)

$$\mathcal{L}_Y(D(\overline{a},0,0) - D_{Q_i}) \otimes_{\mathcal{O}_Y} \mathcal{O}_{V_\tau}$$
$$\cong \mathcal{L}_{Y^\tau}(D(\overline{a}|_\tau,0,0) - D_{T(W)\cap\mathcal{N}^\tau}) \otimes_{\mathcal{O}_{Y^\tau}} \mathcal{O}_{V_\tau} \otimes_{\mathcal{O}_{Y^{\tau^c}}} \mathcal{L}_{Y^{\tau^c}}(D(\overline{a}|_{\tau^c},0,0) - D_{S\cap\mathcal{N}^{\tau^c}})$$

(with $\overline{a}|_\tau$ and $\overline{a}|_{\tau^c}$ as defined in the proof of Theorem 1.4). Since S is stable (with respect to $\mathcal{V}\cong\mathcal{N}$) also $S\cap\mathcal{N}^{\tau^c}$ is stable with respect to $\mathcal{N}^{\tau^c}$. On the other hand, $T(W)\cap\mathcal{N}^\tau$ is stable with respect to $\mathcal{N}^\tau$. Therefore, our induction hypothesis says $H^t(Y^\tau,\mathcal{L}_{Y^\tau}(D(\overline{a}|_\tau,0,0)-D_{T(W)\cap\mathcal{N}^\tau}))=0$ and $H^t(Y^{\tau^c},\mathcal{L}_{Y^{\tau^c}}(D(\overline{a}|_{\tau^c},0,0)-D_{S\cap\mathcal{N}^{\tau^c}}))=0$ for all $t\in\mathbb{Z}_{>0}$. We conclude by the Künneth formula. □

For subsets S of $\mathcal{V}$ let $W_S=\bigcap_{V\in S}V$; in particular, let $W_\emptyset=Y$.

Corollary 1.6. *Suppose $\overline{a}$ is as in Theorem 1.5.*

(a) *Let S,S' be subsets of $\mathcal{V}$ (possibly empty) with $W_{S\cup S'}\neq\emptyset$ and $S\cap S'=\emptyset$. Then*

$$H^t(Y,\mathcal{L}_Y(D(\overline{a},0,0)-D_{S'})\otimes_{\mathcal{O}_Y}\mathcal{O}_{W_S})=0$$

for all $t\in\mathbb{Z}_{>0}$.

(b) *Let S be a subset of $\mathcal{V}$ and let*

$$N=\{V\in\mathcal{V}|\ V_0'\subset V_0\ \ \textit{for all}\ \ V'\in S\}.$$

Let M_0 be a nonempty and stable subset of N. Then the sequence

$$H^0(Y,\mathcal{L}_Y(D(\overline{a},0,0))\otimes_{\mathcal{O}_Y}\mathcal{O}_{W_S})\longrightarrow\prod_{V\in M_0}H^0(Y,\mathcal{L}_Y(D(\overline{a},0,0))\otimes_{\mathcal{O}_Y}\mathcal{O}_{W_{S\cup\{V\}}})$$
$$\longrightarrow\prod_{\substack{V\neq V'\in M_0\\ W_{S\cup\{V,V'\}}\neq\emptyset}}H^0(Y,\mathcal{L}_Y(D(\overline{a},0,0))\otimes_{\mathcal{O}_Y}\mathcal{O}_{W_{S\cup\{V,V'\}}})$$

is exact.

Proof. (a) The condition $W_S\cap W_{S'}\neq\emptyset$ implies that all subsets of $S\cup S'$ are stable. Therefore we can use Theorem 1.5 for an induction on $|S|$.

(b) The kernel of the second arrow is $H^0(Y,\mathcal{L}_Y(D(\overline{a},0,0))\otimes_{\mathcal{O}_Y}\mathcal{O}_W)$ with $W=\bigcup_{V\in M_0}W_{S\cup\{V\}}$ (union inside Y). Looking at the exact sequence

$$0\longrightarrow\mathcal{L}_Y(D(\overline{a},0,0)-D_{M_0})\otimes_{\mathcal{O}_Y}\mathcal{O}_{W_S}\longrightarrow\mathcal{L}_Y(D(\overline{a},0,0))\otimes_{\mathcal{O}_Y}\mathcal{O}_{W_S}$$
$$\longrightarrow\mathcal{L}_Y(D(\overline{a},0,0))\otimes_{\mathcal{O}_Y}\mathcal{O}_W\longrightarrow 0$$

we see that it is enough to prove

$$H^t(Y,\mathcal{L}_Y(D(\overline{a},0,0)-D_{M_0})\otimes_{\mathcal{O}_Y}\mathcal{O}_{W_S})=0$$

for $t=1$. We do this for all $t>0$ by induction on $|S|$. The case $S=\emptyset$ was settled in Theorem 1.5. If $S\neq\emptyset$, we pick an element $s\in S$, let $S'=S-\{s\}$ and consider the exact sequence

$$0\longrightarrow\mathcal{L}_Y(D(\overline{a},0,0)-D_{M_0\cup\{s\}})\otimes_{\mathcal{O}_Y}\mathcal{O}_{W_{S'}}\longrightarrow\mathcal{L}_Y(D(\overline{a},0,0)-D_{M_0})\otimes_{\mathcal{O}_Y}\mathcal{O}_{W_{S'}}$$
$$\longrightarrow\mathcal{L}_Y(D(\overline{a},0,0)-D_{M_0})\otimes_{\mathcal{O}_Y}\mathcal{O}_{W_S}\longrightarrow 0.$$

The induction hypothesis applies to the first two terms (note that also $M_0\cup\{s\}$ is stable!), hence also the third term has no higher cohomology. □

2. The logarithmic de Rham complex on Y

The results of this section are not needed in the remainder of this paper. The following lemma is an easy exercise (count the number of pairs (ij) such that a_{ij} for $(a_{ij})_{ij}\in U_\tau^\sigma$ is not yet forced to be zero by the requirements defining U_τ^σ).

Lemma 2.1. *Let* $\tau = \{a_0, \ldots, a_r\}$ *with* $a_0 < \ldots < a_r$, *and let* $\tau \subset \sigma$. *Then*

$$|U_\tau^\sigma| = q^{\sum_{i=0}^r a_i - |\sigma^c \cap [0,a_i]| - i}.$$

□

For $\tau \subset \{1, \ldots, d\}$ we define $\overline{a}(\tau) = (\overline{a}_1(\tau), \ldots, \overline{a}_d(\tau))$ by

$$\overline{a}_i(\tau) = \begin{cases} -1 & : \quad i \in \tau, \\ 0 & : \quad i \in \{1, \ldots, d\} - \tau. \end{cases} \tag{6}$$

Lemma 2.2.

$$\dim_k H^0(Y, \mathcal{L}_Y(D(\overline{a}(\tau), 0, 0))) = q^{\sum_{i \in \tau} i}.$$

Proof. First we claim

$$\dim_k H^0(Y, \mathcal{L}_Y(D(\overline{a}(\tau), -\mathbf{1}, 0))) = \begin{cases} 0 & : \quad \tau \neq \{1, \ldots, d\}, \\ \prod_{j=1}^d (q^j - 1) & : \quad \tau = \{1, \ldots, d\} \end{cases}$$

for $-\mathbf{1} = (-1, \ldots, -1)$. The argument for this is by outer induction on d and inner induction on $\overline{a}_0(\tau) = -\sum_{j=1}^d \overline{a}_j(\tau) = |\tau|$. For $j \in \tau$ we have

$$\begin{aligned} &\dim_k H^0(Y, \mathcal{L}_Y(D(\overline{a}(\tau), -\mathbf{1}, 0))) \\ &\quad = \dim_k H^0(Y, \mathcal{L}_Y(D(\overline{a}(\tau - \{j\}), -\mathbf{1}, 0))) \\ &\qquad + \sum_{j \in \sigma \in \mathcal{Y}} |U_\sigma - U_\sigma^{\{j\}}| \dim_k H^0(Y^\sigma, \mathcal{L}_{Y^\sigma}(D(\overline{a}(\tau - \{j\})|_\sigma, -\mathbf{1}, 0))) \\ &\qquad\qquad \dim_k H^0(Y^{\sigma^c}, \mathcal{L}_{Y^{\sigma^c}}(D(\overline{a}(\tau)|_{\sigma^c}, -\mathbf{1}, 0))). \end{aligned}$$

Observing $\overline{a}(\tau - \{j\}) = \overline{a}(\tau)^{[j]}$ and $\overline{a}(\tau - \{j\})|_\sigma = \overline{a}(\tau)|_\sigma^{[\iota_\sigma^{-1}(j)]}$ this follows from the proof of Theorem 1.4. By induction hypothesis this already settles the case $\tau \neq \{1, \ldots, d\}$. If $\tau = \{1, \ldots, d\}$, the easiest way is to choose $j = d$. In that case, by induction hypothesis, all summands on the right-hand side except for $\sigma = \{d\}$ vanish, so we get

$$\begin{aligned} &\dim_k H^0(Y, \mathcal{L}_Y(D(\overline{a}(\tau), -\mathbf{1}, 0))) \\ &\quad = |U_{\{d\}} - U_{\{d\}}^{\{d\}}| \dim_k H^0(Y^{\{0,\ldots,d-1\}}, \mathcal{L}_{Y^{\{0,\ldots,d-1\}}}(D(\overline{a}(\tau)|_{\{0,\ldots,d-1\}}, -\mathbf{1}, 0))). \end{aligned}$$

Since $|U_{\{d\}} - U_{\{d\}}^{\{d\}}| = q^d - 1$, the induction hypothesis gives the claim. The claim established, the proof of the theorem itself is again by induction on d and works precisely as the proof of Theorem 1.5, with $S = \emptyset$ there. Namely, taking $W = V_{\{0\}}$, hence $T(W) = \{u.V_\sigma;\ (\sigma, u) \in \mathcal{N}, 0 \in \sigma\}$ so that $D_{T(W)} = D(0, \mathbf{1}, 0)$, we saw that

$$\begin{aligned} &\dim_k H^0(Y, \mathcal{L}_Y(D(\overline{a}(\tau), 0, 0))) \\ &\quad = \dim_k H^0(Y, \mathcal{L}_Y(D(\overline{a}(\tau), -\mathbf{1}, 0))) \\ &\qquad + \sum_{0 \in \sigma} |\{u;\ (\sigma, u) \in \mathcal{N}\}| \dim_k H^0(Y^\sigma, \mathcal{L}_{Y^\sigma}(D(\overline{a}(\tau)|_\sigma, -\mathbf{1}, 0))) \\ &\qquad\qquad \dim_k H^0(Y^{\sigma^c}, \mathcal{L}_{Y^{\sigma^c}}(D(\overline{a}(\tau)|_{\sigma^c}, 0, 0))). \end{aligned}$$

All the terms are known by induction hypothesis, resp. the above claim, and we just have to sum up. □

Denote by $\Omega_Y^\bullet$ the de Rham complex on Y with logarithmic poles along the normal crossings divisor $\sum_{V\in\mathcal{V}} V$ on Y. For $0\leq r\leq d$ we write

$$z_r=\frac{\Xi_r}{\Xi_0},$$

a rational function on Y. For $0\leq s\leq d$ denote by $\mathcal{P}_s$ the set of subsets of $\{1,\ldots,d\}$ consisting of s elements. For a subset $\tau\subset\{1,\ldots,d\}$ let

$$U(\tau)=\{(a_{ij})_{0\leq i,j\leq d}\in U(k)\quad|\quad a_{ij}=0 \text{ if } j\notin\{i\}\cup\tau\}.$$

Theorem 2.3. *(a) For each $0\leq s\leq d$ we have $H^t(Y,\Omega_Y^s)=0$ for all $t>0$.*

(b) The following set is a k-basis of $H^0(Y,\Omega_Y^s)$:

$$\{A.\bigwedge_{t\in\tau}\mathrm{dlog}(z_t)\quad|\quad\tau\in\mathcal{P}_s, A\in U(\tau)\}.$$

In particular,

$$\dim_k(H^0(Y,\Omega_Y^s))=\sum_{\tau\in\mathcal{P}_s}q^{\sum_{i\in\tau}i}.$$

Proof. For elements $\tau^1=\{t_1^1<\ldots<t_s^1\}$ and $\tau^2=\{t_1^2<\ldots<t_s^2\}$ of $\mathcal{P}_s$ we write $\tau^1>\tau^2$ if and only if there is a $1\leq b\leq s$ such that $t_r^1=t_r^2$ for $1\leq r\leq b-1$ and $t_b^1>t_b^2$. In other words, we use the lexicographical ordering on $\mathcal{P}_s$. Let

$$Y'=Y-\bigcup_{\substack{(\sigma,u)\in\mathcal{N}\\ u\neq 1}}u.V_\sigma.$$

Writing $\Omega_{Y'}^s=\Omega_Y^s|_{Y'}$ we consider the following filtration of $\Omega_{Y'}^s$, indexed by $\mathcal{P}_s$:

$$F^\tau\Omega_{Y'}^s=\bigoplus_{\substack{\tau'\in\mathcal{P}_s\\ \tau'\leq\tau}}\mathcal{O}_{Y'}\bigwedge_{t\in\tau'}dz_t.$$

In particular, $F^{\{d+1-s,\ldots,d\}}\Omega_{Y'}^s=\Omega_{Y'}^s$ if $s>0$. One checks that for open subsets W_1 and W_2 of Y' and elements $g\in U(k)$ such that $gW_1=W_2$, the isomorphism $g:\Omega_{Y'}^s|_{W_1}\to\Omega_{Y'}^s|_{W_2}$ restricts to an isomorphism $F^\tau\Omega_{Y'}^s|_{W_1}\to F^\tau\Omega_{Y'}^s|_{W_2}$ for any τ. Since $U(k)Y'=Y$ we may therefore extend this filtration from Y' to all of Y, obtaining a $U(k)$-stable filtration

$$(F^\tau\Omega_Y^s)_{\tau\in\mathcal{P}_s}$$

of Ω_Y^s, with $F^{\{d+1-s,\ldots,d\}}\Omega_Y^s=\Omega_Y^s$ if $s>0$. We have

$$Gr^\tau\Omega_Y^s=\frac{F^\tau\Omega_Y^s}{\bigoplus_{\tau'<\tau}F^{\tau'}\Omega_Y^s}\cong\mathcal{L}_Y(D(\overline{a}(\tau),0,0))$$

$$f\bigwedge_{t\in\tau}dz_t\mapsto f.$$

Hence we get $H^t(Y,Gr^\tau\Omega_Y^s)=0$ for all $t>0$ from Corollary 1.6. Since the $Gr^\tau\Omega_Y^s$ are isomorphic to the graded pieces of the filtration $(F^\tau\Omega_Y^s)_{\tau\in\mathcal{P}_s}$ of Ω_Y^s, this implies statement (a). Moreover, the classes of the elements $A.\bigwedge_{t\in\tau}\mathrm{dlog}(z_t)$ for $A\in U(\tau)$ form a k-basis of $H^0(Y,Gr^\tau\Omega_Y^s)$. This can be proven by tracing back the proof of Lemma 2.2; or by showing the linear independence of the $A.\bigwedge_{t\in\tau}\mathrm{dlog}(z_t)$ and *applying* Lemma 2.2. We get statement (b). □

Corollary 2.4. *The Hodge spectral sequence $E_1^{st}=H^t(Y,\Omega_Y^s)\Longrightarrow H^{s+t}(Y,\Omega_Y^\bullet)$ degenerates in E_1. We have $H^s(Y,\Omega_Y^\bullet)=H^0(Y,\Omega_Y^s)$ for all s.*

We wish to determine $H^0(Y,\Omega_Y^s)$ as a $\mathrm{GL}_{d+1}(k)$-representation. For this we recall the classification of irreducible representations of $\mathrm{GL}_{d+1}(k)$ on k-vector spaces according to Carter and Lusztig. For $1\le r\le d$ let $t_r\in\mathrm{GL}_{d+1}(k)$ denote the permutation matrix obtained by interchanging the $(r-1)$-st and the r-th row (or equivalently: column) of the identity matrix (recall that we start counting with 0). Then $S=\{t_1,\ldots,t_d\}$ is a set of Coxeter generators for the Weyl group of $\mathrm{GL}_{d+1}(k)$.

Theorem 2.5 ([3])**.** *(i) For an irreducible representation ρ of $\mathrm{GL}_{d+1}(k)$ on a k-vector space, the subspace $\rho^{U(k)}$ of $U(k)$-invariants is one-dimensional. If the action of the group $B(k)$ of upper triangular matrices on $\rho^{U(k)}$ is given by the character $\chi: B(k)/U(k)\to k^\times$ and if $J=\{t\in S;\, t.\rho^{U(k)}=\rho^{U(k)}\}$, then the pair (χ,J) determines ρ up to isomorphism.*

(ii) Conversely, given a character $\chi: B(k)/U(k)\to k^\times$ and a subset J of $\{t\in S;\,\chi^t=\chi\}$, there exists an irreducible representation $\Theta(\chi,J)$ of $\mathrm{GL}_{d+1}(k)$ on a k-vector space whose associated pair (as above) is (χ,J).

For $1\le j\le d$ we need the rational function

$$\gamma_j=\prod_{(a_0,\ldots,a_{j-1})\in k^j}(z_j+a_{j-1}z_{j-1}+\ldots+a_1z_1+a_0) \tag{7}$$

on Y, and if in addition $0\le s\le d$, we define the integer

$$m_j^s=\max\{0,s-j\}q-\max\{0,s-j+1\}.$$

Theorem 2.6. *For $0\le s\le d$, the $\mathrm{GL}_{d+1}(k)$-representation on $H^0(Y,\Omega_Y^s)$ is equivalent to $\Theta(1,\{t_{s+1},\ldots,t_d\})$. The subspace of $U(k)$-invariants of $H^0(Y,\Omega_Y^s)$ is generated by*

$$\omega_s=(\prod_{j=1}^d\gamma_j^{m_j^s})dz_1\wedge\ldots\wedge dz_s.$$

Proof. (i) We check that ω_s is indeed an element of $H^0(Y,\Omega_Y^s)$. In the notation of section 1 let us abbreviate

$$W_\sigma=\sum_{u\in U_\sigma}u.V_\sigma$$

for $\sigma\in\mathcal{Y}$, a reduced and $U(k)$-stable divisor. For $1\le j\le d$ one finds that the zero-pole divisor of γ_j^{-1} is

$$\sum_{\substack{\sigma\\ j\in\sigma,0\notin\sigma}}(q^{|\sigma\cap[0,j-1]|})W_\sigma+\sum_{\substack{\sigma\\ j\in\sigma,0\in\sigma}}(q^{|\sigma\cap[0,j-1]|}-q^j)W_\sigma+\sum_{\substack{\sigma\\ j\notin\sigma,0\in\sigma}}-q^jW_\sigma. \tag{8}$$

Hence the zero pole divisor of $\prod_{j=1}^d\gamma_j^{m_j^s}$ is

$$\sum_{\substack{\sigma\\ 0\notin\sigma}}\sum_{j\in\sigma}-m_j^sq^{|\sigma\cap[0,j-1]|}W_\sigma+\sum_{\substack{\sigma\\ 0\in\sigma}}(\sum_{j\ge1,j\in\sigma}-m_j^sq^{|\sigma\cap[0,j-1]|}+\sum_{j=1}^dm_j^sq^j)W_\sigma.$$

This divisor is smaller (for the usual order on the set of Cartier divisors) than

$$D_s=\sum_{\substack{\sigma\\ 0\notin\sigma}}|\sigma\cap[1,s]|W_\sigma+\sum_{\substack{\sigma\\ 0\in\sigma}}(|\sigma\cap[1,s]-s)W_\sigma.$$

Now we have the ($U(k)$-equivariant) embedding

$$\mathcal{L}_Y(D_s) \longrightarrow \Omega^s_Y, \quad f \mapsto f dz_1 \wedge \ldots \wedge dz_s.$$

By the above, $\prod_{j=1}^d \gamma_j^{m_j^s}$ is a global section of $\mathcal{L}_Y(D_s)$; its image in $H^0(Y, \Omega^s_Y)$ is ω_s.

(ii) We check that ω_s is fixed by $B(k)$. The rational function γ_j is $U(k)$-stable, and similarly the s-form $dz_1 \wedge \ldots \wedge dz_s$ is $U(k)$-stable. On the other hand, the torus $T(k)$ of diagonal matrices in $\mathrm{GL}_{d+1}(k)$ acts as follows: if $t \in T(k)$ has diagonal entries $a_{00}, \ldots, a_{dd}$, then t acts on γ_j by permuting its factors different from z_j, and on the factor z_j it acts by multiplication with a_{jj}/a_{00}. Hence $t.\gamma_j^{m_j^s} = (a_{00}/a_{jj})\gamma_j^{m_j^s}$ — because $m_j^s \equiv -1$ (modulo q) — and $t.dz_j = (a_{jj}/a_{00})dz_j$. Multiplying together we get our claim.

(iii) It is clear that $\{t_{s+1}, \ldots, t_d\} = \{t \in S;\, t.\omega_s = \omega_s\}$.

(iv) From (i), (ii), (iii) it follows that $H^0(Y, \Omega^s_Y)$ contains $\Theta(1, \{t_{s+1}, \ldots, t_d\})$. We have

$$\dim_k(\Theta(1, \emptyset)) = q^{1+2+\ldots+d} = \dim_k(H^0(Y, \Omega^d_Y))$$

where the first equality is [2], Theorem 6.12(ii) and the second one follows from Theorem 2.3; hence our statement in the case $s = d$. To go further we point out the following trivial consequence: the $k[\mathrm{GL}_{d+1}(k)]$-module generated by ω_d contains a nonzero logarithmic differential d-form (because $H^0(Y, \Omega^d_Y)$ contains such a form, by Theorem 2.3). For $s < d$ we know from Theorem 2.3 that $H^0(Y, \Omega^s_Y)$ is generated as a k-vector space by logarithmic differential s-forms. Since $\mathrm{GL}_{d+1}(k)$ acts transitively on the set of $k^\times$-homothety classes of nonzero logarithmic differential s-forms, each nonzero logarithmic differential s-form generates $H^0(Y, \Omega^s_Y)$ as a $k[\mathrm{GL}_{d+1}(k)]$-module. Hence we need to show that the sub-$k[\mathrm{GL}_{d+1}(k)]$-module generated by ω_s contains a nonzero logarithmic differential s-form. As a formal expression, ω_s is independent of d, as long as $d \geq s$; this follows immediately from the definition of the numbers m_j^s. Moreover, the action of the subgroup $\mathrm{GL}_{s+1}(k)$ on ω_s is independent of d if we use the obvious embedding $\mathrm{GL}_{s+1}(k) \to \mathrm{GL}_{d+1}(k)$ (into the upper left square, followed by entries $= 1$ on the rest of the diagonal). Therefore, the consequence pointed out above (when d there takes the value of our present s) says that the $k[\mathrm{GL}_{s+1}(k)]$-module generated by ω_s contains a nonzero logarithmic differential s-form; in particular, the $k[\mathrm{GL}_{d+1}(k)]$-module generated by ω_s contains a nonzero logarithmic differential s-form. We are done. □

We draw representation theoretic consequences. Fix $0 \leq s \leq d$, let $n_1 = \ldots = n_s = 1$ and $n_{s+1} = d + 1 - s$ and define the parabolic subgroup

$$P_s = \{(a_{ij})_{0 \leq i,j \leq d} \in \mathrm{GL}_{d+1}; \quad a_{ij} = 0 \text{ if } j + 1 \leq n_1 + \ldots + n_\ell \text{ and } i + 1 > n_1 + \ldots + n_\ell \text{ for some } \ell\}$$

of GL_{d+1}. For any group H let $\mathbf{1}$ denote the trivial $k[H]$-module.

Corollary 2.7. *The following generalized Steinberg representation is irreducible:*

$$\mathrm{ind}_{P_s(k)}^{\mathrm{GL}_{d+1}(k)} \mathbf{1} / \sum_{P \supsetneq P_s} \mathrm{ind}_{P(k)}^{\mathrm{GL}_{d+1}(k)} \mathbf{1}$$

where the sum runs over all parabolic subgroups P of GL_{d+1} strictly containing P_s.

Proof. This representation clearly contains $\Theta(1,\{t_{s+1},\ldots,t_d\})$ and its k-dimension is $\sum_{\tau\in\mathcal{P}_s}q^{\sum_{i\in\tau}i}$ (to see this, use for example, the formula [2], exc. 6.3). But this is also the k-dimension of $\Theta(1,\{t_{s+1},\ldots,t_d\})$ as follows from Theorem 2.6. □

Let W denote the ring of Witt vectors with coefficients in k. Endow Spec(k) and Spf(W) with the trivial log structure and endow Y with the log structure corresponding to the normal crossings divisor $\bigcup_{V\in\mathcal{V}}V$ on Y. We ask for the logarithmic crystalline cohomology $H^*_{crys}(Y/W)$ of Y/k relative to the divided power thickening Spf(W) of Spec(k). (Note that $H^*_{crys}(Y/W)\otimes\mathbb{Q}$ is the rigid cohomology of the open subscheme $Y-\bigcup_{V\in\mathcal{V}}V$ of Y.)

Theorem 2.8. *$H^s_{crys}(Y/W)$ is torsion free for any s, and*

$$H^s_{crys}(Y/W)\otimes_W k=H^s(Y,\Omega^\bullet_Y)=H^0(Y,\Omega^s_Y).$$

Proof. We already remarked that $H^s(Y,\Omega^\bullet_Y)=H^0(Y,\Omega^s_Y)$. The base change property $H^s_{crys}(Y/W)\otimes_W k=H^s(Y,\Omega^\bullet_Y)$ is a consequence of the torsion freeness of $H^s_{crys}(Y/W)$ (for all s). To prove it it suffices by [1], (7.3) to prove that the log scheme Y is ordinary, i.e. that the Newton polygon of $H^s_{crys}(Y/W)\otimes\mathbb{Q}$ coincides with the Hodge polygon of $H^s(Y,\Omega^\bullet_Y)$. But the endpoints of these two polygons are the same, and both have only a single slope: for the Hodge polygon this is our equality $H^s(Y,\Omega^\bullet_Y)=H^0(Y,\Omega^s_Y)$, for the Newton polygon this was verified in [6], Theorem 6.3. □

Remark. (a) See sections 3 and 4 for the notations in this remark which have not yet been defined. Let $\Omega^s_{\mathfrak{X}}$ be the degree s term of the relative logarithmic de Rham complex of $\mathfrak{X}$ over Spf$(\mathcal{O}_K)$ (with respect to the log structures defined by the special fibres). Then $\Omega^s_{\mathfrak{X}}\otimes_{\mathcal{O}_K}\mathcal{O}_{\widehat{K}}\cong\mathcal{V}_{\mathcal{O}_{\widehat{K}}}$ for the irreducible rational representation V of L_1 with heighest weight

$$\sum_{i=1}^{s}(s+1)\epsilon_i+\sum_{i=s+1}^{d}s\epsilon_i\in X^*(T_1).$$

(In fact, the remark following the proof of Proposition 4.3 applies and we have $\Omega^s_{\mathfrak{X}}\cong\mathcal{V}_{\mathcal{O}_K}$). On the other hand, we have $\Omega^s_{\mathfrak{X}}\otimes_{\mathcal{O}_K}\mathcal{O}_Y=\Omega^s_Y$. Via these isomorphisms, the filtration appearing in the proof of Theorem 2.3 is just the one appearing in the proof of 4.5, and statement (a) in Theorem 2.3 is statement (32) in the proof of 4.5 (for this V).

(b) Theorem 2.3 is used in [8] to prove some new cases of Schneider's conjecture (see [14]) concerning p-adic analytic splittings of Hodge filtrations of the de Rham cohomology of certain local systems on projective varieties uniformized by X.

3. Lattices in line bundles on the symmetric space

Let T be the torus of diagonal matrices in G and let $X_*(T)$, resp. $X^*(T)$, denote the group of algebraic cocharacters, resp. characters, of T. Put $A=X_*(T)\otimes\mathbb{R}$. For $0\leq i\leq d$ define the cocharacters

$$e_i:\mathbb{G}_m\to\mathrm{GL}_{d+1},\quad t\mapsto\mathrm{diag}(1,\ldots,1,t,1,\ldots,1)$$

with t as the i-th diagonal entry, i.e. $e_i(t)_{ii}=t$, $e_i(t)_{jj}=1$ for $i\neq j$ and $e_i(t)_{j_1j_2}=0$ for $j_1\neq j_2$. The e_i form a basis of the $\mathbb{R}$-vector space A, hence an identification $A=\mathbb{R}^{d+1}$. The pairing $X_*(T)\times X^*(T)\to\mathbb{Z}$ which sends (x,μ) to the integer $\mu(x)$

such that $\mu(x(y)) = y^{\mu(x)}$ for any $y \in \mathbb{G}_m$ extends to a duality between $\mathbb{R}$-vector spaces

$$A \times (X^*(T) \otimes \mathbb{R}) \longrightarrow \mathbb{R}$$
$$(x, \mu) \mapsto \mu(x).$$

Let $\epsilon_0, \ldots, \epsilon_d \in X^*(T)$ denote the basis dual to $e_0, \ldots, e_d$.

Let $\mathbb{A}((K^{d+1})^*)$, resp. $\mathbb{P}((K^{d+1})^*)$, denote the affine, resp. the projective space spanned by $(K^{d+1})^* = \mathrm{Hom}_K(K^{d+1}, K)$. The action of $G = \mathrm{GL}_{d+1}(K) = \mathrm{GL}(K^{d+1})$ on $(K^{d+1})^*$ defines an action of G on the Drinfel'd symmetric spaces

$$X^{cone} = \mathbb{A}((K^{d+1})^*) - (\text{the union of all } K\text{-rational hyperplanes through } 0),$$
$$X = \mathbb{P}((K^{d+1})^*) - (\text{the union of all } K\text{-rational hyperplanes}).$$

There is a natural G-equivariant projection of K-rigid spaces $X^{cone} \to X$. Let $\Xi_0, \ldots, \Xi_d$ be the standard coordinate functions on X^{cone} corresponding to the canonical basis of $(K^{d+1})^*$; they induce a set of projective coordinate functions on X.

Let $\mathfrak{X}$ be the strictly semistable formal $\mathcal{O}_K$-scheme with generic fibre X introduced in [13]. The action of G on X extends naturally to $\mathfrak{X}$. For open formal subschemes $\mathfrak{U}$ of $\mathfrak{X}$ we put

$$\mathcal{O}_{\widehat{\mathfrak{U}}} = \mathcal{O}_{\mathfrak{U}} \otimes_{\mathcal{O}_K} \mathcal{O}_{\widehat{K}}.$$

For $j \geq 0$ let F^j be the set of nonempty intersections of $(j+1)$-many pairwise distinct irreducible components of $\mathfrak{X} \otimes_{\mathcal{O}_K} k$. (Thus F^j is in natural bijection with the set of j-simplices of the Bruhat Tits building of PGL_{d+1}/K.) For $Z \in F^0$ let $\mathfrak{U}_Z$ be the maximal open formal subscheme of $\mathfrak{X}$ such that $\mathfrak{U}_Z \otimes_{\mathcal{O}_K} k$ is contained in Z. Let $Y \in F^0$ be the central irreducible component of $\mathfrak{X} \otimes_{\mathcal{O}_K} k$ with respect to $\Xi_0, \ldots, \Xi_d$, characterized by the following condition. By construction we may view $\Xi_i \Xi_0^{-1}$ for $0 \leq i \leq d$ as a global section of $\mathcal{O}_{\mathfrak{X}} \otimes_{\mathcal{O}_K} K$ (i.e. as a rigid analytic function on X). Now $Y \in F^0$ is the unique irreducible component such that for all unimodular $d+1$-tupel $(a_0, \ldots, a_d) \in \mathcal{O}_K^{d+1}$ (at least one a_i is a unit in $\mathcal{O}_K$) the linear combination $\sum_{i=0}^d a_i \Xi_i \Xi_0^{-1}$, when restricted to $\mathfrak{U}_Y$, in fact lies in the subalgebra $\mathcal{O}_{\mathfrak{X}}(\mathfrak{U}_Y)$ of $\mathcal{O}_{\mathfrak{X}} \otimes_{\mathcal{O}_K} K(\mathfrak{U}_Y)$ and is even invertible in $\mathcal{O}_{\mathfrak{X}}(\mathfrak{U}_Y)$. By [13] we may identify this k-scheme Y with the one from section 1. The subgroup $K^\times.\mathrm{GL}_{d+1}(\mathcal{O}_K)$ of G is the stabilizer of Y. We define the subset

$$F_A^0 = T.Y,$$

the orbit of Y for the action of T on the set F^0. (This set corresponds to the set of vertices in an apartment of the Bruhat Tits building of PGL_{d+1}/K.) We denote by $\mathfrak{Y}$ the maximal open formal subscheme of $\mathfrak{X}$ such that $\mathfrak{Y} \otimes k$ is contained in the closed subscheme $\bigcup_{Z \in F_A^0} Z$ of $\mathfrak{X} \otimes k$. Note that the open subscheme

$$Y' = \mathfrak{Y} \cap Y$$

of Y is the complement of all divisors $u.V_\sigma$ for $(\tau, u) \in \mathcal{N}$ with $u \neq 1$. Also observe that $U(K).\mathfrak{Y} = \mathfrak{X}$ and $U(k).Y' = Y$. Let $\overline{T} = T/K^\times$. For $\mu = \sum_{j=0}^d a_j \epsilon_j \in X^*(T)$ let

$$\overline{\mu} = \Big(\frac{1}{d+1} \sum_{j=0}^d a_j\Big)\Big(\sum_{j=0}^d \epsilon_j\Big) - \mu, \tag{9}$$

an element of the subspace $X^*(\overline{T})\otimes\mathbb{R}$ of $X^*(T)\otimes\mathbb{R}$. Letting

$$\overline{a}_j(\mu)=\frac{(\sum_{i\neq j}a_i)-da_j}{d+1} \tag{10}$$

we have

$$\overline{\mu}=\sum_{j=0}^{d}\overline{a}_j(\mu)\epsilon_j.$$

There is a $\delta(\mu)\in\frac{1}{d+1}.\mathbb{Z}\cap[0,1[$ such that $\overline{a}_j(\mu)+\delta(\mu)\in\mathbb{Z}$ for all $0\leq j\leq d$. If $\delta(\mu)=0$ — this is equivalent with $\overline{\mu}\in X^*(\overline{T})$ — we define $\overline{a}(\mu)=(\overline{a}(\mu)_i)_{1\leq i\leq d}\in\mathbb{Z}^d$ by setting

$$\overline{a}(\mu)_i=\overline{a}_i(\mu)$$

and then let, as in section 1, $\overline{a}(\mu)_0=-\sum_{i=1}^d\overline{a}(\mu)_i=\overline{a}_0(\mu)$. If $\delta(\mu)\neq 0$ — this is equivalent with $\overline{\mu}\notin X^*(\overline{T})$ — we define $\lceil\overline{a}(\mu)\rceil=(\lceil\overline{a}(\mu)\rceil_i)_{1\leq i\leq d}\in\mathbb{Z}^d$ by setting

$$\lceil\overline{a}(\mu)\rceil_i=\lceil\overline{a}_i(\mu)\rceil$$

and then let, as in section 1, $\lceil\overline{a}(\mu)\rceil_0=-\sum_{i=1}^d\lceil\overline{a}(\mu)\rceil_i=-d\delta(\mu)-\sum_{i=1}^d\overline{a}_i(\mu)$.

We define $m(\mu)=(m(\mu)_i)_{1\leq i\leq d}\in\mathbb{Z}^d$ and $n(\mu)=(n(\mu)_i)_{1\leq i\leq d}\in\mathbb{Z}^d$ by setting

$$m(\mu)_i=\lfloor i\delta(\mu)\rfloor\qquad\text{and}\qquad n(\mu)_i=\lfloor(i-1-d)\delta(\mu)\rfloor.$$

For $Z\in F_A^0$ and $\gamma\in X^*(\overline{T})\otimes\mathbb{R}$ we define $\gamma(Z)\in\mathbb{R}$ as

$$\gamma(Z)=\gamma(t)\quad\text{with } t\in T \text{ such that } t.Y=Z.$$

For $Z\in F_A^0$ let $\mathcal{J}_Z\subset\mathcal{O}_{\mathfrak{Y}}$ be the ideal defining $Z\cap\mathfrak{Y}$ inside $\mathfrak{Y}$. Note that $\mathcal{J}_Z$ is invertible inside $\mathcal{O}_{\mathfrak{Y}}\otimes_{\mathcal{O}_K}K$: indeed, small open formal subschemes of $\mathfrak{Y}$ admit open embeddings into the π-adic completion of $\operatorname{Spec}(\mathcal{O}_K[X_0,\ldots,X_d]/(X_0\ldots X_d-\pi))$, and there a typical generator of $\mathcal{J}_Z$ is of the form X_0 (for an appropriate numbering of $X_0,\ldots,X_d$); in $K[X_0,\ldots,X_d]/(X_0\ldots X_d-\pi)$ its inverse is $\pi^{-1}X_1\ldots X_d$. Thus we may speak of negative integral powers of $\mathcal{J}_Z$ as $\mathcal{O}_{\mathfrak{Y}}$-submodules of $\mathcal{O}_{\mathfrak{Y}}\otimes_{\mathcal{O}_K}K$. Also note that on small open formal subschemes of $\mathfrak{Y}$ we have $\mathcal{J}_Z=\mathcal{O}_{\mathfrak{Y}}$ for almost all Z; therefore, the infinite products of $\mathcal{O}_{\widehat{\mathfrak{Y}}}$-submodules inside $\mathcal{O}_{\mathfrak{Y}}\otimes_{\mathcal{O}_K}\widehat{K}$ below make sense. On $\mathfrak{Y}$ we define the subsheaf

$$(\mathcal{O}_{\widehat{\mathfrak{Y}}})^{\overline{\mu}}=\sum_{s=0}^{d}\widehat{\pi}^s\prod_{Z\in F_A^0}\mathcal{J}_Z^{\lceil\overline{\mu}(Z)-\frac{s}{d+1}\rceil}, \tag{11}$$

of $\mathcal{O}_{\mathfrak{Y}}\otimes_{\mathcal{O}_K}\widehat{K}$, i.e. $(\mathcal{O}_{\widehat{\mathfrak{Y}}})^{\overline{\mu}}$ is the $\mathcal{O}_{\widehat{\mathfrak{Y}}}$-submodule of $\mathcal{O}_{\mathfrak{Y}}\otimes_{\mathcal{O}_K}\widehat{K}$ generated by the $\mathcal{O}_{\widehat{\mathfrak{Y}}}$-submodules

$$\widehat{\pi}^s\prod_{Z\in F_A^0}\mathcal{J}_Z^{\lceil\overline{\mu}(Z)-\frac{s}{d+1}\rceil}\qquad(s=0,\ldots,d).$$

Let $(\mathcal{O}_{\widehat{\mathfrak{X}}})^{\overline{\mu}}$ be the unique $U(K)$-equivariant subsheaf of $\mathcal{O}_{\mathfrak{X}}\otimes_{\mathcal{O}_K}\widehat{K}$ (with its $U(K)$-action induced by that of $U(K)\subset G$ on $\mathfrak{X}$) whose restriction to $\mathfrak{Y}$ is $(\mathcal{O}_{\widehat{\mathfrak{Y}}})^{\overline{\mu}}$.

Lemma 3.1. *(a) If $\overline{\mu}\in X^*(\overline{T})$, then $(\mathcal{O}_{\widehat{\mathfrak{X}}})^{\overline{\mu}}$ is a line bundle on $\mathfrak{X}$. We have an isomorphism*

$$\mathcal{L}_Y(D(\overline{a}(\mu),0,0))\cong(\mathcal{O}_{\widehat{\mathfrak{X}}})^{\overline{\mu}}\otimes_{\mathcal{O}_{\widehat{\mathfrak{X}}}}\mathcal{O}_Y. \tag{12}$$

(b) If $\overline{\mu} \notin X^*(\overline{T})$, we have isomorphisms

$$(\mathcal{O}_{\widehat{\mathfrak{X}}})^{\overline{\mu}} \otimes_{\mathcal{O}_{\widehat{K}}} k \cong \prod_{x \in F^0} \frac{(\mathcal{O}_{\widehat{\mathfrak{X}}})^{\overline{\mu}} \otimes_{\mathcal{O}_{\widehat{\mathfrak{X}}}} \mathcal{O}_Z}{\mathcal{O}_Z\textit{-torsion}}, \tag{13}$$

$$\mathcal{L}_Y(D(\lceil \overline{a}(\mu) \rceil, n(\mu), m(\mu))) \cong \frac{(\mathcal{O}_{\widehat{\mathfrak{X}}})^{\overline{\mu}} \otimes_{\mathcal{O}_{\widehat{\mathfrak{X}}}} \mathcal{O}_Y}{\mathcal{O}_Y\textit{-torsion}}. \tag{14}$$

Proof. (a) Here $\overline{\mu}(Z) \in \mathbb{Z}$ for all $Z \in F_A^0$ and it follows from formula (11) that

$$(\mathcal{O}_{\widehat{\mathfrak{Y}}})^{\overline{\mu}} = \prod_{Z \in F_A^0} \mathcal{J}_Z^{\overline{\mu}(Z)}. \tag{15}$$

In particular, $(\mathcal{O}_{\widehat{\mathfrak{Y}}})^{\overline{\mu}}$ is a line bundle on $\mathfrak{Y}$ in that case. Now $(\mathcal{O}_{\widehat{\mathfrak{X}}})^{\overline{\mu}} \otimes_{\mathcal{O}_{\widehat{\mathfrak{X}}}} \mathcal{O}_Y$ with its $U(k)$-action is the unique $U(k)$-equivariant subsheaf of the constant sheaf with value the function field $k(Y)$ on Y whose restriction to Y' is

$$(\mathcal{O}_{\widehat{\mathfrak{Y}}})^{\overline{\mu}} \otimes_{\mathcal{O}_{\widehat{\mathfrak{Y}}}} \mathcal{O}_{Y'}$$

(for the uniqueness note that $U(k).Y' = Y$). Write $(\mathcal{O}_{\widehat{\mathfrak{Y}}})^{\overline{\mu}} \otimes_{\mathcal{O}_{\widehat{\mathfrak{Y}}}} \mathcal{O}_{Y'} = \mathcal{L}_{Y'}(D)$ (as subsheaves of the constant sheaf $k(Y)$ on Y') with a divisor D on Y'. By $U(k)$-equivariance of its both sides and $U(k).Y' = Y$, to prove formula (12) we only need to prove $D = D(\overline{a}(\mu), 0, 0)|_{Y'}$. This holds because for $\emptyset \neq \sigma \subsetneq \Upsilon = \{0, \ldots, d\}$ the divisor V_σ on Y as defined in section 1 is the divisor $Z_\sigma \cap Y$ on Y. Here we write $Z_\sigma = t_\sigma Y \in F_A^0$ with $t_\sigma = \mathrm{diag}(t_{\sigma,0}, \ldots, t_{\sigma,d}) \in T \subset G$ defined as $t_{\sigma,j} = 1$ if $j \notin \sigma$ and $t_{\sigma,j} = \pi$ if $j \in \sigma$. We then apply equation (15) which tells us that the prime divisor $V_\sigma \cap Y'$ occurs with multiplicity $-\overline{\mu}(Z_\sigma) = -\sum_{j \in \sigma} \overline{a}(\mu)_j$ in D.

(b) We proceed similarly as in case (a). For $Z \in F_A^0$ define the number $s(Z) \in [0, d] \cap \mathbb{Z}$ by requiring $\overline{\mu}(Z) - \frac{s(Z)}{d+1} \in \mathbb{Z}$. The definition of $\overline{\mu}$ together with our assumption $\overline{\mu} \notin X^*(\overline{T})$ implies $s(Z) \neq s(Z')$ for any two distinct but neighbouring Z, Z' in F_A^0. Since restricted to $\mathfrak{U}_Z$, the relevant summand in formula (11) is the one for $s = s(Z)$, we obtain: the reduction $(\mathcal{O}_{\widehat{\mathfrak{Y}}})^{\overline{\mu}} \otimes_{\mathcal{O}_{\widehat{K}}} k$ of $(\mathcal{O}_{\widehat{\mathfrak{Y}}})^{\overline{\mu}}$ decomposes into a product, indexed by the set F_A^0 whose factor for $Z \in F_A^0$ is the image of the map

$$\widehat{\pi}^{s(Z)} \prod_{Z' \in F_A^0} \mathcal{J}_{Z'}^{\lceil \overline{\mu}(Z') - \frac{s(Z)}{d+1} \rceil} \longrightarrow (\mathcal{O}_{\widehat{\mathfrak{Y}}})^{\overline{\mu}} \longrightarrow (\mathcal{O}_{\widehat{\mathfrak{Y}}})^{\overline{\mu}} \otimes_{\mathcal{O}_{\widehat{\mathfrak{Y}}}} \mathcal{O}_{Z \cap \mathfrak{Y}}.$$

This is a line bundle on $Z \cap \mathfrak{Y}$ and maps isomorphically to the quotient of $(\mathcal{O}_{\widehat{\mathfrak{Y}}})^{\overline{\mu}} \otimes_{\mathcal{O}_{\widehat{\mathfrak{Y}}}} \mathcal{O}_{Z \cap \mathfrak{Y}}$ divided by its $\mathcal{O}_{Z \cap \mathfrak{Y}}$-torsion. Thus

$$(\mathcal{O}_{\widehat{\mathfrak{Y}}})^{\overline{\mu}} \otimes_{\mathcal{O}_{\widehat{K}}} k \cong \prod_{Z \in F_A^0} \frac{(\mathcal{O}_{\widehat{\mathfrak{Y}}})^{\overline{\mu}} \otimes_{\mathcal{O}_{\widehat{\mathfrak{Y}}}} \mathcal{O}_{Z \cap \mathfrak{Y}}}{\mathcal{O}_{Z \cap \mathfrak{Y}}\text{-torsion}} \tag{16}$$

and each

$$\frac{(\mathcal{O}_{\widehat{\mathfrak{Y}}})^{\overline{\mu}} \otimes_{\mathcal{O}_{\mathfrak{Y}}} \mathcal{O}_{Z \cap \mathfrak{Y}}}{\mathcal{O}_{Z \cap \mathfrak{Y}}\text{-torsion}}$$

is an invertible $\mathcal{O}_{Z \cap \mathfrak{Y}}$-module. Hence formula (13). If we define the divisor D on Y' by requiring

$$\frac{(\mathcal{O}_{\widehat{\mathfrak{Y}}})^{\overline{\mu}} \otimes_{\mathcal{O}_{\widehat{\mathfrak{Y}}}} \mathcal{O}_{Y'}}{\mathcal{O}_{Y'}\text{-torsion}} = \mathcal{L}_{Y'}(D),$$

then to prove formula (14) we need to prove $D = D(\lceil \overline{a}(\mu) \rceil, n(\mu), m(\mu))|_{Y'}$. This holds because for $\emptyset \neq \sigma \subset \Upsilon - \{0\}$ the prime divisor V_σ occurs in D with multiplicity

$$-\lceil \overline{\mu}(Z_\sigma) \rceil = -\lceil \sum_{j \in \sigma} \overline{a}_j(\mu) \rceil = m(\mu)_{|\sigma|} - \sum_{j \in \sigma} \lceil \overline{a}(\mu) \rceil_j$$

and for $\sigma \subsetneq \Upsilon$ with $0 \in \sigma$ the prime divisor V_σ occurs in D with multiplicity

$$-\lceil \overline{\mu}(Z_\sigma) \rceil = -\lceil \sum_{j \in \sigma} \overline{a}_j(\mu) \rceil = n(\mu)_{|\sigma|} - \sum_{j \in \sigma} \lceil \overline{a}(\mu) \rceil_j$$

with Z_σ as defined before. □

4. The holomorphic discrete series

Let

$$\Phi = \{\epsilon_i - \epsilon_j;\ 0 \leq i, j \leq d \text{ and } i \neq j\} \subset X^*(\overline{T}).$$

For $0 \leq i, j \leq d$ and $i \neq j$ define the morphism of algebraic groups over $\mathbb{Z}$

$$\widetilde{\alpha}_{ij} : \mathbb{G}_a \longrightarrow \mathrm{GL}_{d+1}, \quad u \mapsto I_{d+1} + u.e_{ij} \tag{17}$$

where $I_{d+1} + u.e_{ij}$ is the matrix (u_{rs}) with $u_{rr} = 1$ (all r), with $u_{ij} = u$ and with $u_{rs} = 0$ for all other pairs (r, s). For the root $\alpha = \epsilon_i - \epsilon_j \in \Phi$ and $r \in \mathbb{R}$ let

$$U_{\alpha,r} = \widetilde{\alpha}_{ij}(\{u \in K;\ \omega(u) \geq r\}) \subset G.$$

For $x \in A$ let

$$U_x = \text{the subgroup of } G \text{ generated by all } U_{\alpha,-\alpha(x)} \text{ for } \alpha \in \Phi.$$

We recall definitions from [15]. Note that many conventions are opposite to those in [14]. Define the GL_{d+1}-subgroups

$$P_1 = \text{matrices of the form} \quad \begin{pmatrix} 1 & 0 & \cdots & 0 \\ * & * & \cdots & * \\ \vdots & \vdots & & \vdots \\ * & * & \cdots & * \end{pmatrix},$$

$$L_1 = \text{matrices of the form} \quad \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & * & \cdots & * \\ \vdots & \vdots & & \vdots \\ 0 & * & \cdots & * \end{pmatrix},$$

$$U_1 = \text{matrices of the form} \quad \begin{pmatrix} 1 & 0 \\ * & I_d \end{pmatrix},$$

$$T_1 = \text{matrices of the form} \quad \mathrm{diag}(1, *, \ldots, *).$$

Then $P_1 = L_1 U_1$ and T_1 is a maximal torus in $L_1 \cong \mathrm{GL}_d$. We view the character group $X^*(T_1)$ of T_1 as the subgroup of $X^*(T)$ generated by $\epsilon_1, \ldots, \epsilon_d$. The morphism $\mathrm{GL}_{d+1} \to \mathbb{A}^{d+1}$, $g \mapsto (g(1, 0, \ldots, 0))$ induces an isomorphism $\mathrm{GL}_{d+1}/P_1 \cong \mathbb{A}^{d+1}$. Over X^{cone} it has the section

$$u : X^{cone} \to \mathrm{GL}_{d+1}, \quad z \mapsto u(z) = \begin{pmatrix} \Xi_0(z) & \Xi_1(z) & \cdots & \Xi_d(z) \\ 0 & & I_d & \end{pmatrix}^{-1}.$$

This gives rise to the automorphy factor

$$\mu : G \to P_1(H^0(X^{cone}, \mathcal{O}_{X^{cone}}))$$
$$g \mapsto u(g(z))^{-1}.g.u(z).$$

As a matrix valued function on X^{cone} it satisfies the automorphy factor relation

$$\mu(gh)(z) = \mu(g)(hz).\mu(h)(z) \quad \text{for}\, g, h \in G$$

(the three factors are viewed as elements in $\mathrm{GL}_{d+1}(H^0(X^{cone}, \mathcal{O}_{X^{cone}}))$, and their product there turns out to lie in $P_1(H^0(X^{cone}, \mathcal{O}_{X^{cone}}))$). Let us describe it explicitly. Let

$$g = \begin{pmatrix} a_{00} & \cdots & a_{0d} \\ \vdots & & \vdots \\ a_{d0} & \cdots & a_{dd} \end{pmatrix} \in G$$

and put

$$A_j = a_{0j}\Xi_0 + \ldots + a_{dj}\Xi_d \quad \text{for}\, 0 \leq j \leq d; \tag{18}$$

then

$$\mu(g^{-1})^{-1} = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ a_{10}A_0^{-1} & a_{11} - a_{10}A_1A_0^{-1} & \cdots & a_{1d} - a_{10}A_dA_0^{-1} \\ \vdots & \vdots & & \vdots \\ a_{d0}A_0^{-1} & a_{d1} - a_{d0}A_1A_0^{-1} & \cdots & a_{dd} - a_{d0}A_dA_0^{-1} \end{pmatrix}. \tag{19}$$

In particular, we see that the image of $\mu(g^{-1})^{-1}$ under the projection

$$P_1(H^0(X^{cone}, \mathcal{O}_{X^{cone}})) \to L_1(H^0(X^{cone}, \mathcal{O}_{X^{cone}}))$$

in fact, lies in $L_1(H^0(X, \mathcal{O}_X))$. We denote it by $\nu(g)$.

Now let $V \neq 0$ be an irreducible K-rational representation of L_1. For $\mu \in X^*(T_1)$ let V_μ be the maximal subspace of V on which T_1 acts through μ. Choose a $\mu = \sum_{i=1}^d a_i\epsilon_i \in X^*(T_1)$ such that $V_\mu \neq 0$ and set $|V| = \sum_{i=1}^d a_i$; this is independent of the choice of μ, as all μ with $V_\mu \neq 0$ differ by linear combinations of elements of Φ (see [12], II.2.2). Viewing V as the constant sheaf with value V on X we define the coherent $\mathcal{O}_X \otimes_K \widehat{K}$-module

$$\mathcal{V}_{\widehat{K}} = V \otimes_K \mathcal{O}_X \otimes_K \widehat{K}.$$

Applying $H^0(X, \mathcal{O}_X \otimes_K \widehat{K}) \otimes_K (.)$ the K-linear action of $L_1(K)$ on V gives rise to an action of $L_1(H^0(X, \mathcal{O}_X \otimes_K \widehat{K}))$ on $\mathcal{V}_{\widehat{K}}$. By the automorphy factor relation we get a G-action on $\mathcal{V}_{\widehat{K}}$ by setting

$$g(f \otimes v) = \widehat{\pi}^{-|V|\omega(\det(g))} f(g^{-1}(.))\nu(g)(1 \otimes v) \tag{20}$$

for $g \in G$, $v \in V$ and any section f of $\mathcal{O}_X \otimes_K \widehat{K}$.

Fix a $L_1/\mathcal{O}_K$-invariant $\mathcal{O}_K$-lattice V_0 in V (see [12], I.10.4). (When we write V_0 we always refer to this lattice in V; to refer to the weight space in V for the weight $\mu = 0$ we reserve the phrase "V_μ for $\mu = 0$ ".)

Lemma 4.1. *We have* $V_0 = \bigoplus_{\mu \in X^*(T_1)} V_{\mu,0}$ *with* $V_{\mu,0} = V_0 \cap V_\mu$.

Proof. We reproduce a proof of Schneider and Teitelbaum. Fix $\mu \in X^*(T_1)$. It suffices to construct an element Π_μ in the algebra of distributions $\mathrm{Dist}(L_1/\mathbb{Z})$ (i.e. defined over $\mathbb{Z}$) which on V acts as a projector onto V_μ. For $1 \le i \le d$ let $H_i = (de_i)(1) \in \mathrm{Lie}(L_1/\mathbb{Z})$; then $d\mu'(H_i) \in \mathbb{Z}$ (inside $\mathrm{Lie}(\mathbb{G}_m/\mathbb{Z})$) for any $\mu' \in X^*(T_1)$. According to [10], Lemma 27.1 we therefore find a polynomial $\Pi \in \mathbb{Q}[X_1, \ldots, X_d]$ such that $\Pi(\mathbb{Z}^d) \subset \mathbb{Z}$, $\Pi(d\mu(H_1), \ldots, d\mu(H_d)) = 1$ and $\Pi(d\mu'(H_1), \ldots, d\mu'(H_d)) = 0$ for any $\mu' \in X^*(T_1)$ such that $\mu' \neq \mu$ and $V_{\mu'} \neq 0$. Moreover, [10], Lemma 26.1 says that Π is a $\mathbb{Z}$-linear combination of polynomials of the form

$$\binom{X_1}{b_1} \cdots \binom{X_d}{b_d} \quad \text{with integers} \quad b_1, \ldots, b_d \geq 0.$$

Thus [12], II.1.12 implies that

$$\Pi_\mu = \Pi(H_1, \ldots, H_d)$$

lies in $\mathrm{Dist}(L_1/\mathbb{Z})$. By construction it acts on V as a projector onto V_μ. □

We denote the pushforward $V \otimes_{\mathcal{O}_K} \mathcal{O}_{\widehat{\mathfrak{X}}}$ of $\mathcal{V}_{\widehat{K}}$ via the specialization map $X \to \mathfrak{X}$ again by $\mathcal{V}_{\widehat{K}}$. It is a G-equivariant (via formula (20)) coherent $\mathcal{O}_{\mathfrak{X}} \otimes_{\mathcal{O}_K} \widehat{K}$-module sheaf on $\mathfrak{X}$.

Theorem 4.2. *There exists a G-equivariant coherent $\mathcal{O}_{\widehat{\mathfrak{X}}}$-submodule $\mathcal{V}_{\mathcal{O}_{\widehat{K}}}$ of $\mathcal{V}_{\widehat{K}}$ such that $\mathcal{V}_{\mathcal{O}_{\widehat{K}}} \otimes_{\mathcal{O}_{\widehat{K}}} \widehat{K} = \mathcal{V}_{\widehat{K}}$. For $Z \in F_A^0$ its restriction to $\mathfrak{U}_Z$ is*

$$\mathcal{V}_{\mathcal{O}_{\widehat{K}}}|_{\mathfrak{U}_Z} = \bigoplus_{\mu \in X^*(T_1)} \widehat{\pi}^{(d+1)\overline{\mu}(Z)} V_{\mu,0} \otimes_{\mathcal{O}_K} \mathcal{O}_{\widehat{\mathfrak{U}}_Z}. \tag{21}$$

Proof. We set

$$\mathcal{V}_{\mathcal{O}_{\widehat{K}}}|_{\mathfrak{Y}} = \bigoplus_{\mu \in X^*(T_1)} (\mathcal{O}_{\widehat{\mathfrak{Y}}})^{\overline{\mu}} \otimes_{\mathcal{O}_K} V_{\mu,0}. \tag{22}$$

Clearly $(\mathcal{V}_{\mathcal{O}_{\widehat{K}}}|_{\mathfrak{Y}})|_{\mathfrak{U}_Z}$ satisfies formula (21) for any $Z \in F_A^0$, because $\mathcal{J}_Z|_{\mathfrak{U}_Z} = \pi \mathcal{O}_{\mathfrak{U}_Z}$ and $\mathcal{J}_{Z'}|_{\mathfrak{U}_Z} = \mathcal{O}_{\mathfrak{U}_Z}$ for $Z' \neq Z$. Since $\mathfrak{X} = G\mathfrak{Y}$, the proof of Theorem 4.2 is complete once we have Proposition 4.3 below. □

Proposition 4.3. *Let $\mathfrak{W}_1$, $\mathfrak{W}_2$ be open formal subschemes of $\mathfrak{Y}$, let $g \in G$ such that $g\mathfrak{W}_1 = \mathfrak{W}_2$. Then the isomorphism*

$$g : \mathcal{V}_{\widehat{K}}|_{\mathfrak{W}_1} \cong \mathcal{V}_{\widehat{K}}|_{\mathfrak{W}_2}$$

induces an isomorphism

$$g : (\mathcal{V}_{\mathcal{O}_{\widehat{K}}}|_{\mathfrak{Y}})|_{\mathfrak{W}_1} \cong (\mathcal{V}_{\mathcal{O}_{\widehat{K}}}|_{\mathfrak{Y}})|_{\mathfrak{W}_2}.$$

Proof. The crucial arguments for (i)–(iii) below are due to Schneider and Teitelbaum (who considered a similar situation on X^{cone} rather than on X or $\mathfrak{X}$). From Lemma 3.1 we deduce: For any open $V \subset \mathfrak{Y} \otimes k$ and any $f \in ((\mathcal{O}_{\widehat{\mathfrak{Y}}})^{\overline{\mu}} \otimes_{\mathcal{O}_{\widehat{K}}} k)(V)$, we have $f = 0$ if and only if $f|_{V \cap \mathfrak{U}_Z} = 0$ for any $Z \in F_A^0$. It follows that for any open $\mathfrak{V} \subset \mathfrak{Y}$ and any $f \in ((\mathcal{O}_{\widehat{\mathfrak{Y}}})^{\overline{\mu}} \otimes_{\mathcal{O}_{\widehat{K}}} \widehat{K})(\mathfrak{V})$, we have $f \in (\mathcal{O}_{\widehat{\mathfrak{Y}}})^{\overline{\mu}}(\mathfrak{V})$ if and only if $f|_{\mathfrak{V} \cap \mathfrak{U}_Z} \in (\mathcal{O}_{\widehat{\mathfrak{Y}}})^{\overline{\mu}}(\mathfrak{V} \cap \mathfrak{U}_Z)$ for any $Z \in F_A^0$. Since the sum over the μ's in the definition of $\mathcal{V}_{\mathcal{O}_{\widehat{K}}}|_{\mathfrak{Y}}$ is direct, this last statement holds verbatim also with $\mathcal{V}_{\mathcal{O}_{\widehat{K}}}|_{\mathfrak{Y}}$ instead of $(\mathcal{O}_{\widehat{\mathfrak{Y}}})^{\overline{\mu}}$.

Assume first that we know Proposition 4.3 whenever $\mathfrak{W}_1 = \mathfrak{U}_Z$, $\mathfrak{W}_2 = \mathfrak{U}_{gZ}$ for some $Z \in F_A^0$ such that $gZ \in F_A^0$. Then consider arbitrary $\mathfrak{W}_1$, $\mathfrak{W}_2$. By construction, $\mathcal{F}_1 = g((\mathcal{V}_{\mathcal{O}_{\widehat{K}}}|_{\mathfrak{Y}})|_{\mathfrak{W}_1})$ and $\mathcal{F}_2 = (\mathcal{V}_{\mathcal{O}_{\widehat{K}}}|_{\mathfrak{Y}})|_{\mathfrak{W}_2}$ are coherent $\mathcal{O}_{\widehat{\mathfrak{W}}_2}$-submodules of $\mathcal{V}_{\widehat{K}}|_{\mathfrak{W}_2}$ such that $\mathcal{F}_1 \otimes_{\mathcal{O}_{\widehat{K}}} \widehat{K} = \mathcal{F}_2 \otimes_{\mathcal{O}_{\widehat{K}}} \widehat{K} = \mathcal{V}_{\widehat{K}}|_{\mathfrak{W}_2}$. From our assumption it follows that $\mathcal{F}_1|_{\mathfrak{U}_Z \cap \mathfrak{W}_2} = \mathcal{F}_2|_{\mathfrak{U}_Z \cap \mathfrak{W}_2}$ for all $Z \in F_A^0$. But this indeed implies $\mathcal{F}_1 = \mathcal{F}_2$ by our remark above.

Now we treat the case $\mathfrak{W}_1 = \mathfrak{U}_Z$, $\mathfrak{W}_2 = \mathfrak{U}_{gZ}$ for some $Z \in F_A^0$ such that $gZ \in F_A^0$. Let us write $Z = t.Y$ with some $t = \operatorname{diag}(t_0, \ldots, t_d) \in T$. Let $x = -\sum_{i=0}^d \omega(t_i)e_i \in A$; this may depend on the choice of t, but the group U_x does not, it is canonically associated with Z. Also note that $\gamma(x) = \gamma(Z)$ for all $\gamma \in X^*(\overline{T}) \otimes \mathbb{R}$. Similarly we choose a $gx \in A$ corresponding to gZ in the same manner. (In fact, we may view A as an apartment in the *extended* building associated with $\mathrm{GL}_{d+1}(K)$; it is acted on by $\mathrm{GL}_{d+1}(K)$ and we may take gx to be the image of x under the action of g: that this lies again in A follows from our hypothesis $gZ \in F_A^0$.) If W denotes the subgroup of permutation matrices, then $N = T \rtimes W$ is the normalizer of T in G. By the Bruhat decomposition, there exist $h_x \in U_x$, $h_{gx} \in U_{gx}$ and $n \in N$ such that $g = h_x n h_{gx}$. Therefore, we may split up our task into the following cases (i)–(iii):

(i) $g \in T$, (ii) $g \in W$, and (iii) $x = gx$ and $g \in U_x$.

(i) Suppose $g = \operatorname{diag}(t_0, \ldots, t_d)$. We claim that in this case g even respects weight spaces; in view of formula (21) this means we must prove that g induces for any $\mu \in X^*(T_1)$ with $V_\mu \neq 0$ an isomorphism

$$g : \widehat{\pi}^{(d+1)\overline{\mu}(x)} V_{\mu,0} \otimes_{\mathcal{O}_K} \mathcal{O}_{\widehat{\mathfrak{U}}_Z} \cong \widehat{\pi}^{(d+1)\overline{\mu}(gx)} V_{\mu,0} \otimes_{\mathcal{O}_K} \mathcal{O}_{\widehat{\mathfrak{U}}_{gZ}}.$$

From formula (19) we deduce $\nu(g) = \operatorname{diag}(1, t_1, \ldots, t_d)$. Write $\mu = \sum_{i=1}^d a_i \epsilon_i$. Then $\nu(g)$ acts on V_μ as $v \mapsto (\prod_{i=1}^d t_i^{a_i})v$, hence induces an isomorphism

$$\nu(g) : V_{\mu,0} \cong \pi^k V_{\mu,0} \quad \text{with} \quad k = \sum_{i=1}^d a_i \omega(t_i).$$

But on the other hand, $k = -\omega(\mu(g)) = |V|\omega(\det(g)^{\frac{1}{d+1}}) - \omega(\mu(g)) - |V|\omega(\det(g)^{\frac{1}{d+1}}) = \omega(\overline{\mu}(g)) - |V|\omega(\det(g)^{\frac{1}{d+1}})$ and therefore

$$\overline{\mu}(gx) = \overline{\mu}(x) + \omega(\overline{\mu}(g)) = \overline{\mu}(Z) + k + |V|\omega(\det(g)^{\frac{1}{d+1}}).$$

Together the claim follows by inspecting formula (20).

(ii) Now $g \in W$. First consider the case $g(e_0) = e_0$. Then $\mu(g^{-1})^{-1} = g$, hence $\nu(g)$ is a permutation matrix acting on V by isomorphisms $\nu(g) : V_\mu \cong V_{\nu(g)\mu}$ which restrict to isomorphisms $\nu(g) : V_{\mu,0} \cong V_{\nu(g)\mu,0}$ since by assumption $V_0 \subset V$ is stable for $L_1(\mathcal{O}_K)$. On the other hand, $\nu(g)$ acts on $X^*(T_1)$ such that $\mu(x) = (\nu(g)\mu)(gx)$ and hence $\overline{\mu}(x) = \overline{(\nu(g)\mu)}(gx)$ for $\mu \in X^*(T_1)$. It follows that g induces isomorphisms

$$\widehat{\pi}^{(d+1)\overline{\mu}(x)} V_{\mu,0} \otimes_{\mathcal{O}_K} \mathcal{O}_{\widehat{\mathfrak{U}}_Z} \cong \widehat{\pi}^{(d+1)\overline{(\nu(g)\mu)}(gx)} V_{\nu(g)\mu,0} \otimes_{\mathcal{O}_K} \mathcal{O}_{\widehat{\mathfrak{U}}_{gZ}}$$

for any $\mu \in X^*(T_1)$ and we are done for such g. Next consider the case where $g \in W$ is the transposition $g = (0i)$ for some $1 \leq i \leq d$. For $0 \leq j \leq d$ let $\Upsilon_j = -\frac{\Xi_j}{\Xi_i}$.

Then

(23)

$$\nu(g)=\begin{pmatrix} 1 &&&&&&&& \\ & 1 &&&&&&& \\ && \ddots &&&&&& \\ &&& 1 &&&&& \\ & \Upsilon_1 & \cdots & \Upsilon_{i-1} & \Upsilon_0 & \Upsilon_{i+1} & \cdots & \Upsilon_d \\ &&&&& 1 &&& \\ &&&&&& \ddots && \\ &&&&&&& 1 \end{pmatrix} \in L_1(\mathcal{O}_{\mathfrak{X}}\otimes_{\mathcal{O}_K} K).$$

Let $J=\{1\le j\le d;\ j\ne i\}$. We may factorize $\nu(g)$ as

$$\nu(g)=e_i(-\Upsilon_0)\prod_{j\in J}\widetilde{\alpha}_{ij}(\Theta_j)$$

with $\Theta_j=\frac{\Xi_j}{\Xi_0}=\frac{\Upsilon_j}{\Upsilon_0}$ and with $\widetilde{\alpha}_{ij}$ as defined by formula (17). For the $\mathcal{O}_{\mathfrak{X}}\otimes_{\mathcal{O}_K}K$-linear extension of the $L_1(K)$-action on V to an $L_1(\mathcal{O}_{\mathfrak{X}}\otimes_{\mathcal{O}_K}K)$-action on $\mathcal{O}_{\mathfrak{X}}\otimes_{\mathcal{O}_K}V$ we have

$$\widetilde{\alpha}_{ij}(f)(1\otimes v)\in\sum_{m\ge 0}f^m.V_{\mu+m(\epsilon_i-\epsilon_j),0} \tag{24}$$

for $f\in\mathcal{O}_{\mathfrak{X}}\otimes_{\mathcal{O}_K}K$, $v\in V_{\mu,0}$ and $j\in J$. To see this define $X_\alpha=(d\widetilde{\alpha}_{ij})(1)\in\operatorname{Lie}(L_1/\mathbb{Z})$ for $\alpha=\epsilon_i-\epsilon_j$ and then

$$X_{\alpha,m}=\frac{X_\alpha^m}{m!}\in\operatorname{Dist}(L_1/\mathbb{Z})\quad\text{for}\quad m\ge 0$$

(compare [12], II.1.11 and 1.12). By [12], II.1.19 we have

$$X_{\alpha,m}V_\mu\subset V_{\mu+m\alpha}\quad\text{and}\quad\widetilde{\alpha}_{ij}(f)(1\otimes v)=\sum_{m\ge 0}f^mX_{\alpha,m}(1\otimes v)$$

(with $\alpha=\epsilon_i-\epsilon_j$). Since $X_{\alpha,m}$ is defined over $\mathbb{Z}$ we in turn have $X_{\alpha,m}V_{\mu,0}\subset V_{\mu+m\alpha,0}$ and formula (24) follows. By the above factorization of $\nu(g)$ and formula (24) we get

$$\nu(g)(1\otimes v)\in\sum_{(m_j)_{j\in J}\in\mathbb{N}_0^J}(\mu+\sum_{j\in J}m_j(\epsilon_i-\epsilon_j))(e_i(-\Upsilon_0))(\prod_{j\in J}\Theta_j^{m_j}).V_{\mu+\sum_{j\in J}m_j(\epsilon_i-\epsilon_j),0}$$

for $v\in V_{\mu,0}$. Here $\prod_{j\in J}\Theta_j^{m_j}\in\pi^{\sum_{j\in J}m_j(\epsilon_j-\epsilon_0)(gx)}\mathcal{O}_{\mathfrak{X}}(\mathfrak{U}_{gZ})$ and

$$(\mu+\sum_{j\in J}m_j(\epsilon_i-\epsilon_j))(e_i(-\Upsilon_0))=(-\Upsilon_0)^{a_i+\sum_{j\in J}m_j}\in\pi^{(a_i+\sum_{j\in J}m_j)(\epsilon_0-\epsilon_i)(gx)}\mathcal{O}_{\mathfrak{X}}(\mathfrak{U}_{gZ})$$

if we write $\mu=\sum_{s=1}^d a_s\epsilon_s$. Together

$$\nu(g)(1\otimes v)\in\sum_{(m_j)_{j\in J}\in\mathbb{N}_0^J}\pi^{\sum_{j\in J}m_j(\epsilon_j-\epsilon_0)(gx)+(a_i+\sum_{j\in J}m_j)(\epsilon_0-\epsilon_i)(gx)}$$
$$V_{\mu+\sum_{j\in J}m_j(\epsilon_i-\epsilon_j),0}\otimes_{\mathcal{O}_K}\mathcal{O}_{\mathfrak{X}}(\mathfrak{U}_{gZ})$$

for $v \in V_{\mu,0}$. We may rewrite the exponent as $\sum_{j\in J} m_j(\epsilon_j - \epsilon_0)(gx) + (a_i + \sum_{j\in J} m_j)(\epsilon_0 - \epsilon_i)(gx) = a_i(\epsilon_0 - \epsilon_i)(gx) - \sum_{j\in J} m_j(\epsilon_i - \epsilon_j)(gx)$. On the other hand,

$$\begin{aligned}\overline{\mu}(x) &= \overline{\mu}(gx) - a_i(\epsilon_0 - \epsilon_i)(gx)\\ &= \overline{\mu + \sum_{j\in J} m_j(\epsilon_i - \epsilon_j)}(gZ) - a_i(\epsilon_0 - \epsilon_i)(gx) + \sum_{j\in J} m_j(\epsilon_i - \epsilon_j)(gx),\end{aligned}$$

and it follows that g maps the submodule $\widehat{\pi}^{(d+1)\overline{\mu}(x)} V_{\mu,0} \otimes_{\mathcal{O}_K} \mathcal{O}_{\widehat{\mathfrak{U}}_Z}$ of $\mathcal{V}_{\widehat{K}}|_{\mathfrak{U}_Z}$ into

$$\bigoplus_{\mu'\in X^*(T_1)} \widehat{\pi}^{(d+1)\overline{\mu'}(gx)} V_{\mu',0} \otimes_{\mathcal{O}_K} \mathcal{O}_{\widehat{\mathfrak{U}}_{gZ}},$$

for any $\mu \in X^*(T_1)$. By a symmetry argument (consider g^{-1}) we are done for this kind of g.

(iii) Now consider the case $x = gx$ and $g \in U_x$. We may assume $g \in U_{\alpha,-\alpha(x)}$ for some $\alpha = \epsilon_i - \epsilon_t \in \Phi$. Thus $g = \widetilde{\alpha}_{it}(u)$ for some $u \in K$ with $\omega(u) \geq -\alpha(x)$. If $i = 0$, then $\nu(g) = I_{d+1}$ and our claim is obvious. If $i \geq 1$ and $t \geq 1$, then $\nu(g) = g$ and it suffices to show that the automorphism g of V induces an automorphism

$$g : \bigoplus_{\mu\in X^*(T_1)} \widehat{\pi}^{(d+1)\overline{\mu}(x)} V_{\mu,0} \cong \bigoplus_{\mu\in X^*(T_1)} \widehat{\pi}^{(d+1)\overline{\mu}(x)} V_{\mu,0}.$$

But $\omega(u) \geq -\alpha(x)$ implies $\overline{\mu + m\alpha}(x) \leq \overline{\mu}(x) + m\omega(u)$ for all $\mu \in X^*(T_1)$, all $m \in \mathbb{N}$ and we conclude as in the proof of formula (24). Finally, assume $t = 0$. For $1 \leq j \leq d$ now let $\Upsilon_j = -\frac{u\Xi_j}{\Xi_0 + u\Xi_i}$ and $\Upsilon_0 = \frac{\Xi_0}{\Xi_0 + u\Xi_i}$. Then formula (23) holds also in this context. Letting $J = \{1 \leq j \leq d;\ j \neq i\}$ and this time $\Theta_j = -\frac{u\Xi_j}{\Xi_0} = \Upsilon_0^{-1}\Upsilon_j$ we may factorize $\nu(g)$ as

$$\nu(g) = e_i(\Upsilon_0) \prod_{j\in J} \widetilde{\alpha}_{ij}(\Theta_j).$$

Similarly as in (ii) we get

$$\nu(g)(1 \otimes v) \in \sum_{(m_j)_{j\in J}\in\mathbb{N}_0^J} (\mu + \sum_{j\in J} m_j(\epsilon_i - \epsilon_j))(e_i(\Upsilon_0))(\prod_{j\in J} \Theta_j^{m_j}).V_{\mu+\sum_{j\in J} m_j(\epsilon_i-\epsilon_j),0}$$

for $v \in V_{\mu,0}$ and $\mu \in X^*(T_1)$. Here $\prod_{j\in J} \Theta_j^{m_j} \in \pi^{\sum_{j\in J} m_j(\omega(u)+(\epsilon_j-\epsilon_0)(x))} \mathcal{O}_{\mathfrak{X}}(\mathfrak{U}_Z)$. On the other hand,

$$(\mu + \sum_{j\in J} m_j(\epsilon_i - \epsilon_j))(e_i(\Upsilon_0)) \in \mathcal{O}_{\mathfrak{X}}(\mathfrak{U}_Z)$$

because $\Upsilon_0 = \frac{\Xi_0}{\Xi_0 + u\Xi_i}$ is a unit in $\mathcal{O}_{\mathfrak{X}}(\mathfrak{U}_Z)$ (because $g = \widetilde{\alpha_{i0}}(u)$ and $g\mathfrak{U}_Z = \mathfrak{U}_Z$). Together, observing $\omega(u) \geq -(\epsilon_i - \epsilon_0)(x)$, we obtain

$$\nu(g)(1 \otimes v) \in \sum_{(m_j)_{j\in J}\in\mathbb{N}_0^J} \pi^{-\sum_{j\in J} m_j(\epsilon_i-\epsilon_j)} V_{\mu+\sum_{j\in J} m_j(\epsilon_i-\epsilon_j),0} \otimes_{\mathcal{O}_K} \mathcal{O}_{\mathfrak{X}}(\mathfrak{U}_Z)$$

for $v \in V_{\mu,0}$. This concludes the proof of Proposition 4.3 and thus of Theorem 4.2. □

Remark. If $|V| \in (d+1)\mathbb{Z}$ (equivalently: if $\overline{\mu} \in X^*(\overline{T})$ for all $\mu \in X^*(T^1)$ with $V_\mu \neq 0$), then we could replace the $\widehat{K}$-valued character $\widehat{\pi}^{-|V|\omega(\det(g))}$ in definition (20) with the K-valued character $\det(g)^{-(d+1)^{-1}|V|}$. Then the scalar extension $K \to \widehat{K}$

could be completely avoided and we would obtain a $\mathrm{PGL}_{d+1}(K)$-equivariant locally free coherent $\mathcal{O}_{\mathfrak{X}}$-module $\mathcal{V}_{\mathcal{O}_K}$ such that $\mathcal{V}_{\mathcal{O}_K}\otimes_{\mathcal{O}_K} K = V\otimes_{\mathcal{O}_K}\mathcal{O}_{\mathfrak{X}}$.

A lattice in K^{d+1} is a free $\mathcal{O}_K$-submodule of K^{d+1} of rank $d+1$. Two lattices L, L' are homothetic if $L' = \lambda L$ for some $\lambda\in K^{\times}$. We denote the homothety class of L by $[L]$. The set of vertices of the Bruhat-Tits building $\mathcal{B}T$ of $\mathrm{PGL}_{d+1}(K)$ is the set of homothety classes of lattices (always: in K^{d+1}). For a lattice chain

$$\pi L_s \subsetneq L_1 \subsetneq \ldots \subsetneq L_s$$

we declare the ordered s-tuple $([L_1],\ldots,[L_s])$ to be a pointed $s-1$-simplex (with underlying $s-1$-simplex the unordered set $\{[L_1],\ldots,[L_s]\}$). Call it $\widehat{\eta}$ and consider the set

$$N_{\widehat{\eta}} = \{L \text{ a lattice in } K^{d+1} \quad | \quad \pi L_s \subsetneq L \subsetneq L_1\}.$$

A subset M_0 of $N_{\widehat{\eta}}$ is called *stable* if for all $L, L'\in M_0$ also $L\cap L'$ lies in M_0. We may identify $N_{\widehat{\eta}}$ (and hence M_0) with the set of homothety classes which its elements represent; this is independent of the choice of representing lattices for $[L_1],\ldots,[L_s]$. We recall a result from [7]. Let $\mathcal{F}$ denote a cohomological coefficient system on $\mathcal{B}T$.

Proposition 4.4. *Let $1\le s\le d$. Suppose that for any pointed $s-1$-simplex $\widehat{\eta}$ with underlying $s-1$-simplex η and for any stable subset M_0 of $N_{\widehat{\eta}}$ the following subquotient complex of the cochain complex $C^\bullet(\mathcal{B}T,\mathcal{F})$ with values in $\mathcal{F}$ is exact:*

$$\mathcal{F}(\eta)\longrightarrow \prod_{z\in M_0}\mathcal{F}(\{z\}\cup\eta)\longrightarrow \prod_{\substack{z,z'\in M_0\\ \{z,z'\}\in F^1}}\mathcal{F}(\{z,z'\}\cup\eta).$$

Then the s-th cohomology group $H^s(\mathcal{B}T,\mathcal{F})$ of $C^\bullet(\mathcal{B}T,\mathcal{F})$ vanishes. □

A homothety class $[L]$ (= a vertex of $\mathcal{B}T$) corresponds to an irreducible component $Z_{[L]}$ of $\mathfrak{X}\otimes k$. If L represents $[L]$ and $(L/\pi L)^*$ denotes the k-vector space dual to $L/\pi L$, then we may regard $Z_{[L]}$ as the successive blowing up of the projective space $\mathbb{P}((L/\pi L)^*)$ in all its k-linear subspaces. Now let again $\pi L_s\subsetneq L_1\subsetneq\ldots\subsetneq L_s$ define a pointed $s-1$-simplex $\widehat{\eta}$ and suppose that $L_s=\mathcal{O}_K^{d+1}$, the standard lattice. Then $Z_{[L_s]} = Y$, the central component of $\mathfrak{X}\otimes k$ considered in section 1. The lattices L with $\pi L_s\subsetneq L\subsetneq L_s$ correspond bijectively to the irreducible components Z of $\mathfrak{X}\otimes k$ with $Z\cap Y\neq\emptyset$, or equivalently to the elements of the set $\mathcal{V}$ from section 1. Namely, L defines a k-linear subspace in $L_s/\pi L_s$, hence a quotient of $(L_s/\pi L_s)^*$, hence (taking the kernel) a k-linear subspace of $(L_s/\pi L_s)^*$, hence a k-linear subspace of $\mathbb{P}((L_s/\pi L_s)^*)$; its strict transform under $Y\to\mathbb{P}((L_s/\pi L_s)^*)$ is the element of $\mathcal{V}$ corresponding to L. It is clear that a subset M_0 of $N_{\widehat{\eta}}$ is stable if and only if the corresponding subset of $\mathcal{V}$ is stable in the sense of section 1.

Theorem 4.5. *Suppose that for every $\mu=\sum_{i=1}^d a_i\epsilon_i\in X^*(T_1)$ with $V_\mu\neq 0$ and for every $1\le j\le d$ we have $\sum_{i\neq j}a_i\le da_j$. Then*

$$H^t(\mathfrak{X},\mathcal{V}_{\mathcal{O}_{\widehat{K}}})=0, \tag{25}$$

$$H^t(\mathfrak{X},\mathcal{V}_{\mathcal{O}_{\widehat{K}}}\otimes_{\mathcal{O}_{\widehat{K}}}k)=0 \tag{26}$$

for all $t>0$, and

$$H^0(\mathfrak{X},\mathcal{V}_{\mathcal{O}_{\widehat{K}}})\otimes_{\mathcal{O}_{\widehat{K}}}k = H^0(\mathfrak{X},\mathcal{V}_{\mathcal{O}_{\widehat{K}}}\otimes_{\mathcal{O}_{\widehat{K}}}k). \tag{27}$$

Proof. We use the following ordering on $X^*(T)$: Define

$$\sum_{i=0}^{d} a_i\epsilon_i > \sum_{i=0}^{d} a_i'\epsilon_i \tag{28}$$

if and only if there exists a $0 \leq i_0 \leq d$ such that $a_i = a_i'$ for all $i < i_0$, and $a_{i_0} > a_{i_0}'$. In particular, we get an ordering on $X^*(T_1)$. Then by [12], II.1.19 the filtration $(F^\mu V)_{\mu\in X^*(T_1)}$ of our K-rational L_1-representation V defined by

$$F^\mu V = \sum_{\substack{\mu'\in X^*(T_1)\\ \mu'\geq\mu}} V_{\mu'}$$

is stable for the action of $U(K)\cap L_1(K)$. Define the filtration $(F^\mu\mathcal{V}_{\mathcal{O}_{\widehat{K}}})_{\mu\in X^*(T_1)}$ of $\mathcal{V}_{\mathcal{O}_{\widehat{K}}}$ by

$$F^\mu\mathcal{V}_{\mathcal{O}_{\widehat{K}}} = \mathcal{V}_{\mathcal{O}_{\widehat{K}}} \cap (F^\mu V \otimes_{\mathcal{O}_K} \mathcal{O}_{\widehat{\mathfrak{X}}})$$

$(\mu \in X^*(T_1))$. Let

$$Gr^\mu\mathcal{V}_{\mathcal{O}_{\widehat{K}}} = \frac{F^\mu\mathcal{V}_{\mathcal{O}_{\widehat{K}}}}{F^{\mu+\epsilon_d}\mathcal{V}_{\mathcal{O}_{\widehat{K}}}} = \frac{F^\mu\mathcal{V}_{\mathcal{O}_{\widehat{K}}}}{\bigoplus_{\mu'>\mu} F^{\mu'}\mathcal{V}_{\mathcal{O}_{\widehat{K}}}}.$$

Since for any $g \in U(K)$ the automorphy factor $\nu(g)$ is just the image of g under the natural projection $U(K) \to U(K)\cap L_1(K)$ we deduce that the coherent $\mathcal{O}_{\widehat{\mathfrak{X}}}$-modules $F^\mu\mathcal{V}_{\mathcal{O}_{\widehat{K}}}$ and $Gr^\mu\mathcal{V}_{\mathcal{O}_{\widehat{K}}}$ are $U(K)$-equivariant. Note that the $U(K)$-action on $Gr^\mu\mathcal{V}_{\mathcal{O}_{\widehat{K}}}$ is with *trivial* automorphy factors. We have by construction a canonical isomorphism

$$Gr^\mu\mathcal{V}_{\mathcal{O}_{\widehat{K}}}|_{\mathfrak{Y}} \cong (\mathcal{O}_{\widehat{\mathfrak{Y}}})^{\overline{\mu}} \otimes_{\mathcal{O}_K} V_{\mu,0}. \tag{29}$$

This is because the composition

$$V_{\mu,0} \longrightarrow V_0 \cap F^\mu V \longrightarrow \frac{V_0\cap F^\mu V}{V_0\cap F^{\mu+\epsilon_d}V}$$

is an isomorphism.

Now we prove Theorem 4.5. The hypothesis means that for all $\mu \in X^*(T_1)$ with $V_\mu \neq 0$ we have $\overline{a}_j(\mu) \leq 0$ for all $1 \leq j \leq d$. We claim

$$H^t(\mathfrak{X}, Gr^\mu\mathcal{V}_{\mathcal{O}_{\widehat{K}}} \otimes_{\mathcal{O}_{\widehat{K}}} k) = 0 \qquad (t>0) \tag{30}$$

for all μ. First consider the case $\overline{\mu} \in X^*(\overline{T})$. Then Lemma 3.1 and equation (29) imply that $Gr^\mu\mathcal{V}_{\mathcal{O}_{\widehat{K}}} \otimes_{\mathcal{O}_{\widehat{K}}} k$ is a $U(K)$-equivariant vector bundle on $\mathfrak{X}\otimes k$, of rank $\mathrm{rk}_{\mathcal{O}_K} V_{\mu,0} = \dim_K V_\mu$, and that

$$\mathcal{L}_Y(D(\overline{a}(\mu),0,0)) \otimes_k (V_{\mu,0}\otimes_{\mathcal{O}_K} k) \cong Gr^\mu\mathcal{V}_{\mathcal{O}_{\widehat{K}}} \otimes_{\mathcal{O}_{\widehat{\mathfrak{X}}}} \mathcal{O}_Y. \tag{31}$$

With Corollary 1.6 (and using $U(K)$-equivariance which tells us that we may assume $W \subset Y$) we get

$$H^t(\mathfrak{X}, Gr^\mu\mathcal{V}_{\mathcal{O}_{\widehat{K}}} \otimes_{\mathcal{O}_{\widehat{\mathfrak{X}}}} \mathcal{O}_W) = 0 \tag{32}$$

for all $t > 0$, for every component intersection $W \in F^s$ (any $0 \leq s \leq d$). On the other hand, since $Gr^\mu\mathcal{V}_{\mathcal{O}_{\widehat{K}}} \otimes_{\mathcal{O}_{\widehat{K}}} k$ is locally free over $\mathcal{O}_{\mathfrak{X}\otimes k}$ we have an exact sequence

$$0 \longrightarrow Gr^\mu \mathcal{V}_{\mathcal{O}_{\widehat{K}}} \otimes_{\mathcal{O}_{\widehat{K}}} k \longrightarrow \prod_{W \in F^0} Gr^\mu \mathcal{V}_{\mathcal{O}_{\widehat{K}}} \otimes_{\mathcal{O}_{\widehat{\mathfrak{X}}}} \mathcal{O}_W$$
$$\longrightarrow \prod_{W \in F^1} Gr^\mu \mathcal{V}_{\mathcal{O}_{\widehat{K}}} \otimes_{\mathcal{O}_{\widehat{\mathfrak{X}}}} \mathcal{O}_W \longrightarrow \ldots.$$

Therefore, (32) tells us that to prove (30) we need to prove $H^s(\mathcal{BT}, \mathcal{F}) = 0$ for all $s > 0$, where $\mathcal{F}$ is the coefficient system on $\mathcal{BT}$ defined by

$$\mathcal{F}(W) = H^0(\mathfrak{X}, Gr^\mu \mathcal{V}_{\mathcal{O}_{\widehat{K}}} \otimes_{\mathcal{O}_{\widehat{\mathfrak{X}}}} \mathcal{O}_W)$$

for $W \in F^s$, where we identify F^s with the set of s-simplices of $\mathcal{BT}$ (any s). We apply Proposition 4.4. Given a pointed $s-1$-simplex $\widehat{\eta} = ([L_1], \ldots, [L_s])$ we may, by G-equivariance, assume that $[L_s]$ is represented by $\mathcal{O}_K^{d+1}$. Then (31) and Corollary 1.6 tell us that the hypothesis of Propositon 4.4 is satisfied, proving (30).

The case $\overline{\mu} \notin X^*(\overline{T})$ is easier: there we find

$$Gr^\mu \mathcal{V}_{\mathcal{O}_{\widehat{K}}} \otimes_{\mathcal{O}_{\widehat{K}}} k \cong \prod_{Z \in F^0} \frac{Gr^\mu \mathcal{V}_{\mathcal{O}_{\widehat{K}}} \otimes_{\mathcal{O}_{\widehat{\mathfrak{X}}}} \mathcal{O}_Z}{\mathcal{O}_Z\text{-torsion}}, \tag{33}$$

$$\mathcal{L}_Y(D(\lceil \overline{a}(\mu) \rceil, n(\mu), m(\mu))) \otimes_k (V_{\mu,0} \otimes_{\mathcal{O}_K} k) \cong \frac{Gr^\mu \mathcal{V}_{\mathcal{O}_{\widehat{K}}} \otimes_{\mathcal{O}_{\widehat{\mathfrak{X}}}} \mathcal{O}_Y}{\mathcal{O}_Y\text{-torsion}}. \tag{34}$$

Combining with Theorem 1.4 (and using $U(K)$-equivariance which tells us that it suffices to look at $Z = Y$) we get (30).

By the obvious devissage argument we get (26) from (30). The base change formula (27) is a consequence of (25) for $t = 1$, and (25) for all $t > 0$ is a consequence of (26) for all $t > 0$, as one easily shows by using of the exact sequences

$$0 \longrightarrow \mathcal{V}_{\mathcal{O}_{\widehat{K}}} \otimes_{\mathcal{O}_{\widehat{K}}} k \xrightarrow{\pi^r} \mathcal{V}_{\mathcal{O}_{\widehat{K}}}/(\widehat{\pi}^{r+1}) \longrightarrow \mathcal{V}_{\mathcal{O}_{\widehat{K}}}/(\widehat{\pi}^r) \longrightarrow 0.$$

□

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Mathematisches Institut der Universität Münster, Einsteinstrasse 62, 48149 Münster, Germany
E-mail address: klonne@math.uni-muenster.de