**HARISH-CHANDRA MODULES FOR YANGIANS**

VYACHESLAV FUTORNY, ALEXANDER MOLEV, AND SERGE OVSIENKO

**Abstract.** We study Harish-Chandra representations of the Yangian $Y(\mathfrak{gl}_2)$ with respect to a natural maximal commutative subalgebra. We prove an analogue of the Kostant theorem showing that the restricted Yangian $Y_p(\mathfrak{gl}_2)$ is a free module over the corresponding subalgebra $\Gamma$ and show that every character of $\Gamma$ defines a finite number of irreducible Harish-Chandra modules over $Y_p(\mathfrak{gl}_2)$. We study the categories of generic Harish-Chandra modules, describe their simple modules and indecomposable modules in tame blocks.

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1. **Introduction**

Throughout the paper we fix an algebraically closed field $k$ of characteristic 0. Consider the pair $(U, \Gamma)$ where $U$ is an associative $k$-algebra and $\Gamma$ is a subalgebra of $U$. Denote by $\text{cfs} \Gamma$ the **cofinite spectrum** of $\Gamma$, i.e.,

$$\text{cfs} \Gamma = \{\text{maximal two-sided ideals } m \text{ of } \Gamma \mid \dim \Gamma/m < \infty\}.$$
A finitely generated module $M$ over $U$ is called a *Harish-Chandra module* (with respect to $\Gamma$) if

$$M = \bigoplus_{m \in \text{cfs } \Gamma} M(m),$$

where

$$M(m) = \{ x \in M \mid m^k x = 0 \text{ for some } k \geq 0 \}.$$ 

Harish-Chandra modules play a central role in the classical representation theory; see e.g. Dixmier [Di]. In particular, weight modules over a semisimple Lie algebra are Harish-Chandra modules with respect to the universal enveloping algebra of a Cartan subalgebra. Another important example is provided by the Gelfand–Tsetlin modules [DFO1] over the universal enveloping algebra $U(\mathfrak{gl}_n)$. They are Harish-Chandra modules with respect to the Gelfand–Tsetlin subalgebra of $U(\mathfrak{gl}_n)$. The latter is the commutative subalgebra generated by the centers of $U(\mathfrak{gl}_k)$, $k = 1, \ldots, n$. A theory of Harish-Chandra modules for general pairs $(U, \Gamma)$ is developed in [DFO2].

An irreducible Harish-Chandra module $M$ is said to be *extended from* $m \in \text{cfs } \Gamma$ if $M(m) \neq 0$. A central problem in the theory of Harish-Chandra modules is to investigate the existence and uniqueness conditions for such an extension. In the case where the extension is unique, the irreducible Harish-Chandra modules are parametrized by some equivalence classes of the elements of cfs $\Gamma$. It has been recently proved in [Ov] that for the case of Gelfand–Tsetlin modules over $\mathfrak{gl}_n$ the number of pairwise non-isomorphic irreducible modules extended from a given $m \in \text{cfs } \Gamma$ is nonzero and finite.

In this paper we begin a detailed study of Harish-Chandra modules over the Yangians. The *Yangian for $\mathfrak{gl}_n$* is a unital associative algebra $Y(\mathfrak{gl}_n)$ over $k$ with countably many generators $t_{ij}^{(1)}, t_{ij}^{(2)}, \ldots$ where $1 \leq i, j \leq n$, and the defining relations

$$(1.1) \quad (u - v) [t_{ij}(u), t_{kl}(v)] = t_{kj}(u) t_{il}(v) - t_{kj}(v) t_{il}(u),$$

where

$$t_{ij}(u) = \delta_{ij} + t_{ij}^{(1)} u^{-1} + t_{ij}^{(2)} u^{-2} + \cdots$$

and $u, v$ are formal variables. This algebra originally appeared in the works on the quantum inverse scattering method; see e.g. Takhtajan–Faddeev [TF], Kulish–Sklyanin [KS]. The term “Yangian” and generalizations of $Y(\mathfrak{gl}_n)$ to an arbitrary simple Lie algebra were introduced by Drinfeld [D1]. He then classified finite-dimensional irreducible modules over the Yangians in [D2] using earlier results of Tarasov [T1, T2]. An explicit construction of every such module over $Y(\mathfrak{gl}_n)$ is given in those papers by Tarasov and also in the work by Chari and Pressley [CP]. Apart from this case, the structure of a general finite-dimensional irreducible representation of the Yangian remains unknown. In the case of $Y(\mathfrak{gl}_n)$ a description of generic modules was given in [M1] via Gelfand–Tsetlin bases. A more general class of tame representations of $Y(\mathfrak{gl}_n)$ was introduced and explicitly constructed by Nazarov and Tarasov [NT]. Another family of representations has been constructed in [M3] via tensor products of the so-called evaluation modules. An important role in these works is played by the *Drinfeld generators* [D2]

$$a_i(u), \quad i = 1, \ldots, n, \quad b_i(u), \quad c_i(u), \quad i = 1, \ldots, n - 1$$
of the algebra \( Y(\mathfrak{g}_n) \) which are defined as certain \( \textit{quantum minors} \) of the matrix \( T(u) = (t_{ij}(u)) \). The coefficients of the series \( a_i(u) \), \( i = 1, \ldots, n \) form a commutative subalgebra of \( Y(\mathfrak{g}_n) \) which can be regarded as an analogue of the Gelfand–Tsetlin subalgebra of \( U(\mathfrak{g}_n) \). We shall consider the Harish-Chandra modules for \( Y(\mathfrak{g}_n) \) with respect to this particular subalgebra. So, the Harish-Chandra modules for \( Y(\mathfrak{g}_n) \) are natural analogues of the Gelfand–Tsetlin modules for \( \mathfrak{gl}_n \) [DFO1]. Note also that the tame modules over \( Y(\mathfrak{g}_n) \) [NT] is a particular case of Harish-Chandra modules.

In this paper we are concerned with Harish-Chandra modules for the Yangian \( Y(\mathfrak{g}_2) \). Recall that every irreducible finite-dimensional \( Y(\mathfrak{g}_2) \)-module contains a unique vector \( \xi \) annihilated by \( t_{12}(u) \) and which is an eigenvector for the Drinfeld generators \( a_1(u) \) and \( a_2(u) \) defined by

\[
(1.2) \quad a_1(u) = t_{11}(u) t_{22}(u-1) - t_{21}(u) t_{12}(u-1), \quad a_2(u) = t_{22}(u);
\]

see [T1] and [T2]. Moreover, there exists an automorphism \( t_{ij}(u) \to c(u) t_{ij}(u) \) of \( Y(\mathfrak{g}_2) \), where \( c(u) \in 1 + u^{-1} k[[u^{-1}]] \), such that the eigenvalues of \( \xi \) become polynomials in \( u^{-1} \) under the corresponding twisted action of the Yangian. This prompts the introduction of the class of \( \textit{Harish-Chandra polynomial} \) modules over \( Y(\mathfrak{g}_2) \), i.e., such Harish-Chandra modules where the operators \( a_1(u) \) and \( a_2(u) \) are polynomials. More precisely, due to (1.2), it is natural to require that for some positive integer \( p \) the polynomials \( a_1(u) \) and \( a_2(u) \) have degrees \( 2p \) and \( p \), respectively. Note that \( a_1(u) \) is the \( \textit{quantum determinant} \) of the matrix \( T(u) \) [IK], [KS]. Its coefficients are algebraically independent generators of the center of \( Y(\mathfrak{g}_2) \).

We can interpret the definition of Harish-Chandra polynomial modules using the algebra \( Y_p(\mathfrak{g}_2) \) called the \( \textit{Yangian of level} p \); see Cherednik [C1, C2]. It is defined as the quotient of \( Y(\mathfrak{g}_2) \) by the ideal generated by the elements \( t_{ij}^{(r)} \) with \( r \geq p + 1 \). A Harish-Chandra polynomial module over \( Y(\mathfrak{g}_2) \) is just a Harish-Chandra module over \( Y_p(\mathfrak{g}_2) \) for some positive integer \( p \). In what follows we shall consider Harish-Chandra modules over \( Y_p(\mathfrak{g}_2) \) with respect to the commutative subalgebra \( \Gamma \) generated by the coefficients of the polynomials \( a_1(u) \) and \( a_2(u) \).

Let us now describe our main results. First, we prove that \( Y_p(\mathfrak{g}_2) \) is free as a left (right) \( \Gamma \)-module (Theorem 3.3). This is an analogue of the well-known Kostant theorem [K]. Each character of \( \Gamma \) can therefore be extended to an irreducible \( Y_p(\mathfrak{g}_2) \)-module. An important role in our study is played by certain universal Harish-Chandra modules over \( Y_p(\mathfrak{g}_2) \) (see Section 4) such that every irreducible module in \( \mathbb{H}(Y_p(\mathfrak{g}_2), \Gamma) \) is a quotient of the corresponding universal module.

Further, we show that \( \Gamma \) is a Harish-Chandra subalgebra (Theorem 5.3) in the sense of [DFO2] which allows us to apply the general theory of [DFO2] to the study of Harish-Chandra modules for \( Y_p(\mathfrak{g}_2) \). In particular, it provides an equivalence between the category \( \mathbb{H}(Y_p(\mathfrak{g}_2), \Gamma) \) of Harish-Chandra modules and the category of finitely generated modules over a certain category \( \mathcal{A} \) whose objects are the maximal ideals of \( \Gamma \). We then use this to prove that the number of pairwise non-isomorphic extensions of a character of \( \Gamma \) to an irreducible \( Y_p(\mathfrak{g}_2) \)-module is finite (Theorem 7.2). The full subcategory \( \mathbb{H}(Y_p(\mathfrak{g}_2), \Gamma) \) of \( \mathbb{H}(Y_p(\mathfrak{g}_2), \Gamma) \) which consists of modules with diagonalizable action of \( \Gamma \) turns out to be equivalent to the category of finitely generated modules over a certain quotient category of \( \mathcal{A} \) (Section 2.1). In Section 8 we study a full subcategory in \( \mathbb{H}(Y_p(\mathfrak{g}_2), \Gamma) \) of generic modules, this imposes a certain condition on the eigenvalues of \( a_2(u) \) while those of \( a_1(u) \)
are arbitrary. In particular, we give a complete description of irreducible modules (Theorem 5.3) and indecomposable modules in tame blocks of this category (Theorem 5.9).

2. Preliminaries

2.1. Harish-Chandra subalgebras. In this paper we shall only consider the pairs \((U, \Gamma)\) where the subalgebra \(\Gamma\) of \(U\) is commutative. In this case \(\text{cfs} \Gamma\) coincides with the set \(\text{Specm} \Gamma\) of all maximal ideals in \(\Gamma\). We endow this set with the Zariski topology.

We let \(U\text{-mod}\) denote the category of finitely generated left modules over an associative algebra \(U\). The Harish-Chandra modules for the pair \((U, \Gamma)\) form a full abelian subcategory in \(U\text{-mod}\) which we denote by \(\mathbb{H}(U, \Gamma)\). A Harish-Chandra module \(M\) is called weight if the following condition holds: for all \(m \in \text{Specm} \Gamma\) and all \(x \in M(m)\) one has \(mx = 0\). The full subcategory of \(\mathbb{H}(U, \Gamma)\) consisting of weight modules will be denoted \(\mathbb{H}(U, \Gamma)\). The support of a Harish-Chandra module \(M\) is the subset \(\text{Supp} M \subseteq \text{Specm} \Gamma\) which consists of those \(m\) which have the property \(M(m) \neq 0\). If for a given \(m\) there exists an irreducible Harish-Chandra module \(M\) with \(M(m) \neq 0\), then we say that \(m\) extends to \(M\).

A commutative subalgebra \(\Delta \subseteq U\) is called a Harish-Chandra subalgebra of \(U\) if for any \(a \in U\) the \(\Gamma\)-bimodule \(\Gamma a \Gamma\) is finitely generated both as left and as right \(\Gamma\)-module. The property of \(\Gamma\) to be a Harish-Chandra subalgebra is important for the effective study of the category \(\mathbb{H}(U, \Gamma)\). In this case, for any finite-dimensional \(\Gamma\)-module \(X\) the module \(U \otimes_{\Gamma} X\) is a Harish-Chandra module. For any \(a \in U\) set

\[ X_a = \{(m, n) \in \text{Specm} \Gamma \times \text{Specm} \Gamma \mid \Gamma/a \Gamma \cap \Gamma m = 0\}. \]

Equivalently, \((m, n) \in X_a\) if and only if \((\Gamma/a \Gamma) \otimes_{\Gamma} \Gamma a \Gamma \otimes_{\Gamma} (\Gamma/m) \neq 0\). Denote by \(\Delta\) the minimal equivalence on \(\text{Specm} \Gamma\) containing all \(X_a, a \in U\) and by \(\Delta(A, \Gamma)\) the set of the \(\Delta\)-equivalence classes on \(\text{Specm} \Gamma\). Then for any \(a \in U\) and \(m \in \text{Specm} \Gamma\) we have

\[ aM(m) \subseteq \sum_{(m, n) \in X_a} M(n), \quad \mathbb{H}(U, \Gamma) = \bigoplus_{D \in \Delta(U, \Gamma)} \mathbb{H}(U, \Gamma, D), \]

where the subcategory \(\mathbb{H}(U, \Gamma, D)\) consists of the Harish-Chandra modules \(M\) such that \(\text{Supp} M \subseteq D\). Define a category \(\mathcal{A} = \mathcal{A}_{U, \Gamma}\) with the set of objects \(\text{Ob} \mathcal{A} = \text{Specm} \Gamma\) and with the space of morphisms \(\mathcal{A}(m, n)\) from \(m\) to \(n\), where

\[ \mathcal{A}(m, n) = \lim_{\longrightarrow n, m} U/(n^\alpha U + Um^\alpha) \]

(equivalently, \(\lim_{\longrightarrow n, m} \Gamma/n^\alpha \otimes_{\Gamma} U \otimes_{\Gamma} \Gamma/m^\alpha\)).

Consider the completion \(\Gamma_m = \lim_{\longrightarrow n, m} \Gamma/m^n\) of \(\Gamma\) by an ideal \(m \in \text{Specm} \Gamma\). Then the space \(\mathcal{A}(m, n)\) has a natural structure of \(\Gamma_n - \Gamma_m\)-bimodule. We have the decomposition

\[ \mathcal{A} = \bigoplus_{D \in \Delta(U, \Gamma)} \mathcal{A}(D), \]

where \(\mathcal{A}(D)\) is the restriction of \(\mathcal{A}\) on \(D\). The category \(\mathcal{A}\) is endowed with the topology of the inverse limit while the category of \(k\)-vector spaces \((k\text{-mod})\) is endowed with the discrete topology. Consider the category \(\mathcal{A}\text{-mod}_d\) of continuous
functors $M : \mathcal{A} \to \mathcal{k}\text{-mod}$. We call them discrete modules following the terminology of [DFO2 Section 1.5]. For any discrete $\mathcal{A}$-module $N$ define a Harish-Chandra $U$-module

$$F(N) = \bigoplus_{m \in \text{Specm} \Gamma} N(m).$$

Furthermore, for $x \in N(m)$ and $a \in U$ set

$$ax = \sum_{n \in \text{Specm} \Gamma} a_n x$$

where $a_n$ is the image of $a$ in $\mathcal{A}(m,n)$. For any morphism $f : M \to N$ in the category $\mathcal{A}\text{-mod}_d$ set

$$F(f) = \bigoplus_{m \in \text{Specm} \Gamma} f(m).$$

We thus have a functor $F : \mathcal{A}\text{-mod}_d \to \mathcal{H}(U,\Gamma)$.

**Proposition 2.1** ([DFO2 Theorem 17]). The functor $F$ is an equivalence.

We will identify a discrete $\mathcal{A}$-module $N$ with the corresponding Harish-Chandra module $F(N)$. For $m \in \text{Specm} \Gamma$ denote by $\hat{m}$ the completion of $m$. Clearly, $\hat{m} \subseteq \Gamma_m$. Consider the two-sided ideal $I \subseteq \mathcal{A}$ generated by the completions $\hat{m}$ for all $m \in \text{Specm} \Gamma$ and set $\mathcal{A}_W = \mathcal{A}/I$. Proposition 2.1 implies the following.

**Corollary 2.2.** The categories $\mathcal{H}_W(U,\Gamma)$ and $\mathcal{A}_W\text{-mod}$ are equivalent.

The subalgebra $\Gamma$ is called big in $m \in \text{Specm} \Gamma$ if $\mathcal{A}(m,m)$ is finitely generated as a left (or, equivalently, right) $\Gamma_m$-module.

**Lemma 2.3** ([DFO2 Corollary 19]). If $\Gamma$ is big in $m \in \text{Specm} \Gamma$, then there exist finitely many non-isomorphic irreducible Harish-Chandra $U$-modules $M$ such that $M(m) \neq 0$. For any such module $\dim M(m) < \infty$.

### 2.2. Special PBW algebras

Let $U$ be an associative algebra over $\mathcal{k}$ endowed with an increasing filtration $\{U_i\}_{i \in \mathbb{Z}}, U_{-1} = \{0\}, U_0 = \mathcal{k}$, $U_iU_j \subseteq U_{i+j}$. For $u \in U_i \setminus U_{i-1}$ set $\deg u = i$. Let $\overline{U} = \mathcal{gr} U$ be the associated graded algebra

$$\overline{U} = \bigoplus_{i=0}^{\infty} U_i/U_{i-1}.$$ 

For $u \in U$ denote by $\overline{u}$ its image in $\overline{U}$ and for a subset $S \subseteq U$ set $\overline{S} = \{\overline{s} | s \in S\} \subseteq \overline{U}$. The algebra $\overline{U}$ is called a special PBW algebra if any element of $U$ can be written uniquely as a linear combination of ordered monomials in some fixed generators of $\overline{U}$ and if $\overline{U}$ is a polynomial algebra. Such algebras were introduced in [FO].

Let $\Lambda = \mathcal{k}[X_1,\ldots,X_n]$ be a polynomial algebra. A sequence $g_1,\ldots,g_t \in \Lambda$ is called regular (in $\Lambda$) if the class of $g_i$ in $\Lambda/(g_1,\ldots,g_{i-1})$ is non-invertible and is not a zero divisor for any $i = 1,\ldots,t$.

The next proposition contains some simple properties of regular sequences which will be used in the sequel.

**Proposition 2.4.** (1) A sequence of the form $X_1,\ldots,X_r,G_1,\ldots,G_t$, where $G_1,\ldots,G_t$ are homogeneous elements of $\Lambda$, is regular in $\Lambda$ if and only if the sequence $g_1,\ldots,g_t$ is regular in $\mathcal{k}[X_{r+1},\ldots,X_n]$, where $g_i(X_{r+1},\ldots,X_n) = G_i(0,\ldots,0,X_{r+1},\ldots,X_n)$. 


(2) A sequence \( g_1, g_1', g_2, \ldots, g_t, g_1, g_1' \notin k \), of homogeneous elements of \( \Lambda \) is regular if and only if both sequences \( g_1, g_2, \ldots, g_t \) and \( g_1', g_2, \ldots, g_t \) are regular.

The following analogue of Kostant theorem \([K]\) is valid for special PBW algebras.

**Theorem 2.5** \((\text{FO})\). Let \( U \) be a special PBW algebra and let \( g_1, \ldots, g_t \in U \) be mutually commuting elements such that \( \bar{g}_1, \ldots, \bar{g}_t \) is a regular sequence in \( \overline{U} \), and let \( \Gamma = \mathbb{k}[g_1, \ldots, g_t] \). Then \( U \) is a free left (right) \( \Gamma \)-module. Moreover, \( \Gamma \) is a direct summand of \( U \).

### 3. Freeness of \( Y_p(\mathfrak{gl}_2) \) over \( \Gamma \)

Let \( p \) be a positive integer. The level \( p \) Yangian \( Y_p(\mathfrak{gl}_2) \) for the Lie algebra \( \mathfrak{gl}_2 \) \([\text{C2}]\) can be defined as the algebra over \( k \) with generators \( t^{(1)}_{ij}, \ldots, t^{(p)}_{ij}, i, j = 1, 2 \), subject to the relations

\[
[T_{ij}(u), T_{kl}(v)] = \frac{1}{u-v}(T_{kj}(u)T_{il}(v) - T_{kj}(v)T_{il}(u)),
\]

where \( u, v \) are formal variables and

\[
T_{ij}(u) = \delta_{ij}u^p + \sum_{k=1}^{p} t^{(k)}_{ij} u^{p-k} \in Y_p(\mathfrak{gl}_2)[u].
\]

Explicitly, (3.1) reads

\[
[t^{(r)}_{ij}, t^{(s)}_{kl}] = \sum_{a=1}^{\min(r,s)} (t^{(a-1)}_{kj} t^{(r+s-a)}_{il} - t^{(r+s-a)}_{kj} t^{(a-1)}_{il}),
\]

where \( t^{(0)}_{ij} = \delta_{ij} \) and \( t^{(r)}_{ij} = 0 \) for \( r \geq p + 1 \). Note that the level 1 Yangian \( Y_1(\mathfrak{gl}_2) \) coincides with the universal enveloping algebra \( U(\mathfrak{gl}_2) \). Set

\[
deg t^{(k)}_{ij} = k \quad \text{for} \quad i, j = 1, 2 \quad \text{and} \quad k = 1, \ldots, p.
\]

This defines a natural filtration on the Yangian \( Y_p(\mathfrak{gl}_2) \). The corresponding graded algebra will be denoted by \( \mathfrak{Y}_p(\mathfrak{gl}_2) \). We have the following analogue of the Poincaré–Birkhoff–Witt theorem for the algebra \( Y_p(\mathfrak{gl}_2) \).

**Proposition 3.1** \((\text{C2}; \text{see also} \text{M2})\). Given an arbitrary linear ordering on the set of generators \( t^{(k)}_{ij} \), any element of the algebra \( Y_p(\mathfrak{gl}_2) \) is uniquely written as a linear combination of ordered monomials in these generators. Moreover, the algebra \( \mathfrak{Y}_p(\mathfrak{gl}_2) \) is a polynomial algebra in generators \( t^{(k)}_{ij} \).

Proposition 3.1 implies that \( Y_p(\mathfrak{gl}_2) \) is a special PBW algebra. Denote by \( D(u) \) the quantum determinant

\[
D(u) = T_{11}(u) T_{22}(u-1) - T_{12}(u) T_{21}(u-1)
= T_{11}(u-1) T_{22}(u) - T_{12}(u-1) T_{21}(u)
= T_{22}(u) T_{11}(u-1) - T_{12}(u) T_{21}(u-1)
= T_{22}(u-1) T_{11}(u) - T_{21}(u-1) T_{12}(u).
\]

Clearly, \( D(u) \) is a monic polynomial in \( u \) of degree \( 2p \),

\[
D(u) = u^{2p} + d_1 u^{2p-1} + \cdots + d_{2p}, \quad d_i \in Y_p(\mathfrak{gl}_2).
\]
It was shown in [C1, C2] (see also [M2] for a different proof) that the coefficients \( d_1, \ldots, d_{2p} \) are algebraically independent generators of the center of the algebra \( Y_p(g_l) \). Denote by \( \Gamma \) the subalgebra of \( Y_p(g_l) \) generated by the coefficients of \( D(u) \) and by the elements \( t_{22}^{(k)}, k = 1, \ldots, p \). This algebra is obviously commutative. We will show later (Corollary 5.3) that \( \Gamma \) is a Harish-Chandra subalgebra in \( Y_p(g_l) \).

**Lemma 3.2.** The sequence \( t_{22}^{(1)}, \ldots, t_{22}^{(p)}, \overline{a}_1, \ldots, \overline{a}_{2p} \) of the images of the generators of \( \Gamma \) is regular in \( Y_p(g_l) \).

**Proof.** Let us set

\[
t_i = \overline{\tau}_{11}^{(i)} + \overline{\tau}_{22}^{(i)}, \quad i = 1, \ldots, p \quad \text{and} \quad \Delta_{i,j} = \overline{\tau}_{11}^{(i)}\overline{\tau}_{22}^{(j)} - \overline{\tau}_{21}^{(i)}\overline{\tau}_{12}^{(j)}, \quad i, j = 1, \ldots, p.
\]

It follows from (3.3) that

\[
\overline{D}(u) = u^{2p} + \sum_{i=1}^{2p} \overline{d}_i u^{2p-i},
\]

with

\[
\overline{d}_i = t_i + \sum_{j=1}^{i-1} \Delta_{j,i-j} \quad \text{for} \quad i = 1, \ldots, p \quad \text{and} \quad \overline{d}_i = \sum_{j=-i}^{p} \Delta_{j,i-j} \quad \text{for} \quad i = p + 1, \ldots, 2p.
\]

Hence we need to show that the sequence

\[
\overline{\tau}_{22}^{(1)}, \ldots, \overline{\tau}_{22}^{(p)}, t_1, t_2 + \Delta_{11}, \ldots, t_p + \sum_{i=1}^{p-1} \Delta_{i,p-1}, \sum_{i=1}^{p} \Delta_{i,p+1}, \ldots, \Delta_{pp}
\]

is regular. We will denote by \( \nabla_i \) the result of the substitution \( \overline{\tau}_{22}^{(1)} = \cdots = \overline{\tau}_{22}^{(p)} = 0 \) in \( \overline{d}_i, i = 1, \ldots, 2p \). By Proposition 2.11, we only need to show the regularity of the sequence

\[
\nabla_1, \ldots, \nabla_{2p}.
\]

Consider the automorphism \( \phi \) of \( Y_p(g_l)/I \) given by

\[
\overline{\tau}_{11}^{(i)} \mapsto \nabla_i, \quad \overline{\tau}_{21}^{(i)} \mapsto \overline{\tau}_{21}^{(i)}, \quad \overline{\tau}_{12}^{(i)} \mapsto \overline{\tau}_{12}^{(i)} \quad \text{for} \quad i = 1, \ldots, p,
\]

where \( I \) is the ideal generated by \( \overline{\tau}_{22}^{(1)}, \ldots, \overline{\tau}_{22}^{(p)} \). Note that the elements \( \nabla_i \) with \( i = p + 1, \ldots, 2p \) are stable under \( \phi \). Since the regularity of a sequence is preserved by automorphisms, it is sufficient to demonstrate the regularity of the sequence

\[
\overline{\tau}_{11}^{(1)}, \ldots, \overline{\tau}_{11}^{(p)}, \overline{\nabla}_{p+1}, \ldots, \overline{\nabla}_{2p}.
\]

Since the elements \( \nabla_i \) do not depend on the \( \overline{\tau}_{11}^{(k)} \), Proposition 2.11 implies that this is equivalent to the regularity of the sequence \( \overline{\nabla}_{p+1}, \ldots, \overline{\nabla}_{2p} \). For each pair of indices \( k, l \in \{1, \ldots, p\} \) and any index \( 1 \leq a \leq \max\{k, l\} \), consider the sequence of
a elements which occupy the rows of the table \( s(k, l, a) \) below
\[
\begin{pmatrix}
\mathcal{Z}^{(k)}_{21} \mathcal{Z}^{(l)}_{12} \\
\mathcal{Z}^{(k-1)}_{21} \mathcal{Z}^{(l)}_{12} + \mathcal{Z}^{(k)}_{21} \mathcal{Z}^{(l-1)}_{12} \\
\mathcal{Z}^{(k-2)}_{21} \mathcal{Z}^{(l)}_{12} + \mathcal{Z}^{(k-1)}_{21} \mathcal{Z}^{(l-1)}_{12} + \mathcal{Z}^{(k)}_{21} \mathcal{Z}^{(l-2)}_{12} \\
\vdots \\
\mathcal{Z}^{(k-a+1)}_{21} \mathcal{Z}^{(l)}_{12} + \mathcal{Z}^{(k-a+2)}_{21} \mathcal{Z}^{(l+1)}_{12} + \cdots + \mathcal{Z}^{(k)}_{21} \mathcal{Z}^{(l-a+1)}_{12}
\end{pmatrix}.
\]

Note that when \( k = l = a = p \) the rows of the table are exactly the elements \( \nabla_i \), \( i = p + 1, \ldots, 2p \). We will show by induction on \( a \) that the sequence of rows of \( s(k, l, a) \) is regular. Note that \( s(k, l, 1) \) consists of the single element \( t_{21}^2 t_{12}^1 \) and is obviously regular. Now let \( a > 1 \). Consider the following two tables which we denote by \( s'(k, l, a) \) and \( s''(k, l, a) \), respectively.
\[
\begin{pmatrix}
\mathcal{Z}^{(k)}_{21} \\
\mathcal{Z}^{(k-1)}_{21} \mathcal{Z}^{(l)}_{12} + \mathcal{Z}^{(k)}_{21} \mathcal{Z}^{(l-1)}_{12} \\
\vdots \\
\mathcal{Z}^{(k-a+1)}_{21} \mathcal{Z}^{(l)}_{12} + \mathcal{Z}^{(k-a+2)}_{21} \mathcal{Z}^{(l+1)}_{12} + \cdots + \mathcal{Z}^{(k)}_{21} \mathcal{Z}^{(l-a+1)}_{12}
\end{pmatrix},
\begin{pmatrix}
\mathcal{Z}^{(l)}_{12} \\
\mathcal{Z}^{(k-1)}_{21} \mathcal{Z}^{(l)}_{12} + \mathcal{Z}^{(k)}_{21} \mathcal{Z}^{(l-1)}_{12} \\
\vdots \\
\mathcal{Z}^{(k-a+1)}_{21} \mathcal{Z}^{(l)}_{12} + \mathcal{Z}^{(k-a+2)}_{21} \mathcal{Z}^{(l+1)}_{12} + \cdots + \mathcal{Z}^{(k)}_{21} \mathcal{Z}^{(l-a+1)}_{12}
\end{pmatrix}.
\]

Due to Proposition \( 2.4(2) \), it is sufficient to verify the regularity of both \( s'(k, l, a) \) and \( s''(k, l, a) \). Using again Proposition \( 2.4(1) \), substitute \( \mathcal{Z}^{(k)}_{21} = 0 \) in \( s'(k, l, a) \) and \( \mathcal{Z}^{(l)}_{12} = 0 \) in \( s''(k, l, a) \). It is easy to see that after this substitution we obtain the tables \( s(k - 1, l, a - 1) \) and \( s(k, l - 1, a - 1) \), respectively. By the induction hypothesis, both of them are regular and so is \( s(k, l, a) \). In particular, the sequence \( s(p, p, p) \) is regular which completes the proof. \( \square \)

Using the regularity of the sequence \( \mathcal{Z}^{(1)}_{22}, \ldots, \mathcal{Z}^{(p)}_{22}, d_1, \ldots, d_{2p} \) we immediately obtain the following.

**Corollary 3.3.** The generators \( t_{22}^{(1)}, \ldots, t_{22}^{(p)}, d_1, \ldots, d_{2p} \) of \( \Gamma \) are algebraically independent.

Combining Lemma \( 3.2 \) with Theorem \( 2.5 \) we obtain our first main result.

**Theorem 3.4.**
1. \( Y_p(\mathfrak{gl}_2) \) is free as a left (right) module over \( \Gamma \). Moreover, \( \Gamma \) is a direct summand of \( Y_p(\mathfrak{gl}_2) \).
2. Any \( \mathfrak{m} \in \text{Spec} \Gamma \) extends to an irreducible \( Y_p(\mathfrak{gl}_2) \)-module.

For a subset \( P \subseteq Y_p(\mathfrak{gl}_2) \) denote by \( \mathbb{D}(P) \) the set of all \( x \in Y_p(\mathfrak{gl}_2) \) such that there exists \( z \in \Gamma \), \( z \neq 0 \) for which \( zx \in P \).

**Corollary 3.5.** Let \( P \subseteq Y_p(\mathfrak{gl}_2) \) be a finitely generated left \( \Gamma \)-module, then \( \mathbb{D}(P) \) is a finitely generated left \( \Gamma \)-module.

**Proof.** Since \( \Gamma \) is a domain, then \( \mathbb{D}(P) \) is a \( \Gamma \)-submodule in \( Y_p(\mathfrak{gl}_2) \). Using the fact that \( Y_p(\mathfrak{gl}_2) \) is a free left \( \Gamma \)-module we conclude that \( Y_p(\mathfrak{gl}_2) \cong F_p \otimes F \) where \( F_p \) and \( F \) are free left \( \Gamma \)-modules, \( F_p \) has a finite rank and \( P \subseteq F_p \). Then \( \mathbb{D}(P) \subseteq F_p \) and hence it is finitely generated as a submodule of a finitely generated module over a noetherian ring. \( \square \)
4. Harish-Chandra modules for \(\mathfrak{gl}_2\) Yangians

In this section we introduce universal Harish-Chandra modules \(M(\ell)\). We also describe their structure in an explicit form in the case of generic parameters \(\ell\).

Let \(L\) be a polynomial algebra in the variables \(b_1, \ldots, b_p, g_1, \ldots, g_{2p}\). Define a \(k\)-homomorphism \(i : \Gamma \rightarrow L\) by

\[
\iota(x^{(k)}) = \sigma_{k,p}(b_1, \ldots, b_p), \quad \iota(d_i) = \sigma_{i,2p}(g_1, \ldots, g_{2p}),
\]

where \(\sigma_{i,j}\) is the \(i\)-th elementary symmetric polynomial in \(j\) variables. Due to Corollary 3.3, \(i\) is injective. We will identify the elements of \(\Gamma\) with their images in \(L\) and treat them as polynomials in the variables \(b_1, \ldots, b_p, g_1, \ldots, g_{2p}\) invariant under the action of the group \(S_p \times S_{2p}\). Set \(L = \text{Spec}m L\). We will identify \(L\) with \(k^{[p]}\). If

\[
\beta = (\beta_1, \ldots, \beta_p), \quad \gamma = (\gamma_1, \ldots, \gamma_{2p}) \quad \text{and} \quad \ell = (\beta_1, \ldots, \beta_p, \gamma_1, \ldots, \gamma_{2p}),
\]

then we shall write \(\ell = (\beta, \gamma)\). The monomorphism \(i\) induces the epimorphism

\[
i^* : L \rightarrow \text{Spec}m \Gamma.
\]

If \(\ell \in L\) and \(m = i^*(\ell)\), then \(D(\ell)\) will denote the equivalence class of \(m\) in \(\Delta(Y_p(\mathfrak{gl}_2), \Gamma)\); see Section 2.4.

Let \(P_0 \subseteq L\), \(P_0 \cong \mathbb{Z}^p\), be the lattice generated by the elements \(\delta_i \in k^3\) for \(i = 1, \ldots, p\), where \(\delta_i\) denotes the 3p-tuple with 1 on the \(i\)-th position and zeros elsewhere. Then \(P_0\) acts on \(L\) by shifts \(\delta_i(\ell) := \ell + \delta_i\). Furthermore, the group \(S_p \times S_{2p}\) acts on \(L\) by permutations. Thus the semidirect product \(\mathcal{W}\) of the groups \(S_p \times S_{2p}\) and \(P_0\) acts on \(L\) and \(\mathcal{W}\). Denote by \(S\) a multiplicative set in \(L\) generated by the elements \(b_i - b_j - m\) for all \(i \neq j\) and all \(m \in \mathbb{Z}\) and by \(L\) the localization of \(L\) by \(S\). Note that \(S\) is invariant under the action of \(\mathcal{W}\) and hence \(\mathcal{W}\) acts on \(L\) as well.

For arbitrary 3p-tuple \(\ell = (\beta, \gamma) \in L\) set

\[
\beta(u) = (u + \beta_1) \cdots (u + \beta_p), \quad \gamma(u) = (u + \gamma_1) \cdots (u + \gamma_{2p}).
\]

Let \(I_\ell\) be the left ideal of \(Y_p(\mathfrak{gl}_2)\) generated by the coefficients of the polynomials \(T_{22}(u) - \beta(u)\) and \(D(u) - \gamma(u)\). Define the corresponding quotient module over \(Y_p(\mathfrak{gl}_2)\) by

\[
M(\ell) = Y_p(\mathfrak{gl}_2)/I_\ell.
\]

We shall call it the universal module (corresponding to \(\ell\)). It follows from Theorem 3.3 that \(I_\ell\) is a proper ideal of \(Y_p(\mathfrak{gl}_2)\) and so \(M(\ell)\) is a nontrivial module. It is clear that if \(V\) is an arbitrary Harish-Chandra \(Y_p(\mathfrak{gl}_2)\)-module generated by a nonzero \(\eta \in V\) such that \(D(u)\eta = \gamma(u)\eta\) and \(T_{22}(u)\eta = \beta(u)\eta\), then \(V\) is a homomorphic image of \(M(\ell)\).

Set \(P_1 = \text{Spec}m L \subseteq L\), i.e. \(P_1\) consists of generic 3p-tuples \(\ell = (\beta, \gamma)\) such that

\[
\beta_i - \beta_j \notin \mathbb{Z} \quad \text{for all} \quad i \neq j.
\]

If \(\ell \in P_1\), then the modules from the category \(\mathcal{H}(Y_p(\mathfrak{gl}_2), \Gamma, D(\ell))\) are called generic Harish-Chandra modules.
4.1. Weight modules. For $\ell = (\beta, \gamma) \in \mathcal{L}$ the category $\mathbb{H} W(Y_p(\mathfrak{gl}_2), \Gamma, D(\ell))$ consists of finitely generated weight modules $V$ with central character $\gamma$ and with $\text{Supp} V \subset D(\ell)$. We shall denote this category by $R_{\ell}$ for brevity. If $\ell \in \mathcal{P}_1$, then the modules from $R_{\ell}$ will be called generic weight modules.

A $Y_p(\mathfrak{gl}_2)$-module $V$ is an object of $R_{\ell}$ if $V$ is a direct sum of its weight subspaces:

$$V = \bigoplus_{\ell \in \mathcal{L}} V_{\ell}, \quad V_{\ell} = \{ \eta \in V \mid T_{22}(u)\eta = \beta(u)\eta, \quad D(u)\eta = \gamma(u)\eta \}.$$

If $V \in R_{\ell}$, then we shall simply write $V_{\beta}$ instead of $V_{\ell}$ and identify $\text{Supp} V$ with the set of all $\beta$ such that the subspace $V_{\beta}$ is nonzero. The next lemma describes the action of the Yangian generators on the weight subspaces; cf. (2.1).

**Lemma 4.1.** Let $V$ be a generic weight $Y_p(\mathfrak{gl}_2)$-module and let $\beta = (\beta_1, \ldots, \beta_p) \in \text{Supp} V$. Then

$$T_{21}(u)\text{Supp } V_{\beta} \subseteq \sum_{i=1}^{p} V_{\beta + \delta_i}, \quad \text{and} \quad T_{12}(u)\text{Supp } V_{\beta} \subseteq \sum_{i=1}^{p} V_{\beta - \delta_i},$$

where $\beta \pm \delta_i = (\beta_1, \ldots, \beta_i \pm 1, \ldots, \beta_p)$.

**Proof.** First we show that $T_{21}(\beta_i)\text{Supp } V_{\beta} \subseteq \text{Supp } V_{\beta + \delta_i}$ for all $i = 1, \ldots, p$. Since

$$T_{22}(u - 1)T_{21}(u) = T_{21}(u - 1)T_{22}(u)$$

we have

$$T_{22}(\beta_i - 1)T_{21}(\beta_i)\eta = T_{21}(\beta_i - 1)T_{22}(\beta_i)\eta = 0$$

for all $\eta \in V_{\beta_i}$. Also,

$$T_{22}(\beta_j)T_{21}(\beta_i)\eta = (\beta_i - \beta_j)^{-1}(T_{21}(\beta_i)T_{22}(\beta_j) - T_{21}(\beta_j)T_{22}(\beta_i))\eta$$

$$+ T_{21}(\beta_i)T_{22}(\beta_j)\eta = 0$$

since $T_{22}(\beta_k)\eta = 0$ for all $k = 1, \ldots, p$. Using the fact that $\beta_i - \beta_j \notin \mathbb{Z}$ we conclude that $T_{21}(\beta_i)\text{Supp } V_{\beta} \subseteq \text{Supp } V_{\beta + \delta_i}$ for all $i = 1, \ldots, p$. Since $T_{21}(u)$ is a polynomial of degree $p - 1$ in $u$ and $\beta_i \neq \beta_j$ if $i \neq j$, we thus get the first containment of (4.5). The second is verified in the same way with the use of the identity $T_{22}(u)T_{12}(u - 1) = T_{12}(u)T_{22}(u - 1)$.

**Corollary 4.2.** If $V$ is indecomposable generic weight module over $Y_p(\mathfrak{gl}_2)$ and $\beta \in \text{Supp } V$, then $\text{Supp } V \subseteq \beta + \mathbb{Z}^p$.

**Lemma 4.3.** If $V$ is a generic weight $Y_p(\mathfrak{gl}_2)$-module with the central character $\gamma$, then for any $\beta = (\beta_1, \ldots, \beta_p) \in \text{Supp} V$ and any $\eta \in V_{\beta}$, we have

$$T_{12}(\beta_s)T_{21}(\beta_s)\eta = T_{21}(\beta_s)T_{12}(\beta_s)\eta,$$

if $s \neq r$, and

$$T_{12}(\beta_s - 1)T_{21}(\beta_s)\eta = -\gamma(\beta_s)\eta,$$

$$T_{21}(\beta_s + 1)T_{12}(\beta_s)\eta = -\gamma(\beta_s + 1)\eta.$$

**Proof.** The first equality follows from the defining relations (1.1). The two remaining follow from (3.2).

The following corollary is immediate from Lemma 4.3.

**Corollary 4.4.** Let $V$ be a generic weight $Y_p(\mathfrak{gl}_2)$-module with the central character $\gamma$ and let $\beta = (\beta_1, \ldots, \beta_p) \in \text{Supp } V$. 
(1) If $\gamma(-\beta_i) \neq 0$ then $\ker T_{21}(-\beta_i) \cap V_\beta = 0$.

(2) If $\gamma(-\beta_i + 1) \neq 0$ then $\ker T_{12}(-\beta_i) \cap V_\beta = 0$.

(3) If $V$ is indecomposable and $\gamma(-\beta_i + k) \neq 0$ for all $k \in \mathbb{Z}$ then

$$\ker T_{21}(-\psi_i) \cap V_\psi = \ker T_{12}(-\psi_i) \cap V_\psi = 0$$

for all $\psi = (\psi_1, \ldots, \psi_p) \in \text{Supp } V$.

Since the universal module $M(\ell)$ is nontrivial, the image of 1 in $M(\ell)$ is nonzero. We shall denote this image by $\xi$. Assume that $\beta$ satisfies the genericity condition (4.4). For any $(k) = (k_1, \ldots, k_p) \in \mathbb{Z}^p$ define the corresponding vector of the module $M(\ell)$ by

$$\xi^{(k)} = \prod_{i, k_i > 0} T_{21}(-\beta_i - k_i + 1) \cdots T_{21}(-\beta_i - 1) T_{21}(-\beta_i)$$

$$\times \prod_{i, k_i < 0} T_{12}(-\beta_i - k_i - 1) \cdots T_{12}(-\beta_i + 1) T_{12}(-\beta_i) \xi.$$  

(4.6)

**Theorem 4.5.** The vectors $\xi^{(k)}, (k) \in \mathbb{Z}^p$ form a basis of $M(\ell)$. Moreover, we have the formulas

$$T_{22}(u) \xi^{(k)} = \prod_{i=1}^p (u + \beta_i + k_i) \xi^{(k)}.$$  

(4.7)

$$T_{21}(u) \xi^{(k)} = \sum_{i=1}^p A_i(k) \frac{(u + \beta_1 + k_1) \cdots \wedge_i \cdots (u + \beta_p + k_p)}{(-\beta_i + 1 - k_i) \cdots (-\beta_i + 1 - k_i)} \xi^{(k+\delta_i)}.$$  

(4.8)

$$T_{12}(u) \xi^{(k)} = \sum_{i=1}^p B_i(k) \frac{(u + \beta_1 + k_1) \cdots \wedge_i \cdots (u + \beta_p + k_p)}{(-\beta_i + 1 - k_i) \cdots (-\beta_i + 1 - k_i)} \xi^{(k-\delta_i)},$$

where the symbol $\wedge_i$ indicates that the $i$-th factor in the product is skipped,

$$A_i(k) = \begin{cases} 1 & \text{if } k_i \geq 0, \\ -\gamma(-\beta_i - k_i) & \text{if } k_i < 0 \end{cases}$$

and

$$B_i(k) = \begin{cases} -\gamma(-\beta_i - k_i + 1) & \text{if } k_i > 0, \\ 1 & \text{if } k_i \leq 0. \end{cases}$$

The action of $T_{11}(u)$ is found from the relation

$$T_{11}(u) T_{22}(u - 1) - T_{21}(u) T_{12}(u - 1) \xi^{(k)} = \gamma(u) \xi^{(k)}.$$  

(4.9)

**Proof.** We start by proving the formulas for the action of the generators of $Y_\mu(\mathfrak{gl}_2)$. Formula (4.7) follows by induction with the use of the relations

$$T_{22}(u) T_{21}(v) = \frac{u - v + 1}{u - v} T_{21}(v) T_{22}(u) - \frac{1}{u - v} T_{21}(u) T_{22}(v)$$  

(4.10)

and

$$T_{22}(u) T_{12}(v) = \frac{u - v - 1}{u - v} T_{12}(v) T_{22}(u) + \frac{1}{u - v} T_{12}(u) T_{22}(v)$$  

(4.11)
implied by (5.13). By Lemma 4.3 we have: if \( k_i > 0 \), then
\[
\begin{align*}
T_{21}(-\beta_i - k_i) \xi^{(k)} &= \xi^{(k+\delta_i)}, \\
T_{12}(-\beta_i - k_i) \xi^{(k)} &= -\gamma(-\beta_i - k_i + 1) \xi^{(k-\delta_i)};
\end{align*}
\]
if \( k_i < 0 \), then
\[
\begin{align*}
T_{12}(-\beta_i - k_i) \xi^{(k)} &= \xi^{(k-\delta_i)}, \\
T_{21}(-\beta_i - k_i) \xi^{(k)} &= -\gamma(-\beta_i - k_i) \xi^{(k+\delta_i)};
\end{align*}
\]
and if \( k_i = 0 \), then
\[
\begin{align*}
T_{12}(-\beta_i) \xi^{(k)} &= \xi^{(k-\delta_i)}, \\
T_{21}(-\beta_i) \xi^{(k)} &= \xi^{(k+\delta_i)}.
\end{align*}
\]
Applying the Lagrange interpolation formula we obtain the remaining formulas.

It is implied by the formulas, that the module \( M(\ell) \) is spanned by the vectors \( \xi^{(k)} \). By (4.7) and the genericity assumption, the \( \xi^{(k)} \) are eigenvectors for \( T_{22}(u) \) with distinct eigenvalues. In order to verify their linear independence, suppose first that \( \gamma(u) \) satisfies the condition
\[
(4.15) \quad \gamma(-\beta_i - k) \neq 0 \quad \text{for all } k \in \mathbb{Z} \text{ and all } i.
\]
In this case the linear independence of the \( \xi^{(k)} \) follows from the fact that each of them is nonzero. This is implied by (4.12)–(4.13) since \( \xi \neq 0 \) in \( M(\ell) \).

In the case of general \( \gamma(u) \) let us define a \( Y_p(\mathfrak{gl}_2) \)-module \( \tilde{M}(\ell) \) as follows. As a vector space, \( \tilde{M}(\ell) \) is the \( k \)-linear span of the basis vectors \( \tilde{\xi}^{(k)} \) with \( (k) \) running over \( \mathbb{Z}^p \) and the action of \( Y_p(\mathfrak{gl}_2) \) is given by the formulas (4.7)–(4.9), where the \( \xi^{(k)} \) should be replaced with \( \tilde{\xi}^{(k)} \). We have to verify that the operators \( T_{ij}(u) \) do satisfy the Yangian defining relations (3.1). However, the application of both sides of (3.1) to a basis vector \( \tilde{\xi}^{(k)} \) amounts to polynomial relations on the coefficients of \( \gamma(u) \). By the previous argument, if \( \gamma(u) \) satisfies (4.15), then these relations are identities. Therefore, these identities hold for an arbitrary \( \gamma(u) \) and thus \( \tilde{M}(\ell) \) is well defined.

Finally, consider the \( Y_p(\mathfrak{gl}_2) \)-module homomorphism
\[
\varphi : Y_p(\mathfrak{gl}_2) \to \tilde{M}(\ell), \quad 1 \mapsto \tilde{\xi}^{(0)}.
\]
Obviously, the ideal \( I_p \) is contained in the kernel \( \text{Ker} \varphi \) and so, this defines a homomorphism \( M(\ell) \to \tilde{M}(\ell) \) which takes \( \xi^{(k)} \) to the corresponding vector \( \tilde{\xi}^{(k)} \). Since the vectors \( \tilde{\xi}^{(k)} \) form a basis of \( \tilde{M}(\ell) \), this proves that the vectors \( \xi^{(k)} \) are linearly independent. \( \square \)

Let us fix a \( p \)-tuple \( \beta \) satisfying the genericity condition (4.4) and introduce the elements of \( Y_p(\mathfrak{gl}_2) \) by
\[
\tau^{(k)} = \prod_{i, k_i > 0} T_{21}(-\beta_i - k_i + 1) \cdots T_{21}(-\beta_i - 1) T_{21}(-\beta_i) \\
\times \prod_{i, k_i < 0} T_{12}(-\beta_i - k_i - 1) \cdots T_{12}(-\beta_i) T_{12}(-\beta_i),
\]
where \( (k) \) runs over \( \mathbb{Z}^p \).
Corollary 4.6. The elements $\tau^{(k)}$ are linearly independent over $\Gamma$ in the right $\Gamma$-module $Y_p(\mathfrak{gl}_2)$.

Proof. Suppose that a linear combination of the elements $\tau^{(k)}$ with coefficients in $\Gamma$ is zero:

$$\sum_{(k)} \tau^{(k)} c_{(k)} = 0, \quad c_{(k)} \in \Gamma.$$  \hspace{1cm} (4.16)

Apply the left-hand side to the vector $\xi$ in a module $M(\ell)$ with $\ell = (\beta, \gamma)$ satisfying the assumptions of Theorem 4.5. We get the relation

$$\sum_{(k)} c_{(k)}(\ell) \xi^{(k)} = 0,$$

where $c_{(k)}(\ell)$ is the evaluation of the polynomial $c_{(k)}$ at $T_{22}(u) = \beta(u)$ and $D(u) = \gamma(u)$. Since the vectors $\xi^{(k)}$ form a basis of $M(\ell)$ this implies that $c_{(k)}(\ell) = 0$ for any choice of the parameters $\gamma$. Therefore, each $c_{(k)}$ does not depend on the generators $d_i$ and so it is a polynomial in the $\tau^{(i)}$. However, due to the Poincaré–Birkhoff–Witt theorem for the algebra $Y_p(\mathfrak{gl}_2)$ (Proposition 3.1), a nontrivial relation $\tau^{(i)}$ can only hold if the elements $\tau^{(k)}$ are linearly dependent over $k$. But this is not the case because the vectors $\xi^{(k)} = \tau^{(k)} \xi$ are linearly independent in $M(\ell)$ by Theorem 4.5.

Remark 4.7. One can also produce a family of $\Gamma$-linearly independent elements for the left $\Gamma$-module $Y_p(\mathfrak{gl}_2)$. They can be obtained as images of the $\tau^{(k)}$ under the anti-automorphism of the algebra $Y_p(\mathfrak{gl}_2)$ given by

$$t_{ij}^{(r)} \mapsto t_{ji}^{(r)}.$$  \hspace{1cm} (4.17)

For the proof we observe that every generator of $\Gamma$ is stable under this anti-automorphism. With the exception of the case $p = 1$, the elements $\tau^{(k)}$ do not apparently constitute a basis of $Y_p(\mathfrak{gl}_2)$ as a right $\Gamma$-module.

Remark 4.8. Given two monic polynomials $\alpha(u)$ and $\beta(u)$ of degree $p$ define the corresponding Verma module $V(\alpha(u), \beta(u))$ as the quotient of $Y_p(\mathfrak{gl}_2)$ by the left ideal generated by the coefficients of the polynomials $T_{12}(u) - \alpha(u), \ T_{22}(u) - \beta(u)$ and $T_{12}(u)$; cf. [11, 12]. Then the same argument as above shows that $V(\alpha(u), \beta(u))$ has a basis $\{\xi^{(k)}\}$ parametrized by $p$-tuples of nonnegative integers $(k)$. The formulas of Theorem 4.5 hold for the basis vectors $\xi^{(k)}$, where $\gamma(u)$ should be taken to be $\alpha(u) \beta(u - 1)$ which defines the central character $\gamma$ of $V(\alpha(u), \beta(u))$. In fact, $V(\alpha(u), \beta(u))$ is isomorphic to the quotient of the corresponding universal module $M(\ell)$, $\ell = (\beta, \gamma)$ by the submodule spanned by the vectors $\{\xi^{(k)}\}$ such that $(k)$ contains at least one negative component $k_i$.

Corollary 4.9. Let $\ell = (\beta, \gamma) \in \mathcal{P}_1$.

1. The module $M(\ell)$ is a generic weight $Y_p(\mathfrak{gl}_2)$-module with central character $\gamma$, $\text{Supp} M(\ell) = \mathbb{Z}^p$ and all weight spaces are 1-dimensional.

2. The module $M(\ell)$ has a unique maximal submodule and hence a unique irreducible quotient.

3. The equivalence class $D(\ell)$ coincides with the set $\ell + \mathcal{P}_0$. 

...
Proof. Statement (1) follows immediately from Theorem 4.5. The sum of all proper submodules of \(M(\ell)\) is again a proper submodule implying (2). Statement (3) follows immediately from (1).

We will denote the unique irreducible quotient of \(M(\ell)\) by \(L(\ell)\). It follows from Corollary 4.9 that all weight spaces of \(L(\ell)\) are 1-dimensional. We can now describe all irreducible generic weight \(Y_p(\mathfrak{gl}_2)\)-modules.

**Corollary 4.10.** Let \(\ell = (\beta, \gamma) \in \mathcal{P}_1\).

1. There exists an irreducible generic weight \(Y_p(\mathfrak{gl}_2)\)-module \(L(\ell)\) with \(L(\ell)_\beta \neq 0\) and with central character \(\gamma\). Moreover, \(\dim L(\ell)_\psi = 1\) for all \(\psi \in \text{Supp } L(\ell)\).

2. Any irreducible weight module over \(Y_p(\mathfrak{gl}_2)\) with central character \(\gamma\) generated by a nonzero vector of weight \(\beta\) is isomorphic to \(L(\ell)\).

5. \(\Gamma\) is a Harish-Chandra subalgebra

In this section we adapt the results from [DFO2] and [OV] for the Yangians. In particular, we show that \(\Gamma\) is a Harish-Chandra subalgebra.

We have the following analogue of the Harish-Chandra theorem for Lie algebras [D1].

**Proposition 5.1.** Let \(x \in Y_p(\mathfrak{gl}_2)\) be such that \(xM(\ell) = 0\) for any \(\ell \in \mathcal{P}_1\). Then \(x = 0\).

Proof. Since \(M(\ell) = Y_p(\mathfrak{gl}_2)/I_\ell\), it will be sufficient to show that the intersection \(\bigcap_\ell I_\ell\) over all \(\ell \in \mathcal{P}_1\) is zero. By Theorem 3.3, the Yangian \(Y_p(\mathfrak{gl}_2)\) is free as a right module over \(\Gamma\). Let \(x_i, i \in \mathcal{I}\) be a basis of \(Y_p(\mathfrak{gl}_2)\) over \(\Gamma\). If \(x = \sum_{i \in \mathcal{I}} x_i z_i\) for some \(z_i \in \Gamma\), then \(x \in I_\ell\) if and only if \(z_i(\ell) = 0\) for all \(i \in \mathcal{I}\). Since \(\mathcal{P}_1\) is dense in \(\mathcal{L}\) in Zariski topology it follows immediately that if \(x \in \bigcap_\ell I_\ell\) with \(\ell\) running over \(\mathcal{P}_1\), then \(z_i = 0\) for all \(i \in \mathcal{I}\) and thus \(x = 0\). This completes the proof.

For any \(\ell_0 \in \mathcal{P}_1\) the module \(M(\ell_0)\) has a basis \(\xi^{(k)}, (k) \in \mathbb{Z}_p\) with the action of generators of \(Y(\mathfrak{gl}_2)\) defined by formulas (1.7)—(1.9). We will relabel the basis elements of \(M(\ell_0)\) as \(\xi_\ell, \ell \in \ell_0 + \mathcal{P}_0\). It follows from Theorem 1.10 that for every \(x \in Y_p(\mathfrak{gl}_2)\) there exists a finite subset \(\mathcal{L}_x \subseteq \mathcal{P}_0\) consisting of elements \(\delta\) such that

\[
x \xi_\ell = \sum_{\delta \in \mathcal{L}_x} \theta(x, \ell, \delta) \xi_{\ell + \delta}
\]

for some nonzero coefficients \(\theta(x, \ell, \delta) \in \mathbb{K}\). We can also regard these coefficients as the values of the rational functions \(\theta(x, \ell, \delta) \in \mathbb{L}\) at \(b = \ell\), where \(b = (b_1, \ldots, b_p, g_1, \ldots, g_{2p})\). Clearly, the set \(\mathcal{L}_x\) is \(S_p \times S_{2p}\)-invariant. Note that for a given \(x\) this formula does not depend on \(\ell_0\).

We identify the \((\Gamma-\Gamma)\)-bimodule structure on \(Y_p(\mathfrak{gl}_2)\) with the corresponding \(\Gamma \otimes \Gamma\)-module structure. For any \(z \in \Gamma\) and any finite \(S \subseteq \mathcal{L}\) introduce the following polynomial

\[
F_{S,z} = F_{S,z}(z, b) = \prod_{\delta \in S} (z \otimes 1 - 1 \otimes z(b + \delta)) = \sum_{i=0}^{|S|} z^i \otimes a_i, \quad a_i \in \mathbb{L}
\]
Proposition 5.2 (cf. [DFO2, Lemma 25]). Let $S$ be a finite $S_p \times S_{2p}$-invariant subset in $L$, $q = |S|$, $z \in \Gamma$ and $F_{S,z} = \sum_{i=0}^{q} z^i \otimes a_i$, $a_i \in L$. Then:

1. $a_i \in \Gamma$, $i = 0, \ldots, q$.

2. For any $x \in Y_p(\mathfrak{gl}_2)$ such that $L_x \subseteq S$ we have $\sum_{i=0}^{q} z^i x a_i = 0$.

Proof. Since $S$ is $S_p \times S_{2p}$-invariant, the coefficients of the polynomial $F_{S,z}$ are $S_p \times S_{2p}$-invariant and hence belong to $\Gamma$ which proves (1). It is sufficient to check the statement (2) for $S = L_x$ since $F_{S,z} = F_{S \setminus L_x,z} F_{L_x,z}$. Let $\ell \in P_1$ and let $\xi_\ell$ be a basis element of $M(\ell)$. Then

$$\sum_{i=0}^{q} z^i x a_i(\xi_\ell) = \sum_{i=0}^{q} z^i x a_i(\ell)(\xi_\ell)$$

$$= \sum_{i=0}^{q} z^i a_i(\ell) \sum_{\delta \in L_x} \theta(x, \ell, \delta) \xi_{\ell+\delta}$$

$$= \sum_{\delta \in L_x} \theta(x, \ell, \delta) \sum_{i=0}^{q} a_i(\ell)(z^i \xi_{\ell+\delta})$$

$$= \sum_{\delta \in L_x} \theta(x, \ell, \delta) \sum_{i=0}^{q} a_i(\ell)(z \delta)^i \xi_{\ell+\delta}$$

$$= \sum_{\delta \in L_x} \theta(x, \ell, \delta) F_{L_x,z}(z(\ell + \delta), \ell) \xi_{\ell+\delta} = 0$$

since $F_{L_x,z}(z(\ell + \delta), \ell) = 0$ for every $\delta \in L_x$. Applying Proposition 5.1 we obtain the statement of the proposition.

The main result of this section is the following theorem.

Theorem 5.3. $\Gamma$ is a Harish-Chandra subalgebra of $Y_p(\mathfrak{gl}_2)$.

Proof. Following [DFO2, Proposition 8], it is sufficient to show that a $\Gamma$-bimodule $\Gamma_{t_{ij}^{(k)}}$ is finitely generated both as left and as right module for every possible choice of indices $i, j, k$. This is obvious for $i = j = k$ since $t_{ij}^{(k)} \in \Gamma$. Suppose now that $i = 2, j = 1$. We have $d_i t_{21}^{(k)} = t_{21}^{(k)} d_i$ by the centrality of $d_i$. It follows from (1.3) that $L_{t_{21}^{(k)}} = \{ \delta_i \mid i = 1, \ldots, p \}$. Then for $x = t_{21}^{(k)}$ we have

$$F_{L_x,t_{21}^{(k)}} = z^p \otimes 1 + \sum_{l=0}^{p-1} z^l \otimes a_l, \quad a_l \in \Gamma$$

and

$$(t_{22}^{(i)})^p t_{21}^{(k)} + \sum_{l=0}^{p-1} (t_{22}^{(i)})^l t_{21}^{(k)} a_l = 0$$

by Proposition 5.2(2). Hence the elements

$$\prod_{i=1}^{p} (t_{22}^{(i)})^{k_i} t_{21}^{(k)} , \quad 0 \leq k_i < p$$
are generators of $\Gamma_{21}^{(k)}$ as a right $\Gamma$-module. The cases $i = 1, j = 2$ and $i = j = 1$ are treated similarly since

$$L_{ij}^{(k)} = \{ -\delta_i \mid i = 1, \ldots, p \} \text{ and } L_{ji}^{(k)} = \{ \delta_i - \delta_j \mid i, j = 1, \ldots, p \}.$$ 

Thus, $\Gamma_{ij}^{(k)} \Gamma$ is finitely generated as a right $\Gamma$-module. The claim for the left module is proved by the application of the anti-automorphism of the algebra $Y_p(\mathfrak{gl}_2)$ defined in (4.47) where we note that every element of $\Gamma$ is stable under this anti-automorphism. 

\[\square\]

\textbf{Example 5.4.} We give an explicit form of the relation (6.2) for the particular case $i = k = p = 2$. It reads

$$(w^{-1} \cdot X)_{\ell, \ell'} = w^{-1} \cdot X_{w(\ell)w(\ell')} \quad \text{for} \quad w \in \mathbb{W}.$$ 

Define the map $G : Y_p(\mathfrak{gl}_2) \to M_p(\mathbb{L})$ such that for any $x \in Y_p(\mathfrak{gl}_2)$ and any $\ell \in P_0$, $G(x)_{\ell, \ell} = \theta(x, b + \ell, \delta)$ if $\ell' = \ell = \delta$ and 0 otherwise; see (5.1).

\textbf{Lemma 6.1.} \quad (1) $G$ is a representation of $Y_p(\mathfrak{gl}_2)$ over $\mathbb{L}$.

(2) $G(x)$ is $\mathbb{W}$-invariant for any $x \in Y_p(\mathfrak{gl}_2)$. In particular, $G(x)_{\pi\pi} \in K(\Gamma)$.

(3) If $x = x (b_1, \ldots, b_p, g_1, \ldots, g_{2p}) \in \Gamma$ and $\ell = (l_1, \ldots, l_p, 0, \ldots, 0) \in P_0$, then $G(x)_{\ell\ell} = x (b_1 + l_1, \ldots, b_p + l_p, g_1, \ldots, g_{2p})$.

(4) $G(\Gamma)$ consists of $\mathbb{W}$-invariant diagonal matrices $X$ such that $X_{\pi\pi} \in \Gamma$. In particular, any such matrix $X$ is determined by $X_{\pi\pi} \in \Gamma$.

\textbf{Proof.} Let $T$ be the free associative algebra with generators $\tilde{t}_{ij}^{(k)}$, where $i, j = 1, 2$ and $k = 1, \ldots, p$, and let

$$\pi : T \to Y_p(\mathfrak{gl}_2), \quad \tilde{t}_{ij}^{(k)} \mapsto t_{ij}^{(k)},$$

be the canonical projection. Define a homomorphism $g : T \to M_{p_0}(\mathbb{L})$ by $g(\tilde{t}_{ij}^{(k)}) = G(t_{ij}^{(k)})$ for all suitable $i, j, k$. To prove (1) it is sufficient to show that $g(\ker \pi) = 0$. Suppose that $f \in \ker \pi$. Then $\Phi(\ell)(g(f)) = 0$ and thus $g(f)_{\ell, \ell'}(\ell) = 0$ for any $\ell \in P_1$. Since $P_1$ is dense in $\text{Specm} \mathbb{L}$ we conclude that $g(f) = 0$ implying (1).

The image of $G$ is $\mathbb{W}$-invariant since this holds for the generators of $Y_p(\mathfrak{gl}_2)$; see (4.7)–(4.9). For any $\sigma \in S_p \times S_{2p}$ we have

$$(\sigma^{-1} \cdot G(x)_{\pi\pi} \sigma^{-1} G(x, \sigma, 0, \sigma(0))) = \sigma^{-1}(G(x)_{\pi\pi}).$$
Hence \( G(x)_{\mathbb{P}} \) is \( S_p \times S_{2p} \)-invariant proving (2). The statement (3) follows from (2) if we apply a shift by \( \ell \in \mathbb{P}_0 \) to an arbitrary \( x \in Y_p(\mathfrak{gl}_2) \). The statement (4) follows immediately from (2) and (3).

The composition \( r_\ell = \Phi(\ell) \circ G \) defines a representation of \( Y_p(\mathfrak{gl}_2) \). By the construction, this representation provides a matrix realization of the module \( M(\ell) \); see Theorem 4.5.

**Proposition 6.2.** The representation \( G : Y_p(\mathfrak{gl}_2) \rightarrow M_{\mathbb{P}_0}(\mathbb{L}) \) is faithful.

**Proof.** It is clear that

\[
\ker G = \bigcap_{\ell \in \mathbb{P}_1} \ker r_\ell.
\]

Hence it is sufficient to prove that the intersection of the kernels is zero. Let \( \ell \in \mathbb{P}_1 \). Then \( \ker r_\ell = \text{Ann} M(\ell) \) by definition and so \( \ker r_\ell \subseteq I_\ell \). However, the intersection \( I_\ell \) over all \( \ell \in \mathbb{P}_1 \) is zero, as was shown in the proof of Proposition 5.1.

**Corollary 6.3.**

1. \( \Gamma \) is a maximal commutative subalgebra in \( Y_p(\mathfrak{gl}_2) \).
2. If \( x \in Y_p(\mathfrak{gl}_2) \) the matrix \( G(x) \) is diagonal, then \( x \in \Gamma \).

**Proof.** Consider an element \( x \in Y_p(\mathfrak{gl}_2) \) which commutes with every \( z \in \Gamma \) and such that \( x \notin \Gamma \). Suppose that there exist \( \ell, \ell' \in \mathbb{P}_0 \), \( \ell \neq \ell' \) such that \( G(x)_{\ell\ell'} \neq 0 \). There exists \( z \in \Gamma \) such that \( z(\ell) \neq z(\ell') \) and so \( G(z)_{\ell\ell'} \neq G(z)_{\ell'\ell} \) by Lemma 6.1(3). Then we have

\[
G(xz)_{\ell\ell'} = G(x)_{\ell\ell'} G(z)_{\ell'\ell} = G(zx)_{\ell\ell'} = G(z)_{\ell\ell'} G(x)_{\ell\ell'}
\]

which contradicts to the assumption. Therefore, \( G(x) \) is diagonal. To prove the maximality of \( \Gamma \) it is now sufficient to verify part (2) of the corollary. By Lemma 6.1(2), we have \( G(x)_{\mathbb{P}} = f/g \in K(\Gamma) \) with relatively prime \( f, g \in \Gamma \). Suppose that \( g \notin \mathbb{L} \). By Lemma 6.1(2) and (4), we derive that \( G(x)G(g) = G(f) \) and hence \( xg = f \) by Proposition 6.2. This shows that \( x \in \Gamma \) due to Theorem 3.4(1).

Denote by \( X_0 \) the column matrix defined by

\[
X_0 = \sum_{\delta \in \mathbb{P}_0} \mathbb{L} e_{\delta, \overline{0}},
\]

where \( \overline{0} \) is the zero element of \( \mathbb{P}_0 \). Note that the \( \mathbb{W} \)-action \( (6.1) \) induces an action of \( S_p \times S_{2p} \), on the free \( \mathbb{L} \)-module \( X_0 \).

**Corollary 6.4.** Let \( p : M_{\mathbb{P}_0}(\mathbb{L}) \rightarrow X_0 \) be the natural projection. Then the composition \( r = p \circ G : Y_p(\mathfrak{gl}_2) \rightarrow \) is injective. Moreover, the map \( p \) commutes with the action of \( S_p \times S_{2p} \) and, in particular, \( r(Y_p(\mathfrak{gl}_2)) \) is \( S_p \times S_{2p} \)-invariant.

**Proof.** Note that for any \( x \in Y_p(\mathfrak{gl}_2) \) the matrix \( G(x) \in M_{\mathbb{P}_0}(\mathbb{L}) \) is determined completely by its column \( p(G(x)) \). Thus \( r(x) = 0 \) implies \( G(x) = 0 \) and hence \( x = 0 \) since \( G \) is faithful. This proves that \( r \) is injective. The other statements follow immediately from the definitions and Lemma 6.1(2).

As in Section 5, we identify the \( (\Gamma \otimes \Gamma) \)-bimodule structure on \( Y_p(\mathfrak{gl}_2) \) with the corresponding action of \( \Gamma \otimes \Gamma \). Using the embedding \( (6.1) \), we can regard the elements of \( \Gamma \otimes \Gamma \) as polynomials in two families of variables \( b \) and \( b' \) which are \( S_p \times S_{2p} \)-invariant.
Lemma 6.5. Suppose that \( x \in Y_p(\mathfrak{gl}_2), f \in \Gamma \otimes \Gamma, \) and \( \ell, \ell' \in \mathcal{P}_0. \) Then
\[
G(f \cdot x)_{\ell \ell'} = f(b + \ell, b + \ell')G(x)_{\ell \ell'}.
\]

Proof. Let \( f = \sum_i z_i \otimes z'_i \in \Gamma \otimes \Gamma. \) Then \( G(f \cdot x) = \sum_i G(z_i)G(x)G(z'_i) \) and hence, by Lemma 6.1(4),
\[
G(f \cdot x)_{\ell \ell'} = \sum_i G(z_i)_{\ell \ell'}G(x)_{\ell \ell'}G(z'_i)_{\ell \ell'} = G(x)_{\ell \ell'}\sum_i G(z_i)_{\ell \ell'}G(z'_i)_{\ell \ell'} = G(x)_{\ell \ell'}\sum_i z_i(b + \ell)z'_i(b + \ell').
\]
This proves (2).

Lemma 6.6. Let \( S \subseteq \mathcal{L} \) be an \( S_p \times S_{2p} \)-invariant set. Suppose that \( z \in \Gamma \) and \( x \in Y_p(\mathfrak{gl}_2) \) is such that \( G(x)_{\ell \ell'} = 0 \) for all \( \ell, \ell' \in \mathcal{P}_0. \) Then \( F_{S,z} : x = 0. \)

Proof. Let \( F = F_{S,z} = \sum_i z^i \otimes a_i, \) where \( a_i \in \Gamma \) by Proposition 5.2. If \( \ell - \ell' \in S, \) then
\[
G(F \cdot x)_{\ell \ell'} = G(z(b + \ell), b + \ell')G(x)_{\ell \ell'}
\]
by Proposition 6.1(1) and Lemma 6.5. Furthermore, observe that \( h(z, b) = z \otimes 1 - 1 \otimes z(b + \ell - \ell') \) divides \( F \) and that \( h(z(b + \ell), b + \ell') = 0. \) Here we regard the result of the evaluation of the product of type \( z \otimes z'(b') \) at \( b \) as the polynomial \( z(b)z'(b'). \) This gives \( F(z(b + \ell), b + \ell') = 0. \) Hence, \( G(F \cdot x) = 0 \) implying \( F \cdot x = 0 \) by Proposition 6.2.

Let \( S \subseteq \mathcal{P}_0 \) be a finite \( S_p \times S_{2p} \)-invariant set. Define \( Y^S = \{ x \in Y_p(\mathfrak{gl}_2) \mid L_x \subseteq S \}. \) Clearly \( Y^S \) is a \( \Gamma \)-sub-bimodule of \( Y_p(\mathfrak{gl}_2). \) We have the following characterization of the bimodule \( Y^S. \)

Lemma 6.7. Let \( x \in Y_p(\mathfrak{gl}_2). \) Then
\begin{enumerate}
\item \( x \in Y^S \) if and only if the condition \( G(x)_{\ell \ell'} \neq 0, \) for some \( \ell, \ell' \in \mathcal{P}_0, \) implies that \( \ell - \ell' \in S. \)
\item \( F_{L_x \setminus S,z} : x \in Y^S \) for any \( z \in \Gamma. \)
\item \( Y^S \) is a finitely generated left (right) \( \Gamma \)-module and \( Y^S = D(Y^S). \)
\item \( Y^{(0)} = \Gamma. \)
\end{enumerate}

Proof. Statement (1) follows from the definition of \( Y^S. \) Let \( F = F_{L_x \setminus S,z} \) and \( y = F \cdot x. \) To prove (2) calculate the matrix element \( G(y)_{\ell \ell'} \) provided that \( \ell - \ell' \notin S. \) By Lemma 6.3,
\[
G(y)_{\ell \ell'} = G(F \cdot x)_{\ell \ell'} = F(z(b + \ell), b + \ell')G(x)_{\ell \ell'}.
\]
If \( \ell - \ell' \notin L_x, \) then \( G(x)_{\ell \ell'} = 0 \) and hence \( G(y)_{\ell \ell'} = 0. \) Suppose now that \( \ell - \ell' \in L_x \setminus S. \) Then
\[
F(z(b + \ell), b + \ell') = \prod_{\delta \in L_x \setminus S} (z(b + \ell) - z(b + \ell + \delta)) = 0.
\]
This proves (2).

Let \( x \in D(Y^S) \) and suppose that \( z \in \Gamma \) is such that \( z \neq 0 \) and \( zx \in Y^S. \) Since \( G(zx)_{\ell \ell'} = z(b + \ell)G(x)_{\ell \ell'} \) by Lemma 6.5 we have \( G(zx)_{\ell \ell'} = 0 \) if and only if \( G(x)_{\ell \ell'} = 0 \) implying that \( x \in Y^S. \) Hence \( Y^S = D(Y^S). \)

Consider \( r(Y^S) \) as a \( \Gamma \)-submodule of \( X_0 \) where \( r : Y_p(\mathfrak{gl}_2) \to X_0 \) is defined in Corollary 6.3. Then \( r(Y^S) \) belongs to the free \( \mathbb{L} \)-submodule \( \sum_{\ell \in S} L_{\ell \ell'} \) of
of generality, we can assume that this module is generated by the elements $r(x_1), \ldots, r(x_s) \in r(Y^S)$. Since $D(Y^S) = Y^S$, we have

$$D\left(\sum_{i=1}^s x_i\right) \subseteq Y^S.$$  

Fix $x \in Y^S$. Then

$$r(x) = \sum_{i=1}^s t_i r(x_i), \quad t_i \in \mathbb{L}.$$  

Note that for any $y \in Y^S$ and any $\sigma \in S_p \times S_{2p}$ we have $\sigma \cdot r(y) = r(y)$ since $S$ is $S_p \times S_{2p}$-invariant. Hence

$$p! \, (2p)! \, r(x) = \sum_{\sigma \in S_p \times S_{2p}} \sigma \cdot r(x) = \sum_{\sigma \in S_p \times S_{2p}} \sum_{i=1}^s (\sigma \cdot t_i) \sigma \cdot r(x_i)$$

which can be rewritten as

$$(6.2) \quad r(x) = \frac{1}{p! \,(2p)!} \sum_{i=1}^s u_i r(x_i), \quad \text{where} \quad u_i = \sum_{\sigma \in S_p \times S_{2p}} \sigma \cdot t_i.$$  

Since each $u_i$ is $S_p \times S_{2p}$-invariant, it belongs to the field of fractions $K(\Gamma)$ for all $i = 1, \ldots, s$. Multiplying both parts of $(6.2)$ by the common denominator of the $u_i$ we obtain from Corollary 6.4 that

$$x \in D\left(\sum_{i=1}^s x_i\right), \quad \text{implying} \quad D\left(\sum_{i=1}^s x_i\right) = Y^S.$$  

Due to Corollary 3.5 we can conclude that $Y^S$ is finitely generated over $\Gamma$. This proves (3). By the definition of $Y^S$, $x \in Y^{(0)}$ if and only if $G(x)$ is diagonal. Hence $x \in \Gamma$ by Corollary 6.3.2.

7. Category of Harish-Chandra Modules

We will show in this section that each character of $\Gamma$ extends to a finite number of irreducible Harish-Chandra modules over $Y_p(\mathfrak{gl}_2)$. This is an analogue of the corresponding result in the case of a Lie algebra $\mathfrak{gl}_n$ which was conjectured in [DFO1] and proved in [OVl]. In this section we use the techniques of [DFO2] and [Ov].

Due to Theorem 5.3 $\Gamma$ is a Harish-Chandra subalgebra of $Y_p(\mathfrak{gl}_2)$ so that we can apply all the statements from Section 2.1. Set $A = A_{Y_p(\mathfrak{gl}_2), \Gamma}$. Then by Proposition 2.1, the categories $A$-mod and $\mathbb{H}(Y_p(\mathfrak{gl}_2), \Gamma)$ are equivalent. Also the full subcategory $\mathbb{H}(Y_p(\mathfrak{gl}_2), \Gamma)$ consisting of weight modules is equivalent to the module category $A_W$-mod. If $\ell \in \mathcal{L}$, then the category $R_\ell = \mathbb{H}(Y_p(\mathfrak{gl}_2), \Gamma, D(\ell))$ is equivalent to the block $A_W(D(\ell))$-mod of the category $A_W$-mod.

Let $\mathfrak{m}, \mathfrak{n} \in \text{Spec} \Gamma$, $\ell_\mathfrak{m}, \ell_\mathfrak{n} \in \mathcal{L}$ are such that $r(\ell_\mathfrak{m}) = \mathfrak{m}$ and $r^*(\ell_\mathfrak{n}) = \mathfrak{n}$; see (4.2). Set

$$S(\mathfrak{m}, \mathfrak{n}) = \{\sigma_1 \ell_\mathfrak{n} - \sigma_2 \ell_\mathfrak{m} \mid |\sigma_1, \sigma_2| \in S_p \times S_{2p}\} \cap \mathcal{P}_0.$$  

Consider the following subset in $\mathcal{L}$,

$$\mathcal{P}_2 = \{\ell \in \mathcal{L} \mid \ell_i - \ell_j \not\in \mathbb{Z} \setminus \{0\}, \quad i, j = 1, \ldots, p\}$$

and put $\Omega = r^*(\mathcal{P}_2)$. We shall also be using the set $A(\mathfrak{m}, \mathfrak{n})$ introduced in (2.2).
Proposition 7.1. (1) For any \( m, n \in \text{Specm} \Gamma \) and any \( m, n \geq 0 \) we have
\[
Y_p(\mathfrak{gl}_2) = Y^S + n^nY_p(\mathfrak{gl}_2) + Y_p(\mathfrak{gl}_2)m^n,
\]
where \( S = S(m, n) \).

(2) \( A(m, n) \) is finitely generated as a left \( \Gamma_n \)-module and as a right \( \Gamma_m \)-module.

(3) If \( S(m, n) = \{0\} \), then \( A(m, n) \) is generated as a left \( \Gamma_n \)-module and as a
right \( \Gamma_m \)-module by the image of 1 in \( A(m, n) \).

(4) If \( S(m, m) = \{0\} \), then \( m \in \Omega \). Moreover, \( A(m, m) \) is a quotient algebra
of \( \Gamma_m \) and \( m \) extends uniquely to an irreducible \( Y_p(\mathfrak{gl}_2) \)-module.

(5) If \( \ell_m \in \mathcal{P}_1 \), then \( A(m, m) = \Gamma_m \).

(6) Let \( \ell \in \mathcal{P}_1 \), \( m = \tau^*(\ell) \) and \( n = \tau^*(\ell + \delta_i) \), \( i \in \{1, \ldots, p\} \). Then \( A(m, n) \) is
a free of rank 1 as a right \( \Gamma_m \)-module and as a left \( \Gamma_n \)-module.

Proof. (1) We shall show that for any \( x \in Y_p(\mathfrak{gl}_2) \) and any \( k \geq 1 \) there exists \( x_k \in Y^S \)

such that
\[
x = x_k + \sum_{i=0}^{k} u^{k-i}x^i m^i.
\]

The statement will follow if we choose \( k = m + n + 1 \). We will use induction on \( k \). Suppose that \( k = 1 \). If \( \mathcal{L}_x \subseteq S \), then \( x \in Y^S \) and there is nothing to prove.

Furthermore, by the definition of the set \( S \) for any \( \ell \in \mathcal{L}_x \setminus S \) the \( S_p \times S_2p \)-orbits of \( \ell_n \) and \( \ell_m + \ell \) are disjoint. Hence there exists \( z \in \Gamma \) such that \( z(\ell_n) \neq z(\ell_m + \ell) \)
for any \( \ell \in \mathcal{L}_x \setminus S \). Let \( F = F_{\mathcal{L}_x \setminus S}.z \). Then
\[
F(z(\ell_n), \ell_m) = \prod_{\ell \in \mathcal{L}_x \setminus S} (z(\ell_n) - z(\ell_m + \ell)) \neq 0.
\]

We can assume that \( F(z(\ell_n), \ell_m) = 1 \). Hence we obtain that \( F = 1 + u \) where \( u \in n \otimes \Gamma + \Gamma \otimes m \).

It follows from Lemma \( \mathcal{S}(2) \), that \( x_1 = F \cdot x \) belongs to \( Y^S \).

Hence we have
\[
x_1 = (1 + u) \cdot x \in x + nx \Gamma + \Gamma x m \quad \text{and thus} \quad x \in x_1 + nx \Gamma + \Gamma x m.
\]

This proves the assertion in the case \( k = 1 \). Assume that \( \mathcal{S}(1) \) holds for some \( k \geq 1 \).

Then
\[
x \in x_k + \sum_{i=0}^{k} u^{k-i}(x_k + \sum_{j=0}^{k} u^{k-j}x^j m^j)m^i \subseteq x_k + \sum_{i=0}^{k} u^{k-i}x^i m^i + \sum_{i=0}^{k+1} u^{k+1-i}x^i m^i.
\]

Since \( Y^S \) is a \( \Gamma \)-bimodule we conclude that \( x_k + \sum_{i=0}^{k} u^{k-i}x^i m^i \subseteq Y^S \) which implies
the statement \( \mathcal{S}(1) \). In particular, we have proved that
\[
x_{k+1} - x_k \in \sum_{i=0}^{k} u^{k-i}Y^n S^i m^i.
\]

(2) We prove the statement for the case of left module, the case of right module
can be treated analogously. By \( \mathcal{S}(1) \) the image \( \mathcal{P} \) of every \( x \in Y_p(\mathfrak{gl}_2) \) in \( A(m, m) \)
is the limit of the sequence \( \{\mathcal{P}_k\}_{k \geq 1} \), \( x_k \in Y^S \). Let \( y_1, \ldots, y_m \) be a finite system of
generators of $Y^S$ as a left $\Gamma$-module which exists by Lemma 6.7 (3). Then for every $N > 1$ and every $i = 1, \ldots, m$ there exists $N_i$ such that

$$y_i m^N \subseteq \sum_{j=1}^{m} n^{N_i} y_j.$$  

Since $\Gamma$ is noetherian we have that $\bigcap_k \mathbb{n}^k Y^S = 0$ and hence there exists the maximum value $d_N$ such that

$$y_i m^N \subseteq \sum_{j=1}^{m} n^{d_N} y_j$$

for all $i = 1, \ldots, m$. Moreover, $d_N \to \infty$ while $N \to \infty$ since $Y^S$ is a finitely generated right $\Gamma$-module and $\bigcap_k Y^S m^k = 0$. By (7.2), $x_{k+1} - x_k \in \mathbb{n}^{R_k} Y^S$ where $R_k = \min \{ \lfloor k/2 \rfloor, d_{\lfloor k/2 \rfloor} \}$. We have

$$\bar{x} = \bar{x}_1 + \sum_{k=1}^{\infty} (x_{k+1} - x_k)$$

and thus

$$\bar{x} \in \bigcap_{k=1}^{\infty} \mathbb{n}^{R_k} Y^S \subseteq \sum_{l=1}^{m} \Gamma n \bar{y}_l.$$  

Note that the first sum is well defined since $R_k \to \infty$ when $k \to \infty$. We conclude that $A(m, m)$ is finitely generated as a left $\Gamma_n$-module. This completes the proof of

(3) By Lemma 6.7 (1), $Y^{(0)} = \Gamma$. Hence $A(m, m)$ is generated (both as a left and as a right module) by the image of $1 \in \Gamma$ by (1).

(4) By (3), $A(m, m)$ is 1-generated as a left $\Gamma_m$-module. Then the $k$-algebra homomorphism

$$i_m : \Gamma_m \to A(m, m), \quad z \mapsto z \cdot 1_m,$$

where $1_m$ is a unit morphism, is an epimorphism which shows that $A(m, m)$ is a quotient algebra of $\Gamma_m$. The uniqueness of the extension follows from the uniqueness of the simple $A(m, m)$-module and [DFO2, Theorem 18].

(5) Let $\ell = \ell_m \in P_1$. Then $S(m, m) = \emptyset$ and $A(m, m) \simeq \Gamma_m / J_m$ by (3) and thus $J_m$ acts trivially on $M(m)$ in any Harish-Chandra module $M$. Since $\ell \in P_1$, then for any $k > 0$ there exists a canonical projection $\bar{\pi}_k : L \to L/(\ell)^k$, where $(\ell)^k = \ell^k L$. It induces a homomorphism of the matrix algebras $\pi_k : M_{P_0}(L) \to M_{P_0}(L/(\ell)^k)$ and defines a Harish-Chandra module by the following composition

$$Y_p(g_{l_2}) \xrightarrow{G} M_{P_0}(L) \xrightarrow{\pi_k} M_{P_0}(L/(\ell)^k).$$

For any nonzero $x \in \Gamma$ there exists $k > 0$ such that $x \not\in (\ell)^k$ and hence $\pi_k(G(x)_{\bar{y}_l}) = x + (\ell)^k \neq 0$. Therefore, there exists a Harish-Chandra module $M$ where $x$ acts nontrivially on $M(m)$ implying that $J_m = 0$. This completes the proof.

(6) The proof is analogous to the proof of (5). Let $z \in \Gamma$, $z \neq 0$. Suppose $A(m, m)z = 0$. Then by the construction of the equivalence $F : \mathcal{A} \mod \ell \to \mathcal{H}(U, \Gamma)$ for any Harish-Chandra module $M$ and any $x \in Y_p(g_{l_2})$ the linear operator $xz$ on $M$ induces the zero map between $M(m)$ and $M(n)$. It is sufficient to construct a Harish-Chandra module where this has failed. For $k \geq 1$ consider as in (5) the.
composition \( \pi_k \circ G : Y_p(\mathfrak{gl}_2) \to M_{p_0}(L/\ell)^k \). It defines a Harish-Chandra module structure on a free \( L/(\ell)^k \)-module

\[
\mathcal{X} = \sum_{\delta \in \mathcal{P}_0} L/(\ell)^k e_{\delta, \mathcal{P}}.
\]

Consider \( x \in Y_p(\mathfrak{gl}_2) \) such that \( G(x)_{\delta, \mathcal{P}} \neq 0 \) for some \( i \). Then
\[
G(xz)_{\delta, \mathcal{P}} = G(x)_{\delta, \mathcal{P}} G(z)_{00} = G(x)_{\delta, \mathcal{P}} z \neq 0.
\]

Choose \( k \) such that \( G(xz)_{\delta, \mathcal{P}} \notin (\ell)^k \). Hence \( (\pi_k \cdot G)(xz)_{\delta, \mathcal{P}} \neq 0 \) and the linear operator \( xz \) induces a nonzero map between \( \mathcal{X}(m) = L/(\ell)^k \) and \( \mathcal{X}(n) = L/(\ell + \delta_j)^k \). The contradiction shows that \( A(m, n)z \neq 0 \). The case \( zA(m, n) = 0 \) is treated in a similar manner.

Now we are in a position to state the main result of this section which follows immediately from Lemma 6.3 and Proposition 6.1.2.

**Theorem 7.2.** Let \( m \in \text{Spec} \Gamma \). Then the left ideal \( Y_p(\mathfrak{gl}_2)m \) is contained in finitely many maximal left ideals of \( Y_p(\mathfrak{gl}_2) \). In particular, \( m \) extends to a finitely many (up to an isomorphism) irreducible \( Y_p(\mathfrak{gl}_2) \)-modules and for each such module \( M \), \( \dim M(n) < \infty \) for all \( n \in \text{Spec} \Gamma \).

8. **Category of generic Harish-Chandra modules**

In this section we study a full subcategory of generic modules in \( \mathcal{H}W(Y_p(\mathfrak{gl}_2), \Gamma) \). We give a complete description of irreducible modules and indecomposable modules in tame blocks of this category.

**Lemma 8.1.** Let \( \ell \in \mathcal{P}_1 \), \( \ell = (\beta, \gamma) \), \( m = \iota^*(\ell) \in \text{Spec} \Gamma \), \( n = \iota^*(\ell + \delta_i) \), \( i \in \{1, \ldots, p\} \). If \( \beta_i \notin \{\gamma_1, \ldots, \gamma_2p\} \), then the objects of \( A \) represented by \( m \) and \( n \) are isomorphic.

**Proof.** Choose \( z_1, z_2 \in \Gamma \) such that
\[
z_1(\ell + \delta_i) = \delta_{ij}, \quad z_2(\ell + \delta_i - \delta_j) = \delta_{ij}, \quad j = 1, \ldots, p.
\]

Set \( z = z_2 t_{12}^{(1)} z_1 t_{21}^{(1)} \). Then \( G(z) \) is diagonal by Lemma 6.5 and hence \( z \in \Gamma \) by Corollary 6.3.2. We will show that the image of \( z \) in \( \Gamma_m \) is invertible. Clearly, this is equivalent to the fact that \( z(\ell) \neq 0 \). Formulas (1.7)–(4.9) imply that \( z(\ell) = \gamma(-\beta_i) \neq 0 \) by assumption. Denote by \( T_1 \) (respectively \( T_2 \)) the generator of \( \Gamma_m - \Gamma_n \) (respectively, \( \Gamma_n - \Gamma_m \))-bimodule \( A(m, n) \) (respectively, \( A(n, m) \)); see Proposition 7.1.6. Then
\[
z_2 t_{12}^{(1)} = z_m T_2, \quad z_1 t_{21}^{(1)} = T_1 z_m'
\]
for some \( z_m, z_m' \in \Gamma_m \) and hence \( z = z_m T_2 T_1 z_m' \). Since \( z(\ell) \neq 0 \) it follows that \( z_m(\ell) \neq 0 \), \( z_m(\ell) \neq 0 \) and so \( T_2 T_1 = z_m^{-1} z_m' \) is invertible in \( \Gamma_m \). A similar argument shows that \( T_2 T_1 \) is invertible in \( \Gamma_n \). Therefore the objects \( m \) and \( n \) are isomorphic.

**Corollary 8.2.** Let \( \ell \in \mathcal{P}_1 \), \( \ell = (\beta, \gamma) \), \( \beta_i - \gamma_j \notin \mathbb{Z} \) for all \( i, j \). Then the category \( \mathcal{H}(Y_p(\mathfrak{gl}_2), \Gamma, D(\ell)) \) is hereditary. Moreover,
\[
\dim \text{Ext}^1_{\mathcal{H}(Y_p(\mathfrak{gl}_2), \Gamma, D(\ell))}(L(\ell), L(\ell)) = 3p.
\]
Proof. Let \( m = i^*(\ell) \in \text{Specm} \Gamma \). By Lemma \( 8.4 \) and our assumptions all objects of the category \( A(D(\ell)) \) are isomorphic and hence the category \( A(D(\ell))\)-mod, is equivalent to the category of finite-dimensional modules over \( \Gamma_m \). Applying Proposition \( 8.2 \) we conclude that the category \( \mathbb{H}(\mathfrak{y}_p(\mathfrak{g}_L), \Gamma, D(\ell)) \) is hereditary. Since \( \Gamma_m \) is an algebra of power series in \( 3p \) variables \( \{\ell_1, \ldots, \ell_p\} \), the statement about \( \dim \text{Ext}^1 \) follows. \( \square \)

8.1. Category of generic weight modules. Let us fix

\[ \ell \in \mathcal{P}_1, \quad m = i^*(\ell), \quad n = i^*(\ell + \delta_i) \in \text{Specm} \Gamma, \quad i \in \{1, \ldots, p\}. \]

Then \( A_{W}(m, m) \simeq \Gamma_m/\Gamma_m m \simeq k \) by Proposition \( 7.1(5) \) and so, \( \dim A_{W}(m, n) = 1 \) by Proposition \( 7.1(6) \). We will give a direct construction of the category \( A_{W}(D(\ell)) \).

We shall keep the notation

\[ \ell = (\beta, \gamma), \quad \beta = (\beta_1, \ldots, \beta_p) \in \mathbb{k}^p, \quad \gamma = (\gamma_1, \ldots, \gamma_{2p}) \in \mathbb{k}^{2p}. \]

Since \( \ell \in \mathcal{P}_1 \), then \( \beta_i \neq \beta_j \) for \( i \neq j \). Consider the following category \( K_{\ell} \):

\[ \text{Ob}(K_{\ell}) = \mathbb{Z}^p \text{ and the morphisms are generated by } \]

\[ f_i(a) : a \mapsto a + \delta_i \quad \text{and} \quad e_i(a) : a \mapsto a - \delta_i, \]

where \( i = 1, \ldots, p \) and \( a = (k_1, \ldots, k_p) \in \mathbb{Z}^p \) with the following relations:

\[

typloneq,f_2(a + \delta_i) f_2(a) = f_2(a + \delta_j) f_2(a),
\]

\[
typlotwoneq,e_i(a - \delta_i) e_i(a) = e_i(a - \delta_j) e_j(a),
\]

\[
typethreoneq,e_i(a + \delta_j) f_2(a) = f_2(a - \delta_i) e_i(a) \quad \text{for} \quad i \neq j,
\]

\[
typethreetwo,e_i(a + \delta_i) f_2(a) = -\gamma(-\beta_i - k_i) 1(a),
\]

\[
typethreethree,f_i(a - \delta_i) e_i(a) = -\gamma(-\beta_i - k_i + 1) 1(a).
\]

It follows immediately from Lemmas \( 4.1 \) and \( 4.3 \) that any module in the category \( R_{\ell} \) defined in 4.1 can be naturally viewed as a module over the category \( K_{\ell} \) which defines a functor \( F : R_{\ell} \to K_{\ell}\)-mod. For any \( a \in \mathbb{Z}^p \) consider the subalgebra \( C_{\ell}(a) = \text{Hom}_{K_{\ell}}(a, a) \) of the path algebra. Clearly, \( C_{\ell}(a) \simeq k \) for any \( a \in \mathbb{Z}^p \) due to the defining relations of \( K_{\ell} \). Note also that \( F \) is an exact functor. For any \( a = (k_1, \ldots, k_p) \in \mathbb{Z}^p \) we can construct a universal module \( M(\ell, a) \in K_{\ell}\)-mod. Consider \( k \) as a \( C_{\ell}(a) \)-module with

\[ e_i(a + \delta_i) f_2(a) 1 = -\gamma(-\beta_i - k_i), \]

\[ f_i(a - \delta_i) e_i(a) 1 = -\gamma(-\beta_i - k_i + 1). \]

Let \( A_{\ell, a} \) consist of all paths in \( K_{\ell} \) originating in \( a \). Then \( A_{\ell, a} \) is naturally a \( K_{\ell} \)-\( C_{\ell}(a) \)-bimodule, where the action of \( C_{\ell}(a) \) on \( k \) is determined by the defining relations in \( K_{\ell} \). Now construct a \( \mathbb{Z}^p \)-graded \( K_{\ell}\)-module

\[ M(\ell, a) = A_{\ell, a} \otimes C_{\ell}(a) k. \]

Clearly, all graded components of \( M(\ell, a) \) are 1-dimensional and \( M(\ell, a)_a = 1_a \otimes k \). A module \( M(\ell, a) \) contains a unique maximal \( \mathbb{Z}^p \)-graded submodule which intersects \( M(\ell, a)_a \) trivially and hence has a unique irreducible quotient \( L(\ell, a) \) with \( L(\ell, a)_a \simeq k \) and \( \dim L(\ell, a)_b \leq 1 \) for all \( b \in \mathbb{Z}^p \). If \( V \) is another irreducible \( K_{\ell}\)-module with \( V_0 \neq 0 \), then there exists a nontrivial \( C_{\ell}(a) \)-homomorphism from \( k \) to \( V_0 \) which can be extended to an epimorphism from \( M(\ell, a) \) to \( V \). Since \( V \) is irreducible we conclude that \( V \simeq L(\ell, a) \).
Obviously, we can view $M(\ell)$ as a module over the category $K_\ell$ with a natural action of the morphisms of $K_\ell$ and $F(M(\ell)) = M(\ell, \beta)$. Thus a $K_\ell$-module $M(\ell, \beta)$ can be extended to a $Y_p(\mathfrak{gl}_2)$-module $M(\ell)$. Moreover, the functor $F$ preserves the submodule structure of $M(\ell)$. In particular, $F(L(\ell)) = L(\ell, \beta)$.

**Proposition 8.3.** If $\ell \in \mathcal{P}_1$, then the categories $K_\ell$-mod and $R_\ell$ are equivalent.

**Proof.** Let $\ell = (\beta, \gamma)$. We already have the functor $F : R_\ell \to K_\ell$-mod. Suppose that $V \in K_\ell$-mod. We want to show that $V$ can be extended to a $Y_p(\mathfrak{gl}_2)$-module. Let $a = (k_1, \ldots, k_p)$ and $v \in V_a \setminus \{0\}$. Consider a submodule $W \subseteq V$ generated by $v$. Then $W_a = kv$ and there is an epimorphism from $M(\ell, a)$ to $W$, which maps $1_a \otimes 1$ to $v$. Since $F(M(\ell')) = M(\ell, a)$, where $\ell' = (\beta + a, \gamma)$, then $W$ can be extended to a corresponding quotient of $M(\ell')$. Since $v$ was an arbitrary element of $V$, we conclude that $V$ can be extended to a $Y_p(\mathfrak{gl}_2)$-module and will denote that module by $G(V)$. This way $G$ defines a functor from $K_\ell$-mod to $R_\ell$ (action on morphisms is obvious). One can easily see that the functors $F$ and $G$ define an equivalence between the categories $K_\ell$-mod and $R_\ell$. \hfill \Box

### 8.2. Support of irreducible generic weight modules

To complete the classification of irreducible modules we have to know when two irreducible modules $L(\ell)$ and $L(\ell')$ are isomorphic. For that we need to describe the support Supp $L(\ell)$.

We shall say that the weight subspaces $M(\ell, \psi)$ and $M(\ell, \psi')$ are strongly isomorphic if $\gamma(-\psi_i) \neq 0$ where $\psi = (\psi_1, \ldots, \psi_p)$. This implies

$$f_i(\psi_1, \ldots, \psi_p) M(\ell, \psi) = 0 \quad \text{or} \quad e_i(\psi_1, \ldots, \psi_p) M(\ell, \psi) = 0,$$

then

$$f_i(\psi_1, \ldots, \psi_j \pm 1, \ldots, \psi_p) M(\ell, \psi) = 0 \quad \text{or} \quad e_i(\psi_1, \ldots, \psi_j \pm 1, \ldots, \psi_p) M(\ell, \psi) = 0,$$

respectively, for all $j \neq i$.

Let $a_i, a_i' \in \mathbb{Z} \cup \{\pm \infty\}, a_i \leq a_i'$, $i \in \{1, \ldots, p\}$. Denote

$$P(a_1, \ldots, a_p, a_1', \ldots, a_p') = \{(x_1, \ldots, x_p) \in \mathbb{Z}^p \mid a_i \leq x_i \leq a_i', \quad i = 1, \ldots, p\},$$

a parallelepiped in $\mathbb{Z}^p$. Note that some faces of the parallelepiped can be infinite in some directions. In particular, in the case $a_i = -\infty$, $a_i' = \infty$ for all $i$, the parallelepiped coincides with $\mathbb{Z}^p$.

**Theorem 8.5.** For any irreducible weight module $L(\ell)$ over $Y_p(\mathfrak{gl}_2)$ there exist elements $a_i, a_i' \in \mathbb{Z} \cup \{\pm \infty\}, a_i \leq a_i'$, $i \in \{1, \ldots, p\}$ such that

$$\text{Supp } L(\ell) = P(a_1, \ldots, a_p, a_1', \ldots, a_p').$$

**Proof.** Let $\ell = (\beta, \gamma) \in \mathcal{P}_1$. Fix $i \in \{1, \ldots, p\}$. If $\gamma(-\beta_i + k) \neq 0$ for all $k \in \mathbb{Z}$, then $(k_1, \ldots, k_i + m, \ldots, k_p) \in \text{Supp } L(\ell)$.
as soon as \((k_1, \ldots, k_p) \in \text{Supp } L(\ell)\). This follows immediately from Lemma 8.4. In this case we set \(a_i = -\infty\) and \(a'_i = \infty\). Now let \(\gamma(-\beta + k) = 0\) for some \(k \in \mathbb{Z}\). Let \(m \geq 0\) be the smallest integer (if it exists) such that \(\gamma(-\beta - m) = 0\) and let \(n \leq 0\) be the largest integer (if it exists) such that \(\gamma(-\beta_1 - n + 1) = 0\). It follows from Lemma 8.4 that

\[
\text{Supp } L(\ell) \cap \{\beta + n\delta_i \mid k \in \mathbb{Z}\} = \{\beta + n\delta_i, \ldots, \beta + m\delta_i\}.
\]

If \(\beta + s\delta_j \in \text{Supp } L(\ell)\), \(j \neq i\), then

\[
\text{Supp } L(\ell) \cap \{\beta + s\delta_j + n\delta_i \mid k \in \mathbb{Z}\} = \{\beta + s\delta_j + n\delta_i, \ldots, \beta + s\delta_j + m\delta_i\}.
\]

In this case we set \(a_i = \beta_i + n\) and \(a'_i = \beta_i + m\). The statement of the theorem now follows. \(\square\)

### 8.3. Indecomposable generic weight modules.

Fix \(\ell = (\beta, \gamma) \in \mathcal{P}_1\). A full subcategory \(\mathcal{S} \subseteq K_\ell\) is called a skeleton of \(K_\ell\) provided the objects of \(\mathcal{S}\) are pairwise non-isomorphic and any object of \(K_\ell\) is isomorphic to some object of \(\mathcal{S}\). In this case the categories of \(K_\ell\text{-mod}\) and \(\mathcal{S}\text{-mod}\) are equivalent.

For each \(i \in \{1, \ldots, p\}\) consider a set \(I_i = \{k \in \mathbb{Z} \mid \gamma(-\beta_i - k) = 0\}\). Define a category \(S_\ell\) as a \(k\)-category with the set of objects

\[
S_0 = \{0, \ldots, |I_1|\} \times \cdots \times \{0, \ldots, |I_p|\}
\]

and with morphisms generated by

\[
\begin{align*}
r_{i_1, \ldots, i_p}^k : (i_1, \ldots, i_p) & \mapsto (i_1, \ldots, i_k + 1, \ldots, i_p), \\
s_{j_1, \ldots, j_p}^k : (j_1, \ldots, j_p) & \mapsto (j_1, \ldots, j_k - 1, \ldots, j_p),
\end{align*}
\]

where \(k \in \{1, \ldots, p\}\) is such that \(I_k \neq \emptyset\), \(i_k < |I_k|\), \(j_k > 0\), subject to the relations

\[
s_{i_1, \ldots, i_k + 1, \ldots, i_p}^k r_{i_1, \ldots, i_p}^k = r_{i_1, \ldots, i_p}^k s_{i_1, \ldots, i_k + 1, \ldots, i_p}^k = 0
\]

and

\[
\begin{align*}
x_{i_1, \ldots, i_p}^k y_{e_1, \ldots, e_p}^r = y_{e_1, \ldots, e_p}^r x_{i_1, \ldots, i_p}^k
\end{align*}
\]

for all \(k \neq r\) and all possible \(x, y \in \{r, s\}\) and all \(a_i, e_i, c_i\), with \(1 \leq i \leq p\) for which this equality makes sense.

It follows from the construction that \(S_\ell\) is the skeleton of the category \(K_\ell\). Note that the corresponding algebra is finite-dimensional. In particular, \(S_\ell\) is semisimple when \(I_k = \emptyset\) for all \(1 \leq k \leq p\), i.e., when \(\gamma(-\beta_k + r) \neq 0\) for all \(r \in \mathbb{Z}'\) and all \(k = 1, \ldots, p\). Hence it is sufficient to describe all indecomposable modules over \(S_\ell\).

Fix \(a \in S_0\) and define a simple \(S_\ell\)-module \(S_a\) such that \(S_a(b) = \delta_{a,b}k\) for all \(b \in S_0\) and all morphisms are trivial. Since \(S_\ell\) defines a finite-dimensional algebra we have the following

**Proposition 8.6.** Any simple module over \(S_\ell\) is isomorphic to \(S_a\) for some \(a \in S_0\).

This is another confirmation of the fact that all weight spaces in any irreducible generic weight \(V_\ell(\mathfrak{gl}_2)\)-module are 1-dimensional. But this need not be the case for indecomposable modules. We restrict ourselves to a full subcategory \(R_\ell^f \subseteq R_\ell\) which consists of weight modules \(V\) with \(\dim V_\psi < \infty\) for all \(\psi \in \text{Supp } V\). We will establish the representation type of the category \(R_\ell^f\) (finite, tame or wild). For the necessary definitions we refer the reader to [De].
To establish the representation type of the category $R^f_\ell$ it is sufficient to consider the category $S_\ell \text{-mod}$ of modules over the category $S_\ell$ with finite-dimensional weight spaces. Denote $X_\ell = \{k \in \{1, \ldots, p\} \mid I_k \neq \emptyset\}$.

### 8.3.1. Indecomposable modules in the case $|X_\ell| = 1$.

In this section we describe all indecomposable modules over $S_\ell$ in the case $|X_\ell| = 1$. Let $X_\ell = \{i\}$ and let $|I_i| = r > 0$. In this case the category $S_\ell$ has the following quiver $A$ with the relations

$$
1 \xrightarrow{a_1} 2 \xrightarrow{a_r} r+1 \xrightarrow{a_i} b_1 \xrightarrow{b_r} \cdots \xrightarrow{b_1} r \quad a_i b_i = b_i a_i = 0.
$$

We denote by $S_i, i \in \{1, \ldots, r+1\}$, the simple module corresponding to the point $i$. These modules correspond to all irreducible modules in $R^f_\ell$ by Proposition 8.6.

Now describe the remaining indecomposable modules for the quiver above. Fix integers $1 \leq k_1 < k_2 \leq r+1$ and a function

$$
\xi_{k_1,k_2} : \{k_1, k_1+1, \ldots, k_2\} \to \{0,1\}.
$$

Define the module $M = M(k_1, k_2, \xi_{k_1,k_2})$ as follows: $M(i) = ke_i$, $k_1 \leq i \leq k_2$ and $M(i) = 0$ otherwise. Furthermore,

$$
a_i e_i = e_{i+1}, \quad b_i e_{i+1} = 0 \quad \text{if} \quad \xi_{k_1,k_2}(i) = 1
$$

and

$$
a_i e_i = 0, \quad b_i e_{i+1} = e_i \quad \text{if} \quad \xi_{k_1,k_2}(i) = 0,
$$

for all $1 \leq i < k_2$. The proof of the following proposition is standard; see e.g. [GR].

**Proposition 8.7.** The modules $S_i$ for $1 \leq i \leq r+1$ and $M(k_1, k_2, \xi_{k_1,k_2})$ with $1 \leq k_1 < k_2 \leq r+1$ and

$$
\xi_{k_1,k_2} : \{k_1, k_1+1, \ldots, k_2\} \to \{0,1\},
$$

exhaust all non-isomorphic indecomposable modules for $A$.

### 8.3.2. Indecomposable modules in the case $|X_\ell| = 2$.

In this section we describe the indecomposable modules for $S_\ell$ when $|X_\ell| = 2$ and $|I_k| = 1$ for each $k \in X_\ell$. Then $S_\ell$ is isomorphic to the following category $B$ considered in [BB].

$$
\begin{align*}
B : & \quad 1 \xrightarrow{a_1} 2 \xrightarrow{a_2} 3 \quad a_i b_l = b_l a_i = 0, \quad i = 0, \ldots, 3, \\
& \quad b_0 \xrightarrow{b_1} a_0 \xrightarrow{b_2} a_1 \xrightarrow{b_3} a_2 \xrightarrow{b_4} a_3, \quad a_i a_j = b_l b_m \quad \text{for any} \quad i, j, l, m \in \{0,1,2,3\}, \quad \text{where possible}.
\end{align*}
$$

By Proposition 8.6 this category has four non-isomorphic simple modules $S_i$ where $i \in \{0,1,2,3\}$, with the support in the given point $i$. The indecomposable modules were described in [BB]. For the sake of completeness we repeat here this classification.

We will treat the objects of $B$ as elements of $\mathbb{Z}/4\mathbb{Z}$. Consider the following three families of non-simple indecomposable modules.
Finite family. Fix \( i \in \{0, 1, 2, 3\} \) and define the \( \mathbf{B} \)-module \( M_i \) such that \( M_i(j) = \mathbb{K} e_j \) for each \( j = 0, 1, 2, 3 \) and
\[
a_i e_i = e_{i+1}, \quad a_{i+1} e_{i+1} = e_{i+2}, \quad b_{i-1} e_i = e_{i-1}, \quad b_{i-2} e_{i-1} = e_{i-2}
\]
while \( u_j e_k = 0 \) for all other cases of \( u \in \{a, b\} \) and \( j, k = 0, \ldots, 3 \). Obviously, \( M_i \) is an indecomposable module for any \( i \).

Infinite discrete families. Let \( n \in \mathbb{N} \), \( n > 1 \), and \( j \in \mathbb{Z}_4 \). Define a \( \mathbf{B} \)-module \( \mathbf{M}_{n,j,1} \) (respectively, \( \mathbf{M}_{n,j,2} \)) as follows. Consider \( n \) elements \( e_1, \ldots, e_n \). A \( \mathbb{K} \)-basis of the vector space \( \mathbf{M}_{n,j,1}(l) \) (respectively, \( \mathbf{M}_{n,j,2}(l) \)) is the set of \( e_k \) such that \( j + k - 1 \equiv l (\bmod 4) \). The elements \( a_l \) and \( b_{l-1} \) act as follows:
\[
a_l e_k = \begin{cases} 
eq 0, & \text{if } l \text{ is even (resp., odd), } k < n \text{ and } j + k - 1 \equiv l (\bmod 4); \\
0, & \text{otherwise.}
\end{cases}
\]
\[
b_{l-1} e_k = \begin{cases} 
eq 0, & \text{if } l \text{ is even (resp., odd), } k > 1 \text{ and } j + k - 1 \equiv l (\bmod 4); \\
0, & \text{otherwise.}
\end{cases}
\]
All modules \( \mathbf{M}_{n,j,1} \) and \( \mathbf{M}_{n,j,2} \), \( n > 1, 0 \leq j \leq 3 \) are non-isomorphic indecomposable \( \mathbf{B} \)-modules.

Infinite continuous families. For each \( \lambda \in \mathbb{K}, \lambda \neq 0 \), and \( d \in \mathbb{Z}, d > 0 \) define the \( \mathbf{B} \)-modules \( \mathbf{M}_{d,\lambda,1} \) and \( \mathbf{M}_{d,\lambda,2} \) as follows. Set
\[
\mathbf{M}_{d,\lambda,1}(i) = \mathbb{K}^d, \\
\mathbf{M}_{d,\lambda,1}(a_0) = \mathbf{M}_{d,\lambda,1}(a_2) = \mathbf{M}_{d,\lambda,1}(b_1) = I_d, \\
\mathbf{M}_{d,\lambda,1}(b_0) = \mathbf{M}_{d,\lambda,1}(b_2) = \mathbf{M}_{d,\lambda,1}(a_1) = \mathbf{M}_{d,\lambda,1}(a_3) = 0, \\
\mathbf{M}_{d,\lambda,1}(b_3) = J_{d,\lambda}
\]
and
\[
\mathbf{M}_{d,\lambda,2}(i) = \mathbb{K}^d, \\
\mathbf{M}_{d,\lambda,2}(b_0) = \mathbf{M}_{d,\lambda,2}(b_2) = \mathbf{M}_{d,\lambda,2}(a_1) = I_d, \\
\mathbf{M}_{d,\lambda,2}(a_0) = \mathbf{M}_{d,\lambda,2}(a_2) = \mathbf{M}_{d,\lambda,2}(b_1) = \mathbf{M}_{d,\lambda,2}(b_3) = 0, \\
\mathbf{M}_{d,\lambda,2}(a_3) = J_{d,\lambda},
\]
where \( J_{d,\lambda} \) is the Jordan cell of dimension \( d \) with the eigenvalue \( \lambda \).

All modules \( \mathbf{M}_{d,\lambda,k} \), \( k = 1, 2 \) are indecomposable and corresponding indecomposable modules in \( \mathbf{R}_X^f \) have all weight spaces of dimension \( d \).

**Proposition 8.8** ([BB], Proposition 3.3.1). The modules \( \mathbf{S}_i, \mathbf{M}_i, \mathbf{M}_{n,i,1}, \mathbf{M}_{n,i,2}, \mathbf{M}_{d,\lambda,1}, \mathbf{M}_{d,\lambda,2} \) where \( 0 \leq i \leq 3, d \) is a positive integer, \( \lambda \in \mathbb{K}, \lambda \neq 0 \), and \( n \geq 2 \) is an integer, constitute an exhaustive list of pairwise non-isomorphic indecomposable \( \mathbf{B} \)-modules.

The following theorem describes the representation type of \( \mathbf{R}_X^f \).

**Theorem 8.9.**
1. If \( |X| = 0 \), then \( \mathbf{R}_X^f \) is a semisimple category with a unique indecomposable (=irreducible) module;
2. If \( |X| = 1 \), then \( \mathbf{R}_X^f \) has finite representation type;
3. If \( |X| = 2 \), then \( \mathbf{R}_X^f \) has tame representation type if and only if \( |I_k| = 1 \) for all \( k \in X \). Otherwise, \( \mathbf{R}_X^f \) has wild representation type;
4. If \( |X| > 2 \), then \( \mathbf{R}_X^f \) has wild representation type.
Proof. In the case when $|X| = 1$ all indecomposable modules for $S$ are described in Proposition 8.7. Hence $R$ has the finite representation type. If $|X| = 2$ and $|I_k| = 1$ for each $k \in X$, then all indecomposable modules for $S$ are described in Proposition 8.8. It follows from the definition that $R$ has the tame representation type in this case. If $|I_k| > 1$ for at least one $k$, then it is easy to construct a family of indecomposable modules that depends on two continuous parameters. Hence, in this case $R$ has the wild representation type. Suppose now that $|X| > 2$. Then $S$ contains a full subcategory of wild representation type considered in [BB, Theorem 1]. We immediately conclude that $R$ has the wild representation type. This completes the proof. □

Corollary 8.10. (1) If $|X| = 0$, then the category $R$ is a semisimple category with a unique indecomposable module.

(2) If $|X| = 1$, then $R$ has finite representation type with indecomposable modules as in Proposition 8.7.

Proof. Since cases $|X| \leq 1$ correspond to the finite representation type, then the corresponding categories do not admit infinite-dimensional indecomposable modules by [A] and hence every indecomposable module belongs to $R$.

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References


Institute of Mathematics and Statistics, University of São Paulo, Caixa Postal 66281-CEP 05315-970, São Paulo, Brazil

E-mail address: futorny@ime.usp.br

School of Mathematics and Statistics, University of Sydney, NSW 2006, Australia

E-mail address: alexm@maths.usyd.edu.au

Faculty of Mechanics and Mathematics, Kiev Taras Shevchenko University, Vladimirskaya 64, 00133, Kiev, Ukraine

E-mail address: ovsienko@sita.kiev.ua