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HARISH-CHANDRA MODULES FOR YANGIANS

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ABSTRACT. We study Harish-Chandra representations of the Yangian $\mathrm{Y}(\mathfrak{gl}_2)$ with respect to a natural maximal commutative subalgebra. We prove an analogue of the Kostant theorem showing that the restricted Yangian $\mathrm{Y}_p(\mathfrak{gl}_2)$ is a free module over the corresponding subalgebra Γ and show that every character of Γ defines a finite number of irreducible Harish-Chandra modules over $\mathrm{Y}_p(\mathfrak{gl}_2)$. We study the categories of generic Harish-Chandra modules, describe their simple modules and indecomposable modules in tame blocks.

CONTENTS

1. INTRODUCTION

Throughout the paper we fix an algebraically closed field $\mathbb{k}$ of characteristic 0. Consider the pair (U, Γ) where U is an associative $\mathbb{k}$-algebra and Γ is a subalgebra of U. Denote by $\operatorname{cfs}\Gamma$ the *cofinite spectrum* of Γ, i.e.,

$$\operatorname{cfs}\Gamma = \{\text{maximal two-sided ideals } \mathbf{m} \text{ of } \Gamma \mid \dim \Gamma/\mathbf{m} < \infty\}.$$

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A finitely generated module M over U is called a *Harish-Chandra module* (with respect to Γ) if

$$M = \bigoplus_{\mathbf{m}\in \operatorname{cfs}\Gamma} M(\mathbf{m}),$$

where

$$M(\mathbf{m}) \ = \ \{x \in M \mid \mathbf{m}^k x = 0 \quad \text{for some} \quad k \geq 0\}.$$

Harish-Chandra modules play a central role in the classical representation theory; see e.g. Dixmier [Di]. In particular, weight modules over a semisimple Lie algebra are Harish-Chandra modules with respect to the universal enveloping algebra of a Cartan subalgebra. Another important example is provided by the Gelfand–Tsetlin modules [DFO1] over the universal enveloping algebra $U(\mathfrak{gl}_n)$ of the general linear Lie algebra $\mathfrak{gl}_n$. They are Harish-Chandra modules with respect to the Gelfand–Tsetlin subalgebra of $U(\mathfrak{gl}_n)$. The latter is the commutative subalgebra generated by the centers of $U(\mathfrak{gl}_k)$, $k = 1, \dots, n$. A theory of Harish-Chandra modules for general pairs (U, Γ) is developed in [DFO2].

An irreducible Harish-Chandra module M is said to be *extended* from $\mathbf{m} \in \operatorname{cfs}\Gamma$ if $M(\mathbf{m}) \neq 0$. A central problem in the theory of Harish-Chandra modules is to investigate the existence and uniqueness conditions for such an extension. In the case where the extension is unique, the irreducible Harish-Chandra modules are parametrized by some equivalence classes of the elements of $\operatorname{cfs}\Gamma$. It has been recently proved in [Ov] that for the case of Gelfand–Tsetlin modules over $\mathfrak{gl}_n$ the number of pairwise non-isomorphic irreducible modules extended from a given $\mathbf{m} \in \operatorname{cfs}\Gamma$ is nonzero and finite.

In this paper we begin a detailed study of Harish-Chandra modules over the Yangians. The *Yangian for* $\mathfrak{gl}_n$ is a unital associative algebra $\mathrm{Y}(\mathfrak{gl}_n)$ over $\Bbbk$ with countably many generators $t_{ij}^{(1)},\ t_{ij}^{(2)},\dots$ where $1 \leq i,j \leq n$, and the defining relations

$$(u-v)\,[t_{ij}(u), t_{kl}(v)] = t_{kj}(u)\,t_{il}(v) - t_{kj}(v)\,t_{il}(u), \tag{1.1}$$

where

$$t_{ij}(u) = \delta_{ij} + t_{ij}^{(1)} u^{-1} + t_{ij}^{(2)} u^{-2} + \cdots$$

and u, v are formal variables. This algebra originally appeared in the works on the *quantum inverse scattering method*; see e.g. Takhtajan–Faddeev [TF], Kulish–Sklyanin [KS]. The term "Yangian" and generalizations of $\mathrm{Y}(\mathfrak{gl}_n)$ to an arbitrary simple Lie algebra were introduced by Drinfeld [D1]. He then classified finite-dimensional irreducible modules over the Yangians in [D2] using earlier results of Tarasov [T1, T2]. An explicit construction of every such module over $\mathrm{Y}(\mathfrak{gl}_2)$ is given in those papers by Tarasov and also in the work by Chari and Pressley [CP]. Apart from this case, the structure of a general finite-dimensional irreducible representation of the Yangian remains unknown. In the case of $\mathrm{Y}(\mathfrak{gl}_n)$ a description of *generic* modules was given in [M1] via Gelfand–Tsetlin bases. A more general class of *tame* representations of $\mathrm{Y}(\mathfrak{gl}_n)$ was introduced and explicitly constructed by Nazarov and Tarasov [NT]. Another family of representations has been constructed in [M3] via tensor products of the so-called *evaluation modules*. An important role in these works is played by the *Drinfeld generators* [D2]

$$a_i(u), \quad i = 1, \dots, n, \qquad b_i(u), \ \ c_i(u), \quad i = 1, \dots, n-1$$

of the algebra $\mathrm{Y}(\mathfrak{gl}_n)$ which are defined as certain *quantum minors* of the matrix $T(u) = \big(t_{ij}(u)\big)$. The coefficients of the series $a_i(u)$, $i = 1, \dots, n$ form a commutative subalgebra of $\mathrm{Y}(\mathfrak{gl}_n)$ which can be regarded as an analogue of the Gelfand–Tsetlin subalgebra of $\mathrm{U}(\mathfrak{gl}_n)$. We shall consider the Harish-Chandra modules for $\mathrm{Y}(\mathfrak{gl}_n)$ with respect to this particular subalgebra. So, the Harish-Chandra modules for $\mathrm{Y}(\mathfrak{gl}_n)$ are natural analogues of the Gelfand–Tsetlin modules for $\mathfrak{gl}_n$ [DFO1]. Note also that the tame modules over $\mathrm{Y}(\mathfrak{gl}_n)$ [NT] is a particular case of Harish-Chandra modules.

In this paper we are concerned with Harish-Chandra modules for the Yangian $\mathrm{Y}(\mathfrak{gl}_2)$. Recall that every irreducible finite-dimensional $\mathrm{Y}(\mathfrak{gl}_2)$-module contains a unique vector ξ annihilated by $t_{12}(u)$ and which is an eigenvector for the Drinfeld generators $a_1(u)$ and $a_2(u)$ defined by

$$a_1(u) = t_{11}(u)\, t_{22}(u-1) - t_{21}(u)\, t_{12}(u-1), \qquad a_2(u) = t_{22}(u); \tag{1.2}$$

see [T1, T2] and [CP]. Moreover, there exists an automorphism $t_{ij}(u) \mapsto c(u)\, t_{ij}(u)$ of $\mathrm{Y}(\mathfrak{gl}_2)$, where $c(u) \in 1 + u^{-1}\, \Bbbk[[u^{-1}]]$, such that the eigenvalues of ξ become polynomials in u^{-1} under the corresponding twisted action of the Yangian. This prompts the introduction of the class of *Harish-Chandra polynomial* modules over $\mathrm{Y}(\mathfrak{gl}_2)$, i.e., such Harish-Chandra modules where the operators $a_1(u)$ and $a_2(u)$ are polynomials. More precisely, due to (1.2), it is natural to require that for some positive integer p the polynomials $a_1(u)$ and $a_2(u)$ have degrees $2p$ and p, respectively. Note that $a_1(u)$ is the *quantum determinant* of the matrix $T(u)$ [IK], [KS]. Its coefficients are algebraically independent generators of the center of $\mathrm{Y}(\mathfrak{gl}_2)$.

We can interpret the definition of Harish-Chandra polynomial modules using the algebra $\mathrm{Y}_p(\mathfrak{gl}_2)$ called the *Yangian of level* p; see Cherednik [C1, C2]. It is defined as the quotient of $\mathrm{Y}(\mathfrak{gl}_2)$ by the ideal generated by the elements $t_{ij}^{(r)}$ with $r \geq p+1$. A Harish-Chandra polynomial module over $\mathrm{Y}(\mathfrak{gl}_2)$ is just a Harish-Chandra module over $\mathrm{Y}_p(\mathfrak{gl}_2)$ for some positive integer p. In what follows we shall consider Harish-Chandra modules over $\mathrm{Y}_p(\mathfrak{gl}_2)$ with respect to the commutative subalgebra Γ generated by the coefficients of the polynomials $a_1(u)$ and $a_2(u)$.

Let us now describe our main results. First, we prove that $\mathrm{Y}_p(\mathfrak{gl}_2)$ is free as a left (right) Γ-module (Theorem 3.4). This is an analogue of the well-known Kostant theorem [K]. Each character of Γ can therefore be extended to an irreducible $\mathrm{Y}_p(\mathfrak{gl}_2)$-module. An important role in our study is played by certain universal Harish-Chandra modules over $\mathrm{Y}_p(\mathfrak{gl}_2)$ (see Section 4) such that every irreducible module in $\mathbb{H}(\mathrm{Y}_p(\mathfrak{gl}_2), \Gamma)$ is a quotient of the corresponding universal module.

Further, we show that Γ is a Harish-Chandra subalgebra (Theorem 5.3) in the sense of [DFO2] which allows us to apply the general theory of [DFO2] to the study of Harish-Chandra modules for $\mathrm{Y}_p(\mathfrak{gl}_2)$. In particular, it provides an equivalence between the category $\mathbb{H}(\mathrm{Y}_p(\mathfrak{gl}_2), \Gamma)$ of Harish-Chandra modules and the category of finitely generated modules over a certain category $\mathcal{A}$ whose objects are the maximal ideals of Γ. We then use this to prove that the number of pairwise non-isomorphic extensions of a character of Γ to an irreducible $\mathrm{Y}_p(\mathfrak{gl}_2)$-module is finite (Theorem 7.2). The full subcategory $\mathbb{H}W(\mathrm{Y}_p(\mathfrak{gl}_2), \Gamma)$ of $\mathbb{H}(\mathrm{Y}_p(\mathfrak{gl}_2), \Gamma)$ which consists of modules with diagonalizable action of Γ turns out to be equivalent to the category of finitely generated modules over a certain quotient category of $\mathcal{A}$ (Section 2.1). In Section 8 we study a full subcategory in $\mathbb{H}W(\mathrm{Y}_p(\mathfrak{gl}_2), \Gamma)$ of generic modules, this imposes a certain condition on the eigenvalues of $a_2(u)$ while those of $a_1(u)$

are arbitrary. In particular, we give a complete description of irreducible modules (Theorem 8.5) and indecomposable modules in tame blocks of this category (Theorem 8.9).

2. Preliminaries

2.1. **Harish-Chandra subalgebras.** In this paper we shall only consider the pairs (U,Γ) where the subalgebra Γ of U is commutative. In this case $\operatorname{cfs}\Gamma$ coincides with the set $\operatorname{Specm}\Gamma$ of all maximal ideals in Γ. We endow this set with the Zariski topology.

We let U-mod denote the category of finitely generated left modules over an associative algebra U. The Harish-Chandra modules for the pair (U,Γ) form a full abelian subcategory in U-mod which we denote by $\mathbb{H}(U,\Gamma)$. A Harish-Chandra module M is called *weight* if the following condition holds: for all $\mathbf{m}\in\operatorname{Specm}\Gamma$ and all $x\in M(\mathbf{m})$ one has $\mathbf{m}\,x=0$. The full subcategory of $\mathbb{H}(U,\Gamma)$ consisting of weight modules will be denoted $\mathbb{H}W(U,\Gamma)$. The *support* of a Harish-Chandra module M is the subset $\operatorname{Supp}M\subseteq\operatorname{Specm}\Gamma$ which consists of those $\mathbf{m}$ which have the property $M(\mathbf{m})\neq 0$. If for a given $\mathbf{m}$ there exists an irreducible Harish-Chandra module M with $M(\mathbf{m})\neq 0$, then we say that $\mathbf{m}$ *extends* to M.

A commutative subalgebra $\Gamma\subseteq U$ is called a *Harish-Chandra subalgebra* of U [DFO2] if for any $a\in U$ the Γ-bimodule $\Gamma a\Gamma$ is finitely generated both as left and as right Γ-module. The property of Γ to be a Harish-Chandra subalgebra is important for the effective study of the category $\mathbb{H}(U,\Gamma)$. In this case, for any finite-dimensional Γ-module X the module $U\otimes_\Gamma X$ is a Harish-Chandra module. For any $a\in U$ set

$$X_a = \{(\mathbf{m},\mathbf{n})\in\operatorname{Specm}\Gamma\times\operatorname{Specm}\Gamma \mid \Gamma/\mathbf{n}\quad\text{is a subquotient of}\quad \Gamma a\Gamma/\Gamma a\mathbf{m}\}.$$

Equivalently, $(\mathbf{m},\mathbf{n})\in X_a$ if and only if $(\Gamma/\mathbf{n})\otimes_\Gamma\Gamma a\Gamma\otimes_\Gamma(\Gamma/\mathbf{m})\neq 0$. Denote by Δ the minimal equivalence on $\operatorname{Specm}\Gamma$ containing all X_a, $a\in U$ and by $\Delta(A,\Gamma)$ the set of the Δ-equivalence classes on $\operatorname{Specm}\Gamma$. Then for any $a\in U$ and $\mathbf{m}\in\operatorname{Specm}\Gamma$ we have

$$(2.1)\qquad aM(\mathbf{m})\subseteq\sum_{(\mathbf{m},\mathbf{n})\in X_a}M(\mathbf{n}),\qquad \mathbb{H}(U,\Gamma)=\bigoplus_{D\in\Delta(U,\Gamma)}\mathbb{H}(U,\Gamma,D),$$

where the subcategory $\mathbb{H}(U,\Gamma,D)$ consists of the Harish-Chandra modules M such that $\operatorname{Supp}M\subseteq D$. Define a category $\mathcal{A}=\mathcal{A}_{U,\Gamma}$ with the set of objects $\operatorname{Ob}\mathcal{A}=\operatorname{Specm}\Gamma$ and with the space of morphisms $\mathcal{A}(\mathbf{m},\mathbf{n})$ from $\mathbf{m}$ to $\mathbf{n}$, where

$$(2.2)\quad \mathcal{A}(\mathbf{m},\mathbf{n}) = \lim_{\substack{\longleftarrow\\ n,m}} U/(\mathbf{n}^nU+U\mathbf{m}^m)$$
$$\text{(equivalently, } \lim_{\substack{\longleftarrow\\ n,m}}\Gamma/\mathbf{n}^n\otimes_\Gamma U\otimes_\Gamma\Gamma/\mathbf{m}^m).$$

Consider the completion $\Gamma_{\mathbf{m}}=\lim\limits_{\longleftarrow n}\Gamma/\mathbf{m}^n$ of Γ by an ideal $\mathbf{m}\in\operatorname{Specm}\Gamma$. Then the space $\mathcal{A}(\mathbf{m},\mathbf{n})$ has a natural structure of $\Gamma_{\mathbf{n}}-\Gamma_{\mathbf{m}}$-bimodule. We have the decomposition

$$\mathcal{A}=\bigoplus_{D\in\Delta(U,\Gamma)}\mathcal{A}(D),$$

where $\mathcal{A}(D)$ is the restriction of $\mathcal{A}$ on D. The category $\mathcal{A}$ is endowed with the topology of the inverse limit while the category of $\Bbbk$-vector spaces ($\Bbbk$-mod) is endowed with the discrete topology. Consider the category $\mathcal{A}$-mod_d of continuous functors $M : \mathcal{A} \to \Bbbk\text{-mod}$. We call them *discrete modules* following the terminology of [DFO2, Section 1.5]. For any discrete $\mathcal{A}$-module N define a Harish-Chandra U-module

$$\mathbb{F}(N) = \bigoplus_{\mathbf{m} \in \operatorname{Specm} \Gamma} N(\mathbf{m}).$$

Furthermore, for $x \in N(\mathbf{m})$ and $a \in U$ set

$$ax = \sum_{\mathbf{n} \in \operatorname{Specm} \Gamma} a_{\mathbf{n}} x$$

where $a_{\mathbf{n}}$ is the image of a in $\mathcal{A}(\mathbf{m}, \mathbf{n})$. For any morphism $f : M \to N$ in the category $\mathcal{A}\text{-mod}_d$ set

$$\mathbb{F}(f) = \bigoplus_{\mathbf{m} \in \operatorname{Specm} \Gamma} f(\mathbf{m}).$$

We thus have a functor $\mathbb{F} : \mathcal{A}\text{-mod}_d \to \mathbb{H}(U, \Gamma)$.

Proposition 2.1 ([DFO2, Theorem 17])**.** *The functor* $\mathbb{F}$ *is an equivalence.*

We will identify a discrete $\mathcal{A}$-module N with the corresponding Harish-Chandra module $\mathbb{F}(N)$. For $\mathbf{m} \in \operatorname{Specm} \Gamma$ denote by $\hat{\mathbf{m}}$ the completion of $\mathbf{m}$. Clearly, $\hat{\mathbf{m}} \subseteq \Gamma_{\mathbf{m}}$. Consider the two-sided ideal $I \subseteq \mathcal{A}$ generated by the completions $\hat{\mathbf{m}}$ for all $\mathbf{m} \in \operatorname{Specm} \Gamma$ and set $\mathcal{A}_W = \mathcal{A}/I$. Proposition 2.1 implies the following.

Corollary 2.2. *The categories* $\mathbb{H}W(U, \Gamma)$ *and* $\mathcal{A}_W$*-mod are equivalent.*

The subalgebra Γ is called *big in* $\mathbf{m} \in \operatorname{Specm} \Gamma$ if $\mathcal{A}(\mathbf{m}, \mathbf{m})$ is finitely generated as a left (or, equivalently, right) $\Gamma_{\mathbf{m}}$-module.

Lemma 2.3 ([DFO2, Corollary 19])**.** *If* Γ *is big in* $\mathbf{m} \in \operatorname{Specm} \Gamma$*, then there exist finitely many non-isomorphic irreducible Harish-Chandra* U*-modules* M *such that* $M(\mathbf{m}) \neq 0$*. For any such module* $\dim M(\mathbf{m}) < \infty$*.*

2.2. **Special PBW algebras.** Let U be an associative algebra over $\Bbbk$ endowed with an increasing filtration $\{U_i\}_{i \in \mathbb{Z}}$, $U_{-1} = \{0\}$, $U_0 = \Bbbk$, $U_i U_j \subseteq U_{i+j}$. For $u \in U_i \setminus U_{i-1}$ set $\deg u = i$. Let $\overline{U} = \operatorname{gr} U$ be the associated graded algebra

$$\overline{U} = \bigoplus_{i=0}^{\infty} U_i / U_{i-1}.$$

For $u \in U$ denote by $\overline{u}$ its image in $\overline{U}$ and for a subset $S \subseteq U$ set $\overline{S} = \{\overline{s} \mid s \in S\} \subseteq \overline{U}$. The algebra U is called a *special* PBW algebra if any element of U can be written uniquely as a linear combination of ordered monomials in some fixed generators of U and if $\overline{U}$ is a polynomial algebra. Such algebras were introduced in [FO].

Let $\Lambda = \Bbbk[X_1, \dots, X_n]$ be a polynomial algebra. A sequence $g_1, \dots, g_t \in \Lambda$ is called *regular* (in Λ) if the class of g_i in $\Lambda/(g_1, \dots, g_{i-1})$ is non-invertible and is not a zero divisor for any $i = 1, \dots, t$.

The next proposition contains some simple properties of regular sequences which will be used in the sequel.

Proposition 2.4. (1) *A sequence of the form* $X_1, \dots, X_r, G_1, \dots G_t$*, where* $G_1, \dots, G_t$ *are homogeneous elements of* Λ*, is regular in* Λ *if and only if the sequence* $g_1, \dots, g_t$ *is regular in* $\Bbbk[X_{r+1}, \dots, X_n]$*, where* $g_i(X_{r+1}, \dots, X_n) = G_i(0, \dots, 0, X_{r+1}, \dots, X_n)$*.*

(2) *A sequence* $g_1 g_1', g_2, \ldots, g_t$ $g_1, g_1' \notin \Bbbk$, *of homogeneous elements of* Λ *is regular if and only if both sequences* $g_1, g_2, \ldots, g_t$ *and* $g_1', g_2, \ldots, g_t$ *are regular.*

The following analogue of Kostant theorem [K] is valid for special PBW algebras.

Theorem 2.5 ([FO])**.** *Let* U *be a special PBW algebra and let* $g_1, \ldots, g_t \in U$ *be mutually commuting elements such that* $\overline{g}_1, \ldots, \overline{g}_t$ *is a regular sequence in* $\overline{U}$, *and let* $\Gamma = \Bbbk[g_1, \ldots, g_t]$. *Then* U *is a free left (right)* Γ*-module. Moreover,* Γ *is a direct summand of* U.

3. Freeness of $\mathrm{Y}_p(\mathfrak{gl}_2)$ over Γ

Let p be a positive integer. The *level* p *Yangian* $\mathrm{Y}_p(\mathfrak{gl}_2)$ for the Lie algebra $\mathfrak{gl}_2$ [C2] can be defined as the algebra over $\Bbbk$ with generators $t_{ij}^{(1)}, \ldots, t_{ij}^{(p)}$, $i, j = 1, 2$, subject to the relations

$$[T_{ij}(u), T_{kl}(v)] = \frac{1}{u-v}(T_{kj}(u)\,T_{il}(v) - T_{kj}(v)\,T_{il}(u)), \tag{3.1}$$

where u, v are formal variables and

$$T_{ij}(u) = \delta_{ij}\, u^p + \sum_{k=1}^{p} t_{ij}^{(k)}\, u^{p-k} \in \mathrm{Y}_p(\mathfrak{gl}_2)[u].$$

Explicitly, (3.1) reads

$$[t_{ij}^{(r)}, t_{kl}^{(s)}] = \sum_{a=1}^{\min(r,s)} \big(t_{kj}^{(a-1)} t_{il}^{(r+s-a)} - t_{kj}^{(r+s-a)} t_{il}^{(a-1)}\big),$$

where $t_{ij}^{(0)} = \delta_{ij}$ and $t_{ij}^{(r)} = 0$ for $r \geq p+1$. Note that the level 1 Yangian $\mathrm{Y}_1(\mathfrak{gl}_2)$ coincides with the universal enveloping algebra $\mathrm{U}(\mathfrak{gl}_2)$. Set

$$\deg t_{ij}^{(k)} = k \qquad \text{for} \quad i, j = 1, 2 \quad \text{and} \quad k = 1, \ldots, p.$$

This defines a natural filtration on the Yangian $\mathrm{Y}_p(\mathfrak{gl}_2)$. The corresponding graded algebra will be denoted by $\overline{\mathrm{Y}}_p(\mathfrak{gl}_2)$. We have the following analogue of the Poincaré–Birkhoff–Witt theorem for the algebra $\mathrm{Y}_p(\mathfrak{gl}_2)$.

Proposition 3.1 ([C2]; see also [M2])**.** *Given an arbitrary linear ordering on the set of generators* $t_{ij}^{(k)}$, *any element of the algebra* $\mathrm{Y}_p(\mathfrak{gl}_2)$ *is uniquely written as a linear combination of ordered monomials in these generators. Moreover, the algebra* $\overline{\mathrm{Y}}_p(\mathfrak{gl}_2)$ *is a polynomial algebra in generators* $\overline{t}_{ij}^{(k)}$.

Proposition 3.1 implies that $\mathrm{Y}_p(\mathfrak{gl}_2)$ is a special PBW algebra. Denote by $D(u)$ the *quantum determinant*

$$\begin{aligned} D(u) &= T_{11}(u)\,T_{22}(u-1) - T_{21}(u)\,T_{12}(u-1) \\ &= T_{11}(u-1)\,T_{22}(u) - T_{12}(u-1)\,T_{21}(u) \\ &= T_{22}(u)\,T_{11}(u-1) - T_{12}(u)\,T_{21}(u-1) \\ &= T_{22}(u-1)\,T_{11}(u) - T_{21}(u-1)\,T_{12}(u). \end{aligned} \tag{3.2}$$

Clearly, $D(u)$ is a monic polynomial in u of degree $2p$,

$$D(u) = u^{2p} + d_1\, u^{2p-1} + \cdots + d_{2p}, \qquad d_i \in \mathrm{Y}_p(\mathfrak{gl}_2). \tag{3.3}$$

It was shown in [C1, C2] (see also [M2] for a different proof) that the coefficients $d_1, \dots, d_{2p}$ are algebraically independent generators of the center of the algebra $\mathrm{Y}_p(\mathfrak{gl}_2)$. Denote by Γ the subalgebra of $\mathrm{Y}_p(\mathfrak{gl}_2)$ generated by the coefficients of $D(u)$ and by the elements $t_{22}^{(k)}$, $k = 1, \dots, p$. This algebra is obviously commutative. We will show later (Corollary 5.3) that Γ is a Harish-Chandra subalgebra in $\mathrm{Y}_p(\mathfrak{gl}_2)$.

Lemma 3.2. *The sequence* $\overline{t}_{22}^{(1)}, \dots, \overline{t}_{22}^{(p)}, \overline{d}_1, \dots, \overline{d}_{2p}$ *of the images of the generators of* Γ *is regular in* $\overline{\mathrm{Y}}_p(\mathfrak{gl}_2)$.

Proof. Let us set

$$t_i = \overline{t}_{11}^{(i)} + \overline{t}_{22}^{(i)}, \quad i = 1, \dots, p \qquad \text{and} \qquad \Delta_{i,j} = \overline{t}_{11}^{(i)} \overline{t}_{22}^{(j)} - \overline{t}_{21}^{(i)} \overline{t}_{12}^{(j)}, \quad i, j = 1, \dots, p.$$

It follows from (3.3) that

$$\overline{D}(u) = u^{2p} + \sum_{i=1}^{2p} \overline{d}_i \, u^{2p-i},$$

with

$$\overline{d}_i = t_i + \sum_{j=1}^{i-1} \Delta_{j,i-j} \qquad \text{for} \quad i = 1, \dots, p \qquad \text{and}$$

$$\overline{d}_i = \sum_{j=i-p}^{p} \Delta_{j,i-j} \qquad \text{for} \quad i = p+1, \dots, 2p.$$

Hence we need to show that the sequence

$$\overline{t}_{22}^{(1)}, \dots, \overline{t}_{22}^{(p)}, t_1, t_2 + \Delta_{11}, \dots, t_p + \sum_{i=1}^{p-1} \Delta_{i,p-i}, \sum_{i=1}^{p} \Delta_{i,p+1-i}, \dots, \Delta_{pp}$$

is regular. We will denote by ∇_i the result of the substitution $\overline{t}_{22}^{(1)} = \dots = \overline{t}_{22}^{(p)} = 0$ in $\overline{d}_i$, $i = 1, \dots, 2p$. By Proposition 2.4 (1), we only need to show the regularity of the sequence

$$\nabla_1, \dots, \nabla_{2p}.$$

Consider the automorphism ϕ of $\overline{\mathrm{Y}}_p(\mathfrak{gl}_2)/I$ given by

$$\overline{t}_{11}^{(i)} \mapsto \nabla_i, \qquad \overline{t}_{21}^{(i)} \mapsto \overline{t}_{21}^{(i)}, \qquad \overline{t}_{12}^{(i)} \mapsto \overline{t}_{12}^{(i)}, \qquad \text{for} \quad i = 1, \dots, p,$$

where I is the ideal generated by $\overline{t}_{22}^{(1)}, \dots, \overline{t}_{22}^{(p)}$. Note that the elements ∇_i with $i = p+1, \dots, 2p$ are stable under ϕ. Since the regularity of a sequence is preserved by automorphisms, it is sufficient to demonstrate the regularity of the sequence

$$\overline{t}_{11}^{(1)}, \dots, \overline{t}_{11}^{(p)}, \nabla_{p+1}, \dots, \nabla_{2p}.$$

Since the elements ∇_i do not depend on the $\overline{t}_{11}^{(k)}$, Proposition 2.4 (1) implies that this is equivalent to the regularity of the sequence $\nabla_{p+1}, \dots, \nabla_{2p}$. For each pair of indices $k, l \in \{1, \dots, p\}$ and any index $1 \le a \le \max\{k, l\}$, consider the sequence of

a elements which occupy the rows of the table $s(k,l,a)$ below

$$\begin{pmatrix} \bar{t}_{21}^{(k)}\bar{t}_{12}^{(l)} \\ \bar{t}_{21}^{(k-1)}\bar{t}_{12}^{(l)} + \bar{t}_{21}^{(k)}\bar{t}_{12}^{(l-1)} \\ \bar{t}_{21}^{(k-2)}\bar{t}_{12}^{(l)} + \bar{t}_{21}^{(k-1)}\bar{t}_{12}^{(l-1)} + \bar{t}_{21}^{(k)}\bar{t}_{12}^{(l-2)} \\ \vdots \\ \bar{t}_{21}^{(k-a+1)}\bar{t}_{12}^{(l)} + \bar{t}_{21}^{(k-a+2)}\bar{t}_{12}^{(l+1)} + \cdots + \bar{t}_{21}^{(k)}\bar{t}_{12}^{(l-a+1)} \end{pmatrix}.$$

Note that when $k = l = a = p$ the rows of the table are exactly the elements ∇_i, $i = p+1, \dots, 2p$. We will show by induction on a that the sequence of rows of $s(k,l,a)$ is regular. Note that $s(k,l,1)$ consists of the single element $\bar{t}_{21}^{(k)}\bar{t}_{12}^{(l)}$ and is obviously regular. Now let $a > 1$. Consider the following two tables which we denote by $s'(k,l,a)$ and $s''(k,l,a)$, respectively.

$$\begin{pmatrix} \bar{t}_{21}^{(k)} \\ \bar{t}_{21}^{(k-1)}\bar{t}_{12}^{(l)} + \bar{t}_{21}^{(k)}\bar{t}_{12}^{(l-1)} \\ \vdots \\ \bar{t}_{21}^{(k-a+1)}\bar{t}_{12}^{(l)} + \cdots + \bar{t}_{21}^{(k)}\bar{t}_{12}^{(l-a+1)} \end{pmatrix}, \qquad \begin{pmatrix} \bar{t}_{12}^{(l)} \\ \bar{t}_{21}^{(k-1)}\bar{t}_{12}^{(l)} + \bar{t}_{21}^{(k)}\bar{t}_{12}^{(l-1)} \\ \vdots \\ \bar{t}_{21}^{(k-a+1)}\bar{t}_{12}^{(l)} + \cdots + \bar{t}_{21}^{(k)}\bar{t}_{12}^{(l-a+1)} \end{pmatrix}.$$

Due to Proposition 2.4 (2), it is sufficient to verify the regularity of both $s'(k,l,a)$ and $s''(k,l,a)$. Using again Proposition 2.4 (1), substitute $\bar{t}_{21}^{(k)} = 0$ in $s'(k,l,a)$ and $\bar{t}_{12}^{(k)} = 0$ in $s''(k,l,a)$. It is easy to see that after this substitution we obtain the tables $s(k-1,l,a-1)$ and $s(k,l-1,a-1)$, respectively. By the induction hypothesis, both of them are regular and so is $s(k,l,a)$. In particular, the sequence $s(p,p,p)$ is regular which completes the proof. □

Using the regularity of the sequence $\bar{t}_{22}^{(1)}, \dots, \bar{t}_{22}^{(p)}, \bar{d}_1, \dots, \bar{d}_{2p}$ we immediately obtain the following.

Corollary 3.3. *The generators $t_{22}^{(1)}, \dots, t_{22}^{(p)}, d_1, \dots, d_{2p}$ of Γ are algebraically independent.*

Combining Lemma 3.2 with Theorem 2.5 we obtain our first main result.

Theorem 3.4. (1) *$\mathrm{Y}_p(\mathfrak{gl}_2)$ is free as a left (right) module over Γ. Moreover, Γ is a direct summand of $\mathrm{Y}_p(\mathfrak{gl}_2)$.*

(2) *Any $\mathbf{m} \in \operatorname{Specm}\Gamma$ extends to an irreducible $\mathrm{Y}_p(\mathfrak{gl}_2)$-module.*

For a subset $P \subseteq \mathrm{Y}_p(\mathfrak{gl}_2)$ denote by $\mathbb{D}(P)$ the set of all $x \in \mathrm{Y}_p(\mathfrak{gl}_2)$ such that there exists $z \in \Gamma$, $z \neq 0$ for which $zx \in P$.

Corollary 3.5. *Let $P \subseteq \mathrm{Y}_p(\mathfrak{gl}_2)$ be a finitely generated left Γ-module, then $\mathbb{D}(P)$ is a finitely generated left Γ-module.*

Proof. Since Γ is a domain, then $\mathbb{D}(P)$ is a Γ-submodule in $\mathrm{Y}_p(\mathfrak{gl}_2)$. Using the fact that $\mathrm{Y}_p(\mathfrak{gl}_2)$ is a free left Γ-module we conclude that $\mathrm{Y}_p(\mathfrak{gl}_2) \simeq F_P \oplus F$ where F_P and F are free left Γ-modules, F_P has a finite rank and $P \subseteq F_P$. Then $\mathbb{D}(P) \subseteq F_P$ and hence it is finitely generated as a submodule of a finitely generated module over a noetherian ring. □

4. Harish-Chandra modules for $\mathfrak{gl}_2$ Yangians

In this section we introduce universal Harish-Chandra modules $M(\ell)$. We also describe their structure in an explicit form in the case of generic parameters ℓ.

Let L be a polynomial algebra in the variables $b_1, \dots, b_p, g_1, \dots g_{2p}$. Define a $\Bbbk$-homomorphism $\imath : \Gamma \to \mathrm{L}$ by

$$\imath(t_{22}^{(k)}) = \sigma_{k,p}(b_1, \dots, b_p), \qquad \imath(d_i) = \sigma_{i,2p}(g_1, \dots, g_{2p}), \tag{4.1}$$

where $\sigma_{i,j}$ is the i-th elementary symmetric polynomial in j variables. Due to Corollary 3.3, $\imath$ is injective. We will identify the elements of Γ with their images in L and treat them as polynomials in the variables $b_1, \dots, b_p, g_1, \dots g_{2p}$ invariant under the action of the group $S_p \times S_{2p}$. Set $\mathcal{L} = \operatorname{Specm} \mathrm{L}$. We will identify $\mathcal{L}$ with $\Bbbk^{3p}$. If

$$\beta = (\beta_1, \dots, \beta_p), \qquad \gamma = (\gamma_1, \dots, \gamma_{2p}) \qquad \text{and} \qquad \ell = (\beta_1, \dots, \beta_p, \gamma_1, \dots, \gamma_{2p}),$$

then we shall write $\ell = (\beta, \gamma)$. The monomorphism $\imath$ induces the epimorphism

$$\imath^* : \mathcal{L} \to \operatorname{Specm} \Gamma. \tag{4.2}$$

If $\ell \in \mathcal{L}$ and $\mathbf{m} = \imath^*(\ell)$, then $D(\ell)$ will denote the equivalence class of $\mathbf{m}$ in $\Delta(\mathrm{Y}_p(\mathfrak{gl}_2), \Gamma)$; see Section 2.1.

Let $\mathcal{P}_0 \subseteq \mathcal{L}$, $\mathcal{P}_0 \simeq \mathbb{Z}^p$, be the lattice generated by the elements $\delta_i \in \Bbbk^{3p}$ for $i = 1, \dots, p$, where δ_i denotes the $3p$-tuple with 1 on the i-th position and zeros elsewhere. Then $\mathcal{P}_0$ acts on $\mathcal{L}$ by shifts $\delta_i(\ell) := \ell + \delta_i$. Furthermore, the group $S_p \times S_{2p}$ acts on $\mathcal{L}$ by permutations. Thus the semidirect product $\mathbb{W}$ of the groups $S_p \times S_{2p}$ and $\mathcal{P}_0$ acts on $\mathcal{L}$ and L. Denote by S a multiplicative set in L generated by the elements $b_i - b_j - m$ for all $i \neq j$ and all $m \in \mathbb{Z}$ and by $\mathbb{L}$ the localization of L by S. Note that S is invariant under the action of $\mathbb{W}$ and hence $\mathbb{W}$ acts on $\mathbb{L}$ as well.

For arbitrary $3p$-tuple $\ell = (\beta, \gamma) \in \mathcal{L}$ set

$$\beta(u) = (u + \beta_1) \cdots (u + \beta_p), \qquad \gamma(u) = (u + \gamma_1) \cdots (u + \gamma_{2p}).$$

Let I_ℓ be the left ideal of $\mathrm{Y}_p(\mathfrak{gl}_2)$ generated by the coefficients of the polynomials $T_{22}(u) - \beta(u)$ and $D(u) - \gamma(u)$. Define the corresponding quotient module over $\mathrm{Y}_p(\mathfrak{gl}_2)$ by

$$M(\ell) = \mathrm{Y}_p(\mathfrak{gl}_2)/I_\ell. \tag{4.3}$$

We shall call it the *universal module* (corresponding to ℓ). It follows from Theorem 3.4 that I_ℓ is a proper ideal of $\mathrm{Y}_p(\mathfrak{gl}_2)$ and so $M(\ell)$ is a nontrivial module. It is clear that if V is an arbitrary Harish-Chandra $\mathrm{Y}_p(\mathfrak{gl}_2)$-module generated by a nonzero $\eta \in V$ such that $D(u)\eta = \gamma(u)\eta$ and $T_{22}(u)\eta = \beta(u)\eta$, then V is a homomorphic image of $M(\ell)$.

Set $\mathcal{P}_1 = \operatorname{Specm} \mathbb{L} \subseteq \mathcal{L}$, i.e. $\mathcal{P}_1$ consists of *generic* $3p$-tuples $\ell = (\beta, \gamma)$ such that

$$\beta_i - \beta_j \notin \mathbb{Z} \qquad \text{for all} \quad i \neq j. \tag{4.4}$$

If $\ell \in \mathcal{P}_1$, then the modules from the category $\mathbb{H}(\mathrm{Y}_p(\mathfrak{gl}_2), \Gamma, D(\ell))$ are called *generic* Harish-Chandra modules.

4.1. **Weight modules.** For $\ell = (\beta, \gamma) \in \mathcal{L}$ the category $\mathbb{H}W(\mathrm{Y}_p(\mathfrak{gl}_2), \Gamma, D(\ell))$ consists of finitely generated weight modules V with central character γ and with $\operatorname{Supp} V \subseteq D(\ell)$. We shall denote this category by R_ℓ for brevity. If $\ell \in \mathcal{P}_1$, then the modules from R_ℓ will be called *generic* weight modules.

A $\mathrm{Y}_p(\mathfrak{gl}_2)$-module V is an object of R_ℓ if V is a direct sum of its *weight* subspaces:

$$V = \bigoplus_{\ell \in \mathcal{L}} V_\ell, \qquad V_\ell = \{\eta \in V \mid T_{22}(u)\,\eta = \beta(u)\,\eta, \quad D(u)\,\eta = \gamma(u)\,\eta\}.$$

If $V \in R_\ell$, then we shall simply write V_β instead of V_ℓ and identify $\operatorname{Supp} V$ with the set of all β such that the subspace V_β is nonzero. The next lemma describes the action of the Yangian generators on the weight subspaces; cf. (2.1).

Lemma 4.1. *Let V be a generic weight $\mathrm{Y}_p(\mathfrak{gl}_2)$-module and let $\beta = (\beta_1, \ldots, \beta_p) \in \operatorname{Supp} V$. Then*

$$T_{21}(u)V_\beta \subseteq \sum_{i=1}^{p} V_{\beta+\delta_i} \qquad \text{and} \qquad T_{12}(u)V_\beta \subseteq \sum_{i=1}^{p} V_{\beta-\delta_i} \tag{4.5}$$

where $\beta \pm \delta_i = (\beta_1, \ldots, \beta_i \pm 1, \ldots, \beta_p)$.

Proof. First we show that $T_{21}(-\beta_i)V_\beta \subseteq V_{\beta+\delta_i}$ for all $i = 1, \ldots, p$. Since

$$T_{22}(u-1)T_{21}(u) = T_{21}(u-1)T_{22}(u)$$

we have

$$T_{22}(-\beta_i - 1)T_{21}(-\beta_i)\,\eta = T_{21}(-\beta_i - 1)T_{22}(-\beta_i)\,\eta = 0$$

for all $\eta \in V_\beta$. Also,

$$\begin{aligned} T_{22}(-\beta_j)T_{21}(-\beta_i)\,\eta = (\beta_i - \beta_j)^{-1}(T_{21}(-\beta_i)T_{22}(-\beta_j) - T_{21}(-\beta_j)T_{22}(-\beta_i))\,\eta \\ + T_{21}(-\beta_i)T_{22}(-\beta_j)\,\eta = 0 \end{aligned}$$

since $T_{22}(-\beta_k)\,\eta = 0$ for all $k = 1, \ldots, p$. Using the fact that $\beta_i - \beta_j \notin \mathbb{Z}$ we conclude that $T_{21}(-\beta_i)V_\beta \subseteq V_{\beta+\delta_i}$ for all $i = 1, \ldots, p$. Since $T_{21}(u)$ is a polynomial of degree $p - 1$ in u and $\beta_i \neq \beta_j$ if $i \neq j$, we thus get the first containment of (4.5). The second is verified in the same way with the use of the identity $T_{22}(u)T_{12}(u-1) = T_{12}(u)T_{22}(u-1)$. □

Corollary 4.2. *If V is indecomposable generic weight module over $\mathrm{Y}_p(\mathfrak{gl}_2)$ and $\beta \in \operatorname{Supp} V$, then $\operatorname{Supp} V \subseteq \beta + \mathbb{Z}^p$.* □

Lemma 4.3. *If V is a generic weight $\mathrm{Y}_p(\mathfrak{gl}_2)$-module with the central character γ, then for any $\beta = (\beta_1, \ldots, \beta_p) \in \operatorname{Supp} V$ and any $\eta \in V_\beta$, we have*

$$T_{12}(-\beta_r)T_{21}(-\beta_s)\,\eta = T_{21}(-\beta_s)T_{12}(-\beta_r)\,\eta,$$

if $s \neq r$, and

$$\begin{aligned} T_{12}(-\beta_i - 1)T_{21}(-\beta_i)\,\eta &= -\gamma(-\beta_i)\,\eta, \\ T_{21}(-\beta_i + 1)T_{12}(-\beta_i)\,\eta &= -\gamma(-\beta_i + 1)\,\eta. \end{aligned}$$

Proof. The first equality follows from the defining relations (1.1). The two remaining follow from (3.2). □

The following corollary is immediate from Lemma 4.3.

Corollary 4.4. *Let V be a generic weight $\mathrm{Y}_p(\mathfrak{gl}_2)$-module with the central character γ and let $\beta = (\beta_1, \ldots, \beta_p) \in \operatorname{Supp} V$.*

(1) *If* $\gamma(-\beta_i) \neq 0$ *then* $\operatorname{Ker} T_{21}(-\beta_i) \cap V_\beta = 0$.
(2) *If* $\gamma(-\beta_i + 1) \neq 0$ *then* $\operatorname{Ker} T_{12}(-\beta_i) \cap V_\beta = 0$.
(3) *If* V *is indecomposable and* $\gamma(-\beta_i + k) \neq 0$ *for all* $k \in \mathbb{Z}$ *then*

$$\operatorname{Ker} T_{21}(-\psi_i) \cap V_\psi = \operatorname{Ker} T_{12}(-\psi_i) \cap V_\psi = 0$$

for all $\psi = (\psi_1, \dots, \psi_p) \in \operatorname{Supp} V$. □

Since the universal module $M(\ell)$ is nontrivial, the image of 1 in $M(\ell)$ is nonzero. We shall denote this image by ξ. Assume that β satisfies the genericity condition (4.4). For any $(k) = (k_1, \dots, k_p) \in \mathbb{Z}^p$ define the corresponding vector of the module $M(\ell)$ by

$$\begin{aligned} \xi^{(k)} = &\prod_{i,\ k_i>0} T_{21}(-\beta_i - k_i + 1) \cdots T_{21}(-\beta_i - 1)\, T_{21}(-\beta_i) \\ &\times \prod_{i,\ k_i<0} T_{12}(-\beta_i - k_i - 1) \cdots T_{12}(-\beta_i + 1)\, T_{12}(-\beta_i)\, \xi. \end{aligned} \tag{4.6}$$

Theorem 4.5. *The vectors* $\xi^{(k)}, (k) \in \mathbb{Z}^p$ *form a basis of* $M(\ell)$. *Moreover, we have the formulas*

$$T_{22}(u)\, \xi^{(k)} = \prod_{i=1}^{p} (u + \beta_i + k_i)\, \xi^{(k)}, \tag{4.7}$$

(4.8)

$$T_{21}(u)\, \xi^{(k)} = \sum_{i=1}^{p} A_i(k)\, \frac{(u + \beta_1 + k_1) \cdots \wedge_i \cdots (u + \beta_p + k_p)}{(\beta_1 - \beta_i + k_1 - k_i) \cdots \wedge_i \cdots (\beta_p - \beta_i + k_p - k_i)}\, \xi^{(k+\delta_i)},$$

$$T_{12}(u)\, \xi^{(k)} = \sum_{i=1}^{p} B_i(k)\, \frac{(u + \beta_1 + k_1) \cdots \wedge_i \cdots (u + \beta_p + k_p)}{(\beta_1 - \beta_i + k_1 - k_i) \cdots \wedge_i \cdots (\beta_p - \beta_i + k_p - k_i)}\, \xi^{(k-\delta_i)},$$

where the symbol $\wedge_i$ *indicates that the* i*-th factor in the product is skipped,*

$$A_i(k) = \begin{cases} 1 & \text{if} \quad k_i \geq 0, \\ -\gamma(-\beta_i - k_i) & \text{if} \quad k_i < 0 \end{cases}$$

and

$$B_i(k) = \begin{cases} -\gamma(-\beta_i - k_i + 1) & \text{if} \quad k_i > 0, \\ 1 & \text{if} \quad k_i \leq 0. \end{cases}$$

The action of $T_{11}(u)$ *is found from the relation*

$$\Big(T_{11}(u)\, T_{22}(u-1) - T_{21}(u)\, T_{12}(u-1)\Big)\, \xi^{(k)} = \gamma(u)\, \xi^{(k)}. \tag{4.9}$$

Proof. We start by proving the formulas for the action of the generators of $\mathrm{Y}_p(\mathfrak{gl}_2)$. Formula (4.7) follows by induction with the use of the relations

$$T_{22}(u)\, T_{21}(v) = \frac{u - v + 1}{u - v}\, T_{21}(v)\, T_{22}(u) - \frac{1}{u - v}\, T_{21}(u)\, T_{22}(v) \tag{4.10}$$

and

$$T_{22}(u)\, T_{12}(v) = \frac{u - v - 1}{u - v}\, T_{12}(v)\, T_{22}(u) + \frac{1}{u - v}\, T_{12}(u)\, T_{22}(v) \tag{4.11}$$

implied by (3.1). By Lemma 4.3 we have: if $k_i > 0$, then

$$T_{21}(-\beta_i - k_i)\,\xi^{(k)} = \xi^{(k+\delta_i)},\qquad T_{12}(-\beta_i - k_i)\,\xi^{(k)} = -\gamma(-\beta_i - k_i + 1)\,\xi^{(k-\delta_i)}; \tag{4.12}$$

if $k_i < 0$, then

$$T_{12}(-\beta_i - k_i)\,\xi^{(k)} = \xi^{(k-\delta_i)},\qquad T_{21}(-\beta_i - k_i)\,\xi^{(k)} = -\gamma(-\beta_i - k_i)\,\xi^{(k+\delta_i)}; \tag{4.13}$$

and if $k_i = 0$, then

$$T_{12}(-\beta_i)\,\xi^{(k)} = \xi^{(k-\delta_i)},\qquad T_{21}(-\beta_i)\,\xi^{(k)} = \xi^{(k+\delta_i)}. \tag{4.14}$$

Applying the Lagrange interpolation formula we obtain the remaining formulas.

It is implied by the formulas, that the module $M(\ell)$ is spanned by the vectors $\xi^{(k)}$. By (4.7) and the genericity assumption, the $\xi^{(k)}$ are eigenvectors for $T_{22}(u)$ with distinct eigenvalues. In order to verify their linear independence, suppose first that $\gamma(u)$ satisfies the condition

$$\gamma(-\beta_i - k) \neq 0 \qquad \text{for all} \quad k \in \mathbb{Z} \quad \text{and all} \quad i. \tag{4.15}$$

In this case the linear independence of the $\xi^{(k)}$ follows from the fact that each of them is nonzero. This is implied by (4.12)–(4.13) since $\xi \neq 0$ in $M(\ell)$.

In the case of general $\gamma(u)$ let us define a $\mathrm{Y}_p(\mathfrak{gl}_2)$-module $\widetilde{M}(\ell)$ as follows. As a vector space, $\widetilde{M}(\ell)$ is the $\Bbbk$-linear span of the basis vectors $\widetilde{\xi}^{(k)}$ with (k) running over $\mathbb{Z}^p$ and the action of $\mathrm{Y}_p(\mathfrak{gl}_2)$ is given by the formulas (4.7)–(4.9), where the $\xi^{(k)}$ should be replaced with $\widetilde{\xi}^{(k)}$. We have to verify that the operators $T_{ij}(u)$ do satisfy the Yangian defining relations (3.1). However, the application of both sides of (3.1) to a basis vector $\widetilde{\xi}^{(k)}$ amounts to polynomial relations on the coefficients of $\gamma(u)$. By the previous argument, if $\gamma(u)$ satisfies (4.15), then these relations are identities. Therefore, these identities hold for an arbitrary $\gamma(u)$ and thus $\widetilde{M}(\ell)$ is well defined.

Finally, consider the $\mathrm{Y}_p(\mathfrak{gl}_2)$-module homomorphism

$$\varphi : \mathrm{Y}_p(\mathfrak{gl}_2) \to \widetilde{M}(\ell), \qquad 1 \mapsto \widetilde{\xi}^{(0)}.$$

Obviously, the ideal I_ℓ is contained in the kernel $\operatorname{Ker}\varphi$ and so, this defines a homomorphism $M(\ell) \to \widetilde{M}(\ell)$ which takes $\xi^{(k)}$ to the corresponding vector $\widetilde{\xi}^{(k)}$. Since the vectors $\widetilde{\xi}^{(k)}$ form a basis of $\widetilde{M}(\ell)$, this proves that the vectors $\xi^{(k)}$ are linearly independent. □

Let us fix a p-tuple β satisfying the genericity condition (4.4) and introduce the elements of $\mathrm{Y}_p(\mathfrak{gl}_2)$ by

$$\tau^{(k)} = \prod_{i,\ k_i>0} T_{21}(-\beta_i - k_i + 1)\cdots T_{21}(-\beta_i - 1)\,T_{21}(-\beta_i) \times \prod_{i,\ k_i<0} T_{12}(-\beta_i - k_i - 1)\cdots T_{12}(-\beta_i + 1)\,T_{12}(-\beta_i),$$

where (k) runs over $\mathbb{Z}^p$.

Corollary 4.6. *The elements $\tau^{(k)}$ are linearly independent over Γ in the right Γ-module $\mathrm{Y}_p(\mathfrak{gl}_2)$.*

Proof. Suppose that a linear combination of the elements $\tau^{(k)}$ with coefficients in Γ is zero:

$$\sum_{(k)} \tau^{(k)}\, c_{(k)} = 0, \qquad c_{(k)} \in \Gamma. \tag{4.16}$$

Apply the left-hand side to the vector ξ in a module $M(\ell)$ with $\ell = (\beta, \gamma)$ satisfying the assumptions of Theorem 4.5. We get the relation

$$\sum_{(k)} c_{(k)}(\ell)\, \xi^{(k)} = 0,$$

where $c_{(k)}(\ell)$ is the evaluation of the polynomial $c_{(k)}$ at $T_{22}(u) = \beta(u)$ and $D(u) = \gamma(u)$. Since the vectors $\xi^{(k)}$ form a basis of $M(\ell)$ this implies that $c_{(k)}(\ell) = 0$ for any choice of the parameters γ. Therefore, each $c_{(k)}$ does not depend on the generators d_i and so it is a polynomial in the $t_{22}^{(i)}$. However, due to the Poincaré–Birkhoff–Witt theorem for the algebra $\mathrm{Y}_p(\mathfrak{gl}_2)$ (Proposition 3.1), a nontrivial relation (4.16) can only hold if the elements $\tau^{(k)}$ are linearly dependent over $\Bbbk$. But this is not the case because the vectors $\xi^{(k)} = \tau^{(k)}\,\xi$ are linearly independent in $M(\ell)$ by Theorem 4.5. □

Remark 4.7. One can also produce a family of Γ-linearly independent elements for the left Γ-module $\mathrm{Y}_p(\mathfrak{gl}_2)$. They can be obtained as images of the $\tau^{(k)}$ under the anti-automorphism of the algebra $\mathrm{Y}_p(\mathfrak{gl}_2)$ given by

$$t_{ij}^{(r)} \mapsto t_{ji}^{(r)}. \tag{4.17}$$

For the proof we observe that every generator of Γ is stable under this anti-automorphism. With the exception of the case $p = 1$, the elements $\tau^{(k)}$ do not apparently constitute a basis of $\mathrm{Y}_p(\mathfrak{gl}_2)$ as a right Γ-module.

Remark 4.8. Given two monic polynomials $\alpha(u)$ and $\beta(u)$ of degree p define the corresponding *Verma module* $V(\alpha(u), \beta(u))$ as the quotient of $\mathrm{Y}_p(\mathfrak{gl}_2)$ by the left ideal generated by the coefficients of the polynomials $T_{11}(u) - \alpha(u)$, $T_{22}(u) - \beta(u)$ and $T_{12}(u)$; cf. [T1, T2]. Then the same argument as above shows that $V(\alpha(u), \beta(u))$ has a basis $\{\xi^{(k)}\}$ parametrized by p-tuples of nonnegative integers (k). The formulas of Theorem 4.5 hold for the basis vectors $\xi^{(k)}$, where $\gamma(u)$ should be taken to be $\alpha(u)\,\beta(u-1)$ which defines the central character γ of $V(\alpha(u), \beta(u))$. In fact, $V(\alpha(u), \beta(u))$ is isomorphic to the quotient of the corresponding universal module $M(\ell)$, $\ell = (\beta, \gamma)$ by the submodule spanned by the vectors $\{\xi^{(k)}\}$ such that (k) contains at least one negative component k_i.

Corollary 4.9. *Let $\ell = (\beta, \gamma) \in \mathcal{P}_1$.*

1. *The module $M(\ell)$ is a generic weight $\mathrm{Y}_p(\mathfrak{gl}_2)$-module with central character γ, $\operatorname{Supp} M(\ell) = \mathbb{Z}^p$ and all weight spaces are 1-dimensional.*
2. *The module $M(\ell)$ has a unique maximal submodule and hence a unique irreducible quotient.*
3. *The equivalence class $D(\ell)$ coincides with the set $\ell + \mathcal{P}_0$.*

Proof. Statement (1) follows immediately from Theorem 4.5. The sum of all proper submodules of $M(\ell)$ is again a proper submodule implying (2). Statement (3) follows immediately from (1). □

We will denote the unique irreducible quotient of $M(\ell)$ by $L(\ell)$. It follows from Corollary 4.9 that all weight spaces of $L(\ell)$ are 1-dimensional. We can now describe all irreducible generic weight $\mathrm{Y}_p(\mathfrak{gl}_2)$-modules.

Corollary 4.10. *Let $\ell = (\beta, \gamma) \in \mathcal{P}_1$.*

(1) *There exists an irreducible generic weight $\mathrm{Y}_p(\mathfrak{gl}_2)$-module $L(\ell)$ with $L(\ell)_\beta \neq 0$ and with central character γ. Moreover, $\dim L(\ell)_\psi = 1$ for all $\psi \in \operatorname{Supp} L(\ell)$.*
(2) *Any irreducible weight module over $\mathrm{Y}_p(\mathfrak{gl}_2)$ with central character γ generated by a nonzero vector of weight β is isomorphic to $L(\ell)$.*

5. Γ is a Harish-Chandra subalgebra

In this section we adapt the results from [DFO2] and [Ov] for the Yangians. In particular, we show that Γ is a Harish-Chandra subalgebra.

We have the following analogue of the Harish-Chandra theorem for Lie algebras [Di].

Proposition 5.1. *Let $x \in \mathrm{Y}_p(\mathfrak{gl}_2)$ be such that $xM(\ell) = 0$ for any $\ell \in \mathcal{P}_1$. Then $x = 0$.*

Proof. Since $M(\ell) = \mathrm{Y}_p(\mathfrak{gl}_2)/I_\ell$, it will be sufficient to show that the intersection $\bigcap_\ell I_\ell$ over all $\ell \in \mathcal{P}_1$ is zero. By Theorem 3.4 (1), the Yangian $\mathrm{Y}_p(\mathfrak{gl}_2)$ is free as a right module over Γ. Let $x_i, i \in \mathcal{I}$ be a basis of $\mathrm{Y}_p(\mathfrak{gl}_2)$ over Γ. If $x = \sum_{i\in\mathcal{I}} x_i z_i$ for some $z_i \in \Gamma$, then $x \in I_\ell$ if and only if $z_i(\ell) = 0$ for all $i \in \mathcal{I}$. Since $\mathcal{P}_1$ is dense in $\mathcal{L}$ in Zariski topology it follows immediately that if $x \in \bigcap_\ell I_\ell$ with ℓ running over $\mathcal{P}_1$, then $z_i = 0$ for all $i \in \mathcal{I}$ and thus $x = 0$. This completes the proof. □

For any $\ell_0 \in \mathcal{P}_1$ the module $M(\ell_0)$ has a basis $\xi^{(k)}$, $(k) \in \mathbb{Z}^p$ with the action of generators of $\mathrm{Y}(\mathfrak{gl}_2)$ defined by formulas (4.7)–(4.9). We will relabel the basis elements of $M(\ell_0)$ as ξ_ℓ, $\ell \in \ell_0 + \mathcal{P}_0$. It follows from Theorem 4.5 that for every $x \in \mathrm{Y}_p(\mathfrak{gl}_2)$ there exists a finite subset $\mathcal{L}_x \subseteq \mathcal{P}_0$ consisting of elements δ such that

$$x\, \xi_\ell = \sum_{\delta \in \mathcal{L}_x} \theta(x, \ell, \delta)\, \xi_{\ell+\delta} \tag{5.1}$$

for some nonzero coefficients $\theta(x, \ell, \delta) \in \Bbbk$. We can also regard these coefficients as the values of the rational functions $\theta(x, \mathsf{b}, \delta) \in \mathbb{L}$ at $\mathsf{b} = \ell$, where $\mathsf{b} = (b_1, \ldots, b_p, g_1, \ldots, g_{2p})$. Clearly, the set $\mathcal{L}_x$ is $S_p \times S_{2p}$-invariant. Note that for a given x this formula does not depend on ℓ_0.

We identify the $(\Gamma{-}\Gamma)$-bimodule structure on $\mathrm{Y}_p(\mathfrak{gl}_2)$ with the corresponding $\Gamma\otimes\Gamma$-module structure. For any $z \in \Gamma$ and any finite $S \subseteq \mathcal{L}$ introduce the following polynomial

$$F_{S,z} = F_{S,z}(z, \mathsf{b}) = \prod_{\delta\in S} (z \otimes 1 - 1 \otimes z(\mathsf{b} + \delta)) = \sum_{i=0}^{|S|} z^i \otimes a_i, \qquad a_i \in \mathrm{L}.$$

Proposition 5.2 (cf. [DFO2, Lemma 25])**.** *Let S be a finite $S_p \times S_{2p}$-invariant subset in $\mathcal{L}$, $q = |S|$, $z \in \Gamma$ and $F_{S,z} = \sum_{i=0}^{q} z^i \otimes a_i, \quad a_i \in \mathrm{L}$. Then:*

(1) $a_i \in \Gamma$, $i = 0, \ldots, q$.

(2) *For any $x \in \mathrm{Y}_p(\mathfrak{gl}_2)$ such that $\mathcal{L}_x \subseteq S$ we have $\sum_{i=0}^{q} z^i x a_i = 0$.*

Proof. Since S is $S_p \times S_{2p}$-invariant, the coefficients of the polynomial $F_{S,z}$ are $S_p \times S_{2p}$-invariant and hence belong to Γ which proves (1). It is sufficient to check the statement (2) for $S = \mathcal{L}_x$ since $F_{S,z} = F_{S\setminus\mathcal{L}_x,z} F_{\mathcal{L}_x,z}$. Let $\ell \in \mathcal{P}_1$ and let ξ_ℓ be a basis element of $M(\ell)$. Then

$$\begin{aligned}\sum_{i=0}^{q} z^i x a_i(\xi_\ell) &= \sum_{i=0}^{q} z^i x a_i(\ell)(\xi_\ell)\\ &= \sum_{i=0}^{q} z^i a_i(\ell) \sum_{\delta\in\mathcal{L}_x} \theta(x,\ell,\delta)\xi_{\ell+\delta}\\ &= \sum_{\delta\in\mathcal{L}_x} \theta(x,\ell,\delta) \sum_{i=0}^{q} a_i(\ell)(z^i\xi_{\ell+\delta})\\ &= \sum_{\delta\in\mathcal{L}_x} \theta(x,\ell,\delta) \sum_{i=0}^{q} a_i(\ell) z(\ell+\delta)^i \xi_{\ell+\delta}\\ &= \sum_{\delta\in\mathcal{L}_x} \theta(x,\ell,\delta) F_{\mathcal{L}_x,z}(z(\ell+\delta),\ell)\xi_{\ell+\delta} = 0\end{aligned}$$

since $F_{\mathcal{L}_x,z}(z(\ell+\delta),\ell) = 0$ for every $\delta \in \mathcal{L}_x$. Applying Proposition 5.1 we obtain the statement of the proposition. □

The main result of this section is the following theorem.

Theorem 5.3. *Γ is a Harish-Chandra subalgebra of $\mathrm{Y}_p(\mathfrak{gl}_2)$.*

Proof. Following [DFO2, Proposition 8], it is sufficient to show that a Γ-bimodule $\Gamma\, t_{ij}^{(k)}\, \Gamma$ is finitely generated both as left and as right module for every possible choice of indices i, j, k. This is obvious for $i = j = 2$ since $t_{22}^{(k)} \in \Gamma$. Suppose now that $i = 2, j = 1$. We have $d_i\, t_{21}^{(k)} = t_{21}^{(k)} d_i$ by the centrality of d_i. It follows from (4.8) that $\mathcal{L}_{t_{21}^{(k)}} = \{\delta_i \mid i = 1, \ldots, p\}$. Then for $x = t_{21}^{(k)}$ we have

$$F_{\mathcal{L}_x, t_{22}^{(i)}} = z^p \otimes 1 + \sum_{l=0}^{p-1} z^l \otimes a_l, \qquad a_l \in \Gamma$$

and

$$(t_{22}^{(i)})^p t_{21}^{(k)} + \sum_{l=0}^{p-1} (t_{22}^{(i)})^l t_{21}^{(k)} a_l = 0 \tag{5.2}$$

by Proposition 5.2 (2). Hence the elements

$$\prod_{i=1}^{p} (t_{22}^{(i)})^{k_i}\, t_{21}^{(k)}, \qquad 0 \le k_i < p$$

are generators of $\Gamma t_{21}^{(k)}\Gamma$ as a right Γ-module. The cases $i=1, j=2$ and $i=j=1$ are treated similarly since

$$\mathcal{L}_{t_{12}^{(k)}} = \{-\delta_i \mid i=1,\dots,p\} \qquad \text{and} \qquad \mathcal{L}_{t_{11}^{(k)}} = \{\, \delta_i - \delta_j \mid i,j=1,\dots,p\}.$$

Thus, $\Gamma t_{ij}^{(k)}\Gamma$ is finitely generated as a right Γ-module. The claim for the left module is proved by the application of the anti-automorphism of the algebra $\mathrm{Y}_p(\mathfrak{gl}_2)$ defined in (4.17) where we note that every element of Γ is stable under this anti-automorphism. □

Example 5.4. We give an explicit form of the relation (5.2) for the particular case $i=k=p=2$. It reads

$$t_{22}^{(2)\,2}\, t_{21}^{(2)} - t_{22}^{(2)}\, t_{21}^{(2)}\Big(2\, t_{22}^{(2)} + t_{22}^{(1)}\Big) + t_{21}^{(2)}\Big(t_{22}^{(2)\,2} + t_{22}^{(2)}\, t_{22}^{(1)} + t_{22}^{(2)}\Big) = 0.$$

6. Universal representation of the Yangian

We will denote by $K(\Gamma)$ the field of fractions of Γ.

Let $\mathrm{M}_{\mathcal{P}_0}(\mathbb{L})$ be the ring of locally finite matrices over $\mathbb{L}$ (with a finite number of nonzero elements in each row and each column) with the entries indexed by the elements of $\mathcal{P}_0$. Any $\ell \in \mathcal{P}_1$ defines the evaluation homomorphism $\chi_\ell : \mathbb{L} \to \Bbbk$, which induces the homomorphism of matrix algebras $\Phi(\ell) : \mathrm{M}_{\mathcal{P}_0}(\mathbb{L}) \to \mathrm{M}_{\mathcal{P}_0}(\Bbbk)$. For $\ell, \ell' \in \mathcal{P}_0$ denote by $e_{\ell\,\ell'}$ the corresponding matrix unit in $\mathrm{M}_{\mathcal{P}_0}(\mathbb{L})$. The group $\mathbb{W}$ acts on $\mathrm{M}_{\mathcal{P}_0}(\mathbb{L})$ by the rule: if $X = (X_{\ell\,\ell'})_{\ell,\ell' \in \mathcal{P}_0}$, then

$$(w^{-1}\cdot X)_{\ell,\ell'} = w^{-1}\cdot X_{w(\ell)w(\ell')} \qquad \text{for} \quad w \in \mathbb{W}. \tag{6.1}$$

Define the map

$$\mathrm{G} : \mathrm{Y}_p(\mathfrak{gl}_2) \to \mathrm{M}_{\mathcal{P}_0}(\mathbb{L})$$

such that for any $x \in \mathrm{Y}_p(\mathfrak{gl}_2)$ and any $\ell \in \mathcal{P}_0$, $\mathrm{G}(x)_{\ell'\,\ell} = \theta(x, \mathsf{b}+\ell, \delta)$ if $\ell' - \ell = \delta$ and 0 otherwise; see (5.1).

Lemma 6.1. (1) G *is a representation of* $\mathrm{Y}_p(\mathfrak{gl}_2)$ *over* $\mathbb{L}$.

(2) $\mathrm{G}(x)$ *is* $\mathbb{W}$*-invariant for any* $x \in \mathrm{Y}_p(\mathfrak{gl}_2)$*. In particular,* $\mathrm{G}(x)_{\overline{0}\,\overline{0}} \in K(\Gamma)$.

(3) *If* $x = x(b_1,\dots,b_p,g_1,\dots,g_{2p}) \in \Gamma$ *and* $\ell = (l_1,\dots,l_p,0,\dots,0) \in \mathcal{P}_0$*, then*

$$\mathrm{G}(x)_{\ell\ell} = x(b_1+l_1,\dots,b_p+l_p,g_1,\dots,g_{2p}).$$

(4) $\mathrm{G}(\Gamma)$ *consists of* $\mathbb{W}$*-invariant diagonal matrices* X *such that* $X_{\overline{0}\,\overline{0}} \in \Gamma$*. In particular, any such matrix* X *is determined by* $X_{\overline{0}\,\overline{0}} \in \Gamma$.

Proof. Let T be the free associative algebra with generators $\widetilde{t}_{ij}^{(k)}$, where $i,j=1,2$ and $k=1,\dots,p$, and let

$$\pi : \mathrm{T} \to \mathrm{Y}_p(\mathfrak{gl}_2), \qquad \widetilde{t}_{ij}^{(k)} \mapsto t_{ij}^{(k)},$$

be the canonical projection. Define a homomorphism $g : \mathrm{T} \to \mathrm{M}_{\mathcal{P}_0}(\mathbb{L})$ by $g(\widetilde{t}_{ij}^{(k)}) = \mathrm{G}(t_{ij}^{(k)})$ for all suitable i,j,k. To prove (1) it is sufficient to show that $g(\operatorname{Ker}\pi) = 0$. Suppose that $f \in \operatorname{Ker}\pi$. Then $\Phi(\ell)(g(f)) = 0$ and thus $g(f)_{\ell'\ell''}(\ell) = 0$ for any $\ell \in \mathcal{P}_1$. Since $\mathcal{P}_1$ is dense in $\operatorname{Specm} L$ we conclude that $g(f) = 0$ implying (1). The image of G is $\mathbb{W}$-invariant since this holds for the generators of $\mathrm{Y}_p(\mathfrak{gl}_2)$; see (4.7)–(4.9). For any $\sigma \in S_p \times S_{2p}$ we have

$$(\sigma^{-1}\cdot \mathrm{G})(x)_{\overline{0}\,\overline{0}} = \sigma^{-1}(\mathrm{G}(x)_{\sigma(\overline{0})\,\sigma(\overline{0})}) = \sigma^{-1}(\mathrm{G}(x)_{\overline{0}\,\overline{0}}).$$

Hence $\mathrm{G}(x)_{\overline{0}\,\overline{0}}$ is $S_p \times S_{2p}$-invariant proving (2). The statement (3) follows from (2) if we apply a shift by $\ell \in \mathcal{P}_0$ to an arbitrary $x \in \mathrm{Y}_p(\mathfrak{gl}_2)$. The statement (4) follows immediately from (2) and (3). □

The composition $r_\ell = \Phi(\ell) \circ \mathrm{G}$ defines a representation of $\mathrm{Y}_p(\mathfrak{gl}_2)$. By the construction, this representation provides a matrix realization of the module $M(\ell)$; see Theorem 4.5.

Proposition 6.2. *The representation* $\mathrm{G} : \mathrm{Y}_p(\mathfrak{gl}_2) \longrightarrow \mathrm{M}_{\mathcal{P}_0}(\mathbb{L})$ *is faithful.*

Proof. It is clear that

$$\operatorname{Ker} \mathrm{G} \subseteq \bigcap_{\ell \in \mathcal{P}_1} \operatorname{Ker} r_\ell .$$

Hence it is sufficient to prove that the intersection of the kernels is zero. Let $\ell \in \mathcal{P}_1$. Then $\operatorname{Ker} r_\ell = \operatorname{Ann} M(\ell)$ by definition and so $\operatorname{Ker} r_\ell \subseteq I_\ell$. However, the intersection $\bigcap_\ell I_\ell$ over all $\ell \in \mathcal{P}_1$ is zero, as was shown in the proof of Proposition 5.1. □

Corollary 6.3. (1) Γ *is a maximal commutative subalgebra in* $\mathrm{Y}_p(\mathfrak{gl}_2)$.
(2) *If for* $x \in \mathrm{Y}_p(\mathfrak{gl}_2)$ *the matrix* $\mathrm{G}(x)$ *is diagonal, then* $x \in \Gamma$.

Proof. Consider an element $x \in \mathrm{Y}_p(\mathfrak{gl}_2)$ which commutes with every $z \in \Gamma$ and such that $x \notin \Gamma$. Suppose that there exist $\ell, \ell' \in \mathcal{P}_0$, $\ell \neq \ell'$ such that $\mathrm{G}(x)_{\ell\ell'} \neq 0$. There exists $z \in \Gamma$ such that $z(\ell) \neq z(\ell')$ and so $\mathrm{G}(z)_{\ell\ell} \neq \mathrm{G}(z)_{\ell'\ell'}$ by Lemma 6.1 (3). Then we have

$$\mathrm{G}(xz)_{\ell\ell'} = \mathrm{G}(x)_{\ell\ell'}\mathrm{G}(z)_{\ell'\ell'} = \mathrm{G}(zx)_{\ell\ell'} = \mathrm{G}(z)_{\ell\ell}\mathrm{G}(x)_{\ell\ell'}$$

which contradicts to the assumption. Therefore, $\mathrm{G}(x)$ is diagonal. To prove the maximality of Γ it is now sufficient to verify part (2) of the corollary. By Lemma 6.1 (2), we have $\mathrm{G}(x)_{\overline{0}\,\overline{0}} = f/g \in K(\Gamma)$ with relatively prime $f, g \in \Gamma$. Suppose that $g \notin \Bbbk$. By Lemma 6.1 (2) and (4), we derive that $\mathrm{G}(x)\mathrm{G}(g) = \mathrm{G}(f)$ and hence $xg = f$ by Proposition 6.2. This shows that $x \in \Gamma$ due to Theorem 3.4 (1). □

Denote by X_0 the column matrix defined by

$$X_0 = \sum_{\delta \in \mathcal{P}_0} \mathbb{L}\, e_{\delta,\overline{0}},$$

where $\overline{0}$ is the zero element of $\mathcal{P}_0$. Note that the $\mathbb{W}$-action (6.1) induces an action of $S_p \times S_{2p}$ on the free $\mathbb{L}$-module X_0.

Corollary 6.4. *Let* $p : \mathrm{M}_{\mathcal{P}_0}(\mathbb{L}) \rightarrow X_0$ *be the natural projection. Then the composition* $r = p \circ \mathrm{G} : \mathrm{Y}_p(\mathfrak{gl}_2) \rightarrow X_0$ *is injective. Moreover, the map* p *commutes with the action of* $S_p \times S_{2p}$ *and, in particular,* $r(\mathrm{Y}_p(\mathfrak{gl}_2))$ *is* $S_p \times S_{2p}$*-invariant.*

Proof. Note that for any $x \in \mathrm{Y}_p(\mathfrak{gl}_2)$ the matrix $\mathrm{G}(x) \in \mathrm{M}_{\mathcal{P}_0}(\mathbb{L})$ is determined completely by its column $p(\mathrm{G}(x))$. Thus $r(x) = 0$ implies $\mathrm{G}(x) = 0$ and hence $x = 0$ since G is faithful. This proves that r is injective. The other statements follow immediately from the definitions and Lemma 6.1 (2). □

As in Section 5, we identify the $(\Gamma{-}\Gamma)$-bimodule structure on $\mathrm{Y}_p(\mathfrak{gl}_2)$ with the corresponding action of $\Gamma \otimes \Gamma$. Using the embedding (4.1), we can regard the elements of $\Gamma \otimes \Gamma$ as polynomials in two families of variables b and b' which are $S_p \times S_{2p}$-invariant.

Lemma 6.5. *Suppose that* $x \in \mathrm{Y}_p(\mathfrak{gl}_2)$, $f \in \Gamma \otimes \Gamma$, *and* ℓ, $\ell' \in \mathcal{P}_0$. *Then*

$$\mathrm{G}(f \cdot x)_{\ell\ell'} \;=\; f(\mathsf{b}+\ell, \mathsf{b}+\ell')\mathrm{G}(x)_{\ell\ell'}.$$

Proof. Let $f = \sum_i z_i \otimes z_i' \in \Gamma \otimes \Gamma$. Then $\mathrm{G}(f \cdot x) = \sum_i \mathrm{G}(z_i)\mathrm{G}(x)\mathrm{G}(z_i')$ and hence, by Lemma 6.1 (4),

$$\begin{aligned}\mathrm{G}(f \cdot x)_{\ell\ell'} &= \sum_i \mathrm{G}(z_i)_{\ell\,\ell}\mathrm{G}(x)_{\ell\,\ell'}\mathrm{G}(z_i')_{\ell'\,\ell'} = \mathrm{G}(x)_{\ell\,\ell'} \sum_i \mathrm{G}(z_i)_{\ell\,\ell}\mathrm{G}(z_i')_{\ell'\,\ell'} \\ &= \mathrm{G}(x)_{\ell\,\ell'} \sum_i z_i(\mathsf{b}+\ell)z_i'(\mathsf{b}+\ell') = \mathrm{G}(x)_{\ell\,\ell'} f(\mathsf{b}+\ell, \mathsf{b}+\ell').\end{aligned}$$

□

Lemma 6.6. *Let* $S \subseteq \mathcal{L}$ *be an* $S_p \times S_{2p}$*-invariant set. Suppose that* $z \in \Gamma$ *and* $x \in \mathrm{Y}_p(\mathfrak{gl}_2)$ *is such that* $\mathrm{G}(x)_{\ell\,\ell'} = 0$ *for all* ℓ, ℓ', $\ell - \ell' \notin S$. *Then* $F_{S,z} \cdot x = 0$.

Proof. Let $F = F_{S,z} = \sum_i z^i \otimes a_i$, where $a_i \in \Gamma$ by Proposition 5.2. If $\ell - \ell' \in S$, then

$$\mathrm{G}(F \cdot x)_{\ell\ell'} = F(z(\mathsf{b}+\ell), \mathsf{b}+\ell')\,\mathrm{G}(x)_{\ell\,\ell'}$$

by Proposition 5.2(1) and Lemma 6.5. Furthermore, observe that $h(z, \mathsf{b}) = z \otimes 1 - 1 \otimes z(\mathsf{b}+\ell-\ell')$ divides F and that $h(z(\mathsf{b}+\ell), \mathsf{b}+\ell') = 0$. Here we regard the result of the evaluation of the product of type $z \otimes z'(\mathsf{b}')$ at b as the polynomial $z(\mathsf{b})z'(\mathsf{b}')$. This gives $F(z(\mathsf{b}+\ell), \mathsf{b}+\ell') = 0$. Hence, $\mathrm{G}(F \cdot x) = 0$ implying $F \cdot x = 0$ by Proposition 6.2. □

Let $S \subseteq \mathcal{P}_0$ be a finite $S_p \times S_{2p}$-invariant set. Define $\mathrm{Y}^S = \{x \in \mathrm{Y}_p(\mathfrak{gl}_2) \,|\, \mathcal{L}_x \subseteq S\}$. Clearly Y^S is a Γ-sub-bimodule of $\mathrm{Y}_p(\mathfrak{gl}_2)$. We have the following characterization of the bimodule Y^S.

Lemma 6.7. *Let* $x \in \mathrm{Y}_p(\mathfrak{gl}_2)$. *Then*

(1) $x \in \mathrm{Y}^S$ *if and only if the condition* $\mathrm{G}(x)_{\ell,\ell'} \neq 0$, *for some* ℓ, $\ell' \in \mathcal{P}_0$, *implies that* $\ell - \ell' \in S$.
(2) $F_{\mathcal{L}_x \setminus S, z} \cdot x \in \mathrm{Y}^S$ *for any* $z \in \Gamma$.
(3) Y^S *is a finitely generated left (right)* Γ*-module and* $\mathrm{Y}^S = \mathbb{D}(\mathrm{Y}^S)$.
(4) $\mathrm{Y}^{\{0\}} = \Gamma$.

Proof. Statement (1) follows from the definition of Y^S. Let $F = F_{\mathcal{L}_x \setminus S, z}$ and $y = F \cdot x$. To prove (2) calculate the matrix element $\mathrm{G}(y)_{\ell\ell'}$ provided that $\ell - \ell' \notin S$. By Lemma 6.5,

$$\mathrm{G}(y)_{\ell\ell'} = \mathrm{G}(F \cdot x)_{\ell\ell'} = F(z(\mathsf{b}+\ell), \mathsf{b}+\ell')\mathrm{G}(x)_{\ell\ell'}.$$

If $\ell - \ell' \notin \mathcal{L}_x$, then $\mathrm{G}(x)_{\ell\ell'} = 0$ and hence $\mathrm{G}(y)_{\ell\ell'} = 0$. Suppose now that $\ell - \ell' \in \mathcal{L}_x \setminus S$. Then

$$F(z(\mathsf{b}+\ell), \mathsf{b}+\ell') = \prod_{\delta \in \mathcal{L}_x \setminus S} (z(\mathsf{b}+\ell) - z(\mathsf{b}+\ell'+\delta)) = 0.$$

This proves (2).

Let $x \in \mathbb{D}(\mathrm{Y}^S)$ and suppose that $z \in \Gamma$ is such that $z \neq 0$ and $zx \in \mathrm{Y}^S$. Since $\mathrm{G}(zx)_{\ell\ell'} = z(\mathsf{b}+\ell)\mathrm{G}(x)_{\ell\ell'}$ by Lemma 6.5, we have $\mathrm{G}(zx)_{\ell\ell'} = 0$ if and only if $\mathrm{G}(x)_{\ell\ell'} = 0$ implying that $x \in \mathrm{Y}^S$. Hence $\mathrm{Y}^S = \mathbb{D}(\mathrm{Y}^S)$.

Consider $r(\mathrm{Y}^S)$ as a Γ-submodule of X_0 where $r : \mathrm{Y}_p(\mathfrak{gl}_2) \to X_0$ is defined in Corollary 6.4. Then $r(\mathrm{Y}^S)$ belongs to the free $\mathbb{L}$-submodule $\sum_{\ell \in S} \mathbb{L} e_{\ell\,\overline{0}}$ of

X_0 of finite rank. Hence $\mathbb{L} \cdot r(\mathrm{Y}^S)$ is a finitely generated $\mathbb{L}$-module. Without loss of generality, we can assume that this module is generated by the elements $r(x_1), \dots, r(x_s) \in r(\mathrm{Y}^S)$. Since $\mathbb{D}(\mathrm{Y}^S) = \mathrm{Y}^S$, we have

$$\mathbb{D}\Big(\sum_{i=1}^{s} \Gamma x_i\Big) \subseteq \mathrm{Y}^S.$$

Fix $x \in \mathrm{Y}^S$. Then

$$r(x) = \sum_{i=1}^{s} t_i\, r(x_i), \qquad t_i \in \mathbb{L}.$$

Note that for any $y \in \mathrm{Y}^S$ and any $\sigma \in S_p \times S_{2p}$ we have $\sigma \cdot r(y) = r(y)$ since S is $S_p \times S_{2p}$-invariant. Hence

$$p!\,(2p)!\, r(x) = \sum_{\sigma \in S_p \times S_{2p}} \sigma \cdot r(x) = \sum_{\sigma \in S_p \times S_{2p}} \sum_{i=1}^{s} (\sigma \cdot t_i)\, \sigma \cdot r(x_i)$$

which can be rewritten as

$$r(x) = \frac{1}{p!\,(2p)!} \sum_{i=1}^{s} u_i\, r(x_i), \qquad \text{where} \quad u_i = \sum_{\sigma \in S_p \times S_{2p}} \sigma \cdot t_i. \tag{6.2}$$

Since each u_i is $S_p \times S_{2p}$-invariant, it belongs to the field of fractions $K(\Gamma)$ for all $i = 1, \dots, s$. Multiplying both parts of (6.2) by the common denominator of the u_i we obtain from Corollary 6.4 that

$$x \in \mathbb{D}\Big(\sum_{i=1}^{s} \Gamma x_i\Big), \qquad \text{implying} \qquad \mathbb{D}\Big(\sum_{i=1}^{s} \Gamma x_i\Big) = \mathrm{Y}^S.$$

Due to Corollary 3.5, we can conclude that Y^S is finitely generated over Γ. This proves (3). By the definition of Y^S, $x \in \mathrm{Y}^{\{0\}}$ if and only if $\mathrm{G}(x)$ is diagonal. Hence $x \in \Gamma$ by Corollary 6.3 (2). □

7. Category of Harish-Chandra modules

We will show in this section that each character of Γ extends to a finite number of irreducible Harish-Chandra modules over $\mathrm{Y}_p(\mathfrak{gl}_2)$. This is an analogue of the corresponding result in the case of a Lie algebra $\mathfrak{gl}_n$ which was conjectured in [DFO1] and proved in [Ov]. In this section we use the techniques of [DFO2] and [Ov].

Due to Theorem 5.3, Γ is a Harish-Chandra subalgebra of $\mathrm{Y}_p(\mathfrak{gl}_2)$ so that we can apply all the statements from Section 2.1. Set $\mathcal{A} = \mathcal{A}_{\mathrm{Y}_p(\mathfrak{gl}_2),\Gamma}$. Then by Proposition 2.1, the categories $\mathcal{A}$-mod$_d$ and $\mathbb{H}(\mathrm{Y}_p(\mathfrak{gl}_2), \Gamma)$ are equivalent. Also the full subcategory $\mathbb{H}W(\mathrm{Y}_p(\mathfrak{gl}_2), \Gamma)$ consisting of weight modules is equivalent to the module category $\mathcal{A}_W$-mod. If $\ell \in \mathcal{L}$, then the category $R_\ell = \mathbb{H}W(\mathrm{Y}_p(\mathfrak{gl}_2), \Gamma, D(\ell))$ is equivalent to the block $\mathcal{A}_W(D(\ell))$-mod of the category $\mathcal{A}_W$-mod.

Let $\mathbf{m}$, $\mathbf{n} \in \operatorname{Specm}\Gamma$, $\ell_{\mathbf{m}}$, $\ell_{\mathbf{n}} \in \mathcal{L}$ are such that $\imath^*(\ell_{\mathbf{m}}) = \mathbf{m}$ and $\imath^*(\ell_{\mathbf{n}}) = \mathbf{n}$; see (4.2). Set

$$S(\mathbf{m}, \mathbf{n}) = \{\sigma_1 \ell_{\mathbf{n}} - \sigma_2 \ell_{\mathbf{m}} \mid \sigma_1, \sigma_2 \in S_p \times S_{2p}\} \cap \mathcal{P}_0.$$

Consider the following subset in $\mathcal{L}$,

$$\mathcal{P}_2 = \{\ell \in \mathcal{L} \mid \ell_i - \ell_j \notin \mathbb{Z} \setminus \{0\}, \qquad i, j = 1, \dots, p\}$$

and put $\Omega = \imath^*(\mathcal{P}_2)$. We shall also be using the set $\mathcal{A}(\mathbf{m}, \mathbf{n})$ introduced in (2.2).

Proposition 7.1. (1) *For any* $\mathbf{m}$, $\mathbf{n} \in \operatorname{Specm}\Gamma$ *and any* $m, n \geq 0$ *we have*

$$\mathrm{Y}_p(\mathfrak{gl}_2) = \mathrm{Y}^S + \mathbf{n}^n \mathrm{Y}_p(\mathfrak{gl}_2) + \mathrm{Y}_p(\mathfrak{gl}_2)\mathbf{m}^m,$$

where $S = S(\mathbf{m}, \mathbf{n})$.

(2) $\mathcal{A}(\mathbf{m}, \mathbf{n})$ *is finitely generated as a left* $\Gamma_{\mathbf{n}}$*-module and as a right* $\Gamma_{\mathbf{m}}$*-module. In particular, the algebra* Γ *is big in every* $\mathbf{n} \in \operatorname{Ob}\mathcal{A}$.

(3) *If* $S(\mathbf{m}, \mathbf{n}) = \{0\}$*, then* $\mathcal{A}(\mathbf{m}, \mathbf{n})$ *is generated as a left* $\Gamma_{\mathbf{n}}$*-module and as a right* $\Gamma_{\mathbf{m}}$*-module by the image of* 1 *in* $\mathcal{A}(\mathbf{m}, \mathbf{n})$.

(4) *If* $S(\mathbf{m}, \mathbf{m}) = \{0\}$*, then* $\mathbf{m} \in \Omega$*. Moreover,* $\mathcal{A}(\mathbf{m}, \mathbf{m})$ *is a quotient algebra of* $\Gamma_{\mathbf{m}}$ *and* $\mathbf{m}$ *extends uniquely to an irreducible* $\mathrm{Y}_p(\mathfrak{gl}_2)$*-module.*

(5) *If* $\ell_{\mathbf{m}} \in \mathcal{P}_1$*, then* $\mathcal{A}(\mathbf{m}, \mathbf{m}) = \Gamma_{\mathbf{m}}$.

(6) *Let* $\ell \in \mathcal{P}_1$*,* $\mathbf{m} = \imath^*(\ell)$ *and* $\mathbf{n} = \imath^*(\ell + \delta_i)$*,* $i \in \{1, \dots, p\}$*. Then* $\mathcal{A}(\mathbf{m}, \mathbf{n})$ *is a free of rank* 1 *as a right* $\Gamma_{\mathbf{m}}$*-module and as a left* $\Gamma_{\mathbf{n}}$*-module.*

Proof. (1) We shall show that for any $x \in \mathrm{Y}_p(\mathfrak{gl}_2)$ and any $k \geqslant 1$ there exists $x_k \in \mathrm{Y}^S$ such that

$$x \in x_k + \sum_{i=0}^{k} \mathbf{n}^{k-i} x \mathbf{m}^i. \tag{7.1}$$

The statement will follow if we choose $k = m + n + 1$. We will use induction on k. Suppose that $k = 1$. If $\mathcal{L}_x \subseteq S$, then $x \in \mathrm{Y}^S$ and there is nothing to prove. Furthermore, by the definition of the set S for any $\ell \in \mathcal{L}_x \setminus S$ the $S_p \times S_{2p}$-orbits of $\ell_{\mathbf{n}}$ and $\ell_{\mathbf{m}} + \ell$ are disjoint. Hence there exists $z \in \Gamma$ such that $z(\ell_{\mathbf{n}}) \neq z(\ell_{\mathbf{m}} + \ell)$ for any $\ell \in \mathcal{L}_x \setminus S$. Let $F = F_{\mathcal{L}_x \setminus S, z}$. Then

$$F(z(\ell_{\mathbf{n}}), \ell_{\mathbf{m}}) = \prod_{\ell \in \mathcal{L}_x \setminus S} (z(\ell_{\mathbf{n}}) - z(\ell_{\mathbf{m}} + \ell)) \neq 0.$$

We can assume that $F(z(\ell_{\mathbf{n}}), \ell_{\mathbf{m}}) = 1$. Hence we obtain that $F = 1 + u$ where $u \in \mathbf{n} \otimes \Gamma + \Gamma \otimes \mathbf{m}$. It follows from Lemma 6.7 (2), that $x_1 = F \cdot x$ belongs to Y^S. Hence we have

$$x_1 = (1 + u) \cdot x \in x + \mathbf{n}\, x\, \Gamma + \Gamma\, x\, \mathbf{m} \qquad \text{and thus} \qquad x \in x_1 + \mathbf{n}\, x\, \Gamma + \Gamma\, x\, \mathbf{m}.$$

This proves the assertion in the case $k = 1$. Assume that (7.1) holds for some $k \geq 1$. Then

$$x \in x_k + \sum_{i=0}^{k} \mathbf{n}^{k-i} (x_k + \sum_{j=0}^{k} \mathbf{n}^{k-j} x \mathbf{m}^j) \mathbf{m}^i \subseteq x_k + \sum_{i=0}^{k} \mathbf{n}^{k-i} x_k \mathbf{m}^i + \sum_{i=0}^{k+1} \mathbf{n}^{k+1-i} x \mathbf{m}^i.$$

Since Y^S is a Γ-bimodule we conclude that $x_k + \sum_{i=0}^{k} \mathbf{n}^{k-i} x_k \mathbf{m}^i \subseteq \mathrm{Y}^S$ which implies the statement (1). In particular, we have proved that

$$x_{k+1} - x_k \in \sum_{i=0}^{k} \mathbf{n}^{k-i} \mathrm{Y}^S \mathbf{m}^i. \tag{7.2}$$

(2) We prove the statement for the case of left module, the case of right module can be treated analogously. By (1) the image $\overline{x}$ of every $x \in \mathrm{Y}_p(\mathfrak{gl}_2)$ in $\mathcal{A}(\mathbf{n}, \mathbf{m})$ is the limit of the sequence $(\overline{x}_k)_{k \geqslant 1}$, $x_k \in \mathrm{Y}^S$. Let $y_1, \dots, y_m$ be a finite system of

generators of Y^S as a left Γ-module which exists by Lemma 6.7 (3). Then for every $N > 1$ and every $i = 1, \ldots, m$ there exists N_i such that

$$y_i \mathbf{m}^N \subseteq \sum_{j=1}^{m} \mathbf{n}^{N_i} y_j.$$

Since Γ is noetherian we have that $\bigcap_k \mathbf{n}^k \mathrm{Y}^S = 0$ and hence there exists the maximum value d_N such that

$$y_i \mathbf{m}^N \subseteq \sum_{j=1}^{m} \mathbf{n}^{d_N} y_j$$

for all $i = 1, \ldots, m$. Moreover, $d_N \to \infty$ while $N \to \infty$ since Y^S is a finitely generated right Γ-module and $\bigcap_k \mathrm{Y}^S \mathbf{m}^k = 0$. By (7.2), $x_{k+1} - x_k \in \mathbf{n}^{R_k} \mathrm{Y}^S$ where $R_k = \min\{[k/2], d_{[k/2]}\}$. We have

$$\overline{x} = \overline{x}_1 + \sum_{k=1}^{\infty} \overline{(x_{k+1} - x_k)}$$

and thus

$$\overline{x} \in \sum_{k=1}^{\infty} \overline{\mathbf{n}^{R_k} \mathrm{Y}^S} \subseteq \sum_{l=1}^{m} \Gamma_{\mathbf{n}} \, \overline{y_l}.$$

Note that the first sum is well defined since $R_k \to \infty$ when $k \to \infty$. We conclude that $\mathcal{A}(\mathbf{n}, \mathbf{m})$ is finitely generated as a left $\Gamma_{\mathbf{n}}$-module. This completes the proof of (2).

(3) By Lemma 6.7 (4), $\mathrm{Y}^{\{0\}} = \Gamma$. Hence $\mathcal{A}(\mathbf{m}, \mathbf{n})$ is generated (both as a left and as a right module) by the image of $1 \in \Gamma$ by (1).

(4) By (3), $\mathcal{A}(\mathbf{m}, \mathbf{m})$ is 1-generated as a left $\Gamma_{\mathbf{m}}$-module. Then the $\mathbb{k}$-algebra homomorphism

$$\hat{\imath}_{\mathbf{m}} : \Gamma_{\mathbf{m}} \to \mathcal{A}(\mathbf{m}, \mathbf{m}), \qquad z \mapsto z \cdot \mathbf{1}_{\mathbf{m}},$$

where $\mathbf{1}_{\mathbf{m}}$ is a unit morphism, is an epimorphism which shows that $A(\mathbf{m}, \mathbf{m})$ is a quotient algebra of $\Gamma_{\mathbf{m}}$. The uniqueness of the extension follows from the uniqueness of the simple $A(\mathbf{m}, \mathbf{m})$-module and [DFO2, Theorem 18].

(5) Let $\ell = \ell_{\mathbf{m}} \in \mathcal{P}_1$. Then $S(\mathbf{m}, \mathbf{m}) = \emptyset$ and $A(\mathbf{m}, \mathbf{m}) \simeq \Gamma_{\mathbf{m}} / J_{\mathbf{m}}$ by (4) and thus $J_{\mathbf{m}}$ acts trivially on $M(\mathbf{m})$ in any Harish-Chandra module M. Since $\ell \in \mathcal{P}_1$, then for any $k > 0$ there exists a canonical projection $\tilde{\pi}_k : \mathbb{L} \longrightarrow \mathbb{L}/(\ell)^k$, where $(\ell)^k = \ell^k \mathbb{L}$. It induces a homomorphism of the matrix algebras $\pi_k : \mathrm{M}_{\mathcal{P}_0}(\mathbb{L}) \longrightarrow \mathrm{M}_{\mathcal{P}_0}(\mathbb{L}/(\ell)^k)$ and defines a Harish-Chandra module by the following composition

$$\mathrm{Y}_p(\mathfrak{gl}_2) \xrightarrow{\ \mathrm{G}\ } \mathrm{M}_{\mathcal{P}_0}(\mathbb{L}) \xrightarrow{\ \pi_k\ } \mathrm{M}_{\mathcal{P}_0}(\mathbb{L}/(\ell)^k).$$

For any nonzero $x \in \Gamma$ there exists $k > 0$ such that $x \notin (\ell)^k$ and hence $\pi_k(\mathrm{G}(x)_{\overline{0},\overline{0}}) = x + (\ell)^k \neq 0$. Therefore, there exists a Harish-Chandra module M where x acts nontrivially on $M(\mathbf{m})$ implying that $J_{\mathbf{m}} = 0$. This completes the proof.

(6) The proof is analogous to the proof of (5). Let $z \in \Gamma$, $z \neq 0$. Suppose $\mathcal{A}(\mathbf{m}, \mathbf{n}) z = 0$. Then by the construction of the equivalence $\mathbb{F} : \mathcal{A}\text{-mod}_d \longrightarrow \mathbb{H}(U, \Gamma)$ for any Harish-Chandra module M and any $x \in \mathrm{Y}_p(\mathfrak{gl}_2)$ the linear operator xz on M induces the zero map between $M(\mathbf{m})$ and $M(\mathbf{n})$. It is sufficient to construct a Harish-Chandra module where this has failed. For $k \geq 1$ consider as in (5) the

composition $\pi_k \circ \mathrm{G} : \mathrm{Y}_p(\mathfrak{gl}_2) \to \mathrm{M}_{\mathcal{P}_0}(\mathbb{L}/(\ell)^k)$. It defines a Harish-Chandra module structure on a free $\mathbb{L}/(\ell)^k$-module

$$\overline{X} = \sum_{\delta \in \mathcal{P}_0} \mathbb{L}/(\ell)^k e_{\delta,\overline{0}}.$$

Consider $x \in \mathrm{Y}_p(\mathfrak{gl}_2)$ such that $\mathrm{G}(x)_{\delta_i \overline{0}} \neq 0$ for some i. Then

$$\mathrm{G}(xz)_{\delta_i \overline{0}} = \mathrm{G}(x)_{\delta_i \overline{0}} \mathrm{G}(z)_{\overline{0}\overline{0}} = \mathrm{G}(x)_{\delta_i \overline{0}} z \neq 0.$$

Choose k such that $\mathrm{G}(xz)_{\delta_i \overline{0}} \notin (\ell)^k$. Hence $(\pi_k \cdot \mathrm{G})(xz)_{\delta_i,\overline{0}} \neq 0$ and the linear operator xz induces a nonzero map between $\overline{X}(\mathbf{m}) = \mathbb{L}/(\ell)^k$ and $\overline{X}(\mathbf{n}) = \mathbb{L}/(\ell + \delta_i)^k$. The contradiction shows that $\mathcal{A}(\mathbf{m},\mathbf{n})z \neq 0$. The case $z\mathcal{A}(\mathbf{m},\mathbf{n}) = 0$ is treated in a similar manner. □

Now we are in a position to state the main result of this section which follows immediately from Lemma 2.3 and Proposition 7.1 (2).

Theorem 7.2. *Let* $\mathbf{m} \in \operatorname{Specm}\Gamma$. *Then the left ideal* $\mathrm{Y}_p(\mathfrak{gl}_2)\mathbf{m}$ *is contained in finitely many maximal left ideals of* $\mathrm{Y}_p(\mathfrak{gl}_2)$. *In particular,* $\mathbf{m}$ *extends to a finitely many (up to an isomorphism) irreducible* $\mathrm{Y}_p(\mathfrak{gl}_2)$*-modules and for each such module* M, $\dim M(\mathbf{n}) < \infty$ *for all* $\mathbf{n} \in \operatorname{Specm}\Gamma$.

8. Category of generic Harish-Chandra modules

In this section we study a full subcategory of generic modules in $\mathbb{H}W(\mathrm{Y}_p(\mathfrak{gl}_2), \Gamma)$. We give a complete description of irreducible modules and indecomposable modules in tame blocks of this category.

Lemma 8.1. *Let* $\ell \in \mathcal{P}_1$, $\ell = (\beta,\gamma)$, $\mathbf{m} = \imath^*(\ell) \in \operatorname{Specm}\Gamma$, $\mathbf{n} = \imath^*(\ell + \delta_i)$, $i \in \{1,\dots,p\}$. *If* $\beta_i \notin \{\gamma_1,\dots,\gamma_{2p}\}$, *then the objects of* $\mathcal{A}$ *represented by* $\mathbf{m}$ *and* $\mathbf{n}$ *are isomorphic.*

Proof. Choose $z_1, z_2 \in \Gamma$ such that

$$z_1(\ell + \delta_j) = \delta_{ij}, \qquad z_2(\ell + \delta_i - \delta_j) = \delta_{ij}, \qquad j = 1,\dots,p.$$

Set $z = z_2\, t_{12}^{(1)} z_1\, t_{21}^{(1)}$. Then $G(z)$ is diagonal by Lemma 6.5 and hence $z \in \Gamma$ by Corollary 6.3 (2). We will show that the image of z in $\Gamma_{\mathbf{m}}$ is invertible. Clearly, this is equivalent to the fact that $z(\ell) \neq 0$. Formulas (4.7)–(4.9) imply that $z(\ell) = \gamma(-\beta_i) \neq 0$ by assumption. Denote by T_1 (respectively T_2) the generator of $\Gamma_{\mathbf{m}} - \Gamma_{\mathbf{n}}$(respectively, $\Gamma_{\mathbf{n}} - \Gamma_{\mathbf{m}}$)-bimodule $\mathcal{A}(\mathbf{m},\mathbf{n})$ (respectively, $\mathcal{A}(\mathbf{n},\mathbf{m})$); see Proposition 7.1 (6). Then

$$z_2\, t_{12}^{(1)} = z_{\mathbf{m}} T_2, \qquad z_1\, t_{21}^{(1)} = T_1 z'_{\mathbf{m}}$$

for some $z_{\mathbf{m}}, z'_{\mathbf{m}} \in \Gamma_{\mathbf{m}}$ and hence $z = z_{\mathbf{m}} T_2 T_1 z'_{\mathbf{m}}$. Since $z(\ell) \neq 0$ it follows that $z'_{\mathbf{m}}(\ell) \neq 0, z_{\mathbf{m}}(\ell) \neq 0$ and so $T_2T_1 = z_{\mathbf{m}}^{-1} z (z'_{\mathbf{m}})^{-1}$ is invertible in $\Gamma_{\mathbf{m}}$. A similar argument shows that T_1T_2 is invertible in $\Gamma_{\mathbf{n}}$. Therefore the objects $\mathbf{m}$ and $\mathbf{n}$ are isomorphic. □

Corollary 8.2. *Let* $\ell \in \mathcal{P}_1$, $\ell = (\beta,\gamma)$, $\beta_i - \gamma_j \notin \mathbb{Z}$ *for all* i,j. *Then the category* $\mathbb{H}(\mathrm{Y}_p(\mathfrak{gl}_2), \Gamma, D(\ell))$ *is hereditary. Moreover,*

$$\dim \operatorname{Ext}^1_{\mathbb{H}(\mathrm{Y}_p(\mathfrak{gl}_2),\Gamma,D(\ell))}(L(\ell), L(\ell)) = 3p.$$

Proof. Let $\mathbf{m} = \imath^*(\ell) \in \operatorname{Specm}\Gamma$. By Lemma 8.1 and our assumptions all objects of the category $\mathcal{A}(D(\ell))$ are isomorphic and hence the category $\mathcal{A}(D(\ell))$-mod$_d$ is equivalent to the category of finite-dimensional modules over $\Gamma_{\mathbf{m}}$. Applying Proposition 2.1 we conclude that the category $\mathbb{H}(\mathrm{Y}_p(\mathfrak{gl}_2), \Gamma, D(\ell))$ is hereditary. Since $\Gamma_{\mathbf{m}}$ is an algebra of power series in $3p$ variables (7.1, (5)), the statement about $\dim \operatorname{Ext}^1$ follows. □

8.1. **Category of generic weight modules.** Let us fix

$$\ell \in \mathcal{P}_1, \qquad \mathbf{m} = \imath^*(\ell), \qquad \mathbf{n} = \imath^*(\ell + \delta_i) \in \operatorname{Specm}\Gamma, \qquad i \in \{1, \dots, p\}.$$

Then $\mathcal{A}_W(\mathbf{m}, \mathbf{m}) \simeq \Gamma_{\mathbf{m}}/\Gamma_{\mathbf{m}}\mathbf{m} \simeq \Bbbk$ by Proposition 7.1 (5) and so, $\dim \mathcal{A}_W(\mathbf{m}, \mathbf{n}) = 1$ by Proposition 7.1 (6). We will give a direct construction of the category $\mathcal{A}_W(D(\ell))$.

We shall keep the notation

$$\ell = (\beta, \gamma), \qquad \beta = (\beta_1, \dots, \beta_p) \in \Bbbk^p, \qquad \gamma = (\gamma_1, \dots, \gamma_{2p}) \in \Bbbk^{2p}.$$

Since $\ell \in \mathcal{P}_1$, then $\beta_i - \beta_j \notin \mathbb{Z}$ for $i \neq j$. Consider the following category K_ℓ: $\operatorname{Ob}(K_\ell) = \mathbb{Z}^p$ and the morphisms are generated by

$$f_i(a) : a \mapsto a + \delta_i \qquad \text{and} \qquad e_i(a) : a \mapsto a - \delta_i,$$

where $i = 1, \dots, p$ and $a = (k_1, \dots, k_p) \in \mathbb{Z}^p$ with the following relations:

$$\begin{aligned}
f_j(a + \delta_i)\, f_i(a) &= f_i(a + \delta_j)\, f_j(a), \\
e_j(a - \delta_i)\, e_i(a) &= e_i(a - \delta_j)\, e_j(a), \\
e_i(a + \delta_j)\, f_j(a) &= f_j(a - \delta_i)\, e_i(a) \qquad \text{for} \quad i \neq j, \\
e_i(a + \delta_i)\, f_i(a) &= -\gamma(-\beta_i - k_i)\, 1_{(a)}, \\
f_i(a - \delta_i)\, e_i(a) &= -\gamma(-\beta_i - k_i + 1)\, 1_{(a)}.
\end{aligned}$$

It follows immediately from Lemmas 4.1 and 4.3 that any module in the category R_ℓ defined in 4.1 can be naturally viewed as a module over the category K_ℓ which defines a functor $F : R_\ell \to K_\ell$-mod. For any $a \in \mathbb{Z}^p$ consider the subalgebra $C_\ell(a) = \operatorname{Hom}_{K_\ell}(a, a)$ of the path algebra. Clearly, $C_\ell(a) \simeq \Bbbk$ for any $a \in \mathbb{Z}^p$ due to the defining relations of K_ℓ. Note also that F is an exact functor. For any $a = (k_1, \dots, k_p) \in \mathbb{Z}^p$ we can construct a universal module $M(\ell, a) \in K_\ell$-mod. Consider $\Bbbk$ as a $C_\ell(a)$-module with

$$\begin{aligned}
e_i(a + \delta_i)\, f_i(a)\, 1 &= -\gamma(-\beta_i - k_i), \\
f_i(a - \delta_i)\, e_i(a)\, 1 &= -\gamma(-\beta_i - k_i + 1).
\end{aligned}$$

Let $A_{\ell,a}$ consist of all paths in K_ℓ originating in a. Then $A_{\ell,a}$ is naturally a $K_\ell - C_\ell(a)$-bimodule, where the action of $C_\ell(a)$ on $\Bbbk$ is determined by the defining relations in K_ℓ. Now construct a $\mathbb{Z}^p$-graded K_ℓ-module

$$M(\ell, a) = A_{\ell,a} \otimes_{C_\ell(a)} \Bbbk.$$

Clearly, all graded components of $M(\ell, a)$ are 1-dimensional and $M(\ell, a)_a = 1_a \otimes \Bbbk$. A module $M(\ell, a)$ contains a unique maximal $\mathbb{Z}^p$-graded submodule which intersects $M(\ell, a)_a$ trivially and hence has a unique irreducible quotient $L(\ell, a)$ with $L(\ell, a)_a \simeq \Bbbk$ and $\dim L(\ell, a)_b \leq 1$ for all $b \in \mathbb{Z}^p$. If V is another irreducible K_ℓ-module with $V_a \neq 0$, then there exists a nontrivial $C_\ell(a)$-homomorphism from $\Bbbk$ to V_a which can be extended to an epimorphism from $M(\ell, a)$ to V. Since V is irreducible we conclude that $V \simeq L(\ell, a)$.

Obviously, we can view $M(\ell)$ as a module over the category K_ℓ with a natural action of the morphisms of K_ℓ and $F(M(\ell)) = M(\ell,\beta)$. Thus a K_ℓ-module $M(\ell,\beta)$ can be extended to a $\mathrm{Y}_p(\mathfrak{gl}_2)$-module $M(\ell)$. Moreover, the functor F preserves the submodule structure of $M(\ell)$. In particular, $F(L(\ell)) = L(\ell,\beta)$.

Proposition 8.3. *If $\ell \in \mathcal{P}_1$, then the categories K_ℓ-mod and R_ℓ are equivalent.*

Proof. Let $\ell = (\beta,\gamma)$. We already have the functor $F : R_\ell \to K_\ell$-mod. Suppose that $V \in K_\ell$-mod. We want to show that V can be extended to a $\mathrm{Y}_p(\mathfrak{gl}_2)$-module. Let $a = (k_1,\dots,k_p)$ and $v \in V_a \setminus \{0\}$. Consider a submodule $W \subseteq V$ generated by v. Then $W_a = \Bbbk v$ and there is an epimorphism from $M(\ell,a)$ to W, which maps $1_a \otimes 1$ to v. Since $F(M(\ell')) = M(\ell,a)$, where $\ell' = (\beta + a,\gamma)$, then W can be extended to a corresponding quotient of $M(\ell')$. Since v was an arbitrary element of V, we conclude that V can be extended to a $\mathrm{Y}_p(\mathfrak{gl}_2)$-module and will denote that module by $G(V)$. This way G defines a functor from K_ℓ-mod to R_ℓ (action on morphisms is obvious). One can easily see that the functors F and G define an equivalence between the categories K_ℓ-mod and R_ℓ. □

8.2. **Support of irreducible generic weight modules.** To complete the classification of irreducible modules we have to know when two irreducible modules $L(\ell)$ and $L(\ell')$ are isomorphic. For that we need to describe the support $\operatorname{Supp} L(\ell)$.

We shall say that the weight subspaces $M(\ell)_\psi$ and $M(\ell)_{\psi+\delta_i}$ are *strongly isomorphic* if $\gamma(-\psi_i) \neq 0$ where $\psi = (\psi_1,\dots,\psi_p)$. This implies

$$f_i(\psi_1,\dots,\psi_p)\, M(\ell)_\psi \neq 0 \quad \text{and} \quad e_i(\psi_1,\dots,\psi_i+1,\dots,\psi_p)\, M(\ell)_{\psi+\delta_i} \neq 0.$$

The statement below follows immediately from the relations in K_ℓ (cf. also Corollary 4.4).

Lemma 8.4. *If $M(\ell)_\psi$ and $M(\ell)_{\psi+\delta_i}$ are strongly isomorphic, then $M(\ell)_{\psi\pm\delta_j}$ and $M(\ell)_{\psi+\delta_i\pm\delta_j}$ are strongly isomorphic for all $i,j = 1,\dots,p$ such that $i \neq j$. Moreover, if*

$$f_i(\psi_1,\dots,\psi_p)\, M(\ell)_\psi = 0 \qquad \text{or} \qquad e_i(\psi_1,\dots,\psi_p)\, M(\ell)_\psi = 0,$$

then

$$f_i(\psi_1,\dots,\psi_j\pm 1,\dots,\psi_p)\, M(\ell)_{\psi\pm\delta_j} = 0 \qquad \text{or}$$
$$e_i(\psi_1,\dots,\psi_j\pm 1,\dots,\psi_p)\, M(\ell)_{\psi\pm\delta_j} = 0,$$

respectively, for all $j \neq i$.

Let $a_i, a_i' \in \mathbb{Z} \cup \{\pm\infty\}, a_i \leq a_i'$, $i \in \{1,\dots,p\}$. Denote

$$P(a_1,\dots,a_p,a_1',\dots,a_p') = \{(x_1,\dots,x_p) \in \mathbb{Z}^p \mid a_i \leq x_i \leq a_i',\ i = 1,\dots,p\},$$

a parallelepiped in $\mathbb{Z}^p$. Note that some faces of the parallelepiped can be infinite in some directions. In particular, in the case $a_i = -\infty$, $a_i' = \infty$ for all i, the parallelepiped coincides with $\mathbb{Z}^p$.

Theorem 8.5. *For any irreducible weight module $L(\ell)$ over $\mathrm{Y}_p(\mathfrak{gl}_2)$ there exist elements $a_i, a_i' \in \mathbb{Z} \cup \{\pm\infty\}$, $a_i \leq a_i'$, $i \in \{1,\dots,p\}$ such that*

$$\operatorname{Supp} L(\ell) = P(a_1,\dots,a_p,a_1',\dots,a_p').$$

Proof. Let $\ell = (\beta,\gamma) \in \mathcal{P}_1$. Fix $i \in \{1,\dots,p\}$. If $\gamma(-\beta_i + k) \neq 0$ for all $k \in \mathbb{Z}$, then

$$(k_1,\dots,k_i+m,\dots,k_p) \in \operatorname{Supp} L(\ell)$$

as soon as $(k_1,\dots,k_p)\in \operatorname{Supp} L(\ell)$. This follows immediately from Lemma 8.4. In this case we set $a_i=-\infty$ and $a_i'=\infty$. Now let $\gamma(-\beta_i+k)=0$ for some $k\in\mathbb{Z}$. Let $m\geq 0$ be the smallest integer (if it exists) such that $\gamma(-\beta_i-m)=0$ and let $n\leq 0$ be the largest integer (if it exists) such that $\gamma(-\beta_i-n+1)=0$. It follows from Lemma 8.4 that

$$\operatorname{Supp} L(\ell)\cap\{\beta+k\delta_i \mid k\in\mathbb{Z}\}=\{\beta+n\delta_i,\dots,\beta,\dots,\beta+m\delta_i\}.$$

If $\beta+s\delta_j\in\operatorname{Supp} L(\ell)$, $j\neq i$, then

$$\operatorname{Supp} L(\ell)\cap\{\beta+s\delta_j+k\delta_i \mid k\in\mathbb{Z}\}=\{\beta+s\delta_j+n\delta_i,\dots,\beta+s\delta_j,\dots,\beta+s\delta_j+m\delta_i\}.$$

In this case we set $a_i=\beta_i+n$ and $a_i'=\beta_i+m$. The statement of the theorem now follows. $\square$

8.3. **Indecomposable generic weight modules.** Fix $\ell=(\beta,\gamma)\in\mathcal{P}_1$. A full subcategory $\mathcal{S}\subseteq K_\ell$ is called a *skeleton* of K_ℓ provided the objects of $\mathcal{S}$ are pairwise non-isomorphic and any object of K_ℓ is isomorphic to some object of $\mathcal{S}$. In this case the categories of K_ℓ-mod and $\mathcal{S}$-mod are equivalent.

For each $i\in\{1,\dots,p\}$ consider a set $I_i=\{k\in\mathbb{Z}\mid \gamma(-\beta_i-k)=0\}$. Define a category S_ℓ as a $\Bbbk$-category with the set of objects

$$S_0=\{0,\dots,|I_1|\}\times\cdots\times\{0,\dots,|I_p|\}$$

and with morphisms generated by

$$r^k_{(i_1,\dots,i_p)}:(i_1,\dots,i_p)\mapsto(i_1,\dots,i_k+1,\dots,i_p),$$
$$s^k_{(j_1,\dots,j_p)}:(j_1,\dots,j_p)\mapsto(j_1,\dots,j_k-1,\dots,j_p),$$

where $k\in\{1,\dots,p\}$ is such that $I_k\neq\emptyset$, $i_k<|I_k|$, $j_k>0$, subject to the relations

$$s^k_{(i_1,\dots,i_k+1,\dots,i_p)}r^k_{(i_1,\dots,i_p)}=r^k_{(i_1,\dots,i_p)}s^k_{(i_1,\dots,i_k+1,\dots,i_p)}=0$$

and

$$x^k_{(a_1,\dots,a_p)}y^r_{(e_1,\dots,e_p)}=y^r_{(c_1,\dots,c_p)}x^k_{(e_1,\dots,e_p)}$$

for all $k\neq r$ and all possible $x,y\in\{r,s\}$ and all a_i,e_i,c_i, with $1\leq i\leq p$ for which this equality makes sense.

It follows from the construction that S_ℓ is the skeleton of the category K_ℓ. Note that the corresponding algebra is finite-dimensional. In particular, S_ℓ is semisimple when $I_k=\emptyset$ for all $1\leq k\leq p$, i.e., when $\gamma(-\beta_k+r)\neq 0$ for all $r\in z'$ and all $k=1,\dots,p$. Hence it is sufficient to describe all indecomposable modules over S_ℓ.

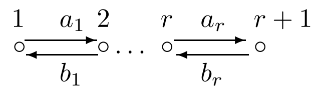
Fix $a\in S_0$ and define a simple S_ℓ-module S_a such that $S_a(b)=\delta_{a,b}\Bbbk$ for all $b\in S_0$ and all morphisms are trivial. Since S_ℓ defines a finite-dimensional algebra we have the following

Proposition 8.6. *Any simple module over S_ℓ is isomorphic to S_a for some $a\in S_0$.*

This is another confirmation of the fact that all weight spaces in any irreducible generic weight $\mathrm{Y}_p(\mathfrak{gl}_2)$-module are 1-dimensional. But this need not be the case for indecomposable modules. We restrict ourselves to a full subcategory $R^f_\ell\subseteq R_\ell$ which consists of weight modules V with $\dim V_\psi<\infty$ for all $\psi\in\operatorname{Supp} V$. We will establish the representation type of the category R^f_ℓ (finite, tame or wild). For the necessary definitions we refer the reader to [Dr].

To establish the representation type of the category R_ℓ^f it is sufficient to consider the category S_ℓ-mod^f of modules over the category S_ℓ with finite-dimensional weight spaces. Denote $X_\ell = \{k \in \{1, \dots, p\} \mid I_k \neq \emptyset\}$.

8.3.1. *Indecomposable modules in the case* $|X_\ell| = 1$. In this section we describe all indecomposable modules over S_ℓ in the case $|X_\ell| = 1$. Let $X_\ell = \{i\}$ and let $|I_i| = r > 0$. In this case the category S_ℓ has the following quiver $\mathbf{A}$ with the relations



$$a_i\, b_i = b_i\, a_i = 0.$$

We denote by S_i, $i \in \{1, \dots, r+1\}$, the simple module corresponding to the point i. These modules correspond to all irreducible modules in R_ℓ^f by Proposition 8.6. Now describe the remaining indecomposable modules for the quiver above. Fix integers $1 \leq k_1 < k_2 \leq r+1$ and a function

$$\xi_{k_1,k_2} : \{k_1, k_1+1, \dots, k_2\} \to \{0, 1\}.$$

Define the module $M = M(k_1, k_2, \xi_{k_1,k_2})$ as follows: $M(i) = \Bbbk e_i$, $k_1 \leq i \leq k_2$ and $M(i) = 0$ otherwise. Furthermore,

$$a_i e_i = e_{i+1}, \qquad b_i e_{i+1} = 0 \qquad \text{if} \qquad \xi_{k_1,k_2}(i) = 1$$

and

$$a_i e_i = 0, \qquad b_i e_{i+1} = e_i \qquad \text{if} \qquad \xi_{k_1,k_2}(i) = 0,$$

for all $1 \leq i < k_2$. The proof of the following proposition is standard; see e.g. [GR].

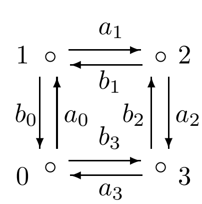
Proposition 8.7. *The modules S_i for $1 \leq i \leq r+1$ and $M(k_1, k_2, \xi_{k_1,k_2})$ with $1 \leq k_1 < k_2 \leq r+1$ and*

$$\xi_{k_1,k_2} : \{k_1, k_1+1, \dots, k_2\} \to \{0, 1\},$$

exhaust all non-isomorphic indecomposable modules for $\mathbf{A}$.

8.3.2. *Indecomposable modules in the case* $|X|_\ell = 2$. In this section we describe the indecomposable modules for S_ℓ when $|X|_\ell = 2$ and $|I_k| = 1$ for each $k \in X_\ell$. Then S_ℓ is isomorphic to the following category $\mathbf{B}$ considered in [BB].

$\mathbf{B}$:



$a_i b_i = b_i a_i = 0, \qquad i = 0, \dots, 3,$
$a_i a_j = b_l b_m$ for any $i, j, l, m \in \{0, 1, 2, 3\}$,
where possible.

By Proposition 8.6, this category has four non-isomorphic simple modules S_i where $i \in \{0, 1, 2, 3\}$, with the support in the given point i. The indecomposable modules were described in [BB]. For the sake of completeness we repeat here this classification.

We will treat the objects of $\mathbf{B}$ as elements of $\mathbb{Z}/4\mathbb{Z}$. Consider the following three families of non-simple indecomposable modules.

Finite family. Fix $i \in \{0,1,2,3\}$ and define the $\mathbf{B}$-module M_i such that $M_i(j) = \Bbbk e_j$ for each $j = 0,1,2,3$ and

$$a_i e_i = e_{i+1}, \qquad a_{i+1}e_{i+1} = e_{i+2}, \qquad b_{i-1}e_i = e_{i-1}, \qquad b_{i-2}e_{i-1} = e_{i-2}$$

while $u_j e_k = 0$ for all other cases of $u \in \{a,b\}$ and $j,k = 0,\dots,3$. Obviously, M_i is an indecomposable module for any i.

Infinite discrete families. Let $n \in \mathbb{N}$, $n > 1$, and $j \in \mathbb{Z}_4$. Define a $\mathbf{B}$-module $M_{n,j,1}$ (respectively, $M_{n,j,2}$) as follows. Consider n elements $e_1,\dots,e_n$. A $\Bbbk$-basis of the vector space $M_{n,j,1}(l)$ (respectively, $M_{n,j,2}(l)$) is the set of e_k such that $j+k-1 \equiv l (\operatorname{mod} 4)$. The elements a_l and b_{l-1} act as follows:

$$a_l e_k = \begin{cases} e_{k+1}, & \text{if } l \text{ is even (resp., odd), } k < n \text{ and } j+k-1 \equiv l (\operatorname{mod} 4); \\ 0, & \text{otherwise.} \end{cases}$$

$$b_{l-1} e_k = \begin{cases} e_{k-1}, & \text{if } l \text{ is even (resp., odd), } k > 1 \text{ and } j+k-1 \equiv l (\operatorname{mod} 4); \\ 0, & \text{otherwise.} \end{cases}$$

All modules $M_{n,j,1}$ and $M_{n,j,2}$, $n > 1$, $0 \le j \le 3$ are non-isomorphic indecomposable $\mathbf{B}$-modules.

Infinite continuous families. For each $\lambda \in \Bbbk$, $\lambda \ne 0$, and $d \in \mathbb{Z}$, $d > 0$ define the $\mathbf{B}$-modules $M_{d,\lambda,1}$ and $M_{d,\lambda,2}$ as follows. Set

$$\begin{aligned} M_{d,\lambda,1}(i) &= \Bbbk^d, \\ M_{d,\lambda,1}(a_0) &= M_{d,\lambda,1}(a_2) = M_{d,\lambda,1}(b_1) = \mathbf{I}_d, \\ M_{d,\lambda,1}(b_0) &= M_{d,\lambda,1}(b_2) = M_{d,\lambda,1}(a_1) = M_{d,\lambda,1}(a_3) = 0, \\ M_{d,\lambda,1}(b_3) &= J_{d,\lambda} \end{aligned}$$

and

$$\begin{aligned} M_{d,\lambda,2}(i) &= \Bbbk^d, \\ M_{d,\lambda,2}(b_0) &= M_{d,\lambda,2}(b_2) = M_{d,\lambda,2}(a_1) = \mathbf{I}_d, \\ M_{d,\lambda,2}(a_0) &= M_{d,\lambda,2}(a_2) = M_{d,\lambda,2}(b_1) = M_{d,\lambda,2}(b_3) = 0, \\ M_{d,\lambda,2}(a_3) &= J_{d,\lambda}, \end{aligned}$$

where $J_{d,\lambda}$ is the Jordan cell of dimension d with the eigenvalue λ.

All modules $M_{d,\lambda,k}$, $k = 1,2$ are indecomposable and corresponding indecomposable modules in R_ℓ^f have all weight spaces of dimension d.

Proposition 8.8 ([BB], Proposition 3.3.1)**.** *The modules S_i, M_i, $M_{n,i,1}$, $M_{n,i,2}$, $M_{d,\lambda,1}$, $M_{d,\lambda,2}$ where $0 \le i \le 3$, d is a positive integer, $\lambda \in \Bbbk$, $\lambda \ne 0$, and $n \ge 2$ is an integer, constitute an exhaustive list of pairwise non-isomorphic indecomposable $\mathbf{B}$-modules.*

The following theorem describes the representation type of R_ℓ^f.

Theorem 8.9.
(1) *If $|X_\ell| = 0$, then R_ℓ^f is a semisimple category with a unique indecomposable (=irreducible) module;*
(2) *If $|X_\ell| = 1$, then R_ℓ^f has finite representation type;*
(3) *If $|X_\ell| = 2$, then R_ℓ^f has tame representation type if and only if $|I_k| = 1$ for all $k \in X$. Otherwise, R_ℓ^f has wild representation type;*
(4) *If $|X_\ell| > 2$, then R_ℓ^f has wild representation type.*

Proof. In the case when $|X_\ell| = 1$ all indecomposable modules for S_ℓ are described in Proposition 8.7. Hence R_ℓ^f has the finite representation type. If $|X_\ell| = 2$ and $|I_k| = 1$ for each $k \in X$, then all indecomposable modules for S_ℓ are described in Proposition 8.8. It follows from the definition that R_ℓ^f has the tame representation type in this case. If $|I_k| > 1$ for at least one k, then it is easy to construct a family of indecomposable modules that depends on two continuous parameters. Hence, in this case R_ℓ^f has the wild representation type. Suppose now that $|X_\ell| > 2$. Then S_ℓ contains a full subcategory of wild representation type considered in [BB, Theorem 1]. We immediately conclude that R_ℓ^f has the wild representation type. This completes the proof. □

Corollary 8.10. (1) *If* $|X_\ell| = 0$*, then the category* R_ℓ *is a semisimple category with a unique indecomposable module.*

(2) *If* $|X_\ell| = 1$*, then* R_ℓ *has finite representation type with indecomposable modules as in Proposition* 8.7.

Proof. Since cases $|X_\ell| \leq 1$ correspond to the finite representation type, then the corresponding categories do not admit infinite-dimensional indecomposable modules by [A] and hence every indecomposable module belongs to R_ℓ^f. □

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