

## FINITE DIMENSIONAL REPRESENTATIONS OF SYMPLECTIC REFLECTION ALGEBRAS ASSOCIATED TO WREATH PRODUCTS

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ABSTRACT. Using deformation theory of representations of algebras, we construct families of finite dimensional representations of symplectic reflection algebras associated to wreath products.

### 1. INTRODUCTION

In this paper we study finite dimensional representations of the wreath product symplectic reflection algebra  $H_{1,k,c}(\Gamma_N)$  of rank  $N$  attached to the group  $\Gamma_N = S_N \ltimes \Gamma^N$  ([EG], page 6), where  $\Gamma \subset SL(2, \mathbf{C})$  is a finite subgroup, and  $(k, c) \in C(\mathcal{S})$ , where  $C(\mathcal{S})$  is the space of (complex valued) class functions on the set  $\mathcal{S}$  of symplectic reflections of  $\Gamma_N$ .

In the rank 1 case, there is no parameter  $k$ , and finite dimensional representations of the wreath product algebra have been classified in [CBH] by Crawley-Boevey and Holland, by establishing a Morita equivalence between the algebra  $H_{1,c}(\Gamma)$  and the deformed preprojective algebra  $\Pi_{\lambda(c)}(Q)$  attached to the (extended Dynkin) quiver  $Q$  associated to  $\Gamma$  via the McKay correspondence.

We consider the higher rank case. When  $k=0$ , then  $H_{1,k,c}(\Gamma_N) = S_N \sharp H_{1,c}(\Gamma)^{\otimes N}$ , so the finite dimensional representations of  $H_{1,k,c}(\Gamma_N)$  are known. Using a cohomological approach, we investigate the possibility of deforming some of these representations to values of the parameters with  $k \neq 0$ . This allows us to produce the first nontrivial examples of finite dimensional representations of  $H_{1,k,c}(\Gamma_N)$  for noncyclic  $\Gamma$  and  $k \neq 0$ .

Specifically, we show that if  $W$  is an irreducible representation of  $S_N$  whose Young diagram is a rectangle, and  $Y$  an irreducible finite dimensional representation of  $H_{1,c}(\Gamma)$ , then the representation  $M = W \otimes Y^{\otimes N}$  of  $H_{1,0,c}(\Gamma_N)$  can be deformed along a hyperplane in  $C(\mathcal{S})$ . On the other hand, if  $\dim Y = 1$  and the Young diagram of  $W$  is not a rectangle, such a deformation does not exist.

### 2. PRELIMINARIES

**2.1. The wreath product construction.** In this subsection we will briefly recall the wreath product construction. Let  $L$  be a 2-dimensional complex vector space with a symplectic form  $\omega_L$ , and consider the space  $V = L^{\oplus N}$ , endowed with the

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induced symplectic form  $\omega_V = \omega_L^{\oplus N}$ . Let  $\Gamma$  be a finite subgroup of  $Sp(L)$ , and let  $S_N$  be the symmetric group acting on  $V$  by permuting the factors. The group  $\mathbf{\Gamma}_N := S_N \ltimes \Gamma^N \subset Sp(V)$  acts naturally on  $V$ . In the sequel we will write  $\gamma_i \in \mathbf{\Gamma}_N$  for any element  $\gamma \in \Gamma$  seen as an element in the  $i$ -th factor  $\Gamma$  of  $\mathbf{\Gamma}_N$ . The symplectic reflections in  $\mathbf{\Gamma}_N$  are the elements  $s$  such that  $\text{rk}(Id - s)|_V = 2$ .  $\mathbf{\Gamma}_N$  acts by conjugation on the set  $\mathcal{S}$  of its symplectic reflections. It is easy to see that there are symplectic reflections of two types in  $\mathbf{\Gamma}_N$ :

- (S) the elements  $s_{ij}\gamma_i\gamma_j^{-1}$  where  $i, j \in [1, N]$ ,  $s_{ij}$  is the transposition  $(ij) \in S_N$ , and  $\gamma \in \Gamma$ ,
- ( $\Gamma$ ) the elements  $\gamma_i$ , for  $i \in [1, N]$  and  $\gamma \in \Gamma \setminus \{1\}$ .

Elements of type (S) are all in the same conjugacy class, while elements of type ( $\Gamma$ ) form one conjugacy class for any conjugacy class of  $\gamma$  in  $\Gamma$ . Thus elements  $f \in C[\mathcal{S}]$  can be written as pairs  $(k, c)$ , where  $k$  is a number (the value of  $f$  on elements of type (S)), and  $c$  is a conjugation invariant function on  $\Gamma \setminus \{1\}$  (encoding the values of  $f$  on elements of type ( $\Gamma$ )).

For any  $s \in \mathcal{S}$  we write  $\omega_s$  for the bilinear form on  $V$  that coincides with  $\omega_V$  on  $\text{Im}(Id - s)$  and has  $\text{Ker}(Id - s)$  as radical. Denote by  $TV$  the tensor algebra of  $V$ .

**Definition 2.1.** For any  $t \in \mathbf{C}$  and  $f = (k, c) \in C[\mathcal{S}]$ , the symplectic reflection algebra  $H_{t,k,c}(\mathbf{\Gamma}_N)$  is the quotient

$$(\mathbf{\Gamma}_N \# TV) / \langle [u, v] - \kappa(u, v) \rangle_{u,v \in V}$$

where

$$\kappa : V \otimes V \longrightarrow \mathbf{C}[\mathbf{\Gamma}_N] : (u, v) \mapsto t \cdot \omega(u, v) \cdot 1 + \sum_{s \in \mathcal{S}} f_s \cdot \omega_s(u, v) \cdot s$$

with  $f_s = f(s)$ , and  $\langle \dots \rangle$  is the two-sided ideal in the smash product  $\mathbf{\Gamma}_N \# TV$  generated by the elements  $[u, v] - \kappa(u, v)$  for  $u, v \in V$ .

We will be interested in the case  $t \neq 0$ , and it will be enough to consider the case  $t = 1$  since  $H_{t,k,c}(\mathbf{\Gamma}_N) \cong H_{1,k/t,c/t}(\mathbf{\Gamma}_N)$  for any  $t \neq 0$  (cf. [EG], page 14). We recall that the case  $t = 0$  is remarkably different and the corresponding representation theory has been studied in [EG], Section 3 and in [GS].

It is clear that choosing a symplectic basis  $x, y$  for  $L$  we can consider  $\Gamma$  as a subgroup of  $SL(2, \mathbf{C})$ . We will denote by  $x_i, y_i$  the corresponding vectors in the  $i$ -th  $L$ -factor of  $V$ . Following [GG] we will now give a more explicit representation of the algebra  $H_{1,k,c}(\mathbf{\Gamma}_N)$ .

**Lemma 2.2** ([GG]). *The algebra  $H_{1,k,c}(\mathbf{\Gamma}_N)$  is the quotient of  $\mathbf{\Gamma}_N \# TV$  by the following relations:*

(R1) For any  $i \in [1, N]$ :

$$[x_i, y_i] = 1 + \frac{k}{2} \sum_{j \neq i} \sum_{\gamma \in \Gamma} s_{ij}\gamma_i\gamma_j^{-1} + \sum_{\gamma \in \Gamma \setminus \{1\}} c_\gamma \gamma_i.$$

(R2) For any  $u, v \in L$  and  $i \neq j$ :

$$[u_i, v_j] = -\frac{k}{2} \sum_{\gamma \in \Gamma} \omega_L(\gamma u, v) s_{ij}\gamma_i\gamma_j^{-1}.$$

□

In the case  $N = 1$ , there is no parameter  $k$  (there are no symplectic reflections of type (S)) and

$$H_{1,c}(\Gamma) = \mathbf{C}\Gamma \sharp \mathbf{C} \langle x, y \rangle / (xy - yx = \lambda)$$

where  $\lambda = \lambda(c) = 1 + \sum_{\gamma \in \Gamma \setminus \{1\}} c_\gamma \gamma \in Z(\mathbf{C}[\Gamma])$  is the central element corresponding to  $c$ .

We end this subsection by recalling an important result that we will need later. It is stated in [EG] (Theorem 1.3) and is called the *Poincaré-Birkhoff-Witt (PBW-) property* for  $H_{1,k,c}(\Gamma_N)$ . Consider the increasing filtration on  $TV \sharp \Gamma_N$  obtained by assigning degree zero to the elements of the group algebra  $\mathbf{C}[\Gamma_N]$  and degree one to the vectors in  $V$ . This filtration induces a filtration on  $H_{1,k,c}(\Gamma_N)$ . The following theorem holds:

**Theorem 2.3** (PBW). *The associated graded algebra to  $H_{1,k,c}(\Gamma_N)$  with respect to the above increasing filtration is  $\Gamma_N \sharp SV$ , where  $SV$  is the symmetric algebra of  $V$ .  $\square$*

**2.2. Representations of  $S_N$  with a rectangular Young diagram.** We will use the following standard results from representation theory of the symmetric group. The proofs are well known, but we recall them for the reader's convenience. Denote by  $\mathfrak{h}$  the reflection representation of  $S_N$ . For a Young diagram  $\mu$  we denote by  $\pi_\mu$  the corresponding irreducible representation of  $S_N$ .

**Lemma 2.4.** (i)  $\text{Hom}_{S_N}(\mathfrak{h} \otimes \pi_\mu, \pi_\mu) = \mathbf{C}^{m-1}$ , where  $m$  is the number of corners of the Young diagram  $\mu$ . In particular,  $\text{Hom}_{S_N}(\mathfrak{h} \otimes \pi_\mu, \pi_\mu) = 0$  if and only if  $\mu$  is a rectangle.

(ii) *The element  $C = s_{12} + s_{13} + \cdots + s_{1N}$  acts by a scalar in  $\pi_\mu$  if and only if  $\mu$  is a rectangle.*

*Proof.* Let  $S_{N-1} \subset S_N$  be the subgroup of permutations fixing the index 1. It is well known that  $\pi_\mu|_{S_{N-1}} = \sum \pi_{\mu-j}$ , where the sum is taken over the corners of  $\mu$  and  $\mu-j$  is the Young diagram obtained from  $\mu$  by cutting off the corner  $j$ . Since  $\mathfrak{h} \oplus \mathbf{C} = \text{Ind}_{S_{N-1}}^{S_N} \mathbf{C}$ , the assertion (i) follows from the Frobenius reciprocity. To prove (ii), observe that  $C$  commutes with  $S_{N-1}$ , so it acts by a scalar on each  $\pi_{\mu-j}$ . Thus, if  $\mu$  is a rectangle,  $C$  acts as a scalar (as we have only one summand), and the “if” part of the statement is proved. To prove the “only if” part, let  $Z_N$  be the sum of all transpositions in  $S_N$ .  $Z_N$  is a central element in the group algebra, and it is known to act in  $\pi_\mu$  by the scalar  $\mathbf{c}(\mu)$ , where  $\mathbf{c}(\mu)$  is the content of  $\mu$ , i.e. the sum over all cells of the signed distances from these cells to the diagonal. Now,  $C = Z_N - Z_{N-1}$ , so it acts on  $\pi_{\lambda-j}$  by the scalar  $\mathbf{c}(j)$ , the signed distance from the cell  $j$  to the diagonal. The numbers  $\mathbf{c}(j)$  are clearly different for all corners  $j$ , so if there are 2 or more corners, then  $C$  cannot act by a scalar. This finishes the proof of (ii).  $\square$

### 3. THE MAIN THEOREM

Let  $Y$  be a finite dimensional irreducible representation of the algebra  $H_{1,c}(\Gamma)$  for some<sup>1</sup>  $c$ . Let  $W$  be an irreducible representation of  $S_N$ . Since the algebra  $H_{1,0,c}(\Gamma_N)$  is naturally isomorphic to  $S_N \sharp H_{1,c}(\Gamma)^{\otimes N}$ , there is a natural action of  $H_{1,0,c}(\Gamma_N)$  on the vector space  $M := W \otimes Y^{\otimes N}$ . Namely, each copy of  $H_{1,c}(\Gamma)$  acts

<sup>1</sup>Such representations exist only for special  $c$ ; as for generic  $c$ , the algebra  $H_{1,c}(\Gamma)$  is simple; see [CBH], Theorem 0.3, Corollary 7.6.

in the corresponding copy of  $Y$ , while  $S_N$  acts in  $W$  and simultaneously permutes the factors in the product  $Y^{\otimes N}$ . We will denote this representation by  $M_c$ . The main theorem tells us when such a representation can be deformed to nonzero values of  $k$ .

Assume that the Young diagram of  $W$  is a rectangle of height  $l$  and width  $m = N/l$  (the trivial representation corresponds to the horizontal strip of height 1).

Let  $\mathcal{H}_{Y,m,l}$  be the hyperplane in  $C(\mathcal{S})$  consisting of all pairs  $(k, c)$  satisfying the equation

$$(1) \quad \dim Y + \frac{k}{2} |\Gamma|(m-l) + \sum_{\gamma \in \Gamma \setminus \{1\}} c_\gamma \chi_Y(\gamma) = 0,$$

where  $\chi_Y$  is the character of  $Y$ .

Let  $X = X(Y, m, l)$  be the moduli space of irreducible representations of  $H_{1,k,c}(\mathbf{\Gamma}_N)$  isomorphic to  $M$  as  $\mathbf{\Gamma}_N$ -modules (where  $(k, c)$  are allowed to vary). This is a quasi-affine algebraic variety: it is the quotient of the quasi-affine variety  $\tilde{X}(Y, m, l)$  of extensions of the  $\mathbf{\Gamma}_N$ -module  $M$  to an irreducible  $H_{1,k,c}(\mathbf{\Gamma}_N)$ -module by a free action of the reductive group  $G$  of basis changes in  $M$  compatible with  $\mathbf{\Gamma}_N$  modulo scalars. Let  $f : X \rightarrow C(\mathcal{S})$  be the morphism which sends a representation to the corresponding values of  $(k, c)$ .

The main result of this paper is the following theorem.

**Theorem 3.1.** (i) *For any  $c_0$  the representation  $M_{c_0}$  of  $H_{1,0,c_0}(\mathbf{\Gamma}_N)$  can be formally deformed to a representation of  $H_{1,k,c}(\mathbf{\Gamma}_N)$  along the hyperplane  $\mathcal{H}_{Y,m,l}$ , but not in other directions. This deformation is unique.*

(ii) *The morphism  $f$  maps  $X$  to  $\mathcal{H}_{Y,m,l}$  and is étale at  $M_{c_0}$  for all  $c_0$ . Its restriction to the formal neighborhood of  $M_{c_0}$  is the deformation from (i).*

(iii) *There exists a nonempty Zariski open subset  $\mathcal{U}$  of  $\mathcal{H}_{Y,m,l}$  such that for  $(k, c) \in \mathcal{U}$ , the algebra  $H_{1,k,c}(\mathbf{\Gamma}_N)$  admits a finite dimensional irreducible representation isomorphic to  $M$  as a  $\mathbf{\Gamma}_N$ -module.*

The proof of this theorem occupies the remaining sections of the paper.

*Remark.* In the case of cyclic  $\Gamma$  and trivial  $W$  Theorem 3.1 was proved in [CE]. In this case, the deformation of the representation  $M$  can be constructed explicitly.

We expect that the condition that the Young diagram of  $W$  is a rectangle is essential to obtain the deformation of Theorem 3.1 (i). For example, this is the case if  $Y$  is 1-dimensional. This follows from the following more general statement.

**Proposition 3.2.** *Let  $W$  be an irreducible  $S_N$ -module. If  $W$  extends to a representation of  $H_{1,k,c}(\mathbf{\Gamma}_N)$  for some  $(k, c)$  with  $k \neq 0$ , then the Young diagram of  $W$  is a rectangle.*

*Proof.* Suppose that  $W$  extends to a representation of  $H_{1,k,c}(\mathbf{\Gamma}_N)$ . Such an extension is, first of all, an extension of  $W$  to a representation of the wreath product group  $S_N \rtimes \Gamma^N$ . This can only be done by making  $\Gamma^N$  act by an  $S_N$ -invariant character  $\xi$ , i.e.,  $\xi(\gamma_1, \dots, \gamma_N) = \chi(\gamma_1) \dots \chi(\gamma_N)$ , where  $\chi : \Gamma \rightarrow \mathbf{C}^*$  is a character. But in this case  $\Gamma^N$  acts trivially on  $\text{End}_{\mathbf{C}}(W)$ , and hence  $x_i, y_i$  must act by 0 on  $W$  for each  $i = 1, \dots, N$ . So, denoting by  $\rho$  the possible extended representation,

we obtain from relation (R1) for  $i = 1$ ,

$$\rho(s_{12} + \dots + s_{1N}) = -2 \frac{1 + \sum_{\gamma \in \Gamma \setminus \{1\}} c_\gamma \chi(\gamma)}{k |\Gamma|},$$

i.e.,  $C = s_{12} + \dots + s_{1N}$  acts by a constant on  $W$ . Now applying Lemma 2.4, part (ii), we get that  $W$  must correspond to a rectangular Young diagram.  $\square$

#### 4. PROOF OF THEOREM 3.1

**4.1. Deformation theory.** In this section we recall deformation theory of representations of algebras. This theory is well known, but we give the details for the reader’s convenience.

Let  $A$  be an associative algebra over  $\mathbf{C}$ . In what follows, for each  $A$ -bimodule  $E$ , we write  $H^n(A, E)$  for the  $n$ -th Hochschild cohomology group of  $A$  with coefficients in  $E$ . We recall that  $H^n(A, E)$  is defined to be the  $i$ -th cohomology group of the Hochschild complex:

$$0 \longrightarrow C^0(A, E) \xrightarrow{d} \dots \xrightarrow{d} C^n(A, E) \xrightarrow{d} C^{n+1}(A, E) \xrightarrow{d} \dots$$

where  $C^n(A, E) = \text{Hom}_{\mathbf{C}}(A^{\otimes n}, E)$  is the space of  $n$ -linear maps from  $A^{\otimes n}$  to  $E$ , and the differential  $d$  is defined as follows:

$$\begin{aligned} (d\varphi)(a_1, \dots, a_{n+1}) : &= a_1 \varphi(a_2, \dots, a_{n+1}) \\ &+ \sum_{i=1}^n (-1)^i \varphi(a_1, \dots, a_{i-1}, a_i a_{i+1}, a_{i+2}, \dots, a_{n+1}) \\ &- (-1)^n \varphi(a_1, \dots, a_n) a_{n+1}. \end{aligned}$$

We remark that  $H^i(A, E)$  coincides with the vector space  $\text{Ext}_{A \otimes A^o}^i(A, E)$ , where  $A^o$  is the opposite algebra of  $A$ .

Let  $A_U$  be a flat formal deformation of  $A$  over the formal neighborhood of zero in a finite dimensional vector space  $U$  with coordinates  $t_1, \dots, t_n$ . This means that  $A_U$  is an algebra over  $\mathbf{C}[[U]] = \mathbf{C}[[t_1, \dots, t_n]]$  which is *topologically free* as a  $\mathbf{C}[[U]]$ -module (i.e.,  $A_U$  is isomorphic as a  $\mathbf{C}[[U]]$ -module to  $A[[U]]$ ), together with a fixed isomorphism of algebras  $A_U/JA_U \cong A$ , where  $J$  is the maximal ideal in  $\mathbf{C}[[U]]$ . Given such a deformation, we have a natural linear map  $\phi : U \longrightarrow H^2(A, A)$  defined as follows.

We can think of  $A_U$  as  $A[[t_1, \dots, t_n]]$  equipped with a new  $\mathbf{C}[[t_1, \dots, t_n]]$ -linear (and continuous) associative product defined by

$$a * b = \sum_{p_1, \dots, p_n} c_{p_1, \dots, p_n}(a, b) \prod_j t_j^{p_j} \quad a, b \in A$$

where  $c_{p_1, \dots, p_n} : A \times A \longrightarrow A$  are  $\mathbf{C}$ -bilinear functions and  $c_{0, \dots, 0}(a, b) = ab$ , for any  $a, b \in A$ .

Imposing the associativity condition on  $*$ , one obtains that  $c_{0, \dots, 1_j, \dots, 0}$  must be Hochschild 2-cocycles for each  $j$ . The map  $\phi$  is given by the assignment  $(t_1, \dots, t_n) \rightarrow \sum_j t_j [c_{0, \dots, 1_j, \dots, 0}]$  for any  $(t_1, \dots, t_n) \in U$ , where  $[C]$  stands for the cohomology class of a cocycle  $C$ .

Now let  $M$  be a representation of  $A$ . In general it does not deform to a representation of  $A_U$ . However, we have the following standard proposition. Let  $\eta : U \rightarrow H^2(A, \text{End } M)$  be the composition of  $\phi$  with the natural map  $\psi : H^2(A, A) \rightarrow H^2(A, \text{End } M)$ .

**Proposition 4.1.** *Assume that  $\eta$  is surjective with kernel  $K$ , and  $H^1(A, \text{End } M) = 0$ . Then:*

(i) *There exists a unique smooth formal subscheme  $S$  of the formal neighborhood of the origin in  $U$ , with tangent space  $K$  at the origin, such that  $M$  deforms to a representation of the algebra  $A_S := A_U \hat{\otimes}_{\mathbf{C}[[U]]} \mathbf{C}[S]$  (where  $\hat{\otimes}$  is the completed tensor product).*

(ii) *The deformation of  $M$  over  $S$  is unique.*

*Proof.* Let us realize  $A_U$  explicitly as  $A[[t_1, \dots, t_n]]$  equipped with a product  $*$  as above. We may assume that  $K$  is the space of all vectors  $(t_1, \dots, t_n)$  such that  $t_{m+1} = \dots = t_n = 0$ .

Let  $D$  be the formal neighborhood of the origin in  $K$ , with coordinates  $h_1 = t_1, \dots, h_m = t_m$ . Let  $\theta : D \rightarrow U$  be a map given by the formula  $\theta(h_1, \dots, h_m) = (t_1, \dots, t_n)$ , where  $t_i = h_i$  for  $i \leq m$ , and

$$t_k = \sum_{p_1, \dots, p_m} t_{k, p_1, \dots, p_m} h_1^{p_1} \dots h_m^{p_m}, k > m,$$

where  $t_{k, p_1, \dots, p_m} \in \mathbf{C}$ . More briefly, we can write  $t_k = \sum_P t_{kP} h^P$ , where  $P$  is a multi-index. We will use the notation  $|P|$  for the sum of indices in a multi-index  $P$ . For brevity we also let  $e_j$  to be the multi-index  $(0, \dots, 1_j, \dots, 0)$ .

We claim that there exist unique formal functions  $t_k = t_k(h)$ ,  $k > m$ , for which we can deform  $M$  over  $D$ . Indeed, such a deformation would be defined by a series

$$\tilde{r}(a) = \sum_P r_P(a) h^P,$$

where  $r_0(a) = r(a)$ , and  $r$  is the homomorphism giving the representation  $M$ . The condition that  $\tilde{r}$  is a representation gives, for each  $P$ ,

$$(2) \quad dr_P = \sum_j t_{jP} r(c_{e_j}) + C_P,$$

where for  $j \leq m$ ,  $t_{jP} = 1$  if  $P = e_j$  and zero otherwise, and  $C_P$  is a 2-cocycle whose expression may involve  $r_Q$  and  $t_{kQ}$  **only** with  $|Q| < |P|$ . Since the map  $\eta$  is surjective, there are (unique)  $t_{d+1, P}, \dots, t_{nP}$  for which the right-hand side is a coboundary. For such  $t_{d+1, P}, \dots, t_{nP}$  (and only for them), we can solve (2) for  $r_P$ .

This shows the existence of the functions  $t_j(h)$ ,  $j > m$ , such that the deformation of  $M$  over  $D$  is possible. To show the uniqueness of these functions, let  $t_j$  and  $t'_j$  be two sets of functions for which the deformation exists. Let  $r_P, r'_P$  be the coefficients of the corresponding representations  $\tilde{r}, \tilde{r}'$ . Let  $N$  be the maximal number such that  $t_{jP} = t'_{jP}$  for  $|P| < N$ . Since  $H^1(A, \text{End } M) = 0$ , the solution  $r_P$  of (2) is unique up to adding a coboundary. Thus we can use changes of basis in  $M$  to modify  $\tilde{r}$  so that  $r_P = r'_P$  for  $|P| < N$  (note that this does not affect  $t_j$ ). Then for any  $Q$  with  $|Q| = N$ ,  $C_Q(\tilde{r}) = C_Q(\tilde{r}')$ , and hence  $t_{jQ} = t'_{jQ}$ . This contradicts the maximality of  $N$ .

Thus, we have shown that the functions  $t_j$  exist and are unique; they define a parametrization of the desired subscheme  $S$  by  $D$ . Our proof also implies that the deformation of  $M$  over  $S$  is unique, so we are done.  $\square$

We end this section by recalling the following fact from algebraic geometry that will guarantee that the representations we found in Theorem 3.1 are actually irreducible.

Let  $X$  be an affine irreducible algebraic variety over  $\mathbf{C}$ ,  $R = \mathbf{C}[X]$ . Let  $A$  be an algebra over  $R$  and  $M$  an  $A$ -module, such that  $A$  and  $M$  are free as  $R$ -modules and  $M$  is of finite rank. For  $x \in X$ , let  $A_x, M_x$  be the fibers of  $A, M$  at  $x$ ; so  $A_x$  is a  $\mathbf{C}$ -algebra and  $M_x$  a finite dimensional module over  $A_x$ .

**Proposition 4.2.** *The set of  $x$  for which  $M_x$  is irreducible is open in  $X$ .*

*Proof.* Let  $x$  be a point of  $X$  where  $M_x$  is irreducible. Then the map  $f_x : A \rightarrow \text{End } M_x$  is surjective. This means that there exist elements  $a_1, \dots, a_{N^2}$  in  $A$ ,  $N = \dim_R M$ , such that  $f_x(a_i)$  is a basis of  $\text{End } M_x$ . The set  $U$  of points  $z$  of  $X$  such that  $f_z(a_i)$  are a basis of  $\text{End } M_z$  is open and contains  $x$ . We found a neighborhood  $U$  of  $x$  such that, for all  $z$  in  $U$ ,  $M_z$  is an irreducible  $A_z$ -module, as desired.  $\square$

**4.2. Homological properties of the algebra  $H_{1,c}(\Gamma)$ .** We recall the following definition (see [VB1, VB2, EO]):

**Definition 4.3.** An algebra  $A$  is defined to be in the class  $VB(d)$  if it is of finite Hochschild dimension (i.e. there exists  $n \in \mathbf{N}$  s.t.  $H^i(A, E) = 0$  for any  $i > n$  and any  $A$ -bimodule  $E$ ) and  $H^*(A, A \otimes A^o)$  is concentrated in degree  $d$ , where it equals  $A$  as an  $A$ -bimodule.

The meaning of this definition is clarified by the following result by van den Bergh ([VB1, VB2]).

**Theorem 4.4.** *If  $A \in VB(d)$ , then for any  $A$ -bimodule  $E$ , the Hochschild homology  $H_i(A, E)$  is naturally isomorphic to the Hochschild cohomology  $H^{d-i}(A, E)$ .*  $\square$

Now let  $B = H_{1,c}(\Gamma)$ .

**Proposition 4.5.** *The algebra  $B$  belongs to the class  $VB(2)$ .*

*Proof.* If  $\Gamma = \{1\}$ , the statement is well known ([VB1, VB2]; see also [EO]). Let us consider the case  $\Gamma \neq \{1\}$ . We have to show that  $B$  has finite Hochschild dimension and that

$$\begin{aligned} H^i(B, B \otimes B^o) &= 0 && \text{for } i \neq 2 \\ H^2(B, B \otimes B^o) &\cong B && \text{as } B\text{-bimodules.} \end{aligned}$$

The algebra  $\mathbf{C}\Gamma \sharp \mathbf{C}\langle x, y \rangle$  has a natural increasing filtration obtained by putting  $x, y$  in degree 1 and the elements of  $\Gamma$  in degree 0. This filtration clearly induces a filtration on  $B$ :  $B = \bigcup_{n \geq 0} F^n B$ , and the associated graded algebra is  $B_0 = gr B = \mathbf{C}\Gamma \sharp \mathbf{C}[x, y]$  (by the PBW theorem), which has Hochschild dimension 2. So by a deformation argument we have that  $B$  has finite Hochschild dimension (equal to 2) and  $H^i(B, B \otimes B^o) = 0$  for  $i \neq 2$ , as this is true for  $B_0$  (which is easily checked since  $B_0$  is a semidirect product of a finite group with a polynomial algebra).

It remains to show the  $B$ -bimodule  $E := H^2(B, B \otimes B^o)$  is isomorphic to  $B$ . Using again a deformation argument (cf. [VB1]), we can see that  $E$  is invertible and free as a right and left  $B$ -module, because this is true for  $B_0$ . So  $E = B\phi$  where  $\phi$  is an automorphism of  $B$  such that  $gr\phi = 1$ . We will now show that  $\phi = 1$ , which will conclude the proof.

Define a linear map  $\xi : B_0 \rightarrow B_0$  as follows: if  $z \in B_0$  is a homogeneous element of degree  $n$ , and  $\tilde{z}$  is its lifting to  $B$ , then  $\xi(z)$  is defined to be the projection of the element  $\phi(\tilde{z}) - \tilde{z}$  (which has filtration degree  $n - 1$ ) to  $(gr B)_{n-1}$ . It is easy to

check that  $\xi$  is well defined (i.e., independent on the choice of the lifting), and is a derivation of  $B_0$  of degree  $-1$ .

Our job is to show that  $\xi = 0$ . This would imply that  $\phi = 1$ , since  $B$  is generated by  $F^1 B$ .

It is clear that any homogeneous inner derivation of  $B_0$  has nonnegative degree. Hence, it suffices to show that the degree  $-1$  part of  $H^1(B_0, B_0)$  is zero. But it is easy to compute using Koszul complexes that  $H^1(B_0, B_0) = \text{Vect}(L)^\Gamma$ , the space of  $\Gamma$ -invariant vector fields on  $L$ . In particular, vector fields of degree  $-1$  are those with constant coefficients. But such a vector field cannot be  $\Gamma$ -invariant unless it is zero, since the space  $L$  has no nonzero vectors fixed by  $\Gamma$ . Thus,  $\xi = 0$  and we are done.  $\square$

**Corollary 4.6.**  $H^2(B, \text{End } Y) = H_0(B, \text{End } Y) = \mathbf{C}$ .

*Proof.* We apply Theorem 4.4 to obtain the first identity. Furthermore,  $H_0(B, \text{End } Y) = \text{End } Y/[B, \text{End } Y] = \mathbf{C}$  as  $Y$  is irreducible, so the second identity follows.  $\square$

**Proposition 4.7.**  $H^1(B, \text{End } Y) = 0$ .

*Proof.* We have  $H^1(B, \text{End } Y) = \text{Ext}_{B \otimes B^o}^1(B, \text{End } Y) = \text{Ext}_B^1(Y, Y)$ . But it is known ([CBH], Corollary 7.6) that  $B$  contains only one minimal ideal  $J$  among all the nonzero ideals, and  $\text{Ext}_{B/J}^1(Y', Y') = 0$  for any irreducible module  $Y'$  over the (finite dimensional) quotient algebra  $B/J$ . Since any finite dimensional  $B$ -module must factor through  $B/J$ , we get  $\text{Ext}_B^1(Y, Y) = 0$ , as desired.  $\square$

**4.3. Homological properties of  $A = H_{1,0,c}(\Gamma_N)$ .** We now let  $A$  denote the algebra  $H_{1,0,c_0}(\Gamma_N)$ . The algebra  $A$  has a flat deformation over  $U = C(\mathcal{S})$ , which is given by the algebra  $H_{1,k,c_0+c'}(\Gamma_N)$ . The fact that this deformation is flat follows from Theorem 2.3.

**Proposition 4.8.** *If the Young diagram of  $W$  is a rectangle, then*

$$H^2(A, \text{End } M) = H^2(B, \text{End } Y) = \mathbf{C}.$$

*Proof.* The second equality follows from Corollary 4.6. Let us prove the first equality. We have

$$\begin{aligned} H^*(A, \text{End } M) &= \text{Ext}_{A \otimes A^o}^*(A, \text{End } M) \\ &= \text{Ext}_{S_N \sharp B^{\otimes N} \otimes S_N \sharp B^{o \otimes N}}^*(S_N \sharp B^{\otimes N}, \text{End } W \otimes \text{End } Y^{\otimes N}) \\ &= \text{Ext}_{S_N \times S_N \sharp (B^{\otimes N} \otimes B^{o \otimes N})}^*(S_N \sharp B^{\otimes N}, \text{End } W \otimes \text{End } Y^{\otimes N}). \end{aligned}$$

Now, the  $S_N \times S_N \sharp (B^{\otimes N} \otimes B^{o \otimes N})$ -module  $S_N \sharp B^{\otimes N}$  is induced from the module  $B^{\otimes N}$  over the subalgebra  $S_N \sharp B^{\otimes N} \otimes B^{o \otimes N}$ , in which  $S_N$  acts simultaneously permuting the factors of  $B^{\otimes N}$  and  $B^{o \otimes N}$  (note that  $S_N \sharp (B^{\otimes N} \otimes B^{o \otimes N})$  is indeed a subalgebra of  $S_N \times S_N \sharp (B^{\otimes N} \otimes B^{o \otimes N})$  as it can be identified with the subalgebra  $D \sharp (B^{\otimes N} \otimes B^{o \otimes N})$  where  $D = \{(\sigma, \sigma), \sigma \in S_N\} \subset S_N \times S_N$ ). Applying the Shapiro Lemma, we get

$$\begin{aligned} &\text{Ext}_{S_N \times S_N \sharp (B^{\otimes N} \otimes B^{o \otimes N})}^*(S_N \sharp B^{\otimes N}, \text{End } W \otimes \text{End } Y^{\otimes N}) \\ &= \text{Ext}_{S_N \sharp (B^{\otimes N} \otimes B^{o \otimes N})}^*(B^{\otimes N}, \text{End } W \otimes \text{End } Y^{\otimes N}) \\ &= (\text{Ext}_{B^{\otimes N} \otimes B^{o \otimes N}}^*(B^{\otimes N}, \text{End } W \otimes \text{End } Y^{\otimes N}))^{S_N}. \end{aligned}$$

But since  $B^{\otimes N} \otimes B^{\circ \otimes N}$  does not act on  $\text{End } W$ , the latter module equals

$$\left( \text{Ext}_{B^{\otimes N} \otimes B^{\circ \otimes N}}^*(B^{\otimes N}, \text{End } Y^{\otimes N}) \otimes \text{End } W \right)^{S_N}.$$

Using Proposition 4.7 and the Künneth formula in degree 2, we get that as an  $S_N$ -module,  $\text{Ext}_{B^{\otimes N} \otimes B^{\circ \otimes N}}^2(B^{\otimes N}, \text{End } Y^{\otimes N}) = \text{Ext}_{B \otimes B^{\circ}}^2(B, \text{End } Y) \otimes \mathbf{C}^N$  where  $S_N$  acts only on  $\mathbf{C}^N$  permuting the factors. But as an  $S_N$ -module,  $\mathbf{C}^N = \mathbf{C} \oplus \mathfrak{h}$ , where  $\mathbf{C}$  is the trivial representation. As a result we get

$$\begin{aligned} \text{Ext}_{A \otimes A^{\circ}}^2(A, \text{End } M) &= \text{Ext}_{B \otimes B^{\circ}}^2(B, \text{End } Y) \otimes (\mathbf{C}^N \otimes \text{End } W)^{S_N} \\ &= \text{Ext}_{B \otimes B^{\circ}}^2(B, \text{End } Y) \otimes (\mathbf{C} \otimes \text{End } W \oplus \mathfrak{h} \otimes \text{End } W)^{S_N} \\ &= \text{Ext}_{B \otimes B^{\circ}}^2(B, \text{End } Y) \otimes (\text{Hom}_{S_N}(W, W) \oplus \text{Hom}_{S_N}(\mathfrak{h} \otimes W, W)) \\ &= \text{Ext}_{B \otimes B^{\circ}}^2(B, \text{End } Y) \end{aligned}$$

as  $\text{Hom}_{S_N}(\mathfrak{h} \otimes W, W) = 0$  by Lemma 2.4 part (i).  $\square$

**Corollary 4.9.** *The map  $\eta : U \rightarrow H^2(A, \text{End } M)$  is surjective.*

*Proof.* Let  $U_0 \subset U$  be the subspace of vectors  $(0, c')$ . It is sufficient to show that the restriction of  $\eta$  to  $U_0$  is surjective. But this restriction is a composition of three natural maps:

$$U_0 \rightarrow H^2(B, B) \rightarrow H^2(A, A) \rightarrow H^2(A, \text{End } M).$$

Here the first map  $\eta_0 : U_0 \rightarrow H^2(B, B)$  is induced by the deformation of  $B$  along  $U_0$ , the second map  $\xi : H^2(B, B) \rightarrow H^2(A, A)$  comes from the Künneth formula, and the third map  $\psi : H^2(A, A) \rightarrow H^2(A, \text{End } M)$  is induced by the homomorphism  $A \rightarrow \text{End } M$ .

Now, by Proposition 4.8, the map  $\psi \circ \xi$  coincides with the map  $\psi_0 : H^2(B, B) \rightarrow H^2(B, \text{End } Y)$  induced by the homomorphism  $B \rightarrow \text{End } Y$ . We claim that this map is surjective. Indeed, since by Proposition 4.5,  $B$  is in  $VB(2)$ , by Theorem 4.4 there is a natural identification of  $H^2(B, E)$  with  $H_0(B, E)$  for any  $B$ -bimodule  $E$ ; hence  $\psi_0$  can be viewed as the natural map  $\psi_0 : H_0(B, B) \rightarrow H_0(B, \text{End } Y)$ . But  $H_0(B, E) = E/[B, E]$  for any  $B$ -bimodule  $E$ . Hence,  $\psi_0$  can be viewed as the natural map

$$\psi_0 : B/[B, B] \rightarrow \text{End } Y/[B, \text{End } Y].$$

This map is clearly nonzero: the representation  $Y$  is irreducible, and hence the map  $B \rightarrow \text{End } Y$  is surjective. Thus  $\psi_0$  is surjective, as claimed (as the space  $\text{End } Y/[B, \text{End } Y]$  is 1-dimensional).

Let  $K$  be the kernel of  $\psi_0$ . It remains to show that the map  $\eta_0$  does not land in  $K$ . To show this, recall that by Proposition 4.1, the representation  $Y$  of  $B$  can be deformed along  $K$ . Thus it remains to show that  $Y$  does not admit a first order deformation along the entire  $U_0$ . But this follows easily by computing the trace of both sides of the commutation relation  $xy - yx = \lambda$  in a deformation of  $Y$ . We are done.  $\square$

**Proposition 4.10.**  $H^1(A, \text{End } M) = 0$ .

*Proof.* Arguing as in the proof of Proposition 4.8, we get that  $H^1(A, \text{End } M) = H^1(B, \text{End } Y)$ , which is zero. This proves the proposition.  $\square$

We have thus proved the following result.

**Proposition 4.11.** *If the Young diagram corresponding to  $W$  is a rectangle, then there exists a unique smooth codimension one formal subscheme  $S$  of the formal neighborhood of the origin in  $U$  such that the representation  $M = W \otimes Y^{\otimes N}$  of  $H_{1,0,c_0}(\mathbf{\Gamma}_N)$  formally deforms to a representation of  $H_{1,k,c_0+c'}(\mathbf{\Gamma}_N)$  along  $S$  (i.e., abusing the language, for  $(k, c') \in S$ ). Furthermore, the deformation of  $M$  over  $S$  is unique.*

*Proof.* Corollary 4.9 and Proposition 4.10 show that our case satisfies all the hypotheses of Proposition 4.1. Moreover, from  $H^2(A, \text{End } M) = \mathbf{C}$  we deduce  $\dim \text{Ker } \eta = \dim U - 1$ , and the proposition follows.  $\square$

**4.4. The trace condition and the proof of Theorem 3.1.** Now we would like to find the subscheme  $S$  of Proposition 4.11. For this we take the trace in  $M$  of the commutation relation (TR), and obtain a necessary condition on the parameters  $(k, c)$  for the algebra  $H_{1,k,c}(\mathbf{\Gamma}_N)$  to admit a representation isomorphic to  $M$  as a  $\mathbf{\Gamma}_N$ -module:

(TR) For any  $i \in [1, n]$ ,

$$\dim M + \frac{k}{2} \sum_{j \neq i} \sum_{\gamma \in \Gamma} \text{tr}|_M(s_{ij} \gamma_i \gamma_j^{-1}) + \sum_{\gamma \in \Gamma \setminus \{1\}} c_\gamma \text{tr}|_M(\gamma_i) = 0.$$

This relation can be easily rewritten in terms of the characters  $\chi_Y$  of  $Y$  as a representation of  $\Gamma$  and  $\psi_W$  of  $W$  as a representation of  $S_N$ . Indeed, one can check:

(3)  $\text{tr}|_M(\gamma_i) = \dim W \dim Y^{N-1} \chi_Y(\gamma),$

(4)  $\text{tr}|_M(s_{ij} \gamma_i \gamma_j^{-1}) = \psi_W(s_{ij}) \dim Y^{N-1}.$

Namely, (3) is an easy consequence of the fact that the group  $\Gamma^{\times N} \subset \mathbf{\Gamma}_N$  acts only on  $Y^{\otimes N}$  with character  $\chi_Y^{\otimes N}$ , and  $\gamma_i$  is by definition the element  $(1, \dots, \overset{i}{\gamma}, \dots, 1) \in \Gamma^{\times N}$ . To obtain (4), we observe that  $s_{ij} \gamma_i \gamma_j^{-1}$  is conjugate in  $\mathbf{\Gamma}_N$  to  $s_{ij}$  and that the character of  $S_N$  on  $M$  is simply the product of the characters on  $W$  and  $Y^{\otimes N}$ . An easy computation gives  $\text{tr}|_{Y^{\otimes N}} s_{ij} = \dim Y^{N-1}$ , hence the formula.

We now recall that, for any transposition  $\sigma \in S_N$ ,  $\psi|_W(\sigma) = \frac{\dim W}{N(N-1)/2} \mathbf{c}(\mu)$ , where  $\mathbf{c}(\mu)$  is the content of the Young diagram  $\mu$  attached to  $W$ . In particular, if  $\mu$  is a rectangular diagram of size  $l \times m$  with  $lm = N$ , it can be easily computed that

$$\mathbf{c}(\mu) = \frac{N(m-l)}{2},$$

so we have

(5)  $\text{tr}|_M(s_{ij} \gamma_i \gamma_j^{-1}) = \frac{(m-l) \dim W}{N-1} \dim Y^{N-1}.$

Finally, substituting (3), (5) in (TR) and dividing the relation by  $\dim Y^{N-1} \dim W$ , we obtain

(TR') If the Young diagram of  $W$  is of size  $l \times m$ , then

$$\dim Y + \frac{k}{2} |\Gamma|(m-l) + \sum_{\gamma \in \Gamma \setminus \{1\}} c_\gamma \chi_Y(\gamma) = 0.$$

The condition  $(TR')$  defines exactly the hyperplane  $\mathcal{H}_{Y,m,l}$ .

Thus we have shown that  $(0, c_0) + S \subset \mathcal{H}_{Y,m,l}$ . But  $S$  and  $\mathcal{H}_{Y,m,l}$  have the same dimension, which implies that  $S$  is the formal neighborhood of zero in  $\mathcal{H}_{Y,m,l} - (0, c_0)$ . This proves part (i) of Theorem 3.1.

We now conclude the proof of Theorem 3.1. Let  $X'$  be the formal neighborhood of  $M_{c_0}$  in  $X$ . We have shown that the morphism  $f : X \rightarrow U$  lands in  $\mathcal{H}_{Y,m,l}$ , and that  $f|_{X'} : X' \rightarrow (0, c_0) + S$  is an isomorphism. This implies that the map  $f : X \rightarrow \mathcal{H}_{Y,m,l}$  is étale at  $M_{c_0}$ . This proves part (ii) of Theorem 3.1, and, together with Proposition 4.2, also implies (iii), since a map which is étale at one point is dominant.

#### REFERENCES

- [CBH] W. Crawley-Boevey, M. Holland, *Noncommutative deformations of Kleinian singularities*, Duke Math. J. **92** (1998), no. 3, 605–635. MR1620538 (99f:14003)
- [CE] T. Chmutova and P. Etingof, *On some representations of the rational Cherednik algebra*, Represent Theory **7** (electronic) (2003), 641–650. MR2017070 (2005a:16019)
- [EG] P. Etingof, V. Ginzburg, *Symplectic reflection algebras, Calogero-Moser space, and deformed Harish-Chandra homomorphism*, Invent. Math. **147** (2002), no. 2, 243–348. MR1881922 (2003b:16021)
- [EO] P. Etingof, A. Oblomkov, *Quantization, orbifold cohomology, and Cherednik algebras*, preprint, [math.QA/0311005](https://arxiv.org/abs/math/0311005).
- [GG] W. L. Gan, V. Ginzburg *Deformed preprojective algebras and symplectic reflection algebras for wreath products*, J. Algebra **283** (2005), no. 1, 350–363. MR2102087
- [GS] I. Gordon, S.P. Smith *Representations of symplectic reflection algebras and resolutions of deformations of symplectic quotient singularities*, Math. Ann. **330** (2004), no. 1, 185–200. MR2091684
- [VB1] M. van den Bergh *A relation between Hochschild homology and cohomology for Gorenstein rings*, Proc. Amer. Math. Soc. **126** (1998), no. 5, 1345–1348. MR1443171 (99m:16013)
- [VB2] M. van den Bergh *Erratum to: “A relation between Hochschild homology and cohomology for Gorenstein rings”*, Proc. Amer. Math. Soc. **130** (2000), no. 9, 2809–2810. MR1900889 (2003f:16016)

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