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EQUIVARIANT DERIVED CATEGORY OF A COMPLETE SYMMETRIC VARIETY

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Abstract. Let G be a complex algebraic semi-simple adjoint group and X a smooth complete symmetric G-variety. Let $L = \bigoplus_\alpha L_\alpha$ be the direct sum of all irreducible G-equivariant intersection cohomology complexes on X, and let $\mathcal{E} = \mathrm{Ext}^{\cdot}_{\mathrm{D}_G(X)}(L, L)$ be the extension algebra of L, computed in the G-equivariant derived category of X. We considered $\mathcal{E}$ as a dg-algebra with differential $d_\mathcal{E} = 0$, and the $\mathcal{E}_\alpha = \mathrm{Ext}^{\cdot}_{\mathrm{D}_G(X)}(L, L_\alpha)$ as $\mathcal{E}$-dg-modules. We show that the bounded equivariant derived category of sheaves of $\mathbf{C}$-vector spaces on X is equivalent to $\mathrm{D}_\mathcal{E}\langle\mathcal{E}_\alpha\rangle$, the subcategory of the derived category of $\mathcal{E}$-dg-modules generated by the $\mathcal{E}_\alpha$.

1. Introduction

The aim of this paper is to give a description of the equivariant derived category of a smooth complete symmetric variety. Let G be a complex algebraic semi-simple adjoint group, σ an automorphism of G of order 2 and $H = G^\sigma$. Let X be a complete symmetric variety containing G/H as open orbit, as defined in [7], section 5: this is a smooth compactification of G/H with a G-action (extending the action on G/H) and with a G-equivariant morphism to the canonical compactification described in [6]. From [6] and [7] we have the following results on the G-orbits of X: $X \setminus (G/H)$ is the union of irreducible smooth G-stable divisors with normal crossings, say $D_1, \ldots, D_m$; any non-empty intersection $D_{i_1} \cap \ldots \cap D_{i_n}$ is the closure of a single G-orbit, and, conversely, for any G-orbit $\mathcal{O}$ in X, $\overline{\mathcal{O}}$ is the intersection of the D_i containing $\mathcal{O}$. We consider X as a complex analytic variety, with its transcendental topology.

We denote by $\mathrm{D}^b_G(X)$ the bounded equivariant derived category of sheaves of $\mathbf{C}$-vector spaces on X; we let $\mathrm{D}^b_{G,c}(X)$ be the subcategory formed by constructible objects (it is introduced in [3]—we recall the points we need in section 2.1). Let $\mathcal{S}$ be the set of G-orbits of X. For a G-orbit $\mathcal{O}$, let $\tau_\mathcal{O}$ be the group of components of the stabiliser. A representation ρ of $\tau_\mathcal{O}$ induces a G-equivariant local system on $\mathcal{O}$ and we let $L^\rho_\mathcal{O}$ be the corresponding intersection cohomology complex. Since $\overline{\mathcal{O}} \setminus \mathcal{O}$ consists of normal crossings divisors, $L^\rho_\mathcal{O}$ is in fact a sheaf. It is known that, for any orbit $\mathcal{O}$, there exists s with $\tau_\mathcal{O} \simeq (\mathbf{Z}/2\mathbf{Z})^s$ (we recall this in section 4). In particular, for ρ irreducible, the local system $L^\rho_\mathcal{O}|_\mathcal{O}$ is of rank 1. The category $\mathrm{D}^b_{G,c}(X)$ is generated by the $L^\rho_\mathcal{O}$, for all $\mathcal{O}$, ρ as above. We set $L = \bigoplus L^\rho_\mathcal{O} \in \mathrm{D}^b_{G,c}(X)$, where $\mathcal{O}$ runs over $\mathcal{S}$ and ρ runs over the irreducible representations of

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$\tau_{\mathcal{O}}$. We consider the graded ring $\mathcal{E} = \bigoplus_{i\in\mathbf{N}} \mathrm{Ext}^i(L, L)$ (here $\mathrm{Ext}^i(L, L)$ denotes $\mathrm{Hom}_{\mathrm{D}^b_{G,c}(X)}(L, L[i])$). We view $\mathcal{E}$ as a differential graded algebra ("dg-algebra"), with differential $d_{\mathcal{E}} = 0$, and we denote by $\mathrm{D}_{\mathcal{E}}$ the derived category of $\mathcal{E}$-dg-modules, introduced in [3] (we recall its definition in section 2.1). We let $\mathrm{D}_{\mathcal{E}}\langle\mathcal{E}^{\rho}_{\mathcal{O}}\rangle$ be the subcategory generated by the $\mathcal{E}$-modules $\mathcal{E}^{\rho}_{\mathcal{O}} = \bigoplus_{i\in\mathbf{Z}} \mathrm{Ext}^i(L, L^{\rho}_{\mathcal{O}})$, for all $\mathcal{O}$ in $\mathcal{S}$ and all irreducible representations ρ of $\tau_{\mathcal{O}}$.

On general grounds recalled below (a kind of derived version of the Freyd-Mitchell embedding theorem) there exists a dg-algebra R with cohomology $H^{\cdot}(R) \simeq \mathcal{E}$ such that the category $\mathrm{D}^b_{G,c}(X)$ is equivalent to a subcategory of D_R. We will prove that we can actually take $R = \mathcal{E}$.

Theorem 1.1. *With the above notation, the categories $\mathrm{D}^b_{G,c}(X)$ and $\mathrm{D}_{\mathcal{E}}\langle\mathcal{E}^{\rho}_{\mathcal{O}}\rangle$ are equivalent.*

The statement of this theorem is inspired by questions of Soergel (see [14], [15] where it is asked whether it holds for a Langlands parameter space, instead of a symmetric variety here). The case of general (possibly singular) toric varieties was done by Lunts in [13] and we follow the principle of his proof. Some difficulties appear: unlike the toric case, for a G-orbit $\mathcal{O}$, the smallest open G-stable set containing $\mathcal{O}$ is in general not homotopically equivalent to $\mathcal{O}$; moreover, we may have non-connected isotropy groups.

Let us say a word about the algebra $\mathcal{E}$. For two G-orbits $\mathcal{O}$ and $\mathcal{O}'$, $\overline{\mathcal{O}} \cap \overline{\mathcal{O}'}$ is a (smooth) orbit closure, say $\overline{\mathcal{O}} \cap \overline{\mathcal{O}'} = \overline{\mathcal{O}''}$. Let c be its complex codimension in $\overline{\mathcal{O}'}$. If ρ and ρ' are the trivial representations of $\tau_{\mathcal{O}}$ and $\tau_{\mathcal{O}'}$, then $\mathrm{Ext}^{\cdot}(L^{\rho}_{\mathcal{O}}, L^{\rho'}_{\mathcal{O}'}) \simeq H^{\cdot+2c}_G(\overline{\mathcal{O}''})$ (in this paper for a group G and a topological space Y with a G-action, we denote by $H^{\cdot}_G(Y) = H^{\cdot}_G(Y; \mathbf{C})$ the G-equivariant cohomology of Y, with coefficients in $\mathbf{C}$). Hence, in the case where all $\tau_{\mathcal{O}}$ are trivial, the computation of $\mathcal{E}$ reduces to the computation of $H^{\cdot}_G(\overline{\mathcal{O}})$, for all G-orbits $\mathcal{O}$ of X. But $\overline{\mathcal{O}}$ is a "regular embedding" in the sense of Definition 5 of [2] (this means that $\overline{\mathcal{O}}$ has a finite number of G-orbits, that each G-orbit closure in $\overline{\mathcal{O}}$ is the transversal intersection of the codimension 1 orbits containing it, and that, for each $p \in \overline{\mathcal{O}}$, G_p has a dense orbit in the normal space at p to the G-orbit containing p). In [2] there is a description of the equivariant cohomology of a regular embedding, say Y, as a subalgebra of the product of the equivariant cohomology algebras of the G-orbits of Y. Note that, for a G-orbit $G \cdot p$, $p \in Y$, $H^{\cdot}_G(G \cdot p) \simeq H^{\cdot}_{G_p}(\{pt\})$ only depends on the stabiliser of p. For a symmetric variety, these stabilisers G_p can be determined from the action of the involution θ on the root system of G with respect to a suitable maximal torus. Hence we have a method to compute the $H^{\cdot}_G(\overline{\mathcal{O}})$, for all G-orbits $\mathcal{O}$ of X. This could lead to a combinatorial description of the algebra $\mathcal{E}$ and the modules $\mathcal{E}^{\rho}_{\mathcal{O}}$, at least when the groups $\tau_{\mathcal{O}}$ are trivial. We note that in the case of toric varieties the computation of the equivariant intersection cohomology from combinatorial data has been carried out in [1] and [4].

The plan of the proof mostly follows that of Lunts in the toric case (see [13]). The principle is the following. We first show that $\mathrm{D}^b_{G,c}(X)$ is equivalent to $\mathrm{D}_{\mathcal{H}}\langle H^{\rho}_{\mathcal{O}}\rangle$, where $\mathcal{H}$ is a sheaf of dg-algebras on a finite set I, $\mathrm{D}_{\mathcal{H}}$ denotes the derived category of sheaves of $\mathcal{H}$-dg-modules, and the $H^{\rho}_{\mathcal{O}}$ are $\mathcal{H}$-modules corresponding to the $L^{\rho}_{\mathcal{O}}$ (I and $\mathcal{H}$ are described below). On a finite set, the category of sheaves has enough projectives, so that, for each $H^{\rho}_{\mathcal{O}}$, we may choose a projective resolution $P^{\rho}_{\mathcal{O}} \to H^{\rho}_{\mathcal{O}}$. We set $P^{\cdot} = \bigoplus P^{\rho}_{\mathcal{O}}$, where the sum is over all G-orbits $\mathcal{O}$ and irreducible

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representations ρ of $\tau_{\mathcal{O}}$. We consider the dg-algebra $R^{\cdot} = \mathrm{Hom}(P^{\cdot}, P^{\cdot})$ and the left R-modules $R^{\rho}_{\mathcal{O}} = \mathrm{Hom}(P^{\cdot}, P^{\rho}_{\mathcal{O}})$. Since $\mathrm{D}^b_{G,c}(X)$ and $\mathrm{D}_{\mathcal{H}}\langle H^{\rho}_{\mathcal{O}}\rangle$ are equivalent, we have $H^{\cdot}(R) \simeq \mathcal{E}$ and $H^{\cdot}(R^{\rho}_{\mathcal{O}}) \simeq \mathcal{E}^{\rho}_{\mathcal{O}}$. We let D_R be the derived category of left R-dg-modules. By general arguments, the functor $\mathrm{D}_{\mathcal{H}}\langle H^{\rho}_{\mathcal{O}}\rangle \to \mathrm{D}_R\langle R^{\rho}_{\mathcal{O}}\rangle$, $F \mapsto \mathrm{Hom}(P^{\cdot}, F)$ is an equivalence of categories. Now, any quasi-isomorphism $R \to R'$ between two dg-algebras induces an equivalence of categories between D_R and $\mathrm{D}_{R'}$, by restriction and extension of scalars. So we conclude the proof of the theorem by showing that R is quasi-isomorphic to its cohomology (such dg-algebras are called "formal").

We give the details first assuming that all $\tau_{\mathcal{O}}$ are trivial.

1) We recall the following facts about $\mathrm{D}^b_{G,c}(X)$ (see section 2.1). Let E be a universal bundle for G. By the construction of Bernstein and Lunts in [3], the category $\mathrm{D}^b_{G,c}(X)$ is the subcategory of $\mathrm{D}(E \times_G X)$ generated by the sheaves induced by G-equivariant constructible sheaves on X. More precisely, in our situation it is generated by the sheaves $E \times_G L^{\rho}_{\mathcal{O}}$, which are equal to $E \times_G \mathbf{C}_{\overline{\mathcal{O}}} = \mathbf{C}_{E \times_G \overline{\mathcal{O}}}$, since we assume that the $\tau_{\mathcal{O}}$ are trivial. However, there is no slice theorem for a G-action and we find it easier to work in the equivariant derived category for the action of a compact group. In fact, if K is a maximal compact subgroup of G, the restriction functor $\mathrm{D}^b_G(X) \to \mathrm{D}^b_K(X)$ is fully faithful. Since $\mathrm{D}^b_K(X)$ is itself a subcategory of $\mathrm{D}(E \times_K X)$, this identifies $\mathrm{D}^b_{G,c}(X)$ with the subcategory of $\mathrm{D}(E \times_K X)$ generated by the $\mathbf{C}_{E \times_K \overline{\mathcal{O}}}$.

2) Following [13], we obtain a category equivalent to $\mathrm{D}(E \times_G X)$ as follows. From now on, we assume that the maximal compact subgroup K is compatible with σ (i.e., σ commutes with the conjugation on G induced by K). We decompose X according to the K-orbit types, and take the connected components: $X = \bigsqcup_{i \in I} X_i$. This is a stratification of X, which is precisely described in [2]. We let $\phi : X \to I$ be the map such that $\phi(X_i) = \{i\}$ and endow I with the quotient topology. We also denote by $\psi : E \times_K X \to I$ the induced map. Since G is linear, we may take for E an increasing union of G-manifolds (e.g. Stiefel manifolds), $E = \bigcup_{k \in \mathbf{N}} E_k$. Then the sheaves of $\mathcal{C}^{\infty}$-forms of degree i on $E_k \times_K X$, $\Omega^i_{E_k \times_K X}$, form a projective system, and we define $\Omega^i_{E \times_K X}$ as the projective limit of the $\Omega^i_{E_k \times_K X}$. The complex $\Omega^{\cdot}_{E \times_K X}$ has a natural structure of sheaf of dg-algebras and it gives a soft resolution of $\mathbf{C}_{E \times_K X}$.

We set $\mathcal{A} = \psi_*(\Omega_{E \times_K X})$. This is a sheaf of dg-algebras on I. We consider the direct image functor $\gamma : \mathrm{D}^b_{G,c}(X) \to \mathrm{D}_{\mathcal{A}}$, $F^{\cdot} \mapsto \psi_*(\Omega_{E \times_K X} \otimes F^{\cdot})$. We prove that $\gamma(\mathbf{C}_{\overline{\mathcal{O}}}) \simeq \mathcal{A}_{\phi(\overline{\mathcal{O}})}$, for any G-orbit $\mathcal{O}$, and that γ gives an equivalence between $\mathrm{D}^b_{G,c}(X)$ and $\mathrm{D}_{\mathcal{A}}\langle \mathcal{A}_{\phi(\overline{\mathcal{O}})}\rangle$. This point uses the following property of our stratification. For $j \in I$, we let $V_j \subset X$ be the smallest open subset of X containing X_j and constructible with respect to the stratification $X = \bigsqcup_{i \in I} X_i$. Then, there exists a K-equivariant homotopy contracting V_j to $K \cdot x_j$, for some $x_j \in X_j$.

3) Let $\mathcal{H}$ be the cohomology sheaf of $\mathcal{A}$, i.e., the sheaf on I associated to $U \mapsto H^{\cdot}(\mathcal{A}(U))$. We consider $\mathcal{H}$ as a sheaf of dg-algebras with differential 0. We prove that there exists a sequence of quasi-isomorphisms $\mathcal{A} \leftarrow \mathcal{A}' \to \mathcal{A}'' \leftarrow \cdots \to \mathcal{H}$ (actually there are 5 steps in this sequence). This implies that the categories $\mathrm{D}_{\mathcal{A}}\langle \mathcal{A}_{\phi(\overline{\mathcal{O}})}\rangle$ and $\mathrm{D}_{\mathcal{H}}\langle \mathcal{H}_{\phi(\overline{\mathcal{O}})}\rangle$ are equivalent.

Let us remark that the sheaf $\mathcal{A}$ is determined by the stalks $\mathcal{A}_i$, $i \in I$, since we are on a finite set. For a given $i \in I$, the formality of the dg-algebra $\mathcal{A}_i$ is easy: we

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have $\mathcal{A}_i = \mathcal{A}(\phi(V_i))$, with V_i as above, and $H^\cdot(\mathcal{A}_i)$ is the K-equivariant cohomology of $K \cdot x_i$, since V_i has a retraction to $K \cdot x_i$. Since $H^\cdot_K(K \cdot x_i) = H^\cdot_{K_{x_i}}(\{pt\})$ is a polynomial algebra (because K_{x_i} is connected), any choice of representatives for its generators gives a quasi-isomorphism $H^\cdot_{K_{x_i}}(\{pt\}) \to \mathcal{A}_i$. But, of course, to obtain the formality of the sheaf of dg-algebras $\mathcal{A}$, we need additionally that these quasi-isomorphisms commute with the restriction maps $\mathcal{A}_i \to \mathcal{A}_j$, for $i \in \overline{\{j\}}$.

The construction of the sequence of quasi-isomorphisms above makes use of the description of the stabilisers given in [2]. Let us briefly recall it. Let S be a maximal split torus of G and denote by $x_0 \in X$ the class of 1_G, $x_0 \in G/H \subset X$. Then $\overline{S \cdot x_0}$ is a smooth toric variety for the action of $S' = S/(S \cap H) = S/\{t \in S; t^2 = 1\}$ and contains a toric subvariety Z, with the following properties. Taking intersection with Z gives a bijection between the set of G-orbits of X and the set of S-orbits of Z. Moreover, the action of K in X has a compact fundamental domain $C_X \subset Z$. We set $S^c = S \cap K$; this is a maximal compact subgroup of S. For $x_i \in X_i \cap C_X$, we have, modulo a finite group, the decomposition $K_{x_i} = S^c_i \times K_i$, where $S^c_i = S^c_{x_i}$ and $K_i = K_{x_i} \cap K^\sigma$, only depend on the stratum X_i. Hence $\mathcal{H}_i = H^\cdot_{K_{x_i}}(\{pt\}) \simeq H^\cdot_{S^c_i}(\{pt\}) \otimes H^\cdot_{K_i}(\{pt\})$.

Now we build a morphism from $H^\cdot_{S^c_i}(\{pt\})$ to $\mathcal{A}_i$. Let D_v, $v \in V$, be the irreducible G-stable divisors of X and $\mathcal{O}_v$ the G-orbits such that $D_v = \overline{\mathcal{O}_v}$. We denote by S^c_v the stabiliser in S^c of any point of $\mathcal{O}_v \cap Z$. Then $S^c_i \simeq \prod_{v \in \Delta_i} S^c_v$, where $\Delta_i = \{v \in V;\ X_i \subset D_v\}$, and $H^\cdot_{S^c_i}(\{pt\}) \simeq \mathbf{C}[\Xi_v;\ v \in \Delta_i]$, with $\deg \Xi_v = 2$. Let δ_v be the G-equivariant fundamental class of D_v in X and $\xi_v \in \Omega^2_{E \times_K X}$ a representative of δ_v. For $i \in I$, we define $f_i : H^\cdot_{S^c_i}(\{pt\}) \to \mathcal{A}_i$, $\Xi_v \mapsto \xi_v|_{E \times_K V_i}$.

For the factor K_i in the decomposition, using the Cartan model for the K-equivariant cohomology, we prove that we have quasi-isomorphisms:

$$\Gamma(E/K_i; \Omega_{E/K_i}) \xleftarrow{g_i} W_i \xrightarrow{h_i} H^\cdot_{K_i}(\{pt\}),$$

where the W_i are intermediate dg-algebras, which are subalgebras of the Weil algebra of the Lie algebra of K. These quasi-isomorphisms only depend on the choice of a connection on E. They are compatible with the natural maps from $H^\cdot_{K_i}(\{pt\})$ to $H^\cdot_{K_j}(\{pt\})$ induced by inclusions $K_j \subset K_i$ (and the similar maps between the de Rham algebras). Finally, the maps $f_i \otimes g_i$ and $f_i \otimes h_i$ give compatible quasi-isomorphisms between the $\mathcal{A}_i$ and the $\mathcal{H}_i$.

4) By steps 2 and 3, the categories $\mathrm{D}^b_{G,c}(X)$ and $\mathrm{D}_{\mathcal{H}}\langle \mathcal{H}_{\phi(\overline{\mathcal{O}})} \rangle$ are equivalent. Now we can build explicit projective resolutions, using Čech coverings of I. Let $P_{\mathcal{O}} \to \mathcal{H}_{\phi(\overline{\mathcal{O}})}$ be such a resolution, $P^\cdot = \bigoplus P_{\mathcal{O}}$ and $R^\cdot = \mathrm{Hom}(P^\cdot, P^\cdot)$. To conclude the proof of the theorem it is sufficient to show that $R^\cdot$ is quasi-isomorphic to its cohomology. The sheaf $\mathcal{H}$ itself has differential 0 and this fact can be used, as in [13], to endow $R^\cdot$ with a graduation different from the canonical one. With this graduation, we prove that $H^i(R^\cdot)$ vanishes for $i \neq 0$. Hence $R^\cdot$ is concentrated in degree 0, and quasi-isomorphic to its cohomology, by the natural morphisms $R \leftarrow \tau_{\leq 0} R \to H(R)$.

This was the idea for the case where all isotropy groups are connected. In general, simply taking the direct image to I as above would send some local systems to objects quasi-isomorphic to 0 (for example, let H be a finite group, V a nontrivial irreducible representation of H, L the H-equivariant local system on the

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point corresponding to V, then we have $H^{\cdot}_H(\{pt\};L)=0$; recall that we work with coefficients in $\mathbf{C}$). So we modify step 2 as follows.

2'.a) Let L_E be the sheaf induced by $L=\bigoplus_{(\mathcal{O},\rho)} L^\rho_{\mathcal{O}}$ on $E\times_K X$. We will replace the dg-algebra $\mathcal{A}=\psi_*(\Omega_{E\times_K X})$, representing $R\psi_*(\mathbf{C}_{E\times_K X})$, by a dg-algebra representing $R\psi_* R\mathcal{H}om(L_E,L_E)$. Let us consider a variety Y endowed with two sheaves L, L' which are local systems on subvarieties Z, Z' and 0 outside (here $Y=E_k\times_K V_i$, for some k and i, and $L=E_k\times_K L^\rho_{\mathcal{O}}|_{V_i}$, $L'=E_k\times_K L^{\rho'}_{\mathcal{O}'}|_{V_i}$). We assume that $Z\cap Z'$ is a smooth subvariety of Z'. Here is how we represent $\mathrm{RHom}(L,L')$. We consider tubular neighbourhoods: T_1 of $Z\cap Z'$ in Y, T of $Z\setminus T_1$ in $Y\setminus T_1$, T' of $Z'\setminus T_1$ in $Y\setminus T_1$. We choose them so that $T\cap T'=\emptyset$ and $T_1\cap \overline{T}$ is a tubular neighbourhood of $T_1\cap Z$ (and the same for T'). We extend L to a local system L_1 on $T_1\sqcup T$ and extend L_1 by 0 outside $T_1\sqcup T$; we define L'_1 from L' similarly. Then we show that $\mathrm{RHom}(L,L')\simeq \mathrm{RHom}(L_1,L'_1)$ and that the complex of sheaves $R\mathcal{H}om(L_1,L'_1)$ is actually a sheaf (concentrated in degree 0). Hence we may represent $\mathrm{RHom}(L,L')$ by the complex $\Gamma(Y;\Omega_Y\otimes\mathcal{H}om(L_1,L'_1))$.

We want to make this procedure work not only for the sheaves L, L' as above, but for all pairs $L^\rho_{\mathcal{O}}$, $L^{\rho'}_{\mathcal{O}'}$, simultaneously. For this, we prove that we can decompose X into K-invariant "tubes" T_i, such that $X=\bigsqcup_{i\in I} T_i$, with the following properties. For a G-orbit $\mathcal{O}$ and a representation ρ of $\tau_{\mathcal{O}}$, we set $Z^\rho_{\mathcal{O}}=\{x\in X;(L^\rho_{\mathcal{O}})_x\neq 0\}$ and $T^\rho_{\mathcal{O}}=\bigsqcup_{\{i;Y_i\subset Z^\rho_{\mathcal{O}}\}} T_i$. Then $L^\rho_{\mathcal{O}}$ has an extension, $L^\rho_{\mathcal{O}'}$, to $T^\rho_{\mathcal{O}}$ and we have, for any other pair $(\mathcal{O}',\rho')$:

$$\mathrm{RHom}(L^\rho_{\mathcal{O}},L^{\rho'}_{\mathcal{O}'})\simeq \mathrm{RHom}(L^\rho_{\mathcal{O}'},L'^{\rho'}_{\mathcal{O}'}) \quad\text{and}\quad H^i(R\mathcal{H}om(L^\rho_{\mathcal{O}'},L'^{\rho'}_{\mathcal{O}'}))=0,\quad \text{for } i\neq 0.$$

We set $L'_E=E\times_K\bigoplus_{(\mathcal{O},\rho)} L^\rho_{\mathcal{O}'}$ and define $\psi':E\times_K X\to I$ by $\psi'(E\times_K T_i)=\{i\}$. Then $\mathcal{A}'=\psi'_*(\Omega_{E\times_K X}\otimes\mathcal{H}om(L'_E,L'_E))$ is a sheaf of dg-algebras on I, such that $H^{\cdot}(\mathcal{A}'_i)\simeq \mathrm{Ext}^{\cdot}_{\mathrm{D}_K(V_i)}(L|_{V_i},L|_{V_i})$. We also have a direct image functor $\gamma:\mathrm{D}^+_K(X)\to \mathrm{D}_{\mathcal{A}'}$, $F\mapsto \psi'_*(\Omega_{E\times_K X}\otimes\mathcal{H}om(L'_E,J))$, for an injective resolution $F\to J$ of F. Setting $M^\rho_{\mathcal{O}}=\gamma(L^\rho_{\mathcal{O}})$, γ gives an equivalence of categories between $\mathrm{D}^b_{G,c}(X)$ and $\mathrm{D}_{\mathcal{A}'}\langle M^\rho_{\mathcal{O}}\rangle$.

2'.b) This procedure almost replaces step 2 above: we have built a sheaf of dg-algebras on I, whose derived category is equivalent to $\mathrm{D}^b_{G,c}(X)$. But our sheaf $\mathcal{A}'$ is not so easy to handle, because the "tubes" T_i are not intrinsically related to the data. We first built a second sheaf $\mathcal{B}$, quasi-isomorphic to $\mathcal{A}'$ as follows. Let us consider again the variety Y endowed with local systems L, L' on subvarieties Z, Z' extended to L_1, L'_1 on $T_1\sqcup T$, $T_1\sqcup T'$. Let us assume moreover that $Z\cap Z'$ has a neighbourhood T_2 containing $\overline{T_1}$ and such that the inclusion $Z\cap Z'\subset T_2$ is a homotopy equivalence. Hence we may extend $L_1|_{\overline{T_1}}$ and $L'_1|_{\overline{T_1}}$ to local systems, L_2 and L'_2 on T_2. For $c=\mathrm{codim}^{\mathbf{C}}_{Z'} Z\cap Z'$, we have a "Gysin isomorphism" $\mathrm{Ext}^{\cdot}(L|_{Z\cap Z'},L'|_{Z\cap Z'})\xrightarrow{\sim}\mathrm{Ext}^{\cdot+2c}(L,L')$ given by the multiplication by the fundamental class, δ, of $Z\cap Z'$ in Z'. Let $\xi\in\Gamma(Y;(\Omega^{2c}_Y)_{T_1})$ be a representative of δ. Then the multiplication by ξ gives a quasi-isomorphism $\Gamma(T_2;\Omega_{T_2}\otimes\mathcal{H}om(L_2,L'_2))\to\Gamma(Y;\Omega_Y\otimes\mathcal{H}om(L_1,L'_1))[2c]$. Our sheaf $\mathcal{B}$ is defined as follows. For $i\in I$, recall that the inclusion $X_i\subset V_i$ is a homotopy equivalence, so that the $L^\rho_{\mathcal{O}}|_{X_i}$ extend as local systems, $L^\rho_{\mathcal{O},i}$, to V_i. We set $L''_i=E\times_K\bigoplus_{(\mathcal{O},\rho)} L^\rho_{\mathcal{O},i}$ and $\mathcal{B}_i=\Gamma(E\times_K V_i;\Omega_{E\times_K V_i}\otimes\mathcal{H}om(L''_i,L''_i))$. The above Gysin isomorphism implies that we have a quasi-isomorphism $\mathcal{B}\to\mathcal{A}'$. Now, as a sheaf, $\mathcal{B}$ is intrinsically defined from the data of X, the local systems $L^\rho_{\mathcal{O}}$ and the stratification. However,

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we also have to understand what the multiplicative structure becomes through the Gysin isomorphism: for a third local system L'' on a subvariety Z'', we have a composition $\mathrm{Ext}^{\cdot}(L,L')\times\mathrm{Ext}^{\cdot}(L',L'')\to\mathrm{Ext}^{\cdot}(L,L'')$. Its counterpart on the extensions groups $\mathrm{Ext}^{\cdot}(L|_{Z\cap Z'},L'|_{Z\cap Z'})$,... is also given by the composition, but twisted by fundamental classes of some subvarieties obtained from intersections of Z, Z', Z''. More precisely, let us consider $\Delta,\Delta',\Delta''\subset V$, such that $Z=\bigcap_{v\in\Delta}E\times_K D_v$, Z', Z'' being obtained in the same way from Δ',Δ''. For $W\subset V$, we set $\delta(W)=\prod_{v\in W}\delta_v$. The Gysin isomorphism, from $\mathrm{Ext}^{\cdot}(L|_{Z\cap Z'},L'|_{Z\cap Z'})$ to $\mathrm{Ext}^{\cdot+2c}(L,L')$ is then given by the multiplication by $\delta(\Delta\setminus\Delta')$, and the product

$$\mathrm{Ext}^{\cdot}(L|_{Z\cap Z'},L'|_{Z\cap Z'})\times\mathrm{Ext}^{\cdot}(L'|_{Z'\cap Z''},L''|_{Z'\cap Z''})\to\mathrm{Ext}^{\cdot}(L|_{Z\cap Z''},L''|_{Z\cap Z''})$$

is defined by the composition of the cup-product on $Z\cap Z'\cap Z''$ and the multiplication by $\delta((\Delta'\setminus(\Delta\cup\Delta''))\cup((\Delta\cap\Delta'')\setminus\Delta'))$. The product in $\mathcal{B}$ is defined by a similar formula, where the classes δ_v are replaced by their representatives ξ_v already introduced above. We prove that we can choose a chain of quasi-isomorphisms, from $\mathcal{B}$ to $H(\mathcal{B})$, compatible with the product, as in step 3. The final step 4 is the same as in the case of connected isotropy groups.

Here is the plan of the paper. In section 2 we recall some facts about equivariant derived categories, Weil algebras and constructible sheaves. In section 3 we construct the dg-algebras $\mathcal{A}'$ and $\mathcal{B}$ of steps 2'.a and 2'.b above. The main result of this section is Proposition 3.7. In section 4 we recall some results of [2] on symmetric varieties and use them to prove that the hypothesis of Proposition 3.7 are satisfied. Sections 5 and 6 are devoted to the proofs of steps 3 and 4.

Notations. Notations for functors on sheaves are taken from [10]. For a topological space X, we denote by $\mathrm{D}(X)$ (resp. $\mathrm{D}^b(X)$) the (resp. bounded) derived category of sheaves of $\mathbf{C}$-vector spaces on X. If X is a real analytic manifold, we denote by $\mathrm{D}^b_{\mathbf{R}-c}(X)$ the subcategory of $\mathrm{D}^b(X)$ formed by complexes with real constructible cohomology. The constant sheaf of group M on X is denoted M_X. The direct and inverse images by a map $i:X\to Y$ are denoted i_* and i^{-1}. If X and Y are separated and locally compact, $i_!$ denotes the direct image with proper supports. If i is the embedding of a locally closed subset X of Y, and $F\in\mathrm{D}(Y)$, we set $F_X=i_!i^{-1}F$ and for a group M, $M_X=(M_Y)_X$. We will also use $\Gamma_X(F)$, which is the subsheaf of F given by the sections with support in X, when X is closed, and the sheaf $U\mapsto F(U\cap X)$ when X is open; in general, we have $\Gamma_{X\cap X'}F=\Gamma_X\Gamma_{X'}F$. The homomorphisms sheaf is denoted $\mathcal{H}om(\cdot,\cdot)$. We recall that $R\Gamma_X(F)\simeq R\mathcal{H}om(\mathbf{C}_X,F)$. For $F\in\mathrm{D}^b(X)$, we set $F^*=R\mathcal{H}om(F,\mathbf{C}_X)$. We will sometimes use the notation, for a subset $Z\subset Y$ and $F\in\mathrm{D}(Y)$, $R\Gamma(Z;F)=R\Gamma(Z;F|_Z)$.

For a triangulated category D and objects $M_\alpha\in\mathrm{D}$, we denote by $\mathrm{D}\langle M_\alpha\rangle$ the triangulated subcategory generated by the M_α, i.e., the smallest full triangulated subcategory of D such that: (i) $\mathrm{D}\langle M_\alpha\rangle$ contains the M_α, and (ii) if $N\simeq M$ and $M\in\mathrm{D}\langle M_\alpha\rangle$, then $N\in\mathrm{D}\langle M_\alpha\rangle$.

For a complex $M^{\cdot}$, $n\in\mathbf{Z}$, $M[n]$ denotes the complex $M^i[n]=M^{i+n}$ with differential $(-1)^n d_M$.

For a manifold X, we denote by Ω_X the de Rham complex of X. If not specified, the cohomology of a space is taken with coefficients in $\mathbf{C}$.

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2. Preliminaries

2.1. **Equivariant derived categories.** In this section we recall some results of [3] and [13] about equivariant derived categories and (sheaves of) dg-algebras. We consider, for the convenience of exposition, a linear Lie group G with finitely many connected components (in section 4, G will be complex semi-simple). We let K be a maximal compact subgroup of G. Hence the variety G/K is isomorphic to an affine space $\mathbf{R}^k$.

We consider a sequence of embeddings of G-manifolds $E_i \subset E_{i+1}$ with free actions such that $H^k(E_i) = 0$ for $0 < k < i$ (since G is a linear group, one may choose Stiefel varieties for the E_i). We set $E = \bigcup_{i\in\mathbf{N}} E_i$, endowed with the limit topology. The bounded equivariant derived category of X, $\mathrm{D}^b_G(X)$, is the category formed by the triples $F = (F_X, \overline{F}, \beta)$ where $F_X \in \mathrm{D}^b(X)$, $\overline{F} \in \mathrm{D}^b(E \times_G X)$ and β is an isomorphism between the inverse images of F_X and $\overline{F}$ on $E \times X$. The morphisms from F to $F' = (F'_X, \overline{F'}, \beta')$ are the pairs of morphisms $(u_X, \bar{u})$, $u_X : F_X \to F'_X$, $\bar{u} : \overline{F} \to \overline{F'}$ commuting with β and β'. It is shown in [3] that $\mathrm{D}^b_G(X)$ is independent of the choice of E. (The reason to assume that E is a limit of manifolds is to be able to define functors such as the proper direct image or the extraordinary inverse image.) If X is a real analytic manifold, we denote by $\mathrm{D}^b_{G,c}(X)$ the subcategory of $\mathrm{D}^b_G(X)$ formed by the triples F above such that F_X has real constructible cohomology sheaves. By [3], Lemma 2.9.2, the forgetful functor $F = (F_X, \overline{F}, \beta) \mapsto \overline{F}$ identifies $\mathrm{D}^b_{G,c}(X)$ as a full subcategory of $\mathrm{D}^b(E \times_G X)$.

Let $\bar{q} : E \times_K X \to E \times_G X$ be the quotient map. The restriction functor $R_{K,G} : \mathrm{D}^b_{G,c}(X) \to \mathrm{D}^b_{K,c}(X)$, $(M_X, \overline{M}, \beta) \mapsto (M_X, \bar{q}^{-1}(\overline{M}), \beta)$, is fully faithful. Indeed, for F, $F' \in \mathrm{D}^b_{G,c}(X)$ we have an isomorphism $\mathrm{Hom}_{\mathrm{D}^b_{G,c}(X)}(F, F') \simeq H^0(E \times_G X; R\mathcal{H}om(\overline{F}, \overline{F'}))$, and the corresponding isomorphism in $\mathrm{D}^b_{K,c}(X)$. Since the fibres of $\bar{q}$ are acyclic, we have, $\forall M \in \mathrm{D}^b_{G,c}(X)$, $H^0(E \times_K X; \bar{q}^{-1}(\overline{M})) \simeq H^0(E \times_G X; \overline{M})$. This implies the claim because of the isomorphism $R_{K,G}(R\mathcal{H}om(F, F')) \simeq R\mathcal{H}om(R_{K,G}(F), R_{K,G}(F'))$.

For a G-orbit, $\mathcal{O} \simeq G/H$, of X, we denote by $\tau_{\mathcal{O}} = H/H^0$ the group of connected components of the isotropy group H. The G-equivariant sheaves with support $\mathcal{O}$ are in correspondence with the representations of $\tau_{\mathcal{O}}$. (Let us recall that the objects of $\mathrm{D}^b_G(X)$ concentrated in degree 0 correspond to the G-equivariant sheaves on X.) For a representation ρ of $\tau_{\mathcal{O}}$, let $L^\rho_{\mathcal{O}}$ denote the corresponding local system on $\mathcal{O}$. Let us assume that G has finitely many orbits in X. Let $i_{\mathcal{O}} : \mathcal{O} \to X$ be the inclusion of an orbit $\mathcal{O}$. For $F \in \mathrm{D}^b_{G,c}(X)$, we see, by induction on the length of $i_{\mathcal{O}}^{-1}F$, that $i_{\mathcal{O}}^{-1}F$ belongs to the triangulated subcategory of $\mathrm{D}^b_{G,c}(\mathcal{O})$ generated by the $L^\rho_{\mathcal{O}}$. We consider the morphism $u : F \to R(i_{\mathcal{O}})_* i_{\mathcal{O}}^{-1}F$ and denote by F_u the third object of a distinguished triangle built on u. If $\mathcal{O}$ is open in $\operatorname{supp} F$, then $\operatorname{supp} F_u$ contains less orbits than $\operatorname{supp} F$. We deduce that the category $\mathrm{D}^b_{G,c}(X)$ is generated by the $Ri_{\mathcal{O}*}L^\rho_{\mathcal{O}}$, where $\mathcal{O}$ runs over the G-orbits and ρ over the irreducible representations of $\tau_{\mathcal{O}}$. Hence the restriction functor identifies $\mathrm{D}^b_{G,c}(X)$ with the subcategory of $\mathrm{D}^b_{K,c}(X)$ generated by the $Ri_{\mathcal{O}*}L^\rho_{\mathcal{O}}$ (viewed as objects of $\mathrm{D}^b_{K,c}(X)$).

As in the introduction, we let $\Omega^d_{E_i \times_K X}$ denote the sheaf of $\mathcal{C}^\infty$-forms of degree d on $E_i \times_K X$, and we set $\Omega^d_{E\times_G X} = \varprojlim_i \Omega^d_{E_i \times_G X}$. The complex $\Omega^{\cdot}_{E\times_K X}$ is a soft resolution of $\mathbf{C}_{E\times_G X}$ by a sheaf of differential graded anti-commutative algebras.

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We call dg-algebra a non-negatively graded algebra over $\mathbf{C}$, $\mathcal{A} = \bigoplus_{i\in\mathbf{N}} \mathcal{A}^i$, endowed with a differential d of degree 1 such that, for any homogeneous elements a, b, $d(ab) = (da)b + (-1)^{\deg a} a\, db$. In [13] Lunts considers a sheaf of dg-algebras, $\mathcal{A}$, over a topological set I with finitely many points and defines the derived category $\mathrm{D}_{\mathcal{A}}$ as follows. We denote by $\mathrm{M}_{\mathcal{A}}$ the category of sheaves of dg-modules over $\mathcal{A}$. A morphism $f : M \to M'$ is called a quasi-isomorphism if $\forall i \in I$, $f_i : M_i \to M'_i$ is a quasi-isomorphism. We consider $\mathrm{K}_{\mathcal{A}}$, the category with the same objects as $\mathrm{M}_{\mathcal{A}}$ and with sets of morphisms quotiented by the null-homotopic morphisms. Then $\mathrm{D}_{\mathcal{A}}$ is the localisation of $\mathrm{K}_{\mathcal{A}}$ by the quasi-isomorphisms. There is a substitute for the notion of projective object in this framework: an object $P \in \mathrm{M}_{\mathcal{A}}$ is said to be K-projective (see [16] and [3], p. 74) if $\mathrm{Hom}_{\mathrm{K}_{\mathcal{A}}}(P, \cdot) = \mathrm{Hom}_{\mathrm{D}_{\mathcal{A}}}(P, \cdot)$.

Let A be the graded algebra underlying $\mathcal{A}$ and let M_A be the category of (non-differential) graded A-modules. For $M, N \in \mathrm{M}_{\mathcal{A}}$, we set $\mathrm{Hom}^n(M,N) = \mathrm{Hom}_{\mathrm{M}_A}(M, N[n])$ and, for $f \in \mathrm{Hom}^n(M,N)$, $df = d_N \circ f - (-1)^n f \circ d_M$. This turns $\mathrm{Hom}^n(M,N)$ into a complex, and we obtain a bifunctor from $\mathrm{M}_{\mathcal{A}}^{op} \times \mathrm{M}_{\mathcal{A}}$ to the category of complexes of abelian groups. Denoting by RHom its derived functor, we have $\mathrm{Hom}_{\mathrm{D}_{\mathcal{A}}}(M,N) \simeq H^0\, \mathrm{RHom}(M,N)$.

Here is how to obtain K-projectives in $\mathrm{M}_{\mathcal{A}}$. For an open subset U of I and an $\mathcal{A}$-module F, let F_U be the extension by 0 of $F|_U$. For $M \in \mathrm{M}_{\mathcal{A}}$, we have $\mathrm{Hom}^{\cdot}(\mathcal{A}_U, M) = \Gamma(U; M)$. For a point $i \in I$, we denote by U_i the smallest open subset of I containing i. These fundamental open sets generate the topology of I. For a sheaf F on I we have $F_i = F(U_i)$. Hence the functor of sections over U_i is exact and the $\mathcal{A}$-module $\mathcal{A}_{U_i}$ is K-projective. One may deduce that the category $\mathrm{K}_{\mathcal{A}}$ has enough K-projectives and hence also enough K-flat objects (see [13], Proposition 1.7.4). Let $\phi : \mathcal{A} \to \mathcal{B}$ be a morphism of sheaves of dg-algebras on I such that $\forall i \in I$, $H(\phi_i) : H(\mathcal{A}_i) \to H(\mathcal{B}_i)$ is an isomorphism. By [13], Proposition 1.11.2, the functors of restriction and extension of scalars induce an equivalence of categories $\mathrm{D}_{\mathcal{A}} \simeq \mathrm{D}_{\mathcal{B}}$.

2.2. **Formality of classifying spaces.** An important point in the proof of Theorem 1.1 is the fact that some de Rham algebras are formal, i.e., quasi-isomorphic to their cohomology algebras. We will use, in particular, the following consequence of the results of [5]: for a compact Lie group K with universal bundle E (given as above by a sequence of K-manifolds) the de Rham algebras $\Gamma(E/H; \Omega^{\cdot}_{E/H})$ are formal in a compatible way for all subgroups $H \subset K$ (see Lemma 2.3 below).

Let us first recall the definition of the Weil algebra $W(\mathfrak{k})$ of a Lie algebra $\mathfrak{k}$, as explained in [5] (see also [9]). As a graded $\mathbf{C}$-algebra, $W(\mathfrak{k}) = \Lambda^{\cdot}(\mathfrak{k}^*) \otimes S^{\cdot}(\mathfrak{k}^*)$, where $\Lambda^{\cdot}(\mathfrak{k}^*)$ denotes the exterior algebra of $\mathfrak{k}^*$ and elements of $\Lambda^1(\mathfrak{k}^*) \simeq \mathfrak{k}^*$ have degree 1, and $S^{\cdot}(\mathfrak{k}^*)$ denotes the symmetric algebra of $\mathfrak{k}^*$ and elements of $S^1(\mathfrak{k}^*) \simeq \mathfrak{k}^*$ have degree 2. The algebra $W(\mathfrak{k})$ is endowed with a differential, δ, of degree 1, and derivations, for any $x \in \mathfrak{k}$, $i(x)$ of degree -1, $\theta(x)$ of degree 0. They satisfy the relations, for $x, y \in \mathfrak{k}$:

$$\theta([x,y]) = \theta(x)\,\theta(y) - \theta(y)\,\theta(x), \tag{1}$$

$$i([x,y]) = \theta(x)\, i(y) - i(y)\,\theta(x), \tag{2}$$

$$\theta(x) = i(x)\,\delta + \delta\, i(x). \tag{3}$$

They are defined as follows. First we note that, for a connected Lie group K, with Lie algebra $\mathfrak{k}$, $\Lambda^{\cdot}(\mathfrak{k}^*)$ is identified with the left invariant subalgebra of $\Omega^{\cdot}(K)$

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and inherits the differential d_Λ, the contraction $i_\Lambda(x)$ by the vector field associated to $x \in \mathfrak{k}$, and the Lie derivative $\theta_\Lambda(x)$. Explicitly, for $(x, x') \in \mathfrak{k} \times \mathfrak{k}^*$, we have $i_\Lambda(x)(x') = \langle x, x' \rangle$, $\theta_\Lambda(x)(x') = -ad_x^t(x')$, and, for dual basis (x_i) of $\mathfrak{k}$, (x'_i) of $\mathfrak{k}^*$, we have $d_\Lambda = \frac{1}{2} \sum_i x'_i \theta_\Lambda(x_i)$.

Now we define $i(x)$, $\theta(x)$ and δ on $W(\mathfrak{k})$. Since they are derivations, they are uniquely determined by their values on the generators of $W(\mathfrak{k})$. In the following formulas, $x \in \mathfrak{k}$, $x' \in \Lambda^1(\mathfrak{k}^*) \simeq \mathfrak{k}^*$, $\tilde{x}' \in S^1(\mathfrak{k}^*) \simeq \mathfrak{k}^*$ (recall that $\deg(x') = 1$, $\deg(\tilde{x}') = 2$). We let $h : \Lambda^1(\mathfrak{k}^*) \xrightarrow{\sim} S^1(\mathfrak{k}^*)$ be the natural isomorphism and we consider dual basis (x_i) of $\mathfrak{k}$, (x'_i) of $\mathfrak{k}^*$. With these notations we have:

$$i(x)(x' \otimes 1) = i_\Lambda(x)(x') = \langle x, x' \rangle, \qquad i(x)(1 \otimes \tilde{x}') = 0,$$
$$\theta(x)(x' \otimes 1) = -ad_x^t(x') \otimes 1, \qquad \theta(x)(1 \otimes \tilde{x}') = -1 \otimes ad_x^t(\tilde{x}'),$$
$$\delta(x' \otimes 1) = d_\Lambda(x') \otimes 1 + 1 \otimes h(x'), \qquad \delta(1 \otimes \tilde{x}') = \textstyle\sum_i x'_i \otimes \theta(x_i)(\tilde{x}').$$

By [5], Theorem 1, we have:

$$H^0(W(\mathfrak{k}), \delta) = \mathbf{C}, \qquad \forall i > 0 \quad H^i(W(\mathfrak{k}), \delta) = 0. \tag{4}$$

Definition 2.1. Let (A, d_A) be a dg-algebra. One says that $\mathfrak{k}$ acts on A, if A is endowed with two linear maps, $i, \theta : \mathfrak{k} \to \mathrm{Der}(A)$, from $\mathfrak{k}$ to the space of derivations of A, such that, $\forall x \in \mathfrak{k}$, $i(x)$ is of degree -1, $\theta(x)$ of degree 0, $i(x)^2 = 0$ and i, θ, d_A satisfy the relations (1) to (3), with d_A instead of δ. In this case, the subspace of "$\mathfrak{k}$-basic" elements,

$$A_{\mathfrak{k}-b} = \{a \in A;\ \forall x \in \mathfrak{k},\ i(x)(a) = \theta(x)(a) = 0\},$$

is a sub-dg-algebra.

We note that, if θ is given by differentiation of a K-action in A, and K is connected, then the subalgebra of K-invariants is $A^K = \{a \in A;\ \forall x \in \mathfrak{k},\ \theta(x)(a) = 0\}$ (in general A^K is not stable by d_A). The main example is given by the de Rham algebra $\Omega^\cdot(T)$ of the total space of a K-principal fibre bundle, $\tau : T \to B$: for $x \in \mathfrak{k}$, $i(x)$ and $\theta(x)$, are the usual contraction and Lie derivative associated to the vector field on T induced by x. We have $\Omega^\cdot(T)_{\mathfrak{k}-b} \simeq \Omega^\cdot(B)$. For $W(\mathfrak{k})$, the elements annihilated by all contractions $i(x)$, $x \in \mathfrak{k}$, are the elements of $S(\mathfrak{k}^*)$; hence, if K is connected, $W(\mathfrak{k})_{\mathfrak{k}-b} \simeq (S(\mathfrak{k}^*))^K$.

For a K-principal fibre bundle T as above, recall that a connection on T is the data of projections, $\forall P \in T$, $\phi_P : T_P T \to T_P(\tau^{-1}\tau(P))$, such that $\forall k \in K$, ϕ_{kP} is conjugate to ϕ_P by the action of k (and the ϕ_P vary differentiably). Since the derivative of the K-action naturally identifies $\mathfrak{k}$ with $T_P(\tau^{-1}\tau(P))$, a connection corresponds to a morphism $f : \mathfrak{k}^* \to \Omega^1(T)$. More generally, for a dg-algebra (A, d_A), with a $\mathfrak{k}$-action, a "connection" on A is a linear morphism $f : \mathfrak{k}^* \to A^1$ satisfying:

$$\forall x \in \mathfrak{k}, \forall x' \in \mathfrak{k}^*, \qquad i(x)(f(x')) = \langle x, x' \rangle, \quad \theta(x)(f(x')) = f(-ad_x^t(x')). \tag{5}$$

We extend naturally f to an algebras morphism, still denoted by f, from $\Lambda^\cdot(\mathfrak{k}^*)$ to A. But, in general, f does not commute with the differential. The algebra $W(\mathfrak{k})$ has the following universal property: we may extend f to an algebras morphism, $\bar{f} : W(\mathfrak{k}) \to \Omega^\cdot(T)$, with the following values on the generators:

$$\bar{f}(x' \otimes 1) = f(x'), \qquad \bar{f}(1 \otimes h(x')) = d_A(f(x')) - f(d_\Lambda(x')),$$

commuting with the differentials, the "contractions", $i(x)$, and the "Lie derivatives", $\theta(x)$. In particular, for the K-principal fibre bundle T above, we obtain a

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morphism $\bar{f} : W(\mathfrak{k}) \to \Omega(T)$, and it induces a morphism on the basic sub-algebras $\tilde{f} : (S(\mathfrak{k}^*))^K \to \Omega^{\cdot}(B)$.

The following result can be found in [5], though not explicitly stated. This is also a particular case of Theorem 4.3.1 of [9].

Theorem 2.2 ([5], [9])**.** *Let H be a connected compact Lie group with Lie algebra $\mathfrak{h}$, A a dg-algebra with $\mathfrak{h}$-action and a connection. We assume that $H^i(A) = 0$ for $i > 0$ and $H^0(A) = \mathbf{C}$. Then $H^{\cdot}(A_{\mathfrak{h}-b}) \simeq S(\mathfrak{h}^*)^H$.*

We return to the situation of a compact connected Lie group K, acting on a universal bundle E which is an increasing union of K-manifolds, $E = \bigcup_i E_i$. We choose compatible connections on the E_i (i.e. the connection on E_i is the restriction of the one on E_{i+1}). This gives a connection, in the algebraic sense of (5), $f : \mathfrak{k}^* \to \Gamma(E; \Omega^1_E)$. It induces a dg-algebras morphism $\bar{f} : W(\mathfrak{k}) \to \Gamma(E; \Omega^{\cdot}_E)$, compatible with the contraction i and the Lie derivative θ. For a connected subgroup $H \subset K$, with Lie algebra $\mathfrak{h}$, the action of $\mathfrak{k}$ on $W(\mathfrak{k})$ obviously restricts to an action of $\mathfrak{h}$.

Lemma 2.3. *With the notations H, K, E, f, introduced above, the space of $\mathfrak{h}$-basic elements of $W(\mathfrak{k})$ is $W(\mathfrak{k})_{\mathfrak{h}-b} \simeq (\Lambda^{\cdot}(\mathfrak{h}^{\perp}) \otimes S^{\cdot}(\mathfrak{k}^*))^H$, where $\mathfrak{h}^{\perp} \subset \mathfrak{k}^*$ denotes the subspace orthogonal to $\mathfrak{h}$. The projection $W(\mathfrak{k})_{\mathfrak{h}-b} \to S^{\cdot}(\mathfrak{h}^*)^H$ and the morphism induced by the connection, $W(\mathfrak{k})_{\mathfrak{h}-b} \to \Gamma(E; \Omega^{\cdot}_E)_{\mathfrak{h}-b} \simeq \Gamma(E/H; \Omega^{\cdot}_{E/H})$, are quasi-isomorphisms.*

The normaliser of H, $N_K(H)$, acts on $W(\mathfrak{k})_{\mathfrak{h}-b}$, $S^{\cdot}(\mathfrak{k}^*)^H$ and $\Gamma(E/H; \Omega^{\cdot}_{E/H})$, and the above morphisms are $N_K(H)$-equivariant. For another connected subgroup $H_1 \subset H \subset K$, with Lie algebra $\mathfrak{h}_1$, we have a commutative diagram

$$\begin{array}{ccccc}
\Gamma(E/H; \Omega^{\cdot}_{E/H}) & \longleftarrow & W(\mathfrak{k})_{\mathfrak{h}-b} & \longrightarrow & S^{\cdot}(\mathfrak{h}^*)^H \\
\downarrow & & \downarrow & & \downarrow \\
\Gamma(E/H_1; \Omega^{\cdot}_{E/H_1}) & \longleftarrow & W(\mathfrak{k})_{\mathfrak{h}_1-b} & \longrightarrow & S^{\cdot}(\mathfrak{h}_1^*)^{H_1},
\end{array}$$

where the horizontal arrows are quasi-isomorphisms.

Proof. The $\mathfrak{h}$-basic elements of $W(\mathfrak{k})$ are the elements annihilated by all $i(x)$ and $\theta(x)$ for $x \in \mathfrak{h}$. Since $i(x)$ is a derivation and acts trivially on $S^{\cdot}(\mathfrak{k}^*)$, the set of elements of $W(\mathfrak{k})$ annihilated by all $i(x)$, $x \in \mathfrak{h}$, is $\Lambda^{\cdot}(\mathfrak{h}^{\perp}) \otimes S^{\cdot}(\mathfrak{k}^*)$. Since H is connected, the elements annihilated by the $\theta(x)$ are the H-invariants. Hence we have the description of $W(\mathfrak{k})_{\mathfrak{h}-b}$ given in the lemma. By this description, $W(\mathfrak{k})_{\mathfrak{h}-b}$ admits a projection to $S^{\cdot}(\mathfrak{k}^*)^H$ and hence to $S^{\cdot}(\mathfrak{h}^*)^H$.

Let us choose an H-stable decomposition $\mathfrak{k} = \mathfrak{h} \oplus V$. It induces an H-equivariant splitting $g : \mathfrak{h}^* \to \mathfrak{k}^* \simeq W^1(\mathfrak{k})$. This is a connection on $W(\mathfrak{k})$, for the $\mathfrak{h}$-action, in the sense of (5). Hence it gives a morphism of dg-algebras $\bar{g} : W(\mathfrak{h}) \to W(\mathfrak{k})$. By (4) and Theorem 2.2, the induced morphism $\tilde{g} : S^{\cdot}(\mathfrak{h}^*)^H \simeq W(\mathfrak{h})_{\mathfrak{h}-b} \to W(\mathfrak{k})_{\mathfrak{h}-b}$ is a quasi-isomorphism. We note that, by definition, $\tilde{g}$ is also a splitting of the projection $q : W(\mathfrak{k})_{\mathfrak{h}-b} \to S^{\cdot}(\mathfrak{h}^*)^H$, so that q is a quasi-isomorphism too.

The composition $f_1 = f \circ g : \mathfrak{h}^* \to \Gamma(E; \Omega^1_E)$ is also a connection on $\Gamma(E; \Omega^{\cdot}_E)$, for the $\mathfrak{h}$-action. Hence it gives a morphism $\bar{f}_1 : W(\mathfrak{h}) \to \Gamma(E; \Omega^{\cdot}_E)$, and we have $\bar{f}_1 = \bar{f} \circ \bar{g}$. By Theorem 2.2 again, the induced morphism on the $\mathfrak{h}$-basic elements, $(\bar{f}_1)_{\mathfrak{h}-b}$ is a quasi-isomorphism. Since $(\bar{g})_{\mathfrak{h}-b}$ is also, $(\bar{f})_{\mathfrak{h}-b} : W(\mathfrak{k})_{\mathfrak{h}-b} \to \Gamma(E; \Omega^{\cdot}_E)_{\mathfrak{h}-b}$ is a quasi-isomorphism, as claimed.

The compatibility of the above morphisms with the $N_K(H)$-action and the commutativity of the diagram follows from the functoriality of the construction. □

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2.3. **Constructible sheaves.** Here we recall some results of [10] on constructible (complex of) sheaves on real analytic manifolds. Let Y be a real analytic manifold. We say that a locally finite partition of Y by locally closed real analytic manifolds, $Y = \bigsqcup_{i\in I} Y_i$, is a stratification if $\forall i,j \in I$, $Y_i \cap \overline{Y_j} \neq \emptyset$ implies $Y_i \subset \overline{Y_j}$. For two closed subsets A, B of T^*Y, which are conic, i.e., stable by the action of $\mathbf{R}_{>0}$ in the fibres, we let $A \widehat{+} B$ be the subset of T^*Y defined as follows (see [10] Definition 6.2.3 and Remark 6.2.8): in a local chart $U \simeq \mathbf{R}^d$ of Y, $(x,\xi) \in T^*U \simeq \mathbf{R}^d \times \mathbf{R}^d$ belongs to $A \widehat{+} B$ iff there exists sequences $(x_n, \xi_n) \in A$, $(y_n, \eta_n) \in B$ such that $x_n \to x$, $y_n \to x$, $\xi_n + \eta_n \to \xi$ and $|x_n - y_n||\xi_n| \to 0$. We let $\pi_Y : T^*Y \to Y$ be the projection and set $T^*_{Y_i}Y = \{(y,\xi) \in T^*Y;\ y \in Y_i,\ \langle \xi, T_yY_i\rangle = 0\}$. We say that the stratification is a μ-stratification if $\forall i \neq j \in I$ such that $Y_i \subset \overline{Y_j}$ we have $(T^*_{Y_j}Y \widehat{+} T^*_{Y_i}Y) \cap \pi_Y^{-1}(Y_i) \subset T^*_{Y_i}Y$. We note that if $Y = \bigsqcup_{i\in I} Y_i$ is a μ-stratification, then so is the product $Y \times \mathbf{R}^d = \bigsqcup_{i\in I} Y_i \times \mathbf{R}^d$, and if $S = \bigsqcup_{i\in I} S_i$ is a μ-stratification of the d-sphere, then so is the cone over it: $\mathbf{R}^{d+1} = \{0\} \sqcup (\bigsqcup_{i\in I} \mathbf{R}_{>0} \cdot S_i)$ (the condition is trivial at the vertex and at other strata the stratification is diffeomorphic to a product). Finally, if a submanifold Z of $Y = \bigsqcup_{i\in I} Y_i$ intersects all strata transversally and the partition is a μ-stratification, then so is the partition $Z = \bigsqcup_{i\in I} Z \cap Y_i$.

For a complex of sheaves $F \in \mathrm{D}^b(Y)$, we have the notion of micro-support, $SS(F)$, which is a closed conic subset of T^*Y. We refer to Definition 5.1.1 of [10] and just recall that, if $Y = \bigsqcup_{i\in I} Y_i$ is a μ-stratification, and F is constructible with respect to this stratification, then $SS(F) \subset \bigsqcup_i T^*_{Y_i}Y$ (see Proposition 8.4.1 of [10]). We denote by $\mathrm{D}^b_{\mathbf{R}-c}(Y)$ the subcategory of $\mathrm{D}^b(Y)$ formed by complexes with real constructible cohomology (with respect to any stratification). We will use several times the following results of [10].

Lemma 2.4 ([10], Lemma 5.4.14)**.** *Let Y be a real analytic manifold, $F \in \mathrm{D}^b_{\mathbf{R}-c}(Y)$, $G \in \mathrm{D}^b(Y)$ and assume that $SS(F) \cap SS(G) \subset T^*_YY$. Then the natural morphism $R\mathcal{H}om(F, \mathbf{C}_Y) \otimes G \to R\mathcal{H}om(F,G)$ is an isomorphism.*

Lemma 2.5 ([10], Lemma 8.4.7)**.** *Let Y be a real analytic manifold, $F \in \mathrm{D}^b_{\mathbf{R}-c}(Y)$, $f : Y \to \mathbf{R}$ a real analytic function such that $f|_{\operatorname{supp} F}$ is proper. For $\varepsilon > 0$ we set $Z = f^{-1}(0)$, $Z_\varepsilon = f^{-1}([0,\varepsilon])$, $U_\varepsilon = f^{-1}([0,\varepsilon[)$. Then there exists $\varepsilon_0 > 0$ such that, $\forall \varepsilon$, $0 < \varepsilon < \varepsilon_0$, we have the isomorphisms*

$$H^{\cdot}_Z(Y;F) \xrightarrow{\sim} H^{\cdot}_{Z_\varepsilon}(Y;F), \qquad H^{\cdot}(Z_\varepsilon;F) \xrightarrow{\sim} H^{\cdot}(U_\varepsilon;F) \xrightarrow{\sim} H^{\cdot}(Z;F).$$

2.4. **Local systems outside normal crossings divisors.** Here we make some easy remarks on local systems defined outside normal crossings divisors. Let Y be a smooth complex manifold and $(D_v)_{v\in V}$ a finite family of smooth normal crossings divisors. We set $U = Y \setminus \bigcup_{v\in V} D_v$. Local systems (over $\mathbf{C}$) on U are in bijective correspondence with complex representations of $\pi_1(U)$. For such a representation, ρ, we denote by L^ρ the associated local system.

We fix $v \in V$ and set $Y_v = Y \setminus \bigcup_{w\neq v} D_w$, $D'_v = D_v \cap Y_v$. We let T_v be a tubular neighbourhood of D'_v in Y_v, homeomorphic to the normal bundle of D'_v in Y_v, and with a projection $\pi_v : T_v \to D'_v$. For $x \in T_v \cap U = T_v \setminus D'_v$, the fibre $\pi_v^{-1}\pi_v(x) \simeq \mathbf{C}$ is oriented, and we let γ_x be a loop in $\pi_v^{-1}\pi_v(x)$ with base-point x and turning $+1$ time around 0. Now, let $b \in U$ be a base-point, τ a path from b to x. The conjugacy class of the image of $\tau^{-1}\gamma_x\tau$ in $\pi_1(U)$ is well-defined. We denote it by C_x. If x' is another point of $T_v \cap U$, and $\gamma : [0,1] \to T_v \cap U$ a path from x to x', the loops γ_x and $\gamma^{-1}\gamma_{x'}\gamma$ are homotopic. It follows that C_x is independent of x. Hence the

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image of C_x by ρ also is well-defined up to conjugacy. We call it the monodromy of L^ρ around D_v. We quote the following facts for later use.

Lemma 2.6. *In the above situation, let L be a local system of finite rank on U.*

(i) *If the monodromy of L around D_v is Id, then L extends as a local system, L', to Y_v. For $w \neq v$, the monodromy of L' around D_w is the monodromy of L around D_w.*

(ii) *Let $j : U \to Y$ be the inclusion. We assume that ρ factors through a finite quotient of $\pi_1(U)$. If, for each $v \in V$, the monodromy of L around D_v has no eigenvalue equal to 1, then $Rj_*L \simeq j_*L \simeq j_!L$.*

Proof. (i) Let $j_v : U \to Y_v$ be the inclusion. It follows from the definition that $L' = (j_v)_*L$ has the required properties.

(ii) The assertion is equivalent to $(Rj_*L)_x = 0$ for any $x \in Y \setminus U$. Since this is a local problem around x, we may assume that $Y = \mathbf{C}^n$, and we have coordinates $(x_1, \dots, x_n)$ such that $x = (0, \dots, 0)$, $D_v = \{x_v = 0\}$, $v = 1, \dots, m$, $U = X \setminus \bigcup_{v=1,\dots,m} D_v$. Then $(R^i j_*L)_x = \varinjlim_V H^i(V \cap U; L)$, where V runs over neighbourhoods of 0. We may assume V of the type $V = \{(\underline{x});\ \forall i,\ |x_i| < \varepsilon\}$. Then $V \cap U$ decomposes as a product $V \cap U \simeq \mathbf{R}^{2n-m} \times (S^1)^m$ and $\pi_1(V \cap U)$ acts on $L|_{V\cap U}$ through a finite abelian group. Hence we may decompose $L|_{V\cap U}$ into a sum of irreducible components, L_k, which are local systems of rank 1. Then $L_k \simeq \mathbf{C}_{\mathbf{R}^{2n-m}} \boxtimes L_k^1 \boxtimes \cdots \boxtimes L_k^m$, for rank 1 local systems L_k^j on S^1. The monodromy of L_k around D_j is the monodromy of L_k^j around S^1. By hypothesis, it is not 1, so that L_k^j is non-trivial and we have $H^0(S^1; L_k^j) = H^1(S^1; L_k^j) = 0$. The Künneth formula yields $\forall i$, $H^i(V \cap U; L) = 0$, as desired. □

3. Categories of sheaves and dg-algebras

We consider a manifold Y endowed with a finite stratification $Y = \bigsqcup_{i\in I} Y_i$ by locally closed submanifolds. We denote by $\phi : Y \to I$ the natural map and endow I with the quotient topology. We consider sheaves, $(L_\alpha)_{\alpha\in A}$ on Y, constructible with respect to this stratification and which are local systems of finite rank on $Z_\alpha = \{x \in Y;\ (L_\alpha)_x \neq 0\}$. We will realize $\mathrm{D}(Y)\langle L_\alpha\rangle$ as a derived category of dg-modules over a sheaf of dg-algebras, $\mathcal{A}$, on the finite set I (see Proposition 3.7 below). This sheaf $\mathcal{A}$ will be quasi-isomorphic to $R\phi_* R\mathcal{H}om(\oplus L_\alpha, \oplus L_\alpha)$. We make the following hypothesis on the stratification and the L_α.

Assumptions 3.1. Let Y be a complex manifold endowed with a finite μ-stratification, $Y = \bigsqcup_{i\in I} Y_i$, by real analytic submanifolds. We assume that Y is an analytic open subset of an analytic manifold, X, such that $\overline{Y}$ is compact and has a stratification, $\overline{Y} = \bigsqcup_{i\in I} Y_i'$, satisfying: $\forall i \in I$, $Y_i = Y \cap Y_i'$ (note that $\overline{Y}$ has no additional stratum). For $i \in I$, we define, as in section 2.1, U_i to be the smallest open subset of I containing i:

$$U_i = \{j \in I;\ Y_i \subset \overline{Y_j}\}. \tag{6}$$

We consider a finite family of (complex) smooth, connected, normal crossings divisors, $(D_v)_{v\in V}$, on Y. We assume that the divisors are a union of strata: $D_v = \bigsqcup_{i\in I_v} Y_i$, for some $I_v \subset I$. We define:

$$\text{for } \Delta \subset V, \quad Z_\Delta = \textstyle\bigcap_{v\in\Delta} D_v, \qquad \mathcal{S} = \{\Delta \subset V;\ Z_\Delta \neq \emptyset\}. \tag{7}$$

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We also consider a finite family of constructible sheaves, $(L_\alpha)_{\alpha\in A}$ on Y, and set $Z_\alpha = \{x \in Y;\ (L_\alpha)_x \neq 0\}$. We make the following hypothesis on these data:

(i) $\forall i, i' \in I$, $\exists j \in I$, $U_i \cap U_{i'} = U_j$.

(ii) $\forall i \in I$ there exists a homotopy $h : [0,1] \times \phi^{-1}(U_i) \to \phi^{-1}(U_i)$ contracting $\phi^{-1}(U_i)$ to a submanifold Y_i'' of Y_i and preserving the closures of strata: $\forall j \in U_i$, $h([0,1] \times Y_j) \subset \overline{Y_j}$.

(iii) $\forall i \in I$, $\forall v \in V$, $\exists j \in I$, $U_i \setminus \phi(D_v) = U_j$.

(iv) $\forall \alpha \in A$, $\exists \Delta_\alpha \in \mathcal{S}$, $\exists \Delta'_\alpha \subset (V \setminus \Delta_\alpha)$ such that $Z_\alpha = Z_{\Delta_\alpha} \setminus \bigcup_{v\in\Delta'_\alpha} D_v$.

(v) $\forall \alpha \in A$, $L_\alpha|_{Z_\alpha}$ is a local system on Z_α with monodromy $-Id$ around each $Z_{\Delta_\alpha} \cap D_v$, for $v \in \Delta'_\alpha$ (see section 2.4).

Example 3.2. We will verify in section 4 that the decomposition of a symmetric variety given in [2] satisfies these assumptions. A more simple example is given by smooth toric varieties: let $T = (\mathbf{C}^*)^l$ be a torus, $D \subset T$ a subgroup consisting of order 2 elements and $T' = T/D$. Let Y be a smooth T'-toric variety, with the action of T through T'. We let $(Y_i)_{i\in I}$ be the stratification given by the T'-orbits, and let $(D_v)_{v\in V}$ be the set of T'-stable irreducible divisors. For a T'-orbit $\mathcal{O}$ and $x \in \mathcal{O}$, we set $\tau_{\mathcal{O}} = T_x/T_x^0$; we have $\tau_{\mathcal{O}} \simeq (\mathbf{Z}/2\mathbf{Z})^{c_{\mathcal{O}}}$, for some $c_{\mathcal{O}} \in \mathbf{N}$. The irreducible T-equivariant local systems on $\mathcal{O}$ correspond to irreducible representations of $\tau_{\mathcal{O}}$. Let A be the set of pairs $\alpha = (\mathcal{O}, \rho)$, where $\mathcal{O}$ is an orbit and ρ an irreducible representation of $\tau_{\mathcal{O}}$. We let $\Delta_\alpha \in \mathcal{S}$ be such that $\overline{\mathcal{O}} = \bigcap_{v\in\Delta_\alpha} D_v$ and let L'_α be the local system on $\mathcal{O}$ given by ρ. Since $\tau_{\mathcal{O}}$ is a 2-group, the irreducible representations are one-dimensional and the elements of $\tau_{\mathcal{O}}$ act by 1 or -1. In particular, L'_α has monodromy Id or $-Id$ around any divisor $D_v \cap \overline{\mathcal{O}}$ (for v such that $\mathcal{O} \not\subset D_v$ and $\overline{\mathcal{O}} \cap D_v \neq \emptyset$). We let Δ'_α be the set of $v \in V$ for which this monodromy is $-Id$. Then L'_α extend as a local system to $Z_{\Delta_\alpha} \setminus \bigcup_{v\in\Delta'_\alpha} D_v$ and we let L_α be the extension by 0 of this local system. Then the assumptions above are satisfied in this situation.

Remarks 3.3. 1) In fact, we do not use the complex structure; only the geometry of the intersections of the D_v matters. In particular, the strata Y_i are not assumed to be complex.

2) In view of Lemma 2.6, hypothesis (iv) and (v) have the following consequences: let $j_\alpha : Z_\alpha \to Y$ be the inclusion. Then $L_\alpha \simeq R(j_\alpha)_*(L_\alpha|_{Z_\alpha}) \simeq (j_\alpha)_*(L_\alpha|_{Z_\alpha}) \simeq (j_\alpha)_!(L_\alpha|_{Z_\alpha})$ (or, with different notation, $L_\alpha \simeq R\Gamma_{Z_\alpha}(L_\alpha) \simeq (L_\alpha)_{Z_\alpha}$). For $\alpha, \beta \in A$, we will give representatives for the complex $R\mathcal{H}om(L_\alpha, L_\beta)$. We already note that

$$R\mathcal{H}om(L_\alpha, L_\beta) \simeq R\mathcal{H}om((L_\alpha)_{Z_\alpha}, R\Gamma_{Z_\beta}(L_\beta)) \simeq R\mathcal{H}om(L_\alpha, R\Gamma_{Z_\alpha\cap Z_\beta}(L_\beta)).$$

In particular, if $Z_\alpha \cap Z_\beta = \emptyset$, then $R\mathcal{H}om(L_\alpha, L_\beta) = 0$.

3) We note that Y_i is closed in $\phi^{-1}(U_i)$ (because the strata contained in $\overline{Y_i} \setminus Y_i$ cannot be in $\phi^{-1}(U_i)$) so that $U_i \setminus \{i\}$ is open in I. In particular, if $U_i = U_j$, then $i = j$. Hence the j in hypothesis (i) and (iii) are unique.

4) For any $i \in I$ and any closed subset J of U_i, the homotopy h of (ii), also contracts $\phi^{-1}(J)$ to Y_i''. Hence the inclusions $Y_i'' \subset \phi^{-1}(J) \subset \phi^{-1}(U_i)$ are homotopy equivalences (i.e. induce isomorphisms on all homotopy groups). In particular, the inclusion $Y_i \subset \phi^{-1}(U_i)$ is a homotopy equivalence. Hence, for any $\alpha \in A$ and $i \in I$ with $Y_i \subset Z_\alpha$, the local system $L_\alpha|_{Y_i}$ has a unique extension to a local system defined on $\phi^{-1}(U_i)$. We denote this extension by $L_{\alpha,i}$. We have $L_{\alpha,i}|_{\phi^{-1}(U_i)\cap Z_\alpha} \simeq L_\alpha|_{\phi^{-1}(U_i)\cap Z_\alpha}$ and, for j such that $Y_j \subset Z_\alpha$ and $Y_i \subset \overline{Y_j}$, we have an isomorphism $u_{ij} : L_{\alpha,i}|_{\phi^{-1}(U_j)} \simeq L_{\alpha,j}$. Since u_{ij} is determined by its restriction

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to Y_j, the u_{ij}, $(i,j) \in I^2$, satisfy the same relations as their restrictions to Z_α. In particular, they satisfy the cocycle condition which says that the $L_{\alpha,i}$ glue together into a local system, say L^1_α, on $\bigcup_{\{i;Y_i \subset Z_\alpha\}} U_i$.

5) Since the U_i form a basis of the topology of I, a sheaf F on I is determined by its stalks $F_i = F(U_i)$, for all $i \in I$, and the restriction maps, $F_i \to F_j$, for all $i,j \in I$ with $i \in \overline{\{j\}}$. Conversely, the data of groups F_i, for all $i \in I$, and restriction maps, $f_{ji} : F_i \to F_j$, for all $i,j \in I$ with $i \in \overline{\{j\}}$, satisfying $f_{ij} \circ f_{jk} = f_{ik}$ (whenever it makes sense) define a sheaf on I.

Notations 3.4. We introduce the following notations, for $\alpha, \beta \in A$:

$$Z_{\alpha\beta} = Z_\alpha \cap Z_\beta, \qquad d_{\alpha\beta} = \operatorname{codim}^{\mathbf{C}}_{Z_\beta} Z_{\alpha\beta}, \qquad I_{\alpha\beta} = \phi(Z_{\alpha\beta}),$$
$$\Delta'_{\alpha\beta} = (\Delta'_\alpha \setminus \Delta'_\beta) \cup (\Delta'_\beta \setminus \Delta'_\alpha), \qquad I'_{\alpha\beta} = \phi\big(Z_{\Delta_\alpha \cup \Delta_\beta} \setminus (\textstyle\bigcup_{v \in \Delta'_{\alpha\beta}} D_v)\big) \setminus I_{\alpha\beta},$$

and, for $i \in I_{\alpha\beta}$, $L_{\alpha,i}$, $L_{\beta,i}$ as in Remark 3.3 (4), we introduce the following sheaf on $\phi^{-1}(U_i)$: $\Omega_{\alpha\beta,i} = \Omega_{\phi^{-1}(U_i)} \otimes \mathcal{H}om(L_{\alpha,i}, L_{\beta,i})$.

Let us prove the following facts:

(a) If $I_{\alpha\beta} = \emptyset$, then $I'_{\alpha\beta} = \emptyset$; in any case $I'_{\alpha\beta} \subset \overline{I_{\alpha\beta}}$.
(b) $I_{\alpha\beta} \sqcup I'_{\alpha\beta}$ is open in $\overline{I_{\alpha\beta}}$.
(c) $\forall i \in I'_{\alpha\beta}$, $\exists! j \in I_{\alpha\beta}$, such that $U_i \setminus \phi(\bigcup_{v \in \Delta'_\alpha \cap \Delta'_\beta} D_v) = U_j$.

For (a) we note that $I_{\alpha\beta} = \emptyset$ means $(\bigcap_{v \in \Delta_\alpha} D_v \setminus \bigcup_{w \in \Delta'_\alpha} D_w) \cap (\bigcap_{v \in \Delta_\beta} D_v \setminus \bigcup_{w \in \Delta'_\beta} D_w) = \emptyset$, which is equivalent to $\bigcap_{v \in \Delta_\alpha \cup \Delta_\beta} D_v \subset \bigcup_{w \in \Delta'_\alpha \cup \Delta'_\beta} D_w$, or also to $(\Delta_\alpha \cup \Delta_\beta) \cap (\Delta'_\alpha \cup \Delta'_\beta) \neq \emptyset$. Since $\Delta_\alpha \cap \Delta'_\alpha = \emptyset$ and $\Delta_\beta \cap \Delta'_\beta = \emptyset$, this implies that $(\Delta_\alpha \cup \Delta_\beta) \cap \Delta'_{\alpha\beta} \neq \emptyset$, and then $Z_{\Delta_\alpha \cup \Delta_\beta} \setminus (\bigcup_{v \in \Delta'_{\alpha\beta}} D_v) = \emptyset$. In particular, $I'_{\alpha\beta} = \emptyset$ as claimed, and $I'_{\alpha\beta} \subset \overline{I_{\alpha\beta}}$. Now we note that $Z_{\alpha\beta}$ is open in $Z_{\Delta_\alpha \cup \Delta_\beta}$. Arguing locally around each connected component of $Z_{\Delta_\alpha \cup \Delta_\beta}$, we deduce that, if $I_{\alpha\beta} \neq \emptyset$, we also have $\overline{I_{\alpha\beta}} = \phi(Z_{\Delta_\alpha \cup \Delta_\beta}) \supset I'_{\alpha\beta}$.

For (b), we have $\overline{I_{\alpha\beta}} \setminus (I_{\alpha\beta} \sqcup I'_{\alpha\beta}) = \phi(Z_{\Delta_\alpha \cup \Delta_\beta} \cap (\bigcup_{v \in \Delta'_{\alpha\beta}} D_v))$ and this is closed.

For (c), applying hypothesis (iii) of Assumptions 3.1 several times, we know that there exists a unique $j \in I$ such that $U_i \setminus \phi(\bigcup_{v \in \Delta'_\alpha \cap \Delta'_\beta} D_v) = U_j$. We note that $U_i \cap \overline{I_{\alpha\beta}} \neq \emptyset$ and $\overline{I_{\alpha\beta}} \not\subset \phi(\bigcup_{v \in \Delta'_\alpha \cap \Delta'_\beta} D_v)$ (recall that $I_{\alpha\beta} \neq \emptyset$), hence $U_j \cap \overline{I_{\alpha\beta}} \neq \emptyset$. This implies $j \in \overline{I_{\alpha\beta}}$. But $i \notin \overline{I_{\alpha\beta}} \setminus (I_{\alpha\beta} \sqcup I'_{\alpha\beta})$ which is closed, hence $j \notin \overline{I_{\alpha\beta}} \setminus (I_{\alpha\beta} \sqcup I'_{\alpha\beta})$ as well. We thus obtain $j \in I_{\alpha\beta} \sqcup I'_{\alpha\beta}$. Since $j \notin \phi(\bigcup_{v \in \Delta'_\alpha \cap \Delta'_\beta} D_v)$, this implies $j \in I_{\alpha\beta}$.

Definition 3.5. For $i \in I'_{\alpha\beta}$, we denote by $i(\alpha,\beta)$ the element $j \in I_{\alpha\beta}$ given by assertion (c) above. We define a sheaf $\mathcal{A}^{\alpha\beta}$ on I by its stalks at $i \in I$:

$$\mathcal{A}^{\alpha\beta}_i = \begin{cases} \Gamma(\phi^{-1}(U_i); \Omega_{\alpha\beta,i}))\,[-2d_{\alpha\beta}] & \text{if } i \in I_{\alpha\beta}, \\ \mathcal{A}^{\alpha\beta}_j & \text{if } i \in I'_{\alpha\beta} \text{ and } j = i(\alpha,\beta), \\ 0 & \text{if } i \notin I_{\alpha\beta} \sqcup I'_{\alpha\beta}, \end{cases} \tag{8}$$

and the natural restriction maps. Let us check that this is indeed a sheaf. The first case ($i \in I_{\alpha\beta}$) defines a sheaf, say $\mathcal{A}' = \phi_*(\Omega_Y \otimes \mathcal{H}om(L^1_\alpha, L^1_\beta))$ (with L^1_α, L^1_β as in Remark 3.3 (4)), on $I_{\alpha\beta}$. Then the second case defines a sheaf, say $\mathcal{A}''$, on $I_{\alpha\beta} \sqcup I'_{\alpha\beta}$, as $u_*\mathcal{A}''$, where u is the inclusion $u : I_{\alpha\beta} \to I_{\alpha\beta} \sqcup I'_{\alpha\beta}$. Finally, the third case defines $\mathcal{A}^{\alpha\beta}$ as the extension by 0 of $\mathcal{A}''$.

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We also define $\mathcal{A} = \bigoplus_{\alpha,\beta\in A} \mathcal{A}^{\alpha\beta}$.

Remarks 3.6. 1) The stalks $\mathcal{A}_i^{\alpha\beta}$ are defined to be 0 when the stratum Y_i is included in a divisor D_w such that the local system $\mathcal{H}om(L_\alpha, L_\beta)$ (on $Z_{\alpha\beta}$) has monodromy $-Id$ around D_w. This definition is justified in view of Lemma 2.6, (ii).

2) For $\mathcal{A}_i^{\alpha\beta} \neq 0$ (i.e., $i \in I_{\alpha\beta} \sqcup I'_{\alpha\beta}$) and $Y_i \subset D_v$, we have $v \in \Delta'_\alpha \iff v \in \Delta'_\beta$. Hence, setting $V_i = \phi^{-1}(U_i)$:

$$\text{(9)} \qquad \text{for } \mathcal{A}_i^{\alpha\beta} \neq 0, \qquad V_i \cap \bigcup_{v\in\Delta'_\alpha} D_v = V_i \cap \bigcup_{v\in\Delta'_\beta} D_v = V_i \cap \bigcup_{v\in\Delta'_\alpha\cap\Delta'_\beta} D_v.$$

We will introduce an algebra structure on $\mathcal{A}$. For $v \in V$, D_v has a fundamental class, $\delta_v \in H^2_{D_v}(Y; \mathbf{C}_Y)$. We choose representatives, $\xi_v \in \Gamma(Y; \Omega^2_Y)$, of the δ_v. For $\Delta, \Delta', \Delta'' \in \mathcal{S}$, we set

$$\nabla(\Delta, \Delta', \Delta'') = \big(\Delta' \setminus (\Delta \cup \Delta'')\big) \cup \big((\Delta \cap \Delta'') \setminus \Delta'\big) \tag{10}$$

and for $\alpha, \beta, \gamma \in A$, $\eta_{\alpha\beta\gamma} = \prod_{v\in\nabla} \xi_v$, where $\nabla = \nabla(\Delta_\alpha, \Delta_\beta, \Delta_\gamma)$. For $\alpha, \beta, \gamma \in A$, we define a morphism $m^{\alpha\beta\gamma} : \mathcal{A}^{\beta\gamma} \otimes \mathcal{A}^{\alpha\beta} \to \mathcal{A}^{\alpha\gamma}$ as follows. For $i \in \phi(Z_\alpha \cap Z_\beta \cap Z_\gamma)$, we define a sheaf morphism

$$\begin{aligned} n^i_{\alpha\beta\gamma} : \Omega_{\beta\gamma,i} \otimes \Omega_{\alpha\beta,i} &\to \Omega_{\alpha\gamma,i} \\ (\tau \otimes v) \otimes (\sigma \otimes u) &\mapsto (\eta_{\alpha\beta\gamma}\, \tau\, \sigma) \otimes (v \circ u), \end{aligned}$$

where σ, τ are sections of $\Omega_{\phi^{-1}(U_i)}$ and u, v sections of $\mathcal{H}om$ sheaves. We set $m_i^{\alpha\beta\gamma} = \Gamma(\phi^{-1}(U_i); n^i_{\alpha\beta\gamma})$. This definition extends to other $i \in I$, either by restriction to the case $i \in \phi(Z_\alpha \cap Z_\beta \cap Z_\gamma)$ or, trivially, when one of the terms is 0. Indeed, if $i \in I \setminus \phi(Z_\alpha \cap Z_\beta \cap Z_\gamma)$ satisfies $\mathcal{A}_i^{\beta\gamma} \neq 0$, $\mathcal{A}_i^{\alpha\beta} \neq 0$ and $\mathcal{A}_i^{\alpha\gamma} \neq 0$, then we have, by (9),

$$U_i \setminus \phi(\textstyle\bigcup_{v\in\Delta'_\beta\cap\Delta'_\gamma} D_v) = U_i \setminus \phi(\bigcup_{v\in\Delta'_\alpha\cap\Delta'_\beta} D_v) = U_i \setminus \phi(\bigcup_{v\in\Delta'_\alpha\cap\Delta'_\gamma} D_v).$$

It follows that, for the same $j \in \phi(Z_\alpha \cap Z_\beta \cap Z_\gamma)$, we have $\mathcal{A}_i^{\beta\gamma} = \mathcal{A}_j^{\beta\gamma}$, $\mathcal{A}_i^{\alpha\beta} = \mathcal{A}_j^{\alpha\beta}$, $\mathcal{A}_i^{\alpha\gamma} = \mathcal{A}_j^{\alpha\gamma}$. This allows one to define $m_i^{\alpha\beta\gamma}$. By definition, these morphisms $m_i^{\alpha\beta\gamma}$ commute with the restriction maps and we obtain a sheaves morphism $m^{\alpha\beta\gamma} : \mathcal{A}^{\beta\gamma} \otimes \mathcal{A}^{\alpha\beta} \to \mathcal{A}^{\alpha\gamma}$, as claimed. (The justification for the definition of this product is given in section 3.3.2.)

Now we define a product m on $\mathcal{A} = \bigoplus_{\alpha,\beta\in A} \mathcal{A}^{\alpha\beta}$ by $m = \bigoplus m^{\alpha\beta\gamma}$. One checks that m is an associative product using the straightforward identity:

$$\eta_{\alpha\beta\gamma}\, \eta_{\alpha\gamma\delta} = \eta_{\beta\gamma\delta}\, \eta_{\alpha\beta\delta}. \tag{11}$$

Hence $\mathcal{A}$ is a sheaf of dg-algebras on I. For $\alpha \in A$, $N_\alpha = \bigoplus_{\alpha'\in A} \mathcal{A}^{\alpha'\alpha}$ has a natural structure of $\mathcal{A}$-module defined in the same way as the product of $\mathcal{A}$. The result of this section is the following equivalence of categories.

Proposition 3.7. *Let $Y = \bigsqcup_{i\in I} Y_i$ be a stratified complex analytic manifold, endowed with normal crossings divisors D_v, $v \in V$, and sheaves L_α, $\alpha \in A$, satisfying Assumptions* 3.1. *For a choice of forms $\xi_v \in \Gamma(Y; \Omega^2_Y)$, we define a sheaf of dg-algebras $\mathcal{A}$ on I, and $\mathcal{A}$-modules N_α as above.*

Then, there exists a choice of ξ_v such that we have an equivalence of categories between $\mathrm{D}(Y)\langle L_\alpha\rangle$ *and* $\mathrm{D}_{\mathcal{A}}\langle N_\alpha\rangle$ *sending L_α to N_α.*

The proof is given at the end of this section.

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Remarks 3.8. 1) In fact, one could prove that two choices of representatives ξ_v, ξ'_v of the δ_v give quasi-isomorphic sheaves of dg-algebras, $\mathcal{A}$, $\mathcal{A}'$: with ζ_v such that $\xi_v - \xi'_v = d\zeta_v$, and replacing Y by $Y \times \mathbf{C}$, endowed with $\xi^+_v = \xi_v + d(t\zeta_v)$ (t is the coordinate on $\mathbf{C}$) and the data $Y_i \times \mathbf{C}$, $\mathrm{D}_v \times \mathbf{C}$, $L_\alpha \boxtimes \mathbf{C}_{\mathbf{C}}$, we could build a third sheaf $\mathcal{A}^+$ on I, quasi-isomorphic to $\mathcal{A}$ and $\mathcal{A}'$. Hence the conclusion of the proposition is valid for any choice of ξ_v, but we will not use this result.

2) The results of this section will be applied to $Y = E \times_K X$, where X is a symmetric variety under the action of a semi-simple complex algebraic group G, K a suitable maximal compact subgroup of G and E a universal bundle for K. Of course, E is not a manifold, but we may assume that it is an increasing union of K-manifolds, $E = \bigcup_k E_k$, and consider the de Rham algebra of Y, $\Omega_Y = \varprojlim_k \Omega_{E_k \times_K X}$. The stratification $Y = \bigsqcup_i Y_i$ and the divisors D_v will be given by a K-invariant stratification of X and K-invariant divisors. All constructions in this section can be made K-invariant (if we choose K-invariant functions f_i in Notations 3.9 below, by averaging under the action of K) and transposed to $Y = E \times_K X$.

3.1. **System of tubes.** Our first task in the proof of Proposition 3.7 is to "compute" the global sections $\Gamma_{\alpha\beta i} = R\Gamma(\phi^{-1}(U_i); R\mathcal{H}om(L_\alpha, L_\beta))$, for $i \in I$, $\alpha, \beta \in A$ (to compute just means to find suitable representatives). For this we replace the strata Y_i by a system of "tubes", T_i (with T_i closed enough to Y_i so that the local systems $L_\alpha|_{Y_i}$ extend to T_i) with the properties:

(i) replacing L_α by its extension, say L'_α, to the union of tubes $\bigcup_{\{i; Y_i \subset Z_\alpha\}} T_i$ doesn't change the global sections $\Gamma_{\alpha\beta i}$,
(ii) the complexes $R\mathcal{H}om(L'_\alpha, L'_\beta)$ are in fact sheaves.

The precise statement is given in Proposition 3.11 below. The first property implies that the category $\mathrm{D}(Y)\langle L_\alpha\rangle$ is equivalent to the $\mathrm{D}(Y)\langle L'_\alpha\rangle$ (see Corollary 3.13 below). The second property will be used to define a sheaf of dg-algebras, $\mathcal{B}$, on I, such that $\mathrm{D}(Y)\langle L'_\alpha\rangle$ is equivalent to a subcategory of $\mathrm{D}_{\mathcal{B}}$ (Definition 3.18 and Proposition 3.19). The proof of Proposition 3.7 will then be achieved by showing that $\mathcal{B}$ and $\mathcal{A}$ are quasi-isomorphic.

Notations 3.9. First we assume that the finite set indexing the stratification is $I = \{1, \ldots, n\}$, ordered such that $\dim Y_i \leq \dim Y_{i+1}$, for $i = 1, \ldots, n-1$. Recall that Y is open in an analytic manifold X and $\overline{Y}$ is compact, with a stratification $\overline{Y} = \bigsqcup_{i \in I} Y'_i$, satisfying: $\forall i \in I$, $Y_i = Y \cap Y'_i$. For $i = 1, \ldots, n-1$, we choose a neighbourhood of Y_i, $\tilde{Y}_i$, whose closure is a neighbourhood of Y'_i in $\overline{Y}$. We also choose real analytic functions $f_i : \tilde{Y}_i \to \mathbf{R}$, such that $f_i(\tilde{Y}_i) \subset \mathbf{R}_{\geq 0}$ and $Y_i = f_i^{-1}(0) \cap \tilde{Y}_i$. For $k < n$ and $\varepsilon_1, \ldots, \varepsilon_k > 0$, we define

$$T_1(\varepsilon_1) = \{y \in \tilde{Y}_1;\ f_1(y) \leq \varepsilon_1\}, \ldots, T_k(\varepsilon_1, \ldots, \varepsilon_k) = \{y \in \tilde{Y}_k;\ f_k(y) \leq \varepsilon_k\} \setminus \bigsqcup_{i<k} T_i.$$

By abuse of notation we will write $T_i(\underline{\varepsilon}) = T_i(\varepsilon_1, \ldots, \varepsilon_i)$ for any $\underline{\varepsilon}$ of length greater than i. We also set $T_n(\underline{\varepsilon}) = Y \setminus \bigsqcup_{i<n} T_i(\underline{\varepsilon})$. For a union of strata, Z, we set:

$$T_Z(\underline{\varepsilon}) = \textstyle\bigsqcup_{i \in J} T_i(\underline{\varepsilon}), \quad \text{where } J \text{ satisfies} \quad Z = \bigsqcup_{i \in J} Y_i. \tag{12}$$

Definition 3.10. We call "set of bounds" a subset $B \in]0, +\infty[^k$ such that $\exists \varepsilon_1^0 > 0$, $\forall \varepsilon_1 < \varepsilon_1^0$, $\exists \varepsilon_2^0 > 0$, $\forall \varepsilon_2 < \varepsilon_2^0$, $\ldots$, $\exists \varepsilon_k^0 > 0$, $\forall \varepsilon_k < \varepsilon_k^0$, $(\varepsilon_1, \ldots, \varepsilon_k) \in B$.

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We note that a set of bounds is non-empty and that the intersection of two sets of bounds is a set of bounds. The aim of this paragraph is to prove the following result.

Proposition 3.11. *Let $Y = \bigsqcup_{i=1,\dots,n} Y_i$ be a stratified analytic manifold as above. Let $Z_1, Z_2 \subset Y$ be locally closed subsets of Y which are unions of strata. Let $\mathcal{L}^1$, $\mathcal{L}^2$ be local systems respectively defined on neighbourhoods of Z_1 and Z_2. Then there exists a set of bounds $B \subset]0,+\infty[^{n-1}$ such that $\forall \underline{\varepsilon} \in B$, setting $T_Z = T_Z(\underline{\varepsilon})$, $\mathcal{L}^i$ is defined on T_{Z_i} and we have:*

(i) There exist natural morphisms $\mathcal{L}^i_{T_{Z_i}} \to \mathcal{L}^i_{Z_i}$ and they induce isomorphisms

$$\mathrm{RHom}(\mathcal{L}^1_{T_{Z_1}}, \mathcal{L}^2_{T_{Z_2}}) \xrightarrow{\sim} \mathrm{RHom}(\mathcal{L}^1_{T_{Z_1}}, \mathcal{L}^2_{Z_2}) \xleftarrow{\sim} \mathrm{RHom}(\mathcal{L}^1_{Z_1}, \mathcal{L}^2_{Z_2}). \tag{13}$$

(ii) For any locally closed union of strata $Z \subset Y$ such that Z_1, $Z_2 \subset Y \setminus \overline{Z}$, we have an isomorphism

$$R\Gamma(Y; (R\mathcal{H}om(\mathcal{L}^1_{T_{Z_1}}, \mathcal{L}^2_{T_{Z_2}}))_{T_Z}) \simeq R\Gamma(Y; (R\mathcal{H}om(\mathcal{L}^1_{Z_1}, \mathcal{L}^2_{Z_2}))_Z). \tag{14}$$

(iii) If $Z_1 \subset Z_2$, then $R\mathcal{H}om(\mathcal{L}^1_{T_{Z_1}}, \mathcal{L}^2_{T_{Z_2}})$ is concentrated in degree 0.

(iv) Let us assume that Z_2 and $Z_1 \cap Z_2$ are smooth and let $\mu : Z_2 \hookrightarrow \overline{Z_2}$, $\nu : Z_1 \cap Z_2 \hookrightarrow \overline{Z_1 \cap Z_2}$ be the inclusions. We assume that $R\mu_*(\mathcal{L}^2|_{Z_2}) = \mu_!(\mathcal{L}^2|_{Z_2})$ and $R\nu_*(\mathcal{L}^2|_{Z_1\cap Z_2}) = \nu_!(\mathcal{L}^2|_{Z_1\cap Z_2})$. Then, for any open union of strata $V \subset Y$,

$$\mathrm{RHom}(\mathcal{L}^1_{Z_1}|_V, \mathcal{L}^2_{Z_2}|_V) \simeq R\Gamma(T_V; \mathcal{H}om(\mathcal{L}^1_{T_{Z_1}}, \mathcal{L}^2_{T_{Z_2}})). \tag{15}$$

The proof will be given at the end of the paragraph. We first deduce the following corollary, which gives a category equivalent to $\mathrm{D}(Y)\langle L_\alpha\rangle$.

We consider Y and L_α, $\alpha \in A$, as in Assumptions 3.1. We choose neighbourhoods of the Z_α on which the local systems L_α may be extended to local systems L^+_α. For $\alpha, \beta \in A$, we set $Z_1 = Z_\alpha$, $\mathcal{L}^1 = L^+_\alpha$, $Z_2 = Z_\beta$, $\mathcal{L}^2 = L^+_\beta$. Then, Z_2 and $Z_1 \cap Z_2$ are open subsets of intersections of some D_v, hence smooth. Moreover, by Assumptions 3.1 (v), $\mathcal{L}^2$ has monodromy $-Id$ around each irreducible divisor of $\overline{Z_2}\setminus Z_2$ and the similar property holds a fortiori for $\mathcal{L}^2|_{Z_1\cap Z_2}$. Hence, by Lemma 2.6, the hypothesis of Proposition 3.11, (iv), are verified.

Notations 3.12. We choose a set of bounds, B, such that the conclusions of Proposition 3.11 hold for $Z_1 = Z_\alpha$, $\mathcal{L}^1 = L^+_\alpha$, $Z_2 = Z_\beta$, $\mathcal{L}^2 = L^+_\beta$, for any pair $(\alpha,\beta) \in A^2$. We fix $\underline{\varepsilon} \in B$ and set:

$$T_i = T_i(\underline{\varepsilon}), \qquad T_\alpha = T_{Z_\alpha}(\underline{\varepsilon}), \qquad L'_\alpha = (L^+_\alpha)_{T_\alpha}.$$

Corollary 3.13. *The categories $\mathrm{D}(Y)\langle L_\alpha\rangle$ and $\mathrm{D}(Y)\langle L'_\alpha\rangle$ are equivalent.*

Proof. This is a consequence of the natural isomorphisms (13). By definition the category $\mathrm{D}(Y)\langle L_\alpha\rangle$ is the union of the full subcategories D_n of $\mathrm{D}(Y)$, for $n \in \mathbf{N}$, where D_0 consists of the $L_\alpha[k]$, $\alpha \in A$, $k \in \mathbf{Z}$, and D_{n+1} is obtained from D_n by adding objects H appearing in distinguished triangles $F \to G \to H \xrightarrow{+1}$, with $F, G \in \mathrm{D}_n$. We write in the same way $\mathrm{D}(Y)\langle L'_\alpha\rangle = \bigcup \mathrm{D}'_n$. We assume by induction that we have an equivalence, δ_n, between D_n and D'_n, together with functorial morphisms $r_n(F) : \delta_n(F) \to F$ such that δ_n is given on the morphisms by composing isomorphisms

$$\mathrm{Hom}_{\mathrm{D}_n}(F, F') \xrightarrow{\sim} \mathrm{Hom}_{\mathrm{D}(Y)}(\delta_n(F), F') \xleftarrow{\sim} \mathrm{Hom}_{\mathrm{D}'_n}(\delta_n(F), \delta_n(F'))$$

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induced by $r_n(F)$, $r_n(F')$. (The first step is given by (13).) Let $F \xrightarrow{u} G \to H \xrightarrow{+1}$ be a distinguished triangle as above, $F' = \delta_n(F)$, $G' = \delta_n(G)$ $u' = \delta_n(u)$ and consider a distinguished triangle $F' \xrightarrow{u'} G' \to H' \xrightarrow{+1}$. We extend the square built on u, u', $r_n(F)$ and $r_n(G)$ to a morphism of triangles:

$$\begin{array}{ccccccc} F' & \longrightarrow & G' & \longrightarrow & H' & \xrightarrow{+1} & \\ \downarrow & & \downarrow & & \downarrow{\scriptstyle r} & & \\ F & \longrightarrow & G & \longrightarrow & H & \xrightarrow{+1} & . \end{array}$$

We set $H' = \delta_{n+1}(H)$, $r_{n+1}(H) = r$ and we have to define the images of the morphisms. First, for $X \in \mathrm{D}_n$, we have long exact sequences of homomorphisms groups:

$$\begin{array}{ccccccc} \mathrm{Hom}(X,F) & \longrightarrow & \mathrm{Hom}(X,G) & \longrightarrow & \mathrm{Hom}(X,H) & \longrightarrow & \\ \downarrow & & \downarrow & & \downarrow & & \\ \mathrm{Hom}(\delta_n(X),F) & \longrightarrow & \mathrm{Hom}(\delta_n(X),G) & \longrightarrow & \mathrm{Hom}(\delta_n(X),H) & \longrightarrow & \\ \uparrow & & \uparrow & & \uparrow & & \\ \mathrm{Hom}(\delta_n(X),\delta_n(F)) & \longrightarrow & \mathrm{Hom}(\delta_n(X),\delta_n(G)) & \longrightarrow & \mathrm{Hom}(\delta_n(X),\delta_n(H)) & \longrightarrow & . \end{array}$$

By the five lemma it gives an isomorphism $\mathrm{Hom}(X,H) \simeq \mathrm{Hom}(\delta_n(X),\delta_n(H))$, which we use to define δ_n on $\mathrm{Hom}(X,H)$. In the same way, we may define δ_n on $\mathrm{Hom}(H,X)$, still for $X \in \mathrm{D}_n$. Then we may assume $X \in \mathrm{D}_{n+1}$ in the above diagram, and this defines δ_n on $\mathrm{Hom}(X,H)$ for $X, H \in \mathrm{D}_{n+1}$, satisfying the compatibility with r_{n+1}. $\square$

Now we give some preliminary results before we prove Proposition 3.11.

Lemma 3.14. *Let $Y = \bigsqcup_{i=1,\dots,n} Y_i$, f_i, be as in Assumptions* 3.1 *and Notations* 3.9. *There exists a set of bounds $B \subset]0,+\infty[^{n-1}$ such that $\forall \underline{\varepsilon} \in B$ and for any union of strata $Z \subset Y$:*

(i) *if Z is closed, then so is $T_Z(\underline{\varepsilon})$, and $Z \subset T_Z(\underline{\varepsilon})$,*
(ii) *if Z is open, then so is $T_Z(\underline{\varepsilon})$,*
(iii) *$\forall i_1 < \dots < i_p < n$, and $y \in Y$ such that $f_{i_l}(y) = \varepsilon_{i_l}$, we have $df_{i_1} \wedge \ldots \wedge df_{i_p}(y) \neq 0$. In particular, locally around any point $y \in Y$, the partition $Y = \bigsqcup T_i(\underline{\varepsilon})$ is homeomorphic to $\mathbf{R}^d = \{x_1 \leq 0\} \sqcup \{x_1 > 0,\ x_2 \leq 0\} \sqcup \{x_1 > 0, \dots, x_{q-1} > 0,\ x_q \leq 0\} \sqcup \{x_1 > 0, \dots, x_q > 0\}$, for some q.*

Proof. Of course (ii) follows from (i) because $T_{Y\setminus Z}(\underline{\varepsilon}) = Y \setminus T_Z(\underline{\varepsilon})$. We prove (i) by induction on n, the case $n = 1$ or 2 being obvious. Recall that Y is open in a manifold X. For i such that $Y_1 \cap \overline{Y_i} = \emptyset$, we also have $\overline{Y_1}^X \cap \overline{Y_i}^X = \emptyset$, because otherwise the stratification of $\overline{Y}^X$ would have additional strata. Since $\overline{Y}^X$ is compact, we deduce $d(Y_1, Y_i) > 0$. Hence we may choose $r > 0$ smaller than $\min\{d(Y_1,Y_i);\ Y_1 \cap \overline{Y_i} = \emptyset\}$ and $\sup\{d(Y_1,y);\ y \in Y_j\}$, for all j. Then, for ε_1^0 such that $T_1(\varepsilon_1^0) \subset \{y \in Y; d(Y_1,y) < r\}$ and for $0 < \varepsilon_1 < \varepsilon_1^0$, we have $T_1(\varepsilon_1) \cap \overline{Y_i} \neq \emptyset$ if and only if $Y_1 \subset \overline{Y_i}$ and, moreover, $\forall j \neq 1$, $Y_j \not\subset T_1(\varepsilon_1)$.

The induction hypothesis applied to $Y' = Y \setminus T_1(\varepsilon_1)$ stratified by the $Y_i' = Y' \cap Y_i$ gives a set of bounds $B'(\varepsilon_1) \subset]0,+\infty[^{n-2}$ for which (i) holds in Y'. For i such that $Y_1 \cap \overline{Y_i}^Y = \emptyset$ we may choose ε_i small enough so that $T_1(\varepsilon_1) \cap \{y \in \tilde{Y}_i; f_i(y) \leq \varepsilon_i\} = \emptyset$. In particular, restricting to a smaller set of bounds $B''(\varepsilon_1)$, we may assume that $T_1(\varepsilon_1) \cap \overline{T_i(\underline{\varepsilon})}^Y = \emptyset$. Let $Z \subset Y$ be closed.

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If $Y_1 \subset Z$, then $T_Z = T_1(\varepsilon_1) \sqcup T_{Z\cap Y'}$. Since $T_{Z\cap Y'}$ is closed in Y', T_Z is closed in $Y = T_1(\varepsilon_1) \sqcup Y'$. By induction we also have $Z \cap Y' \subset T_Z \cap Y'$ and this implies $Z \subset T_Z$.

If $Y_1 \not\subset Z$, then Z only contains strata Y_i such that $Y_1 \cap \overline{Y_i} = \emptyset$, so that $T_1(\varepsilon_1) \cap \overline{Y_i} = \emptyset$. It follows that $Z \subset Y'$ (and Z is closed in Y'). This also gives $T_1(\varepsilon_1) \cap \overline{(T_Z)}^Y = \emptyset$ and, since T_Z is closed in Y', it is closed in Y too. Finally, $Z \subset T_Z$ since this is already true in Y'. In conclusion, the set of bounds $B = \{(\varepsilon_1, \dots, \varepsilon_{n-1}); 0 < \varepsilon_1 < \varepsilon_1^0,\ (\varepsilon_2, \dots, \varepsilon_{n-1}) \in B''(\varepsilon_1)\}$ has the required property.

Now we prove by induction on p that there exists a set of bounds B_p such that the conclusion of (iii) holds for any $i_1 < \dots < i_p < n$. For $p = 1$ this is a consequence of the curve selection lemma: by contradiction, if the closure of $\{y \in Y_{i_1}; df_{i_1}(y) = 0\}$ intersects $\overline{Y_1}$, then there exists a real analytic curve $\gamma :]-1,1[\to \overline{Y}$ such that $\gamma(t) \in Y \setminus Y_1$ for $t \neq 0$ and $\gamma(0) \in \overline{Y_1}$. But this implies $df_{i_1}(\gamma(t)) = 0$ so that $f_{i_1}(\gamma(t))$ is constant, which is impossible. Hence there exists $\varepsilon_{i_1}^0 > 0$ such that $0 < f_{i_1}(y) < \varepsilon_{i_1}^0$ implies $df_{i_1}(y) \neq 0$. We take $B_1 = \prod_i]0, \varepsilon_i^0[$.

Assuming (iii) holds for p, we consider, for $\underline{\varepsilon} \in B_p$ and $i_1 < \dots < i_p$, the smooth subvariety of Y, $Y' = \{y \in Y;\ f_{i_l}(y) = \varepsilon_{i_l},\ l = 1, \dots, p\}$. For $i_{p+1} > i_p$, the function $f_{i_{p+1}}$ is not constant on Y' and the proof of the first step gives $\varepsilon^0_{i_1,\dots,i_p}(\varepsilon_{i_1}, \dots, \varepsilon_{i_p}) > 0$ such that the conclusion holds for $(\varepsilon_{i_1}, \dots, \varepsilon_{i_{p+1}})$ with $\varepsilon_{i_{p+1}} < \varepsilon^0_{i_1,\dots,i_p}(\varepsilon_{i_1}, \dots, \varepsilon_{i_p})$. We set $B_p^p = B_p$ and, for $k = p+1, \dots, n$, $B_p^k = \{\underline{\varepsilon} \in B_p^{k-1};\ \forall i_1 < \dots < i_p < k,\ \varepsilon_k < \varepsilon^0_{i_1,\dots,i_p}(\varepsilon_{i_1}, \dots, \varepsilon_{i_p})\}$. Then $B_{p+1} = B_p^n$ is a set of bounds with the required property for step $p+1$ and we take $B = B_{n-1}$.

Now, for $\varepsilon \in B$ and $y \in Y$, we let $i_1 < \dots < i_q$ be the indices such that $f_{i_l}(y) = \varepsilon_{i_l}$. Since $df_{i_1} \wedge \ldots \wedge df_{i_q}(y) \neq 0$ the functions $x_l = f_{i_l} - \varepsilon_{i_l}$, $l = 1, \dots, q$, may be extended to a coordinates system around y and in any such system the description of the partition is the one given in the lemma. ☐

Lemma 3.15. *Let $Y = \bigsqcup_{i=1,\dots,n} Y_i$, f_i, be as in Assumptions* 3.1 *and Notations* 3.9. *Let $F_1, \dots, F_m \in \mathrm{D}^b_{\mathbf{R}-c}(Y)$ (i.e., F_j is constructible for some stratification of Y, not a priori $(Y_i)_{i\in I}$). Then there exists a set of bounds $B \subset]0, +\infty[^{n-1}$ such that $\forall \underline{\varepsilon} \in B$, setting for short $T_i = T_i(\underline{\varepsilon})$, we have isomorphisms $\forall i = 1, \dots, n$, $\forall j = 1, \dots, m$:*

$$H_{Y_i}(Y; F_j) \xrightarrow{\sim} H_{T_i}(Y; F_j), \qquad H(Y; (F_j)_{T_i}) \xrightarrow{\sim} H(Y; (F_j)_{Y_i}).$$

Proof. We prove by induction on k that there exists a set of bounds $B \subset]0, +\infty[^k$ such that the first isomorphism holds for any j and for $i = 1, \dots, k$. For $k = 1$, this is a consequence of Lemma 2.5.

Let us assume the result is true for k, and apply it to F_j and $F'_j = R\Gamma_{Y_{k+1}} F_j$, $j = 1, \dots, m$. Let $B \subset]0, +\infty[^k$ be the set of bounds obtained and $(\varepsilon_1, \dots, \varepsilon_k) \in B$, T_i, $i = 1, \dots, k$ as in the lemma. Let us set $T = T_1 \sqcup \ldots \sqcup T_k$ and $U = Y \setminus T$. By Lemma 2.5 again, applied to $f = f_{k+1}$ and the complexes $R\Gamma_U(F_j)$, there exists $\varepsilon^0_{k+1} > 0$ such that $\forall 0 < \varepsilon_{k+1} < \varepsilon^0_{k+1}$ we have, setting $T_{k+1} = f^{-1}_{k+1}([0, \varepsilon_{k+1}]) \cap \tilde{Y}_{k+1} \cap U$, $H_{Y_{k+1}}(U; F_j) \xrightarrow{\sim} H_{T_{k+1}}(U; F_j)$. Since $T_{k+1} \subset U$, we have $H_{T_{k+1}}(U; F_j) \simeq H_{T_{k+1}}(Y; F_j)$ and we have to prove that $H_{Y_{k+1}}(U; F_j) \simeq H_{Y_{k+1}}(Y; F_j)$. Using an excision exact sequence, we are reduced to proving the vanishing of $A_j = H_{Y_{k+1}\cap T}(Y; F_j)$. Now $A_j \simeq H_T(Y; F'_j)$ and, for $i \leq k$, we have

$$H_{Y_i}(Y; F'_j) \xrightarrow{\sim} H_{T_i}(Y; F'_j) \quad \text{and} \quad H_{Y_i}(Y; F'_j) \simeq H_{Y_i \cap Y_{k+1}}(Y; F_j) = 0.$$

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Let us set $T'_i = T_1 \sqcup \ldots \sqcup T_i$; we have distinguished triangles $R\Gamma_{T'_{i-1}}(\cdot) \to R\Gamma_{T'_i}(\cdot) \to R\Gamma_{T_i}(\cdot) \xrightarrow{+1}$. We deduce by induction on $i \le k$ that $H_{T'_i}(Y; F'_j) = 0$. For $i = k$ we obtain $A_j = 0$ as desired and this concludes the proof of the first isomorphism.

The proof of the second isomorphism is the same, again using Lemma 2.5 (we just note that we apply the induction hypothesis to F_j and $F''_j = (F_j)_{Y_{k+1}}$, and Lemma 2.5 to $(F_j)_U$). □

We still consider Y satisfying Assumptions 3.1 and we keep Notations 3.9. We consider, moreover, local systems L^i defined on neighbourhoods of Y_i. The sheaves $L^i_{Y_i}$ are well-defined and, for $\underline{\varepsilon} \in]0, +\infty[^{n-1}$ small enough, the $L^i_{T_i(\underline{\varepsilon})}$ are also well-defined.

Lemma 3.16. *Let $Y = \bigsqcup_{i=1,\ldots,n} Y_i$, L^i, be as above. Let $U \subset Y$ be an analytic open subset, U' a neighbourhood of $\overline{U}$ and $f : U' \to \mathbf{R}$ an analytic function with 1 as regular value. We assume that the smooth hypersurface $S = \{y \in U;\ f(y) = 1\}$ meets the strata $U \cap Y_i$ transversally. Let L be a sheaf on Y which is a local system in a neighbourhood of S. We set $U_+ = \{y;\ f(y) > 1\}$. Then there exists a set of bounds $B \subset]0, +\infty[^{n-1}$ such that $\forall \underline{\varepsilon} \in B$, setting $T'_i = U \cap T_i(\underline{\varepsilon})$, we have an isomorphism $R\mathcal{H}om(L_S, L^i_{T'_i}|_U) \simeq (L^* \otimes L^i|_U)_{S \cap T'_i}[-1]$ and the morphisms*

$$\mathrm{RHom}(L_S, L^i_{T'_i \cap U_+}) \to \mathrm{RHom}(L_S, L^i_{T'_i}) \to \mathrm{RHom}(L_S, L^i_{Y_i})$$

are isomorphisms.

Proof. We have $R\mathcal{H}om(L_S, \mathbf{C}_U) \simeq L^*_S[-1]$ because S is a smooth, relatively oriented, hypersurface. The micro-support of L_S is $SS(L_S) = T^*_S U$ and we also have the bound $SS(L^i_{Y_i}) \subset \bigcup_i T^*_{Y_i} U$. By the transversality hypothesis, we have $SS(\mathbf{C}_S) \cap SS(L^i_{Y_i}) \subset T^*_U U$. Hence, by Lemma 2.4, we have isomorphisms

$$R\mathcal{H}om(L_S, L^i_{Y_i}|_U) \simeq R\mathcal{H}om(L_S, \mathbf{C}_U) \otimes L^i_{Y_i}|_U \simeq (L^* \otimes L^i|_U)_{S \cap Y_i}[-1].$$

For ε_i small enough, $\partial T'_i$ also is transversal to S and, since $SS(L^i_{T'_i})$ is the outer conormal of $\partial T'_i$ in U, we obtain similarly $R\mathcal{H}om(L_S, L^i_{T'_i}|_U) \simeq (L^* \otimes L^i|_U)_{S \cap T'_i}[-1]$.

Hence, to show the last isomorphism, it is sufficient to find a set of bounds such that the morphisms $H(S; (L^* \otimes L^i|_U)_{S \cap T'_i}) \to H(S; (L^* \otimes L^i|_U)_{S \cap Y_i})$ are isomorphisms. But this follows from Lemma 3.15 applied to the stratification $S = \bigsqcup_i S \cap Y_i$.

Let us prove that the remaining morphism also is an isomorphism. Let us set $U_- = f^{-1}(]-\infty, 1[)$. We have $R\mathcal{H}om(\mathbf{C}_{U_-}, \mathbf{C}_U) \simeq \mathbf{C}_{\overline{U}_-}$ and $SS(\mathbf{C}_{U_-})$ is the inner conormal to U_-. We deduce as above $R\mathcal{H}om(\mathbf{C}_{U_-}, L^i_{T'_i}) \simeq L^i_{T'_i \cap \overline{U}_-}$ and

$$R\mathcal{H}om(L_S, L^i_{T'_i \cap \overline{U}_-}) \simeq R\mathcal{H}om(L_S, R\mathcal{H}om(\mathbf{C}_{U_-}, L^i_{T'_i})) \simeq R\mathcal{H}om(L_{S \cap U_-}, L^i_{T'_i}) = 0.$$

Using the distinguished triangle $L^i_{T'_i \cap U_+} \to L^i_{T'_i} \to L^i_{T'_i \cap \overline{U}_-} \xrightarrow{+1}$, we conclude that $\mathrm{RHom}(L_S, L^i_{T'_i \cap U_+}) \simeq \mathrm{RHom}(L_S, L^i_{T'_i})$ as desired. □

Proposition 3.17. *Let $Y = \bigsqcup_{i=1,\ldots,n} Y_i$, L^i, be as in Lemma 3.16. For $\underline{\varepsilon} \in]0, +\infty[^{n-1}$ and a union of strata $Z \subset Y$, we set:*

$$T_i = T_i(\underline{\varepsilon}), \quad T_Z = T_Z(\underline{\varepsilon}), \qquad \text{for } i < j: \quad T_{ij} = (\overline{T_i} \cap \overline{T_j}) \setminus (\textstyle\bigcup_{i<l<j} \overline{T_l}).$$

There exists a set of bounds $B \subset]0, +\infty[^{n-1}$ such that $\forall \underline{\varepsilon} \in B$:

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(i) for any $i < j$, the morphism $\alpha_{ij} : (L^{i*} \otimes L^j)_{T_{ij}}[-1] \to R\mathcal{H}om(L^i_{T_i}, L^j_{T_j})$ is an isomorphism,

(ii) for any i, j, the natural morphisms

$$\mathrm{RHom}(L^i_{T_i}, L^j_{T_j}) \xrightarrow{a_{ij}} \mathrm{RHom}(L^i_{T_i}, L^j_{Y_j}) \xleftarrow{b_{ij}} \mathrm{RHom}(L^i_{Y_i}, L^j_{Y_j})$$

are isomorphisms,

(iii) for any i, j, p with $i, j > p$, we have an isomorphism

$$R\Gamma(Y; (R\mathcal{H}om(L^i_{T_i}, L^j_{T_j}))_{T_{Y_p}}) \simeq R\Gamma(Y; (R\mathcal{H}om(L^i_{Y_i}, L^j_{Y_j}))_{Y_p}).$$

Proof. (i) and (ii). By Lemma 3.15, we know that the b_{ij} are isomorphisms for $\underline{\varepsilon}$ in a suitable set of bounds. We prove by induction on k that there exists a set of bounds $B \subset]0, +\infty[^{n-1}$ such that, for $i = 1, \dots, k$ and $j = 1, \dots, n$, the morphisms a_{ij} and α_{ij} also are isomorphisms. For $k = 1$, we set for short $K^j = L^{1*} \otimes L^j$ and we note that

$$\mathrm{RHom}(L^1_{T_1}, L^j_{Y_j}) \simeq R\Gamma_{T_1}(Y; K^j_{Y_j}), \qquad \mathrm{RHom}(L^1_{T_1}, L^1_{T_1}) \simeq R\Gamma(Y; K^1_{T_1}),$$
$$\mathrm{RHom}(L^1_{T_1}, L^1_{Y_1}) \simeq R\Gamma(Y; K^1_{Y_1}), \quad \mathrm{RHom}(L^1_{\mathrm{Int}(T_1)}, L^j_{Y_j}) \simeq R\Gamma(\mathrm{Int}(T_1); K^j_{Y_j}).$$

Hence by Lemma 2.5, we may choose ε_1^0 such that $\forall 0 < \varepsilon_1 < \varepsilon_1^0$, a_{11} is an isomorphism and

$$\mathrm{RHom}(L^1_{\mathrm{Int}(T_1)}, L^j_{Y_j}) \simeq R\Gamma(Y_1; K^j_{Y_j}). \tag{16}$$

Let us prove that we may also find a set of bounds $B' \subset]0, +\infty[^{n-2}$ such that for $(\varepsilon_1, \dots, \varepsilon_{n-1}) \in \{\varepsilon_1\} \times B'$ the a_{1j} are isomorphisms. For $j \geq 2$, we have $R\Gamma(Y_1; K^j_{Y_j}) = 0$ because $Y_1 \cap Y_j = \emptyset$. Hence the identity (16) and the distinguished triangle $L^1_{\mathrm{Int}(T_1)} \to L^1_{T_1} \to L^1_{\partial T_1} \xrightarrow{+1}$ imply $\mathrm{RHom}(L^1_{T_1}, L^j_{Y_j}) \simeq \mathrm{RHom}(L^1_{\partial T_1}, L^j_{Y_j})$. We may also assume since the beginning that ε_1^0 is small enough so that ∂T_1 is smooth. Then Lemma 3.16, applied with $S = \partial T_1$, yields a set of bounds $B' \subset]0, +\infty[^{n-2}$ such that $\forall (\varepsilon_2, \dots, \varepsilon_{n-1}) \in B'$, we have isomorphisms:

$$R\mathcal{H}om(L^1_{\partial T_1}, L^j_{T_j}) \simeq K^j_{T_{1j}}[-1], \quad \mathrm{RHom}(L^1_{\partial T_1}, L^j_{T_j}) \simeq \mathrm{RHom}(L^1_{\partial T_1}, L^j_{Y_j}).$$

(With the notation of Lemma 3.16, we have $T_j = T'_j \cap U_+$ and $T_{1j} = S \cap T'_j$.) Now we just have to note that $R\mathcal{H}om(L^1_{\mathrm{Int}(T_1)}, L^j_{T_j}) = 0$, because $T_j \cap \mathrm{Int}(T_1) = \emptyset$, and use the same distinguished triangle as above to conclude.

Now let us assume the conclusion is true for k. Let B be the set of bounds given in the statement and $(\varepsilon_1^0, \dots, \varepsilon_{n-1}^0) \in B$. Let us set $T = T_1 \sqcup \dots \sqcup T_k$, $U = Y \setminus T$. Arguing as in the case $k = 1$ on the open subset U, we find a set of bounds $B' \subset]0, +\infty[^{n-k-1}$ such that $\forall (\varepsilon_1, \dots, \varepsilon_{n-1}) \in B \cap (\{(\varepsilon_1^0, \dots, \varepsilon_k^0)\} \times B')$ we have, with the "new" T_i (but the T_i for $i \leq k$ stay the same):

$$R\mathcal{H}om(L^{k+1}_{T_{k+1}}|_U, L^j_{T_j}|_U) \simeq (L^{k+1*} \otimes L^j)_{T'_{k+1,j}}[-1] \quad \text{for } j > k+1,$$
$$\mathrm{RHom}(L^{k+1}_{T_{k+1}}|_U, L^j_{T_j}|_U) \xrightarrow{\sim} \mathrm{RHom}(L^{k+1}_{T_{k+1}}|_U, L^j_{Y_j}|_U) \quad \text{for } j \geq k+1,$$

where $T'_{k+1,j} = U \cap \partial T_{k+1} \cap (\overline{T_j} \setminus \bigcup_{i<l<j} \overline{T_l})$. Since $T_{k+1} \subset U$, we have in fact $\mathrm{RHom}(L^{k+1}_{T_{k+1}}|_U, L^j_{T_j}|_U) = \mathrm{RHom}(L^{k+1}_{T_{k+1}}, L^j_{T_j})$ and the same with Y_j instead of T_j. Hence $a_{k+1,j}$ is an isomorphism, for $j \geq k+1$. Since we have chosen $(\varepsilon_1, \dots, \varepsilon_{n-1}) \in B$, $a_{i,j}$ is an isomorphism for $i \leq k$ and any j. It just remains to check the case $i = k+1$ and $j \leq k$. But T_{k+1} is contained in the open set U which does not meet

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T_j and Y_j. Hence $\mathrm{RHom}(L^{k+1}_{T_{k+1}}, L^j_{T_j}) = \mathrm{RHom}(L^{k+1}_{T_{k+1}}, L^j_{Y_j}) = 0$ and the assertion is trivially true.

Now let us consider $\alpha_{k+1,j}$. Let $u : U \to Y$ be the embedding. Since $T_{k+1} \subset U$, we have:

$$R\mathcal{H}om(L^{k+1}_{T_{k+1}}, L^j_{T_j}) \simeq Ru_* R\mathcal{H}om(L^{k+1}_{T_{k+1}}|_U, L^j_{T_j}|_U) \simeq Ru_*(L^{k+1*} \otimes L^j)_{T'_{k+1,j}}[-1].$$

We may assume, moreover, up to shrinking B, that the conclusions of Lemma 3.14 hold. Hence we have local coordinates (x_i) such that $U = \{x_i > 0; i = 1, \dots, m\}$ and $C = T'_{k+1,j}$ is a convex locally closed subset of U. It follows that $Ru_*\mathbf{C}_C \simeq \mathbf{C}_D$ for some locally closed subset $D \subset Y$. More precisely, if C is closed in U, we check that $(Ru_*\mathbf{C}_C)_x = \mathbf{C}$ if $x \in \overline{C}^Y$ and is 0 otherwise. Hence $D = \overline{C}^Y$. If we write $C = \overline{C}^U \setminus C_1$, with C_1 closed in U, we obtain, by the distinguished triangle $\mathbf{C}_C \to \mathbf{C}_{\overline{C}^U} \to \mathbf{C}_{C_1} \xrightarrow{+1}$, that $D = \overline{C}^Y \setminus \overline{C_1}^Y$. When applied to $T'_{k+1,j} = \partial T_{k+1} \cap (\overline{T_j} \setminus \bigcup_{i<l<j} \overline{T_l}) = (T_{k+1} \cap \overline{T_j}) \setminus (\bigcup_{i<l<j} \overline{T_l})$, this formula for D gives the expression of the proposition.

(iii) We first consider $p < i < j$, fixed. We set $F = R\mathcal{H}om(L^i_{Y_i}, L^j_{Y_j})$ and $G = R\mathcal{H}om(L^i_{T_i}, L^j_{T_j})$. By Lemma 3.15 applied to F, there exists a set of bounds, B_0, such that, $\forall(\underline{\varepsilon}) \in B_0$, we have $R\Gamma(Y; F_{T_p}) \simeq R\Gamma(Y; F_{Y_p})$. Let us also consider the Verdier dual $DF = R\mathcal{H}om(F, \mathbf{C}_Y[d_Y])$ (where d_Y is the real dimension of Y). In the proof of Lemma 3.15, at the p^{th} step, when $(\varepsilon_1, \dots, \varepsilon_{p-1})$ are fixed, we can choose ε_p so as to have, moreover, $R\Gamma(\mathrm{Int}(T_p); DF) \simeq R\Gamma(T_p \cap Y_p; DF)$ (for this we apply Lemma 2.5 to DF restricted to $Y \setminus \bigsqcup_{k<p} T_k$). We also set $S_p = T_p \setminus \mathrm{Int}(T_p)$. Let us prove that

$$R\Gamma(Y; F_{Y_p}) \simeq R\Gamma(\overline{S_p}; F_{S_p}). \tag{17}$$

We note that $R\Gamma_{Y_p}(F) \simeq R\mathcal{H}om(L^i_{Y_i \cap Y_p}, L^j_{Y_j}) = 0$. Since F is constructible we get $(DF)_{Y_p} \simeq D(R\Gamma_{Y_p}(F)) = 0$, so that $R\Gamma(\mathrm{Int}(T_p); DF) \simeq R\Gamma(T_p \cap Y_p; DF) = 0$. By Poincaré-Verdier duality, this gives the vanishing of the cohomology with compact supports, $R\Gamma_c(\mathrm{Int}(T_p); F) = 0$. Since $\overline{T_p}$ is compact, this is equivalent to $R\Gamma(Y; F_{\mathrm{Int}(T_p)}) = 0$. By the distinguished triangle $F_{\mathrm{Int}(T_p)} \to F_{T_p} \to F_{S_p} \xrightarrow{+1}$, we deduce $R\Gamma(Y; F_{T_P}) \simeq R\Gamma(Y; F_{S_p})$. This yields (17) because $R\Gamma(Y; F_{T_P}) \simeq R\Gamma(Y; F_{Y_P})$ and $R\Gamma(Y; F_{S_p}) \simeq R\Gamma(\overline{S_p}; F_{S_p})$.

On the other hand, $\mathrm{supp}\, G \subset \overline{T_i} \cap \overline{T_j}$ and, for $k > p$ we have $\overline{T_k} \cap T_p \subset S_p$, so that $R\Gamma(Y; G_{T_p}) \simeq R\Gamma(\overline{S_p}; G_{S_p})$. Together with (17), this shows that (iii) for our p, i, j, is equivalent to

$$R\Gamma(\overline{S_p}; F_{S_p}) \simeq R\Gamma(\overline{S_p}; G_{S_p}). \tag{18}$$

We let $\mathcal{S}_p$ be the smallest set of subsets of $\overline{S_p}$ stable by taking intersections and complements and containing the $\overline{S_p} \cap T_k$, for $k < p$. This set $\mathcal{S}_p$ is finite and we denote by $\mathcal{S}_{p,min}$ the set of its minimal elements (for the inclusion relation); then any element of $\mathcal{S}_p$ is a union of elements of $\mathcal{S}_{p,min}$. By Lemma 3.14 (iii), up to shrinking the set of bounds B_0, any $S \in \mathcal{S}_{p,min}$ is a locally closed submanifold of Y, transversal to every stratum Y_i that it meets. Let us first prove:

$$\forall S \in \mathcal{S}_{p,min}, \qquad R\Gamma(S; F) \simeq R\Gamma(S; G). \tag{19}$$

By part (i) of the proposition, we have $G|_S \simeq (L^{i*} \otimes L^j)_{T_{ij} \cap S}[-1]$. Let us stratify S by $S = \bigsqcup_{k>p}(S \cap Y_k)$ and define T^S_k, T^S_{kl}, similarly as T_k, T_{kl}, with the functions

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$f_m|_S$. We have $\overline{T_k^S} = \overline{T_k} \cap S$. Hence (i) applied to S gives

$$R\mathcal{H}om(L^i_{T^S_i}, L^j_{T^S_j}) \simeq (L^{i*} \otimes L^j)_{T^S_{ij}}[-1] \simeq (L^{i*} \otimes L^j)_{T_{ij}}|_S[-1] \simeq G|_S.$$

Now, part (ii) of the proposition applied to S gives, up to shrinking B_0 again:

$$R\Gamma(S;G) \simeq \mathrm{RHom}(L^i_{T^S_i}, L^j_{T^S_j}) \simeq \mathrm{RHom}(L^i_{Y_i \cap S}, L^j_{Y_j \cap S}) \simeq R\Gamma(S;F),$$

where the last isomorphism follows from the transversality of S and Y_i, Y_j. This is (19). We will deduce:

$$\forall V \in \mathcal{S}_p,\ V \text{ closed in } \overline{S_p}, \qquad R\Gamma(V;F) \simeq R\Gamma(V;G). \tag{20}$$

Since F is constructible with respect to the stratification $\bigsqcup Y_i$ and any $S \in \mathcal{S}_{p,min}$ is transversal to every stratum Y_i, we have the isomorphisms, $\forall S \in \mathcal{S}_{p,min}$, $u_S : R\Gamma(\overline{S};F) \xrightarrow{\sim} R\Gamma(S;F)$.

Recall that $G \simeq (L^{i*} \otimes L^j)_{T_{ij}}[-1]$. For any $S \in \mathcal{S}_{p,min}$, the inclusion $(T_{ij} \cap S) \subset T_{ij} \cap \overline{S}$ is an equivalence of homotopy (by Lemma 3.14 (iii)), so that we also have an isomorphism $v_S : R\Gamma(\overline{S};G) \xrightarrow{\sim} R\Gamma(S;G)$.

Now let us prove (20). Let us write $V = V_1 \sqcup \ldots \sqcup V_r$, with $V_i \in \mathcal{S}_{p,min}$, and argue by induction on r. For $r = 1$, our assertion is (19). We may assume that V_r is of maximal dimension (among the V_i), so that $V' = V \setminus V_r$ is closed and the induction hypothesis applies to V'. We have $\overline{V_r} = V$ or $V'' = \overline{V_r} \cap V'$ is a closed union of less than r subsets V_i and the induction hypothesis also applies to V''. By (19) and the isomorphisms u_{V_r}, v_{V_r}, we have $R\Gamma(\overline{V_r};F) \simeq R\Gamma(\overline{V_r};G)$. We conclude by the Mayer-Vietoris distinguished triangle that

$$R\Gamma(V;F) \to R\Gamma(\overline{V_r};F) \oplus R\Gamma(V';F) \to R\Gamma(V'';F) \xrightarrow{+1},$$

and a similar one for G.

Now we can prove (18) (and thus (iii) for our i, j, p). We have an excision distinguished triangle,

$$R\Gamma(\overline{S_p};F_{S_p}) \to R\Gamma(\overline{S_p};F) \to R\Gamma(\overline{S_p} \setminus S_p;F) \xrightarrow{+1},$$

and a similar one for G. The last two terms of these distinguished triangles are isomorphic because of (20) applied to $V = \overline{S_p}$ and $V = \overline{S_p} \setminus S_p$. Hence the first terms are also isomorphic, as desired.

For $i = j$, the same proof works, replacing the isomorphism $G \simeq (L^{i*} \otimes L^j)_{T_{ij}}[-1]$ by $G \simeq (L^{i*} \otimes L^i)_{\overline{T_i}}$: we still have $R\mathcal{H}om(L^i_{T^S_i}, L^i_{\overline{T^S_i}}) \simeq G|_S$ and $(\overline{T_i} \cap S) \subset \overline{T_i} \cap \overline{S}$ is an equivalence of homotopy. For $j < i$, we have $F = R\mathcal{H}om(L^i_{Y_i}, L^j_{Y_j}) = 0$ because Y_i is open in $\overline{Y_i}$ and $Y_j \subset (\overline{Y_i} \setminus Y_i)$. In the same way $G = 0$ and (iii) is trivial.

We let $B_{i,j,p}$ be the subset of B_0 formed by the $\underline{\varepsilon}$ such that (iii) holds for p, i, j. This is a set of bounds and the intersection of all $B_{i,j,p}$, for $i, j > p$, gives us the desired set of bounds. □

Proof of Proposition 3.11. Let us set, for $i = 1, \ldots, n$, $k = 1, 2$, $L^{ki} = \mathcal{L}^k$ if $Y_i \subset Z_k$ and $L^{ki} = 0$ otherwise. We set $L^i = L^{1i} \oplus L^{2i}$; this is a local system defined in a neighbourhood of Y_i. We first choose a set of bounds B such that the conclusions of Lemma 3.14 and Proposition 3.17 hold.

(i) Let us prove (13). We begin with the case where $Z_1 = Y_i$ is a single stratum. Let $Z' \subset Z_2$ be a closed subset of Z_2 (which is a union of strata). Set $W = Z_2 \setminus Z'$.

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We have $T_{Z_2} = T_{Z'} \sqcup T_W$ and $T_{Z'}$ is closed in T_{Z_2}. This gives a morphism of distinguished triangles:

$$\begin{array}{ccccccc} \mathbf{C}_{T_W} & \longrightarrow & \mathbf{C}_{T_{Z_2}} & \longrightarrow & \mathbf{C}_{T_{Z'}} & \xrightarrow{+1} & \\ \downarrow & & \downarrow & & \downarrow & & \\ \mathbf{C}_W & \longrightarrow & \mathbf{C}_{Z_2} & \longrightarrow & \mathbf{C}_{Z'} & \xrightarrow{+1} & \end{array}$$

Tensoring this diagram by $\mathcal{L}^2$ and applying the functor $\mathrm{RHom}(L^i_{T_i}, \cdot)$, where Y_i is any stratum of Y, we obtain the morphism of distinguished triangles:

$$\begin{array}{ccccccc} \mathrm{RHom}(L^i_{T_i}, \mathcal{L}^2_{T_W}) & \longrightarrow & \mathrm{RHom}(L^i_{T_i}, \mathcal{L}^2_{T_{Z_2}}) & \longrightarrow & \mathrm{RHom}(L^i_{T_i}, \mathcal{L}^2_{T_{Z'}}) & \xrightarrow{+1} & \\ \downarrow & & \downarrow f_i & & \downarrow & & \\ \mathrm{RHom}(L^i_{T_i}, \mathcal{L}^2_{W}) & \longrightarrow & \mathrm{RHom}(L^i_{T_i}, \mathcal{L}^2_{Z_2}) & \longrightarrow & \mathrm{RHom}(L^i_{T_i}, \mathcal{L}^2_{Z'}) & \xrightarrow{+1} & . \end{array}$$

It follows that the morphism f_i in this diagram is an isomorphism, for any $i = 1, \dots, n$. Indeed this is true if Z_2 consists of one stratum, say Y_j, by Proposition 3.17: we have here $L^j = L^{1j} \oplus L^{2j}$, and f_i is the L^{2j}-component of isomorphism a_{ij} of this proposition. Then the above diagram allows an induction on the number of strata of Z_2.

The same reasoning, on the first argument of $\mathrm{RHom}(\cdot, \cdot)$, gives the similar isomorphism with $L^i_{T_i}$ replaced by $\mathcal{L}^1_{T_{Z_1}}$. The same proof, using the isomorphisms b_{ij} of Proposition 3.17, yields the second isomorphism of (13).

(ii) We prove (14) first in the case where $Z_1 = Y_i$ and $Z_2 = Y_j$. Let us set $F = R\mathcal{H}om(\mathcal{L}^1_{Y_i}, \mathcal{L}^2_{Y_j})$, $G = R\mathcal{H}om(\mathcal{L}^1_{T_i}, \mathcal{L}^2_{T_j})$ and $Z' = Z \cap \overline{Y_i} \cap \overline{Y_j}$. We see that $F_Z = F_{Z'}$ and $G_{T_Z} = G_{T_{Z'}}$, hence we may assume that Z consists of strata Y_p with $p < i, j$ (recall that Z_1, $Z_2 \subset Y \setminus \overline{Z}$, so that $p \neq i$, $p \neq j$).

Let us argue by induction on the number of strata of Z. By Proposition 3.17, (14) is true when $Z = Y_p$ with $p < i, j$. Let us choose $Y_p \subset Z$ such that it is open in Z. We have an excision distinguished triangle $R\Gamma(Y; F_{Y_p}) \to R\Gamma(Y; F_Z) \to R\Gamma(Y; F_{Z \setminus Y_p}) \xrightarrow{+1}$, and a similar one with G_{T_Z}. By induction $R\Gamma(Y; F_{Z\setminus Y_p}) \simeq R\Gamma(Y; G_{T_{Z\setminus Y_p}})$, and by the first step $R\Gamma(Y; F_{Y_p}) \simeq R\Gamma(Y; G_{T_p})$. Hence we obtain $R\Gamma(Y; F_Z) \simeq R\Gamma(Y; G_{T_Z})$ as desired.

Going from a single stratum to arbitrary locally closed sets Z_1, Z_2, is the same as in the proof of (13).

(iii) Let us first assume that Z_1 consists of a single stratum. Since the statement is local on Y, we may take coordinates as in Lemma 3.14 and assume $\mathcal{L}^1 = \mathcal{L}^2 = \mathbf{C}_{\mathbf{R}^d}$. We set $C_l = \{x_1 > 0, \dots, x_{l-1} > 0,\ x_l \leq 0\}$ and assume that $Z_1 = C_i$. We set $U = \{x_1 > 0, \dots, x_{i-1} > 0\}$ and let $u : U \to \mathbf{R}^d$ be the inclusion. Since Z_2 is locally closed and $Z_1 \subset Z_2$, there exists $l \geq i$ such that $U \cap T_{Z_2} = C_i \sqcup \dots \sqcup C_l$. Since $C_i \subset U$, we have

$$R\mathcal{H}om(\mathbf{C}_{C_i}, \mathbf{C}_{T_{Z_2}}) \simeq Ru_* R\mathcal{H}om(\mathbf{C}_{C_i}, \mathbf{C}_{T_{Z_2}}|_U) \simeq Ru_*(\mathbf{C}_{C_i \setminus C_{il}}),$$

where $C_{il} = \overline{C_{i+1} \sqcup \dots \sqcup C_l}$. Now $Ru_*(\mathbf{C}_{C_i \setminus C_{il}}) \simeq \mathbf{C}_{\overline{C_i} \setminus \overline{C_{il}}}$ is concentrated in degree 0.

We deduce the result for an arbitrary locally closed subset $Z_1 \subset Z_2$ using an excision distinguished triangle.

(iv) We first note that we may assume $V = Y$. Indeed, by Lemma 3.15 applied to the complex $G = R\mathcal{H}om(\mathcal{L}^1_{Z_1}, \mathcal{L}^2_{Z_2})$, we may choose a set of bounds such that

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$\mathrm{RHom}(\mathcal{L}^1_{Z_1}|_V, \mathcal{L}^2_{Z_2}|_V) \simeq \mathrm{RHom}(\mathcal{L}^1_{Z_1}|_{T_V}, \mathcal{L}^2_{Z_2}|_{T_V})$. Hence, if (15) is true for global sections, applying it to $Y = T_V$, with the induced stratification and local systems, gives the result for any open V.

Let us set $F = \mathcal{H}om(\mathcal{L}^1_{T_{Z_1}}, \mathcal{L}^2_{T_{Z_2}})$, $U = Y \setminus (\overline{Z_1} \setminus Z_1)$ and show that we may replace Y by T_U. Since $T_{Z_1} \subset T_U$, we have $F \simeq \Gamma_{T_U}(F)$. But $F|_{T_U} \simeq (\mathcal{L}^{1*} \otimes \mathcal{L}^2)_{T_{Z_1 \cap Z_2}}$ and locally around any point of $\overline{T_U}$, the inclusions $T_{Z_1 \cap Z_2} \subset T_U \subset Y$ are homeomorphic to inclusions of convex subsets of $\mathbf{R}^d$. Hence $\Gamma_{T_U}(F) \simeq R\Gamma_{T_U}(F)$ and $R\Gamma(Y; F) \simeq R\Gamma(T_U; F)$. We also have $G \simeq R\Gamma_U(G)$, hence $\mathrm{RHom}(\mathcal{L}^1_{Z_1}, \mathcal{L}^2_{Z_2}) \simeq R\Gamma(U; G)$. By Lemma 3.15 applied to G, we have, up to shrinking the set of bounds, $R\Gamma(U; G) \simeq R\Gamma(T_U; G) \simeq \mathrm{RHom}(\mathcal{L}^1_{Z_1}|_{T_U}, \mathcal{L}^2_{Z_2}|_{T_U})$. Hence we may replace Y by T_U and assume Z_1 is closed.

By assertion (iii) proved above, we have, setting $F' = R\mathcal{H}om(\mathcal{L}^1_{T_{Z_1}}, \mathcal{L}^2_{T_{Z_2}})$, $F'_{T_{Z_2}} \simeq F_{T_{Z_2}}$. Since Z_1 is closed, we also have an exact sequence $0 \to \Gamma_{T_{Z_1}} \mathcal{L}^2_{T_{Z_2}} \to \mathcal{L}^2_{T_{Z_2}}$, so that $F_x = 0$ for $x \notin T_{Z_2}$, and $F_{T_{Z_2}} \simeq F$. Hence $R\Gamma(Y; F) \simeq R\Gamma(Y; F'_{T_{Z_2}})$ and, setting $W = \overline{Z_2} \setminus Z_2$ (which is closed), we have the distinguished triangles:

$$R\Gamma(Y; F'_{T_W}) \to R\Gamma(Y; F') \to R\Gamma(Y; F) \xrightarrow{+1},$$
$$R\mathcal{H}om(\mathcal{L}^1_{T_{Z_1 \cap W}}, \mathcal{L}^2_{T_{Z_2}}) \to F' \to R\mathcal{H}om(\mathcal{L}^1_{T_{Z_1 \cap (Y \setminus W)}}, \mathcal{L}^2_{T_{Z_2}}) \xrightarrow{+1}.$$

By (13), $R\Gamma(Y; F') \simeq \mathrm{RHom}(\mathcal{L}^1_{Z_1}, \mathcal{L}^2_{Z_2})$, thus the first triangle implies that (15) is equivalent to the vanishing of $R\Gamma(Y; F'_{T_W})$. Using the second triangle, this vanishing follows from the two sequences of isomorphisms below:

$$R\Gamma(Y; R\mathcal{H}om(\mathcal{L}^1_{T_{Z_1 \cap W}}, \mathcal{L}^2_{T_{Z_2}})_{T_W}) \simeq \mathrm{RHom}(\mathcal{L}^1_{T_{Z_1 \cap W}}, \mathcal{L}^2_{T_{Z_2}}) \tag{21}$$
$$\simeq \mathrm{RHom}(\mathcal{L}^1_{Z_1 \cap W}, \mathcal{L}^2_{Z_2}) \tag{22}$$
$$\simeq \mathrm{RHom}(\mathcal{L}^1_{Z_1 \cap W}, R\Gamma_{Z_2}(\mathcal{L}^2_{Z_2})) = 0, \tag{23}$$

$$\begin{aligned} &R\Gamma(Y; R\mathcal{H}om(\mathcal{L}^1_{T_{Z_1 \cap (Y \setminus W)}}, \mathcal{L}^2_{T_{Z_2}})_{T_W}) \\ &\qquad \simeq R\Gamma(Y; R\mathcal{H}om(\mathcal{L}^1_{Z_1 \cap (Y \setminus W)}, \mathcal{L}^2_{Z_2})_W) = 0. \end{aligned} \tag{24}$$

Let us explain these isomorphisms: (21) is true because the application of the functor $(\cdot)_{T_W}$ does not change anything (since the support of the complex of sheaves is included in T_W), (22) follows from (13), (23) comes from the hypothesis $R\mu_*(\mathcal{L}^2|_{Z_2}) = \mu_!(\mathcal{L}^2|_{Z_2})$ and the vanishing follows from $(Z_1 \cap W) \cap Z_2 = \emptyset$.

The first isomorphism in (24) follows from (14). The smoothness hypothesis gives $R\mathcal{H}om(\mathcal{L}^1_{Z_1 \cap (Y \setminus W)}, \mathcal{L}^2_{Z_2}) \simeq R\nu_*((\mathcal{L}^{1*} \otimes \mathcal{L}^2)|_{Z_1 \cap Z_2})$. Since $\mathcal{L}^1$ is defined on Z_1, which is closed, we also have $R\nu_*((\mathcal{L}^{1*} \otimes \mathcal{L}^2)|_{Z_1 \cap Z_2}) \simeq \nu_!((\mathcal{L}^{1*} \otimes \mathcal{L}^2)|_{Z_1 \cap Z_2})$; this implies the desired vanishing and concludes the proof of (15). □

3.2. **Direct image to I.** We consider a stratified analytic manifold $Y = \bigsqcup_{i \in I} Y_i$ endowed with normal crossings divisors $(D_v)_{v \in V}$ and sheaves $(L_\alpha)_{\alpha \in A}$, satisfying Assumptions 3.1. As in Notations 3.12, we fix $\underline{\varepsilon}$ in a set of bounds such that the conclusions of Proposition 3.11 hold for any pair $(\alpha, \beta) \in A^2$. We keep the notations T_i, for $i \in I$, and T_α, L'_α, for $\alpha \in A$. We will give an equivalence between $\mathrm{D}(Y)\langle L'_\alpha \rangle$ and a derived category of dg-modules on I following the construction of [13].

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Definition 3.18. We define a sheaf of dg-algebras on Y, Ω, by

$$\Omega = \bigoplus_{(\alpha,\beta)\in A^2} \Omega_{\alpha\beta}, \quad \text{where, for } \alpha,\, \beta \in A, \quad \Omega_{\alpha\beta} = \Omega_Y \otimes \mathcal{H}om(L'_\alpha, L'_\beta).$$

The product $m_{\alpha\beta\gamma} : \Omega_{\beta\gamma} \otimes \Omega_{\alpha\beta} \to \Omega_{\alpha\gamma}$, for $\alpha, \beta, \gamma \in A$, is induced by the product of forms and the composition in the endomorphisms sheaves.

Considering the partition $Y = \bigsqcup_{i\in I} T_i$, we define $\phi' : Y \to I$ (like $\phi : Y = \bigsqcup_{i\in I} Y_i \to I$) by $\phi'(T_i) = \{i\}$. We set

$$\mathcal{B}^{\alpha\beta} = \phi'_*(\Omega_{\alpha\beta}), \qquad \mathcal{B} = \phi'_*(\Omega) \simeq \bigoplus_{(\alpha,\beta)\in A^2} \mathcal{B}^{\alpha\beta}.$$

These are sheaves of differential graded $\mathbf{C}$-vector spaces on I, and $\mathcal{B}$ is a sheaf of dg-algebras. We also define a direct image functor $\gamma : \mathrm{D}^+(Y) \to \mathrm{D}_{\mathcal{B}}$. For $F \in \mathrm{D}^+(Y)$, we choose an injective resolution of F, say $F \to R_F$ (R_F and the morphism $F \to R_F$ depending functorially on F), and we set

$$\gamma(F) = \phi'_*(\bigoplus_{\alpha\in A} \Omega_Y \otimes \mathcal{H}om(L'_\alpha, R_F)).$$

We note that $\gamma(F)$ has a natural structure of $\mathcal{B}$-dg-module, defined by multiplication of forms and composition of homomorphisms sheaves, like the multiplicative structure of $\mathcal{B}$.

We also have the following description of the cohomology of the sections of $\mathcal{B}$. Since Ω_Y is a soft resolution of $\mathbf{C}_Y$, $\Omega_{\alpha\beta}$ is a soft resolution of $\mathcal{H}om(L'_\alpha, L'_\beta)$. Let $U \subset I$ be an open subset, $V = \phi^{-1}(U)$, $V' = \phi'^{-1}(U) = T_V$, and let $\alpha, \beta \in A$. By (iii) and (iv) of Proposition 3.11, we obtain:

$$(25) \qquad H^i(\phi'_*(\Omega_{\alpha\beta})(U)) \simeq \mathrm{Ext}^i((L'_\alpha)|_{V'}, (L'_\beta)|_{V'}) \simeq \mathrm{Ext}^i((L_\alpha)|_V, (L_\beta)|_V).$$

Proposition 3.19. *With the above notations, we set, for $\alpha \in A$, $M_\alpha = \gamma(L'_\alpha)$ and $M^0_\alpha = \bigoplus_{\alpha'\in A} \mathcal{B}^{\alpha'\alpha}$. Then the natural morphism $M^0_\alpha \to M_\alpha$ is an isomorphism (in $\mathrm{D}(Y)$). The functor γ induces an equivalence of categories between $\mathrm{D}(Y)\langle L'_\alpha\rangle$ and $\mathrm{D}_{\mathcal{B}}\langle M_\alpha\rangle \simeq \mathrm{D}_{\mathcal{B}}\langle M^0_\alpha\rangle$.*

Proof. For $\alpha \in A$, we denote by $L'_\alpha \to R_\alpha = R_{L'_\alpha}$ the chosen injective resolution of L'_α.

The first assertion follows from the definitions of M_α, M^0_α and the isomorphisms

$$R\mathcal{H}om(L'_{\alpha'}, L'_\alpha) \simeq \mathcal{H}om(L'_{\alpha'}, L'_\alpha), \quad R\mathcal{H}om(L'_{\alpha'}, L'_\alpha) \simeq \mathcal{H}om(L'_{\alpha'}, R_\alpha).$$

The first of these isomorphisms follows from Proposition 3.11, (iii), and the second one follows from the injectivity of R_α.

Let us prove the second assertion. Since our categories are respectively generated by the L'_α and the M_α, it is sufficient to prove that γ gives isomorphisms, $\forall \alpha, \beta \in A$, $\forall p \in \mathbf{Z}$:

$$(26) \qquad \mathrm{Hom}_{\mathrm{D}(Y)}(L'_\alpha, L'_\beta[p]) \simeq \mathrm{Hom}_{\mathrm{D}_{\mathcal{B}}}(M_\alpha, M_\beta[p]).$$

Indeed an inductive argument as in the proof of Corollary 3.13 (but easier because, here, the functor giving the equivalence is a priori defined) implies that γ also gives a bijection between $\mathrm{Hom}_{\mathrm{D}(Y)}(L_1, L_2)$ and $\mathrm{Hom}_{\mathrm{D}_{\mathcal{B}}}(\gamma(L_1), \gamma(L_2))$ for any objects L_1, L_2 of $\mathrm{D}(Y)\langle L'_\alpha\rangle$. Let us prove (26). We set $L' = \bigoplus_{\alpha\in A} L'_\alpha$ and $M = \gamma(L') = \bigoplus_{\alpha\in A} M_\alpha$. Then (26) is equivalent to

$$(27) \qquad \forall p \in \mathbf{Z} \qquad \mathrm{Hom}_{\mathrm{D}(Y)}(L', L'[p]) \simeq \mathrm{Hom}_{\mathrm{D}_{\mathcal{B}}}(M, M[p]).$$

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The morphisms $L'_\alpha \to R_\alpha$ induce morphisms of differential graded sheaves, $f^{\alpha\beta}$, from $\mathcal{B}^{\alpha\beta} = \phi'_*(\Omega_{\alpha\beta})$ to $\mathcal{B}'^{\alpha\beta} = \phi'_*(\Omega_Y \otimes \mathcal{H}om(L'_\alpha, R_\beta))$. Since R_β is injective we have the isomorphisms below in $\mathrm{D}(Y)$, which give the cohomology of sections of $\mathcal{B}'^{\alpha\beta}$, for an open set $U \subset I$, and $V' = \phi'^{-1}(U)$:

$$\Omega_Y \otimes \mathcal{H}om(L'_\alpha, R_\beta) \simeq \mathcal{H}om(L'_\alpha, R_\beta) \simeq R\mathcal{H}om(L'_\alpha, R_\beta),$$
$$H^{\cdot}(\mathcal{B}'^{\alpha\beta}(U)) = H^{\cdot}(V'; \Omega_Y \otimes \mathcal{H}om(L'_\alpha, R_\beta)) \simeq \mathrm{Ext}^i((L'_\alpha)|_{V'}, (L'_\beta)|_{V'}).$$

By (25) this means that $f^{\alpha\beta}$ is a quasi-isomorphism of differential graded sheaves. Summing over all pairs (α, β), we obtain a quasi-isomorphism of $\mathcal{B}$-modules between $\mathcal{B}$ and $\gamma(\bigoplus_{\beta\in A} L'_\beta) = M$. Hence we obtain:

$$\mathrm{Hom}_{\mathrm{D}_\mathcal{B}}(M, M[p]) \simeq \mathrm{Hom}_{\mathrm{D}_\mathcal{B}}(\mathcal{B}, \mathcal{B}[p]). \tag{28}$$

We have seen that $\forall i \in I$, $\mathcal{B}_{U_i}$ is K-projective. By (i) of Assumptions 3.1, any intersection of open sets of the type U_i still is of this kind, hence $\forall i_1, \dots, i_n \in I$, $\mathcal{B}_{U_{i_1}\cap\dots\cap U_{i_n}}$ is K-projective. Let us put any total order on I; we obtain a K-projective resolution of $\mathcal{B}$ by taking the total complex of the following Čech-like complex of complexes:

$$\cdots \to \bigoplus_{i_1<i_2<i_3\in I} \mathcal{B}_{U_{i_1}\cap U_{i_2}\cap U_{i_3}} \to \bigoplus_{i_1<i_2\in I} \mathcal{B}_{U_{i_1}\cap U_{i_2}} \to \bigoplus_{i_1\in I} \mathcal{B}_{U_{i_1}} \to 0,$$

with the usual differential $(da)_{i_1,\dots,i_r} = \sum_{i_1<\dots<i_k<j<i_{k+1}<\dots<i_r} (-1)^k a_{i_1,\dots,j,\dots,i_r}$. For an open set $U \subset I$, we have $\mathrm{RHom}(\mathcal{B}_U, \mathcal{B}) \simeq R\Gamma(U; \mathcal{B})$ and, for $U = U_i$, the functor $\Gamma(U_i; \cdot) = (\cdot)_i$ is exact. Hence:

$$\mathrm{Hom}_{\mathrm{D}_\mathcal{B}}(\mathcal{B}_{U_i}, \mathcal{B}[p]) \simeq H^p(\mathrm{RHom}(\mathcal{B}_{U_i}, \mathcal{B})) \simeq H^p(R\Gamma(U_i; \mathcal{B})) \simeq H^p(\Gamma(U_i; \mathcal{B})).$$

By definition $\Gamma(U_i; \mathcal{B}) = \Gamma(\phi'^{-1}(U_i); \Omega)$ and we obtain that $\mathrm{Hom}_{\mathrm{D}_\mathcal{B}}(M, M[p])$ is the pth cohomology group of the total complex of the double complex:

$$0 \to \bigoplus_{i_1\in I} \Gamma(\phi'^{-1}(U_{i_1}); \Omega) \to \bigoplus_{i_1<i_2\in I} \Gamma(\phi'^{-1}(U_{i_1}) \cap \phi'^{-1}(U_{i_2}); \Omega) \to \cdots.$$

This is a Čech resolution of the complex Ω, which is formed by soft sheaves. Hence $\mathrm{Hom}_{\mathrm{D}_\mathcal{B}}(\mathcal{B}, \mathcal{B}[p]) \simeq H^p(Y; \Omega)$, and, by (25), this last group is isomorphic to $\mathrm{Ext}^p(L', L')$. In view of (28), this gives (27) and concludes the proof. □

3.3. **Gysin isomorphism and product.** In the previous paragraph, we have obtained a sheaf of dg-algebras, $\mathcal{B}$, on I such that $\mathrm{D}(Y)\langle L_\alpha\rangle$ is equivalent to a subcategory of $\mathrm{D}_\mathcal{B}$. In section 5, we will construct a sequence of quasi-isomorphisms between $\mathcal{B}$ and its cohomology. For this we will, in particular, replace sets like $U_{i\alpha\beta} = \{x \in \phi'^{-1}(U_i);\ \mathcal{H}om(L'_\alpha, L'_\beta)_x \neq 0\}$ by homotopy equivalent ones. But before that, we note that $\mathcal{B}_i$ is not immediately quasi-isomorphic to sections of $\Omega_{U_{i\alpha\beta}} \otimes \mathcal{H}om(L'_\alpha, L'_\beta)$. Indeed, the cohomology of $\mathcal{B}_i$ is $\mathrm{Ext}^{\cdot}(L'_\alpha|_{\phi'^{-1}(U_i)}, L'_\beta|_{\phi'^{-1}(U_i)}) = \mathrm{Ext}^{\cdot}(L_\alpha|_{\phi^{-1}(U_i)}, L_\beta|_{\phi^{-1}(U_i)})$, whereas the cohomology of $\Omega_{U_{i\alpha\beta}} \otimes \mathcal{H}om(L'_\alpha, L'_\beta)$ is $\mathrm{Ext}^{\cdot}(L'_\alpha|_{U_{i\alpha\beta}}, L'_\beta|_{U_{i\alpha\beta}}) = \mathrm{Ext}^{\cdot}(L_\alpha|_{Y_i}, L_\beta|_{Y_i})$. In our situation they are isomorphic under a "twisted" Gysin isomorphism. We build this isomorphism at the level of the de Rham algebras and describe the algebra structure. Then we use this description to obtain a quasi-isomorphism between $\mathcal{B}$ and the sheaf $\mathcal{A}$ defined in 3.5.

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3.3.1. *Gysin isomorphism.* Let us first consider the usual Gysin isomorphism. If M is an oriented manifold and N a closed oriented submanifold of codimension c, we have an isomorphism $R\Gamma_N(\mathbf{C}_M) \simeq \mathbf{C}_N[-c]$. On the global sections it induces $H^{\cdot}(N;\mathbf{C}_N) \simeq H_N^{\cdot+c}(M;\mathbf{C}_M)$. Let us choose, by Lemma 2.5, two open tubular neighbourhoods U, V of N, such that $\overline{U} \subset V$ and

$$H^{\cdot}(V;\mathbf{C}_V) \xrightarrow{\sim} H^{\cdot}(U;\mathbf{C}_U) \xrightarrow{\sim} H^{\cdot}(N;\mathbf{C}_N), \quad H_N^{\cdot}(M;\mathbf{C}_M) \xrightarrow{\sim} H_{\overline{U}}^{\cdot}(M;\mathbf{C}_M).$$

If we assume, moreover, that the boundary of U is smooth, we have $R\Gamma_{\overline{U}}\mathbf{C}_M \simeq \mathbf{C}_U$, so that $H_N^{\cdot}(M;\mathbf{C}_M) \simeq H^{\cdot}(M;\mathbf{C}_U)$. Let $\delta(N,M) \in H_N^c(M;\mathbf{C}_M)$ be the fundamental class of N in M. We may choose a representative $\xi(N,M) \in \Gamma(M;\Omega_M^c)$ such that $\operatorname{supp}\xi(N,M) \subset U$. Then, the multiplication by $\xi(N,M)$ induces a well-defined quasi-isomorphism of $\Gamma(N;\Omega_N)$-dg-modules between $\Gamma(V;\Omega_V)$ and $\Gamma(M;(\Omega_M)_U)[c]$. More generally, given local systems L_1 on M, and L_2 on V, we obtain isomorphisms between extension groups and a corresponding quasi-isomorphism between de Rham complexes:

$$\cdot\delta(N,M) : \operatorname{Ext}^{\cdot}_{\mathrm{D}(N)}(L_2|_N, L_1|_N) \xrightarrow{\sim} \operatorname{Ext}^{\cdot+c}_{\mathrm{D}(M)}((L_2)_N, L_1), \tag{29}$$

$$\operatorname{Ext}^{\cdot}_{\mathrm{D}(V)}(L_2, L_1|_V) \xrightarrow{\sim} \operatorname{Ext}^{\cdot+c}_{\mathrm{D}(M)}((L_2)_{\overline{U}}, L_1), \tag{30}$$

$$\begin{aligned}\cdot\xi(N,M) : \Gamma(V;\Omega_V \otimes \mathcal{H}om(L_2, L_1|_V))\\ \xrightarrow{qis} \Gamma(M;\Omega_M \otimes \mathcal{H}om((L_2)_{\overline{U}}, L_1))[c].\end{aligned} \tag{31}$$

We will use (31) to build a second sheaf of dg-algebras on I, quasi-isomorphic to $\mathcal{B}$. For this we also need to describe the product structure: the composition $\mathcal{H}om(L_2,L_1) \otimes \mathcal{H}om(L_3,L_2) \to \mathcal{H}om(L_3,L_1)$ induces a product on the right-hand side of the above isomorphisms and we want to understand the corresponding product on the left-hand side.

3.3.2. *Algebra structure.* This paragraph mainly has an heuristic purpose, in order to justify the definition of the product m of the sheaf of dg-algebras $\mathcal{A}$ introduced in 3.5 above. We consider a complex manifold Y, endowed with normal crossings divisors D_v, $v \in V$, and sheaves L_α, $\alpha \in A$, satisfying Assumptions 3.1. We keep the notations of 3.1, in particular, for Z_Δ, $\Delta \in \mathcal{S}$, $Z_\alpha = Z_{\Delta_\alpha} \setminus \bigcup_{v \in \Delta'_\alpha} D_v$, for $\alpha \in A$. We set also:

$$Z_{\alpha\beta} = Z_\alpha \cap Z_\beta, \qquad Z_{\alpha\beta\gamma} = Z_\alpha \cap Z_\beta \cap Z_\gamma.$$

We fix $\alpha, \beta, \gamma \in A$ and, up to restricting ourselves to an open subset of Y, we assume that Z_α, Z_β, Z_γ are closed.

For $v \in V$, D_v has a fundamental class $\delta_v \in H^2_{D_v}(Y;\mathbf{C}_Y)$. For $\Delta \in \mathcal{S}$ the fundamental class of Z_Δ in Y is $\delta_\Delta = \prod_{v\in\Delta} \delta_v$. It belongs to $H^{2|\Delta|}_{Z_\Delta}(Y;\mathbf{C}_Y)$. For $\Delta' \subset \Delta$, we have $Z_\Delta \subset Z_{\Delta'}$ and Z_Δ is the transversal intersection of $Z_{\Delta'}$ and $Z_{\Delta\setminus\Delta'}$. Hence the fundamental class $\delta(Z_\Delta, Z_{\Delta'}) \in H^{2d}_{Z_\Delta}(Z_{\Delta'};\mathbf{C}_{Z_{\Delta'}})$, with $d = |\Delta| - |\Delta'|$, is the image of $\delta_{\Delta\setminus\Delta'} \in H^{2d}_{Z_{\Delta\setminus\Delta'}}(Y;\mathbf{C}_Y)$ in $H^{2d}_{Z_\Delta}(Z_{\Delta'};\mathbf{C}_{Z_{\Delta'}})$. By abuse of notation we will write $\delta_{\Delta\setminus\Delta'}$ for $\delta(Z_\Delta, Z_{\Delta'})$.

We define $\varepsilon_{\alpha\beta} = \delta(Z_{\alpha\beta}, Z_\beta)$. We have:

$$\varepsilon_{\alpha\beta} = \delta_{\Delta_\alpha\setminus\Delta_\beta} \in H^{2d_{\alpha\beta}}_{Z_{\alpha\beta}}(Z_\beta;\mathbf{C}_{Z_\beta}), \qquad \text{with} \quad d_{\alpha\beta} = |\Delta_\alpha \setminus \Delta_\beta|.$$

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We remark that $\mathrm{Ext}^{\cdot}_{\mathrm{D}(Y)}(L_\alpha, L_\beta) \simeq \mathrm{Ext}^{\cdot}_{\mathrm{D}(Y)}((L_\alpha)_{Z_{\alpha\beta}}, L_\beta)$, so that, by (29), multiplication by $\varepsilon_{\alpha\beta}$ gives an isomorphism:

$$\cdot\varepsilon_{\alpha\beta} : \mathrm{Ext}^i_{\mathrm{D}(Z_{\alpha\beta})}(L_\alpha|_{Z_{\alpha\beta}}, L_\beta|_{Z_{\alpha\beta}}) \xrightarrow{\sim} \mathrm{Ext}^{i+2d_{\alpha\beta}}_{\mathrm{D}(Y)}(L_\alpha, L_\beta). \tag{32}$$

From now on we forget the subscripts $\mathrm{D}(Z_\cdot)$ if there is no ambiguity. We want to understand the product of two extension classes in $\mathrm{Ext}^{\cdot}(L_\alpha, L_\beta)$ and $\mathrm{Ext}^{\cdot}(L_\beta, L_\gamma)$ in terms of the corresponding classes on the left-hand side of the above formula (see diagram (37) below). Let us introduce some notation:

$$\Delta_1 = \Delta_\beta \setminus (\Delta_\alpha \cup \Delta_\gamma), \qquad d_1 = |\Delta_1|, \qquad \varepsilon_{\alpha\beta\gamma} = \delta(Z_{\alpha\beta\gamma}, Z_{\alpha\gamma}) = \delta_{\Delta_1}, \tag{33}$$

$$\Delta_2 = (\Delta_\alpha \cap \Delta_\gamma) \setminus \Delta_\beta, \qquad d_2 = |\Delta_2|, \qquad \varepsilon^+_{\alpha\beta\gamma} = \delta_{\Delta_2}. \tag{34}$$

We first restrict the classes in $\mathrm{Ext}^{\cdot}(L_\alpha|_{Z_{\alpha\beta}}, L_\beta|_{Z_{\alpha\beta}})$ and $\mathrm{Ext}^{\cdot}(L_\beta|_{Z_{\beta\gamma}}, L_\gamma|_{Z_{\beta\gamma}})$ to $Z_{\alpha\beta\gamma}$ and make the product, obtaining a morphism:

$$r : \mathrm{Ext}^i(L_\alpha|_{Z_{\alpha\beta}}, L_\beta|_{Z_{\alpha\beta}}) \otimes \mathrm{Ext}^j(L_\beta|_{Z_{\beta\gamma}}, L_\gamma|_{Z_{\beta\gamma}}) \to \mathrm{Ext}^{i+j}(L_\alpha|_{Z_{\alpha\beta\gamma}}, L_\gamma|_{Z_{\alpha\beta\gamma}}).$$

Now L_α and L_γ both restrict to local systems on $Z_{\alpha\gamma}$ (and $Z_{\alpha\beta\gamma}$), so that we may identify $\mathrm{Ext}^{\cdot}(L_\alpha|_{Z_{\alpha\gamma}}, L_\gamma|_{Z_{\alpha\gamma}})$ with $H^{\cdot}(Z_{\alpha\gamma}; L_\alpha^* \otimes L_\gamma)$ (and the same on $Z_{\alpha\beta\gamma}$). Hence multiplication by $\varepsilon_{\alpha\beta\gamma}$ gives a morphism:

$$\cdot\varepsilon_{\alpha\beta\gamma} : \mathrm{Ext}^{\cdot}(L_\alpha|_{Z_{\alpha\beta\gamma}}, L_\gamma|_{Z_{\alpha\beta\gamma}}) \to \mathrm{Ext}^{\cdot+2d_1}(L_\alpha|_{Z_{\alpha\gamma}}, L_\gamma|_{Z_{\alpha\gamma}}).$$

The "correcting term" $\varepsilon^+_{\alpha\beta\gamma}$ is introduced to have the identity (35) below (in $H^{\cdot}_{\mathbf{Z}_{\alpha\beta\gamma}}(Y; \mathbf{C}_Y)$). Multiplication by $\varepsilon^+_{\alpha\beta\gamma}$ gives morphism (36).

$$\varepsilon_{\alpha\beta} \cdot \varepsilon_{\beta\gamma} = \varepsilon^+_{\alpha\beta\gamma} \cdot \varepsilon_{\alpha\beta\gamma} \cdot \varepsilon_{\alpha\gamma}, \tag{35}$$

$$\cdot\varepsilon^+_{\alpha\beta\gamma} : \mathrm{Ext}^{\cdot}(L_\alpha|_{Z_{\alpha\gamma}}, L_\gamma|_{Z_{\alpha\gamma}}) \to \mathrm{Ext}^{\cdot+2d_2}(L_\alpha|_{Z_{\alpha\gamma}}, L_\gamma|_{Z_{\alpha\gamma}}). \tag{36}$$

Because of (35), the composition of (36) with $\varepsilon_{\alpha\beta\gamma} \circ r$ gives the commutative diagram

$$\begin{array}{ccc} \mathrm{Ext}^i(L_\alpha|_{Z_{\alpha\beta}}, L_\beta|_{Z_{\alpha\beta}}) \otimes \mathrm{Ext}^j(L_\beta|_{Z_{\beta\gamma}}, L_\gamma|_{Z_{\beta\gamma}}) & \xrightarrow[p]{} & \mathrm{Ext}^k(L_\alpha|_{Z_{\alpha\gamma}}, L_\gamma|_{Z_{\alpha\gamma}}) \\ \varepsilon_{\alpha\beta}\otimes\varepsilon_{\beta\gamma} \downarrow \wr & & \wr \downarrow \varepsilon_{\alpha\gamma} \\ \mathrm{Ext}^{i+2d_{\alpha\beta}}_{\mathrm{D}(Y)}(L_\alpha, L_\beta) \otimes \mathrm{Ext}^{j+2d_{\beta\gamma}}_{\mathrm{D}(Y)}(L_\beta, L_\gamma) & \longrightarrow & \mathrm{Ext}^l_{\mathrm{D}(Y)}(L_\alpha, L_\gamma), \end{array} \tag{37}$$

where $k = i + j + 2(d_1 + d_2)$, $l = i + j + 2(d_{\alpha\beta} + d_{\beta\gamma})$, $p = (\cdot\varepsilon^+_{\alpha\beta\gamma}) \circ (\cdot\varepsilon_{\alpha\beta\gamma}) \circ r$ and the bottom arrow is the composition of extension classes. This diagram gives us the composition of extension classes, through isomorphism (29). The definition of the product of $\mathcal{A}$ in 3.5 is copied from the definition of p above.

3.4. **Proof of Proposition 3.7.** Recall that quasi-isomorphic sheaves of dg-algebras have equivalent derived categories (see [13], Proposition 1.11.2). Hence, by Corollary 3.13 and Proposition 3.19, the proof of Proposition 3.7 will be achieved if we show that $\mathcal{B}$ and $\mathcal{A}$ are quasi-isomorphic.

We still consider $Y = \bigsqcup_{i\in I} Y_i$, $(D_v)_{v\in V}$, $(L_\alpha)_{\alpha\in A}$ satisfying Assumptions 3.1. We keep the notation $L_{\alpha,i}$ for the local system on $\phi^{-1}(U_i)$ extending $L_\alpha|_{Y_i}$, and also Notations 3.12 for T_i, T_α, L'_α, as well as the notations of paragraph 3.3.2 for δ_v, δ_Δ, Δ_1, Δ_2.

We choose representatives, $\xi_v \in \Gamma(Y; \Omega^2_Y)$, of the δ_v, such that $\operatorname{supp}\xi_v \subset \operatorname{Int}(T_{D_v})$ (remember that $T_{D_v} = \bigsqcup_{\{i,\, Y_i \subset D_v\}} T_i$). For $\Delta \subset V$ we define $\xi_\Delta =$

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$\prod_{v\in\Delta}\xi_v \in \Gamma(Y;\Omega_Y^{2d})$; it is a representative of δ_Δ with $\operatorname{supp}\xi_\Delta\subset \mathrm{Int}(T_{Z_\Delta})$. For $\alpha,\beta,\gamma\in A$ we define forms $\eta_{\alpha\beta}$, $\eta_{\alpha\beta\gamma}$, representing the classes $\varepsilon_{\alpha\beta}$, $(\varepsilon_{\alpha\beta\gamma}\cdot\varepsilon^+_{\alpha\beta\gamma})$:

$$\eta_{\alpha\beta}=\xi_{(\Delta_\alpha\setminus\Delta_\beta)},\qquad \eta_{\alpha\beta\gamma}=\xi_{\Delta_1}\cdot\xi_{\Delta_2}.$$

The forms $\eta_{\alpha\beta\gamma}$ were already introduced when we defined the product of $\mathcal{A}$. By Lemma 3.20 below, the multiplications by the forms $\eta_{\alpha\beta}$ give quasi-isomorphisms of sheaves $g^{\alpha\beta}:\mathcal{A}^{\alpha\beta}\to\mathcal{B}^{\alpha\beta}$. By the definitions and the identity (similar to (35)) $\eta_{\alpha\beta}\cdot\eta_{\beta\gamma}=\eta_{\alpha\beta\gamma}\cdot\eta_{\alpha\gamma}$, the morphism $g=\bigoplus g^{\alpha\beta}:\mathcal{A}\to\mathcal{B}$ is a morphism of sheaves of dg-algebras.

We have thus obtained a quasi-isomorphism between $\mathcal{B}$ and $\mathcal{A}$, and hence an equivalence of categories between $\mathrm{D}_{\mathcal{B}}\langle M^0_\alpha\rangle$ and $\mathrm{D}_{\mathcal{A}}\langle\mathcal{A}\otimes^L_{\mathcal{B}} M^0_\alpha\rangle$. We remark that M^0_α is $\mathcal{B}$-flat because $\mathcal{B}\simeq\bigoplus_{\alpha\in A}M^0_\alpha$. It follows that $\mathcal{A}\otimes^L_{\mathcal{B}}M^0_\alpha\simeq N_\alpha$, for the $\mathcal{A}$-module $N_\alpha=\bigoplus_{\alpha'\in A}\mathcal{A}^{\alpha'\alpha}$. This concludes the proof of the proposition.

Lemma 3.20. *Let us set, for $i\in I$, $V_i=\phi^{-1}(U_i)$, $V'_i=\phi'^{-1}(U_i)=T_{V_i}$.*

(i) For $\alpha,\beta\in A$, $i\in\phi(Z_{\alpha\beta})$, we have a well-defined morphism of sheaves on V'_i:

$$f^i_{\alpha\beta}:\Omega^\cdot_{V'_i}\otimes\mathcal{H}om(L_{\alpha,i},L_{\beta,i})\to\Omega^{\cdot+2d_{\alpha\beta}}_{V'_i}\otimes\mathcal{H}om(L'_\alpha,L'_\beta),\quad(\sigma\otimes u)\mapsto(\eta_{\alpha\beta}\sigma)\otimes u,$$

where $\sigma\in\Omega_{V'_i}$, $u\in\mathcal{H}om(L'_\alpha,L'_\beta)$. On the global sections, it induces a morphism $g^{\alpha\beta}_i:\mathcal{A}^{\alpha\beta}_i\to\mathcal{B}^{\alpha\beta}_i$.

(ii) The morphisms $g^{\alpha\beta}_i$, $i\in\phi(Z_{\alpha\beta})$, extend to a morphism of differential graded sheaves on I, $g^{\alpha\beta}:\mathcal{A}^{\alpha\beta}\to\mathcal{B}^{\alpha\beta}$, which is a quasi-isomorphism.

Proof. Since U_i is open, we have, by Lemma 3.14 (i), $Y\setminus V_i\subset T_{Y\setminus V_i}=Y\setminus V'_i$. By definition of $T_{Y\setminus V_i}$, this implies in fact $Y\setminus V_i\subset\mathrm{Int}(Y\setminus V'_i)$, or, as well, $\overline{V'_i}\subset V_i$.

(i) We consider $i\in\phi(Z_{\alpha\beta})$ and set $T^i_{\alpha\beta}=((T_\alpha\cap T_\beta)\setminus\overline{(T_\beta\setminus T_\alpha)})\cap V'_i$. We first prove the isomorphism:

$$\mathcal{H}om(L'_\alpha,L'_\beta)|_{V'_i}\simeq(\mathcal{H}om(L_{\alpha,i},L_{\beta,i}))_{T^i_{\alpha\beta}}.\tag{38}$$

We have by definition $L'_\alpha|_{V'_i}\simeq(L_{\alpha,i})_{T_\alpha\cap V'_i}$. Since $i\in\phi(Z_{\alpha\beta})$, $Z_\alpha\cap V_i$ is closed in V_i and, by Lemma 3.14 (i), (ii), $T_\alpha\cap V'_i$ is closed in V'_i. The same holds for β. Since (38) can be checked locally, we may assume $L_{\alpha,i}=L_{\beta,i}=\mathbf{C}_{V_i}$. Now, for two closed subsets M, N of a manifold X, and i_N the inclusion of N in X, we have

$$\mathcal{H}om(\mathbf{C}_M,\mathbf{C}_N)\simeq(i_N)_*\mathcal{H}om(i_N^{-1}\mathbf{C}_M,\mathbf{C}_N)\simeq(i_N)_!\mathcal{H}om(\mathbf{C}_{M\cap N},\mathbf{C}_N).$$

One checks that $\mathcal{H}om(\mathbf{C}_{M\cap N},\mathbf{C}_N)\simeq\mathbf{C}_U$, where U is the interior of $M\cap N$ in N. This gives the formula for $T^i_{\alpha\beta}$.

We may write as well $T^i_{\alpha\beta}=((T_\alpha\cap T_\beta)\setminus\overline{(T_\beta\setminus T_{Z_{\Delta_\alpha}})})\cap V'_i$. A form $\omega\in\Gamma(V'_i;\Omega_Y)$ such that $\operatorname{supp}\omega\cap\overline{T_\beta\setminus T_{Z_{\Delta_\alpha}}}=\emptyset$ belongs in fact to $\Gamma(V'_i;(\Omega_Y)_{V'_i\setminus\overline{(T_\beta\setminus T_{Z_{\Delta_\alpha}})}})$. Since $T_\alpha\cap T_\beta$ is closed in V'_i, we have a natural morphism from $(\Omega_Y)_{V'_i\setminus\overline{(T_\beta\setminus T_{Z_{\Delta_\alpha}})}}$ to $(\Omega_Y)_{T^i_{\alpha\beta}}$, and ω induces an element of $\Gamma(V'_i;(\Omega_Y)_{T^i_{\alpha\beta}})$. This condition on the support is satisfied by $\eta_{\alpha\beta}$. Indeed, we have $\operatorname{supp}\eta_{\alpha\beta}\subset\mathrm{Int}(T_{Z_{\Delta_\alpha\setminus\Delta_\beta}})$, hence it is sufficient to check that $T_{Z_{\Delta_\alpha\setminus\Delta_\beta}}\cap T_{Z_\beta\setminus Z_{\Delta_\alpha}}=\emptyset$. This is equivalent to $Z_{\Delta_\alpha\setminus\Delta_\beta}\cap(Z_\beta\setminus Z_{\Delta_\alpha})=\emptyset$, which is obvious. Hence the multiplication by $\eta_{\alpha\beta}$ sends Ω_{V_i} into $(\Omega_{V_i})_{T^i_{\alpha\beta}}[2d_{\alpha\beta}]$. In view of (38), this gives the morphism $f^i_{\alpha\beta}$. Now, remember that

$$\mathcal{B}^{\alpha\beta}_i=\Gamma(V'_i;\Omega_Y\otimes\mathcal{H}om(L'_\alpha,L'_\beta)),$$
$$\mathcal{A}^{\alpha\beta}_i=\Gamma(V_i;\Omega_Y\otimes\mathcal{H}om(L_{\alpha,i},L_{\beta,i}))\,[-2d_{\alpha\beta}].$$

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Hence the restriction from V_i to V'_i, composed with $\Gamma(V'_i; f^i_{\alpha\beta})$ gives $g^{\alpha\beta}_i$.

(ii) Let us first see that the $g^{\alpha\beta}_i$ extend to $i \notin \phi(Z_{\alpha\beta})$. In view of the definition of $\mathcal{A}$ (see 3.5), for $i \notin \phi(Z_{\alpha\beta}) \sqcup \phi(Z_{\Delta_\alpha\cup\Delta_\beta} \cap (\bigcup_{v\in\Delta'_\alpha\cap\Delta'_\beta} D_v))$, we have $\mathcal{A}^{\alpha\beta}_i = 0$ and $g^{\alpha\beta}_i$ is trivially defined. So we assume $i \in \phi(Z_{\Delta_\alpha\cup\Delta_\beta} \cap (\bigcup_{v\in\Delta'_\alpha\cap\Delta'_\beta} D_v)$. We let j be such that $V_i \setminus (\bigcup_{v\in\Delta'_\alpha\cap\Delta'_\beta} D_v) = V_j$.

Let us first extend (38) to the case

$$\mathcal{H}om(L'_\alpha, L'_\beta)|_{V'_i} \simeq (\mathcal{H}om(L_{\alpha,i}, L_{\beta,i}))_{U^i_{\alpha\beta}}, \tag{39}$$

where $U^i_{\alpha\beta} = (\overline{(T_\alpha \cap T_\beta)} \setminus \overline{(T_\beta \setminus T_\alpha)}) \cap V'_i$. By definition, $L'_\alpha|_{T_{D_v}} = 0$ for $v \in \Delta'_\alpha$, hence we have $L'_\alpha|_{V'_i} = (L'_\alpha)_{V'_j}$. Denoting by $u : V'_j \to V'_i$ the inclusion, we deduce, in view of (38):

$$\mathcal{H}om(L'_\alpha, L'_\beta)|_{V'_i} \simeq u_*(\mathcal{H}om(L'_\alpha, L'_\beta)|_{V'_j}) \simeq u_*(\mathcal{H}om(L_{\alpha,i}, L_{\beta,i})_{T^j_{\alpha\beta}}). \tag{40}$$

By Lemma 3.14 (iii), the inclusions $T^j_{\alpha\beta} \subset V'_j \subset V'_i$ are locally homeomorphic to inclusions of convex subsets of $\mathbf{R}^d$, and (39) follows.

Now we define $g^{\alpha\beta}_i$. Since $\mathcal{A}^{\alpha\beta}_i = \mathcal{A}^{\alpha\beta}_j$, we just have to check that $g^{\alpha\beta}_j$ factors through the restriction morphism $\mathcal{B}^{\alpha\beta}_i \to \mathcal{B}^{\alpha\beta}_j$. As in (i), formula (39) implies the existence of a morphism

$$f^i_{\alpha\beta} : \Omega^{\cdot}_{V'_i} \otimes \mathcal{H}om(L_{\alpha,i}, L_{\beta,i}) \to \Omega^{\cdot+2d_{\alpha\beta}}_{V'_i} \otimes \mathcal{H}om(L'_\alpha, L'_\beta).$$

We also note, by (40), that $\operatorname{supp}\mathcal{H}om(L'_\alpha, L'_\beta)|_{V'_i} \subset \overline{V'_j}$. Since $\overline{V'_j} \subset V_j$, we obtain

$$\mathcal{B}^{\alpha\beta}_i = \Gamma(V'_i \cap V_j; \Omega_Y \otimes \mathcal{H}om(L'_\alpha, L'_\beta)).$$

Hence $\Gamma(V'_i \cap V_j; f^i_{\alpha\beta})$ yields a morphism from

$$B'_{\alpha\beta j} = \Gamma(V'_i \cap V_j; \Omega_Y \otimes \mathcal{H}om(L_{\alpha,i}, L_{\beta,i}))\,[-2d_{\alpha\beta}]$$

to $\mathcal{B}^{\alpha\beta}_i$. Composed with the restriction morphism from $\mathcal{A}^{\alpha\beta}_j$ to $B'_{\alpha\beta j}$, it gives the required morphism $\mathcal{A}^{\alpha\beta}_j \to \mathcal{B}^{\alpha\beta}_i$.

Since $g^{\alpha\beta}_i$ is defined by factorising $g^{\alpha\beta}_j$, it is clear that the $g^{\alpha\beta}_{\cdot}$ commute with the restriction maps and define a morphism of sheaves, $g^{\alpha\beta} : \mathcal{A}^{\alpha\beta} \to \mathcal{B}^{\alpha\beta}$.

Now, let us check that $g^{\alpha\beta}$ is a quasi-isomorphism. We have to prove (with the Notations 3.4):

(a) for $i \in I_{\alpha\beta}$, $H^{\cdot}(g^{\alpha\beta}_i)$ is an isomorphism,
(b) for $i \in I'_{\alpha\beta}$ and $j \in I_{\alpha\beta}$ such that $V_i \setminus (\bigcup_{v\in\Delta'_\alpha\cap\Delta'_\beta} D_v) = V_j$, we have $H^{\cdot}(\mathcal{B}^{\alpha\beta}_i) \simeq H^{\cdot}(\mathcal{B}^{\alpha\beta}_j)$,
(c) for $i \notin I_{\alpha\beta} \sqcup I'_{\alpha\beta}$, $H^{\cdot}(\mathcal{B}^{\alpha\beta}_i) = 0$.

By the definition of $g^{\alpha\beta}_i$, (a) follows directly from quasi-isomorphism (31). Let us verify (b). Since $L_\alpha|_{D_v} = 0$ for $v \in \Delta'_\alpha$, we have $L_\alpha|_{V_i} = (L_\alpha)_{V_j}$. Hence, using (25), we have the required isomorphism:

$$H^{\cdot}(\mathcal{B}^{\alpha\beta}_i) \simeq \operatorname{Ext}^{\cdot}(L_\alpha|_{V_i}, L_\beta|_{V_i}) \simeq \operatorname{Ext}^{\cdot}(L_\alpha|_{V_j}, L_\beta|_{V_j}) \simeq H^{\cdot}(\mathcal{B}^{\alpha\beta}_j).$$

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Let us prove (c). We first note that Assumption 3.1 (ii) implies, for $F \in D^b(Y)$, constructible with respect to the stratification $Y = \bigsqcup_{i \in I} Y_i$, $\forall i \in I$, $H^\cdot(V_i; F) \simeq H^\cdot(Y_i; F)$. By (25), it follows that

$$\begin{aligned} H^\cdot(\mathcal{B}_i^{\alpha\beta}) &\simeq \mathrm{Ext}^\cdot(L_\alpha|_{V_i}, L_\beta|_{V_i}) \\ &\simeq H^\cdot(V_i; R\mathcal{H}om(L_\alpha, L_\beta)) \simeq H^\cdot(Y_i; R\mathcal{H}om(L_\alpha, L_\beta)|_{Y_i}). \end{aligned}$$

Now, we have either $i \notin \phi(Z_{\Delta_\alpha \cup \Delta_\beta})$ or there exists $v_0 \in \Delta'_{\alpha\beta} = (\Delta'_\alpha \cup \Delta'_\beta) \setminus (\Delta'_\alpha \cap \Delta'_\beta)$ such that $Y_i \subset D_{v_0}$. In the first case Y_i doesn't meet $\mathrm{supp}(R\mathcal{H}om(L_\alpha, L_\beta))$ and the vanishing of $H^\cdot(\mathcal{B}_i^{\alpha\beta})$ is clear. Let us assume we are in the second case. For $\alpha \in A$, we set $V_\alpha = Y \setminus \bigcup_{v \in \Delta'_\alpha} D_v$. By Lemma 2.6 we have $L_\alpha \simeq (L_\alpha)_{V_\alpha} \simeq \Gamma_{V_\alpha}(L_\alpha)$. Let us set $V = V_\alpha \cap V_\beta$ and let $j : V \to Y$ be the inclusion. We obtain:

$$\begin{aligned} R\mathcal{H}om(L_\alpha, L_\beta) &\simeq R\mathcal{H}om((L_\alpha)_{V_\alpha}, R\Gamma_{V_\beta}(L_\beta)) \\ &\simeq R\Gamma_V R\mathcal{H}om(L_\alpha, L_\beta) \simeq Rj_* R\mathcal{H}om(L_\alpha|_V, L_\beta|_V). \end{aligned}$$

Now $Z_\alpha \cap V$ and $Z_\beta \cap V$ are closed in V and their intersection $Z_{\alpha\beta} \cap V$ is empty or a smooth submanifold of $Z_\beta \cap V$ of codimension $2d_{\alpha\beta}$. Hence $R\mathcal{H}om(L_\alpha|_V, L_\beta|_V)$ is isomorphic to $K_{\alpha\beta}[2d_{\alpha\beta}]$, where $K_{\alpha\beta}$ is a local system on $Z_{\alpha\beta} \cap V$. By Assumption 3.1 (v), the monodromy of $K_{\alpha\beta}$ around D_{v_0} is $-Id$. By Lemma 2.6 we obtain $Rj_* K_{\alpha\beta}|_{D_{v_0}} = 0$ and, a fortiori $R\mathcal{H}om(L_\alpha, L_\beta)|_{Y_i} = 0$ and $H^\cdot(\mathcal{B}_i^{\alpha\beta}) = 0$. □

4. Symmetric varieties

We recall some results of [2] on regular compactifications of homogeneous symmetric varieties, in particular, the structure of the decomposition by the K-orbit types, for a suitable maximal compact subgroup K. Then we show that the hypothesis of Proposition 3.7 are satisfied.

Let G be a semi-simple algebraic group of adjoint type over $\mathbf{C}$, σ an automorphism of order 2 of G and $H = G^\sigma$. Let T be a σ-stable maximal torus of G containing a maximal σ-split torus S (i.e., $\forall t \in S$, $\sigma(t) = t^{-1}$). The corresponding root system $\Phi = \Phi(G, T)$ decomposes as $\Phi = \Phi_0 \sqcup \Phi_1$, where Φ_0 denotes the set of roots fixed by the action of σ. One may choose a basis of simple roots Σ such that σ exchanges the corresponding positive roots of Φ_1 with the negative roots of Φ_1. The non-fixed roots Φ_1 induce a root system on S with basis $\{\gamma_1, \dots, \gamma_l\}$ given by the restriction of $\Sigma \cap \Phi_1$. The corresponding Weyl group is denoted by W^1. We set $D = H \cap S$, the subgroup of S of elements of order 2, and $S' = S/D \simeq T/(T \cap H)$. The natural map $S \to S'$ gives an identification between the Lie algebras $Lie(S)$ and $Lie(S')$. Let H^0 be the identity component of H. By Proposition 1 of [12] (see also Proposition 7 of [17]), we have $H = D \cdot H^0$. In particular, the group of components of H is a quotient of D, hence of the type $H/H^0 \simeq (\mathbf{Z}/2\mathbf{Z})^a$, for some $a \in \mathbf{N}$.

Regular compactifications. Let X be the canonical compactification of G/H described in [6] and [7]. It can be defined as follows: let Gr_n be the Grassmann variety of n-dimensional subspaces of $Lie(G)$, for $n = \dim(H)$, with the G-action induced by the adjoint action. Let $x \in Gr_n$ be the point associated to $Lie(H)$. One can show that $G \cdot x \simeq G/H$ and that X is isomorphic to the closure of $G \cdot x$ in Gr_n. It is proved in [6] that X is smooth, $X \setminus (G/H)$ is the union of l smooth, normal crossings divisors, say D_i, $i = 1, \dots, l$, which are closures of G-orbits, and any G-orbit closure is the intersection of the D_i containing it. More precisely,

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the decomposition into G-orbits is identified with the decomposition of the toric variety $\mathbf{C}^l$ into orbits for the action of $(\mathbf{C}^*)^l$, as follows. The inclusion $S \subset G$ gives an embedding of S' in G/H, and hence in X. The closure of S' in X is a S'-toric variety, whose fan can be identified with the subdivision of $Lie(S')$ into Weyl chambers under the action of W^1. We consider the affine space $\mathbf{C}^l$ associated to the negative Weyl chamber. Then the G-orbits in X correspond bijectively (by taking the intersection with $\mathbf{C}^l$) to the S'-orbits in $\mathbf{C}^l$. In particular, there are 2^l-orbits and one single closed orbit. Let $\mathcal{O} \subset X$ be a G-orbit. Then its closure is fibred over a variety of parabolic subgroups of G, with fibre a symmetric variety. More precisely, there exist a parabolic subgroup $P \subset G$, and a G-equivariant fibration $\overline{\mathcal{O}} \to G/P$, whose fibre, say $X_{\mathcal{O}}$, has the following description. There exists a σ-stable Levi subgroup L of P, such that, denoting by L' the quotient $L/Z(L)$ ($Z(L)$ is the centre of L), $X_{\mathcal{O}}$ is the canonical compactification of L'/L'^{σ}.

In [7] there is a description of the embeddings of G/H over X. If Y is such an embedding, with a map $\pi : Y \to X$, the closure of S' in Y gives a toric variety, say Z', and $Z = Z' \cap \pi^{-1}(\mathbf{C}^l)$ is a toric variety over $\mathbf{C}^l$. It is shown in [7] that this gives a bijection between the embeddings of G/H over X and the toric varieties over $\mathbf{C}^l$. For $Z \to \mathbf{C}^l$ a morphism of toric varieties, let $\pi : X_Z \to X$ be the corresponding embedding of G/H. Then $Z = \pi^{-1}(\mathbf{C}^l)$ and the G-orbits in X_Z correspond bijectively (by taking the intersection with Z) to the S'-orbits in Z. Moreover, X_Z is smooth if, and only if, Z is smooth and X_Z is complete if, and only if, $Z \to \mathbf{C}^l$ is proper. From now on we assume that X_Z is smooth and complete. These are the symmetric varieties we consider here.

Notations 4.1. We denote by V the set of irreducible G-stable divisors, D_v, $v \in V$. Any G-orbit closure is the intersection of the D_v containing it. For $\Delta \subset V$ such that $\bigcap_{v\in\Delta} D_v \neq \emptyset$, we let $\mathcal{O}_\Delta$ be the G-orbit such that $\overline{\mathcal{O}_\Delta} = \bigcap_{v\in\Delta} D_v$. We denote by $\mathcal{S}$ the set of G-orbits; hence $\mathcal{S}$ is identified with a set of subsets of V.

Fundamental domain. In [2], we also have a description of the orbits of a suitable maximal compact subgroup of G. Let K be a compact form of G such that the Cartan involution of G corresponding to K commutes with σ and such that $K \cap T$ is a maximal compact subgroup of T. We set $S^c = K \cap S$. Let $t_i = t^{-2\gamma_i}$, $i = 1, \dots, l$, be the characters of S' associated to the simple roots γ_i, extended to a coordinates system on $\mathbf{C}^l$. We set $C =]0,1]^l \subset S' \subset G/H$. We consider the closure of C in X_Z, $C_{X_Z} = \overline{C}$. In particular, $C_X = [0,1]^l$. Note that $(\pi|_Z)^{-1}([0,1]^l)$ is closed and contains C, so that C_{X_Z} is in fact contained in Z. Hence C_{X_Z} is mapped to C_X by π. This is a fundamental domain for the action of K on X_Z (see [2], Theorem 27).

Stabilisers. The stabiliser in K of a point of C_{X_Z} is described in [2], p. 27, as follows. For a point $q \in C_X = [0,1]^l$, we call its J-support the subset of $\{1, \dots, l\}$ defined by $J(q) = \{i;\ q_i \neq 1\}$. Let $\tau : S \to S'$ denote the quotient map. For $q \in]0,1]^l \subset S'$, let $\tilde{q} \in \tau^{-1}(q)$ be in the connected component of $\tau^{-1}(]0,1]^l)$ containing 1. Then the centraliser of $\tilde{q}$ in G only depends on $J = J(q)$; we set:

$$K_J = K^\sigma \cap Z_G(\tilde{q}).$$

For $J \subset \{1, \dots, l\}$, we denote by $C_{X_Z,J}$, or simply C_J if there is no ambiguity, the subset of C_{X_Z} formed by the p such that $J(\pi(p)) = J$. By definition this decomposition of C_{X_Z} arises from the decomposition of $C_X = [0,1]^l$ according to the set of coordinates equal to 1: $C_{X_Z,J} = (\pi|_{C_{X_Z}})^{-1}(C_{X,J})$.

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We also consider the partition of C_{X_Z} given by the G-orbits: $C_{X_Z} = \bigsqcup_{\Delta \in \mathcal{S}} \mathcal{O}_\Delta \cap C_{X_Z}$. All points $p \in \mathcal{O}_\Delta \cap C_{X_Z}$ have the same stabiliser in S^c. Indeed $C_{X_Z} \subset Z$, and $\mathcal{O}_\Delta \cap Z$ is a single S'-orbit of the toric variety Z, so that all points of $\mathcal{O}_\Delta \cap Z$ have the same stabiliser in S'. We set $S^c_\Delta = S^c_p$ for any $p \in \mathcal{O}_\Delta \cap C_{X_Z}$.

Now we mix the two partitions above and set, for $\Delta \in \mathcal{S}$ and $J \subset \{1, \ldots, l\}$:

$$F_{\Delta,J} = \mathcal{O}_\Delta \cap C_J.$$

The $F_{\Delta,J}$ are called the "faces" of C_{X_Z} (if X_Z is projective, the moment map for the K-action identifies C_{X_Z} with a polytope, and the "faces" are the usual faces of this polytope; see [2], p. 25). We have the following description of the stabilisers:

Theorem 4.2 (Theorem 32 and Corollary 35 of [2])**.** *For $p \in C_{X_Z}$ let $F_{\Delta,J}$ be the face containing p, and let K_p be the stabiliser of p in K. Since $\mathcal{O}_\Delta$ is K-stable, K_p acts on $N_p = T_p X_Z / T_p \mathcal{O}_\Delta$. We have*

$$K_J = \ker(K_p \to GL(N_p)), \tag{41}$$

$$K_p = S^c_\Delta \cdot K_J, \qquad S^c_\Delta \cap K_J = D. \tag{42}$$

In particular, K_p only depends on the face to which p belongs. For a face F, we set $K_F = K_p$, for any $p \in F$. Let us make some remarks on the decomposition of C_{X_Z} into faces.

For $X_Z = X$, we have $Z = \mathbf{C}^l$, $C_X = [0,1]^l$. We may identify V with $\{1, \ldots, l\}$ so that, for $\Delta \subset V$, $\mathcal{O}_\Delta \cap \mathbf{C}^l = \{(t_1, \ldots, t_l);\ t_i = 0 \text{ iff } i \in \Delta\}$. Hence:

$$\text{for } X_Z = X: \quad F_{\Delta,J} = \{(t_1, \ldots, t_l) \in [0,1]^l;\ t_i = 0 \text{ iff } i \in \Delta \text{ and } t_i = 1 \text{ iff } i \notin J\}.$$

The image of a G-orbit of X_Z, say $\mathcal{O}_\Delta$, $\Delta \subset V$, by π is a G-orbit of X, say $\mathcal{O}_\Sigma$, $\Sigma \subset \{1, \ldots, l\}$. The restriction $\pi|_{\mathcal{O}_\Delta} : \mathcal{O}_\Delta \to \mathcal{O}_\Sigma$ is a fibration, and gives a bijection between the faces $F_{\Delta,J}$ of $C_{X_Z} \cap \mathcal{O}_\Delta$ and the faces $F_{\Sigma,J}$ of $C_X \cap \mathcal{O}_\Sigma$. In particular, $C_{X_Z} \cap \mathcal{O}_\Delta$ has a unique closed face, $F_{\Delta,\Sigma}$, and a unique open face, $F_{\Delta,\{1,\ldots,l\}}$. We also deduce that $F_{\Delta,J} \subset \overline{F_{\Delta,J'}}$ if and only if $J \subset J'$.

We have similarly $F_{\Delta,J} \subset \overline{F_{\Delta',J}}$ if and only if $\Delta' \subset \Delta$. Indeed, if $F_{\Delta,J} \subset \overline{F_{\Delta',J}}$, then certainly $\mathcal{O}_\Delta \cap \overline{\mathcal{O}_{\Delta'}} \neq \emptyset$ and hence $\Delta' \subset \Delta$. Conversely, assume $\Delta' \subset \Delta$ and let $p \in F_{\Delta,J}$. Since $p \in \overline{\mathcal{O}_{\Delta'}}$, we may write $p = \lim_n p_n$, with $p_n \in \mathcal{O}_{\Delta'}$. Now each p_n is itself a limit of points of $\mathcal{O}_\emptyset$, which is identified with $(\mathbf{C}^*)^l$ by π. Let us write $p_n = \lim_i q^n_i$, with $q^n_i = (t^n_{1,i}, \ldots, t^n_{l,i})$. Since $p \in C_J$, for any $\varepsilon > 0$, there exists a neighbourhood U of p, such that $q^n_i \in U$ implies $|t^n_{j,i} - 1| < \varepsilon$, $\forall j \notin J$. Hence, up to restricting to a subsequence, we may assume that, for each n great enough, $\forall j \notin J$, $\lim_i t^n_{j,i} = s^n_j$ for some s^n_j, with $|s^n_j - 1| \leq \varepsilon$. We set $s^n_j = 1$ for $j \in J$. Then, for the element $g_n = (\underline{s}^n) \in (\mathbf{C}^*)^l$, the point $p'_n = g_n^{-1} \cdot p_n = \lim_i g_n^{-1} \cdot q^n_i$ is in $\mathcal{O}_{\Delta'} \cap \mathbf{C}_J$. The p'_n converge to p, and we have $p \in \overline{F_{\Delta',J}}$, as required.

Combining both characterisations for the inclusions of closures of faces, we have: $F_{\Delta,J} \subset \overline{F_{\Delta',J'}}$ is equivalent to $\Delta' \subset \Delta$ and $J \subset J'$. Now we summarise the notations and properties introduced so far and add some others.

Properties 4.3. For $\Delta \in \mathcal{S}$, $J \subset \{1, \ldots, l\}$, we have $F_{\Delta,J} = \mathcal{O}_\Delta \cap C_{X_Z,J}$. We let $\mathcal{F}$ denote the set of faces, $\mathcal{F} = \{F_{\Delta,J};\ F_{\Delta,J} \neq \emptyset\}$. We let $\varphi : C_{X_Z} \to \mathcal{F}$ be the natural map induced by this partition and endow $\mathcal{F}$ with the quotient topology. For $\Delta \in \mathcal{S}$, $\mathcal{O}_\Delta \cap C_{X_Z}$ has a unique closed face, which we denote by F_{Δ,J_Δ}, $J_\Delta \subset \{1, \ldots, l\}$. As in section 2.1, for a face F of C_{X_Z}, we let U_F be the smallest open subset of $\mathcal{F}$

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containing F. We set $U'_F = \varphi^{-1}(U_F)$; this is an open subset of C_{X_Z} which contains F as its unique closed face.

$$K_{F_{\Delta,J}} = S^c_\Delta \cdot K_J, \tag{43}$$

$$\Delta' \subset \Delta \implies S^c_{\Delta'} \subset S^c_\Delta, \qquad J \subset J' \implies K_{J'} \subset K_J, \tag{44}$$

$$F_{\Delta,J} \subset \overline{F_{\Delta',J'}} \qquad \iff \qquad \Delta' \subset \Delta \quad \text{and} \quad J \subset J', \tag{45}$$

$$U_F = \{F' \in \mathcal{F};\ F \subset \overline{F'}\}, \qquad U'_F = \textstyle\bigsqcup_{F \subset \overline{F'}} F', \tag{46}$$

$$U'_{F_{\Delta,J}} \ = \textstyle\bigsqcup_{\Delta' \subset \Delta, J' \supset J} F_{\Delta',J'} \ = \ (\bigsqcup_{\Delta' \subset \Delta} \mathcal{O}_{\Delta'}) \cap (\bigsqcup_{J' \supset J} C_{J'}), \tag{47}$$

$$U_{F_{\Delta,J}} \cap U_{F_{\Delta',J'}} = U_{F_{\Delta \cap \Delta', J \cup J'}}. \tag{48}$$

The first equality in (47) follows from (45). The second follows directly by applying the definition of the faces, and it implies (48).

4.1. **Stratification by the faces.**

Lemma 4.4. *We keep the notations introduced above. Let $X'_Z = X_Z/K$ be the topological quotient, $p_Z : X_Z \to X'_Z$ the quotient map and $q_Z = p_Z|_{C_{X_Z}}$. We consider on C_{X_Z} the topology induced by its inclusion in X_Z. We have:*

(i) *the map q_Z is a homeomorphism,*

(ii) *the partition of C_{X_Z} by the faces, $C_{X_Z} = \bigsqcup_{F \in \mathcal{F}} F$, satisfies: if $F \cap \overline{F'} \neq \emptyset$, then $F \subset \overline{F'}$,*

(iii) *the induced partition of X_Z, $X_Z = \bigsqcup_{F \in \mathcal{F}} K \cdot F$, is a μ-stratification; it satisfies the same inclusions relations for the closures of strata as the partition of C_{X_Z} by the faces.*

Proof. (i) Since C_{X_Z} is a fundamental domain for the K-action, q_Z is bijective. It is continuous by definition. For an open subset $U \subset C_{X_Z}$, $C_{X_Z} \setminus U$ is compact because C_{X_Z} is. Hence $q_Z(C_{X_Z} \setminus U)$ is compact too, and $q_Z(U)$ is open. This proves that q_Z^{-1} is continuous too.

(ii) Let $F_{\Delta,J}$, $F_{\Delta',J'}$ be two faces such that $F_{\Delta,J} \cap \overline{F_{\Delta',J'}} \neq \emptyset$. By definition of the faces, this implies $\mathcal{O}_\Delta \cap \overline{\mathcal{O}_{\Delta'}} \neq \emptyset$ and $C_J \cap \overline{C_{J'}} \neq \emptyset$. Hence $\Delta' \subset \Delta$ and $J \subset J'$; we conclude by (45).

(iii) By (i), we have for any subset $C \subset C_{X_Z}$:

$$\overline{K \cdot C} = p_Z^{-1}(\overline{p_Z(K \cdot C)}) = p_Z^{-1}(\overline{q_Z(C)}) = p_Z^{-1}(q_Z(\overline{C})) = K \cdot \overline{C}.$$

This implies that our partition of X_Z is a stratification and the last assertion. Let us verify the "μ-condition" for two strata $K \cdot F$, $K \cdot F'$, with $F \subset \overline{F'}$ (see section 2.3). This condition is local around a point of $K \cdot F$ and we may restrict ourselves to $K \cdot U'_F$. In $K \cdot U'_F$, our stratification coincides, by (43), with the partition by the K-orbit types. By the existence of slices for action of compact groups this partition is locally trivial: a point $x \in K \cdot F$ has a neighbourhood of the type $\mathbf{R}^d \times E_\lambda$, where E_λ is a representation of K_x with 0 as unique fixed point, and the partition is induced by the partition of E_λ in K_x-orbit types. In E_λ, the μ-condition for the strata $\{0\}$ and $(K \cdot F') \cap E_\lambda$ is trivially satisfied; hence it holds for $K \cdot F$ and $K \cdot F'$. □

Notations 4.5. We stratify X_Z as in the above lemma. We denote by $\phi : X_Z \to \mathcal{F}$ the continuous map defined by $\phi(K \cdot F) = F$, for $F \in \mathcal{F}$. We set $V_F = K \cdot U'_F = \phi^{-1}(U_F)$. From now on we denote by E a universal bundle for K which is an increasing union of manifolds, $E = \bigcup_k E_k$. Since $\pi_1(E) = 1$, for any subgroup

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$H \subset K$ we have $\pi_1(E/H) \simeq H/H^0$. We let $\psi : E \times_K X_Z \to \mathcal{F}$ be the map induced by ϕ.

We are in the setting of Assumptions 3.1, with $Y = X_Z$ in a K-equivariant way, or $Y = E \times_K X_Z$ (see Remark 3.8), $I = \mathcal{F}$, and a set of normal crossings divisors $D_v = \overline{\mathcal{O}_{\{v\}}}$, $v \in V$ (we will introduce local systems L_α in the next paragraph). Let us verify conditions (i)–(iii). The first condition follows directly from (48). By (47), we have, for $\Delta \in \mathcal{S}$ and $v \in \Delta$,

$$\begin{aligned} U'_{F_{\Delta,J}} \setminus D_v &= ((\textstyle\bigsqcup_{\Delta' \subset \Delta} \mathcal{O}_{\Delta'}) \setminus D_v) \cap (\textstyle\bigsqcup_{J' \supset J} C_{J'}) \\ &= (\textstyle\bigsqcup_{v \notin \Delta' \subset \Delta} \mathcal{O}_{\Delta'}) \cap (\textstyle\bigsqcup_{J' \supset J} C_{J'}) = U'_{F_{\Delta \setminus \{v\},J}}, \end{aligned} \tag{49}$$

and this gives condition (iii), the case $v \notin \Delta$ being trivial. Condition (ii) of Assumptions 3.1 follows from the next lemma.

Lemma 4.6. *Let F be a face.*

(i) There exists a homotopy $h : [0,1] \times U'_F \to U'_F$, such that $h_1 = id$ and h_0 is the projection of U'_F to a point of F, with the property that the closures of the faces of U'_F are stable under h_t, $\forall t \in [0,1]$.

(ii) It induces a K-equivariant homotopy $\bar{h} : [0,1] \times V_F \to V_F$ contracting V_F to a K-orbit $K/K_F \subset K \cdot F$ and preserving the closures of strata.

Proof. (i) By definition, the faces of C_{X_Z} are contained in S'-orbits of Z. Let Y be the S'-orbit containing F and let U be the union of the S'-orbits whose closure contains Y. Then U'_F is included in U. There exists a one-dimensional subtorus $i : \mathbf{C}^* \hookrightarrow S'$ of S' contracting U to Y, i.e., $\forall y \in U \lim_{t\to 0} i(t) \cdot y \in Y$. Note that for $t \in]0,1]$, $x \in U \cap C_{X_Z}$, $i(t) \cdot x$ still is in $U \cap C_{X_Z}$ and belongs to the same face as x. Hence the homotopy $h_1 : [0,1] \times U'_F \to U'_F$, $(t,x) \mapsto i(t) \cdot x$ if $t \neq 0$, $(0,x) \mapsto \lim_{t\to 0} i(t) \cdot x$, contracts U'_F to $U'_F \cap Y$ and preserves the closure of the faces.

Now Y is fibred over $\pi(Y)$ with fibre $(\mathbf{C}^*)^k$, a quotient of S'. It follows that $C_{X_Z} \cap Y$ is fibred over $C_X \cap \pi(Y)$, with fibre W, where W is a subset of $\mathbf{R}^k_{>0}$ stable by multiplication by $]0,1]^k$. In particular, W is contractible. Moreover, this fibration is a bijection on the faces. Hence there exist sections $s : C_X \cap \pi(Y) \to C_{X_Z} \cap Y$ of $\pi|_{C_{X_Z} \cap Y}$, and for any such s, we may find a homotopy contracting $C_{X_Z} \cap Y$ to $\mathrm{im}(s)$, which preserves the fibres and thus the faces.

Finally, $\mathrm{im}(s) \simeq C_X \cap \pi(Y)$ is a union of standard faces in $[0,1]^l$, isomorphic to $]0,1]^m \times \{0\}^{l-m}$, up to a permutation of coordinates. Then $\mathrm{im}(s) \cap F \simeq$ $]0,1[^n \times \{1\}^{m-n} \times \{0\}^{l-m}$ and $\mathrm{im}(s) \cap U'_F \simeq]0,1[^n \times]0,1]^{m-n} \times \{0\}^{l-m}$. Hence there exists a third homotopy contracting $\mathrm{im}(s) \cap U'_F$ to a point of F also preserving the closure of the faces.

(ii) By (44) and (45), for any faces F_1, F_2 with $F_1 \subset \overline{F_2}$, we have $K_{F_2} \subset K_{F_1}$. Since $\forall F' \in U_F$, $h([0,1] \times F') \subset \overline{F'}$, we obtain $\forall (t,x) \in [0,1] \times U'_F$, $K_x \subset K_{h(t,x)}$. Hence it makes sense to define $\bar{h} : [0,1] \times V_F \to V_F$, $(t, k \cdot x) \mapsto k \cdot h(t,x)$, for $t \in [0,1]$, $k \in K$, $x \in U'_F$. It is clear that $\bar{h}_0 = id$ and $\bar{h}_1$ is a K-equivariant projection to K/K_F. Let us verify that $\bar{h}$ is continuous.

Let (t_n, y_n) be any sequence in $[0,1] \times V_F$ converging to (t,y). Let us see that a subsequence of $\bar{h}(t_n, y_n)$ converges to $\bar{h}(t,y)$. For each n there exists a unique $x_n \in U'_F$ such that $y_n \in K \cdot x_n$. With the notations of Lemma 4.4, we have $x_n = q_Z^{-1}(p_Z(y_n))$. Since these maps are continuous, the sequence x_n converges to $x = q_Z^{-1}(p_Z(y))$. Let us write $y_n = k_n \cdot x_n$, $k_n \in K$. Since K is compact, we

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may assume, up to restriction to a subsequence, that k_n converges to $k \in K$. Then $\bar{h}(t_n, y_n) = k_n \cdot h(t_n, x_n)$ converges to $\bar{h}(t, y) = k \cdot h(t, x)$, as desired. □

Let us give some immediate consequences of this lemma. We consider the map $p_F : V_F \to K/K_F$, $k \cdot x \mapsto \overline{k}$, where $k \in K$, $x \in U'_F$. This makes sense because $\forall x \in U'_F$, $K_x \subset K_F$ (recall that $U'_F = \bigsqcup_{F \subset \overline{F'}} F'$). With the notations of Lemma 4.6, let $x_0 \in F$, be the point of F such that $h(1, U'_F) = \{x_0\}$. By definition, we have in fact $p_F = \bar{h}(1, \cdot)$ modulo the identification $K \cdot x_0 = K/K_F$. In particular, p_F is continuous. Since it is K-equivariant, it is a fibration over K/K_F, with fibre $K_F \cdot U'_F$. The homotopy $\bar{h}$ also is K-equivariant and hence contracts each fibre $p_F^{-1}(y)$, $y \in K/K_F$ to the point y. We let

$$q_F : E \times_K V_F \to E/K_F \tag{50}$$

be the map induced by p_F. This also is a fibration with contractible fibres.

For a G-orbit $\mathcal{O}_\Delta$, with closed face F_{Δ, J_Δ}, and a face $F' = F_{\Delta, J'}$ such that $F' \subset \mathcal{O}_\Delta$ (i.e., $J_\Delta \subset J'$), $\mathcal{O}_\Delta \cap V_{F'}$ is closed in $V_{F'}$: indeed if a face F_{Δ_1, J_1} is included in $\overline{\mathcal{O}_\Delta}$ it satisfies $\Delta \subset \Delta_1$, and if it is in $V_{F'}$ it satisfies $\Delta_1 \subset \Delta$; hence any face of $\overline{\mathcal{O}_\Delta} \cap V_{F'}$ is in $\mathcal{O}_\Delta$. Since the homotopy h of Lemma 4.6 preserves the closures of faces, it follows that h contracts $\mathcal{O}_\Delta \cap V_{F'}$ to K/K_F. Hence the maps

$$E \times_K (\mathcal{O}_\Delta \cap V_{F'}) \to E \times_K V_{F'} \to E/K_{F'} \tag{51}$$

are homotopy equivalences. In particular, the fundamental groups are the same:

$$\pi_1(E \times_K (\mathcal{O}_\Delta \cap V_{F'})) = \pi_1(E/K_{F'}) = K_{F'}/K^0_{F'}. \tag{52}$$

4.2. **Equivariant local systems on X_Z.** Remember that G-equivariant local systems on a homogeneous variety G/G' are in bijective correspondence with representations of the components group G'/G'^0 of G'. For a G-orbit $\mathcal{O}$, we denote by $\tau_{\mathcal{O}}$ the group of components of a stabiliser, $\tau_{\mathcal{O}} = G_p/G_p^0$, for $p \in \mathcal{O}$. We introduce similar notations for the groups of components of the groups defined up to now:

$$S^{c0}_\Delta = (S^c_\Delta)^0, \qquad D_\Delta = D \cap S^{c0}_\Delta,$$
$$\tau_J = K_J/K_J^0, \qquad \tau_\Delta = S^c_\Delta/S^{c0}_\Delta = D/D_\Delta, \qquad \tau_F = K_F/K_F^0.$$

For a G-orbit $\mathcal{O}_\Delta$, remember that F_{Δ, J_Δ} denotes the unique closed face of $\mathcal{O}_\Delta \cap C_{X_Z}$. For $p \in F = F_{\Delta, J_\Delta}$, K_p is a maximal compact subgroup of G_p (see [2], p. 31), hence $\tau_{\mathcal{O}} \simeq \tau_F$. Let us remark that $\tau_J \simeq (\mathbf{Z}/2\mathbf{Z})^a$, for some $a \in \mathbf{N}$. Indeed, by (42), we have $K_F = S^c_\Delta \cdot K_J$, but K_F is connected and S^c_Δ and K_J are compact, so that we have as well $K_F = S^{c0}_\Delta \cdot K_J^0$. Hence, for $k \in K_J$, there exist $s \in S^{c0}_\Delta$, $k' \in K_J^0$ such that $k = sk'$. Applying σ we find $k = s^{-1}k'$, so that $s = s^{-1}$ and $s \in D$. Thus we have proved that $K_J = D \cdot K_J^0$, and the claim follows.

Let us fix a face $F = F_{\Delta, J}$. By (41), K_J is a normal subgroup of K_F. In fact, the identity component S^{c0}_Δ of S^c_Δ centralises K_J. Indeed, $\forall s \in S^c_\Delta$, $\forall k \in K_J$, we have $k' = s\,k\,s^{-1} \in K_J$. Since $K_J \subset K^\sigma$, we deduce $k' = \sigma(k') = s^{-1}\,k\,s$ and then $k = s\,k'\,s^{-1} = s^2\,k\,s^{-2}$. Since any element of the torus S^{c0}_Δ is a square, our claim follows. Then (42) gives the exact sequences (53), (in which $S^{c0}_\Delta \times K_J$ is a direct product). We deduce the exact sequences (54).

$$1 \to D \to S^c_\Delta \ltimes K_J \to K_F \to 1, \qquad 1 \to D_\Delta \to S^{c0}_\Delta \times K_J \to K_F \to 1, \tag{53}$$

$$D \to \tau_\Delta \ltimes \tau_J \to \tau_F \to 1, \qquad D_\Delta \to \tau_J \to \tau_F \to 1. \tag{54}$$

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In particular, the groups τ_F (and then $\tau_{\mathcal{O}}$) are quotients of the τ_J, hence of the type $(\mathbf{Z}/2\mathbf{Z})^a$, for some $a \in \mathbf{N}$.

Let us consider more precisely the group τ_Δ. Remember that S^c_Δ is the stabiliser of a point $p \in C_{X_Z} \subset Z$ in $S^c = K \cap S$, which is the maximal compact subgroup of S (hence connected). Moreover, $D = \{t \in S;\ t^2 = 1\} \simeq (\mathbf{Z}/2\mathbf{Z})^l$, S acts on Z via $S' = S/D$ and Z is toric for S', so that S'_p is connected. Let $S'^c = S^c/D$ be the maximal compact subgroup of S'. Then S'^c_p is connected too and the exact sequences

$$1 \to D \to S^c_p \to S'^c_p \to 1, \qquad 1 \to (\mathbf{Z}/2\mathbf{Z})^{l'} \to S^{c0}_p \to S'^c_p \to 1,$$

where $l' = \dim S^{c0}_p = |\Delta|$, show that $\tau_\Delta \simeq (\mathbf{Z}/2\mathbf{Z})^{l-|\Delta|}$ and $D_\Delta \simeq (\mathbf{Z}/2\mathbf{Z})^{|\Delta|}$. For $\Delta_1 \in \mathcal{S}$, such that $\Delta \subset \Delta_1$, we have $S^c_\Delta \subset S^c_{\Delta_1}$ and a natural morphism $\tau_\Delta \to \tau_{\Delta_1}$ which is surjective with kernel $(\mathbf{Z}/2\mathbf{Z})^{|\Delta_1|-|\Delta|}$.

Lemma 4.7. *Let $\mathcal{O}_\Delta$ be a G-orbit of X_Z and L_ρ a G-equivariant local system on $\mathcal{O}_\Delta$, corresponding to a representation $\rho : \tau_{\mathcal{O}_\Delta} \to GL(V_\rho)$ of $\tau_{\mathcal{O}_\Delta}$. We assume that L_ρ is irreducible. We set $F = F_{\Delta,J_\Delta}$ and let $i_F : \tau_\Delta \to \tau_F = \tau_{\mathcal{O}_\Delta}$ be the morphism induced by* (54)*. For $v \in V \setminus \Delta$ such that $\Delta_1 = \Delta \sqcup \{v\} \in \mathcal{S}$, we have* $\ker(\tau_\Delta \to \tau_{\Delta_1}) \simeq \mathbf{Z}/2\mathbf{Z}$*. We let $s_v \in \tau_\Delta$ be the generator of this kernel.*

Then $\rho(i_F(s_v)) = \pm Id_{V_\rho}$ and this is the monodromy of L_ρ around D_v. If $\rho(i_F(s_v)) = Id_{V_\rho}$, then ρ induces a representation, ρ_1, of $\tau_{\mathcal{O}_{\Delta_1}}$ and L_ρ extends to a local system, L_1, on $\mathcal{O}_\Delta \sqcup \mathcal{O}_{\Delta_1}$, such that $L_1|_{\mathcal{O}_{\Delta_1}}$ corresponds to ρ_1.

Proof. Recall that the variety X_Z comes with a morphism $\pi : X_Z \to X$. The G-orbits of X are parameterised by subsets of $\{1,\dots,l\}$ and $J_\Delta \subset \{1,\dots,l\}$ is determined by $\pi(\mathcal{O}_\Delta) = \mathcal{O}_{J_\Delta}$. The hypothesis gives $\pi(\mathcal{O}_{\Delta_1}) \subset \overline{\pi(\mathcal{O}_\Delta)}$, so that $J_\Delta \subset J_{\Delta_1}$. We set $F_1 = F_{\Delta_1,J_{\Delta_1}}$. By (48) we deduce that $U_{F_1} \cap U_F = U_{F_2}$, where $F_2 = F_{\Delta,J_{\Delta_1}}$. We obtain the following commutative diagram:

$$\begin{array}{ccccc}
E \times_K V_{F_{\Delta_1,J_{\Delta_1}}} & \hookleftarrow & E \times_K V_{F_{\Delta,J_{\Delta_1}}} & \hookrightarrow & E \times_K V_{F_{\Delta,J_\Delta}} \\
\cup & & \cup & & \cup \\
E \times_K \mathcal{O}_{\Delta_1} & \overset{i}{\hookleftarrow} & E \times_K (\mathcal{O}_\Delta \cap V_{F_{\Delta,J_{\Delta_1}}}) & \hookrightarrow & E \times_K \mathcal{O}_\Delta \\
\downarrow & & \downarrow & & \downarrow \\
E/K_{F_{\Delta_1,J_{\Delta_1}}} & \twoheadleftarrow & E/K_{F_{\Delta,J_{\Delta_1}}} & \twoheadrightarrow & E/K_{F_{\Delta,J_\Delta}}\,,
\end{array}$$

where the vertical arrows are homotopy equivalences, by (51). Let us consider a "small" loop γ in $E \times_K \mathcal{O}_\Delta$ around $E \times_K \mathcal{O}_{\Delta_1}$, as in section 2.4. Since γ is small, we may assume that it is included in the neighbourhood $E \times_K V_{F_1}$ of $E \times_K \mathcal{O}_{\Delta_1}$. Since γ doesn't meet $E \times_K D_v$ and $V_{F_1} \setminus D_v = V_{F_2}$, by (49), γ is in fact contained in $E \times_K V_{F_2}$. Hence it represents a generator of the kernel of $\pi_1(i)$, the map induced on the fundamental groups by the inclusion i of the above diagram.

Let $j : \tau_{F_2} \to \tau_{F_1}$ be the morphism induced by $\tau_\Delta \to \tau_{\Delta_1}$. By (52), $j = \pi_1(i)$ and by (54) we have the commutative diagram:

$$\begin{array}{ccccccc}
D & \longrightarrow & \tau_\Delta \ltimes \tau_{J_{\Delta_1}} & \longrightarrow & \tau_{F_2} & \longrightarrow & 1 \\
\| & & \downarrow & & \downarrow{\scriptstyle j} & & \| \\
D & \longrightarrow & \tau_{\Delta_1} \ltimes \tau_{J_{\Delta_1}} & \longrightarrow & \tau_{F_1} & \longrightarrow & 1\,.
\end{array}$$

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We have already seen that $\ker(\tau_\Delta \to \tau_{\Delta_1}) \simeq (\mathbf{Z}/2\mathbf{Z})^{|\Delta_1|-|\Delta|} \simeq \mathbf{Z}/2\mathbf{Z}$. The above diagram implies that $\ker(j)$ is 0 or $\mathbf{Z}/2\mathbf{Z}$ and is generated by $i_F(s_v)$. Since $s_v^2 = 1$ and V_ρ is irreducible it follows that $\rho(i_F(s_v)) = \pm Id_{V_\rho}$.

We consider L_ρ as well as a G-equivariant local system on $\mathcal{O}_\Delta$ or as a local system on $E \times_K \mathcal{O}_\Delta$. Then $L_\rho|_{E\times_K V_{F_2}}$ corresponds to the representation ρ_2 of τ_{F_2} given by j and ρ. The representation ρ_2 gives a representation ρ_1 of $\tau_{F_1} = \tau_{\mathcal{O}_{\Delta_1}}$ if and only if it sends $\ker(j)$ to Id_{V_ρ}, i.e., $\rho(i_F(s_v)) = Id_{V_\rho}$. This also is equivalent to the fact that L_ρ extends to $\mathcal{O}_\Delta \sqcup \mathcal{O}_{\Delta_1}$, with a restriction to $\mathcal{O}_{\Delta_1}$ corresponding to ρ_1. □

Definition 4.8. We let A be the set of pairs $\alpha = (\mathcal{O}, \rho)$, where $\mathcal{O}$ is a G-orbit and $\rho : \tau_\mathcal{O} \to GL(V_\rho)$ an irreducible representation of $\tau_\mathcal{O}$. For $\alpha = (\mathcal{O}, \rho) \in A$, we let $\Delta_\alpha \in \mathcal{S}$ be such that $\mathcal{O} = \mathcal{O}_{\Delta_\alpha}$ and, with the notations of the previous lemma, we set

$$\Delta'_\alpha = \{v \in V \setminus \Delta_\alpha;\ \Delta_\alpha \sqcup \{v\} \in \mathcal{S} \text{ and } \rho(i_F(s_v)) = -Id\}, \quad Z_\alpha = \overline{\mathcal{O}} \setminus \bigcup_{v\in\Delta'_\alpha} D_v.$$

By the lemma, the G-equivariant local system on $\mathcal{O}$ corresponding to ρ extends to a G-equivariant local system L^0_α on Z_α. We extend it by 0 outside Z_α (keeping the notation L^0_α also for the extension). We denote by L_α the corresponding local system on $E \times_K X_Z$.

4.3. **dg-algebras on the set of faces.** By section 2.1 we have the equivalences of categories:

$$\mathrm{D}^b_{G,c}(X_Z) \simeq \mathrm{D}_G(X_Z)\langle L^0_\alpha,\, \alpha\in A\rangle \simeq \mathrm{D}(E\times_K X_Z)\langle L_\alpha,\, \alpha \in A\rangle.$$

We are in the situation of Assumptions 3.1 with $Y = E\times_K X_Z$, stratified by the set of faces $\mathcal{F}$, the subspaces $E\times_K D_v$ and the local systems L_α, $\alpha \in A$. Conditions (i)–(iii) were verified in section 4.1 and (iv), (v) follow from Definition 4.8. Hence we may apply Proposition 3.7: the category $\mathrm{D}_G(X_Z)\langle L^0_\alpha\rangle$ is equivalent to a category of dg-modules over $\mathcal{F}$, $\mathrm{D}_\mathcal{R}\langle N_\alpha\rangle$, where $\mathcal{R}$ is a sheaf of dg-algebras on $\mathcal{F}$, whose description is recalled below, and the N_α are $\mathcal{R}$-modules.

First, for $\alpha = (\mathcal{O}, \rho), \beta = (\mathcal{O}', \rho') \in A$, we define a sheaf $\mathcal{R}^{\alpha\beta}$ on $\mathcal{F}$ by its stalks at any face F. We have

$$\phi(\overline{\mathcal{O}}) = \{F_{\Delta,J} \in \mathcal{F};\ \Delta_\alpha \subset \Delta\} \quad \text{and} \quad \phi(Z_\alpha) = \{F_{\Delta,J}\in\mathcal{F};\ \Delta_\alpha \subset \Delta \subset (V\setminus \Delta'_\alpha)\}.$$

We recall the Notations 3.4 (with $\mathcal{F}_{\alpha\beta}$ instead of $I_{\alpha\beta}$):

$$\begin{aligned}
\mathcal{F}_{\alpha\beta} &= \phi(Z_\alpha\cap Z_\beta) = \{F_{\Delta,J}\in\mathcal{F};\ (\Delta_\alpha\cup\Delta_\beta)\subset\Delta\subset(V\setminus(\Delta'_\alpha\cup\Delta'_\beta))\},\\
d_{\alpha\beta} &= |\Delta_\alpha\setminus\Delta_\beta|, \qquad \Delta'_{\alpha\beta} = (\Delta'_\alpha\setminus\Delta'_\beta)\cup(\Delta'_\beta\setminus\Delta'_\alpha),\\
\mathcal{F}'_{\alpha\beta} &= \phi\big(Z_{\Delta_\alpha\cup\Delta_\beta}\setminus(\textstyle\bigcup_{v\in\Delta'_{\alpha\beta}} D_v)\big)\setminus\mathcal{F}_{\alpha\beta}\\
&= \{F_{\Delta,J}\in\mathcal{F};\ (\Delta_\alpha\cup\Delta_\beta)\subset\Delta\subset(V\setminus\Delta'_{\alpha\beta}) \quad\text{and}\quad \Delta'_\alpha\cap\Delta'_\beta\cap\Delta\neq\emptyset\}.
\end{aligned} \tag{55}$$

For $F \in \phi(Z_\alpha)$, the restriction to $E\times_K K\cdot F$ of the local system L_α has an extension to $E\times_K V_F$ (recall that $V_F = \phi^{-1}(U_F)$—Notations 4.5); we denote it by $L_{\alpha,F}$. According to the defining formula (8) we consider three cases: (i) $F \in \mathcal{F}_{\alpha\beta}$, (ii) $F\in\mathcal{F}'_{\alpha\beta}$, (iii) $F \notin \mathcal{F}_{\alpha\beta}\sqcup\mathcal{F}'_{\alpha\beta}$. For $F\in\mathcal{F}_{\alpha\beta}$, we have

$$\mathcal{R}^{\alpha\beta}_F = \Gamma(E\times_K V_F; \Omega_{E\times_K V_F}\otimes \mathcal{H}om(L_{\alpha,F}, L_{\beta,F}))\,[-2d_{\alpha\beta}].$$

Case (ii) is reduced to (i) as in Definition 3.5, and in case (iii) we have $\mathcal{R}^{\alpha\beta}_F = 0$. We set $\mathcal{R} = \bigoplus_{\alpha,\beta\in A}\mathcal{R}^{\alpha\beta}$.

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We denote by $\delta_v \in H^2_{K,D_v}(X_Z; \mathbf{C}_{X_Z})$ the K-equivariant fundamental class of D_v in X_Z. We choose forms $\xi_v \in \Gamma(E \times_K X_Z; \Omega^2_{E\times_K X_Z})$ representing the δ_v, and use them to define a product on $\mathcal{R}$, as in Definition 3.5, turning $\mathcal{R}$ into a sheaf of dg-algebras on $\mathcal{F}$.

The $\mathcal{R}$-dg-module N_α is $N_\alpha = \bigoplus_{\alpha',\alpha} \mathcal{R}^{\alpha'\alpha}$, with a $\mathcal{R}$-structure defined like the product of $\mathcal{R}$. Let us set $L = \bigoplus_{\alpha\in A} L_\alpha$. By (25) and Lemma 3.20 (ii), we have, for a face $F \in \mathcal{F}$, the isomorphism of algebras:

$$H^\cdot(\mathcal{R}_F) \simeq H^\cdot(\Gamma(U_F; \mathcal{R})) \simeq \mathrm{Ext}^\cdot_{\mathrm{D}_G(V_F)}(L|_{V_F}, L|_{V_F}). \tag{56}$$

We also introduce the sheaf $\mathcal{H}$ on $\mathcal{F}$ given by the cohomology of $\mathcal{R}$, i.e., the sheaf associated to the presheaf $U \mapsto H^\cdot(\Gamma(U; \mathcal{R}))$. This is a sheaf of dg-algebras on $\mathcal{F}$, with differential 0. For a face $F \in \mathcal{F}$ we have $\Gamma(U_F; \mathcal{H}) = \mathcal{H}_F = H^\cdot(\mathcal{R}_F)$. We define in the same way the $\mathcal{H}$-module $\mathcal{H}_\alpha$ associated to $U \mapsto H^\cdot(\Gamma(U; N_\alpha))$.

5. Formality of the de Rham algebra

We keep the notations introduced in the previous section. Our aim is to prove the following result.

Proposition 5.1. *There exists a sequence of quasi-isomorphisms of sheaves of dg-algebras on $\mathcal{F}$, $\mathcal{R} \to \mathcal{RA} \leftarrow \mathcal{RB} \to \mathcal{RC} \leftarrow \mathcal{RD} \to \mathcal{RE} \simeq \mathcal{H}$, relating $\mathcal{R}$ and $\mathcal{H}$. It induces an equivalence of categories between $\mathrm{D}^b_{G,c}(X_Z)$ and $\mathrm{D}_{\mathcal{H}}\langle \mathcal{H}_\alpha,\ \alpha \in A\rangle$.*

We know from the previous section that $\mathrm{D}^b_{G,c}(X_Z)$ is equivalent to $\mathrm{D}_{\mathcal{R}}\langle N_\alpha\rangle$. Now a quasi-isomorphism between sheaves of dg-algebras induces an equivalence between their derived categories of dg-modules. Hence the first part of the proposition implies that $\mathrm{D}^b_{G,c}(X_Z)$ is equivalent to $\mathrm{D}_{\mathcal{H}}\langle M_\alpha\rangle$, where M_α is the image of N_α by the chain of equivalences. The remainder of this section is devoted to the construction of a sequence of quasi-isomorphisms as in the proposition. It will follow from the construction that M_α is indeed isomorphic to $\mathcal{H}_\alpha$.

5.1. Decomposition of the cohomology.

By Theorem 4.2, the isotropy group K_F almost decomposes as a product, up to a finite subgroup. We deduce a decomposition for $H^\cdot(\mathcal{R}^{\alpha\beta}_F)$.

For $F \in \mathcal{F}$, we have defined in (50) a fibration $q_F : E \times_K V_F \to E/K_F$, with contractible fibres $K_F \cdot U'_F$. In particular, q_F gives an isomorphism between the fundamental groups. Hence, for $\alpha \in A$ such that $F \subset Z_\alpha$, the local system $L_{\alpha,F}$ on $E \times_K V_F$ is the inverse image of a local system $L'_{\alpha,F}$ on E/K_F. Setting $M = \mathcal{H}om(L'_{\alpha,F}, L'_{\beta,F})$, for another $\beta \in A$ with $F \subset Z_\beta$, we obtain

$$\begin{aligned} H^\cdot(\mathcal{R}^{\alpha\beta}_F) &\simeq \mathrm{Ext}^\cdot_{\mathrm{D}(E\times_K V_F)}(L_{\alpha,F}, L_{\beta,F}) \\ &\simeq \mathrm{Ext}^\cdot_{\mathrm{D}(E/K_F)}(L'_{\alpha,F}, L'_{\beta,F}) \simeq H^\cdot(E/K_F; M). \end{aligned} \tag{57}$$

The following lemma describes more precisely $H^\cdot(E/K_F; M)$.

Let us introduce some notation. For a face $F = F_{\Delta,J}$, we recall that S^{c0}_Δ and K_J commute and we consider the action of $S^{c0}_\Delta \times K_J$ on E^3 by $(s,k)\cdot(e_1,e_2,e_3) = (sk\cdot e_1, s\cdot e_2, k\cdot e_3)$. Let also a_F be the group morphism $S^{c0}_\Delta \times K_J \to K_F$, $(s,k) \mapsto sk$. The first projection $E^3 \to E$ is a_F-equivariant and induces the morphism r^1_F below. In view of (53), r^1_F is a fibration with fibre E^2/D_Δ, which is acyclic (i.e., $H^0(E^2/D_\Delta; \mathbf{C}) = \mathbf{C}$ and, for $i \neq 0$, $H^i(E^2/D_\Delta; \mathbf{C}) = 0$). The projection to the

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last two factors $E^3 \to E^2$ induces in the same way the morphism r_F^2 below, which is a fibration with acyclic fibre E.

$$E/K_F \xleftarrow[E^2/D_\Delta]{r_F^1} E^3/(S_\Delta^{c0} \times K_J) \xrightarrow[E]{r_F^2} (E/S_\Delta^{c0}) \times (E/K_J). \tag{58}$$

Lemma 5.2. *We consider a face $F = F_{\Delta,J} \in \mathcal{F}$, $\rho : \tau_F \to GL(V_\rho)$ a representation of τ_F, and M the local system on E/K_F corresponding to ρ. We let $\rho_J : \tau_J \to GL(V_\rho)$ be the representation obtained from ρ and the morphism $\tau_J \to \tau_F$. We let M_J be the local system on E/K_J corresponding to ρ_J. Then*

$$H^\cdot(E/K_F; M) \simeq \mathbf{C}[X_v;\, v \in \Delta] \otimes H^\cdot(E/K_J; M_J),$$

where the X_v are indeterminates of degree 2.

Proof. Since r_F^1 is a fibration with acyclic fibres, we have $M \simeq R(r_F^1)_*(r_F^1)^{-1}M$. Hence $H^\cdot(E/K_F; M) \simeq H^\cdot(E^3/(S_\Delta^c \times K_J); (r_F^1)^{-1}M)$. We have $\pi_1(E/K_F) = \tau_F$, $\pi_1(E^3/(S_\Delta^{c0} \times K_J)) = \tau_J$ and the morphism induced by r_F^1 on the fundamental groups is the morphism of the lemma $\tau_J \to \tau_F$. Hence $(r_F^1)^{-1}M$ is the local system corresponding to ρ_J.

Since r_F^2 has a contractible fibre, it gives an isomorphism on the fundamental groups. Hence $(r_F^1)^{-1}M \simeq (r_F^2)^{-1}(\mathbf{C}_{E/S_\Delta^{c0}} \boxtimes M_J)$. This also gives an isomorphism on the cohomology groups of $(r_F^1)^{-1}M$ and $\mathbf{C}_{E/S_\Delta^{c0}} \boxtimes M_J$. We conclude by the Künneth formula and the fact that S_Δ^{c0} is the torus $(\mathbf{C}^*)^{|\Delta|}$, so that $H^\cdot(E/S_\Delta^{c0}; \mathbf{C}) = H^\cdot_{S_\Delta^{c0}}(\{pt\}; \mathbf{C})$ is a polynomial algebra in $|\Delta|$ variables. □

We describe the local systems $L_{\alpha,F}$ on $E \times_K V_F$, $L'_{\alpha,F}$ on E/K_F and $(L'_{\alpha,F})_J$ on E/K_J, in terms of representations.

For $\alpha = (\mathcal{O}, \rho) \in A$, let $F_\alpha = F_{\Delta_\alpha, J_{\Delta_\alpha}}$ be the closed face of $\mathcal{O} \cap C_{X_Z}$ and V_ρ be the representation space of ρ. We have seen that $\tau_{\mathcal{O}} = \tau_{F_\alpha}$, so that we have a morphism $\tau_{J_{\Delta_\alpha}} \to \tau_{\mathcal{O}}$. Let $\mathcal{O}_\Delta$ be a G-orbit such that $\mathcal{O}_\Delta \subset \overline{\mathcal{O}}$ and with closed face F_{Δ,J_Δ}. We recall that $\pi(\mathcal{O}) = \mathcal{O}_{J_{\Delta_\alpha}}$ and $\pi(\mathcal{O}_\Delta) = \mathcal{O}_{J_\Delta}$ (where π is the map from X_Z to X). Hence $J_{\Delta_\alpha} \subset J_\Delta$. If $F = F_{\Delta,J}$ is another face of $\mathcal{O}_\Delta$, we have $J_\Delta \subset J$. Finally, for any face $F = F_{\Delta,J}$ such that $F \subset \overline{\mathcal{O}}$, we have $J_{\Delta_\alpha} \subset J$, so that $K_J \subset K_{J_{\Delta_\alpha}}$ and we obtain a group morphism:

$$\text{for } F_{\Delta,J} \subset \overline{\mathcal{O}}, \qquad t_J^{\mathcal{O}} : \tau_J \to \tau_{\mathcal{O}}. \tag{59}$$

We let ρ_J be the representation of τ_J given by V_ρ and $t_J^{\mathcal{O}}$, and we let $L_{\alpha,F}^1$ be the corresponding local system on E/K_J.

Now we assume, moreover, that $F \subset Z_\alpha$. This means, by Lemma 4.7, that ρ induces a representation, say ρ', of $\tau_{\mathcal{O}_\Delta}$. Then the representation ρ_J of τ_J is given by ρ' and the morphism $\tau_J \to \tau_{J_\Delta} \to \tau_{\mathcal{O}_\Delta}$. Since the morphism induced by r_F^1 on the fundamental groups is $\tau_J \to \tau_F$, we obtain the following relations:

$$L_{\alpha,F} \simeq q_F^{-1}(L'_{\alpha,F}), \qquad (r_F^1)^{-1}(L'_{\alpha,F}) \simeq (r_F^2)^{-1}(\mathbf{C}_{E/S_\Delta^{c0}} \boxtimes L_{\alpha,F}^1). \tag{60}$$

In Lemma 5.2, we have used $H^\cdot_{S_\Delta^{c0}}(\{pt\}; \mathbf{C}) \simeq \mathbf{C}[X_v; v \in \Delta]$. The choice of indexing the indeterminates by Δ is not arbitrary, as explained in the next lemma.

For $v \in V$, we have denoted by δ_v the G- (or K-) equivariant fundamental class of D_v in X_Z, $\delta_v \in H^2_{K,D_v}(X_Z; \mathbf{C})$. Let us also denote by δ_v its "restriction" to any K stable open subset of X_Z. The following lemma describes the image of $\delta_v \in H^2_K(V_F; \mathbf{C})$ by the isomorphism $H^\cdot_K(V_F; \mathbf{C}) \simeq H^\cdot(E/K_F; \mathbf{C})$ composed with the isomorphism of Lemma 5.2.

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Let us first recall the construction of the isomorphism

$$H^{\cdot}(E/S^{c0}_{\Delta};\mathbf{C}) = H^{\cdot}_{S^{c0}_{\Delta}}(\{pt\};\mathbf{C}) \simeq Sym(Lie(S^{c0}_{\Delta})^*), \tag{61}$$

where the elements of $Lie(S^{c0}_{\Delta})^*$ have degree 2. A character $\chi : S^{c0}_{\Delta} \to \mathbf{C}^*$ gives an element $d_\chi \in Lie(S^{c0}_{\Delta})^*$ by differentiation. It also gives a one-dimensional representation of S^{c0}_{Δ}, $\mathbf{C}_\chi$, and a line bundle $l_\chi = E \times_{S^{c0}_{\Delta}} \mathbf{C}_\chi$ over E/S^{c0}_{Δ}. The above isomorphism sends d_χ to the Chern class $c_2(l_\chi)$. We note that this Chern class is nothing but the S^{c0}_{Δ}-equivariant fundamental class of $\{0\}$ in $\mathbf{C}_\chi$.

Lemma 5.3. *Let $v \in V$ and $F = F_{\Delta,J}$ be a face such that $F \subset D_v$. For a point $p \in F$, S^{c0}_{Δ} acts on $T_pX_Z/T_pD_v \simeq \mathbf{C}$. Let χ_v be the corresponding character of S^{c0}_{Δ} and $X_v \in H^2_{S^{c0}_{\Delta}}(\{pt\};\mathbf{C})$ the associated equivariant class. We have*

$$H^{\cdot}_K(V_F;\mathbf{C}) \simeq H^{\cdot}_{K_F}(\{pt\};\mathbf{C}) \simeq H^{\cdot}_{S^{c0}_{\Delta}}(\{pt\};\mathbf{C}) \otimes H^{\cdot}_{K_J}(\{pt\};\mathbf{C})$$

and this isomorphism sends δ_v to $X_v \otimes 1$. Moreover, $H^{\cdot}_{S^{c0}_{\Delta}}(\{pt\};\mathbf{C}) \simeq \mathbf{C}[X_v; v \in \Delta]$.

Proof. We have seen that the first isomorphism follows from the homotopy equivalence q_F (see (50)). The second one is a special case of Lemma 5.2, with $M = \mathbf{C}_{E/K_F}$.

We have an action of K_F on $N_{p,v} = T_pX_Z/T_pD_v$ and natural isomorphisms:

$$H^{\cdot}_K(V_F;\mathbf{C}) \xrightarrow{\simeq} H^{\cdot}_{K_F}(N_{p,v};\mathbf{C}) \xrightarrow{\simeq} H^{\cdot}_{K_F}(\{pt\};\mathbf{C}).$$

The class $\delta_v \in H^{\cdot}_K(V_F;\mathbf{C})$ can be identified with the K_F-equivariant fundamental class of $\{0\}$ in $N_{p,v}$, $\delta_{\{0\}|N_{p,v}} \in H^2_{K_F,\{0\}}(N_{p,v};\mathbf{C})$. Hence its image by the natural morphism from $H^{\cdot}_{K_F}(\{pt\};\mathbf{C})$ to $H^{\cdot}_{S^{c0}_{\Delta}}(\{pt\};\mathbf{C})$ is the S^{c0}_{Δ}-equivariant fundamental class of $\{0\}$ in $N_{p,v}$, i.e., X_v. By (41), K_J acts trivially on N_p, so that the image of δ_v by $H^{\cdot}_{K_F}(\{pt\};\mathbf{C}) \to H^{\cdot}_{K_J}(\{pt\};\mathbf{C})$ is 0. Since $H^2_K(V_F;\mathbf{C})$ only has the components $H^2_{S^{c0}_{\Delta}}(\{pt\};\mathbf{C})$ and $H^2_{K_J}(\{pt\};\mathbf{C})$, we deduce that δ_v is sent to $X_v \otimes 1$, as claimed.

Let us set $N_p = T_pX_Z/T_p\mathcal{O}_\Delta$. By Theorem 4.2, the kernel of $S^{c0}_{\Delta} \to GL(N_p)$ is finite. Since $N_p \simeq \oplus_{v\in\Delta} N_{p,v}$, it follows that the characters χ_v, $v \in \Delta$, are independent. Since $\dim S^{c0}_{\Delta} = |\Delta|$ we obtain the last assertion. □

5.2. **Decomposition of the dg-algebras.** We would like to decompose the de Rham complex $\mathcal{R}^{\alpha\beta}_F$ as we have decomposed its cohomology in Lemma 5.2. However, in the sequence of fibrations,

$$E \times_K V_F \xrightarrow{q_F} E/K_F \xleftarrow{r^1_F} E^3/(S^{c0}_{\Delta} \times K_J) \xrightarrow{r^2_F} (E/S^{c0}_{\Delta}) \times (E/K_J),$$

the morphism r^1_F goes in the wrong direction, i.e., we have no natural map from $\Gamma(E/K_J;\Omega_{E/K_J})$ to $\Gamma(E \times_K V_F;\Omega_{E\times_K V_F})$. Hence we first replace $E \times_K V_F$ by the fibre product built on q_F and r^1_F.

5.2.1. *Pull-back to a fibre product.*

Lemma 5.4. *We keep the notations q_F, r^1_F, r^2_F defined in* (50) *and* (58)*. For a face $F = F_{\Delta,J} \in \mathcal{F}$, we set:*

$$V^+_F = (E \times_K V_F) \times_{E/K_F} (E^3/(S^{c0}_{\Delta} \times K_J)).$$

(i) *For any face $F = F_{\Delta,J} \in \mathcal{F}$, we have fibrations*

$$\nu_F : V^+_F \to E \times_K V_F, \qquad r_F : V^+_F \to (E/S^{c0}_{\Delta}) \times (E/K_J),$$

ν_F has fibres homeomorphic to E^2/D_Δ and r_F has contractible fibres.

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(ii) *For* $F_i = F_{\Delta_i,J_i}$, $i=1,2$, *with* $F_1 \subset \overline{F_2}$, *we have a natural morphism* $v_{F_1F_2} : V^+_{F_2} \to V^+_{F_1}$ *and a commutative diagram*

$$
(62) \qquad
\begin{array}{ccccc}
E \times_K V_{F_2} & \xleftarrow{\nu_{F_2}} & V^+_{F_2} & \xrightarrow{r_{F_2}} & (E/S^{c0}_{\Delta_2}) \times (E/K_{J_2}) \\
\downarrow & & \downarrow & & \downarrow \\
E \times_K V_{F_1} & \xleftarrow{\nu_{F_1}} & V^+_{F_1} & \xrightarrow{r_{F_1}} & (E/S^{c0}_{\Delta_1}) \times (E/K_{J_1})\,.
\end{array}
$$

(iii) *For a third face* F_3 *with* $F_1 \subset \overline{F_2} \subset \overline{F_3}$, *we have* $v_{F_1F_3} = v_{F_1F_2} \circ v_{F_2F_3}$.

Proof. The proof is more or less tautological. By definition, V^+_F comes with two fibrations, $\nu_F : V^+_F \to E \times_K V_F$, with fibre E^2/D_Δ, and $\mu_F : V^+_F \to E^3/(S^{c0}_\Delta \times K_J)$, with contractible fibre $K_F \cdot U'_F$. We set $r_F = r^2_F \circ \mu_F$; since μ_F and r^2_F are fibrations with contractible fibres, so is r_F. This gives (i).

For (ii), we have the inclusions $K_{F_2} \subset K_{F_1}$, $K_{J_2} \subset K_{J_1}$, $S^c_{\Delta_2} \subset S^c_{\Delta_1}$ and they induce commutative squares of fibrations:

$$
(63) \qquad
\begin{array}{ccc}
E^3/(S^c_{\Delta_2} \times K_{J_2}) & \xrightarrow{r^1_{F_2}} & E/K_{F_2} \\
\downarrow & & \downarrow \\
E^3/(S^c_{\Delta_1} \times K_{J_1}) & \xrightarrow{r^1_{F_1}} & E/K_{F_1}
\end{array}
\qquad
\begin{array}{ccc}
E \times_K V_{F_2} & \xrightarrow{q_{F_2}} & E/K_{F_2} \\
\downarrow & & \downarrow \\
E \times_K V_{F_1} & \xrightarrow{q_{F_1}} & E/K_{F_1}\,,
\end{array}
$$

and a similar square corresponding to $r^2_{F_\cdot}$. Now (ii) follows from these diagrams and the definitions. The proof of (iii) is similar. □

Definition of $\mathcal{RA}$. Now we pull back the construction of $\mathcal{R}$ to the V^+_F. For $\alpha = (\mathcal{O}, \rho) \in A$, and $F \in F$, we set $L^+_{\alpha,F} = \nu_F^{-1} L_{\alpha,F}$. For $v \in V$, we set $\xi'_{v,F} = \nu^*_F(\xi_v|_{E\times_K V_F}) \in \Gamma(V^+_F; \Omega^2_{V^+_F})$. For two faces $F_1 \subset \overline{F_2}$, we have

$$
(64) \qquad v^{-1}_{F_1F_2}(L^+_{\alpha,F_1}) = L^+_{\alpha,F_2}, \qquad v^*_{F_1F_2}(\xi'_{v,F_1}) = \xi'_{v,F_2}.
$$

We introduce a sheaf $\mathcal{RA}$ on $\mathcal{F}$, copying the definition of $\mathcal{A}$ in 3.5. For $\alpha, \beta \in A$, we define the sheaf $\mathcal{RA}^{\alpha\beta}$ by its stalks at a face $F \in \mathcal{F}_{\alpha\beta}$:

$$
\mathcal{RA}^{\alpha\beta}_F = \Gamma(V^+_F; \Omega_{V^+_F} \otimes \mathcal{H}om(L^+_{\alpha,F}, L^+_{\beta,F}))\,[-2d_{\alpha\beta}],
$$

and we reduce the case $F \notin \mathcal{F}_{\alpha\beta}$ to this one, as in Definition 3.5. The only difference is that the restriction maps, say from $\mathcal{RA}^{\alpha\beta}_{F_1}$ to $\mathcal{RA}^{\alpha\beta}_{F_2}$, for $F_1 \subset \overline{F_2}$, are induced by $v_{F_1F_2} : V^+_{F_2} \to V^+_{F_1}$, instead of the inclusion $E \times_K V_{F_2} \subset E \times_K V_{F_1}$. This gives a sheaf by (iii) of Lemma 5.4.

We set $\mathcal{RA} = \bigoplus_{(\alpha,\beta)\in A^2} \mathcal{RA}^{\alpha\beta}$ and endow it with an algebra structure as $\mathcal{R}$; more precisely, we define the product on the stalks at a given face F by replacing the ξ_v in the definition of $\mathcal{R}$ by the $\xi'_{v,F}$. The compatibility of the product and the restriction maps follows from (64).

For a face F, the inverse image by ν_F induces a natural morphism $\nu^*_F : \mathcal{R}_F \to \mathcal{RA}_F$. Since $\xi'_{v,F} = \nu^*_F \xi_v$, we see that the ν^*_F are morphisms of dg-algebras. The commutative squares in Lemma 5.4 imply that the ν^*_F induce a morphism of sheaves on $\mathcal{F}$, say $\nu^* : \mathcal{R} \to \mathcal{RA}$. Let us verify that it is a quasi-isomorphism. We have $\mathcal{H}om(L^+_{\alpha,F}, L^+_{\beta,F}) \simeq \nu_F^{-1}\mathcal{H}om(L_{\alpha,F}, L_{\beta,F})$. Since ν_F is a fibration with fibre E^2/D_Δ, which is acyclic over $\mathbf{C}$, we have, for any sheaf L on $E \times_K V_F$, $R(\nu_F)_* \nu_F^{-1} L \simeq L$. Hence $H^\cdot(\mathcal{RA}^{\alpha\beta}_F) = H^\cdot(\mathcal{R}^{\alpha\beta}_F)$, as required.

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5.2.2. *Decomposition.* For $\alpha = (\mathcal{O}, \rho) \in A$ and $F = F_{\Delta,J} \in \mathcal{F}$ with $F \subset Z_\alpha$, let $L^1_{\alpha,F}$ be the local system on (E/K_J), corresponding to the representation of τ_J given by ρ and $t_J^{\mathcal{O}} : \tau_J \to \tau_{\mathcal{O}}$ (see (59) and after). Then, by (60), $L^+_{\alpha,F} \simeq r_F^{-1}(\mathbf{C}_{E/S^{c0}_\Delta} \boxtimes L^1_{\alpha,F})$.

For $\alpha, \beta \in A$ and $F \in \mathcal{F}_{\alpha\beta}$, we have, by Lemma 5.2:

$$H^\cdot(\mathcal{RA}^{\alpha\beta}_F) \simeq H^\cdot(E/S^{c0}_\Delta; \mathbf{C}_{E/S^{c0}_\Delta}) \otimes H^\cdot(E/K_J; \mathcal{H}om(L^1_{\alpha,F}, L^1_{\beta,F})).$$

This isomorphism corresponds to a quasi-isomorphism at the level of de Rham complexes. The product of forms, composed with the inverse image by r_F gives a quasi-isomorphism:

$$\Gamma(E/S^{c0}_\Delta; \Omega_{E/S^{c0}_\Delta}) \otimes \Gamma(E/K_J; \Omega_{E/K_J} \otimes \mathcal{H}om(L^1_{\alpha,F}, L^1_{\beta,F})) \xrightarrow{qis} \mathcal{RA}^{\alpha\beta}_F. \tag{65}$$

However, the forms $\xi'_{v,F} \in \Omega^2_{V_F^+}$ do not have to be pull-backs of forms by r_F and we have no natural algebra structure on the sum over $(\alpha, \beta) \in A^2$ of the groups appearing in the left-hand side of (65). For this we will replace the factor $\Gamma(E/S^{c0}_\Delta; \Omega_{E/S^{c0}_\Delta})$ by a free anti-commutative algebra quasi-isomorphic to it. We will define a sheaf $\mathcal{RB}$ on $\mathcal{F}$ (see (67) below) as the product of two sheaves: $\mathcal{RS}$, quasi-isomorphic to the de Rham algebra of E/S^{c0}_Δ appearing in (65), and $\mathcal{RK}$, given by the twisted de Rham complex on E/K_J.

For $\Delta \subset V$, we introduce the dg-algebras $A(\Delta)$, $B(\Delta)$ below, which are free anti-commutative algebras, and a quasi-isomorphism, $b(\Delta)$, between them. (The reason for introducing $B(\Delta)$ is to be able to define morphism f below, which would be impossible with $A(\Delta)$ instead of $B(\Delta)$.)

$$\begin{gathered} A(\Delta) = \mathbf{C}[X_v;\, v \in \Delta], \quad B(\Delta) = \mathbf{C}[X_v, Y_w;\, v \in V,\, w \in V \setminus \Delta], \\ \deg X_v = 2, \quad \deg Y_w = 1, \qquad dX_v = 0, \quad dY_w = X_w, \\ b(\Delta) : B(\Delta) \to A(\Delta), \qquad \forall v \in \Delta,\ X_v \mapsto X_v, \quad \forall w \in V \setminus \Delta,\ X_w \mapsto 0,\ Y_w \mapsto 0. \end{gathered} \tag{66}$$

For $(\alpha, \beta) \in A^2$, and a face $F = F_{\Delta,J} \in \mathcal{F}_{\alpha\beta}$ we set

$$\mathcal{RS}^{\alpha\beta}_F = B(\Delta)\,[-2d_{\alpha\beta}].$$

For other faces, we reduce to the case $F \in \mathcal{F}_{\alpha\beta}$, as in Definition 3.5. The restriction maps are given by the inclusions $B(\Delta_1) \subset B(\Delta_2)$ if $\Delta_2 \subset \Delta_1$. We have a product similar to the product in $\mathcal{R}$, as follows. For $\alpha, \beta, \gamma \in A$, we set $\varepsilon_{\alpha\beta\gamma} = \prod_{v \in \nabla} X_v$, where ∇ is defined in (10). We define $m_S^{\alpha\beta\gamma} : \mathcal{RS}^{\alpha\beta} \otimes \mathcal{RS}^{\beta\gamma} \to \mathcal{RS}^{\alpha\gamma}$, $P \otimes Q \mapsto \varepsilon_{\alpha\beta\gamma} PQ$.

In a similar way, for $(\alpha, \beta) \in A^2$, and a face $F = F_{\Delta,J} \in \mathcal{F}_{\alpha\beta}$ we set

$$\mathcal{RK}^{\alpha\beta}_F = \Gamma(E/K_J; \Omega_{E/K_J} \otimes \mathcal{H}om(L^1_{\alpha,F}, L^1_{\beta,F})).$$

For other faces, we reduce to the case $F \in \mathcal{F}_{\alpha\beta}$, as in Definition 3.5. The restriction maps are the following ones: for $U_{F_2} \subset U_{F_1}$ we have a map $r_{12} : E/K_{J_2} \to E/K_{J_1}$ and $L^1_{\alpha,F_2} = r_{12}^{-1} L^1_{\alpha,F_1}$, hence an inverse image morphism $\mathcal{RK}^{\alpha\beta}_{F_1} \to \mathcal{RK}^{\alpha\beta}_{F_2}$. We also have an obvious product $m_K^{\alpha\beta\gamma} : \mathcal{RK}^{\alpha\beta} \otimes \mathcal{RK}^{\beta\gamma} \to \mathcal{RK}^{\alpha\gamma}$ given by the product of forms and the composition of morphisms.

Definition of $\mathcal{RB}$. Now we set

$$\forall (\alpha, \beta) \in A^2, \quad \mathcal{RB}^{\alpha\beta} = \mathcal{RS}^{\alpha\beta} \otimes \mathcal{RK}^{\alpha\beta}, \qquad \mathcal{RB} = \bigoplus_{(\alpha,\beta) \in A^2} \mathcal{RB}^{\alpha\beta}. \tag{67}$$

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We let $m^{\alpha\beta\gamma}$ be the tensor product of $m_S^{\alpha\beta\gamma}$ and $m_K^{\alpha\beta\gamma}$. Since, by definition, the $\varepsilon_{\alpha\beta\gamma}$ satisfy the same identity (11) as the $\eta_{\alpha\beta\gamma}$, we obtain a product on $\mathcal{RB}$ defined by the sum of the $m^{\alpha\beta\gamma}$.

Definition of $f : \mathcal{RB} \to \mathcal{RA}$. For $F \in \mathcal{F}$, we note that $\mathcal{RA}_F$ is an algebra over $\Gamma(V_F^+; \Omega_{V_F^+})$. We define a morphism $f_F^{\alpha\beta} : \mathcal{RB}_F^{\alpha\beta} \to \mathcal{RA}_F^{\alpha\beta}$ as the product of $f_F^{S\alpha\beta} : \mathcal{RS}_F^{\alpha\beta} \to \Gamma(V_F^+; \Omega_{V_F^+})$ and $f_F^{K\alpha\beta} : \mathcal{RK}_F^{\alpha\beta} \to \mathcal{RA}_F^{\alpha\beta}$, which are obtained as follows.

For a face F, let $s_F : V_F^+ \to E/K_J$ be the composition of r_F and the projection to E/K_J. For a form σ and a sheaves endomorphism u, we set $f_F^{K\alpha\beta}(\sigma \otimes u) = s_F^*(\sigma) \otimes s_F^{-1}(u)$.

Now we define $f_F^{S\alpha\beta}$. For $v \in V$, the fundamental class, δ_v, of D_v in X_Z restricts to 0 on $X_Z \setminus D_v$. Hence the restriction of ξ_v on $E \times_K (X_Z \setminus D_v)$ is a boundary. Let us choose a form $\zeta_v \in \Gamma(E \times_K (X_Z \setminus D_v); \Omega^1_{E\times_K X_Z})$ such that, on $E \times_K (X_Z \setminus D_v)$ we have $\xi_v = d\zeta_v$. We set $\zeta'_{v,F} = \nu_F^*(\zeta_v|_{E\times_K V_F})$ (we note that, for $F = F_{\Delta,J}$ and $v \in V$ such that $v \notin \Delta$, we have $V_F \cap D_v = \emptyset$, so that ζ_v is defined on $E \times_K V_F$). We set

$$\forall v \in V,\ \ f_F^{S\alpha\beta}(X_v) = \xi'_{v,F}, \qquad \forall w \in V \setminus \Delta,\ \ f_F^{S\alpha\beta}(Y_w) = \zeta'_{w,F}.$$

Using Lemma 5.4, one checks that $f_F^{S\alpha\beta}$ and $f_F^{K\alpha\beta}$ give morphisms of sheaves, say $f^{S\alpha\beta}$ and $f^{K\alpha\beta}$. We define $f^{\alpha\beta} = f^{S\alpha\beta} \otimes f^{K\alpha\beta}$ and $f = \bigoplus f^{\alpha\beta}$. In view of the definitions of the product and the differentials, f is a morphism of dg-algebras. By Lemmas 5.2 and 5.3 it is a quasi-isomorphism.

5.3. **Formality of the toric part.** We define a sheaf quasi-isomorphic to $\mathcal{RS}$ but with differential 0 as follows. For $(\alpha,\beta) \in A^2$, and a face $F = F_{\Delta,J} \in \mathcal{F}_{\alpha\beta}$, we set

$$\mathcal{RT}_F^{\alpha\beta} = A(\Delta)\,[-2d_{\alpha\beta}]. \tag{68}$$

with the following restriction maps. For two faces $F_i = F_{\Delta_i,J_i}$, $i = 1, 2$, such that $F_1 \subset \overline{F_2}$ (i.e., $U_{F_2} \subset U_{F_1}$), we have $\Delta_2 \subset \Delta_1$, and the restriction $A(\Delta_1) \to A(\Delta_2)$ sends X_v to X_v for $v \in \Delta_2$ and to 0 for $v \in \Delta_1 \setminus \Delta_2$. As for $\mathcal{RS}$, other faces are reduced to this case.

We define a product, $m_T^{\alpha\beta\gamma}$, similar to the product of $\mathcal{RS}$. For $\alpha, \beta, \gamma \in A$, we let $e_{\alpha\beta\gamma}$ be the section of $\mathcal{RT}^{\alpha\gamma}$ defined by $(e_{\alpha\beta\gamma})_{F_{\Delta,J}} = b(\Delta)(\varepsilon_{\alpha\beta\gamma})$. We set $m_T^{\alpha\beta\gamma}, \mathcal{RT}^{\alpha\beta} \otimes \mathcal{RT}^{\beta\gamma} \to \mathcal{RT}^{\alpha\gamma}$, $P \otimes Q \mapsto e_{\alpha\beta\gamma}PQ$.

Definition of $\mathcal{RC}$. We set, as in (67),

$$\forall (\alpha,\beta) \in A^2, \quad \mathcal{RC}^{\alpha\beta} = \mathcal{RT}^{\alpha\beta} \otimes \mathcal{RK}^{\alpha\beta}, \qquad \mathcal{RC} = \bigoplus_{(\alpha,\beta)\in A^2} \mathcal{RC}^{\alpha\beta} \tag{69}$$

We let $m_C^{\alpha\beta\gamma}$ be the tensor product of $m_T^{\alpha\beta\gamma}$ and $m_K^{\alpha\beta\gamma}$. The sum $m_C = \bigoplus m_C^{\alpha\beta\gamma}$ defines a product on $\mathcal{RC}$. The $b(\Delta)$ defined in (66) give a quasi-isomorphism of sheaves of dg-algebras $\mathcal{RS} \to \mathcal{RT}$. This induces a quasi-isomorphism of sheaves of dg-algebras $\mathcal{RB} \to \mathcal{RC}$.

5.4. **Formality of $\mathcal{RK}$ and conclusion.** It remains to prove that the factor $\mathcal{RK}$ of $\mathcal{RB}$ also is quasi-isomorphic to its cohomology. For this we first give a more convenient expression for $\mathcal{RK}_F^{\alpha\beta}$ (see formula (72) below).

For $J \subset \{1, \dots, l\}$, let $\pi_J : E/K_J^0 \to E/K_J$ be the covering map, with group τ_J, and set $A_J = (\pi_J)_*(\mathbf{C}_{E/K_J^0})$. Then A_J is a local system on E/K_J, considered as a

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right module over $\mathbf{C}[\tau_J]$, locally free of rank one. It is also a sheaf of algebras. For $\alpha = (\mathcal{O}, \rho) \in A$, $F = F_{\Delta,J} \in \mathcal{F}$, such that $F \subset Z_\alpha$, we have, by (59),

$$L^1_{\alpha,F} \simeq A_J \otimes_{\mathbf{C}[\tau_J]} V_\rho.$$

Hence, for another element $\beta = (\mathcal{O}', \rho') \in A$, with $F \subset Z_\beta$, we have

$$\mathcal{H}om(L^1_{\alpha,F}, L^1_{\beta,F}) \simeq \mathcal{H}om(A_J \otimes_{\mathbf{C}[\tau_J]} V_\rho, A_J \otimes_{\mathbf{C}[\tau_J]} V_{\rho'}).$$

We want to "factorise" the local systems A_J and the representation spaces V_ρ, $V_{\rho'}$ in this last formula. We will use the following definition (see [15]).

Definition 5.5. For a group W and a $\mathbf{C}$-algebra A with a right W-action by algebra automorphisms, we set $A^t[W] = A \otimes \mathbf{C}[W]$ with the product, for $a \in A$, $w \in W$, $(a \otimes w) \cdot (a' \otimes w') = (a(a' \cdot w)) \otimes (w'w)$. We have a natural embedding $\mathbf{C}[W]^{op} \hookrightarrow A^t[W]$, $w \mapsto a_w = 1 \otimes w$. It induces a structure of right $\mathbf{C}[W] \otimes \mathbf{C}[W]^{op}$-module on $A^t[W]$: for $x \in A^t[W]$, $w, w' \in W$, $x \cdot (w \otimes w') = a_w\, x\, a_{w'}$.

We set $B_J = \mathbf{C}[\tau_J] \otimes \mathbf{C}[\tau_J]^{op}$. We consider $\mathcal{H}om(A_J, A_J)$ as a sheaf of algebras, where the product is the composition, and right B_J-module by $(\phi \cdot (t \otimes t'))(a) = (\phi(a \cdot t')) \cdot t$, for ϕ a section of $\mathcal{H}om(A_J, A_J)$, a a section of A_J and $t, t' \in \tau_J$. With these definitions, one can check that the map

$$A^t_J[\tau_J] \to \mathcal{H}om(A_J, A_J), \qquad a \otimes t \mapsto (\alpha \mapsto a(\alpha \cdot t)), \tag{70}$$

where a, α are sections of A_J and $t \in \tau_J$, is an isomorphism of (sheaves of) algebras and right B_J-modules.

Lemma 5.6. *For $J \subset \{1, \dots, l\}$, $\alpha, \beta \in A$, $F = F_{\Delta,J} \in \mathcal{F}_{\alpha\beta}$, we have, with the above notations:*

$$\mathcal{H}om(L^1_{\alpha,F}, L^1_{\beta,F}) \simeq \mathcal{H}om(A_J, A_J) \otimes_{B_J} \mathrm{Hom}(V_\rho, V_{\rho'}), \tag{71}$$

$$\mathcal{R}\mathcal{K}^{\alpha\beta}_F \simeq \Gamma(E/K^0_J; \Omega_{E/K^0_J})^t[\tau_J] \otimes_{B_J} \mathrm{Hom}(V_\rho, V_{\rho'}). \tag{72}$$

Proof. Let us prove (71). We will use the following fact. Set $R = \mathbf{C}[W]$ for a finite group W and consider left and right R-modules, M, N, of finite ranks. We have a right R-module structure on $M^* = \mathrm{Hom}_{\mathbf{C}}(M, \mathbf{C})$ and a left one on N^*. Then the composition of (vector spaces) morphisms

$$(N \otimes_R M)^* \to (N \otimes_{\mathbf{C}} M)^* \xleftarrow{\sim} M^* \otimes_{\mathbf{C}} N^* \to M^* \otimes_R N^* \tag{73}$$

is a canonical isomorphism. Indeed, it is compatible with direct sum, and any R-module is semi-simple. For N, M irreducible with $N \neq M^*$, both $N \otimes_R M$ and $M^* \otimes_R N^*$ are 0. For $N = M^*$, both $N \otimes_R M$ and $M^* \otimes_R N^*$ are canonically identified with $\mathbf{C}$ by the duality contractions $N \otimes M \to \mathbf{C}$ and $M^* \otimes N^* \to \mathbf{C}$. Since the three morphisms in (73) commute with the duality contraction their composition corresponds to $id_{\mathbf{C}}$. Using this we deduce canonical isomorphisms for left and right R-modules, M_i, N_i, $i = 1, 2$:

$$\begin{aligned} \mathrm{Hom}_{\mathbf{C}}(N_1 \otimes_R M_1, N_2 \otimes_R M_2) &\simeq (M_1^* \otimes_R N_1^*) \otimes_{\mathbf{C}} (N_2 \otimes_R M_2) \\ &\simeq (N_1^* \otimes_{\mathbf{C}} N_2) \otimes_{R \otimes R^{op}} (M_1^* \otimes_{\mathbf{C}} M_2) \\ &\simeq \mathrm{Hom}_{\mathbf{C}}(N_1, N_2) \otimes_{R \otimes R^{op}} \mathrm{Hom}_{\mathbf{C}}(M_1, M_2). \end{aligned} \tag{74}$$

Since isomorphism (74) is canonical, it works as well for sheaves and we obtain (71).

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Now we deduce the second isomorphism. We remark that $\Omega_{E/K_J} \otimes A_J$ is isomorphic to $(\pi_J)_*\Omega_{E/K_J^0}$. This isomorphism respects the τ_J-action and we have, by isomorphisms (71) and (70):

$$\Omega_{E/K_J} \otimes \mathcal{H}om(L^1_{\alpha,F}, L^1_{\beta,F}) \simeq ((\pi_J)_*\Omega_{E/K_J^0})^t[\tau_J] \otimes_{B_J} \mathrm{Hom}(V_\rho, V_{\rho'}).$$

Since $\otimes_{B_J}$ is exact for B_J-modules, the constant sheaf $\mathrm{Hom}(V_\rho, V_{\rho'})$ factors out when we take global sections, and we obtain (72). □

Remark 5.7. On the right-hand side of formula (72), the restriction maps are given as follows. For another face $F' = F_{\Delta',J'} \in \mathcal{F}_{\alpha\beta}$ such that $F \subset \overline{F'}$, we have $K_{J'} \subset K_J$, hence a groups morphism $a : \tau_{J'} \to \tau_J$ and a quotient map $p : E/K^0_{J'} \to E/K^0_J$. The inverse image of forms by p is compatible with the action of τ_J, $\tau_{J'}$ (via a) and we obtain a dg-algebras morphism:

$$b : \Gamma(E/K^0_J; \Omega_{E/K^0_J})^t[\tau_J] \to \Gamma(E/K^0_{J'}; \Omega_{E/K^0_{J'}})^t[\tau_{J'}] \otimes_{B_{J'}} B_J,$$

by $b(\omega \otimes t) = (p^*(\omega) \otimes 1) \otimes (1 \otimes t)$, for a form ω and $t \in \tau_J$. Tensorisation with $\mathrm{Hom}(V_\rho, V_{\rho'})$ gives the desired restriction map from $\mathcal{RK}^{\alpha\beta}_F$ to $\mathcal{RK}^{\alpha\beta}_{F'}$.

Let us explain how to recover the product $\mathcal{RK}^{\alpha\beta} \times \mathcal{RK}^{\beta\gamma} \to \mathcal{RK}^{\alpha\gamma}$ in the right-hand side of (72). First we consider isomorphism (74). For $u : N_1 \to N_2$, $v : M_1 \to M_2$, let us denote by $u \cdot v : N_1 \otimes_R M_1 \to N_2 \otimes_R M_2$ the image of $u \otimes v$ by (74). For left and right R-modules, M_3, N_3, $u' : N_2 \to N_3$, $v' : M_2 \to M_3$, we see that $(u' \cdot v') \circ (u \cdot v) = (u' \circ u) \cdot (v' \circ v)$.

Hence the product in the right-hand side of (71) is given by the composition in $\mathcal{H}om(A_J, A_J)$ and in $\mathrm{Hom}(V_\rho, V_{\rho'})$. When we replace $\mathcal{H}om(A_J, A_J)$ by another algebra, say R_1, as in (72), we use the following lemma (with $R_2 = \mathrm{Hom}(V, V)$, V being the sum of irreducible representations of W and ϕ_2 the action of W on V).

Lemma 5.8. *Let W be a finite group, $R = \mathbf{C}[W]$ and R_i, $i = 1, 2$, algebras. Let $\phi_1 : R^{op} \to R_1$, $\phi_2 : R \to R_2$ be algebras morphisms. We consider the right $R \otimes R^{op}$-module structure on R_1 given by $a \cdot (w \otimes w') = \phi_1(w) a \phi_1(w')$, for $w, w' \in W$, $a \in R_1$, and the left structure on R_2, $(w \otimes w') \cdot a' = \phi_2(w) a' \phi_2(w')$.*

Then the formula $(a \otimes a', b \otimes b') \mapsto ab \otimes a'b'$, for $a, b \in R_1$, $a', b' \in R_2$, gives a well-defined product on $(R_1 \otimes_{R \otimes R^{op}} R_2)$.

Proof. By symmetry, it is sufficient to prove, for $a, b \in R_1$, $a', b' \in R_2$, $w\, w' \in W$: $(\phi_1(w) a \phi_1(w') b) \otimes a'b' = ab \otimes (\phi_2(w) a' \phi_2(w') b')$. The tensor product is over $R \otimes R^{op}$, so that we are reduced to

$$(a\phi_1(w')b) \otimes a'b' = ab \otimes (a'\phi_2(w')b'). \tag{75}$$

Let us consider the subgroup $W' \subset W$ generated by w', and $R' = \mathbf{C}[W']$. Then it is sufficient to prove that (75) holds with a tensor product over $R' \otimes R'^{op}$. Hence, replacing W by W', we may assume from the beginning that $W = \langle w' \rangle$ is commutative. We decompose R_i under the action of w': $R_i = \bigoplus_{\lambda,\mu \in \mathbf{C}} R_i^{\lambda\mu}$, where $\forall a \in R_i^{\lambda\mu}$, $\phi_i(w')a = \lambda a$, $a\phi_i(w') = \mu a$. By additivity, we may assume that a, b, a', b' are elements of some $R_i^{\lambda\mu}$. But $R_1^{\lambda\mu} \otimes_{R' \otimes R'^{op}} R_2^{\lambda'\mu'}$ is 0 unless $\lambda = \lambda'$ and $\mu = \mu'$. Similarly, for $a \in R_1^{\lambda\mu}$, $b \in R_1^{\mu'\nu}$, the product ab is 0 unless $\mu\mu' = 1$. Hence we may assume $a \in R_1^{\lambda\mu}$, $b \in R_1^{\mu^{-1}\nu}$, $a' \in R_2^{\lambda\mu}$, $b' \in R_2^{\mu^{-1}\nu}$. In this case formula (75) is obvious. □

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5.4.1. *Formality of* $\mathcal{RK}$. Using (72) we will deduce the formality of $\mathcal{RK}$ from the formality of the de Rham algebras of E/K_J^0 obtained in Lemma 2.3.

Let us denote by $\mathfrak{k}$, $\mathfrak{k}_J$, the Lie algebras of K, K_J, for $J \subset \{1, \dots, l\}$. Let us set for short $H_J^0 = (S^{\cdot}(\mathfrak{k}_J))^{K_J^0}$, viewed as a dg-algebra with differential 0; it is isomorphic to the K_J^0-equivariant cohomology algebra of the point, $H^{\cdot}_{K_J^0}(\{pt\})$.

By Lemma 2.3, the choice of a connection on $\Gamma(E; \Omega_E)$ (in the sense of (5)) gives functorial quasi-isomorphisms:

$$\Gamma(E/K_J^0; \Omega_{E/K_J^0}) \xleftarrow{f_J} W(\mathfrak{k})_{\mathfrak{k}_J - b} \xrightarrow{g_J} H_J^0.$$

The normaliser $N_K(K_J^0)$ acts on the above dg-algebras and K_J^0 acts trivially, so that $\tau_J = K_J/K_J^0$ also acts. The morphisms f_J and g_J are τ_J-equivariant and yield quasi-isomorphisms:

$$\Gamma(E/K_J^0; \Omega_{E/K_J^0})^t[\tau_J] \xleftarrow{f'_J} (W(\mathfrak{k})_{\mathfrak{k}_J - b})^t[\tau_J] \xrightarrow{g'_J} (H_J^0)^t[\tau_J]. \tag{76}$$

Definition of $\mathcal{RD}$. For $(\alpha, \beta) \in A^2$, we define $\mathcal{RL}^{\alpha\beta}$ similarly as $\mathcal{RK}^{\alpha\beta}$, replacing the forms over E/K_J^0 in expression (72) by a quasi-isomorphic dg-algebra. The stalks at a face $F = F_{\Delta,J} \in \mathcal{F}_{\alpha\beta}$ are given by

$$\mathcal{RL}_F^{\alpha\beta} = (W(\mathfrak{k})_{\mathfrak{k}_J - b})^t[\tau_J] \otimes_{B_J} \mathrm{Hom}(V_\rho, V_{\rho'}),$$

with restriction morphisms defined as in Remark 5.7, and product as in Lemma 5.8. As for $\mathcal{RK}$, other faces are reduced to this case. In view of the isomorphism (72) and the quasi-isomorphism f'_J of (76), we have a quasi-isomorphism of sheaves $\mathcal{RL}^{\alpha\beta} \to \mathcal{RK}^{\alpha\beta}$. Setting

$$\mathcal{RD}^{\alpha\beta} = \mathcal{RT}^{\alpha\beta} \otimes \mathcal{RL}^{\alpha\beta}, \qquad \mathcal{RD} = \bigoplus_{(\alpha,\beta) \in A^2} \mathcal{RD}^{\alpha\beta},$$

with the product defined as the product of $\mathcal{RC}$, we deduce a quasi-isomorphism of sheaves of dg-algebras $\mathcal{RD} \to \mathcal{RC}$.

Definition of $\mathcal{RE}$. We define $\mathcal{RM}^{\alpha\beta}$ similarly as $\mathcal{RL}^{\alpha\beta}$, with stalks at a face $F = F_{\Delta,J} \in \mathcal{F}_{\alpha\beta}$:

$$\mathcal{RM}_F^{\alpha\beta} = (H_J^0)^t[\tau_J] \otimes_{B_J} \mathrm{Hom}(V_\rho, V_{\rho'}). \tag{77}$$

Then the g'_J induce a quasi-isomorphism $\mathcal{RL}^{\alpha\beta} \to \mathcal{RM}^{\alpha\beta}$ and, setting

$$\mathcal{RE}^{\alpha\beta} = \mathcal{RT}^{\alpha\beta} \otimes \mathcal{RM}^{\alpha\beta}, \qquad \mathcal{RE} = \bigoplus_{(\alpha,\beta) \in A^2} \mathcal{RE}^{\alpha\beta}, \tag{78}$$

we obtain a quasi-isomorphism of sheaves of dg-algebras $\mathcal{RD} \to \mathcal{RE}$.

5.4.2. *End of proof.* Since the differential in $\mathcal{RE}$ is 0, $\mathcal{RE}$ coincides with its cohomology algebra $\mathcal{H}$. Finally, we have built a sequence of quasi-isomorphic sheaves of dg-algebras $\mathcal{R} \to \mathcal{RA} \leftarrow \mathcal{RB} \to \cdots \to \mathcal{H}$, as required.

To conclude the proof of Proposition 5.1 we remark that in the above sequence of quasi-isomorphisms, each of the intermediate sheaves, say $\mathcal{A}$, is defined as a sum $\mathcal{A} = \bigoplus_{(\alpha,\beta)\in A^2} \mathcal{A}^{\alpha\beta}$, so that, for $\alpha \in A$, $\mathcal{A}_\alpha = \bigoplus_{\alpha_1} \mathcal{A}^{\alpha_1\alpha}$ has a natural structure of $\mathcal{A}$-module. It follows that, for a quasi-isomorphism $\mathcal{A} \to \mathcal{A}'$ in the above sequence, the equivalence of categories between $\mathrm{D}_{\mathcal{A}}$ and $\mathrm{D}_{\mathcal{A}'}$ sends the $\mathcal{A}$-module $\mathcal{A}_\alpha$ to a $\mathcal{A}'$-module isomorphic to $\mathcal{A}'_\alpha$. This shows that N_α corresponds to $\mathcal{H}_\alpha$.

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6. Proof of Theorem 1.1

Now we prove Theorem 1.1 using the equivalence of Proposition 5.1 between $\mathrm{D}^b_G(X_Z)$ and $\mathrm{D}_{\mathcal{H}}\langle \mathcal{H}_\alpha, \alpha \in A\rangle$. In view of this equivalence and the equality $\mathcal{H} = \bigoplus_{\alpha\in A} \mathcal{H}_\alpha$, the algebra $\mathcal{E}$ of the theorem is isomorphic to $\mathrm{Ext}^{\cdot}_{\mathrm{D}_{\mathcal{H}}}(\mathcal{H},\mathcal{H})$ and $\mathcal{E}^\rho_{\mathcal{O}} \simeq \mathrm{Ext}^{\cdot}_{\mathrm{D}_{\mathcal{H}}}(\mathcal{H},\mathcal{H}_\alpha)$, for $\alpha = (\mathcal{O},\rho)$. We have to prove that $\mathrm{D}_{\mathcal{H}}\langle\mathcal{H}_\alpha\rangle$ is equivalent to $\mathrm{D}_{\mathcal{E}}\langle\mathcal{E}^\rho_{\mathcal{O}}\rangle$. We recall the following construction:

1) Let $\mathcal{A}$ be a sheaf of dg-algebras on a finite set I (hence, by [13], $\mathrm{M}_{\mathcal{A}}$ has enough K-projectives). Let $M^1,\dots,M^r$ be $\mathcal{A}$-modules, $P^i \to M^i$ a K-projective resolution of M^i and $P = \bigoplus_i P^i$. The composition of morphisms induces a structure of dg-algebra on $R = \mathrm{Hom}^{\cdot}(P,P)$ (see section 2.1 for the definition of $\mathrm{Hom}^{\cdot}(\cdot,\cdot)$). We have a functor $F : \mathrm{M}_{\mathcal{A}} \to \mathrm{M}_R$, $M \mapsto \mathrm{Hom}^{\cdot}(P,M)$, which sends quasi-isomorphisms to quasi-isomorphisms because P is K-projective. Hence it induces $F : \mathrm{D}_{\mathcal{A}} \to \mathrm{D}_R$. We set $N^i = F(M^i)$. Then F restricts to an equivalence of categories between $\mathrm{D}_{\mathcal{A}}\langle M^i\rangle$ and $\mathrm{D}_R\langle N^i\rangle$ (for example, this is a very special case of [11], Theorem 4.3).

2) We apply this to $\mathrm{D}_{\mathcal{H}}\langle\mathcal{H}_\alpha\rangle$. Since $\mathcal{H}_\alpha$ is a direct summand of $\mathcal{H}$, $(\mathcal{H}_\alpha)_{U_F}$ is K-projective, for any $F \in \mathcal{F}$. Moreover, since $\mathcal{F}$ satisfies (i) of Assumptions 3.1, we see, as in the proof of Proposition 3.19, that the Čech resolution (where we fix a total order on $\mathcal{F}$):

$$(79) \qquad P_\alpha \quad = \quad \cdots \to \bigoplus_{F_1<\cdots<F_k\in\mathcal{F}} (\mathcal{H}_\alpha)_{U_{F_1}\cap\ldots\cap U_{F_k}} \to \cdots \to \bigoplus_{F\in\mathcal{F}} (\mathcal{H}_\alpha)_{U_F} \to 0.$$

is a K-projective resolution of $\mathcal{H}_\alpha$. We set $P = \bigoplus_\alpha P_\alpha$ and $R = \mathrm{Hom}^{\cdot}(P,P)$. Following [13] we use the fact that the differential of $\mathcal{H}$ is 0 and define $\mathcal{H}'$ to be the sheaf of non-graded algebras underlying $\mathcal{H}$. Replacing $\mathcal{H}$ by $\mathcal{H}'$, we define similarly $\mathcal{H}'_\alpha$, P'_α, P', R' (we note that P'_α still is a K-projective resolution of $\mathcal{H}'_\alpha$). The algebras R and R' are canonically isomorphic as differential algebras, but they do not have the same graduation. If we write P as a double complex $P = \bigoplus_{i,j} P^{ij}$ and set $Q^{kl}_{ij} = \mathrm{Hom}_{\mathbf{C}}(P^{ij},P^{kl})$, then $R^d = R \cap (\bigoplus_{k+l=i+j+d} Q^{kl}_{ij})$ and $R'^d = R \cap (\bigoplus_{k=i+d} Q^{kl}_{ij})$.

Claim 6.1. the dg-algebra R' is concentrated in degree 0.

We will prove this below. Let us see why it implies the theorem. We set $R_0 = \tau_{\leq 0}R' = \cdots \oplus R'^{-2} \oplus R'^{-1} \oplus \ker d_0$. This is a differential sub-algebra of R' (or of R as well) and the claim implies that we have quasi-isomorphisms of differential algebras:

$$R' \xleftarrow{u} R_0 \xrightarrow{v} H(R'), \qquad R \xleftarrow{u'} R_0 \xrightarrow{v'} H(R).$$

In view of the decompositions of R and R' above, we have $R'^d = \bigoplus_n (R^n \cap R'^d)$. Hence we may endow R_0 with the graduation induced by the embedding u'. Then v' is a graded morphism: indeed v' is the composition of the projection from R_0 to $R'^0 \cap R_0$ and the projection from $\ker d_R$ to $H(R)$; both morphisms are graded and so is v'. Now we just remark that $H(R) = \mathcal{E}$ by definition. Moreover, the functor from $\mathrm{D}_{\mathcal{H}}$ to D_R sends $P_\alpha \simeq \mathcal{H}_\alpha$ to $\mathrm{Hom}^{\cdot}(P,P_\alpha)$. Since this last object is a summand of R, we see that, in the equivalence from D_R to $\mathrm{D}_{H(R)}$, it is sent to its cohomology, $\mathrm{Ext}^{\cdot}_{\mathrm{D}_{\mathcal{H}}}(P,P_\alpha) \simeq \mathrm{Ext}^{\cdot}_{\mathrm{D}_{\mathcal{H}}}(\mathcal{H},\mathcal{H}_\alpha) = \mathcal{E}^\rho_{\mathcal{O}}$. Summing up we obtain an equivalence between $\mathrm{D}_{\mathcal{H}}\langle\mathcal{H}_\alpha\rangle$ and $\mathrm{D}_{\mathcal{E}}\langle\mathcal{E}^\rho_{\mathcal{O}}\rangle$, as desired.

Proof of Claim 6.1. We have to prove that $\mathrm{Hom}_{\mathrm{D}_{\mathcal{H}'}}(\mathcal{H}',\mathcal{H}'[p])$ is 0, for $p \neq 0$. We use the Čech resolution (79), with $\mathcal{H}'$ instead of $\mathcal{H}_\alpha$. Since $\mathrm{Hom}^{\cdot}(\mathcal{H}'_{U_F},\mathcal{H}') \simeq$

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$\Gamma(U_F;\mathcal{H}')$, we obtain

$$\mathrm{RHom}(\mathcal{H}',\mathcal{H}') \quad \simeq \quad 0 \to \bigoplus_{F\in\mathcal{F}} \Gamma(U_F;\mathcal{H}') \to \bigoplus_{F_1<F_2\in\mathcal{F}} \Gamma(U_{F_1}\cap U_{F_2};\mathcal{H}') \to \cdots .$$

Since the open sets $U_{F_1}\cap\ldots\cap U_{F_r}$ are fundamental open sets, the functor of sections over them is exact. Hence the above resolution computes $H^{\cdot}(\mathcal{F};\mathcal{H}')$ and the claim follows from the next lemma. □

Lemma 6.2. *For any G-stable open subset V of X_Z, and $U=\phi(V)$, we have $H^i(U;\mathcal{H}')=0$ for $i>0$, where $\mathcal{H}'$ is the non-graded sheaf underlying $\mathcal{H}$.*

Proof. By Proposition 5.1, we have an isomorphism $\mathcal{H}'\simeq\mathcal{RE}$, and, by definition, $\mathcal{RE}=\bigoplus_{(\alpha,\beta)\in A^2}\mathcal{RE}^{\alpha\beta}$. Hence it sufficient to prove the vanishing of $H^i(U;\mathcal{RE}^{\alpha\beta})$. We fix $(\alpha,\beta)\in A^2$ for the remainder of the proof and set for short $\mathcal{A}=\mathcal{RE}^{\alpha\beta}$.

1) Let us set $V_\alpha=X_Z\setminus\bigcup_{v\in\Delta'_\alpha}D_v$, $V_\beta=X_Z\setminus\bigcup_{v\in\Delta'_\beta}D_v$, $V_{\alpha\beta}=V_\alpha\cap V_\beta$, $U_{\alpha\beta}=\phi(V_{\alpha\beta})$. We let $j:U_{\alpha\beta}\to\mathcal{F}$ be the inclusion. We first prove that $\mathcal{A}=j_*(\mathcal{A}|_{U_{\alpha\beta}})$. For a face $F\in\mathcal{F}$, repeated applications of hypothesis (iii) of Assumptions 3.1, show that there exists a face F' such that $U_F\cap U_{\alpha\beta}=U_{F'}$. Our claim is equivalent to the fact that for any F, the restriction map $\mathcal{A}_F\to\mathcal{A}_{F'}$ is an isomorphism. We have already seen, by definition of Δ'_α, Δ'_β and Lemma 2.6, that $L_\alpha\simeq(L_\alpha)_{V_\alpha}$, $L_\beta\simeq R\Gamma_{V_\beta}(L_\beta)$. It follows that $R\mathcal{H}om(L_\alpha,L_\beta)\simeq R\Gamma_{V_{\alpha\beta}}R\mathcal{H}om(L_\alpha,L_\beta)$. Using (56), this implies the desired result

$$\begin{aligned}\mathcal{A}_F &\simeq \mathrm{Ext}^{\cdot}_{\mathrm{D}_G(X_Z)}(L_\alpha|_{V_F},L_\beta|_{V_F})\simeq H^{\cdot}_G(V_F;R\mathcal{H}om(L_\alpha,L_\beta))\\ &\simeq H^{\cdot}_G(V_F\cap V_{\alpha\beta};R\mathcal{H}om(L_\alpha,L_\beta))\simeq\mathcal{A}_{F'}.\end{aligned}$$

Now we remark that the functor j_* is exact, because, for any sheaf B on $\mathcal{F}$, any face F, we have $(j_*B)_F\simeq B_{F'}$, for F' satisfying $U_F\cap U_{\alpha\beta}=U_{F'}$ as above. Hence we have $H^i(U;\mathcal{A})\simeq H^i(U\cap U_{\alpha\beta};\mathcal{A})$.

2) This means that we may assume $V\subset V_{\alpha\beta}$. We prove the result by induction on the number of G-orbits in V. If V consists of the open orbit of X_Z, then U is the fundamental open set U_F, with $F=F_{\emptyset,\emptyset}$. Hence $\Gamma(U;\cdot)=(\cdot)_F$ is exact and we are done.

Now let us assume that $V=W\sqcup\mathcal{O}_\Delta$, where $W\subsetneq V$ is a G-stable open subset of X_Z. By induction the result is true for W. Let F_{Δ,J_Δ} be the closed face of $\mathcal{O}_\Delta$. Since $U_{F_{\Delta,J_\Delta}}$ contains $\mathcal{O}_\Delta$, we have $\phi(V)=\phi(W)\cup U_{F_{\Delta,J_\Delta}}$. Setting $U'=\phi(W)\cap U_{F_{\Delta,J_\Delta}}$ and using the Mayer-Vietoris sequence, it is sufficient to prove:

(80) $\quad \forall i>0,\ H^i(U';\mathcal{A})=0 \quad$ and $\quad H^0(U_{F_{\Delta,J_\Delta}};\mathcal{A})\to H^0(U';\mathcal{A})$ is surjective.

3) We compute $H^{\cdot}(U';\mathcal{A})$ with the help of a Čech covering. Remember that $U_{F_{\Delta',J'}}\subset U_{F_{\Delta,J}}$ if and only if $\Delta'\subset\Delta$ and $J\subset J'$. For $\Delta'\subset\Delta$, we have $\mathcal{O}_\Delta\cap U_{F_{\Delta',J'}}=\emptyset$ if and only if $\Delta'\neq\Delta$; but this implicitly assumes that $F_{\Delta',J'}$ is a face, i.e., $\mathcal{O}_{\Delta'}\cap C_{J'}$ is non-empty. For any $\Delta'\subset\Delta$, we have indeed $F_{\Delta',J_\Delta}\neq\emptyset$: this is easily seen if $X_Z=X$, in which case $\mathcal{F}$ is the set of faces of $[0,1]^l$. This implies the general case because $C_{X_Z,J_\Delta}=\pi^{-1}\pi(C_{X,J_\Delta})$ (where π is the map $X_Z\to X$). It follows that

(81) $$U'=\textstyle\bigcup_{\Delta'\subsetneq\Delta}U_{F_{\Delta',J_\Delta}}.$$

By (48), this covering is stable by taking intersections. Since it consists of fundamental open sets, on which the functor of sections is exact, it can be used to

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compute $H^{\cdot}(U';\cdot)$. For this, we have to know $\mathcal{A}(U_F)$. By definition (see (78)), for any face $F = F_{\Delta_1,J_1} \in \mathcal{F}$:

$$\mathcal{A}(U_F) = \mathcal{A}_F = \mathcal{RT}_F^{\alpha\beta} \otimes \mathcal{RM}_F^{\alpha\beta}.$$

Let us describe more precisely the components of the tensor product. Recall that $F \in U_{\alpha\beta}$; hence either $F \notin \mathcal{F}_{\alpha\beta} \cup \mathcal{F}'_{\alpha\beta}$ or $F \in \mathcal{F}_{\alpha\beta}$ (see (55)). In the first case we have $\mathcal{RM}_F^{\alpha\beta} = \mathcal{RT}_F^{\alpha\beta} = 0$. In the second case, by Definition (68) we have

$$\mathcal{RT}^{\alpha\beta}_{F_{\Delta_1,J_1}} = \mathbf{C}[X_v;\, v \in \Delta_1],$$

and, by (77), $\mathcal{RM}_F^{\alpha\beta}$ only depends on J_1, say $\mathcal{RM}_F^{\alpha\beta} = M_{J_1}$. These descriptions of $\mathcal{RT}_F^{\alpha\beta}$ and $\mathcal{RM}_F^{\alpha\beta}$ assume that F_{Δ_1,J_1} is a face. We have seen that this is the case for F_{Δ',J_Δ} with $\Delta' \subset \Delta$.

4) Since, in the covering (81), all faces have the same "J-index", we obtain $H^i(U';\mathcal{A}) = M(J_\Delta) \otimes H^i(C^{\cdot})$, where $C^{\cdot} = C^{\cdot}(S_\Delta)$ is the following complex. We let S_Δ be the set of subsets Δ' of Δ such that $(\Delta_\alpha \cup \Delta_\beta) \subset \Delta' \subsetneq \Delta$ and we consider any total order on S_Δ:

$$C^{\cdot}(S_\Delta) \;=\; 0 \to \bigoplus_{\Delta'_1 \in S_\Delta} \mathbf{C}[X_v;\, v \in \Delta'_1] \to \bigoplus_{\Delta'_1 < \Delta'_2 \in S_\Delta} \mathbf{C}[X_v;\, v \in \Delta'_1 \cap \Delta'_2] \to \cdots .$$

We have a bijection between S_Δ and the set, S_Φ, of strict subsets of $\Phi = \Delta \setminus (\Delta_\alpha \cup \Delta_\beta)$. Setting $C_1 = \mathbf{C}[X_v;\, v \in \Delta_\alpha \cup \Delta_\beta]$ and $C_2^{\cdot} = C^{\cdot}(S_\Phi)$, we obtain $C^{\cdot} = C_1 \otimes C_2^{\cdot}$.

Hence $H^i(U';\mathcal{A}) = M(J_\Delta) \otimes C_1 \otimes H^i(C_2^{\cdot})$. Since $H^0(U_{F_{\Delta,J_\Delta}};\mathcal{A}) \simeq M(J_\Delta) \otimes \mathbf{C}[X_v;\, v \in \Delta]$, (80) will follow from

(82) $\quad \forall i > 0,\ H^i(C_2^{\cdot}) = 0 \quad$ and $\quad \mathbf{C}[X_v;\, v \in \Phi] \to H^0(C_2^{\cdot})$ is surjective.

5) We may interpret $C_2^{\cdot}$ as another Čech complex: we consider the following sheaves on the quadrant $\mathcal{Q} = \mathbf{R}^{\Phi}_{\geq 0}$,

$$M_1 = \bigoplus_{k \in \Phi} \mathbf{C}_{\{x_k = 0\}}, \quad M_2 \text{ associated to } O \mapsto Sym(M_1(O)),$$

where $O \subset \mathcal{Q}$ is open and $Sym(\cdot)$ denotes the symmetric algebra. We also consider the open subsets of $\mathcal{Q}$, for $\Phi' \subset \Phi$, $U_{\Phi'} = \{x_k > 0;\, k \in \Phi \setminus \Phi'\}$. The $U_{\Phi'}$, for $\Phi' \subsetneq \Phi$, give a covering of $\mathcal{Q} \setminus \{0\}$, and

$$\Gamma(U_{\Phi'}; M_2) \simeq \mathbf{C}[X_v;\, v \in \Phi'].$$

Hence $C_2^{\cdot}$ is isomorphic to the Čech complex $\mathcal{C}^{\cdot}((U_{\Phi'})_{\Phi' \subsetneq \Phi}; M_2)$. Now M_2 is constructible for the stratification of $\mathcal{Q}$ by the strata $\mathcal{Q}_{\Phi'} = \{x_k = 0,\, k \in \Phi';\, x_k > 0,\, k \notin \Phi'\}$, $\Phi' \subset \Phi$. Each open $U_{\Phi'}$ is contractible to a point by a homotopy preserving the closures of strata; hence $H^i(U_{\Phi'}; M_2) = 0$, for $i > 0$. It follows that $\forall i$, $H^i(C_2^{\cdot}) \simeq H^i(\mathcal{Q} \setminus \{0\}; M_2)$.

For $\Phi_1 \subset \Phi$ we have of course $\bigotimes_{k \in \Phi_1} \mathbf{C}_{\{x_k=0\}} \simeq \mathbf{C}_{\{x_k = 0; k \in \Phi_1\}}$. Hence M_2 is a sum of sheaves of the type $\mathbf{C}_{\{x_k=0;k\in\Phi_1\}}$. For each of them, say $N = \mathbf{C}_{\{x_k=0;k\in\Phi_1\}}$, we have

(i) $\forall i > 0$, $H^i(\mathcal{Q} \setminus \{0\}; N) = 0$,
(ii) the map $H^0(\mathcal{Q}; N) \to H^0(\mathcal{Q} \setminus \{0\}; N)$ is surjective.

By additivity, both assertions are true with M_2 instead of N. We also have $H^0(\mathcal{Q}; M_2) \simeq \mathbf{C}[X_v;\, v \in \Phi]$. Hence we obtain (82) and the lemma is proved. □

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