

AN AVERAGING PROCESS FOR UNIPOTENT GROUP ACTIONS

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ABSTRACT. We present an averaging process for sections of a torsor under a unipotent group. This process allows one to integrate local sections of such a torsor into a global simplicial section. The results of this paper have applications to deformation quantization of algebraic varieties.

0. INTRODUCTION

Let \mathbb{K} be a field of characteristic 0. For any natural number q let $\mathbb{K}[t_0, \dots, t_q]$ be the polynomial algebra, and let $\Delta_{\mathbb{K}}^q$ be the *geometric q -dimensional simplex*

$$\Delta_{\mathbb{K}}^q := \text{Spec } \mathbb{K}[t_0, \dots, t_q]/(t_0 + \dots + t_q - 1).$$

Our main result is the following theorem.

Theorem 0.1. *Let G be a unipotent algebraic group over \mathbb{K} , let X be a \mathbb{K} -scheme, let Z be a G -torsor over X , and let Y be any X -scheme. Suppose $\mathbf{f} = (f_0, \dots, f_q)$ is a sequence of X -morphisms $f_i : Y \rightarrow Z$. Then there is an X -morphism*

$$\text{wav}_G(\mathbf{f}) : \Delta_{\mathbb{K}}^q \times Y \rightarrow Z$$

called the weighted average, such that the operation wav_G is symmetric, simplicial, functorial in the data (G, X, Y, Z) , and is the identity for $q = 0$.

“Symmetric” means that wav_G is equivariant for the action of the permutation group of $\{0, \dots, q\}$ on the sequence \mathbf{f} and the scheme $\Delta_{\mathbb{K}}^q$. “Simplicial” says that as q varies

$$\text{wav}_G : \text{Hom}_X(Y, Z)^{q+1} \rightarrow \text{Hom}_X(\Delta_{\mathbb{K}}^q \times Y, Z)$$

is a map of simplicial sets. “Functorial in the data (G, X, Y, Z) ” means that, e.g., given a morphism $h : Z \rightarrow Z'$ of G -torsors over X , one has

$$h \circ \text{wav}_G(\mathbf{f}) = \text{wav}_G(h \circ \mathbf{f}),$$

where $h \circ \mathbf{f}$ is the sequence $(h \circ f_0, \dots, h \circ f_q)$. Theorem 0.1 is repeated, in full detail, as Theorem 1.11.

Observe that when we restrict the morphism $\text{wav}_G(\mathbf{f})$ to each of the vertices of $\Delta_{\mathbb{K}}^q$ we recover the original morphisms f_0, \dots, f_q . This is due to the simplicial property of wav_G . Thus, $\text{wav}_G(\mathbf{f})$ interpolates between f_0, \dots, f_q . This is illustrated in Figure 1.

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Here is a naive corollary of Theorem 0.1, which can further explain the result. Let us write $G(\mathbb{K})$ for the group of \mathbb{K} -rational points of G . By a *weight sequence* in \mathbb{K} we mean a sequence $\mathbf{w} = (w_0, \dots, w_q)$ of elements of \mathbb{K} such that $\sum w_i = 1$.

Corollary 0.2. *Let G be a unipotent group over \mathbb{K} . Suppose Z is a set with $G(\mathbb{K})$ -action which is transitive and has trivial stabilizers. Let $\mathbf{z} = (z_0, \dots, z_q)$ be a sequence of points in Z and let \mathbf{w} be a weight sequence in \mathbb{K} . Then there is a point $\text{wav}_{G, \mathbf{w}}(\mathbf{z}) \in Z$ called the weighted average. The operation wav_G is symmetric, functorial, simplicial, and is the identity for $q = 0$.*

The corollary is proved in Section 1. The idea is of course to take $Y = X := \text{Spec } \mathbb{K}$ in the theorem, and to note that \mathbf{w} is a \mathbb{K} -rational point of $\Delta_{\mathbb{K}}^q$.

Another consequence of Theorem 0.1 is a new proof of the fact that a unipotent group in characteristic 0 is special. See Remark 1.12. (This observation is due to Reichstein.)

Theorem 0.1 was discovered in the course of research on deformation quantization in algebraic geometry [Ye], in which we tried to apply ideas of Kontsevich [Ko] to the algebraic context. Here is a brief outline. Suppose X is a smooth n -dimensional \mathbb{K} -scheme. The coordinate bundle of X [GK, Ko] is an infinite dimensional bundle $Z \rightarrow X$ which parameterizes formal coordinate systems on X . The bundle Z is a torsor under an affine group scheme $\text{GL}_{n, \mathbb{K}} \times G$, where G is pro-unipotent. One is interested in sections of the quotient bundle $\bar{Z} := Z/\text{GL}_{n, \mathbb{K}}$. If we are in the differentiable setup (i.e. $\mathbb{K} = \mathbb{R}$ and X is a C^∞ manifold), then the fibers of \bar{Z} are contractible (since they are isomorphic to $G(\mathbb{R})$), and therefore global C^∞ sections exist. However, in the setup of algebraic geometry such an argument is invalid, and we were forced to seek an alternative approach.

Our solution was to use *simplicial sections* of \bar{Z} (see Section 2, and especially Corollary 2.7). This approach was inspired by work of Bott [Bo] on simplicial connections (cf. also [HY]).

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1. THE AVERAGING PROCESS

Throughout the paper \mathbb{K} denotes a fixed base field of characteristic 0. All schemes and all morphisms are over \mathbb{K} .

We begin by recalling some standard facts about the combinatorics of simplicial objects. Let Δ denote the category with objects the ordered sets $[q] := \{0, 1, \dots, q\}$, $q \in \mathbb{N}$. The morphisms $[p] \rightarrow [q]$ are the order-preserving functions, and we write $\Delta_p^q := \text{Hom}_\Delta([p], [q])$. The i -th co-face map $\partial^i : [p] \rightarrow [p+1]$ is the injective function that does not take the value i ; and the i -th co-degeneracy map $s^i : [p] \rightarrow [p-1]$ is the surjective function that takes the value i twice. All morphisms in Δ are compositions of various ∂^i and s^i .

An element of Δ_p^q may be thought of as a sequence $\mathbf{i} = (i_0, \dots, i_p)$ of integers with $0 \leq i_0 \leq \dots \leq i_p \leq q$. Given $\mathbf{i} \in \Delta_q^m$, $\mathbf{j} \in \Delta_m^p$ and $\alpha \in \Delta_p^q$, we sometimes write $\alpha_*(\mathbf{i}) := \mathbf{i} \circ \alpha \in \Delta_p^m$ and $\alpha^*(\mathbf{j}) := \alpha \circ \mathbf{j} \in \Delta_m^q$.

Let \mathbf{C} be some category. A *cosimplicial object* in \mathbf{C} is a functor $C : \Delta \rightarrow \mathbf{C}$. We shall usually refer to the cosimplicial object as $C = \{C^p\}_{p \in \mathbb{N}}$, and for any $\alpha \in \Delta_p^q$ the corresponding morphism in \mathbf{C} will be denoted by $\alpha^* : C^p \rightarrow C^q$. A *simplicial*

object in \mathbf{C} is a functor $C : \Delta^{\text{op}} \rightarrow \mathbf{C}$. The notation for a simplicial object will be $C = \{C_p\}_{p \in \mathbb{N}}$ and $\alpha_* : C_q \rightarrow C_p$.

An important example is the cosimplicial scheme $\{\Delta_{\mathbb{K}}^p\}_{p \in \mathbb{N}}$. The morphisms are defined as follows. For any p we identify the ordered set $[p]$ with the set of vertices of $\Delta_{\mathbb{K}}^p$. Given $\alpha \in \Delta_{\mathbb{K}}^q$, the morphism $\alpha^* : \Delta_{\mathbb{K}}^p \rightarrow \Delta_{\mathbb{K}}^q$ is then the unique linear morphism extending $\alpha : [p] \rightarrow [q]$.

Let G be a unipotent (affine finite type) algebraic group over \mathbb{K} , with (nilpotent) Lie algebra \mathfrak{g} . We write $d(G)$ for the minimal number d such that there exists a chain of closed normal subgroups $G = G_0 \supset G_1 \cdots \supset G_d = 1$ with G_k/G_{k+1} abelian for all $k \in \{0, \dots, d-1\}$. The exponential map $\exp_G : \mathfrak{g} \rightarrow G$ is an isomorphism of schemes, with inverse \log_G ; see [Ho, Theorem VIII.1.1].

Given a \mathbb{K} -scheme X , there is a Lie algebra $\mathfrak{g} \times X$ (in the category of X -schemes), and a group-scheme $G \times X$. There is also an induced exponential map

$$\exp_{G \times X} := \exp_G \times \mathbf{1}_X : \mathfrak{g} \times X \rightarrow G \times X.$$

We will need the following result.

Lemma 1.1. *Let G, G' be two unipotent groups, with Lie algebras $\mathfrak{g}, \mathfrak{g}'$. Let X, X' be schemes, let $X \rightarrow X'$ be a morphism of schemes, and let $\phi : G \times X \rightarrow G' \times X'$ be a morphism of group-schemes over X' . Denote by $d\phi : \mathfrak{g} \times X \rightarrow \mathfrak{g}' \times X'$ the induced Lie algebra morphism (the differential of ϕ). Then the diagram*

$$(1.2) \quad \begin{array}{ccc} \mathfrak{g} \times X & \xrightarrow{\exp_{G \times X}} & G \times X \\ d\phi \downarrow & & \downarrow \phi \\ \mathfrak{g}' \times X' & \xrightarrow{\exp_{G' \times X'}} & G' \times X' \end{array}$$

commutes.

Proof. For the case $X = X' = \text{Spec } \mathbb{K}$ this is contained in the proof of [Ho, Theorem VIII.1.2].

In order to handle the general case we first recall the Campbell-Baker-Hausdorff formula:

$$\exp_G(\gamma_1) \cdot \exp_G(\gamma_2) = \exp_G(F(\gamma_1, \gamma_2)),$$

where

$$F(\gamma_1, \gamma_2) = \gamma_1 + \gamma_2 + \frac{1}{2}[\gamma_1, \gamma_2] + \cdots$$

is a universal power series (see [Ho, Section XVI.2]). Hence if we define $\gamma_1 * \gamma_2 := F(\gamma_1, \gamma_2)$, then $(\mathfrak{g}, *)$ becomes an algebraic group (with unit element 0), and $\exp_G : (\mathfrak{g}, *) \rightarrow (G, \cdot)$ is a group isomorphism. In this way we may eliminate G altogether, and just look at the scheme \mathfrak{g} with its two structures: a Lie algebra and a group-scheme. Note that now \mathfrak{g} is its own Lie algebra, as can be seen from the second-order term in the series $F(\gamma_1, \gamma_2)$.

Consider a morphism $\phi : \mathfrak{g} \times X \rightarrow \mathfrak{g}' \times X'$ of X' -schemes. Then ϕ is a morphism of Lie algebras over X' iff it is a morphism of group-schemes (for the multiplications $*$) and, moreover, $d\phi = \phi$. Therefore, the diagram (1.2) is commutative. \square

From now on we will write \exp_G instead of $\exp_{G \times X}$, for the sake of brevity.

Consider the following setup: X is a \mathbb{K} -scheme, and Y, Z are two X -schemes. Suppose Z is a torsor under the group scheme $G \times X$. We denote the action of G on Z by $(g, z) \mapsto g \cdot z$.

Let $f_0, \dots, f_q : \Delta_{\mathbb{K}}^q \times Y \rightarrow Z$ be X -morphisms. We are going to define a new sequence of X -morphisms $f'_0, \dots, f'_q : \Delta_{\mathbb{K}}^q \times Y \rightarrow Z$. Because Z is a torsor under $G \times X$, for any $i, j \in \{0, \dots, q\}$ there exists a unique morphism $g_{i,j} : \Delta_{\mathbb{K}}^q \times Y \rightarrow G$ such that $f_j(w, y) = g_{i,j}(w, y) \cdot f_i(w, y)$ for all points $w \in \Delta_{\mathbb{K}}^q$ and $y \in Y$. Here w and y are scheme-theoretic points, i.e., $w \in \Delta_{\mathbb{K}}^q(U) = \text{Hom}_{\mathbb{K}}(U, \Delta_{\mathbb{K}}^q)$ and $y \in Y(U) = \text{Hom}_{\mathbb{K}}(U, Y)$ for some \mathbb{K} -scheme U . Define

$$(1.3) \quad f'_i(w, y) := \exp_G \left(\sum_{j=0}^q t_j(w) \cdot \log_G(g_{i,j}(w, y)) \right) \cdot f_i(w, y).$$

In this formula we view t_j as a morphism $t_j : \Delta_{\mathbb{K}}^q \rightarrow \mathbf{A}_{\mathbb{K}}^1$, and we use the fact that \mathfrak{g} is a vector space (in the category of \mathbb{K} -schemes).

For any set S let us write $S^{\Delta_0^q}$ for the set of functions $\Delta_0^q \rightarrow S$, which is the same as the set of sequences (s_0, \dots, s_q) in S . As usual $\text{Hom}_X(\Delta_{\mathbb{K}}^q \times Y, Z)$ is the set of X -morphisms $\Delta_{\mathbb{K}}^q \times Y \rightarrow Z$. In this notation the sequences (f_0, \dots, f_q) and (f'_0, \dots, f'_q) are elements of $\text{Hom}_X(\Delta_{\mathbb{K}}^q \times Y, Z)^{\Delta_0^q}$. We denote by

$$\text{wsym}_G : \text{Hom}_X(\Delta_{\mathbb{K}}^q \times Y, Z)^{\Delta_0^q} \rightarrow \text{Hom}_X(\Delta_{\mathbb{K}}^q \times Y, Z)^{\Delta_0^q}$$

the operation $(f_0, \dots, f_q) \mapsto (f'_0, \dots, f'_q)$ given by the formula (1.3).

We will also need a similar operation

$$w_G : \text{Hom}_X(Y, Z)^{\Delta_0^q} \rightarrow \text{Hom}_X(\Delta_{\mathbb{K}}^q \times Y, Z)^{\Delta_0^q},$$

defined by

$$w_G(f_0, \dots, f_q) := (f'_0, \dots, f'_q)$$

with

$$(1.4) \quad f'_i(w, y) := \exp_G \left(\sum_{j=0}^q t_j(w) \cdot \log_G(g_{i,j}(y)) \right) \cdot f_i(y).$$

It is clear that for $q = 0$ both operations w_G and wsym_G act as the identity, i.e., $w_G(f_0) = \text{wsym}_G(f_0) = f_0$ for all $f_0 \in \text{Hom}_X(Y, Z)$. Both operations w_G and wsym_G are symmetric, namely they are equivariant for the simultaneous action of the permutation group of $\{0, \dots, q\}$ on Δ_0^q and $\Delta_{\mathbb{K}}^q$. Also, if $\mathbf{f} = (f_0, \dots, f_q) \in \text{Hom}_X(\Delta_{\mathbb{K}}^q \times Y, Z)^{\Delta_0^q}$ is a constant sequence, i.e., $f_0 = \dots = f_q$, then $\text{wsym}_G(\mathbf{f}) = \mathbf{f}$.

Lemma 1.5. *Both operations w_G and wsym_G are simplicial. Namely, given $\alpha \in \Delta_p^q$ the diagram*

$$\begin{array}{ccc} \text{Hom}_X(\Delta_{\mathbb{K}}^q \times Y, Z)^{\Delta_0^q} & \xrightarrow{\text{wsym}_G} & \text{Hom}_X(\Delta_{\mathbb{K}}^q \times Y, Z)^{\Delta_0^q} \\ \alpha_* \downarrow & & \alpha_* \downarrow \\ \text{Hom}_X(\Delta_{\mathbb{K}}^p \times Y, Z)^{\Delta_0^p} & \xrightarrow{\text{wsym}_G} & \text{Hom}_X(\Delta_{\mathbb{K}}^p \times Y, Z)^{\Delta_0^p} \end{array}$$

is commutative, and likewise for w_G .

Proof. It suffices to consider $\alpha = \partial^i$ or $\alpha = s^i$. Since wsym_G is symmetric, we may assume that $\alpha = \partial^q : [q-1] \rightarrow [q]$ or $\alpha = s^q : [q+1] \rightarrow [q]$. Fix a sequence $\mathbf{f} = (f_0, \dots, f_q) \in \text{Hom}_X(\Delta_{\mathbb{K}}^q \times Y, Z)^{\Delta_0^q}$. Let $g_{i,j} \in \text{Hom}_{\mathbb{K}}(\Delta_{\mathbb{K}}^q \times Y, G)$ be such that $f_j = g_{i,j} \cdot f_i$, and let $\mathbf{f}' = (f'_0, \dots, f'_q) := \text{wsym}_G(\mathbf{f})$.

First we look at the case $\alpha = \partial^q$. Take $w \in \Delta_{\mathbb{K}}^{q-1}$ and $y \in Y$, and let $v := \alpha^*(w) \in \Delta_{\mathbb{K}}^q$. The coordinates of v are $t_j(v) = t_j(w)$ for $j \leq q - 1$, and $t_q(v) = 0$. Then for any i the i -th term in the sequence $\alpha_*(\mathbf{f}')$, evaluated at (w, y) , equals

$$\begin{aligned} f'_i(v, y) &= \exp_G \left(\sum_{j=0}^q t_j(v) \cdot \log_G(g_{i,j}(v, y)) \right) \cdot f_i(v, y) \\ (1.6) \qquad &= \exp_G \left(\sum_{j=0}^{q-1} t_j(w) \cdot \log_G(g_{i,j}(v, y)) \right) \cdot f_i(v, y). \end{aligned}$$

On the other hand, the i -th term of the sequence $\text{wsym}_G(\alpha_*(\mathbf{f}))$ is

$$\begin{aligned} &\exp_G \left(\sum_{j=0}^{q-1} t_j(w) \cdot \log_G(\alpha_*(g_{i,j})(w, y)) \right) \cdot \alpha_*(f_i)(w, y) \\ &= \exp_G \left(\sum_{j=0}^{q-1} t_j(w) \cdot \log_G(g_{i,j}(v, y)) \right) \cdot f_i(v, y). \end{aligned}$$

So, indeed, $\alpha_* \circ \text{wsym}_G = \text{wsym}_G \circ \alpha_*$ in this case.

Next consider the case $\alpha = s^q$. Take $w \in \Delta_{\mathbb{K}}^{q+1}$ and $y \in Y$, and let $v := \alpha^*(w) \in \Delta_{\mathbb{K}}^q$. The coordinates of v are $t_j(v) = t_j(w)$ for $j \leq q - 1$, and $t_q(v) = t_q(w) + t_{q+1}(w)$. For any $i \leq q$ the i -th term in the sequence $\alpha_*(\mathbf{f})$, evaluated at (w, y) , is $f'_i(v, y)$, which was calculated in (1.6). The $(q + 1)$ -st term is also $f'_q(v, y)$. On the other hand, for any $i \leq q$ the i -th term in the sequence $\text{wsym}_G(\alpha_*(\mathbf{f}'))$, evaluated at (w, y) , is

$$z_i := \exp_G \left(\sum_{j=0}^{q+1} t_j(w) \cdot \log_G(\alpha_*(g_{i,j})(w, y)) \right) \cdot \alpha_*(f_i)(w, y).$$

But $t_q(w) + t_{q+1}(w) = t_q(v)$, $\alpha_*(g_{i,j})(w, y) = g_{i,j}(v, y)$ for $j \leq q$, and $\alpha_*(g_{i,q+1})(w, y) = g_{i,q}(v, y)$. Therefore,

$$z_i = \exp_G \left(\sum_{j=0}^q t_j(v) \cdot \log_G(g_{i,j}(v, y)) \right) \cdot f_i(v, y).$$

For $i = q + 1$ one has $z_{q+1} = z_q$. We conclude that $\alpha_* \circ \text{wsym}_G = \text{wsym}_G \circ \alpha_*$ in this case too.

The proof for w_G is the same. □

Lemma 1.7. *Both operations w_G and wsym_G are functorial in the data (G, X, Y, Z) . Namely, given another such quadruple (G', X', Y', Z') , a morphism of schemes $X \rightarrow X'$, a morphism of schemes $e : Y' \rightarrow Y$ over X' , a morphism of group-schemes $\phi : G \times X \rightarrow G' \times X'$ over X' , and a $G \times X$ -equivariant morphism of schemes $f : Z \rightarrow Z'$ over X' , the diagram*

$$\begin{array}{ccc} \text{Hom}_X(\Delta_{\mathbb{K}}^q \times Y, Z)^{\Delta_0^q} & \xrightarrow{\text{wsym}_G} & \text{Hom}_X(\Delta_{\mathbb{K}}^q \times Y, Z)^{\Delta_0^q} \\ (e, f) \downarrow & & (e, f) \downarrow \\ \text{Hom}_{X'}(\Delta_{\mathbb{K}}^q \times Y', Z')^{\Delta_0^q} & \xrightarrow{\text{wsym}_{G'}} & \text{Hom}_{X'}(\Delta_{\mathbb{K}}^q \times Y', Z')^{\Delta_0^q} \end{array}$$

is commutative, and likewise for w_G .

Proof. This is due to the functoriality of the exponential map; see Lemma 1.1. □

Lemma 1.8. *Assume G is abelian. Then for any $(f_0, \dots, f_q) \in \text{Hom}_X(\Delta_{\mathbb{K}}^q \times Y, Z)^{\Delta_0^q}$ the sequence $\text{wsym}_G(f_0, \dots, f_q)$ is constant.*

Proof. In this case $\exp : \mathfrak{g} \rightarrow G$ is an isomorphism of algebraic groups, where \mathfrak{g} is viewed as an additive group. So we may assume that Z is a torsor under $\mathfrak{g} \times X$. Let $(f'_0, \dots, f'_q) := \text{wsym}_G(f_0, \dots, f_q)$, and let $\gamma_{i,j} : \Delta_{\mathbb{K}}^q \times Y \rightarrow \mathfrak{g}$ be morphisms such that $f_j = \gamma_{i,j} + f_i$. Take $(w, y) \in \Delta_{\mathbb{K}}^q \times Y$. Then

$$f'_i(w, y) = \left(\sum_{j=0}^q t_j(w) \cdot \gamma_{i,j}(w, y) \right) + f_i(w, y)$$

for any i . Because $\gamma_{i,j} = -\gamma_{j,i} = \gamma_{0,j} - \gamma_{0,i}$, $f_i = f_0 + \gamma_{0,i}$ and $\sum_{j=0}^q t_j(w) = 1$, it follows that $f'_i(w, y) = f'_0(w, y)$. \square

We write wsym_G^d for the d -th iteration of the operation wsym_G .

Lemma 1.9. *For any $\mathbf{f} = (f_0, \dots, f_q) \in \text{Hom}_X(\Delta_{\mathbb{K}}^q \times Y, Z)^{\Delta_0^q}$, the sequence $\text{wsym}_G^{d(G)}(\mathbf{f})$ is constant. For any $d \geq d(G)$, one has $\text{wsym}_G^d(\mathbf{f}) = \text{wsym}_G^{d(G)}(\mathbf{f})$.*

Proof. For any k , the orbit of $f_0 \in \text{Hom}_X(\Delta_{\mathbb{K}}^q \times Y, Z)$ under the action of the group $G_k(\Delta_{\mathbb{K}}^q \times Y)$ will be denoted by $G_k(\Delta_{\mathbb{K}}^q \times Y) \cdot f_0$. Let $(f'_0, \dots, f'_q) := \text{wsym}_G(f_0, \dots, f_q)$. We will prove that if $f_1, \dots, f_q \in G_k(\Delta_{\mathbb{K}}^q \times Y) \cdot f_0$, then $f'_1, \dots, f'_q \in G_{k+1}(\Delta_{\mathbb{K}}^q \times Y) \cdot f'_0$. The assertions of the lemma will then follow.

Let $\tilde{Y} = \tilde{X} := \Delta_{\mathbb{K}}^q \times Y$ and $\tilde{Z} := \tilde{X} \times_X Z$. So \tilde{Z} is a torsor under $G \times \tilde{X}$, and f_0 induces a morphism $\tilde{f}_0 \in \text{Hom}_{\tilde{X}}(\tilde{Y}, \tilde{Z})$. The morphism $\tau : G \times \tilde{X} \rightarrow \tilde{Z}$, $(g, \tilde{x}) \mapsto g \cdot \tilde{f}_0(\tilde{x})$, is an isomorphism of \tilde{X} -schemes. Define $\tilde{W} := \tau(G \times \tilde{X}) \subset \tilde{Z}$. Then \tilde{W} is the “geometric orbit” of \tilde{f}_0 under $G_k \times \tilde{X}$; and, in particular, \tilde{W} is a torsor under $G_k \times \tilde{X}$. By assumption $\tilde{f}_1, \dots, \tilde{f}_q \in \text{Hom}_{\tilde{X}}(\tilde{Y}, \tilde{W})$. Define $(\tilde{f}'_0, \dots, \tilde{f}'_q) := \text{wsym}_{G_k}(\tilde{f}_0, \dots, \tilde{f}_q)$. By Lemma 1.7 it suffices to prove that $\tilde{f}'_1, \dots, \tilde{f}'_q \in G_{k+1}(\tilde{Y}) \cdot \tilde{f}'_0$.

Define $\bar{W} := \tilde{W}/G_{k+1}$. This is a torsor under the group scheme $(G_k/G_{k+1}) \times \tilde{X}$. Let $\bar{f}_0, \dots, \bar{f}_q \in \text{Hom}_{\tilde{X}}(\tilde{Y}, \bar{W})$ be the images of $(\tilde{f}_0, \dots, \tilde{f}_q)$. Because the group G_k/G_{k+1} is abelian, Lemma 1.8 says that $\text{wsym}_{G_k/G_{k+1}}(\bar{f}_0, \dots, \bar{f}_q)$ is a constant sequence. Again using Lemma 1.7, we see that, in fact, $\bar{f}'_1, \dots, \bar{f}'_q \in G_{k+1}(\tilde{Y}) \cdot \bar{f}'_0$. \square

Given an X -scheme Y , the collections $\{\text{Hom}_X(\Delta_{\mathbb{K}}^q \times Y, Z)\}_{q \in \mathbb{N}}$ and $\{\text{Hom}_X(Y, Z)^{\Delta_0^q}\}_{q \in \mathbb{N}}$ are simplicial sets. For $q = 0$ there are equalities

$$(1.10) \quad \text{Hom}_X(\Delta_{\mathbb{K}}^0 \times Y, Z) = \text{Hom}_X(Y, Z) = \text{Hom}_X(Y, Z)^{\Delta_0^0}.$$

Theorem 1.11. *Let G be a unipotent algebraic group over \mathbb{K} , let X be a \mathbb{K} -scheme, and let $Z \rightarrow X$ be a G -torsor over X . For any X -scheme Y and natural number q there is a function*

$$\text{wav}_G : \text{Hom}_X(Y, Z)^{\Delta_0^q} \rightarrow \text{Hom}_X(\Delta_{\mathbb{K}}^q \times Y, Z)$$

called the weighted average. The function wav_G enjoys the following properties.

- (1) *Symmetric:* wav_G is equivariant for the action of the permutation group of $\{0, \dots, q\}$ on Δ_0^q and on $\Delta_{\mathbb{K}}^q$.
- (2) *Simplicial:* wav_G is a map of simplicial sets

$$\{\text{Hom}_X(Y, Z)^{\Delta_0^q}\}_{q \in \mathbb{N}} \rightarrow \{\text{Hom}_X(\Delta_{\mathbb{K}}^q \times Y, Z)\}_{q \in \mathbb{N}}.$$

- (3) *Functorial:* given another such quadruple (G', X', Y', Z') , a morphism of schemes $X \rightarrow X'$, a morphism of X' -group-schemes $G \times X \rightarrow G' \times X'$, a $G \times X$ -equivariant morphism of X' -schemes $f : Z \rightarrow Z'$ and a morphism of X' -schemes $e : Y' \rightarrow Y$, the diagram

$$\begin{CD} \mathrm{Hom}_X(Y, Z)^{\Delta_0^q} @>\mathrm{wav}_G>> \mathrm{Hom}_X(\Delta_{\mathbb{K}}^q \times Y, Z) \\ @V(e, f)VV @VV(e, f)V \\ \mathrm{Hom}_{X'}(Y', Z')^{\Delta_0^q} @>\mathrm{wsym}_{G'}>> \mathrm{Hom}_{X'}(\Delta_{\mathbb{K}}^q \times Y', Z') \end{CD}$$

is commutative.

- (4) If $q = 0$, then wav_G is the identity map of $\mathrm{Hom}_X(Y, Z)$.

Proof. Given a sequence $\mathbf{f} = (f_0, \dots, f_q) \in \mathrm{Hom}_X(Y, Z)^{\Delta_0^q}$, define $\mathrm{wav}_G(\mathbf{f}) := f' \in \mathrm{Hom}_X(\Delta_{\mathbb{K}}^q \times Y, Z)$ to be the morphism such that

$$(\mathrm{wsym}_{G'}^{d(G)} \circ \mathrm{w}_G)(f_0, \dots, f_q) = (f', \dots, f');$$

see Lemma 1.9. Properties (1)–(4) follow from the corresponding properties of w_G and wsym_G . \square

Proof of Corollary 0.2. Take $X = Y := \mathrm{Spec} \mathbb{K}$ in Theorem 1.11, and consider the G -torsor $\underline{Z} := G$. Choose any base point $z \in Z$; this defines an isomorphism of left $G(\mathbb{K})$ -sets $\underline{Z}(\mathbb{K}) \cong Z$. The weight sequence \mathbf{w} can be considered as a \mathbb{K} -rational point of $\Delta_{\mathbb{K}}^q$, and we define

$$\mathrm{wav}_{G, \mathbf{w}}(z) := \mathrm{wav}_G(z)(\mathbf{w}) \in Z.$$

If we were to choose another base point $z' \in Z$, this would amount to applying an automorphism of the torsor \underline{Z} , namely right multiplication by some element of $G(\mathbb{K})$. Due to the functoriality of wav_G the point $\mathrm{wav}_{G, \mathbf{w}}(z)$ will be unchanged.

The properties of this set-theoretical averaging process are now immediate consequences of the corresponding properties of the geometric average. \square

Remark 1.12. Z. Reichstein observed that our averaging process provides a new proof (in characteristic 0) of the fact that a unipotent group G is special, namely any G -torsor Z over \mathbb{K} has a \mathbb{K} -rational point. Let us explain the idea.

Let $z_0 \in Z$ be some closed point. Choose a finite Galois extension L of \mathbb{K} containing the residue field $\mathbf{k}(z_0)$. Let Γ be the Galois group of L over \mathbb{K} , which acts on the set $Z(L)$. Let $z_0, \dots, z_q \in Z(L)$ be the Γ -conjugates of z_0 . The group Γ acts on the sequence $\mathbf{z} := (z_0, \dots, z_q)$ by permutations. Thus the simultaneous action of Γ on

$$Z(L)^{\Delta_0^q} = \mathrm{Hom}_{\mathrm{Spec} \mathbb{K}}(\mathrm{Spec} L, Z)^{\Delta_0^q}$$

fixes \mathbf{z} .

We know that the operator wav_G is symmetric. And functoriality says that the action of the Galois group on $\mathrm{Spec} L$ is also respected. Since \mathbf{z} is fixed by the simultaneous action of Γ , so is $\mathrm{wav}_G(\mathbf{z})$. Take the uniform weight sequence $\mathbf{w} := (\frac{1}{q+1}, \dots, \frac{1}{q+1})$ and define $z' := \mathrm{wav}_G(\mathbf{z})(\mathbf{w}) \in Z(L)$. Because \mathbf{w} is fixed by the permutation group we conclude that z' is Γ -invariant, and hence $z' \in Z(\mathbb{K})$.

Remark 1.13. Theorem 1.11 has a rather obvious parallel in differential geometry. Indeed, a simply connected nilpotent Lie group is the same as the group $G(\mathbb{R})$ of rational points of a unipotent algebraic group G over \mathbb{R} .

2. SIMPLICIAL SECTIONS

In this section we show how the averaging process is used to obtain simplicial sections of certain bundles.

Suppose H and G are affine group schemes over \mathbb{K} , and H acts on G by automorphisms. Namely, there is a morphism of schemes $H \times G \rightarrow G$ which for every \mathbb{K} -scheme Y induces a group homomorphism $H(Y) \rightarrow \text{Aut}_{\text{Groups}}(G(Y))$. Then $H \times G$ has a structure of a group scheme, and we denote this group by $H \ltimes G$; it is a geometric semi-direct product.

Recall that an affine group scheme G is called *pro-unipotent* if it is isomorphic to an inverse limit $\lim_{\leftarrow i} G_i$ of an inverse system $\{G_i\}_{i \geq 0}$ of (finite type affine) unipotent groups. One may assume that each of the morphisms $G \rightarrow G_i \rightarrow G_{i-1}$ is surjective. Thus $G_i \cong G/N_i$ where N_i is a normal closed subgroup of G .

We will be concerned with the following geometric situation.

Scenario 2.1. Let $H \ltimes G$ be an affine group scheme over \mathbb{K} . Assume G is pro-unipotent and, moreover, there exists a sequence $\{N_i\}_{i \geq 0}$ of H -invariant closed normal subgroups of G such that $G \cong \lim_{\leftarrow i} G/N_i$ and each G/N_i is unipotent. Let $\pi : Z \rightarrow X$ be an $H \ltimes G$ -torsor over X which is locally trivial for the Zariski topology of X . Define $\bar{Z} := Z/H$ and let $\bar{\pi} : \bar{Z} \rightarrow X$ be the projection.

Theorem 2.2. *Assume Scenario 2.1. Suppose $U \subset X$ is an open set and $\sigma_0, \dots, \sigma_q : U \rightarrow \bar{Z}$ are sections of $\bar{\pi}$. Then there exists a morphism*

$$\sigma : \Delta_{\mathbb{K}}^q \times U \rightarrow \bar{Z}$$

such that the diagram

$$\begin{array}{ccc} \Delta_{\mathbb{K}}^q \times U & \xrightarrow{\sigma} & \bar{Z} \\ \text{p}_2 \downarrow & & \bar{\pi} \downarrow \\ U & \longrightarrow & X \end{array}$$

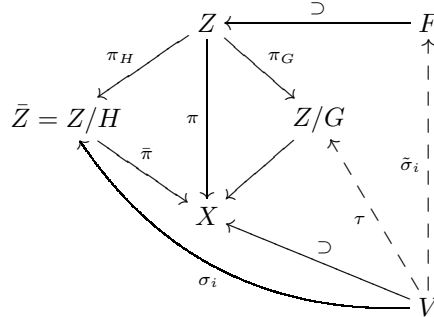
is commutative. The morphism σ depends functorially on U and simplicially on the sequence $(\sigma_0, \dots, \sigma_q)$. If $q = 0$, then $\sigma = \sigma_0$.

Proof. We might as well assume that $U = X$. Consider the quotient Z/G . Since G is normal in $H \ltimes G$ it follows that Z/G is a torsor under $H \times X$. We write $\pi_H : Z \rightarrow \bar{Z} = Z/H$ and $\pi_G : Z \rightarrow Z/G$ for the projections.

Pick an open set $V \subset X$ which trivializes $\pi : Z \rightarrow X$. We write $Z|_V := \pi^{-1}(V)$. Because $\pi_H|_V : Z|_V \rightarrow \bar{Z}|_V$ is a trivial torsor under $H \times \bar{Z}|_V$, we can lift the sections $\sigma_0, \dots, \sigma_q$ to sections $\tilde{\sigma}_0, \dots, \tilde{\sigma}_q : V \rightarrow Z$ such that $\pi_H \circ \tilde{\sigma}_j = \sigma_j$. Furthermore, since $\pi_G : Z \rightarrow Z/G$ is H -equivariant and Z/G is a torsor under $H \times X$, it follows that we can choose $\tilde{\sigma}_0, \dots, \tilde{\sigma}_q$ such that $\pi_G \circ \tilde{\sigma}_j = \tau$ for some section $\tau : V \rightarrow Z/G$.

Let $F \subset Z|_V$ be the fiber over τ , i.e., $F := V \times_{Z/G} Z$ via the morphisms $\pi_G : Z \rightarrow Z/G$ and $\tau : V \rightarrow Z/G$. Then F is a torsor under $G \times V$, and $\tilde{\sigma}_0, \dots, \tilde{\sigma}_q \in$

$\text{Hom}_X(V, F)$. See the diagram below.



For any i define $F_i := F/N_i$, which is a torsor under $(G/N_i) \times V$. Let $\alpha_i : F \rightarrow F_i$ be the projection, so $\alpha_i \circ \tilde{\sigma}_j \in \text{Hom}_X(V, F_i)$. By Theorem 1.11 we get an average

$$(2.3) \quad \rho_i := \text{wav}_{G/N_i}(\alpha_i \circ \tilde{\sigma}_0, \dots, \alpha_i \circ \tilde{\sigma}_q) : \Delta_{\mathbb{K}}^q \times V \rightarrow F_i.$$

The functoriality of wav says that the ρ_i form an inverse system, and we let

$$(2.4) \quad \rho := \lim_{\leftarrow i} \rho_i : \Delta_{\mathbb{K}}^q \times V \rightarrow F$$

and

$$(2.5) \quad \sigma := \pi_H \circ \rho : \Delta_{\mathbb{K}}^q \times V \rightarrow \bar{Z}.$$

We claim that the morphism σ does not depend on the choice of the section $\tau : V \rightarrow Z/G$. Suppose $\tau' : V \rightarrow Z/G$ is another such section. Let F' be the fiber over τ' , and let $\rho' : \Delta_{\mathbb{K}}^q \times V \rightarrow F'$ be the corresponding morphism as in (2.4). Now $\tau' = h \cdot \tau$ for some morphism $h : V \rightarrow H$. Then $F' = h \cdot F$, and $h : F \rightarrow F'$ is a $G \times V$ -equivariant morphism of torsors, with respect to the group-scheme automorphism $\text{Ad}(h) : G \times V \rightarrow G \times V$. The new lift of σ_j is $\tilde{\sigma}'_j := h \cdot \tilde{\sigma}_j : V \rightarrow F'$. Define $F'_i := F'/N_i$, and let $\rho'_i : \Delta_{\mathbb{K}}^q \times V \rightarrow F'_i$ be the morphism as in (2.3). Since $N_i \times V = \text{Ad}(h)(N_i \times V)$, we get a group-scheme automorphism $\text{Ad}(h) : (G/N_i) \times V \rightarrow (G/N_i) \times V$, and a $(G/N_i) \times V$ -equivariant morphism of torsors $h : F_i \rightarrow h \cdot F'_i$. By functoriality of wav (property (3) in Theorem 1.11) it follows that $\rho'_i = h \cdot \rho_i$. Therefore, $\rho' = h \cdot \rho$, and $\pi_H \circ \rho' = \pi_H \circ \rho = \sigma$.

Property (2) in Theorem 1.11 implies that σ depends simplicially on $(\sigma_0, \dots, \sigma_q)$.

Finally, take an open covering $X = \bigcup V_j$ such that each V_j trivializes $\pi : Z \rightarrow X$, and let $\sigma_j : \Delta_{\mathbb{K}}^q \times V_j \rightarrow \bar{Z}|_{V_j}$ be the morphism constructed in (2.5). Since no choices were made, we have $\sigma_j|_{V_j \cap V_k} = \sigma_k|_{V_j \cap V_k}$ for any two indices. Therefore, these sections can be glued to a morphism $\sigma : \Delta_{\mathbb{K}}^q \times X \rightarrow \bar{Z}$. The functorial and simplicial properties of σ are clear from its construction. \square

Let X be a \mathbb{K} -scheme, and let $X = \bigcup_{i=0}^m U_{(i)}$ be an open covering, with inclusions $g_{(i)} : U_{(i)} \rightarrow X$. We denote this covering by \mathbf{U} . For any multi-index $\mathbf{i} = (i_0, \dots, i_q) \in \Delta_q^m$, we write $U_{\mathbf{i}} := \bigcap_{j=0}^q U_{(i_j)}$, and we define the scheme $U_q := \bigsqcup_{\mathbf{i} \in \Delta_q^m} U_{\mathbf{i}}$. Given $\alpha \in \Delta_p^q$ and $\mathbf{i} \in \Delta_q^m$, there is an inclusion of open sets $\alpha_* : U_{\mathbf{i}} \rightarrow U_{\alpha_*(\mathbf{i})}$. These patch to a morphism of schemes $\alpha_* : U_q \rightarrow U_p$, making $\{U_q\}_{q \in \mathbb{N}}$ into a simplicial scheme. The inclusions $g_{(i)} : U_{(i)} \rightarrow X$ induce inclusions $g_{\mathbf{i}} : U_{\mathbf{i}} \rightarrow X$ and morphisms $g_q : U_q \rightarrow X$; and one has the relations $g_p \circ \alpha_* = g_q$ for any $\alpha \in \Delta_p^q$.

Definition 2.6. Let $\pi : Z \rightarrow X$ be a morphism of \mathbb{K} -schemes. A *simplicial section* of π based on the covering U is a sequence of morphisms

$$\sigma = \{\sigma_q : \Delta_{\mathbb{K}}^q \times U_q \rightarrow Z\}_{q \in \mathbb{N}}$$

satisfying the following conditions.

(i) For any q , the diagram

$$\begin{array}{ccc} \Delta_{\mathbb{K}}^q \times U_q & \xrightarrow{\sigma_q} & Z \\ p_2 \downarrow & & \downarrow \pi \\ U_q & \xrightarrow{g_q} & X \end{array}$$

is commutative.

(ii) For any $\alpha \in \Delta_{\mathbb{K}}^q$, the diagram

$$\begin{array}{ccc} & \Delta_{\mathbb{K}}^p \times U_p & \\ \uparrow 1 \times \alpha^* & \searrow \sigma_p & \\ \Delta_{\mathbb{K}}^p \times U_q & & Z \\ \downarrow \alpha_* \times 1 & \nearrow \sigma_q & \\ & \Delta_{\mathbb{K}}^q \times U_q & \end{array}$$

is commutative.

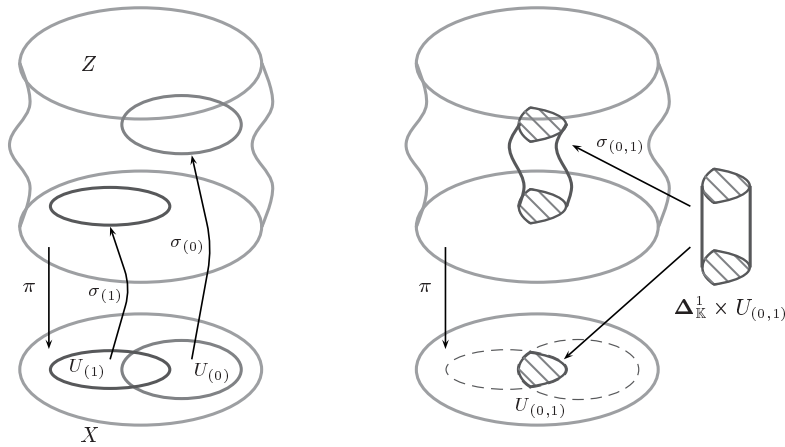


FIGURE 1. Simplicial sections, $q = 1$. We start with sections over two open sets $U_{(0)}$ and $U_{(1)}$ in the left diagram; and we pass to a simplicial section $\sigma_{(0,1)}$ on the right. As can be seen, $\sigma_{(0,1)}$ interpolates between $\sigma_{(0)}$ and $\sigma_{(1)}$.

Corollary 2.7. *Assume Scenario 2.1. Let $\mathbf{U} = \{U_{(i)}\}_{i=0}^m$ be an open covering of X . Suppose that for any $i \in \{0, \dots, m\}$ we are given some section $\sigma_{(i)} : U_{(i)} \rightarrow \bar{Z}$ of $\bar{\pi}$. Then there exists a simplicial section*

$$\sigma = \{\sigma_q : \Delta_{\mathbb{K}}^q \times U_q \rightarrow \bar{Z}\}_{q \in \mathbb{N}}$$

based on \mathbf{U} , such that $\sigma_0|_{U_{(i)}} = \sigma_{(i)}$ for all $i \in \{0, \dots, m\}$.

Proof. For any multi-index $\mathbf{i} = (i_0, \dots, i_q)$, we have sections $\sigma_{(i_0)}, \dots, \sigma_{(i_q)} : U_{\mathbf{i}} \rightarrow \bar{Z}$. Let $\sigma_{\mathbf{i}} : \Delta_{\mathbb{K}}^q \times U_{\mathbf{i}} \rightarrow \bar{Z}$ be the morphism provided by Theorem 2.2. For fixed q these patch to a morphism $\sigma_q : \Delta_{\mathbb{K}}^q \times U_q \rightarrow \bar{Z}$. The functorial and simplicial properties in Theorem 2.2 imply that this is a simplicial section. \square

This result (with H trivial) is illustrated in Figure 1.

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