REPRESENTATION THEORY
An Electronic Journal of the American Mathematical Society
Volume 10, Pages 435–480 (October 25, 2006)
S 1088-4165(06)00301-3

UNITARY $\mathcal{I}$-SPHERICAL REPRESENTATIONS FOR SPLIT p-ADIC $\mathbf{E}_6$

DAN CIUBOTARU

Abstract. The determination of the Iwahori-spherical unitary representations for split p-adic groups can be reduced to the classification of unitary representations with real infinitesimal character for associated graded Hecke algebras. We determine the unitary modules with real infinitesimal character for the graded Hecke algebra of type E_6.

1. Introduction

1.1. **The Iwahori-Hecke algebra.** Let $\mathbb{F}$ denote a p-adic field with the ring of integers $\mathcal{R}$ and unique prime ideal $\mathcal{P}$. Let $\mathcal{G}$ be the $\mathbb{F}$-points of a split reductive algebraic group defined over $\mathbb{F}$. $\mathcal{K} = \mathcal{G}(\mathcal{R})$ is a maximal compact open subgroup in $\mathcal{G}$. Let $\mathcal{B}$ be a Borel subgroup such that $\mathcal{G} = \mathcal{K}\mathcal{B}$. $\mathcal{B} = \mathcal{A}\mathcal{N}$, where $\mathcal{A}$ is a maximal split torus and $\mathcal{N}$ is the unipotent radical.

There is a surjective homomorphism $\mathcal{K} \to \mathcal{G}(\mathcal{R}/\mathcal{P})$. Define the *Iwahori subgroup*, $\mathcal{I} \subset \mathcal{G}$, to be the inverse image in $\mathcal{K}$ of a Borel subgroup in $\mathcal{G}(\mathcal{R}/\mathcal{P})$. Then $\mathcal{I}$ is a compact open subgroup of $\mathcal{G}$.

Definition. A smooth admissible representation (π, V) of $\mathcal{G}$ is called $\mathcal{I}$-spherical (respectively spherical) if $V^{\mathcal{I}} \neq 0$ (respectively $V^{\mathcal{K}} \neq 0$).

Define the *Iwahori-Hecke algebra*, $\mathcal{H} = \mathcal{H}(\mathcal{I}\backslash\mathcal{G}/\mathcal{I})$, to be the set of compactly supported $\mathcal{I}$-bi-invariant functions on $\mathcal{G}$. This is an algebra under the convolution of functions. If (π, V) is an $\mathcal{I}$-spherical representation of $\mathcal{G}$, then $\mathcal{H}$ acts on $V^{\mathcal{I}}$ via:

$$\pi(f)v = \int_{\mathcal{G}} f(g)(\pi(g)v)\, dg, \text{ for } v \in V^{\mathcal{I}} \text{ and } f \in \mathcal{H}. \tag{1.1.1}$$

Let ν denote a character of $\mathcal{A}$. The representation $I(\nu)$ obtained by normalized induction (which preserves unitarity) from ν is called a *principal series*. ν is called *unramified* if its restriction to $\mathcal{A}\cap\mathcal{K}$ is trivial. Consider the following two categories of representations:

$$\begin{aligned} C(\mathcal{I}) = &\text{ admissible finite length representations of } \mathcal{G} \text{ such that all of}\\ &\text{their subquotients are generated by their Iwahori fixed vectors,}\\ C(\mathcal{H})! &\text{ finite dimensional representations of } \mathcal{H}. \end{aligned} \tag{1.1.2}$$

Theorem ([Bo], [Cas])**.** (1) *The functor $V \to V^{\mathcal{I}}$ is an equivalence of categories between $C(\mathcal{I})$ and $C(\mathcal{H})$.*

Received by the editors November 15, 2005 and, in revised form, August 23, 2006.
2000 *Mathematics Subject Classification.* Primary 22E50.

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(2) *The irreducible objects of $C(\mathcal{I})$ are the irreducible subquotients of the unramified principal series $I(\nu)$.*

The algebra $\mathcal{H}$ has a star operation defined as $f \to f^*$,

$$f^*(g) := \overline{f(g^{-1})} \tag{1.1.3}$$

and therefore one can define Hermitian and unitary modules for $\mathcal{H}$. The results of [BM1] and [BM2] reduce the problem of classifying the irreducible unitary representations in $C(\mathcal{I})$ to the determination of the simple unitary modules in $C(\mathcal{H})$ with *real infinitesimal character*. It is known that the characters of the center of $\mathcal{H}$, the *infinitesimal characters*, correspond to Weyl group conjugacy classes of semisimple elements s in the dual complex group ${}^L\mathcal{G}$. An infinitesimal character is called *real* if the corresponding s is a hyperbolic element.

Theorem ([BM1])**.** *Assume $\mathcal{G}$ has connected center. An irreducible Hermitian representation with real infinitesimal character $(\pi, V) \in C(\mathcal{I})$ is unitary if and only if $V^{\mathcal{I}} \in C(\mathcal{H})$ is unitary.*

In [BM2], the restrictions on $\mathcal{G}$ and on the infinitesimal character are removed. This is accomplished via a reduction to the *affine graded Hecke algebra* $\mathbb{H}$ introduced in [Lu1]. Let $C(\mathcal{I}, s)$ be the subcategory of $C(\mathcal{I})$ of representations with (non-real) infinitesimal character conjugate to s. Let s_e be the elliptic part of s and let $C_{{}^L\mathcal{G}}(s_e)$ be the centralizer in the dual complex group. This is a connected group when $\mathcal{G}$ has connected center. In general, the classification of unitary representations in $C(\mathcal{I}, s)$ is reduced to the determination of simple unitary modules with real infinitesimal character for the graded Hecke algebra attached to the identity component of $C_{{}^L\mathcal{G}}(s_e)$.

1.2. **Outline of the paper.** In sections 2.1–2.4, we review the definition of the affine graded Hecke algebra $\mathbb{H}$, the geometric and Langlands classifications for simple $\mathbb{H}$-modules and Springer's correspondence. The main references that we use are [KL], [Lu1]–[Lu4], [Ev] and [BM2]. Geometrically, the simple $\mathbb{H}$-modules are parameterized by ${}^L\mathcal{G}$-conjugacy classes in

$$\begin{aligned}\{(s, e, \psi) : s, e \in {}^L\mathfrak{g},\ s \text{ semisimple},\ e \text{ nilpotent},\ [s, e] = e,\\ \psi \in \widehat{A(s, e)} \text{ subject to certain restrictions}\}.\end{aligned} \tag{1.2.1}$$

(For any subset $\mathcal{S}$ of ${}^L\mathfrak{g}$, we will denote by $Z(\mathcal{S})$ the centralizer of $\mathcal{S}$ in ${}^L\mathcal{G}$, and set $A(\mathcal{S}) = Z(\mathcal{S})/Z(\mathcal{S})^0 Z(G)$, where $Z(\mathcal{S})^0$ is the identity component.)

To every triple (s, e, ψ) as in (1.2.1), one attaches a standard module $X(s, e, \psi)$, and every simple $\mathbb{H}$-module appears as a quotient, $\overline{X}(s, e, \psi)$, of a standard module. Subsequently, we will say that a simple $\mathbb{H}$-module V is attached to a nilpotent orbit $\mathcal{O} \subset {}^L\mathfrak{g}$ if V is parameterized by (s, e, ψ), and $e \in \mathcal{O}$.

In sections 2.5 and 2.6, we recall the construction of intertwining operators and Hermitian forms (as in [BM3]) which we will use for the determination of unitary modules. The intertwining operators induce operators (and Hermitian forms) on W-representations and the main criterion for proving nonunitarity is to compute the signature of these operators on certain W-representations.

Section 2.7 recalls the definition of the Iwahori-Matsumoto involution IM of $\mathbb{H}$. Since IM preserves Hermitian and unitary modules, we will use it in section 3 to prove the unitarity of some isolated representations.

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Sections 2.8 and 2.9 present the special case of spherical $\mathbb{H}$-modules and spherical intertwining operators. We need to recall some results about the spherical unitary dual for Hecke algebras of classical types as in [Ba], particularly Theorems 2.8 and 2.9. We end section 2 with the relation between restrictions to W-representations of Hermitian forms for irreducible modules associated to a fixed nilpotent orbit $\mathcal{O}$, on one hand, and spherical Hermitian forms for Hecke algebras of some smaller subalgebras, on the other (see 2.10; details in [BC2]).

Section 3 presents the classification of unitary $\mathbb{H}$-modules, with real infinitesimal character, when $\mathbb{H}$ is of type E_6. In 3.1, we treat the case of simple modules attached to nilpotent orbits which, in the Bala-Carter classification, correspond to Levi components of maximal parabolic subalgebras.

Sections 3.2–3.5 contain the main results of the paper. Following the geometric classification, the unitary dual of $\mathbb{H}$ is partitioned into pieces $\mathcal{U}(\mathcal{O})$ labeled by nilpotent orbits $\mathcal{O} \subset {}^L\mathfrak{g}$. For each $\mathcal{O}$, fix a Lie triple $\{e, h, f\}$. The centralizer of the triple in ${}^L\mathfrak{g}$ is a reductive subalgebra. We will denote it by $\mathfrak{z}(\mathcal{O})$, and let $\mathbb{H}(\mathfrak{z}(\mathcal{O}))$ be the corresponding graded Hecke algebra.

For each $\phi \in \widehat{A(e)}_0$ (notation as for the Springer correspondence, see section 2.2), define

$$\mathcal{U}(\mathcal{O},\phi) = \{(s,e,\psi) \text{ as in } (1.2.1) : \overline{X}(s,e,\psi) \text{ unitary and } [\psi : \phi|_{A(s,e)}] \neq 0\}. \tag{1.2.2}$$

The centralizer of the semisimple element s in ${}^L\mathcal{G}$ is connected, and $A(s,e) \subset A(e)$.

Clearly, $\mathcal{U}(\mathcal{O}) = \bigcup_{\phi\in\widehat{A(e)}_0} \mathcal{U}(\mathcal{O},\phi)$, but this is not a disjoint union in general. We also denote by $\mathcal{SU}(\mathbb{H})$ the spherical unitary dual of a Hecke algebra $\mathbb{H}$. (Note that $\mathcal{SU}(\mathbb{H}) = \mathcal{U}(0)$, where 0 is the trivial nilpotent orbit.)

The semisimple element s in (1.2.1) can be written as $s = \frac{1}{2}h + \nu$, with $\nu \in \mathfrak{z}(\mathcal{O})$. Then ν parameterizes a spherical simple module in $\mathbb{H}(\mathfrak{z}(\mathcal{O}))$. Our main result, Theorem 3.2, says that, in E_6, if $\mathcal{O} \neq 3A_1$, $(s,e,\psi) \in \mathcal{U}(\mathcal{O},\phi)$ if and only if $\nu \in \mathcal{SU}(\mathbb{H}(\mathfrak{z}(\mathcal{O})))$. When $\mathcal{O} = 3A_1$ (and $\phi = 1$), there are unitary parameters (s,e,ψ) (a continuous family and an isolated point) such that the corresponding ν is not in the spherical unitary dual of the Hecke algebra of the centralizer $\mathfrak{z}(\mathcal{O}) = A_2 + A_1$. We summarize this as follows.

Theorem (1)**.** *In E_6, $\mathcal{U}(\mathcal{O},\phi)$ is the same as the spherical unitary dual $\mathcal{SU}(\mathbb{H}(\mathfrak{z}(\mathcal{O})))$ of the Hecke algebra $\mathbb{H}(\mathfrak{z}(\mathcal{O}))$, except if $\mathcal{O} = 3A_1$, when it is larger.*

Note that this first form of the answer has the inconvenience that, for a fixed $\mathcal{O}$, the sets $\mathcal{U}(\mathcal{O},\phi)$ are not disjoint. In order to refine the answer, we introduce an extension of the Hecke algebra $\mathbb{H}(\mathfrak{z}(\mathcal{O}))$. Denote $Z(\mathcal{O}) = Z(e,h,f)$, and let $Z^0(\mathcal{O})$ be the identity component. Let $A(\mathcal{O}) = A(e,h,f) = A(e)$ denote the component group. Then $A(\mathcal{O})$ acts on the root datum of $Z^0(\mathcal{O})$, and one can form the object

$$\mathbb{H}(\mathcal{O}) = \mathbb{H}(\mathfrak{z}(\mathcal{O})) \ltimes A(\mathcal{O}). \tag{1.2.3}$$

In E_6, for every nilpotent orbit, $\widehat{A(e)}_0 = \widehat{A(\mathcal{O})}$.

A module for $\mathbb{H}(\mathcal{O})$ is called spherical if its restriction to $\mathbb{H}(\mathfrak{z}(\mathcal{O}))$ is spherical. Mackey theory implies that the spherical modules for $\mathbb{H}(\mathcal{O})$ are classified by pairs (ν,ξ), with $\nu \in \mathfrak{z}(\mathcal{O})$, semisimple, and $\xi \in \widehat{A(\mathcal{O})(\nu)}$, where $A(\mathcal{O})(\nu)$ denotes the stabilizer of ν in $A(\mathcal{O})$. If $s = \frac{1}{2}h + \nu$, since $Z(\nu), Z(s)$ are connected, $A(\mathcal{O})(\nu) = A(e,h,\nu) = A(e,h,s) = A(e,s)$. Then a second form of the answer is:

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Theorem (1'). *In* E_6, $\mathcal{U}(\mathcal{O}) \supseteq \mathcal{SU}(\mathbb{H}(\mathcal{O}))$, *i.e., a parameter* (s,e,ψ) *with* $s = \frac{1}{2}h+\nu$ *is in* $\mathcal{U}(\mathcal{O})$ *if* (ν,ψ) *is in* $\mathcal{SU}(\mathbb{H}(\mathcal{O}))$.

Moreover, if $\mathcal{O} \neq 3A_1$, *then* $\mathcal{U}(\mathcal{O}) = \mathcal{SU}(\mathbb{H}(\mathcal{O}))$.

Of course, both these formulations are tautologies in the case $\mathcal{O} = 0$, the trivial nilpotent orbit, which parameterizes the spherical dual of E_6. $\mathcal{SU}(\mathbb{H}(E_6))$ is partitioned into complementary series (see section 2.8), $CS(\mathcal{O})$, attached to nilpotent orbits $\mathcal{O} \subset {}^L\mathfrak{g}$.

Theorem (2). *In* E_6*:*

(a) $s = \frac{1}{2}h+\nu \in CS(\mathcal{O})$ *if and only if* $\nu \in CS(0)$ *in* $\mathbb{H}(\mathfrak{z}(\mathcal{O}))$.

(b) *The set* $CS(0)$ *in* $\mathbb{H}(E_6)$ *is formed of* $s = (\frac{\nu_1-\nu_2}{2}-\nu_3, \frac{\nu_1-\nu_2}{2}-\nu_4, \frac{\nu_1-\nu_2}{2}+\nu_4, \frac{\nu_1-\nu_2}{2}+\nu_3, \frac{\nu_1+\nu_2}{2}, -\frac{\nu_1+\nu_2}{2}, -\frac{\nu_1+\nu_2}{2}, \frac{\nu_1+\nu_2}{2})$, $\nu_1-\nu_2-\nu_3-\nu_4 \geq 0, \nu_2 \geq \nu_3 \geq \nu_4 \geq 0$, *such that*

(i) $2\nu_1 < 1$, *or*

(ii) $\nu_1+\nu_2+\nu_3-\nu_4 < 1 < \nu_1+\nu_2+\nu_3+\nu_4$.

The explicit description of each $\mathcal{U}(\mathcal{O})$ and the relevant calculations (including the determination of hyperplanes of reducibility for standard modules) are in section 3.4. One important consequence of our description of the unitary dual is that it implies the unitarity of certain Arthur parameters of E_6 (section 3.3). Section 3.5 concludes the classification by identifying $CS(0)$ for type E_6. Finally, in section 4, we record in a table the explicit parameterization of the unitary dual for $\mathbb{H}(E_6)$.

Independently, the spherical unitary dual for type E_6 was found using computer calculations by J. Stembridge (see `www.liegroups.org`), and his result agrees completely with our results for spherical modules (although the form of the answer is different). The author would like to thank him for making his calculations available, and J. Adams for verifying that the two answers agree.

The present paper is part of a joint project with D. Barbasch—following ([BM1]–[BM3], [Ba])—to study the $\mathcal{I}$-spherical unitary representations, and in particular, to classify the unitary dual of graded Hecke algebras of exceptional types (see [BC1], [BC2] for results in this sense). The new results in this paper refer mainly to non-generic modules. The author thanks D. Barbasch and D. Vogan for helpful discussions, and the referee for carefully reading this manuscript, and for his (or her) many helpful suggestions.

2. Hermitian forms

2.1. **The affine graded Hecke algebra.** Let G be a reductive simply-connected algebraic group over $\mathbb{C}$ and let $\mathfrak{g}$ be its Lie algebra. Let H be a maximal torus and B a Borel subgroup, $H \subset B \subset G$, and let $(X, R, \check{X}, \check{R}, \Pi)$ be the based root datum associated to G, H and B. More precisely, $X = \mathrm{Hom}(H, G_m)$ is the lattice of rational characters of H (G_m is the multiplicative group of $\mathbb{C}$), $\check{X} = \mathrm{Hom}(G_m, H)$ is the lattice of one-parameter subgroups, $\Pi \subset R^+ \subset R \subset X$ are the simple roots, positive roots and roots determined by B and H. Similarly, for coroots $\check{\Pi} \subset \check{R}^+ \subset \check{R} \subset \check{X}$.

Set $\mathfrak{h}^* = X \otimes_{\mathbb{Z}} \mathbb{C}$ and $\mathfrak{h} = \check{X} \otimes_{\mathbb{Z}} \mathbb{C}$. Denote by W the Weyl group and $\mathbb{C}[W]$ the group algebra of W. The roots will be denoted by α and the corresponding reflections in the Weyl group by s_α. Let $c : R \to \mathbb{Z}^+$ be a function such that $c_\alpha = c_{\alpha'}$ if α and α' are W-conjugate.

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As a vector space,

$$\mathbb{H} = \mathbb{C}[W] \otimes \mathbb{C}[r] \otimes \mathbb{A}, \tag{2.1.1}$$

where $\mathbb{A}$ is the symmetric algebra over $\mathfrak{h}^*$. The generators are $t_w \in \mathbb{C}[W]$, $w \in W$ and $\omega \in \mathfrak{h}^*$. The relations between the generators are:

$$\begin{aligned} &t_w t'_w = t_{ww'}, && \text{for all } w, w' \in W; \\ &t_s^2 = 1, && \text{for any simple reflection } s \in W; \\ &t_s\omega = s(\omega)t_s + 2rc_\alpha\langle\omega, \check{\alpha}\rangle, && \text{for simple reflections } s = s_\alpha. \end{aligned} \tag{2.1.2}$$

When the function c is not identically 1, we will refer to $\mathbb{H}$ as the Hecke algebra with parameter c. In the simply-laced case, the only values of c one needs to consider are $c \equiv 1$ and $c \equiv 0$. We mention that when $c \equiv 0$, the affine graded Hecke algebra degenerates into $(\mathbb{C}[r] \otimes \mathbb{A}) \ltimes \mathbb{C}[W]$.

Theorem ([Lu1])**.** *The center of* $\mathbb{H}$ *is* $\mathbb{C}[r] \otimes \mathbb{A}^W$.

On simple finite dimensional $\mathbb{H}$-module, the center of $\mathbb{H}$ acts by characters, called *infinitesimal character*. Theorem 2.1 implies that the infinitesimal characters correspond to W-conjugacy classes of semisimple elements $(r_0, s) \in \mathbb{C} \oplus \mathfrak{h}$.

Definition. An infinitesimal character is called *real* if $r_0 \in \mathbb{R}$, and the corresponding semisimple element s is hyperbolic.

$\mathbb{H}$ has a star operation (and therefore one can define Hermitian and unitary modules) given on generators as follows (as in [BM2]):

$$\begin{aligned} &t_w^* = t_{w^{-1}},\ w \in W;\ r^* = r; \\ &\omega^* = -\overline{\omega} + \sum_{\alpha \in R^+} \langle\overline{\omega}, \check{\alpha}\rangle t_\alpha,\ \omega \in \mathfrak{h}^*. \end{aligned} \tag{2.1.3}$$

2.2. **Simple $\mathbb{H}$-modules.** As in the introduction, if $\mathcal{S} \subset \mathfrak{g}$, denote by $N(\mathcal{S})$, $Z(\mathcal{S})$, the normalizer, respectively the centralizer of $\mathcal{S}$ in G, and by $A(\mathcal{S})$ the component group $A(\mathcal{S}) = Z(\mathcal{S})/Z(\mathcal{S})^0 Z(G)$. The similar notation is used for other groups, in which case we use subscripts, e.g., $A_M(\mathcal{S})$.

Let e be a nilpotent element in $\mathfrak{g}$. Consider $\mathcal{B}_e$, the complex variety of Borel subalgebras of $\mathfrak{g}$ containing e, and $H^\bullet(\mathcal{B}_e) = H^\bullet(\mathcal{B}_e, \mathbb{C})$, the cohomology of $\mathcal{B}_e$. The component group $A(e)$ of the centralizer in G of e acts on $H^\bullet(\mathcal{B}_e)$, and let

$$H^\bullet(\mathcal{B}_e)^\phi = \mathrm{Hom}_{A(e)}[\phi : H^\bullet(\mathcal{B}_e)] \tag{2.2.1}$$

be the ϕ-isotypic component of $H^\bullet(\mathcal{B}_e)$, $\phi \in \widehat{A(e)}$. There is an action of W on each $H^\bullet(\mathcal{B}_e)^\phi$ ([Sp1], [Sp2]). If d_e is the (complex) dimension of $\mathcal{B}_e$, then $(H^{2d_e}(\mathcal{B}_e))^\phi$ is either zero, or it is an irreducible representation of W. Denote this representation $\sigma(\mathcal{O}, \phi)$. Set $\widehat{A(e)}_0 = \{\phi \in \widehat{A(e)} :\ \sigma(\mathcal{O}, \phi) \neq 0\}$. The resulting correspondence,

$$\widehat{A(e)}_0 \to \widehat{W}, \quad \phi \to \sigma(\mathcal{O}, \phi), \tag{2.2.2}$$

is the *Springer correspondence*. We normalize the correspondence so that $\sigma(\mathcal{O}, 1)$ is the trivial, respectively the sign representation of W, when $\mathcal{O}$ is the trivial, respectively the principal nilpotent orbit of $\mathfrak{g}$.

We will recall some facts about the geometric classification of simple $\mathbb{H}$-module. One approach is to use [KL], where the classification for the affine Hecke algebra using K-theory is obtained, in conjunction with [Lu1]. A summary of the relevant results is in section 3 in [BM1] and in [BM2]. We will follow instead [Lu2] and

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[Lu3], where the classification is realized using equivariant homology. Although the constructions in these papers are more general and apply to geometric Hecke algebras with unequal parameters, we will only consider here the equal parameter case.

For an algebraic group $\mathbf{G}$ and a $\mathbf{G}$-variety X, let $H^\bullet_{\mathbf{G}}(X) = H^\bullet_{\mathbf{G}}(X,\mathbb{C})$, respectively $H^{\mathbf{G}}_\bullet(X) = H^{\mathbf{G}}_\bullet(X,\mathbb{C})$ denote the equivariant cohomology, respectively homology (as in section 1 of [Lu2]). The component group of $\mathbf{G}$ acts naturally on $H^\bullet_{\mathbf{G}^0}(X)$ and $H^{\mathbf{G}^0}_\bullet(X)$. The cup product defines a structure of a graded $H^\bullet_{\mathbf{G}}(X)$-module on $H^{\mathbf{G}}_\bullet(X)$. If pt is a point of X, one uses the notation $H^\bullet_{\mathbf{G}} = H^\bullet_{\mathbf{G}}(\{pt\})$, respectively $H^{\mathbf{G}}_\bullet = H^{\mathbf{G}}_\bullet(\{pt\})$. There is a $\mathbb{C}$-algebra homomorphism $H^\bullet_{\mathbf{G}} \to H^\bullet_{\mathbf{G}}(X)$ induced by the map $X \to \{pt\}$, and therefore $H^\bullet_{\mathbf{G}}(X)$ and $H^{\mathbf{G}}_\bullet(X)$ can both be considered as $H^\bullet_{\mathbf{G}}$-modules.

The group $G\times\mathbb{C}^*$ acts on $\mathfrak{g}$ by $(g_1,\lambda)\cdot x = \lambda^{-2}Ad(g_1)x$, for every $x\in\mathfrak{g}, g_1\in G, \lambda\in\mathbb{C}^*$. The centralizer $Z_{G\times\mathbb{C}^*}(e)$ acts on $\mathcal{B}_e$ by $(g_1,\lambda).gB = (g_1g)B$.

[Lu2] constructs actions of W and $S(\mathfrak{h}^*\oplus\mathbb{C})$ on $H^{Z^0_{G\times\mathbb{C}^*}(e)}_\bullet(\mathcal{B}_e)$, and proves that these are compatible with the relations on $\mathbb{H}$, therefore obtaining a module of $\mathbb{H}$ (Theorem 8.13). The component group $A_{G\times\mathbb{C}^*}(e)$ acts on $H^{Z^0_{G\times\mathbb{C}^*}(e)}_\bullet(\mathcal{B}_e)$, and commutes with the $\mathbb{H}$-action (8.5).

Consider the variety $\mathcal{V}$ of semisimple $Z^0_{G\times\mathbb{C}^*}(e)$-orbits on the Lie algebra $\mathfrak{z}_{G\times\mathbb{C}^*}(e) = \{(x,r_0)\in\mathfrak{g}\oplus\mathbb{C} : [x,e]=2r_0e\}$ of $Z_{G\times\mathbb{C}^*}(e)$. The affine variety $\mathcal{V}$ has $H^\bullet_{Z^0_{G\times\mathbb{C}^*}(e)}$ as the coordinate ring. Define the $\mathbb{H}$-modules

$$X(s,r_0,e) = \mathbb{C}_{(s,r_0)}\otimes_{H^\bullet_{Z^0_{G\times\mathbb{C}^*}(e)}} H^{Z^0_{G\times\mathbb{C}^*}(e)}_\bullet(\mathcal{B}_e), \tag{2.2.3}$$

where $\mathbb{C}_{(s,r_0)}$ denotes the $H^\bullet_{Z^0_{G\times\mathbb{C}^*}(e)}$-module given by the evaluation at $(s,r_0)\in\mathcal{V}$, $H^\bullet_{Z^0_{G\times\mathbb{C}^*}(e)}\to\mathbb{C}$.

For each $\psi\in\widehat{A}_{G\times\mathbb{C}^*}(e,s,r_0)$, define

$$X(s,r_0,e,\psi) = \mathrm{Hom}_{A_{G\times\mathbb{C}^*}(e,s,r_0)}[\psi : X(s,r_0,e)]. \tag{2.2.4}$$

In particular, when $(s,r_0)=\mathbf{0}$, we have (7.2, 8.9 in [Lu2] and 10.12(d) in [Lu3])

$$X(\mathbf{0},e)\cong H^{\{1\}}_\bullet(\mathcal{B}_e) \text{ as } W\times A_{G\times C^*}(e)\text{-modules.} \tag{2.2.5}$$

Let $\widehat{A}^0_{G\times\mathbb{C}^*}(e,s,r_0)$ denote the set of representations ψ which appear in the restriction of the $A_{G\times\mathbb{C}^*}(e)$-module $H^{\{1\}}_\bullet(\mathcal{B}_e)$ to $A_{G\times\mathbb{C}^*}(e,s,r_0)$. Also, $H^{\{1\}}_\bullet(\mathcal{B}_e) = H^\bullet(\mathcal{B}_e)^*$.

It is proved in Proposition 8.10 in [Lu2] that $X(s,r_0,e,\psi)\neq 0$ if and only if $\psi\in\widehat{A}^0_{G\times\mathbb{C}^*}(e,s,r_0)$. Call such modules *standard*. By Proposition 8.15, any simple $\mathbb{H}$-module is the quotient $\overline{X}(s,r_0,e,\psi)$ of a standard $\mathbb{H}$-module.

We summarize the discussion in the following statement.

Theorem ([KL], [Lu1], [Lu2])**.** *The simple $\mathbb{H}$-modules on which r acts by $r_0\neq 0$ are parameterized by G-conjugacy classes (s,e,ψ), where $s\in\mathfrak{g}$ is semisimple, $e\in\mathfrak{g}$ is a nilpotent element, such that $[s,e]=2r_0e$ and $\psi\in\widehat{A(s,e)}$ is an irreducible representation of $A(s,e)$, the component group of the centralizer of s and e in G. The representation ψ has to satisfy $[\psi:\phi|_{A(s,e)}]\neq 0$, for some $\phi\in\widehat{A(e)}_0$.*

(Note that $A(s,e)$ is naturally a subgroup of $A(e)$, since $Z(s)$ is connected.)

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From now on, **the infinitesimal characters will be assumed real** (i.e., hyperbolic). Since we are interested in the unitary dual, by section 4 in [BM2], it is sufficient to consider one particular value of r_0. We will set $r_0 = 1/2$, and consequently drop r and r_0 from the notation.

If (s, e) is a parameter as in Theorem 2.2, then s can be written as

$$s = \frac{h}{2} + \nu, \text{ with } \nu \in \mathfrak{z}(\mathcal{O}), \text{ the centralizer of the triple } \{e, h, f\} \text{ of } \mathcal{O}. \tag{2.2.6}$$

When we want to emphasize the nilpotent orbit $\mathcal{O} = G\cdot e$, we will write $X(s, \mathcal{O}, \psi)$ instead of $X(s, e, \psi)$.

2.3. **Lowest W-types.** Let $X(s, e)$ be as in section 2.2, and $\mathcal{O} = G \cdot e$. We record some facts about the W-structure of standard modules.

The continuation argument in 10.13 in [Lu3] in conjunction with (2.2.5) shows that

$$X(s, e)|_W \cong H^\bullet(\mathcal{B}_e)^*, \text{ as } W \times A(s, e)\text{-representations.} \tag{2.3.1}$$

(In (2.3.1), $A(s, e)$ acts on $H^\bullet(\mathcal{B}_e)$ via the inclusion $A(s, e) \subset A(e)$.) Fix $\psi \in \widehat{A(s, e)}$. If $\phi \in \widehat{A(e)}_0$ and

$$\mathrm{Hom}_W[(\sigma(\mathcal{O}, \phi) : X(s, e, \psi)] \neq 0, \tag{2.3.2}$$

then (following [BM1]), we will call $\sigma(\mathcal{O}, \phi)$ a *lowest W-type* for $X(s, e, \psi)$.

Moreover, if the parameter is tempered (see Definition 2.4 and Proposition 2.4 below), then $A(s, e) = A(e)$ and $X(s, e, \psi)$ has a unique lowest W-type $\sigma(\mathcal{O}, \psi)$ (with multiplicity one).

2.4. **Langlands classification.** Recall that the graded Hecke algebra $\mathbb{H}$ is attached to a root datum $(X, R, \check{X}, \check{R}, \Pi)$. We present the Langlands classification for $\mathbb{H}$ as in [Ev].

If V is a (finite) dimensional irreducible $\mathbb{H}$-module, $\mathbb{A}$ induces a generalized weight space decomposition

$$V = \bigoplus_{\lambda \in \mathfrak{h}} V_\lambda. \tag{2.4.1}$$

Call λ a *weight* if $V_\lambda \neq 0$.

Definition. The irreducible module V is called *tempered* if $\omega_i(\lambda) \leq 0$, for all weights $\lambda \in \mathfrak{h}$ of V and all fundamental weights $\omega_i \in \mathfrak{h}^*$. If V is tempered and $\omega_i(\lambda) < 0$, for all λ, ω_i as above, then V is called a *discrete series*.

Proposition ([KL], [Lu1], [Lu4]). *Let (s, e, ψ), with s real as in* (2.2.6)*, be a geometric parameter corresponding to a simple $\mathbb{H}$-module V.*

a) *V is tempered if and only if $\nu = 0$.*

b) *V is a discrete series if, in addition, $\mathcal{O} = G \cdot e$ is distinguished, i.e., $\mathcal{O}$ does not meet any proper Levi component.*

An essential fact for us is that simple tempered $\mathbb{H}$-modules (as in Proposition 2.4) are formed of the Iwahori-fixed vectors of tempered representations for the p-adic group $\mathcal{G}$ and, therefore (by [BM1]), they are unitary.

For every $\Pi_M \subset \Pi$, define $R_M \subset R$ to be the set of roots generated by Π_M, and $\check{R}_M \subset \check{R}$ the corresponding set of coroots. Let $\mathbb{H}_M$ be the Hecke algebra attached to

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the root datum $(X, R_M, \check{X}, \check{R}_M, \Pi_M)$. The algebra $\mathbb{H}_M$ can be regarded naturally as a subalgebra of $\mathbb{H}$.

Define $\mathfrak{t} = \{\nu \in \mathfrak{h} : \langle \alpha, \nu \rangle = 0, \text{ for all } \alpha \in \Pi_M\}$ and $\mathfrak{t}^* = \{\lambda \in \mathfrak{h}^* : \langle \lambda, \check{\alpha} \rangle = 0, \text{ for all } \alpha \in \Pi_M\}$. Let $X' \subset X$ and $\check{X}' \subset \check{X}$ be the subsets perpendicular to $\mathfrak{t}$, respectively $\mathfrak{t}^*$.

Then $\mathbb{H}_M$ decomposes as

$$\mathbb{H}_M = \mathbb{H}_{M_0} \otimes S(\mathfrak{t}^*),$$

where $\mathbb{H}_{M_0}$ is the Hecke algebra attached to $(X', R_M, \check{X}', \check{R}_M, \Pi_M)$.

We will denote by $I(M, U)$ the induced module $I(M, U) = \mathbb{H} \otimes_{\mathbb{H}_M} U$.

Theorem ([Ev]).

(1) *Every irreducible $\mathbb{H}$-module is a quotient of a standard induced module $X(M, V, \nu) = I(M, V \otimes \mathbb{C}_\nu)$, where V is a tempered module for $\mathbb{H}_{M_0}$, and $\nu \in \mathfrak{t}^+ = \{\nu \in \mathfrak{t} : \alpha(\nu) > 0, \text{ for all } \alpha \in \Pi \setminus \Pi_M\}$.*

(2) *Assume the notation from (1). Then $X(M, V, \nu)$ has a unique irreducible quotient, denoted by $L(M, V, \nu)$.*

(3) *If $L(M, V, \nu) \cong L(M', V', \nu')$, then $M = M'$, $V \cong V'$ as $\mathbb{H}_{M_0}$-modules, and $\nu = \nu'$.*

We will call a triple (M, V, ν) as in Theorem 2.4 a *Langlands parameter*.

We need to explain the connection between the geometric and classical Langlands classifications. Let $P = MN$ be a standard parabolic subgroup of G (i.e., $B \subset P$), with Lie algebra $\mathfrak{p} = \mathfrak{m} \oplus \mathfrak{n}$, and let $\Pi_M \subset \Pi$ be the subset of simple roots which defines M (and P). Suppose $X_G(s, e)$ is a module for $\mathbb{H} = \mathbb{H}_G$, and that $s = \frac{h}{2} + \nu$ and $\{e, h, f\}$ are contained in the Levi subalgebra $\mathfrak{m}$. One can form the standard $\mathbb{H}_M$-module $X_M(s, e)$. Then

$$X_G(s, e) = I(M, X_M(s, e)). \tag{2.4.2}$$

There is a natural map between component groups $\xi : A_M(s, e) \to A_G(s, e)$ induced by the inclusion $Z_M(s, e) \hookrightarrow Z_G(s, e)$. Via ξ, we can regard any representation of $A_G(s, e)$ as a representation of $A_M(s, e)$. For $\phi \in \widehat{A_M(s, e)}$, the induced module $I(M, X_M(s, e, \phi))$ decomposes into a direct sum of standard modules of G:

$$I(M, X_M(s, e, \phi)) = \bigoplus_{\psi \in \widehat{A_G(s,e)}} [\psi|_{A_M(s,e)} : \phi] X_G(s, e, \psi). \tag{2.4.3}$$

If M denotes the centralizer in G of ν, the standard modules $X_G(s, e, \psi)$ can also be seen as induced modules:

$$X_G(s, e, \psi) = I(M, X_M(\frac{h}{2}, e, \phi) \otimes \mathbb{C}_\nu), \tag{2.4.4}$$
$$\text{for some } \phi \text{ appearing in } \psi|_{A_M(s,e)}.$$

By Proposition 2.4, $V = X_M(\frac{h}{2}, e, \phi)$ is a tempered module of $\mathbb{H}_M$.

A factor of $I(M, V \otimes \nu)$ has Langlands parameter (M', V', ν') with $||\nu'|| \leq ||\nu||$, with equality if and only if $(M', V', \nu') = (M, V, \nu)$. This follows from Lemma XI.2.13 in [BW] applied to our context. This fact is important for the method of determination of the unitary modules used in this paper.

Proposition ([KL], [BW]). *All the composition factors of the standard module $X(s, \mathcal{O}, \psi)$ other than $\overline{X}(s, \mathcal{O}, \psi)$ have parameters $(s, \mathcal{O}', \psi')$ with $\mathcal{O}' \neq \mathcal{O}$ and $\mathcal{O} \subset$*

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$\overline{\mathcal{O}'}$. In particular, in the notation of section 2.3, $\overline{X}(s, \mathcal{O}, \psi)$ is characterized by the fact that it contains the Weyl group representation $\sigma(\mathcal{O}, \phi)$ with multiplicity $[\phi\mid_{A(s,e)} : \psi]$.

2.5. **Intertwining operators and Hermitian forms.** We recall the construction of Hermitian forms and intertwining operators from [BM3]. For a module V, let V^h denote the Hermitian dual. Denote by $W(M) \subset W$ the subgroup generated by the simple reflections in Π_M.

Every element $x \in \mathbb{H}$ can be written uniquely as $x = \sum_{w \in W/W(M)} t_w x_w$, with $x_w \in \mathbb{H}_M$. Let $\epsilon_M : \mathbb{H} \to \mathbb{H}_M$ be the map defined by $\epsilon_M(x) = x_1$, that is, the component of the identity element $1 \in W$. In the particular case $M = A$, we will denote the map by $\epsilon : \mathbb{H} \to \mathbb{A}$.

Lemma ([BM3], 1.4)**.** *If U is a module for $\mathbb{H}_M$, and $\langle\ ,\ \rangle_M$ denotes the Hermitian pairing with U^h, then the Hermitian dual of $I(M, U)$ is $I(M, U^h)$, and the Hermitian pairing is given by*

$$\langle t_x \otimes v_x, t_y \otimes v_y\rangle_h = \langle \epsilon_M(t_y^* t_x)v_x, v_y\rangle_M,\ x, y \in W/W(M),\ v_x, v_y \in U.$$

Applying this result to a Langlands parameter (M, V, ν) as in section 2.4, we find that the Hermitian dual of $X(M, V, \nu)$ is $I(M, V \otimes \mathbb{C}_{-\nu})$. (Recall that ν is assumed real.)

Let w_0 denote the longest Weyl group element in W, and let $W(w_0M)$ be the subgroup of W generated by the reflections in w_0R_M. Let w_m denote a shortest element in the double coset $W(w_0M)w_0W(M)$. Then $w_m\Pi_M$ is a subset of Π, which we denote by Π_{w_mM}.

Proposition ([BM3], 1.5)**.** *The Hermitian dual of the irreducible Langlands quotient $L(M, V, \nu)$ is $L(w_mM, w_mV, -w_m\nu)$. In particular, $L(M, V, \nu)$ is Hermitian if and only if*

$$w_mM = M, \quad w_mV \cong V \quad \textit{and} \quad w_m\nu = -\nu.$$

If this is the case, we will denote by a_m the isomorphism between V and w_mV.

Let $w = s_1 \dots s_k$ be a reduced decomposition of w. For each simple root α, define

$$r_{s_\alpha} = t_{s_\alpha}\alpha - 1; \quad r_w = r_{s_{\alpha_1}} \dots r_{s_{\alpha_k}}. \tag{2.5.1}$$

Lemma 1.6 in [BM3] (based on Proposition 5.2 in [Lu1]) proves that r_w does not depend on the reduced expression of w.

Assume $L(M, V, \nu)$ is Hermitian. Define

$$\begin{aligned}\mathcal{A}(M, V, \nu) : X(M, V, \nu) &\to I(M, V \otimes \mathbb{C}_{-\nu}),\\ x \otimes (v \otimes 1_\nu) &\mapsto x r_{w_m} \otimes (a_m(v) \otimes 1_{-\nu}).\end{aligned} \tag{2.5.2}$$

It is shown in [BM3] that this is, in fact, an intertwining operator.

As a $\mathbb{C}[W]$-module,

$$I(M, V \otimes \mathbb{C}_\nu)\mid_W = \mathbb{C}[W] \otimes_{\mathbb{C}[W(M)]} (V\mid_W). \tag{2.5.3}$$

For any W-type (σ, V_σ), $\mathcal{A}(M, V, \nu)$ induces an operator

$$r_\sigma(w, M, \nu) :\ \mathrm{Hom}_W(\sigma, \mathbb{C}[W] \otimes_{\mathbb{C}[W(M)]} V) \to \mathrm{Hom}_W(\sigma, \mathbb{C}[W] \otimes_{\mathbb{C}[W(M)]} V). \tag{2.5.4}$$

By Frobenius reciprocity,

$$\mathrm{Hom}_W(\sigma, \mathbb{C}[W] \otimes_{\mathbb{C}[W(M)]} V) \cong \mathrm{Hom}_{W(M)}(\sigma, V). \tag{2.5.5}$$

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In conclusion, $\mathcal{A}(M,V,\nu)$ gives rise to operators

$$r_\sigma(M,V,\nu):\ \mathrm{Hom}_{\mathbb{C}[W(M)]}(\sigma,V)\to \mathrm{Hom}_{\mathbb{C}[W(M)]}(\sigma,V). \tag{2.5.6}$$

As seen in sections 2.3 and 2.4, each irreducible $\mathbb{H}$-module contains some special W-types, the lowest W-types. It is an empirical fact that in all cases there exists a lowest W-type, call it σ_0, which appears with multiplicity one. For the classical types, all component groups are products of $\mathbb{Z}_2$'s, so this fact is automatic, and for the exceptional types, this is checked case by case when the component group is not abelian. For E_6, the details are in section 3.4.

Then the operator $r_{\sigma_0}(M,V,\nu)$ is a scalar, and we normalize the intertwining operator $\mathcal{A}(M,V,\nu)$ so that this scalar is 1.

Recall the map $\epsilon:\mathbb{H}\to\mathbb{A}$. We denote by $\epsilon(x)(\nu)$, the evaluation of an element $\epsilon(x)\in\mathbb{A}=S(\mathfrak{h}^*)$ at $\nu\in\mathfrak{h}$.

Theorem ([BM3]). *Let (M,V,ν) be a Hermitian Langlands parameter.*

1. *The map $\mathcal{A}(M,V,\nu)$ is an intertwining operator.*
2. *The image of the operator $\mathcal{A}(M,V,\nu)$ is $L(M,V,\nu)$ and the Hermitian form on $L(M,V,\nu)$ is given by*

$$\begin{aligned}\langle t_x\otimes(v_x\otimes 1_\nu),t_y\otimes(v_y\otimes 1_\nu)\rangle &= \langle t_x\otimes(v_x\otimes 1_\nu),t_yr_{w_m}\otimes(a_m(v_y)\otimes 1_{-\nu})\rangle_h,\\ &= \langle \epsilon(t_y^*t_xr_{w_m})(\nu)a_m(v_x),v_y\rangle_M.\end{aligned}$$

The discussion in this section can be summarized in the following corollary.

Corollary. *A Langlands parameter (M,V,ν), ν real, is unitary if and only if the following two conditions are satisfied:*

1. *$w_mM=M$, $w_mV\cong V$, $w_m\nu=-\nu$;*
2. *the normalized operators $r_\sigma(M,V,\nu)$ are positive semidefinite for all $\sigma\in\widehat{W}$, such that* $\mathrm{Hom}_{W(M)}[\sigma:V]\neq 0$.

One of the main tools for showing $\mathbb{H}$-modules are *not* unitary is to compute the signature of certain $r_\sigma(M,V,\nu)$.

2.6. **Decomposition of intertwining operators.** Let (M,V,ν) be a Langlands parameter. Assume that $\alpha\in\Pi\setminus\Pi_M$ is such that $\langle\nu,\alpha\rangle>0$. Let M_α be the Levi component generated by M and the root vectors corresponding to $\pm\alpha$ ($\Pi_{M_\alpha}=\Pi_M\cup\{\alpha\}$). Then there is a shortest Weyl group element $w_\alpha\in W(M_\alpha)$ so that $w_\alpha\nu$ is non-positive on the roots in $R^+_{M_\alpha}$. Using a reduced decomposition for w_α, we can construct an operator

$$\mathcal{A}_\alpha(M,V,\nu):I(M,V\otimes\mathbb{C}_\nu)\longrightarrow I(w_\alpha(M),w_\alpha(V)\otimes\mathbb{C}_{w_\alpha(\nu)}), \tag{2.6.1}$$

which is induced from the corresponding operator for $\mathbb{H}_{M_\alpha}$. Furthermore, M and $w_\alpha(M)$ are Levi components of maximal parabolic subgroups of M_α.

Apply this idea repeatedly to a Langlands parameter (M,V,ν) (with $\langle\nu,\alpha\rangle>0$, $\alpha\in\Pi\setminus\Pi_M$). We find that the element w_m (notation as in section 2.5) decomposes into

$$w_m=w_{\alpha_1}\cdot w_{\alpha_2}\cdot\cdots\cdot w_{\alpha_k},\qquad \ell(w_m)=\sum_{i=1}^k\ell(w_{\alpha_i}). \tag{2.6.2}$$

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Write $w_i = w_{\alpha_{k-i+1}} \dots w_{\alpha_k}$. The intertwining operator A_m decomposes accordingly into a product

$$\begin{aligned}(2.6.3)\qquad \mathcal{A}(M,V,\nu) =&\mathcal{A}_1(w_1(M),w_1(V),w_1(\nu)) \circ \mathcal{A}_2(w_2(M),w_2(V),w_2(\nu))\circ\\ &\cdots \circ \mathcal{A}_i(w_k(M),w_k(V),w_k(\nu)).\end{aligned}$$

Each $\mathcal{A}_i$ in this decomposition is induced from a similar operator for the Hecke algebra of a Levi component.

2.7. The *Iwahori-Matsumoto involution* of $\mathbb{H}$ is defined on generators as follows:

$$(2.7.1)\qquad IM(t_w) = (-1)^{\ell(w)} t_w, \qquad IM(\omega) = -\omega,\ \omega \in \mathfrak{h}^*.$$

IM acts therefore on the modules of $\mathbb{H}$.

The induced action of the Iwahori-Matsumoto involution on W-types is tensoring with the sign representation of W. The use of IM in our context is justified by the following fact.

Remark ([BM1]). The $\mathbb{H}$-module V is Hermitian (respectively unitary) if and only if $IM(V)$ is Hermitian (respectively unitary).

The *Steinberg representation* of $\mathbb{H}$ is defined as:

$$(2.7.2)\qquad St(t_w) = (-1)^{\ell(w)}, \quad St(\omega) = -\omega.$$

Then $IM(St) = triv$. In the geometric classification, $St = X(\rho, e, 1)$, where e belongs to the principal nilpotent orbit, while $triv = \overline{X}(\rho, 0, 1)$. ($\rho$ is the semi-sum of positive roots in R^+.)

2.8. **Spherical $\mathbb{H}$-modules.** We present the special case of spherical modules and the results for classical groups from [Ba].

The module V is called *spherical* (respectively *generic*) if $\mathrm{Hom}_W[triv, V] \neq 0$ (respectively $\mathrm{Hom}_W[sgn, V] \neq 0$).

Note that IM takes spherical modules into generic modules. If a simple generic $\mathbb{H}$-module is parameterized by a Kazhdan-Lusztig triple (s, e, ψ), the representation ψ must be the trivial representation ([KL], [Lu1]). Moreover, the semisimple element s determines the orbit of e uniquely. Fix a semisimple $s \in \mathfrak{g}$ (actually, one can assume $s \in \mathfrak{h}$). The characterization of the nilpotent orbit $\mathcal{O} = G \cdot e$ is the following:

Proposition ([BM3])**.** *Let $s \in \mathfrak{h}$ be a semisimple element and $\mathcal{O}$ the associated nilpotent orbit constructed before. Let $\{e, h, f\}$ be a Lie triple for the orbit $\mathcal{O}$. Then $\mathcal{O}$ has the property that it is unique subject to the following two conditions:*

1. *there exists $w \in W$ such that $ws = \frac{h}{2} + \nu$, where ν is a semisimple element in the Lie algebra $\mathfrak{z}(\mathcal{O})$ of the centralizer of the Lie triple;*
2. *if s satisfies the first property for a different $\mathcal{O}'$, then $\mathcal{O}' \subset \overline{\mathcal{O}}$.*

Equivalently, set

$$(2.8.1)\qquad \mathfrak{g}_1 = \{x \in \mathfrak{g}\ :\ [s,x] = x\}, \qquad \mathfrak{g}_0 = \{x \in \mathfrak{g}\ :\ [s,x] = 0\}.$$

Then the Lie group G_0 corresponding to the Lie algebra $\mathfrak{g}_0$ has an open dense orbit in $\mathfrak{g}_1$. The nilpotent orbit $\mathcal{O}$ in $\mathfrak{g}$ is the one which meets $\mathfrak{g}_1$ in this open dense orbit.

For the spherical case, consider the *principal series module*

$$(2.8.2)\qquad X(s) = \mathbb{H} \otimes_{\mathbb{A}} \mathbb{C}_s,\ s \in \mathfrak{h}.$$

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Since $X(s)$ is isomorphic as a W-representation to $\mathbb{C}[W]$, it follows that the trivial W-representation appears with multiplicity one in $X(s)$ and therefore, there is a unique spherical subquotient $\overline{X}(s)$. Consequently, we will refer to a semisimple element s as unitary if $\overline{X}(s)$ is unitary.

Consider the long intertwining operator $\mathcal{A}(0, St, s)$ (notation as in equation (2.5.2)), given by r_{w_0}. Since this is determined only by s, we will simply denote it by $\mathcal{A}(s)$. The following theorem is a well-known result about spherical modules. In our setting, it is a particular case of the results in section 2.5.

Theorem. *If s is dominant (i.e., $\langle s, \alpha\rangle \geq 0$ for all positive roots $\alpha \in R^+$) the image of $\mathcal{A}(s)$ is $\overline{X}(s)$.*

Moreover, $\overline{X}(s)$ is Hermitian if and only if $w_0 s = -s$.

Note that $r_{w_0} = r_{s_{\alpha_1}} \cdots r_{s_{\alpha_k}}$ acts on the right and therefore, each α_j in $r_{s_{\alpha_j}}$ can be replaced by the scalar $-\langle \alpha_j, s_{\alpha_1} s_{\alpha_2} \cdots s_{\alpha_{j-1}}(\nu)\rangle$ in the intertwining operator $\mathcal{A}(\nu)$. Consequently, we can think of r_{w_0} as an element in $\mathbb{C}[W]$. The next remark is just Corollary 2.5 in this special case.

Remark. Assume $w_0 s = -s$. The long intertwining operator gives rise to operators on the W-types (σ, V_σ):

$$r_\sigma(s) : \sigma^* \to \sigma^*.$$

They are normalized so that $r_{triv} = 1$. The Hermitian form on $\overline{X}(s)$ is positive definite if and only if all the operators $r_\sigma(s)$ are positive semidefinite.

This suggests the following combinatorial description of the spherical unitary dual. One can consider elements s in the (-1)-eigenspace of w_0 in the dominant Weyl chamber. They parameterize spherical $\mathbb{H}$-modules. In order to determine if s is unitary, one would have to compute the operators $r_\sigma(s)$ on W-types σ. An operator $r_\sigma(s)$ has constant signature on each facet of the hyperplane arrangement given by $\langle s, \alpha\rangle = 1$ for $\alpha \in R^+$ and $\langle s, \alpha\rangle = 0$ for $\alpha \in \Pi$ (see Theorem 2.4. in [BC1]).

Therefore, the spherical unitary dual can be viewed as a (bounded) union of closed facets in this arrangement of hyperplanes.

By Proposition 2.8, the spherical unitary dual is partitioned into subsets, each subset being parameterized by a nilpotent orbit $\mathcal{O}$ in $\mathfrak{g}$.

Definition. The set of unitary spherical (equivalently, generic) parameters $s = \frac{1}{2}h + \nu$ associated to the nilpotent orbit $\mathcal{O}$ by Proposition 2.8 are called the *complementary series of $\mathcal{O}$*.

The explicit description of the spherical unitary dual of $\mathbb{H}$ of classical type ([BM3] and [Ba]) can be summarized in the following theorem. The unitary dual for p-adic $GL(n, \mathbb{F})$ was first classified in [Ta].

Theorem ([Ba])**.** *Let $s \in \mathfrak{h}$ be a (real) semisimple element and $\mathcal{O}$ the unique maximal nilpotent orbit such that $s = \frac{1}{2}h + \nu$, with ν a semisimple element in $\mathfrak{z}(\mathcal{O})$.*

(1) *s is in the complementary series of $\mathcal{O}$ if and only if ν is in the complementary series of the trivial nilpotent orbit of $\mathfrak{z}(\mathcal{O})$.*
(2) *The (real) parameters $s = (\nu_1, \nu_2, \ldots, \nu_n)$ in the complementary series associated to the trivial nilpotent orbit are:*
   A: *$s = (\nu_1, \ldots, \nu_k, -\nu_k, \ldots, -\nu_1)$ or $(\nu_1, \ldots, \nu_k, 0, -\nu_k, \ldots, -\nu_1)$, with $0 \leq \nu_1 \leq \nu_2 \leq \cdots \leq \nu_k < \frac{1}{2}$.*

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C: $0 \le \nu_1 \le \nu_2 \le \dots \le \nu_n < \frac{1}{2}$.

B, D: $0 \le \nu_1 \le \dots \le \nu_i < 1 - \nu_{i-1} < \nu_{i+1} < \dots < \nu_n < 1$, *so that between any* $\nu_j < \nu_{j+1}$, $i \le j < n$, *there is an odd number of* $(1-\nu_l)$, $1 \le l < i$.

(The types A-D in the theorem refer to the Hecke algebra)

2.9. **Relevant W-types.** In view of Remark 2.8, the spherical unitary dual for classical groups is determined by the operators restricted to a small set of W-types, as follows from [Ba].

Theorem ([Ba]). *For* $\mathbb{H}$ *of classical type, a spherical parameter* s *is unitary if and only if the operators* $r_\sigma(s)$ *are positive semidefinite for the following representations* σ *of* W*:*

(1) **A:** $(m, n-m)$, $0 \le m \le [\frac{n}{2}]$.
(2) **B, C:** $(n-m) \times (m)$, $0 \le m \le n$ *and* $(m, n-m) \times (0)$, $m \le [\frac{n}{2}]$.
(3) **D:** $(n-m) \times (m)$, $0 \le m \le [\frac{n}{2}]$ *and* $(m, n-m) \times (0)$, $m \le [\frac{n}{2}]$.

We will also need the description of the spherical unitary dual for the Hecke algebra of type G_2. The unitary dual for p-adic G_2 was determined in [Mu], and the following reformulation in terms of the Hecke algebra can be found in [Ci]. We use the simple roots $\alpha_1 = (\frac{2}{3}, -\frac{1}{3}, -\frac{1}{3})$ and $\alpha_2 = (-1, 1, 0)$ for G_2.

Proposition ([Ci]). *Let* G *be of type* G_2 *and* $s = (\nu_1, \nu_1 + \nu_2, -2\nu_1 - \nu_2)$, $\nu_1 \ge 0$, $\nu_2 \ge 0$, *is a spherical parameter.*

(1) *s is unitary if and only if* $\{3\nu_1 + 2\nu_2 \le 1\}$, $\{2\nu_1 + \nu_2 \le 1 \le 3\nu_1 + \nu_2\}$ *or* $(\nu_1, \nu_2) = (1, 1)$.
(2) *s is unitary if and only if the operators* $r_\sigma(s)$ *are positive semidefinite on the W-types* $\{(1,0), (2,1), (2,2)\}$.

The labeling of irreducible $W(G_2)$-representations is as in [Car].

Definition. The sets of W-types appearing in Theorem 2.9 and Proposition 2.9 are called *relevant W-types*.

2.10. **Matching of intertwining operators.** In this section, we provide a brief summary of the relevant constructions from [BC2]. The notation is the same as in sections 2.1–2.6.

Fix a nilpotent orbit $\mathcal{O} \subset \mathfrak{g}$ with a Lie triple $\{e, h, f\}$ and let $s = \frac{h}{2} + \nu$ be a semisimple element as in (2.2.6). Let $Z^0(\mathcal{O})$ be the identity component of $Z(\mathcal{O})$ and $A(\mathcal{O})$ the component group. Let $\mathfrak{a}$ be a Cartan subalgebra in $\mathfrak{z}(\mathcal{O})$ such that $\nu \in \mathfrak{a}$.

As in section 2.4, we define a Levi component M and a nilpotent orbit $\mathcal{O}_M$, such that each standard module of $\mathcal{O}$, $X(s, \mathcal{O}, \psi)$ is obtained as induced from a standard module $X_M(\frac{h}{2}, \mathcal{O}_M, \phi) \otimes \mathbb{C}_\nu$ (see (2.4.4) and the discussion preceding it). To reconcile notation with sections 2.5-refsec:2.5a, put $V = X_M(\frac{h}{2}, \mathcal{O}_M, \phi)$.

The goal is to relate the intertwining operators $r_\sigma(M, V, \nu)$ (see (2.5.6)) for certain W-types σ with spherical operators for the Hecke algebra associated to Z^0.

Lemma. *If* G *is a complex algebraic group with identity component* G^0 *and component group* $A(G)$, *and* $\mathfrak{h} \subset \mathfrak{g}$ *is a Cartan subalgebra, with* W *the Weyl group of* $\mathfrak{h} \subset \mathfrak{g}$, *then*

$$N_G(\mathfrak{h})/Z_G(\mathfrak{h})^0 \cong W \ltimes A(G).$$

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Proof. There is a short exact sequence

$$1 \to W(\mathfrak{g},\mathfrak{h}) \to N_G(\mathfrak{h})/Z_G(\mathfrak{h})^0 \to A(G) \to 1. \tag{2.10.1}$$

Let $\mathfrak{h} \subset \mathfrak{b} \subset \mathfrak{g}$ be a Borel subalgebra. Then the section is given by the map $A(G) \cong N_G(\mathfrak{b},\mathfrak{h})/Z_G(\mathfrak{h})^0 \to N_G(\mathfrak{h})/Z_G(\mathfrak{h})^0$. □

In the following, we denote by $C_Z(\mathfrak{a})$ the centralizer of $\mathfrak{a}$ in $Z(\mathcal{O})$. Applying the lemma to $Z(\mathcal{O})$, we find that

$$N_Z(\mathfrak{a})/C_Z(\mathfrak{a})^0 \cong W(\mathfrak{z}(\mathcal{O})) \ltimes A(\mathcal{O}). \tag{2.10.2}$$

Define

$$\begin{aligned} N(\mathfrak{a}) &= \{w \in W :\ w\cdot\mathfrak{a} = \mathfrak{a}\}, \\ C(\mathfrak{a}) &= \{w \in W :\ w\cdot\mathfrak{a} = \mathfrak{a},\ w(R_M^+) = R_M^+\}. \end{aligned} \tag{2.10.3}$$

Note that

$$N(\mathfrak{a}) = W(M) \ltimes C(\mathfrak{a}). \tag{2.10.4}$$

Proposition ([BC2])**.** *There is a surjective group homomorphism* $W(\mathfrak{z}(\mathcal{O})) \ltimes A(\mathcal{O}) \cong N_Z(\mathfrak{a})/C_Z(\mathfrak{a})^0 \longrightarrow C(\mathfrak{a})$.

Proof. The homomorphism is obtained by the composition of maps

$$N_Z(\mathfrak{a})/C_Z(\mathfrak{a})^0 \twoheadrightarrow N_Z(\mathfrak{a})/C_Z(\mathfrak{a}) \hookrightarrow N_G(\mathfrak{a})/Z_G(\mathfrak{a}) = N_Z(\mathfrak{a})/M \cong C(\mathfrak{a}). \tag{2.10.5}$$

We will skip the proof of the surjectivity for the resulting map (see [BC2] for details). □

For $\sigma \in \widehat{W}$, the operator (2.5.6) is defined on the space $\mathrm{Hom}_{W(M)}(\sigma, V)$. $C(\mathfrak{a})$ acts on $W(M)$ and it acts on σ since $\sigma \in \widehat{W}$ and $C(\mathfrak{a}) \subset W$. Since $C(\mathfrak{a})$ acts on $W(M)$, it will act on $\widehat{W(M)}$. In section 3.4 we check, in each case, that $C(\mathfrak{a})$ also preserves V.

Recall that σ_0 is a lowest W-type of $X(M,V,\nu)$ which appears with multiplicity one. Then let $\sigma_{M,0}$ be the unique lowest W-type of V (V is tempered). $\sigma_{M,0}$ appears in the restriction of σ_0 to V with multiplicity one. In the actual calculations in section 3.4, we will only consider the W-types σ in $X(M,V,\nu)$ with the property that $\mathrm{Hom}_{W(M)}[\sigma : V] = \mathrm{Hom}_{W(M)}[\sigma : \sigma_{M,0}]$, so that the condition we need is that $C(\mathfrak{a})$ preserves $\sigma_{M,0}$. However, since V is tempered, this is equivalent to $C(\mathfrak{a})$ preserves V.

In conclusion, $\mathrm{Hom}_{W(M)}(\sigma, V)$ has a structure of a representation of $C(\mathfrak{a})$ and via the map from Proposition 2.10, it is a $W(\mathfrak{z}(\mathcal{O})) \ltimes A(\mathcal{O})$-representation. It also descends to a representation of $W(\mathfrak{z}(\mathcal{O}))$. Call this representation $\rho(\sigma)$. In 3.2, we will investigate under what conditions the operator $r_\sigma(M,V,\nu)$ (in $\mathbb{H}(G)$) is identical with the spherical operator $r_{\rho(\sigma)}(\nu)$ (in $\mathbb{H}(Z^0)$).

3. The unitary $\mathbb{H}(E_6)$-modules with real infinitesimal character

3.1. **Maximal parabolic cases.** We retain the notation from section 2. In view of section 2.6, we investigate first the case of intertwining operators for induced modules from Levi components of maximal parabolic subalgebras.

Let $\mathcal{O} \subset \mathfrak{g}$ be a nilpotent orbit which is induced from a Levi component M of a maximal parabolic subgroup. The corresponding nilpotent orbit $\mathcal{O}_M$ of $\mathfrak{m}$ is distinguished. Let V be a discrete series module for $\mathbb{H}_M$ attached to $\mathcal{O}_M$ with

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lowest W-type $\sigma_{M,0}$. Form the standard module $X(M, V, \nu)$, where ν can be viewed as a positive scalar.

From Proposition 2.5, it follows that $L(M, V, \nu)$ is Hermitian if and only if $w_0 M = M$ and $w_0 \nu = -\nu$. Note that, since V is a discrete series of M, the condition $w_0 M = M$ automatically implies $w_m V \cong V$. Assume (M, V, ν) is Hermitian data. We need a lemma first (this is Proposition 2.4. in [BM3]).

Lemma ([BM3])**.** *Assume the W-type σ satisfies the condi tions:*

$$\dim Hom_W[\sigma : X(M, V, \nu)] = 1 \quad \text{and} \quad Hom_{W(M)}[\sigma : V] = Hom_{W(M)}[\sigma : \sigma_{M,0}].$$

Then the signature at ∞ of the operator $r_\sigma(M, V, \nu)$ depends only on the lowest harmonic degree of σ.

There are two cases depending on the type of the centralizer $Z(\mathcal{O})$ of the orbit.

Proposition. *Let (M, V, ν) be a Hermitian maximal parabolic data attached to a nilpotent orbit $\mathcal{O}$.*

(1) *If $Z(\mathcal{O})$ is of type A_1, $L(M, V, \nu)$ is unitary if and only if $0 \leq \nu \leq \nu_0$, where ν_0 is the first reducibility point of $X(M, V, \nu)$ on the half-line $\nu \geq 0$.*
(2) *If $Z(\mathcal{O})$ is of type T_1, $L(M, V, \nu)$ is never unitary for $\nu > 0$.*

Proof. We use the same argument as used for classical groups in [BM3] (the details for E_6 are checked in each case in 3.4).

If $Z(\mathcal{O}) = T_1$ and (M, V, ν) (equivalently (M, ν)) is Hermitian, then $Z(\mathcal{O})$ is not connected and the standard module $X(M, V, \nu)$ has two lowest W-types σ_0 and σ_0' both appearing with multiplicity one and having lowest harmonic degrees of opposite parity. At $\nu = 0$, $X(M, V, 0)$ is reducible and each factor is a tempered module, therefore unitary. If $\nu > 0$, σ_0 and σ_0' are always together in $L(M, V, \nu)$ and they have opposite signature (having opposite signature at ∞). This implies $L(M, V, \nu)$ is not unitary for $\nu > 0$.

If $Z(\mathcal{O}) = A_1$, the standard module $X(M, V, \nu)$ has a unique lowest W-type σ_0. $X(M, V, 0)$ is irreducible and tempered, therefore it is unitary until the first point of reducibility, $\nu = \nu_0$. At $\nu = \nu_0$, all factors other than $L(M, V, \nu)$ are parameterized by strictly larger nilpotent orbits (Proposition 2.4). We find a factor corresponding to an orbit $\mathcal{O}'$ (immediately above $\mathcal{O}$) with a lowest W-type σ_1 satisfying the conditions in Lemma 3.1. Moreover, σ_1 should have harmonic degree of opposite parity to σ_0. We verify that for $\nu > \nu_0$, the two W-types σ_0 and σ_1 are always in the factor $L(M, V, \nu)$ and, therefore, $L(M, V, \nu)$ is not unitary for $\nu > \nu_0$. □

Proposition 3.1 holds in all classical cases and also in G_2, F_4 and E_6. We should mention that in E_6 actually only case (1) of the Proposition appears. In E_6, in the maximal parabolic cases, if the centralizer is of type T_1, the data (M, ν) is never Hermitian (for $\nu > 0$).

There is a second remark which follows from the proof of Proposition 3.1.

Corollary. *In the notation of Proposition* 3.1*:*

(1) *If $Z(\mathcal{O})$ is of type A_1, then $r_{\sigma_0}(M, V, \nu) = 1$ and $r_{\sigma_1}(M, V, \nu) = \left(\frac{\nu_0 - \nu}{\nu_0 + \nu}\right)^\ell$, where ℓ is some odd positive integer.*
(2) *If $Z(\mathcal{O})$ is of type T_1, then $r_{\sigma_0}(M, V, \nu) = +1$ and $r_{\sigma_0'}(M, V, \nu) = -1$ for $\nu > 0$.*

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Remark. In fact, for all cases that we need in section 3.4, explicit calculations in the classical groups show that the integer ℓ from Corollary 3.1 is always equal to 1. (Moreover, ν_0 is always 1 or $\frac{1}{2}$.)

We mention that some of the calculations of this type can also be extracted from [Ba]. Although Remark 3.1 (*i.e.*, $\ell = 1$) seems to hold in general, we do not know yet a conceptual proof for it.

3.2. **Main results.** In this section we present the main result of this paper. The explicit case by case calculations are in 3.4. Recall that we only consider modules with real infinitesimal characters.

Let $\mathcal{O}$ be a nilpotent orbit in $\mathfrak{g}$ with Lie triple $\{e,h,f\}$. As in section 2, we attach to $\mathcal{O}$ real semisimple elements $s = \frac{1}{2}h + \nu$ with $\nu \in \mathfrak{z}(\mathcal{O})$ (notation as in section 2.2 and 2.10). Define

$$\begin{aligned} \mathcal{U}(\mathcal{O}) &= \{\overline{X}(s,\mathcal{O},\psi) \text{ unitary}: \ (s,\mathcal{O},\psi) \\ &\qquad\qquad \text{is a Kazhdan-Lusztig parameter}\}, \\ \mathcal{U}(\mathcal{O},\phi) &= \{\overline{X}(s,\mathcal{O},\psi) \in \mathcal{U}(\mathcal{O}): \ [\psi : \phi|_{A(s,e)}] \neq 0\}, \end{aligned} \tag{3.2.1}$$

for all $\phi \in \widehat{A(e)}_0$. Recall that, although in general, not all $\psi \in \widehat{A(s,e)}$ appear, in type E_6, all such ψ appear in Kazhdan-Lusztig parameters.

The unitary dual of $\mathbb{H}$ is the disjoint union of $\mathcal{U}(\mathcal{O})$ for nilpotent orbits $\mathcal{O}$, so we will describe all $\mathcal{U}(\mathcal{O})$. For any Hecke algebra $\mathbb{H}$, let $\mathcal{SU}(\mathbb{H})$ denote its spherical unitary dual.

Lemma. *Let (M,V,ν) be a (Langlands) parameter associated to $(\mathcal{O},\phi)$, $\phi \in \widehat{A(\mathcal{O})}$. Then there exist a set of W-types $\{\sigma_0,\sigma_1,\dots,\sigma_k\}$ in $X(M,V,\nu)$ (σ_0 a lowest W-type with multiplicity one) such that the corresponding $W(\mathfrak{z}(\mathcal{O}))$-types $\{\rho(\sigma_0),\rho(\sigma_1),\dots,\rho(\sigma_k)\}$ are relevant for $\mathbb{H}(Z^0(\mathcal{O}))$ (with $\rho(\sigma_0)$ the trivial representation) and*

$$r_\sigma(M,V,\nu) = r_{\rho(\sigma)}(\nu), \ \textit{for all } \sigma \in \{\sigma_0,\sigma_1,\dots,\sigma_k\}.$$

Proof. The method of calculation is uniform, but the details need to be checked in each case. We will outline the method here.

Let $\mathcal{A}(M,V,\nu)$ be the intertwining operator (for the Weyl group element w_m) in $\mathbb{H}$ which induces the operators $r_\sigma(M,V,\nu)$. Recall from section 2.10, that $\mathfrak{a}$ denotes a Cartan subalgebra of $\mathfrak{z}(\mathcal{O})$ with $\nu \in \mathfrak{a}$ and that $C(\mathfrak{a}) \subset W$ is defined by (2.10.3).

The idea is to decompose $\mathcal{A}(M,V,\nu)$ into a product of factors similar to the usual decomposition of the (spherical) long intertwining operator for $\mathbb{H}(\mathfrak{z})$, such that the restriction of each factor in $\mathcal{A}(M,V,\nu)$ to σ is identical to the corresponding simple factor in the long intertwining operator of $\mathbb{H}(\mathfrak{z})$. It is a case by case observation that $W(\mathfrak{z}(\mathcal{O}))$ embeds in $C(\mathfrak{a})$. We remark that actually in almost all cases in E_6, $A(\mathcal{O}) = 1$ and $C(\mathfrak{a}) \cong W(\mathfrak{z}(\mathcal{O}))$.

For each simple root $\bar{\alpha} \in \mathfrak{z}(\mathcal{O})$, we find an element $\bar{s}_\alpha \in C(\mathfrak{a})$, which induces the corresponding simple reflection on $\mathfrak{a}$. Then the $\bar{s}_\alpha$'s generate a subgroup of $C(\mathfrak{a})$ isomorphic to $W(\mathfrak{z}(\mathcal{O}))$. Let $w_0(\mathfrak{z})$ be the long Weyl group element in $W(\mathfrak{z}(\mathcal{O}))$ and $\overline{w}_0(\mathfrak{z})$ its image in $C(\mathfrak{a})$. Note that by the definition of $C(\mathfrak{a})$, $w_m \in C(\mathfrak{a})$. There are two cases:

a) $w_m \neq \overline{w}_0(\mathfrak{z})$. In this case, we show that $L(M,V,\nu)$ has two lowest W-types with opposite signature (same type of argument as in section 3.1) and therefore it is not unitary.

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b) $w_m = \overline{w}_0(\mathfrak{z})$. Then $\mathcal{A}(M,V,\nu)$ decomposes into a product of the form (same idea as in section 2.6)

$$\prod \mathcal{A}_{\bar{\alpha}}(M,V,\nu_{\bar{\alpha}}), \tag{3.2.2}$$

corresponding to the Weyl group elements $\bar{s}_\alpha$. Each $\bar{s}_\alpha$ preserves (M,V). In order to calculate the factors $\mathcal{A}_{\bar{\alpha}}(M,V,\nu_{\bar{\alpha}})$, we decompose each one further into a product of maximal parabolic factors $\mathcal{A}(w_iM,w_iV,w_i\nu_{\bar{\alpha}})$ (as in section 2.6, equation (2.6.3)).

Let σ_0 be a lowest W-type in $X(M,V,\nu)$ appearing with multiplicity one and let $\sigma_{M,0}$ be the lowest $W(M)$-type of V. Let M' be a Levi component such that $M \subset M'$ and M comes from a maximal parabolic in M'. The nilpotent orbit $\mathcal{O}_M$, which parameterizes V in $\mathbb{H}_M$, has a set of lowest W-types attached to it in $\mathbb{H}_{M'}$. Call them $\sigma_{M',0}$, $\sigma'_{M',0}$, etc., where $\sigma_{M',0}$ is the one appearing in the restriction of σ_0. Let $\sigma_{M',1}$, $\sigma'_{M',1}$, etc., denote the corresponding $W(M')$-types from Corollary 3.1 and Remark 3.1.

Then we find the set $\{\sigma_0,\sigma_1,\dots,\sigma_k\} \subset \widehat{W}$, such that each σ in this set satisfies the following conditions:
(3.2.3)
(i) $\mathrm{Hom}_{W(M)}[\sigma,V] = \mathrm{Hom}_{W(M)}[\sigma,\sigma_{M,0}] \neq 0$;
(ii) if $w_0(M')M \neq M, \mathrm{Hom}_{W(M')}[\sigma,\mu] = 0,$ for all $\mu \neq \sigma_{M',0}$;
(iii) if $w_0(M')M = M, \mathrm{Hom}_{W(M')}[\sigma,\mu] = 0,$ for all $\mu \notin \{\sigma_{M',0},\sigma'_{M',0},\dots,\sigma_{M',1},\sigma'_{M',1},\dots\}$.

Conditions (3.2.3) are set so that in the decomposition of the operators $\mathcal{A}_{\bar{\alpha}}(M,V,\nu_\alpha)$ into maximal parabolic factors, the non-Hermitian factors (as in (ii)) do not contribute for σ. In addition, on the Hermitian factors (as in (iii)), we want the normalized operators to look exactly like a simple reflection factor in the decomposition of the spherical long intertwining operator of $W(\mathfrak{z}(\mathcal{O}))$. This is the content of the third condition, in conjunction with Remark 3.1. It follows that the operators $r_\sigma(M,V,\nu)$ and $r_{\rho(\sigma)}(\nu)$ are identical. □

For a nilpotent orbit $\mathcal{O}$, let $\mathbb{H}(Z^0(\mathcal{O}))$ be the graded Hecke algebra associated to the identity component of the centralizer of $\mathcal{O}$. As in the introduction, we can define the algebra

$$\mathbb{H}(\mathcal{O}) = \mathbb{H}(Z^0(\mathcal{O})) \ltimes A(\mathcal{O}). \tag{3.2.4}$$

The description of the unitary dual for $\mathbb{H}$ is summarized next. The explicit description appears in section 3.4 and also for convenience, the unitary parameters are listed again in the tables of section 4.

Note that the following result is a tautology when $\mathcal{O}$ is the trivial nilpotent orbit, that is for the spherical $\mathbb{H}$-modules. $\mathcal{SU}(\mathbb{H})$ is formed of the complementary series attached to each nontrivial nilpotent orbit $\mathcal{O}$ (see section 2.8) and the spherical complementary series for type E_6, which is determined in section 3.5. The labeling of nilpotent orbits is as in [Car].

Theorem. *Let $\mathcal{O}$ be a nilpotent orbit in the Lie algebra $\mathfrak{g}$ of type E_6, and $(s,\mathcal{O},\psi)$ be a Kazhdan-Lusztig parameter with $s = \frac{h}{2}+\nu$, $\nu \in \mathfrak{z}(\mathcal{O})$.*

(1) *$\overline{X}(s,\mathcal{O},\psi)$ is Hermitian if and only if the parameter ν is (spherical) Hermitian for $\mathbb{H}(\mathcal{O})$. Moreover, if ν is not Hermitian for $\mathbb{H}(\mathfrak{z}(\mathcal{O}))$, then $\overline{X}(s,\mathcal{O},\psi)$ is not unitary.*

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(2) *For every* $\phi \in \widehat{A(e)}_0$, $\mathcal{SU}(\mathbb{H}(\mathfrak{z}(\mathcal{O}))) \subset \mathcal{U}(\mathcal{O},\phi)$, *i.e.,* $\overline{X}(s,\mathcal{O},\psi)$ *is unitary if the corresponding* ν *is a spherical unitary parameter for* $\mathcal{SU}(\mathbb{H}(\mathfrak{z}(\mathcal{O})))$. *In particular, the complementary series for* $\mathcal{O}$ *is identical with the spherical complementary series for* $\mathbb{H}(\mathfrak{z}(\mathcal{O}))$.

(3) *Except in the case of the nilpotent orbit* $3A_1$, $\overline{X}(s,\mathcal{O},\psi)$ *is unitary if and only if* ν *is a spherical unitary parameter for* $\mathcal{SU}(\mathbb{H}(\mathfrak{z}(\mathcal{O})))$.

Proof. The proof is given in the calculations of the next section. We make some remarks about the methods employed.

(1) It is checked case by case. The component group $A(\mathcal{O})$ only makes a difference for two nilpotent orbits, $D_4(a_1)$ and A_2. If ν is not Hermitian for $\mathbb{H}(\mathfrak{z})$, we show that $\overline{X}(s,\mathcal{O},\psi)$ always contains two lowest W-types with multiplicity one and opposite signature at ∞.

(2) The set of representations from Lemma 3.2 always contains the relevant $W(\mathfrak{z}(\mathcal{O}))$-types which are sufficient for the determination of the spherical complementary series of $\mathbb{H}(\mathfrak{z})$. For each ν in the spherical complementary series, we show that the corresponding s can be deformed, so that $X(s,\mathcal{O},\psi)$ stays irreducible, until we reach a point where $X(s,\mathcal{O},\psi)$ is unitarily induced irreducible from a unitary module on a smaller rank algebra. From this argument, it follows that the complementary series attached to $\mathcal{O}$ is the same with the one for $\mathbb{H}(\mathfrak{z})$.

When the set of representations from Lemma 3.2 gives *all* the relevant $W(\mathfrak{z}(\mathcal{O}))$-types (Definition 2.9), we obtain an inclusion $\mathcal{U}(\mathcal{O},\phi) \subset \mathcal{SU}(\mathbb{H}(\mathfrak{z}))$. Then we want to show "equality". We organize the analysis by the nilpotent orbits in $\mathfrak{z}(\mathcal{O})$, since $\mathcal{SU}(\mathbb{H}(\mathfrak{z}))$ is partitioned by them. For $\nu \in \mathcal{SU}(\mathbb{H}(\mathfrak{z}))$, we show that $X(s,\mathcal{O},\psi)$ is unitary by one of the following methods:

(1) deformation arguments as for the complementary series,
(2) IM-dual of a unitary module, or
(3) direct calculation of intertwining operators.

In order to use the Iwahori-Matsumoto involution effectively, we compute decompositions of standard modules, and the W-structure of the unitary irreducible quotients (especially at endpoints of complementary series).

We only use an explicit computation with intertwining operators in the case when a parameter is isolated for a nilpotent orbit and its IM-dual is also isolated. We do not have a good (elegant) way to treat these cases, and therefore to complete the classification, we find their W-structure, and compute the operators $r_\sigma(M,V,\nu)$ on all W-types which appear with nonzero multiplicity. We remark that usually these isolated modules are small and that, in E_6, there are only two places where such calculations are needed.

If the set of W-types from Lemma 3.2 does not give all the relevant $W(\mathfrak{z}(\mathcal{O}))$-types, we need, in addition, to compute the operators on some $\sigma \in \widehat{W}$, such that $\rho(\sigma)$ contains the unmatched relevant $W(\mathfrak{z}(\mathcal{O}))$-types, in order to rule out the remaining nonunitary modules.

The nilpotent orbit $3A_1$ is one of the cases where we cannot match all the relevant $W(\mathfrak{z}(\mathcal{O}))$-types ($\mathfrak{z}$ is of type $A_2 + A_1$ here), but it is the only case where we obtain a larger set of unitary modules than the spherical unitary dual of the $\mathbb{H}(\mathfrak{z})$ (see Figure 1). $\square$

3.3. **Arthur parameters.** We present an important consequence of Theorem 3.2 (and the calculations in the next section). Let s be a semisimple element (attached

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to the nilpotent $\mathcal{O}$) of the form

$$s = \frac{h}{2} + \frac{h_z}{2}, \text{ for } h_z \text{ the middle element of a nilpotent orbit } \mathcal{O}_z \subset \mathfrak{z}(\mathcal{O}). \tag{3.3.1}$$

The question is to decide the unitarity of the associated modules $\overline{X}(s, \mathcal{O}, \psi)$. These are instances of *Arthur parameters*, they correspond to maps

$$\Phi : SL(2, \mathbb{C}) \times SL(2, \mathbb{C}) \to \mathfrak{g}. \tag{3.3.2}$$

When $\mathcal{O}_z$ is the trivial nilpotent orbit of $\mathfrak{z}$ ($s = \frac{h}{2}$), the corresponding modules $X(s, \mathcal{O}, \psi)$ are all irreducible tempered, therefore unitary.

When $\mathcal{O}$ is the trivial nilpotent orbit of $\mathfrak{g}$, the centralizer $\mathfrak{z}$ equals all of $\mathfrak{g}$, and the associated modules are the *spherical unipotent* $\overline{X}(\frac{h_{\mathcal{O}}}{2})$ ($h_{\mathcal{O}}$ varies over the middle elements of all nilpotent orbits $\mathcal{O}$). Since $\overline{X}(\frac{h_{\mathcal{O}}}{2}) = IM(X(\frac{h_{\mathcal{O}}}{2}, \mathcal{O}, trivial))$ (and IM preserves unitarity), it follows that $\overline{X}(\frac{h_{\mathcal{O}}}{2})$ are also unitary.

From Theorem 3.2 it follows that:

Corollary. *For the Hecke algebra of type E_6, if s is a parameter of the form* (3.3.1), *then the associated modules $\overline{X}(s, \mathcal{O}, \psi)$ are unitary.*

3.4. **Calculations for E_6.** Throughout this section, we will use the Bourbaki realization of the root system of type E_6 (α_i are the simple roots and $\check{\omega}_i$ the corresponding simple coweights):

$\alpha_1 = \frac{1}{2}(1, -1, -1, -1, -1, -1, -1, 1)$	$\check{\omega}_1 = (0, 0, 0, 0, 0, -\frac{2}{3}, -\frac{2}{3}, \frac{2}{3})$
$\alpha_2 = (1, 1, 0, 0, 0, 0, 0, 0)$	$\check{\omega}_2 = (\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, \frac{1}{2})$
$\alpha_3 = (-1, 1, 0, 0, 0, 0, 0, 0)$	$\check{\omega}_3 = (-\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, -\frac{5}{6}, -\frac{5}{6}, \frac{5}{6})$
$\alpha_4 = (0, -1, 1, 0, 0, 0, 0, 0)$	$\check{\omega}_4 = (0, 0, 1, 1, 1, -1, -1, 1)$
$\alpha_5 = (0, 0, -1, 1, 0, 0, 0, 0)$	$\check{\omega}_5 = (0, 0, 0, 1, 1, -\frac{2}{3}, -\frac{2}{3}, \frac{2}{3})$
$\alpha_6 = (0, 0, 0, -1, 1, 0, 0, 0)$	$\check{\omega}_6 = (0, 0, 0, 0, 1, -\frac{1}{3}, -\frac{1}{3}, \frac{1}{3})$

The Weyl group representations (*W-types*) for type E_6 were classified by Frame (see [Fr]) and we will use his labeling of the irreducible characters. The W-structure of standard modules is given by the Green polynomials calculated in [BS]; we also used the (unpublished) tables in [Al]. For restrictions of W-types, and for the computation of the associated $W(\mathfrak{z}(\mathcal{O}))$-type $\rho(\sigma)$ to a given W-type σ, we used the software "GAP". For some of the explicit computations with intertwining operators, we used J. Adams' integer matrix models for $W(E_6)$ (`atlas.math.umd.edu/weyl/`) and the software "Mathematica".

1. $\mathbf{E_6}$: distinguished orbit. $X(E_6)$ is St, the Steinberg representation, with lowest W-type $1'_p$ (even degree). The infinitesimal character is

$$\rho(E_6) = (0, 1, 2, 3, 4, -4, -4, 4).$$

2. $\mathbf{E_6(a_1)}$: distinguished orbit. $X(E_6(a_1))$ is a discrete series,

$$X(E_6(a_1) \mid_W = 6'_p + 1'_p,$$

with lowest W-type $6'_p$ (odd degree). The infinitesimal character is

$$(0, 1, 1, 2, 3, -3, -3, 3).$$

3. $\mathbf{D_5}$: centralizer T_1 (connected).

$$X(D_5)|_W = 20'_p + 6'_p + 1'_p,$$

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with lowest W-type $20'_p$ (even degree). The infinitesimal character is

$$s=(0,1,2,3,4,0,0,0)+\nu\check{\omega}_1,\quad \nu\geq 0.$$

For $\nu=0$, the module is tempered irreducible and the infinitesimal character is W-conjugate to $(\frac{1}{2},\frac{1}{2},\frac{3}{2},\frac{3}{2},\frac{5}{2},-\frac{5}{2},-\frac{5}{2},\frac{5}{2})$.

For $\nu>0$, $\overline{X}(D_5,\nu)=L(D_5,St,\nu)$, which is never Hermitian. The points of reducibility are $\nu=3,\ 6$, with generic factors $X(E_6(a_1))$, and $X(E_6)$. At the first point of reducibility

$$X(D_5,3)=X(E_6(a_1))+\overline{X}(D_5,3),\quad \overline{X}(D_5,3)|_W=20'_p.$$

There is *no complementary series.*

4. $\mathbf{E_6(a_3)}$: distinguished orbit, component group $\mathbb{Z}_2$,

$$X(E_6(a_3)=X(E_6(a_3),(2))+X(E_6(a_3),(11))$$

are discrete series.

$$X(E_6(a_3),(2))|_W=30'_p+20'_p+6'_p+1'_p,\quad X(E_6(a_3),(11))|_W=15'_p+6'_p,$$

with lowest W-types $30'_p$ (odd degree), respectively $15'_p$ (odd degree). The infinitesimal character is

$$(0,0,1,1,2,-2,-2,2).$$

5. $\mathbf{D_5(a_1)}$: centralizer T_1 (connected).

$$X(D_5(a_1))|_W=64'_p+30'_p+15'_p+2\cdot 20'_p+2\cdot 6'_p+1'_p,$$

with lowest W-type $64'_p$ (even degree). The infinitesimal character is

$$s=(0,1,1,2,3,0,0,0)+\nu\check{\omega}_1,\quad \nu\geq 0.$$

For $\nu=0$, the module is tempered irreducible and the infinitesimal character is W-conjugate to $(\frac{1}{4},\frac{3}{4},\frac{3}{4},\frac{5}{4},-\frac{7}{4},-\frac{7}{4},\frac{7}{4})$.

For $\nu>0$, $\overline{X}(D_5(a_1),\nu)=L(D_5,(73),\nu)$, where $\sigma=(73)$ denotes the discrete series parameterized by the nilpotent orbit (73) in D_5. Therefore $\overline{X}(D_5(a_1),\nu)$ is never Hermitian for $\nu>0$. The points of reducibility are $\nu=\frac{3}{2},\ \frac{7}{2},\ \frac{9}{2}$, with generic factors $X(E_6(a_3),(2))$, $X(D_5,1)$, and $E_6(a_1)$. At the first reducibility point, $\nu=\frac{3}{2}$,

$$X(D_5(a_1),\frac{3}{2})=X(E_6(a_3))+\overline{X}(D_5(a_1),\frac{3}{2}),\quad \overline{X}(D_5(a_1),\frac{3}{2})|_W=64'_p+20'_p.$$

There is *no complementary series.*

6. $\mathbf{A_5}$: centralizer A_1 (connected).

$$X(A_5)|_W=15'_q+30'_p+20'_p+6'_p+1'_p,$$

with lowest W-type $15'_q$ (even degree). The infinitesimal character is

$$(-\frac{11}{4},-\frac{7}{4},-\frac{3}{4},\frac{1}{4},\frac{5}{4},-\frac{5}{4},-\frac{5}{4},\frac{5}{4})+\nu\check{\omega}_2,\quad \nu\geq 0.$$

For $\nu=0$, the module is tempered irreducible and the infinitesimal character is W-conjugate to $(0,\frac{1}{2},\frac{1}{2},1,2,-2,-2,2)$.

For $\nu>0$, $X(A_5,\nu)=L(A_5,St,\nu)$ is always Hermitian and it has to remain unitary until the first point of reducibility. The points of reducibility are $\nu=\frac{1}{2},\ \frac{5}{2},\ \frac{7}{2},\ \frac{11}{2}$, with generic factors $X(E_6(a_3),(2))$, $X(D_5)$, $X(E_6(a_1))$ and $X(E_6)$. At the first point of reducibility $\nu=\frac{1}{2}$,

$$X(A_5,\frac{1}{2})=X(E_6(a_3),(2))+\overline{X}(A_5,\frac{1}{2}),\quad \overline{X}(A_5,\frac{1}{2})|_W=15'_q.$$

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For $\nu > \frac{1}{2}$, since $E_6(a_3)$ is a distinguished orbit, there cannot be any factors parameterized by $E_6(a_3)$ and therefore $30'_p$, the lowest W-type of $E_6(a_3)$ has to stay in the same factor as $15'_q$, that is, in $\overline{X}(A_5,\nu)$. Since $15'_q$ and $30'_p$ have opposite signature at ∞, it follows that $\overline{X}(A_5,\nu)$ is not unitary for $\nu > \frac{1}{2}$.

Complementary series: $0 \le \nu \le \frac{1}{2}$.

7. $\mathbf{A_4+A_1}$: centralizer T_1 (connected).

$$X(A_4+A_1)|W = 60'_p + 64'_p + 15'_q + 30'_p + 2\cdot 20'_p + 6'_p + 1'_p,$$

with lowest W-type $60'_p$. The infinitesimal character is

$$(\frac{9}{4},-\frac{5}{4},-\frac{1}{4},\frac{3}{4},\frac{7}{4},-\frac{1}{4},-\frac{1}{4},\frac{1}{4}) + \nu\check{\omega}_3, \quad \nu \ge 0.$$

For $\nu = 0$, the module is tempered and irreducible and the infinitesimal character is W-conjugate to $(0,\frac{1}{2},\frac{1}{2},1,\frac{3}{2},-\frac{3}{2},-\frac{3}{2},\frac{3}{2})$.

For $\nu > 0$, $\overline{X}(A_4+A_1,\nu) = L(A_4+A_1,St,\nu)$ is never Hermitian. The points of reducibility are $\nu = \frac{3}{2},\ \frac{5}{2},\ \frac{7}{2},\ \frac{9}{2}$, with generic factors $X(E_6(a_3),(2))$, $X(D_5,1)$, $X(D_5,4)$ and $X(E_6)$.

There is *no complementary series*.

8. $\mathbf{D_4}$: centralizer A_2 (connected).

$$X(D_4)|_W = 24_p + 2\cdot 64'_p + 30'_p + 15'_p + 3\cdot 20'_p + 2\cdot 6'_p + 1'_p,$$

with lowest W-type $24'_p$ (even degree). The Hermitian parameter is $(M,\sigma) = (D_4,St)$ with (Hermitian) infinitesimal character

$$s = (0,1,2,3,0,0,0,0) + \nu(\check{\omega}_1+\check{\omega}_6) = (0,1,2,3,\nu,-\nu,-\nu,\nu), \quad \nu \ge 0.$$

For $\nu = 0$, the module is tempered irreducible and the infinitesimal character is W-conjugate to $(\frac{1}{2},\frac{1}{2},\frac{3}{2},\frac{3}{2},\frac{3}{2},-\frac{3}{2},-\frac{3}{2},\frac{3}{2})$.

For $\nu > 0$, the Hermitian intertwining operator is given by the element r_{w_m}, where $w_m = w_0 \cdot w_0(M)$. Note that $w_m(D_4,St,\nu) = (D_4,St,-\nu)$. The subgroup $W(\mathfrak{z}) \cong W(A_2)$ is generated by

$$\bar{s}_1 = w_m(D'_5,M),$$
$$\bar{s}_2 = w_m(D''_5,M).$$

Here D'_5 denotes the Levi subgroup generated by $\{\alpha_1,\alpha_2,\alpha_3,\alpha_4,\alpha_5\}$, while D''_5 is generated by $\{\alpha_2,\alpha_3,\alpha_4,\alpha_5,\alpha_6\}$.

Note that $w_m = \bar{s}_1\cdot\bar{s}_2\cdot\bar{s}_1$ and the intertwining operator decomposes accordingly:

$$\mathcal{A}_{E_6}(D_4,St,\nu) = \mathcal{A}_{D'_5}(D_4,St)\circ\mathcal{A}_{D''_5}(D_4,St)\circ\mathcal{A}_{D'_5}(D_4,St).$$

The restrictions of W-types are:

Nilpotent	D_4	$D_5(a_1)$	$E_6(a_3)$	$E_6(a_3)$
W-type	$24'_p$	$64'_p$	$30'_p$	$15'_p$
Multiplicity	1	2	1	1
$D_4 \subset D_5$	$21^3\times 0$	$21^3\times 0, 1^4\times 1$	$1^4\times 1$	$1^4\times 1$
$Z = A_2$	(3)	(21)	(1^3)	(1^3)

In the Hecke algebra of type D_5, the nilpotent orbit is $D_4 = (91)$, with lowest W-type $21^3\times 0$. The corresponding (restriction of the) infinitesimal character is $(0,1,2,3,\bar{\nu})$, where $\bar{\nu}$ takes the values ν, 2ν and ν. The reducibility points are $\bar{\nu} = 1,\ 4$. The intertwining operators in D_5 are:

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$21^3 \times 0$	1
$1^4 \times 1$	$\frac{1-\bar{\nu}}{1+\bar{\nu}}$

It follows that the points of reducibility for $X(D_4,\nu)$ are $\nu = \frac{1}{2},\ 1,\ 2,\ 4$.

For the centralizer A_2, the spherical Hermitian infinitesimal character in standard coordinates is $(\nu, 0, -\nu)$. Therefore, the matching of intertwining operators is:

W-type	$W(\mathfrak{z})$-type
$24'_p$:	(3)
$64'_p$:	(21)
$30'_p$:	(1^3)
$15'_p$:	(1^3)

All W-types of A_2 are matched in this way, which implies that $\mathcal{U}(E_6, D_4) \subset \mathcal{SU}(\mathbb{H}(A_2))$. $\mathcal{SU}(\mathbb{H}(A_2))$ is formed of $0 \le \nu < \frac{1}{2}$, attached to the nilpotent orbit (1^3), $\nu = \frac{1}{2}$, attached to (21) and $\nu = 1$, attached to (3).

Clearly, $X(D_4,\nu)$ has to be unitary for $0 \le \nu \le \frac{1}{2}$. The *complementary series* is $0 \le \nu < \frac{1}{2}$.

At the endpoint of the complementary series $\nu = \frac{1}{2}$,

$$X(D_4, \frac{1}{2}) = X(D_5(a_1)) + \overline{X}(D_4, \frac{1}{2}), \quad \overline{X}(D_4, \frac{1}{2})|_W = 60'_p + 64'_p + 20'_p.$$

At the isolated point $\nu = 1$, the decomposition is

$$X(D_4, 1) = X(E_6(a_3), (2)) + X(E_6(a_3), (11)) + 2 \cdot \overline{X}(D_5(a_1), \frac{3}{2}) + \overline{X}(D_4, 1),$$
$$\overline{X}(D_4, 1)|_W = 24'_p.$$

Since $\overline{X}(D_4, 1)$ is Hermitian and its restriction to W is a single W-type, it must be unitary.

In conclusion, $\mathcal{U}(E_6, D_4) = \mathcal{SU}(\mathbb{H}(A_2))$.

9. $\mathbf{A_4}$: centralizer $A_1 + T_1$ (connected).

$$X(A_4)|_W = 81'_p + 60'_p + 2 \cdot 64'_p + 15'_q + 2 \cdot 30'_p + 15'_p + 3 \cdot 20'_p + 2 \cdot 6'_p + 1'_p,$$

with lowest W-type $81'_p$ (even degree). The Hermitian parameter is $(M, \sigma) = (A_5, (51))$, where (51) denotes the tempered representation parameterized by the orbit (51) in A_5, and (Hermitian) infinitesimal character

$$(-2, -1, 0, 1, 2, 0, 0, 0) + \nu\check{\omega}_2, \quad \nu \ge 0.$$

For $\nu = 0$, the module is tempered irreducible and the infinitesimal character is W-conjugate to $(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{3}{2}, -\frac{3}{2}, -\frac{3}{2}, \frac{3}{2})$.

For $\nu > 0$, note that $w_m(A_5, \sigma, \nu) = (A_5, \sigma, -\nu)$, $\sigma = (51)$. The subgroup $W(\mathfrak{z})$ of the Weyl group corresponding to the (semisimple part of) the centralizer $Z = A_1$ is generated by w_m. w_m decomposes as follows:

$$w_m = w_m(D''_5, A_4) \cdot s_1 \cdot w_m(D''_5, A_4).$$

The intertwining operator decomposes:

$$\mathcal{A}_{E_6}(A_5, (51), \nu) = \mathcal{A}_{D''_5}(A_4, St) \circ \mathcal{A}_{A_1}(0, St) \circ \mathcal{A}_{D''_5}(A_4, St).$$

Restrictions of W-types are:

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Nilpotent	A_4	A_4+A_1
W-type	$81'_p$	$60'_p$
Multiplicity	1	1
$A_4 \subset D_5$	$1^3 \times 1^2$	$1^3 \times 1^2$
$0 \subset A_1$	(2)	(11)
$Z = A_1$	(2)	(11)

In the Hecke algebra of type D_5, the nilpotent orbit is $A_5 = (55)$, with lowest W-types $1^3 \times 1^2$. The corresponding (restriction of the) infinitesimal character are $(-2,-1,0,1,2)+\nu(\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2})$ with reducibility at $\nu = 2,\ 4$.

It follows that the points of reducibility for $X(A_4,\nu)$ are $\nu = \frac{1}{2},\ 2,\ 4$.

The matching of intertwining operators is:

W-type	$W(\mathfrak{z})$-type	
$81'_p$:	(2)	$+1$
$60'_p$:	(11)	$\frac{\frac{1}{2}-\nu}{\frac{1}{2}+\nu}$

Therefore, $\mathcal{U}(E_6,A_4) \subset \mathcal{SU}(\mathbb{H}(A_1)) = \{0 \le \nu \le \frac{1}{2}\}$. Clearly, $X(A_4,\nu)$ is unitary for $0 \le \nu < \frac{1}{2}$, being unitary at $\nu = 0$ and irreducible for $0 \le \nu < \frac{1}{2}$. The **complementary series** is $0 \le \nu \le \frac{1}{2}$.

At the endpoint of the complementary series $\nu = \frac{1}{2}$,

$$X(A_4,\frac{1}{2}) = X(A_4+A_1)+\overline{X}(A_4,\frac{1}{2}),\ \overline{X}(A_4,\frac{1}{2})|_W = 81'_p+64'_p+30'_p+15'_p+20'_p+6'_p.$$

In conclusion, $\mathcal{U}(E_6,A_4) = \mathcal{SU}(\mathbb{H}(A_1))$.

10. $\mathbf{D_4(a_1)}$: centralizer T_2 with component group S_3.

$$X(D_4(a_1)) = X(D_4(a_1),(3)) + 2\cdot X(D_4(a_1),(21)) + X(D_4(a_1),(1^3)),$$

$$X(D_4(a_1),(3))|_W : 80_s + 81'_p + 24'_p + 2\cdot 60'_p + 2\cdot 64'_p + 15'_q + 2\cdot 30_p + 6'_p + 1'_p,$$
$$X(D_4(a_1),(21))|_W : 90_s + 81'_p + 2\cdot 64'_p + 30'_p + 15'_p + 20'_p + 6'_p,$$
$$X(D_4(a_1),(1^3))|_W : 20_s + 15'_p$$

with lowest W-types 80_s (odd degree), 90_s (even degree) and 20_s (even degree). The Hermitian parameter is $(M,\sigma) = (D_4,(53))$, where (53) denotes the discrete series parameterized by the nilpotent orbit (53) in D_4, and Hermitian infinitesimal character

$$s = (0,1,1,2,0,0,0,0) + \nu(\check{\omega}_1+\check{\omega}_6) = (0,1,1,2,\nu,-\nu,-\nu,\nu), \quad \nu \ge 0.$$

For $\nu = 0$, the module decomposes into four irreducible tempered modules and the infinitesimal character is W-conjugate to $(0,0,1,1,1,-1,-1,1)$.

For $\nu > 0$, the three lowest W-types always stay in the same factor (the component group $A(s,e)$ doesn't change). $w_m(D_4,(53),\nu) = (D_4,(53),-\nu)$. $W(\mathfrak{z}) = 1$ and $W(Z) = W(A(\mathcal{O})) \cong S_3$ is generated by

$$\bar{s}_1 = w_m(D'_5,M),$$
$$\bar{s}_2 = w_m(D''_5,M),$$

$w_m = \bar{s}_1\cdot\bar{s}_2\cdot\bar{s}_1$ and the intertwining operator decomposes:

$$\mathcal{A}_{E_6}(D_4,(53),\nu) = \mathcal{A}_{D'_5}(D_4,(53)) \circ \mathcal{A}_{D''_5}(D_4,(53)) \circ \mathcal{A}_{D'_5}(D_4,(53)).$$

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The restrictions of W-types are:

Nilpotent	$D_4(a_1)$	$D_4(a_1)$	$D_4(a_1)$
W-type	80_s	90_s	20_s
Multiplicity	1	2	1
$D_4(a_1) \subset D_5$	211×1	$211\times, 1^3 \times 2$	$1^3 \times 2$
$W(Z) = S_3$	(3)	(21)	(1^3)

In the Hecke algebra of type D_5, the nilpotent orbit is $D_4(a_1) = (53)$ with lowest W-types 211×1 and $1^3 \times 2$. The corresponding (restriction of the) infinitesimal character is $(0, 1, 1, 2, \bar{\nu})$, where $\bar{\nu}$ takes the values ν, 2ν and ν. The two lowest W-types are always in the same factor for $\bar{\nu} > 0$. The reducibility points are $\bar{\nu} = 2, 3$. The intertwining operators in D_5 are:

$$\begin{array}{ll} 211 \times 1: & +1 \\ 1^3 \times 2: & -1 \end{array}$$

It follows that the points of reducibility for $X(D_4(a_1), \nu)$ are $\nu = 1,\ \frac{3}{2},\ 2,\ 3.$

We match the intertwining operators for the nilpotent $D_4(a_1)$ with operators for the Hecke algebra of type A_2 with zero parameter:

W-type	$W(Z)$-type	
80_s:	(3)	$+1$
90_s:	(21)	$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$
20_s:	(1^3)	-1

There is *no complementary series.*

11. $\mathbf{A_3 + A_1}$**:** centralizer $A_1 + T_1$ (connected). $X(A_3 + A_1)$ has lowest W-type 60_s (even degree). The Hermitian parameter is $(M, \sigma) = (A_5, (42))$, where (42) is a tempered module of A_5 (realize $(42) = A_3 + A_1 \subset A_5$ by the roots $\{\alpha_1, \alpha_3, \alpha_4, \alpha_6\}$), and (Hermitian) infinitesimal character

$$s = (-\frac{5}{4}, -\frac{1}{4}, \frac{3}{4}, -\frac{5}{4}, -\frac{1}{4}, -\frac{3}{4}, -\frac{3}{4}, \frac{3}{4}) + \nu \check{\omega}_2, \quad \nu \geq 0.$$

For $\nu = 0$, the module is tempered irreducible and the infinitesimal character is W-conjugate to $(0, \frac{1}{2}, \frac{1}{2}, 1, 1, -1, -1, 1)$.

For $\nu > 0$, $w_m(A_5, (42), \nu) = (A_5, (42), -\nu)$. $W(\mathfrak{z}) \cong W(A_1)$ is generated by w_m. w_m decomposes as follows:

$$w_m = w_m(A_4, A_3') \cdot w_m(D_5, D_3 + A_1) \cdot w_m(A_4, A_3),$$

where A_4 is realized by the roots $\{\alpha_1, \alpha_3, \alpha_4, \alpha_2\}$, A_3 by $\{\alpha_1, \alpha_3, \alpha_4\}$ and A_3' by $\{\alpha_3, \alpha_4, \alpha_2\}$.

The intertwining operator decomposes:

$$\mathcal{A}_{E_6}(A_5, (42), \nu) = \mathcal{A}_{A_4}(A_3, St) \circ \mathcal{A}_{D_5}(D_3 + A_1, St) \circ \mathcal{A}_{A_4}(A_3, St).$$

The restrictions of W-types are:

Nilpotent	$A_3 + A_1$	$D_4(a_1)$	$D_4(a_1)$
W-type	60_s	80_s	90_s
Multiplicity	1	1	1
$A_3 \subset A_4$	(21^3)	(21^3)	(21^3)
D_5	221×0	211×1	211×1
$W(\mathfrak{z}) = S_2$	(2)	(11)	(11)

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The reducibility point coming from the factor $\mathcal{A}_{A_4}(A_3, St, \nu)$ is $\nu = \frac{5}{2}$.

In the Hecke algebra of type D_5, the nilpotent is (5221) with lowest W-type 221×0. The corresponding (restriction of the) infinitesimal character is $(0, 1, 2, -\frac{1}{2}, \frac{1}{2}) + \nu(0, 0, 0, 1, 1)$, with reducibility at $\nu = \frac{1}{2},\ \frac{3}{2},\ \frac{5}{2},\ \frac{7}{2}$. The intertwinining operators are:

$$\begin{array}{ll} 221 \times 0 & 1 \\ 211 \times 1 & \frac{\frac{1}{2}-\nu}{\frac{1}{2}+\nu} \end{array}$$

It follows that the reducibility points for $X(A_3 + A_1, \nu)$ are $\nu = \frac{1}{2},\ \frac{3}{2},\ \frac{5}{2},\ \frac{7}{2}$.

The matching of intertwining operators is:

W-type	$W(\mathfrak{z})$-type	
60_s	(2)	1
80_s	(11)	$\frac{\frac{1}{2}-\nu}{\frac{1}{2}+\nu}$
90_s	(11)	$\frac{\frac{1}{2}-\nu}{\frac{1}{2}+\nu}$

Therefore $\mathcal{U}(E_6, A_3 + A_1) \subset \mathcal{SU}(\mathbb{H}(A_1)) = \{0 \leq \nu \leq \frac{1}{2}\}$. $X(A_3 + A_1, \nu)$ is unitary (actually tempered) at $\nu = 0$ and irreducible for $0 \leq \nu < \frac{1}{2}$, so it is unitary in the entire interval. The *complementary series* is $0 \leq \nu < \frac{1}{2}$.

At the endpoint of the complementary series, $\nu = \frac{1}{2}$,

$$X(A_3 + A_1, \frac{1}{2}) = X(D_4(a_1), (3)) + X(D_4(a_1), (21)) + \overline{X}(A_3 + A_1, \frac{1}{2}),$$

$$\overline{X}(A_3 + A_1, \frac{1}{2})|_W = 60_s + 60'_p + 15'_q.$$

In conclusion, $\mathcal{U}(E_6, A_3 + A_1) = \mathcal{SU}(\mathbb{H}(A_1))$.

12. $\mathbf{2A_2 + A_1}$: centralizer A_1 (connected). $X(2A_2 + A_1)$ has lowest W-type 10_s (odd degree). The infinitesimal character is

$$(0, 1, -\frac{3}{2}, -\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}) + \nu\check{\omega}_4, \quad \nu \geq 0.$$

For $\nu = 0$, the module is tempered irreducible and the infinitesimal character is W-conjugate to $(0, 0, \frac{1}{2}, \frac{1}{2}, 1, -1, -1, 1)$.

For $\nu > 0$, $\overline{X}(2A_2 + A_1, \nu) = L(2A_2 + A_1, St, \nu)$ and it is always Hermitian. It has to remain unitary until the first reducibility point. The points of reducibility are $\nu = \frac{1}{2},\ 1,\ \frac{3}{2},\ \frac{5}{2},\ \frac{7}{2}$, with generic factors $X(D_4(a_1), (2))$, $X(A_4 + A_1)$, $X(E_6(a_3), (2))$, $X(E_6(a_1))$ and $X(E_6)$.

At the first reducibility point, $\nu = \frac{1}{2}$,

$$X(2A_2 + A_1, \frac{1}{2}) = X(D_4(a_1), (3)) + \overline{X}(A_3 + A_1, \frac{1}{2}) + \overline{X}(2A_2 + A_1, \frac{1}{2}),$$

$$\overline{X}(2A_2 + A_1, \frac{1}{2})|_W = 10_s.$$

For $\nu > \frac{1}{2}$, any factor parameterized by the orbit $A_3 + A_1$ would also have to contain the W-type 90_s. Since 90_s does not appear in $X(2A_2 + A_1)$, it follows that 60_s, the lowest W-type of $A_3 + A_1$ has to stay in the same factor with 10_s, that is, in $\overline{X}(2A_2 + A_1, \nu)$. Since 10_s and 60_s have opposite signature at ∞, $\overline{X}(2A_2 + A_1, \nu)$ is not unitary for $\nu > \frac{1}{2}$.

The *complementary series* is $0 \leq \nu < \frac{1}{2}$ and $\mathcal{U}(E_6, 2A_2 + A_1) = \mathcal{SU}(\mathbb{H}(A_1))$.

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13. $\mathbf{A_3}$: centralizer B_2+T_1 (connected). $X(A_3)$ has lowest W-type 81_p (even degree). The Hermitian parameter is $(M,\sigma)=(A_3,St)$, (A_3 is realized by $\{\alpha_3,\alpha_4,\alpha_5\}$) with Hermitian infinitesimal character

$$s=(-\frac{3}{2},-\frac{1}{2},\frac{1}{2},\frac{3}{2},0,0,0,0)+\nu_1(0,0,0,0,\frac{1}{2},-\frac{1}{2},-\frac{1}{2},\frac{1}{2})+\nu_2(\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},0,0,0,0),$$

$\nu_1\geq\nu_2\geq 0$.

For $(\nu_1,\nu_2)=(0,0)$, the module is tempered irreducible and the infinitesimal character is conjugate to $(\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},1,1,1,1)$.

For $\nu_1\geq\nu_2>0$, $w_m(A_3,St,\nu)=(A_3,St,-\nu)$. The subgroup $W(\mathfrak{z})=W(B_2)$ is generated by:

$$\begin{array}{lll}\bar{s}_1: & \nu_1\to\nu_2 & w_m(D_4,M)\\ \bar{s}_2: & \nu_1\to-\nu_1 & (w_m(A_4,A_3))\cdot s_6\cdot(w_m(A_4,A_3')),\end{array}$$

where A_4 is realized by the roots $\{\alpha_1,\alpha_3,\alpha_4,\alpha_5\}$ and A_3' by $\{\alpha_1,\alpha_3,\alpha_4\}$.

The intertwining operator decomposes accordingly:

$$\mathcal{A}_{E_6}(A_3,St,(\nu_1,\nu_2))=\mathcal{A}_1(-\nu_2,-\nu_1)\circ\mathcal{A}_2(-\nu_2,\nu_1)\circ\mathcal{A}_1(\nu_1,-\nu_2)\circ\mathcal{A}_2(\nu_1,\nu_2),$$

where $\mathcal{A}_1=\mathcal{A}_{D_4}(A_3,St)$ and $\mathcal{A}_2=\mathcal{A}_{A_4}(A_3',St)\circ\mathcal{A}_{A_1}(0,St)\circ\mathcal{A}_{A_4}(A_3,St)$.

The restrictions of W-types are:

Nilpotent	A_3	A_3+A_1	$D_4(a_1)$	$D_4(a_1)$
W-type	81_p	60_s	80_s	20_s
Multiplicity	1	1	2	1
A_4	(21^3)	(21^3)	$2\cdot(21^3)$	(21^3)
D_4	$11\times 11_+$	$11\times 11_+$	$11\times 11_+, 1^3\times 1$	$1^3\times 1$
A_1	(2)	(11)	$(2),(11)$	(2)
$W(\mathfrak{z})=W(B_2)$	2×0	11×0	1×1	0×2

The reducibility lines coming from the factor $\mathcal{A}_{A_1}(0,St)$ are $\nu_1\pm\nu_2=1$, while the factors $\mathcal{A}_{A_4}(A_3,St)$ give reducibility at $\nu_1\pm\nu_2=5$.

In the Hecke algebra of type D_4, the nilpotent orbit is $(44)_+$ with lowest W-type $11\times 11_+$. The corresponding infinitesimal character is $(-\frac{3}{2},-\frac{1}{2},\frac{1}{2},\frac{3}{2})+\bar{\nu}(\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2})$, where $\bar{\nu}$ can be ν_1 and ν_2. The reducibility points are $\bar{\nu}=1,\ 3$. The intertwining operators in D_4 are:

$11\times 11_+$	1
$1^3\times 1$	$\frac{1-\bar{\nu}}{1+\bar{\nu}}$

It follows that the lines of reducibility for $X(A_3,(\nu_1,\nu_2))$ are $\nu_1=1$, $\nu_2=1$, $\nu_1\pm\nu_2=1$ (as for B_2) and $\nu_1=3$, $\nu_2=3$ and $\nu_1\pm\nu_2=5$.

The matching of intertwining operators with operators in the Hecke algebra of type B_2 is:

W-type	$W(\mathfrak{z})$-type
81_p:	2×0
60_s:	11×0
80_s:	1×1
20_s:	0×2

All the relevant W-types for B_2 are matched, which implies that $\mathcal{U}(E_6,A_3)\subset\mathcal{SU}(\mathbb{H}(B_2))$. $\mathcal{SU}(\mathbb{H}(B_2))$ is partitioned by the B_2-nilpotent orbits: (1^5), (221), (311) and (5). We analyze the unitarity of the corresponding parameters in $A_3\subset E_6$.

(1^5): in $\mathcal{SU}(\mathbb{H}(B_2))$, this is the spherical complementary series $0\leq\nu_2\leq\nu_1<1-\nu_2$. For $A_3\subset E_6$, $X(A_3,(\nu_1,\nu_2))$ is tempered irreducible at $(0,0)$, so it remains

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unitary until the first reducibility line $\nu_1+\nu_2=1$. It follows that the *complementary series* is the same, $0\le\nu_2\le\nu_1<1-\nu_2$.

(221): these are endpoints of the complementary series, $(\nu_1,\nu_2)=(\frac{1}{2}+\nu,-\frac{1}{2}+\nu)$ with $0\le\nu<\frac{1}{2}$, so that they are automatically unitary.

$$X(A_3,(\frac{1}{2}+\nu,-\frac{1}{2}+\nu))=X(A_3+A_1)+\overline{X}(A_3,(\frac{1}{2}+\nu,-\frac{1}{2}+\nu)),\ 0\le\nu<\frac{1}{2},$$
$$\overline{X}(A_3,(\frac{1}{2}+\nu,-\frac{1}{2}+\nu))|_W=81_p+80_s+\cdots+2\cdot 20'_p+6'_p.$$

(311): $(\nu_1,\nu_2)=(1,0)$ is an endpoint of the complementary series, so it is unitary.

$$X(A_3,(1,0))=X(D_4(a_1))+\overline{X}(A_3,(1,0)),$$
$$\overline{X}(A_3,(1,0))=81_p+80_s+\cdots+20'_p.$$

(5): $(\nu_1,\nu_2)=(2,1)$ is isolated.

$$X(A_3,(2,1))=X(A_4)+\overline{X}(D_4(a_1),1)+\overline{X}(A_3+A_1,\frac{3}{2})+\overline{X}(A_3,(2,1)),$$
$$\overline{X}(A_3,(2,1))=81_p+81'_p.$$

Note that $\overline{X}(A_3,(2,1))$ is self IM-dual. We calculate explicitly the intertwining operator $\mathcal{A}_{E_6}(A_3,St,(2,1))$ and find that it is positive on $81'_p$. It follows that $\overline{X}(A_3,(2,1))$ is unitary.

In conclusion, $\mathcal{U}(E_6,A_3)=\mathcal{SU}(\mathbb{H}(B_2))$.

14. $\mathbf{A_2+2A_1}$: the centralizer is A_1+T_1 (connected). $X(A_2+2A_1)$ has lowest W-type 60_p (odd degree). The Hermitian parameter is $(M,\sigma)=(A_2+2A_1,St)$ (A_2+2A_1 is realized by $\{\alpha_1,\alpha_2,\alpha_4,\alpha_6\}$) and Hermitian infinitesimal character

$$s=(\frac{5}{4},-\frac{1}{4},\frac{3}{4},-\frac{3}{4},\frac{1}{4},-\frac{1}{4},-\frac{1}{4},\frac{1}{4})+\nu(-\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{3}{2},\frac{3}{2},-\frac{3}{2},-\frac{3}{2},\frac{3}{2}),\ \nu\ge 0.$$

For $\nu=0$, the module is tempered irreducible and the infinitesimal character is W-conjugate to $(-\frac{1}{4},\frac{1}{4},\frac{1}{4},\frac{3}{4},\frac{3}{4},-\frac{3}{4},-\frac{3}{4},\frac{3}{4})$.

For $\nu>0$, $w_m(A_2+2A_1,St,\nu)=(A_2+2A_1,St,-\nu)$. The subgroup $W(\mathfrak{z})\cong S_2$ of the Weyl group is generated by w_m. w_m decomposes as follows:

$$w_m=w_m(A_4,A_2A_1)\cdot w_m(A_2,A_1)\cdot w_m(D_5,D_2A_2)\cdot w_m(A_2,A'_1)\cdot w_m(A_4,(A_2A_1)'),$$

where in $A_4=\{\alpha_1,\alpha_3,\alpha_4,\alpha_2\}$, $A_2+A_1=\{\alpha_1,\alpha_4,\alpha_2\}$, $(A_2+A_1)'=\{\alpha_1,\alpha_3,\alpha_2\}$, and in $A_2=\{\alpha_5,\alpha_6\}$, $A_1=\{\alpha_6\}$ and $A'_1=\{\alpha_5\}$.

The intertwining operator decomposes:

$$\mathcal{A}_{E_6}(A_2+2A_1,St,\nu)=\mathcal{A}_{A_4}((A_2+A_1)',St)\circ\mathcal{A}_{A_2}(A'_1,St)\circ\mathcal{A}_{D_5}(D_2+A_2,St)$$
$$\circ\mathcal{A}_{A_2}(A_1,St)\circ\mathcal{A}_{A_4}((A_2+A_1),St).$$

The restrictions of W-types are:

Nilpotent	A_2+2A_1	A_3
W-type	$60'_p$	$81'_p$
Multiplicity	1	1
$A_3\subset A_4$	(221)	(221)
$A_1\subset A_2$	(21)	(21)
$D_2+A_2\subset D_5$	22×1	21×11
$W(\mathfrak{z})=S_2$	(2)	(11)

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There are reducibility points $\nu = \frac{3}{2},\ \frac{5}{2}$ from the factors $\mathcal{A}_{A_4}(A_2 + A_1, St)$, and $\nu = \frac{1}{2}$ from the factors $\mathcal{A}_{A_2}(A_1, St)$.

In the Hecke algebra of type D_5, the nilpotent orbit is $(3^3 1)$ with lowest W-type 22×1. The corresponding infinitesimal character is $(0, 1, -1, 0, 1) + (\bar{\nu})(0, 0, 1, 1, 1)$, with $\bar{\nu} = 2\nu$. The reducibility points are $\bar{\nu} = 1,\ 2,\ 3$. The intertwining operators in D_5 are:

$$\begin{array}{ll} 22 \times 1: & 1 \\ 21 \times 11: & \frac{1-\bar{\nu}}{1+\bar{\nu}} \end{array}$$

It follows that the reducibility points for $X(A_2 + 2A_1, \nu)$ are $\nu = \frac{1}{2},\ 1,\ \frac{3}{2},\ \frac{5}{2}$.

The intertwining operators are:

W-type	$W(\mathfrak{z})$-type	
60_p:	(2)	1
81_p:	(11)	$\frac{\frac{1}{2}-\nu}{\frac{1}{2}+\nu}$

Therefore, $\mathcal{U}(E_6, A_2 + 2A_1) \subset \mathcal{SU}(\mathbb{H}(A_1)) = \{0 \le \nu \le \frac{1}{2}\}$. $X(A_2 + 2A_1, \nu)$ is unitary at $\nu = 0$ and irreducible for $0 \le \nu < \frac{1}{2}$. Then the *complementary series* is $0 \le \nu < \frac{1}{2}$.

At the endpoint of the complementary series:

$$\begin{aligned} X(A_2 + 2A_1, \frac{1}{2}) &= X(D_4(a_1), (3)) + X(D_4(a_1), (21)) + \overline{X}(A_3, (1, 0)) \\ &\quad + \overline{X}(2A_2 + A_1, \frac{1}{2}) + 2 \cdot \overline{X}(A_3 + A_1, \frac{1}{2}) + \overline{X}(A_2 + 2A_1, \frac{1}{2}), \\ \overline{X}(A_2 + 2A_1, \frac{1}{2}) &= 60_p + 60_s + 80_s + 60'_p. \end{aligned}$$

$\mathcal{U}(E_6, A_2 + 2A_1) = \mathcal{SU}(\mathbb{H}(A_1))$.

15. $\mathbf{2A_2}$: the centralizer is G_2 (connected). $X(2A_2)$ has lowest W-type 24_p (even degree). The Hermitian parameter is $(M, \sigma) = (2A_2, St)$, ($2A_2$ realized by $\{\alpha_1, \alpha_3, \alpha_5, \alpha_6\}$) and infinitesimal character

$$\begin{aligned} s = (-\frac{1}{2}, \frac{1}{2}, -\frac{3}{2}, -\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}) + \nu_1(0, 0, 1, 1, 1, -1, -1, 1) \\ +\nu_2(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}), \quad \nu_1 \ge 0,\ \nu_2 \ge 0. \end{aligned}$$

For $(\nu_1, \nu_2) = (0, 0)$, the module is tempered irreducible and the infinitesimal character is W-conjugate to $(0, 0, 0, 0, 1, -1, -1, 1)$.

For $(\nu_1, \nu_2) \ne (0, 0)$, $w_m(2A_2, St, (\nu_1, \nu_2)) = (2A_2, St, (-\nu_1, -\nu_2))$. The subgroup $W(\mathfrak{z}) \cong W(G_2)$ of the Weyl group is generated by:

$$\begin{aligned} \bar{s}_1 &= w_m(A_5, 2A_2), \\ \bar{s}_2 &= s_2. \end{aligned}$$

Note that $w_m = (s_1 \cdot s_2)^3$ and the intertwining operator decomposes as

$$\mathcal{A}_{E_6}(2A_2, St, (\nu_1, \nu_2)) = (\mathcal{A}_{A_5}(2A_2, St) \circ \mathcal{A}_{A_1}(0, St))^3.$$

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The restriction of W-types are:

Nilpotent	$2A_2$	$2A_2A_1$	A_3A_1	$D_4(a_1)$	$D_4(a_1)$	D_4
W-type	24_p	10_s	60_s	80_s	90_s	$24'_p$
Multiplicity	1	1	2	2	1	1
$2A_2 \subset A_5$	(2^3)	(2^3)	$(2^3),(2^21^2)$	$(2^3),(2^21^2)$	(2^21^2)	(2^21^2)
A_1	(2)	(11)	$(2),(11)$	$(2),(11)$	(2)	(11)
$W(\mathfrak{z}) = W(G_2)$	$(1,0)$	$(1,3)''$	$(2,2)$	$(2,1)$	$(1,3)'$	$(1,6)$

In the factor $\mathcal{A}_{A_1}(0,St)$, the root α_2 takes values $\nu_2,\ 3\nu_1+\nu_2$ and $3\nu_1+2\nu_2$.

In the Hecke algebra of type A_5, the nilpotent orbit is (33) with lowest W-type (2^3). The corresponding infinitesimal character is

$$(-1,0,1,-1,0,1)+\bar{\nu}(-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2}),$$

where $\bar{\nu}$ can be ν_1, $\nu_1+\nu_2$ and $2\nu_1+\nu_2$. The reducibility points are $\bar{\nu}=1,\ 2,\ 3.$ The intertwining operators in A_5 are:

(222):	1
(2211):	$\frac{1-\bar{\nu}}{1+\bar{\nu}}$

It follows that the lines of reducibility for $X(2A_2,St,(\nu_1,\nu_2))$ are $\nu_1,\nu_1+\nu_2,2\nu_1+\nu_2=1$, $\nu_2,3\nu_1+\nu_2,3\nu_1+2\nu_2=1$ (as for G_2) and $\nu_1,\nu_1+\nu_2,2\nu_1+\nu_2=2,3$.

The matching of intertwining operators with operators from the Hecke algebra of type G_2 is:

W-type	$W(\mathfrak{z})$-type	
24_p	$(1,0)$	(trivial representation)
10_s	$(1,3)''$	(-1 on the long roots)
60_s	$(2,2)$	
80_s	$(2,1)$	(the reflection representation)
90_s	$(1,3)'$	(-1 on the short roots)
$24'_p$	$(1,6)$	(sign representation)

Note that all the W-types for G_2 are matched, which implies $\mathcal{U}(E_6,2A_2)\subset\mathcal{SU}(\mathbb{H}(G_2))$. $\mathcal{SU}(\mathbb{H}(G_2))$ is partitioned by the G_2-nilpotent orbits: 1, A_1, $\widetilde{A}_1$, $G_2(a_1)$ and G_2. We analyze the unitarity of the corresponding parameters in $2A_2\subset E_6$.

1: in $\mathcal{SU}(\mathbb{H}(G_2))$ this is the spherical complementary series $\{3\nu_1+2\nu_2<1\}\cup\{2\nu_1+\nu_2<1<3\nu_1+\nu_2\}$. For $2A_2\subset E_6$, $X(2A_2,(\nu_1,0))$ is irreducible for $0\le\nu_1<\frac{1}{2}$, $\nu_1\neq\frac{1}{3}$ and unitarily induced from a unitary module, and therefore, it is unitary. This implies that the *complementary series* is the same as the one for G_2.

A_1: these are endpoints of the complementary series $(\nu_1,\nu_2)=(-\frac{1}{2}+\nu,1)$, $0\le\nu<\frac{1}{2}$, so they are unitary. The decomposition of the standard module is:

$$X(2A_2,(-\frac{1}{2}+\nu,1)) = X(2A_2+A_1,\nu)+\overline{X}(2A_2,(-\frac{1}{2}+\nu,1)),$$

$$\overline{X}(2A_2,(-\frac{1}{2}+\nu,1))|_W = 24_p+60_s+\cdots+20'_p+6'_p.$$

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$\widetilde{A}_1$: they are endpoints of the complementary series, $(\nu_1,\nu_2) = (1, -\frac{3}{2}+\nu)$, $0 \le \nu < \frac{1}{2}$, so they are unitary. The decomposition of the standard module is:

$$X(2A_2,(1,-\frac{3}{2}+\nu)) = X(A_3+A_1,\nu) + \overline{X}(2A_2,(1,-\frac{3}{2}+\nu)),$$
$$\overline{X}(2A_2,(1,-\frac{3}{2}+\nu))|_W = 24_p + 10_s + 60_s + \cdots + 30'_p + 15'_q.$$

$G_2(a_1)$: $(\nu_1,\nu_2) = (0,1)$ is an endpoint of the complementary series, so it is unitary.

$$X(2A_2,(0,1)) = X(D_4(a_1),(3)) + X(D_4(a_1),(21)) + 2\cdot\overline{X}(A_3+A_1,\tfrac{1}{2}) + \overline{X}(2A_2+A_1,\tfrac{1}{2}) + \overline{X}(2A_2,(0,1)).$$
$$\overline{X}(2A_2,(0,1))|_W = 24_p + 80_s + 81'_p + 30'_p.$$

G_2: $(\nu_1,\nu_2) = (1,1)$ is isolated. $\overline{X}(2A_2,(1,1))$ is the IM-dual of $\overline{X}(D_4,1)$, $\overline{X}(2A_2,(1,1))|_W = 24_p$, and therefore it is unitary.

In conclusion, $\mathcal{U}(E_6,2A_2) = \mathcal{SU}(\mathbb{H}(G_2))$.

16. $\mathbf{A_2+A_1}$: the centralizer is A_2+T_1 (connected). $X(A_2+A_1)$ has lowest W-type 64_p (odd degree). The Hermitian parameter is $(M,\sigma) = (A_5,(3211))$ ($A_2+A_1 = (3211) \subset A_5$ is realized as $\{\alpha_1,\alpha_3,\alpha_5\}$), with Hermitian infinitesimal character

$$s = (-\frac{1}{2},\frac{1}{2},-1,0,-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},\frac{1}{2}) + \nu\check{\omega}_2, \quad \nu \ge 0.$$

For $\nu = 0$, the module is tempered irreducible and the infinitesimal character is W-conjugate to $(\frac{1}{4},\frac{1}{4},\frac{1}{4},\frac{1}{4},\frac{3}{4},-\frac{3}{4},-\frac{3}{4},\frac{3}{4})$.

For $\nu > 0$, $w_m(A_5,(3211),\nu) = (A_5,(3211),-\nu)$. The subgroup $W(\mathfrak{z}) \cong W(A_2)$ of the Weyl group is generated by:

$$\bar{s}_1 = s_2$$
$$\bar{s}_2 = w_m(A_4,A_2+A_1)\cdot w_m(A_3,A_2)\cdot s_2\cdot w_m(A_3,A'_2)\cdot w_m(A_4,(A_2+A_1)'),$$

where in $A_4 = \{\alpha_1,\alpha_3,\alpha_4,\alpha_5\}$, $A_2+A_1 = \{\alpha_1,\alpha_3,\alpha_5\}$ and $(A_2+A_1)' = \{\alpha_1,\alpha_4,\alpha_5\}$, and in $A_3 = \{\alpha_4,\alpha_5,\alpha_6\}$, $A_2 = \{al_4,\alpha_5\}$ and $A'_2 = \{\alpha_5,\alpha_6\}$.

Note that $w_m = \bar{s}_1\cdot\bar{s}_2\cdot\bar{s}_1$. The intertwining operator $\mathcal{A}_{E_6}(A_5,(3211),\nu)$ decomposes accordingly into a product of operators

$$\mathcal{A}_{A_4}(A_2+A_1,St), \quad \mathcal{A}_{A_3}(A_2,St) \quad \text{and} \quad \mathcal{A}_{A_1}(0,St).$$

The restrictions of W-types are:

Nilpotent	A_2+A_1	$2A_2$	$A2+2A_1$	$2A_2+A_1$
W-type	64_p	24_p	60_p	10_s
Multiplicity	1	1	2	1
A_1	(2)	(2)	(2), (11)	(11)
$A_2 \subset A_3$	(211)	(211)	$2\cdot(211)$	(211)
A_2+A_1	(221)	(221)	$2\cdot(221)$	(221)
$W(\mathfrak{z}) = S_3$	(3)	(3)	(21)	(1^3)

There are reducibility points $\nu = \frac{3}{2}, \frac{5}{2}$ from the factors $\mathcal{A}_{A_4}(A_2+A_1,St)$, $\nu = 2$ from $\mathcal{A}_{A_3}(A_2,St)$ and $\nu = \frac{1}{2},\ 1$ from $\mathcal{A}_{A_1}(0,St)$.

Therefore, the points of reducibility for $X(A_2+A_1,\nu)$ are $\nu = \frac{1}{2},\ 1,\ \frac{3}{2},\ 2,\ \frac{5}{2}$.

The intertwining operators match operators from the Hecke algebra of type A_2, as follows:

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W-type	$W(\mathfrak{z})$-type
64_p:	(3)
24_p:	(3)
60_p:	(21)
10_s:	(1^3)

All W-types of A_2 are matches, which implies that $\mathcal{U}(E_6, A_2+A_1) \subset \mathcal{SU}(\mathbb{H}(A_2))$. $\mathcal{SU}(\mathbb{H}(A_2))$ is formed of $0 \leq \nu < \frac{1}{2}$ attached to the A_2-nilpotent orbit (1^3), $\nu = \frac{1}{2}$ attached to (21) and $\nu = 1$ attached to (3).

$X(A_2, \nu)$ is irreducible for $0 \leq \nu < \frac{1}{2}$ and unitary at $\nu = 0$. The *complementary series* is $0 \leq \nu < \frac{1}{2}$.

At the endpoint of the complementary series, $\nu = \frac{1}{2}$,

$$X(A_2 + A_1, \frac{1}{2}) = X(A_2 + 2A_1) + \overline{X}(A_2 + A_1, \frac{1}{2}),$$

$$\overline{X}(A_2 + A_1, \frac{1}{2})|_W = 64_p + 24_p + 60_p + \cdots + 2 \cdot 15'_p + 2 \cdot 20'_p + 6'_p.$$

The point $\nu = 1$ is isolated. The decomposition of the standard module is

$$X(A_2 + A_1, 1) = X(2A_2 + A_1) + 2 \cdot \overline{X}(A_2 + 2A_1, \nu_{nh}) + \overline{X}(A_2 + A_1, 1),$$

$$\overline{X}(A_2+A_1, 1)|_W = 64_p + 24_p + 81_p + 80_s + 2 \cdot 90_s + 20_s + 2 \cdot 81'_p + 64'_p + 30'_p + 15'_p.$$

We calculate explicitly the intertwining operator $\mathcal{A}_{E_6}(A_5, (3211), 1)$ on the W-types of $\overline{X}(A_2+A_1, 1)$ and find that it is positive. It follows that $\overline{X}(A_2+A_1, 1)$ is unitary.

In conclusion, $\mathcal{U}(E_6, A_2 + A_1) = \mathcal{SU}(\mathbb{H}(A_2))$.

17. $\mathbf{A_2}$: the centralizer is $A_2 + A_2$, with component group $\mathbb{Z}_2$. $X(A_2) = X(A_2, (2)) + X(A_2, (11))$, with lowest W-types 30_p, respectively 15_p (both having odd degrees). There are two non-conjugate choices for the Hermitian parameter:

a) $(M, \sigma) = (A_3, (31))$, (where A_3 is realized by $\{\alpha_3, \alpha_4, \alpha_5\}$ and $A_2 = (31)$ by $\{\alpha_4, \alpha_5\}$) with Hermitian infinitesimal character

$$s = (0, -1, 0, 1, 0, 0, 0, 0) + \nu_1(-\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, \frac{1}{2})$$
$$+\nu_2(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}), \quad \nu_1 \geq, \ \nu_2 \geq 0.$$

b) $(M, \sigma) = (A_2, St)$, (A_2 is realized by $\{al_2, \alpha_4\}$) with Hermitian infinitesimal character

$$s = (1, 0, 1, 0, 0, 0, 0, 0) + \nu'_1(0, 0, 0, 0, 1, -1, -1, 1)$$
$$+\nu'_2(-\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{3}{2}, 0, 0, 0, 0), \quad \nu'_1 \geq 0, \ \nu'_2 \geq 0.$$

We analyze these two cases next.

a) The two lowest W-types are separate if $\nu_1 = \nu_2 = \nu \geq 0$. $w_m = w_0(A_3) \cdot w_0(E_6)$, $w_m(A_3, \sigma, (\nu_1, \nu_2)) = (A_3, \sigma, (-\nu_1, -\nu_2))$, where $\sigma = (31)$. The subgroup $W(\mathfrak{z}) \cong W(A_2) \times W(A_2)$ of the Weyl group is generated by $\bar{s}_1, \ \bar{s}_2, \ \bar{s}_3, \ \bar{s}_4$, where:

$\bar{s}_1 = s_1,$
$\bar{s}_2 = (s_3 s_4 s_5) \cdot s_6 \cdot (s_5 s_4 s_3),$
$\bar{s}_3 = (s_2 s_4 s_5) \cdot (s_3 s_4 s_2) \cdot (s_1 s_3 s_4) \cdot s_5 \cdot (s_4 s_3 s_1)(s_2 s_4 s_3)(s_5 s_4 s_2),$
$\bar{s}_4 = (s_2 s_4 s_5) \cdot (s_3 s_4 s_2) \cdot (s_1 s_3 s_4) \cdot s_6 \cdot (s_4 s_3 s_1)(s_2 s_4 s_3)(s_5 s_4 s_2).$

The factors in the parenthesis are all of the form $w_m(A_3, A_2)$, for some $A_2 \subset A_3 \subset E_6$.

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The intertwining operator decomposes accordingly into a product of factors of the form

$$\mathcal{A}_{A_3}(A_2, St) \quad \text{and} \quad \mathcal{A}_{A_1}(0, St).$$

The restrictions of W-types are:

Nilpotent	A_2	A_2	A_2+A_1	$2A_2$
W-type	30_p	15_p	64_p	24_p
Multiplicity	$1+0$	$0+1$	$2+2$	$1+1$
$A_2 \subset A_3$	(211)	(211)	$4\cdot(211)$	$2\cdot(211)$
A_1	(2)	(2)	$3\cdot(2),(11)$	$(2),(11)$
$W(A_2)\times W(A_2)$	$(3)\otimes(3)$	$(3)\otimes(3)$	$(21)\times(3)$	$(1^3)\times(3)$
			$(3)\otimes(21)$	$(3)\otimes(1^3)$

For $\nu_1 \neq \nu_2$, $X(A_2,(\nu_1,\nu_2)$ has the following reducibility lines: $\nu_1 = 2$, $\nu_2 = 2$, $\nu_1 \pm \nu_2 = 2$ and $\nu_2 - \nu_1 = 2$, from the factors $\mathcal{A}_{A_3}(A_2, St)$ and $\nu_1 = \frac{1}{2}, 1$, $\nu_2 = \frac{1}{2}, 1$, from $\mathcal{A}_{A_1}(0, St)$.

When $\nu_1 \neq \nu_2$, the intertwining operators match operators from the Hecke algebra of type $A_2 \times A_2$ as follows:

W-type	$W(A_2)\times W(A_2)$-type
30_p:	$(3)\otimes(3)$
15_p:	$(3)\otimes(3)$
64_p:	$(21)\otimes(3)+(3)\otimes(21)$
24_p:	$(1^3)\otimes(3)+(3)\otimes(1^3)$

If $\nu_1 = \nu_2 = \nu$, then there are two standard modules $X(A_2,(\nu,\nu),(2))$ and $X(A_2,(\nu,\nu),(11)$, (the lowest W-types 30_p and 15_p are separate) and the operators on each factor match operators in the Hecke algebra of type A_2:

$\overline{X}(A_2,(\nu,\nu),(2))$:

W-type	$W(A_2)$-type
30_p :	(3)
64_p :	(21)
24_p :	(1^3)

$\overline{X}(A_2,(\nu,\nu),(11))$:

W-type	$W(A_2)$-type
15_p :	(3)
64_p :	(21)
24_p :	(1^3)

Since all the relevant W-types of $A_2 \times A_2$ are matched, it follows that $\mathcal{U}(A_2) \subset \mathcal{SU}(\mathbb{H}(A_2)) \times \mathcal{SU}(\mathbb{H}(A_2))$. $\mathcal{SU}(\mathbb{H}(A_2)) \times \mathcal{SU}(\mathbb{H}(A_2))$ is partitioned by nilpotent orbits $\mathcal{O}\otimes\mathcal{O}'$, with $\mathcal{O},\mathcal{O}' \in \{(3),(21),(1^3)\}$. We mention that for $\mathcal{O} \neq \mathcal{O}'$, the case $\mathcal{O}'\otimes\mathcal{O}$ is completely analogous to $\mathcal{O}\otimes\mathcal{O}'$.

$(1^3)\otimes(1^3)$: this is the complementary series $0 \le \nu_1 < \frac{1}{2}$, $0 \le \nu_2 < \frac{1}{2}$, with the observation that along $0 \le \nu_1 = \nu_2 < \frac{1}{2}$ there are two standard modules. At $(\nu_1,\nu_2) = (0,0)$, the standard modules $X(A_2,(2))$ and $X(A_2,(11))$ are tempered, thus unitary. For $\nu_1 \neq \nu_2$, close to $(0,0)$, the standard module $X(A_2,(\nu_1,\nu_2))$ is irreducible. Then it is unitary if and only if the signatures of 30_p and 15_p are the same. From the calculation above, the intertwining operators on both 30_p and 15_p are equal to $+1$, so $X(A_2,(\nu_1,\nu_2))$ is unitary in the region (*complementary series*) $0 \le \nu_1 \neq \nu_2 < \frac{1}{2}$ and on the line $\nu_1 = \nu_2$, both standard modules $X(A_2,(\nu,\nu),(2))$ and $X(A_2,(\nu,\nu),(11))$ are unitary for $0 \le \nu_1 = \nu_2 = \nu < \frac{1}{2}$.

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$(1^3)\otimes(21)$: the parameter is $(\nu_1,\nu_2)=(\nu,\frac{1}{2})$, $0\le\nu<\frac{1}{2}$. This is unitary being endpoints of the complementary series. The decomposition of the standard module is:

$$X(A_2,(\nu,\frac{1}{2}))=X(A_2+A_1,\nu)+\overline{X}(A_2,(\nu,\frac{1}{2})),\ 0\le\nu<\frac{1}{2},$$
$$\overline{X}(A_2,(\nu,\frac{1}{2}))|_W=30_p+15_p+3\cdot64_p+24_p+\cdots+3\cdot20'_p+6'_p.$$

$(21)\otimes(21)$: the parameter is $(\nu_1,\nu_2)=(\frac{1}{2},\frac{1}{2})$. This is unitary, being an endpoint of the complementary series. The decompositions of the standard modules are:

$$X(A_2,(\frac{1}{2},\frac{1}{2}),(2))=X(A_2+2A_1)+\overline{X}(A_2+A_1,\frac{1}{2})+\overline{X}(A_2,(\frac{1}{2},\frac{1}{2}),(2)),$$
$$\overline{X}(A_2,(\frac{1}{2},\frac{1}{2}),(2))|_W=30_p+64_p+\cdots+3\cdot64'_p+30'_p+20'_p,$$
$$X(A_2,(\frac{1}{2},\frac{1}{2}),(11))=\overline{X}(A_2+A_1,\frac{1}{2})+\overline{X}(A_2,(\frac{1}{2},\frac{1}{2}),(11)),$$
$$\overline{X}(A_2,(\frac{1}{2},\frac{1}{2}),(11))|_W=15_p+64_p+\cdots+15'_p.$$

$(1^3)\otimes(3)$: the parameter is $(\nu_1,\nu_2)=(\nu,1)$, with $0\le\nu<\frac{1}{2}$. The decomposition of the standard module is

$$X(A_2,(\nu,1))=X(2A_2,(\nu,0))+2\cdot\overline{X}(A_2+A_1,\nu_{nh})+\overline{X}(A_2,(\nu,1)),\ 0\le\nu<\frac{1}{2},$$
$$\overline{X}(A_2,(\nu,1))|_W=30_p+15_p+2\cdot64_p+24_p+3\cdot81_p+2\cdot20_s+4\cdot90_s$$
$$+2\cdot80_s+2\cdot81'_p+24'_p+2\cdot64'_p+15'_p+30'_p.$$

Note that $\overline{X}(A_2,(\nu,1))$, $0\le\nu<\frac{1}{2}$, is self IM-dual. We check by an explicit computation that the intertwining operators on these W-types are positive for $\nu=0$. Since the W-structure of $\overline{X}(A_2,(\nu,1))$ does not change for $0\le\nu<\frac{1}{2}$, it follows that $\overline{X}(A_2,(\nu,1))$ is unitary for $0\le\nu<\frac{1}{2}$.

$(21)\otimes(3)$: the parameter is $(\nu_1,\nu_2)=(\frac{1}{2},1)$. The module $\overline{X}(A_2,(\frac{1}{2},1))$ is the IM-dual of $\overline{X}(A_2+A_1,1)$ which was proved to be unitary. Therefore $\overline{X}(A_2,(\frac{1}{2},1))$ is also unitary.

$(3)\otimes(3)$: the parameter is $(\nu_1,\nu_2)=(1,1)$. The module $\overline{X}(A_2,(1,1),(2))$ is IM-dual to $\overline{X}(2A_2,(0,1))$ and the module $\overline{X}(A_2,(1,1),(11))$ is IM-dual to $X(D_4(a_1),(1^3))$, so they are both unitary.

b) $(M,\sigma)=(A_2,St)$, (A_2 realized by $\{\alpha_2,\alpha_4\}$) and infinitesimal character $s=(1,0,1,0,0,0,0,0)+\nu'_1(0,0,0,0,1,-1,-1,1)+\nu'_2(-\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{3}{2},0,0,0,0)$, $\nu'_1\ge0$, $\nu'_2\ge0$. The two lowest W-types are separate if $\nu'_2=0$. We will show that for $\nu'_2\neq0$, the two lowest W-types have opposite signature always and therefore the factor containing them, $\overline{X}(A_2,(\nu'_1,\nu'_2))$ is not unitary.

$w_m=w_0(A_2)\cdot w_0(E_6)$ and $w_m(A_2,St,(\nu'_1,\nu'_2))=(A_2,St,(-\nu'_1,-\nu'_2))$. w_m decomposes as $w_m=\bar{s}'_1\cdot\bar{s}'_2$, where

$$\begin{aligned}\bar{s}'_1:\quad &\nu_1\to-\nu_1\quad w_m(E_6,D_4)= s_1\cdot(s_3s_4s_2)(s_5s_4s_3)(s_6s_5s_4)\cdot s_2\cdot s_3\cdot s_1\cdot\\ & \qquad\qquad\qquad\qquad s_3\cdot(s_4s_5s_6)\cdot(s_3s_4s_5)\cdot(s_2s_4s_3)\cdot s_1,\\ \bar{s}'_2:\quad &\nu_2\to-\nu_2\quad w_m(D_4,A_2)\end{aligned}$$

The intertwining operator decomposes accordingly into a product of operators of the form

$$\mathcal{A}_{A_1}(0,St),\quad \mathcal{A}_{A_3}(A_2,St)\quad\text{and}\quad \mathcal{A}_{D_4}(A_2,St).$$

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The restrictions of W-types are:

Nilpotent	A_2	A_2
W-type	30_p	15_p
Multiplicity	$1+0$	$0+1$
$A_2 \subset A_3$	(31)	(31)
A_1	(2)	(2)
$A_2 \subset D_4$	21×1	2×11

In the Hecke algebra of type D_4, the nilpotent orbit is (3311) with lowest W-types 21×1 and 2×11. The corresponding restriction of the infinitesimal character is $(1,0,1,0)+\nu_2'(-\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{3}{2})$. For $\nu_2' \neq 0$, the lowest W-types 21×1 and 2×11 are in the same factor and their intertwining operators in D_4 are:

$$\begin{aligned} 21\times 1&: \quad +1\\ 2\times 11&: \quad -1 \end{aligned}$$

It follows that the operators in $A_2 \subset E_6$, for $\nu_2' \neq 0$ are:

$$\begin{aligned} 30_p&: \quad +1\\ 15_p&: \quad -1 \end{aligned}$$

For $\nu_2' = 0$, the module $\overline{X}(A_2,(\nu_1',0))$ is isomorphic to the module $X(A_2,(\nu,\nu))$ from a), with $\nu=\nu_1'$. Therefore, the same discussion applies.

In conclusion, $\mathcal{U}(A_2) = \mathcal{SU}(\mathbb{H}(A_2\times A_2)\rtimes \mathbb{Z}_2)$, the spherical unitary dual of the Hecke algebra $\mathbb{H}(A_2\times A_2)\rtimes\mathbb{Z}_2$.

18. $\mathbf{3A_1}$: the centralizer A_2+A_1 (connected). $X(3A_1)$ has lowest W-type 15_q (even degree). The Hermitian parameter is $(M,\sigma)=(3A_1,St)$ ($3A_1$ is realized as $\{\alpha_2,\alpha_3,\alpha_5\}$), with Hermitian infinitesimal character

$$s=(0,1,-\frac{1}{2},\frac{1}{2},0,0,0,0)+(0,0,\nu_2,\nu_2,\nu_1,-\nu_1,-\nu_1,\nu_1), \quad \nu_1\geq 0,\nu_2\geq 0.$$

For $(\nu_1,\nu_2)=(0,0)$, the module is tempered irreducible and the infinitesimal character is W-conjugate to $(0,0,\frac{1}{2},\frac{1}{2},\frac{1}{2},-\frac{1}{2},-\frac{1}{2},\frac{1}{2})$.

For $(\nu_1,\nu_2)\neq(0,0)$, $w_m(3A_1,St,(\nu_1,\nu_2))=(3A_1,St,(-\nu_1,-\nu_2))$. The subgroup $W(\mathfrak{z})=W(A_2)\times W(A_1)$ is generated by:

$$\begin{aligned} W(A_2) \quad &\bar{s}_1=(s_1s_3)\cdot(s_4s_2s_5s_4)\cdot(s_3s_1),\\ &\bar{s}_2=(s_6s_5)\cdot(s_4s_2s_3s_4)\cdot(s_5s_6),\\ W(A_1) \quad &\bar{s}_3=w_m(D_4,3A_1), \end{aligned}$$

where the factors in the parenthesis are of the form $w_m(A_2,A_1)$ or $w_m(A_3,2A_1)$.

Note that $w_m=(\bar{s}_1\bar{s}_2\bar{s}_1)\cdot(\bar{s}_3)$ and the intertwining operator $\mathcal{A}_{E_6}(3A_1,St,(\nu_1,\nu_2))$ decomposes similarly into a product of operators of the form

$$\mathcal{A}_{A_3}(2A_1,St), \quad \mathcal{A}_{A_2}(A_1,St) \quad \text{and} \quad \mathcal{A}_{D_4}(3A_1,St).$$

The restrictions of W-types are:

Nilpotent	$3A_1$	A_2	A_2+A_1	$2A_2$
W-type	15_q	30_p	64_p	24_p
Multiplicity	1	1	2	1
$3A_1\subset D_4$	22×0	21×1	$2\cdot 21\times 1$	21×1
$A_1 \subset A_2$	(21)	(21)	$2\cdot(21)$	(21)
$2A_1\subset A_3$	(22)	(22)	(22), (211)	(211)
$W(\mathfrak{z})$	$(3)\otimes(2)$	$(3)\otimes(11)$	$(21)\otimes(11)$	$(1^3)\otimes(11)$

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The reducibility lines coming from the factors $\mathcal{A}_{A_2}(A_1, St)$ are $\nu_1 \pm \nu_2 = \frac{3}{2}$, $\nu_2 - \nu_1 = \frac{3}{2}$, $2\nu_1 \pm \nu_2 = \frac{3}{2}$ and $\nu_2 - 2\nu_1 = \frac{3}{2}$.

In the Hecke algebra of type A_3, the nilpotent orbit is (22), with lowest W-type (22). The corresponding infinitesimal character is $(-\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, \frac{1}{2}) + \bar{\nu}(\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, -\frac{1}{2})$, with $\bar{\nu}$ taking values ν_1 and $2\nu_1$. The reducibility points are $\bar{\nu} = 1,\ 2$. The intertwining operators in A_3 are:

$$\begin{array}{ll} (22): & 1 \\ (211): & \frac{1-\bar{\nu}}{1+\bar{\nu}} \end{array}$$

In the Hecke algebra of type D_4, the nilpotent orbit is (3221) with lowest W-type 22×0. The corresponding infinitesimal character is $(0, 1, -\frac{1}{2}, \frac{1}{2}) + \bar{\nu}(0, 0, 1, 1)$, where $\bar{\nu} = \nu_2$. The reducibility points are $\bar{\nu} = \frac{1}{2},\ \frac{3}{2},\ \frac{5}{2}$. The intertwining operators in D_4 are:

$$\begin{array}{ll} 22 \times 0: & 1 \\ 21 \times 1: & \frac{\frac{1}{2}-\bar{\nu}}{\frac{1}{2}+\bar{\nu}}. \end{array}$$

It follows that the lines of reducibility for the standard module $X(3A_1, (\nu_1, \nu_2))$ are:

$$\begin{array}{lll} \nu_2 = \frac{1}{2}, \frac{3}{2}, \frac{5}{2} & \nu_1 \pm \nu_2 = \frac{3}{2} & 2\nu_1 \pm \nu_2 = \frac{3}{2}, \\ \nu_1 = \frac{1}{2}, 1, 2 & \nu_2 - \nu_1 = \frac{3}{2} & \nu_2 - 2\nu_1 = \frac{3}{2}. \end{array}$$

The matching of the intertwining operators with operators in the Hecke algebra of type $A_2 \times A_1$ is:

W-type	$W(\mathfrak{z})$-type
15_q:	$(3) \otimes (2)$
30_p:	$(3) \otimes (11)$
64_p:	$(21) \otimes (11)$
24_p:	$(1^3) \otimes (11)$

Not all relevant W-types of $A_2 \times A_1$ are matched, so $\mathcal{U}(3A_1)$ could be larger than $\mathcal{SU}(\mathbb{H}(A_2 \times A_1)) = \{0 \le \nu_1 \le \frac{1}{2}, 0 \le \nu_2 \le \frac{1}{2}\} \cup \{\nu_1 = 1, 0 \le \nu_2 \le \frac{1}{2}\}$. We organize the analysis of the unitarity of $\overline{X}(3A_1, (\nu_1, \nu_2))$, as before, by the nilpotent orbits in the centralizer $A_2 \times A_1$. The matched W-types are sufficient for proving nonunitarity for all (ν_1, ν_2) with $\nu_2 \ne \frac{1}{2}$, but they do not give any information on the line $\nu_2 = \frac{1}{2}$.

$(1^3) \otimes (1^2)$: this is the complementary series $\{0 \le \nu_1 < \frac{1}{2}, 0 \le \nu_2 < \frac{1}{2}\}$. The module $X(3A_1, (\nu_1, \nu_2))$ is unitary at $(0, 0)$ and irreducible in this region, therefore $X(3A_1)$ has the same *complementary series* as $A_2 + A_1$.

$(21) \otimes (1^2)$: the parameter is $(\nu_1, \nu_2) = (\frac{1}{2}, \nu)$, $0 \le \nu < \frac{1}{2}$. This is unitary, being an endpoint of the complementary series. The decomposition of the standard module is:

$$X(3A_1, (\frac{1}{2}, \nu)) = X(A_2 + A_1, \nu) + \overline{X}(3A_1, (\frac{1}{2}, \nu)), \quad 0 \le \nu < \frac{1}{2},$$
$$\overline{X}(3A_1, (\frac{1}{2}, \nu))|_W = 15_q + 30_p + 64_p + \cdots + 15'_q + 30'_p + 20'_p.$$

$(3) \otimes (1^2)$: the parameter is $(\nu_1, \nu_2) = (1, \nu)$, $0 \le \nu < \frac{1}{2}$. This is IM-dual to $\overline{X}(2A_2, (1, -\frac{3}{2} + \nu))$, which implies it is unitary.

$(1^3) \otimes (2)$: the parameter is $(\nu_1, \nu_2) = (\nu, \frac{1}{2})$. In $\mathcal{SU}(\mathbb{H}(A_2 \times A_1))$, the corresponding unitary parameter is $0 \le \nu < \frac{1}{2}$. For $0 \le \nu < \frac{1}{2}$, $\overline{X}(3A_1, (\nu, \frac{1}{2}))$ is also

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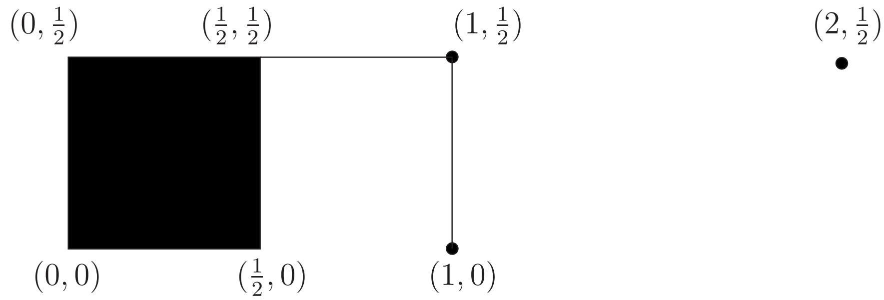

FIGURE 1. Unitary parameters for the nilpotent orbit $3A_1$

unitary, being the endpoint of the complementary series. The decomposition of the standard module is

$$X(3A_1,(\nu,\frac{1}{2}))=X(A_2,(\nu,\nu),(2))+\overline{X}(3A_1,(\nu,\frac{1}{2})),\quad 0\le\nu<\frac{1}{2},$$
$$\overline{X}(3A_1,(\nu,\frac{1}{2}))|_W=15_q+2\cdot 60_p+80_s+3\cdot 60_s+10_s+2\cdot 60'_p+15'_q.$$

Note that $X(3A_1,(\nu,\frac{1}{2}))$ is self IM-dual. We know it is unitary for $0\le\nu<\frac{1}{2}$. The W-structure of $X(3A_1,(\nu,\frac{1}{2}))$ can only change at $\nu=\frac{1}{2}$, 1 and 2. At $\nu=\frac{1}{2}$, the generic factor is parameterized by the nilpotent orbit A_2+2A_1, which has lowest W-type 60_p. This implies that 60_p (also any higher W-type) cannot come out of $X(3A_1,(\nu,\frac{1}{2}))$ at $\nu=\frac{1}{2}$, and therefore, $X(3A_1,(\nu,\frac{1}{2}))$ is unitary for $0\le\nu<1$.

We confirm this by an explicit calculation of the intertwining operator on 60_p along the line $\nu_1=\nu$, $\nu_2=\frac{1}{2}$:

$$60_p:\begin{pmatrix}\frac{2-\nu}{2+\nu} & 0\\ 0 & \frac{(2-\nu)(1-\nu)}{(2+\nu)(1+\nu)}\end{pmatrix}.$$

The operator on 60_p also shows that $X(3A_1,(\nu,\frac{1}{2}))$ is non-unitary for $\nu>1,\nu\ne 2$.

At $(\nu_1,\nu_2)=(2,\frac{1}{2})$, $\overline{X}(3A_1,(2,\frac{1}{2}))$ is the IM-dual of $\overline{X}(A_5,\frac{1}{2})$, therefore, $\overline{X}(3A_1,(2,\frac{1}{2}))|_W=15_q$ and it is unitary.

$(21)\otimes(2)$: this is the point $(\nu_1,\nu_2)=(\frac{1}{2},\frac{1}{2})$ and it is unitary by the previous discussion.

$(3)\otimes(2)$: $(\nu_1,\nu_2)=(1,\frac{1}{2})$. $\overline{X}(3A_1,(1,\frac{1}{2}))$ is the IM-dual of $X(A_3+A_1,\frac{1}{2})$, therefore it is unitary and $\overline{X}(3A_1,(1,\frac{1}{2}))|_W=15_q+60_p+60_s$.

In conclusion, $\mathcal{U}(3A_1)=\mathcal{SU}(\mathbb{H}(A_2\times A_1))\cup\{\nu_2=\frac{1}{2},\frac{1}{2}<\nu_1<1\}\cup\{(2,\frac{1}{2})\}$.

19. $\mathbf{2A_1}$: the centralizer is B_3+T_1 (connected). $X(2A_1)$ has lowest W-type 20_p (even degree). The Hermitian parameter is $(M,\sigma)=(2A_1,St)$ ($2A_1$ is realized as $\{\alpha_3,\alpha_5\}$), and Hermitian infinitesimal character

$$s=(-\frac{1}{2},\frac{1}{2},-\frac{1}{2},\frac{1}{2},0,0,0,0)+\nu_1(0,0,0,0,\frac{1}{2},-\frac{1}{2},-\frac{1}{2},\frac{1}{2})$$
$$+\nu_2(\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},0,0,0,0)+\nu_3(-\frac{1}{2},-\frac{1}{2},\frac{1}{2},\frac{1}{2},0,0,0,0),\quad \nu_1\ge\nu_2\ge\nu_3\ge 0.$$

At $(0,0,0)$, the module is tempered irreducible.

For $\nu_1\ge\nu_2\ge\nu_3\ge 0$, $w_m(2A_1,St,(\nu_1,\nu_2,\nu_3))=(2A_1,St,-(\nu_1,\nu_2,\nu_3))$. The subgroup $W(\mathfrak{z})\cong W(B_3)$ is generated by:

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$$\begin{array}{lll} \bar{s}_1: & \nu_1 \to \nu_2 & (s_1s_3)\cdot(s_4s_5)\cdot s_6\cdot(s_5s_4)\cdot(s_3s_1),\\ \bar{s}_2: & \nu_2 \to \nu_3 & s_2,\\ \bar{s}_3: & \nu_3 \to -\nu_3 & (s_4s_3s_5s_4),\end{array}$$

where the factors in the parenthesis are of the form $w_m(A_2, A_1)$ or $w_m(A_3, 2A_1)$. Note that

$$w_m = \bar{s}_1\bar{s}_2\bar{s}_3\bar{s}_2\bar{s}_1\bar{s}_2\bar{s}_3\bar{s}_2\bar{s}_3,$$

and $\mathcal{A}_{E_6}(2A_1, St)$ decomposes accordingly into a product of operators

$$\mathcal{A}_{A_1}(0, St), \quad \mathcal{A}_{A_2}(A_1, St) \quad \text{and} \quad \mathcal{A}_{A_3}(2A_1, St).$$

The restrictions of W-types are:

Nilpotent	$2A_1$	$3A_1$	A_2	A_2
W-type	20_p	15_q	30_p	15_p
Multiplicity	1	2	3	1
$2A_1 \subset A_3$	(22)	$2\cdot(22)$	$2\cdot(22),(211)$	(211)
$A_1 \subset A_2$	(21)	$2\cdot(21)$	$3\cdot(21)$	(21)
A_1	(2)	$(2),(11)$	$2\cdot(2),(11)$	(2)
$W(\mathfrak{z}) = W(B_3)$	3×0	21×0	2×1	0×3

The reducibility planes for $X(2A_1, (\nu_1, \nu_2, \nu_3))$ are $\pm\nu_1\pm\nu_2\pm\nu_3 = 3$, coming from the factors $\mathcal{A}_{A_2}(A_1, St)$, $\nu_i \pm \nu_j = 1$, $1 \le j < i \le 3$, from the factors $\mathcal{A}_{A_1}(0, St)$ and $\nu_i = 1, 2$, from the factors $\mathcal{A}_{A_3}(2A_1, St)$. So, in addition to the reducibility planes matching the centralizer B_3, the extra reducibility planes are $\nu_i = 2$, $1 \le i \le 3$ and $\pm\nu_1 \pm \nu_2 \pm \nu_3 = 3$.

The matching of the intertwining operators with operators in B_3 is as follows:

W-type	$W(\mathfrak{z})$-type
20_p:	3×0
15_q:	12×0
30_p:	2×1
15_p:	0×3

In addition to these, 1×2 is also a relevant W-type for B_3, but it doesn't have a correspondent among the W-types of $X(2A_1)$. Therefore, we will need to use a calculation with the W-type 64_p, which appears with multiplicity 8 in $X(2A_1)$ and for which the restriction to $W(\mathfrak{z}) = W(B_3)$ contains a copy of 1×2 (but the corresponding intertwining operators do not match).

$\mathcal{U}(2A_1)$ could be larger than $\mathcal{SU}(\mathbb{H}(B_3))$ and we organize the analysis by the nilpotent orbits of B_3.

(1^7): the complementary series for B_3 is

$$\{\nu_1 + \nu_2 < 1\} \cup \{\nu_1 < 1, \nu_2 + \nu_3 < 1, \nu_1 + \nu_3 > 1\},$$

and for its determination 2×1 and 12×0 are sufficient. In the first region any parameter can be deformed irreducibly to $\nu_3 = 0$, while in the second region, any parameter can be deformed irreducibly to $\nu_3 = \nu_2$. For the corresponding parameters in E_6, $X(2A_1)$ is irreducible and unitarily induced from a unitary module, thus unitary. Since, none of the extra hyperplanes of reducibility of $X(2A_1)$ cut the two regions, it follows that $X(2A_1)$ has the same *complementary series* as B_3.

(221^3): the parameter is $(\frac{1}{2} + \nu_1, -\frac{1}{2} + \nu_1, \nu_2)$. In B_3, 2×1 and 12×0 are sufficient for the determination of the unitary region, $\{0 \le \nu_1 < \frac{1}{2}, 0 \le \nu_2 < 1\}$. The corresponding parameters for $X(2A_1)$ are endpoints of the complementary

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series and therefore unitary. The decomposition of the standard module is

$$X(2A_1,\underline{\nu}) = X(3A_1) + \overline{X}(2A_1,\underline{\nu}), \quad \underline{\nu} = (\frac{1}{2}+\nu_1, -\frac{1}{2}+\nu_1, \nu_2),$$
$$\overline{X}(2A_1,\underline{\nu})|_W = 20_p + 2\cdot 30_p + \cdots + 6'_p, \quad 0 \le \nu_1 < \frac{1}{2}, 0 \le \nu_2 < 1.$$

(31^4): the parameter is $(\nu_1,\nu_2,1)$. In B_3, 2×1 and 12×0 are sufficient for the determination of the unitary region, $\{0 \le \nu_2 \le \nu_1 < 1-\nu_2\}$. The corresponding parameters in $X(2A_1)$ are endpoints of the complementary series, thus unitary. The standard module decomposes:

$$X(2A_1,\underline{\nu}) = X(A_2) + \overline{X}(2A_1,\underline{\nu}), \quad \underline{\nu} = (\nu_1,\nu_2,1),$$
$$\overline{X}(2A_1,\underline{\nu})|_W = 20_p + 2\cdot 30_p + \cdots + 2\cdot 30'_p + 20'_p, \quad 0 \le \nu_2 \le \nu_1 < 1-\nu_2.$$

Note that $\overline{X}(2A_1,\underline{\nu})$ is self IM-dual.

(322): the parameter is $(\frac{1}{2}+\nu, -\frac{1}{2}+\nu, 1)$. In B_3, the W-types 2×1 and 12×0 are sufficient for the determination of the unitary region $\{0 \le \nu < \frac{1}{2}\}$. The corresponding parameter for $X(2A_1)$, is on the boundary of the complementary series, therefore unitary. $\overline{X}(2A_1,(\frac{1}{2}+\nu,-\frac{1}{2}+\nu,1))$, $0 \le \nu < \frac{1}{2}$, is also the IM-dual of $X(3A_1,(\frac{1}{2},\nu))$, which confirms it.

(331): the parameter is $(1+\nu,\nu,-1+\nu)$. In B_3, the only unitary point is $\nu = 0$. The operator on 12×0 is 0, and 2×1 and 0×3 rule out $\nu > 0$, except for $1 < \nu < 2$. For this, 1×2 is needed. By an explicit calculation, we find that, in E_6, the corresponding parameter of $X(2A_1)$ on the W-type 64_p is negative. At the remaining point, $\overline{X}(2A_1,(1,1,0))$ is unitary, being on the boundary of the complementary series. Note that $\overline{X}(2A_1,(1,1,0))$ is also the IM-dual of $\overline{X}(A_2,(\frac{1}{2},\frac{1}{2}),(2))$.

(511): the parameter is $(\nu,2,1)$. In B_3, the only unitary point is $\nu = 0$ and 2×1, 12×0 are sufficient to rule out $\nu > 0$. In E_6, for $\nu = 0$, $\overline{X}(2A_1,(2,1,0))$ is the IM-dual of $\overline{X}(A_3,(1,0))$, therefore it is unitary.

(7): the parameter is $(3,2,1)$. $\overline{X}(2A_1,(3,2,1))$ is the IM-dual of $\overline{X}(D_4,\frac{1}{2})$, thus unitary.

In conclusion, $\mathcal{U}(2A_1) = \mathcal{SU}(\mathbb{H}(B_3))$.

20. $\mathbf{A_1}$: the centralizer is A_5 (connected). The Hermitian parameter is $(M,\sigma) = (A_1, St)$ (A_1 is realized by $\{al_2\}$) and Hermitian infinitesimal character:

$$s = (\frac{1}{2},\frac{1}{2},0,0,0,0,0,0) + \nu_1(-\frac{1}{2},\frac{1}{2},-\frac{1}{2},\frac{1}{2},\frac{1}{2},-\frac{1}{2},-\frac{1}{2},\frac{1}{2})$$
$$+\nu_2(\frac{1}{2},-\frac{1}{2},\frac{1}{2},-\frac{1}{2},\frac{1}{2},-\frac{1}{2},-\frac{1}{2},\frac{1}{2}) + \nu_3(0,0,1,1,0,0,0,0),$$

with $\nu_1 \ge \nu_2 \ge \nu_3 \ge 0$. In the centralizer A_5, the corresponding Hermitian parameter in standard coordinates is $(\nu_1,\nu_2,\nu_3,-\nu_3,-\nu_2,-\nu_1)$.

For $(0,0,0)$, the module is tempered and irreducible. $w_m(A_1,St,\underline{\nu}) = (A_1,St,-\underline{\nu})$. The subgroup $W(\mathfrak{z}) \cong W(A_5)$ is generated by:

$\bar{s}_1$: s_3
$\bar{s}_2$: s_1
$\bar{s}_3$: $(s_4s_2)\cdot(s_3s_4)\cdot s_5\cdot(s_4s_3)(s_2s_4)$
$\bar{s}_4$: s_6
$\bar{s}_5$: s_5

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The factors in the parenthesis are of the form $w_m(A_2, A_1)$,

$$w_m = \bar{s}_3\bar{s}_2\bar{s}_4\bar{s}_3\bar{s}_4\bar{s}_2\bar{s}_1\bar{s}_5\bar{s}_2\bar{s}_4\bar{s}_3\bar{s}_4\bar{s}_2\bar{s}_5\bar{s}_1,$$

and the intertwining operator $\mathcal{A}_{E_6}(A_1, St)$ decomposes accordingly into a product of factors of the form

$$\mathcal{A}_{A_2}(A_1, St) \quad \text{and} \quad \mathcal{A}_{A_1}(0, St).$$

The restrictions of W-types are:

Nilpotent	A_1	$2A_1$	$3A_1$	A_2	A_2
W-type	6_p	20_p	15_q	30_p	15_p
Multiplicity	1	5	5	10	5
$A_1 \subset A_2$	(21)	$5 \cdot (21)$	$5 \cdot (21)$	$9 \cdot (21), (1^3)$	$4 \cdot (21), (1^3)$
A_1	(2)	$4 \cdot (2), (11)$	$(2), 4 \cdot (11)$	$7 \cdot (2), 3 \cdot (11)$	$4 \cdot (2), (11)$
$W(\mathfrak{z})$	(6)	(51)	(33)	$(6) + (42)$	(51)

The planes of reducibility for $X(A_1, (\nu_1, \nu_2, \nu_3))$ are $\nu_i = \frac{1}{2}$, $1 \leq i \leq 3$, from factors $\mathcal{A}_{A_1}(0, St)$, $\nu_i \pm \nu_j = 1$, $0 \leq j < i \leq 3$ from factors $\mathcal{A}_{A_2}(A_1, St)$ (matching the centralizer A_5), and $\nu_i = \frac{3}{2}$, $1 \leq i \leq 3$ and $\pm\nu_1 \pm \nu_2 \pm \nu_3 = \frac{3}{2}$ from $\mathcal{A}_{A_2}(A_1, St)$ (extra planes of reducibility).

The matching of intertwining operators with operators from the centralizer A_5 is as follows:

W-type	$W(\mathfrak{z})$-type
6_p:	(6)
20_p:	(51)
15_q:	(33)

In addition to these types, (42) is also a relevant W-type for A_5 and it doesn't have a correspondent in the W-types of $X(A_1)$. The operator on 30_p does not match the operator on $(6) + (42)$. However, we will need to use a calculation with 30_p, and also with 15_p (alternatively, one can use 64_p, which appears with multiplicity 24 in $X(A_1)$).

$\mathcal{U}(A_1)$ could be larger than $\mathcal{SU}(\mathbb{H}(A_5))$. We organize the analysis of $X(2A_1)$, as before, by the nilpotent orbits in the centralizer A_5. In A_5, the W-types (51) and (33) (which are matched in E_6) are sufficient for the determination of the entire spherical unitary dual $\mathcal{SU}(\mathbb{H}(A_5))$, except for the parts attached to the nilpotents (33) and (222), where the W-type (42) is also needed.

(1^6): the parameter is (ν_1, ν_2, ν_3), with $0 \leq \nu_3 \leq \nu_2 \leq \nu_1 < \frac{1}{2}$. $X(2A_1)$ is irreducible and unitary at $(0,0,0)$, therefore it stays unitary until the first plane of reducibility, which is $\nu_1 = \frac{1}{2}$. It follows that the *complementary series* is the same as for A_5.

(21^4): the parameter is $(\nu_1, \nu_2, \frac{1}{2})$, with $0 \leq \nu_2 \leq \nu_1 < \frac{1}{2}$. The corresponding $\overline{X}(A_1)$ are unitary, being endpoints of the complementary series. The decomposition of the standard module is

$$X(A_1, \underline{\nu}) = X(2A_1) + \overline{X}(A_1, \underline{\nu}), \quad \underline{\nu} = (\nu_1, \nu_2, \frac{1}{2}),$$

$$\overline{X}(A_1, \underline{\nu})|_W = 6_p + 4 \cdot 20_p + \cdots + 4 \cdot 20'_p + 6'_p, \quad 0 \leq \nu_2 \leq \nu_1 < \frac{1}{2}.$$

Note that $\overline{X}(A_1, \underline{\nu})$ is self IM-dual.

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(2211): the parameter is $(\frac{1}{2}+\nu_1, -\frac{1}{2}+\nu_1, \nu_2)$, with $0 \le \nu_1 < \frac{1}{2}$, $0 \le \nu_2 < \frac{1}{2}$. The decomposition of the standard module is:

$$\begin{aligned} X(A_1, (\frac{1}{2}+\nu_1, -\frac{1}{2}+\nu_1, \nu_2)) &= X(3A_1, (\nu_1, \nu_2)) \\ &+ 2\cdot \overline{X}(2A_1, (\frac{1}{2}+\nu_2, -\frac{1}{2}+\nu_2, 2\nu_1)) \\ &+ \overline{X}(A_1, (\frac{1}{2}+\nu_1, -\frac{1}{2}+\nu_1, \nu_2)) \quad 0 \le \nu_1, \nu_2 < \frac{1}{2}. \end{aligned}$$

$\overline{X}(A_1, (\frac{1}{2}+\nu_1, -\frac{1}{2}+\nu_1, \nu_2))$ is the IM-dual of $\overline{X}(2A_1, (\frac{1}{2}+\nu_2, -\frac{1}{2}+\nu_2, 2\nu_1))$ for $0 \le \nu_1, \nu_2 < \frac{1}{2}$, and therefore unitary.

(222): the parameter is $(\frac{1}{2}+\nu, -\frac{1}{2}+\nu, \frac{1}{2})$. In A_5, the unitary region is $0 \le \nu < \frac{1}{2}$, but (42) is needed in order to rule out the interval $\frac{1}{2} < \nu < 1$. $\overline{X}(A_1, (\frac{1}{2}+\nu, -\frac{1}{2}+\nu, \frac{1}{2}))$, for $0 \le \nu < \frac{1}{2}$ and $\frac{1}{2} < \nu < 1$ are the IM-dual of the modules $\overline{X}(A_2, (\nu, \nu), (11))$, and therefore, are unitary for $0 \le \nu < \frac{1}{2}$ and non-unitary for $\frac{1}{2} < \nu < 1$. Note also that for $0 \le \nu < \frac{1}{2}$, $\overline{X}(A_1, (\frac{1}{2}+\nu, -\frac{1}{2}+\nu, \frac{1}{2}))$ is an endpoint of the unitary region for $\overline{X}(A_1, (\frac{1}{2}+\nu_1, -\frac{1}{2}+\nu_1, \nu_2))$ (for $\nu_2 = \frac{1}{2}, \nu_1 = \nu$) attached to (2211).

The W-structure for $\overline{X}(A_1, (\frac{1}{2}+\nu, -\frac{1}{2}+\nu, \frac{1}{2}))$ $(0 \le \nu < 1, \nu \ne \frac{1}{2})$ is $6_p + 2\cdot 20_p + 3\cdot 15_p + 3\cdot 30_p + 15_q + \dots + 15'_p$. For this parameter a direct computation with the intertwining operators on 15_p and 30_p shows that in the interval $\frac{1}{2} < \nu < 1$, 15_p is $(++-)$ and 30_p is $(+++)$, and therefore 15_p rules out $\frac{1}{2} < \nu < 1$ (also 64_p is negative in this interval).

(31^3): the parameter is $(\nu, 1, 0)$, with $0 \le \nu < \frac{1}{2}$. In E_6, $\overline{X}(A_1, (\nu, 1, 0))$, $0 \le \nu < \frac{1}{2}$ are unitary being endpoints of the unitary region attached to (2211), $\overline{X}(A_1, (\frac{1}{2}+\nu_1, -\frac{1}{2}+\nu_1, \nu_2))$ (for $\nu_1 = \frac{1}{2}, \nu_2 = \nu$). Also, $\overline{X}(A_1, (\nu, 1, 0))$, $0 \le \nu < \frac{1}{2}$ is the IM-dual of $\overline{X}(A_2, (\nu, \frac{1}{2}))$, which was shown to be unitary.

(321): the parameter is $(1, \frac{1}{2}, 0)$. $\overline{X}(A_1, (1, \frac{1}{2}, 0))$ is the IM-dual of $\overline{X}(A_2+A_1, \frac{1}{2})$ and therefore it is unitary. Note also that it the endpoint of the unitary interval for $\overline{X}(A_1, (\nu, 1, 0))$ attached to (31^3).

(33): the parameter is $(1+\nu, \nu, -1+\nu)$. In A_5, this is unitary for $0 \le \nu < \frac{1}{2}$, but one needs the W-type (42) to rule out the interval $1 < \nu < \frac{3}{2}$. For $0 \le \nu < \frac{1}{2}$ and $1 < \nu < \frac{3}{2}$, $X(A_1, (1+\nu, \nu, -1+\nu))$ is the IM-dual of $\overline{X}(2A_2, (-\frac{1}{2}+\nu))$, which implies that it is unitary for $0 \le \nu < \frac{1}{2}$ and non-unitary for $1 < \nu < \frac{3}{2}$.

The W-structure of $\overline{X}(A_1, (1+\nu, \nu, -1+\nu))$ (on these intervals) is $6_p + 20_p + 15_p + 2\cdot 30_p + \dots + 24'_p$. A direct computation with the intertwining operators on 15_p and 30_p shows that in the interval $1 < \nu < \frac{3}{2}$, 15_p is positive and 30_p is $(+-)$, and therefore 30_p rules out $1 < \nu < \frac{3}{2}$.

(411): the parameter is $(\nu, \frac{3}{2}, \frac{1}{2})$, $0 \le \nu < \frac{1}{2}$. For $0 \le \nu < \frac{1}{2}$, $\overline{X}(A_1, \nu, \frac{3}{2}, \frac{1}{2})$ is IM-dual to $\overline{X}(A_3, (\frac{1}{2}+\nu, -\frac{1}{2}+\nu))$ and therefore it is unitary.

(42): the parameter is $(\frac{3}{2}, \frac{1}{2}, \frac{1}{2})$. $\overline{X}(A_1, (\frac{3}{2}, \frac{1}{2}, \frac{1}{2}))$ is IM-dual to the tempered module $X(D_4(a_1), (21))$, thus unitary.

(51): the parameter is $(2, 1, 0)$. $\overline{X}(A_1, (2, 1, 0))$ is IM-dual to $\overline{X}(A_4, \frac{1}{2})$, therefore unitary.

(6): the parameter is $(\frac{5}{2}, \frac{3}{2}, \frac{1}{2})$. $\overline{X}(A_1, (\frac{5}{2}, \frac{3}{2}, \frac{1}{2}))$ is unitary, being the IM-dual of the tempered module $X(E_6(a_3), (11))$.

In conclusion, $\mathcal{U}(A_1) = \mathcal{SU}(\mathbb{H}(A_5))$.

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21. **0**: this is the trivial nilpotent orbit, so the centralizer is the full E_6. The lowest W-type is the trivial representation, 1_p, and therefore $\mathcal{U}(1)$ is the spherical unitary dual $\mathcal{SU}(\mathbb{H}(E_6))$. Recall that via the Iwahori-Matsumoto involution, the spherical modules are transformed into generic modules, and therefore the spherical unitary dual is precisely the disjoint union of all the (IM-dual of) complementary series attached to nilpotent orbits.

In order to complete the calculation, we only need to determine the spherical complementary series which will be the subject of the next subsection.

3.5. **The spherical complementary series for** E_6**.** The standard module for the trivial nilpotent orbit is the principal series $X(\underline{\nu})$, where $\underline{\nu}$ is real and dominant for the root system of type E_6. We denote by $\overline{X}(\underline{\nu})$ its unique irreducible quotient. The Hermitian parameter has the property that $w_0\underline{\nu} = -\underline{\nu}$, which is equivalent to

$$\langle \alpha_1, \underline{\nu}\rangle = \langle \alpha_6, \underline{\nu}\rangle \text{ and } \langle \alpha_3, \underline{\nu}\rangle = \langle \alpha_5, \underline{\nu}\rangle. \tag{3.5.1}$$

Recall that if α is a positive root, and $\alpha = \sum_{\alpha_i \in \Pi} m_i\alpha_i$ (Π is the set of simple roots), the *height of* α, $ht(\alpha)$ is defined as $ht(\alpha) = \sum m_i$. This gives a partial ordering of the positive roots ($\alpha > \alpha'$ if $\alpha - \alpha'$ is a sum of positive roots) by *levels* (level k is formed of all the positive roots with height k). The simple roots are level 1 and, in E_6, the *highest root* is $\alpha_{36} = \frac{1}{2}(1,1,1,1,1,-1,-1,1)$ on level 11.

Let $\tau : R \to R$ be the outer automorphism of the root system R of type E_6:

$$\tau^2 = 1,\ \tau(\alpha_1) = \alpha_6,\ \tau(\alpha_3) = \alpha_5,\ \tau(\alpha_2) = \alpha_2,\ \tau(\alpha_4) = \alpha_4. \tag{3.5.2}$$

Then R^τ is generated by $\{\frac{1}{2}(\alpha_1+\alpha_6), \frac{1}{2}(\alpha_3+\alpha_5), \alpha_4, \alpha_2\}$ and it is isomorphic to the root system of type F_4. We will use the following coordinates for F_4:

$$\begin{aligned} \gamma_1 &= \epsilon_1 - \epsilon_2 - \epsilon_3 - \epsilon_4,\\ \gamma_2 &= 2\epsilon_4,\\ \gamma_3 &= \epsilon_3 - \epsilon_4,\\ \gamma_4 &= \epsilon_2 - \epsilon_3. \end{aligned} \tag{3.5.3}$$

We write the Hermitian dominant parameter $\underline{\nu}$ so that in R^τ it corresponds to the standard coordinates of F_4. In E_6, it has the form:

$$\begin{aligned} \underline{\nu} = \Big(&\frac{\nu_1-\nu_2}{2} - \nu_3, \frac{\nu_1-\nu_2}{2} - \nu_4, \frac{\nu_1-\nu_2}{2} + \nu_4, \frac{\nu_1-\nu_2}{2} + \nu_3,\\ &\frac{\nu_1+\nu_2}{2}, -\frac{\nu_1+\nu_2}{2}, -\frac{\nu_1+\nu_2}{2}, \frac{\nu_1+\nu_2}{2}\Big),\\ &\nu_1 - \nu_2 - \nu_3 - \nu_4 \geq 0, \nu_2 \geq \nu_3 \geq \nu_4 \geq 0. \end{aligned} \tag{3.5.4}$$

$X(\underline{\nu})$ is irreducible if and only if $\langle \alpha, \underline{\nu}\rangle \neq 1$ for all $\alpha \in R$. We call *regions* the connected components of the complement of all hyperplanes $\langle \alpha, \underline{\nu}\rangle \neq 1$ in the Hermitian parameter space.

The following theorem describes the (real) parameters $\underline{\nu}$ with the property that $X(\underline{\nu})$ is irreducible and unitary.

Theorem. *The spherical complementary series for type* E_6 *is formed of the dominant parameters* $\underline{\nu}$ *as in* (3.5.4) *in the following two regions:*

(1) $2\nu_1 < 1$,
(2) $\nu_1 + \nu_2 + \nu_3 - \nu_4 < 1 < \nu_1 + \nu_2 + \nu_3 + \nu_4$.

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We remark that, in these coordinates, the spherical complementary series for E_6 is identical with the spherical complementary series for F_4 (see Theorem 3.5 in [Ci]).

Proof. If $X(\underline{\nu})$ are unitary for $\underline{\nu}$ in an open region $\mathcal{F}$, then $\overline{X}(\underline{\nu})$ (and all the other irreducible subquotients) are unitary $\underline{\nu}$ on all the walls of the closure of the region $\mathcal{F}$. We analyze first which roots α can give walls of unitary regions.

a) Assume that $\tau(\alpha) \neq \alpha$. Consider a point $\underline{\nu}$ (in general position) on the hyperplane $\langle \alpha, \underline{\nu} \rangle = 1$. Note that the parameter $\underline{\nu}$ is three-dimensional. $\overline{X}(\underline{\nu})$ is the spherical module. Then $IM(\overline{X}(\underline{\nu}))$ is the standard module parameterized by the nilpotent orbit $2A_1$, $X(2A_1, \underline{\nu}')$ for some parameter $\underline{\nu}'$ of $2A_1$. The relation between $\underline{\nu}'$ and $\underline{\nu}$ is that $\underline{\nu}$ is W-conjugate to $\frac{1}{2}h(2A_1) + \underline{\nu}'$.

The only other nilpotent orbit which can parameterize a factor of $X(\underline{\nu})$ is A_1, for some parameter $\underline{\nu}''$ of A_1.

From these considerations and from the W-structure of the modules (also recall that $X(\underline{\nu})$ is isomorphic as an W-module with $\mathbb{C}[W]$), it follows that the decomposition of the standard module at $\underline{\nu}$ is

$$X(\underline{\nu}) = \overline{X}(\underline{\nu}) + X(2A_1, \underline{\nu}') + \overline{X}(A_1, \underline{\nu}'') + IM(\overline{X}(A_1, \underline{\nu}'')). \tag{3.5.5}$$

If $\underline{\nu}$ were on a wall of a unitary region, all the factors in the decomposition (3.5.5) would be unitary. But the irreducible module $\overline{X}(A_1, \underline{\nu}'')$ (which is not a standard module for A_1) is not unitary, since ν'' is three-dimensional and the only three-dimensional unitary parameters for A_1 are in the complementary series of A_1.

It follows that the these roots cannot give walls of unitary regions.

b) Assume that $\tau(\alpha) = \alpha$. Similarly, consider a point $\underline{\nu}$ on the hyperplane $\langle \alpha, \underline{\nu} \rangle = 1$. Then $IM(\overline{X}(\underline{\nu}))$ is the standard module parameterized by the nilpotent orbit A_1, $X(A_1, \underline{\nu}')$ for some parameter $\underline{\nu}'$ of A_1. The decomposition of the standard module $X(\underline{\nu})$ is

$$X(\underline{\nu}) = \overline{X}(\underline{\nu}) + \overline{X}(A_1, \underline{\nu}'). \tag{3.5.6}$$

Assume $\underline{\nu}$ is on a wall of a unitary region. Then $X(A_1, \underline{\nu}')$ has to be in the complementary series of A_1, or equivalently, $\underline{\nu}'$ is in the spherical complementary series for the centralizer A_5. A particular fact about the parameters in the spherical complementary series of type A is that all positive roots are < 1 on these parameters. We mention that this fact does not hold in general (it is also true for type C, but not for B, D, etc.).

This fact implies that, in E_6, a *necessary* condition for a root α, $(\tau(\alpha) = \alpha)$ to give a wall of a unitary region is that

$$\text{there are no positive roots } \beta \text{ such that } \langle \beta, \alpha \rangle = 0 \text{ and } \beta > \alpha. \tag{3.5.7}$$

In E_6, we verify condition (3.5.7) and obtain that there are only three positive roots which satisfy it (on levels $9, 10, 11$):

$$\begin{aligned} \alpha_{36} &= \frac{1}{2}(1,1,1,1,1,-1,-1,1), \\ \alpha_{35} &= \frac{1}{2}(-1,-1,1,1,1,-1,-1,1), \\ \alpha_{34} &= \frac{1}{2}(-1,1,-1,1,1,-1,-1,1). \end{aligned} \tag{3.5.8}$$

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They form three regions inside the dominant Weyl chamber and next we will check if they are unitary. We will use the same argument in all three cases: if $\underline{\nu}$ is a parameter in an open region, deform $\underline{\nu}$ continuously such that $X(\underline{\nu})$ stays irreducible until we reach a point $\underline{\nu}'$ on the wall of the dominant Weyl chamber. Then $X(\underline{\nu})$ is unitary if and only if $X(\underline{\nu}')$ is. But $X(\underline{\nu}')$ is unitarily induced irreducible from a spherical module on a Hecke algebra of smaller rank and therefore we can test its unitarity there.

In these cases, we will need to use the spherical complementary series for D_4. From Theorem 2.8, it follows that in the standard (dominant) coordinates of D_4, $0 \le |\widetilde{\nu}_1| \le \widetilde{\nu}_2 \le \widetilde{\nu}_3 \le \widetilde{\nu}_4$, this is:

$$\begin{aligned} D_4 :&\{\widetilde{\nu}_3 + \widetilde{\nu}_4 < 1\} \text{ and} \\ &\{\widetilde{\nu}_1 + \widetilde{\nu}_4 < 1, -\widetilde{\nu}_1 + \widetilde{\nu}_4 < 1, \widetilde{\nu}_2 + \widetilde{\nu}_3 < 1, \widetilde{\nu}_2 + \widetilde{\nu}_4 > 1\}. \end{aligned} \tag{3.5.9}$$

(i) $\langle \alpha_{36}, \underline{\nu}\rangle < 1$. In coordinates, this is the region $2\nu_1 < 1$. Since $X(\underline{\nu})$ is unitary and irreducible at $\underline{\nu} = 0$, this region has to be unitary, $\langle \alpha_{36}, \underline{\nu}\rangle = 1$ being the first hyperplane of reducibility.

Moreover, one can deform ν_4 continuously to 0 (this is the wall $\alpha_4 = 0$). When ν_4 becomes 0, the corresponding module $X(\underline{\nu}')$ is unitarily induced irreducible from the spherical module in D_4 corresponding to the parameter $(\widetilde{\nu}_1, \widetilde{\nu}_2, \widetilde{\nu}_3, \widetilde{\nu}_4) = (\nu_3, \nu_3, \nu_1 - \nu_2, \nu_1 + \nu_2)$. Since $\widetilde{\nu}_3 + \widetilde{\nu}_4 = 2\nu_1 < 1$, by (3.5.9), the D_4 parameter is unitary.

(ii) $\langle \alpha_{35}, \underline{\nu}\rangle < 1 < \langle \alpha_{36}, \underline{\nu}\rangle$. In coordinates, this is the region $\nu_1 + \nu_2 + \nu_3 + \nu_4 < 1 < 2\nu_1$. As in (i), we can deform ν_4 irreducibly to 0. Now the parameter $(\widetilde{\nu}_1, \widetilde{\nu}_2, \widetilde{\nu}_3, \widetilde{\nu}_4) = (\nu_3, \nu_3, \nu_1 - \nu_2, \nu_1 + \nu_2)$ in D_4 has the property that $\widetilde{\nu}_2 + \widetilde{\nu}_4 < 1 < \widetilde{\nu}_3 + \widetilde{\nu}_4$ and (3.5.9) implies that it is not unitary in D_4.

(iii) $\langle \alpha_{34}, \underline{\nu}\rangle < 1 < \langle \alpha_{35}, \underline{\nu}\rangle$. In coordinates, this is the region $\nu_1 + \nu_2 + \nu_3 - \nu_4 < 1 < \nu_1 + \nu_2 + \nu_3 + \nu_4$. Deform $\nu_1 - \nu_2 - \nu_3 - \nu_4$ continuously to 0 (this is the wall $\alpha_2 = 0$). When $\nu_1 = \nu_2 + \nu_3 + \nu_4$, the corresponding module $X(\underline{\nu}')$ is unitarily induced irreducible from the spherical module in D_4 corresponding to the parameter $(\widetilde{\nu}_1, \widetilde{\nu}_2, \widetilde{\nu}_3, \widetilde{\nu}_4) = (\nu_3 - \nu_4, \nu_3 + \nu_4, \nu_3 + \nu_4, 2\nu_2 + \nu_3 + \nu_4)$. One can easily verify that this parameter is in the second unitary region of D_4 as in (3.5.9). ☐

From the proof and from the discussion about the operators for the minimal nilpotent orbit A_1 in the previous section, the following corollary follows immediately.

Corollary. *The irreducible spherical principal series $X(\underline{\nu})$ in E_6 is unitary if and only if the spherical operators $r_\sigma(\underline{\nu})$ are positive definite for $\sigma \in \{1_p, 6_p, 20_p\}$.*

4. Table

In this section, we record, for the convenience of the reader, the explicit description of the unitary dual for E_6. The table contains the nilpotent orbits (see [Car]), the Hermitian infinitesimal character s, and the coordinates and type of the centralizer.

For all nilpotents except $3A_1$, a parameter (s, e, ψ), $s = \frac{h}{2} + \nu$ is unitary if and only if the corresponding parameter ν is spherical unitary for $\mathfrak{z}(\mathcal{O})$. The description of the spherical unitary dual of the various types $\mathfrak{z}(\mathcal{O})$ can be read from Theorem 2.8 and Proposition 2.9.

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We only list in the table s which parameterize Hermitian modules. The parameter ν is given by a string $(\nu_1, \dots, \nu_k)$, and the order agrees with the way the centralizer $\mathfrak{z}(\mathcal{O})$ is written in the tables. In this table, the ν-string already refers to the semisimple and Hermitian spherical parameter of the centralizer. For example, the nilpotent $A_2 + A_1$ in E_6 has centralizer $A_2 + T_1$, and the corresponding s has a single ν. This ν corresponds to the Hermitian parameter in the A_2 part of $\mathfrak{z}(\mathcal{O})$, so it must satisfy $\nu \in [0, \frac{1}{2}] \cup \{1\}$.

Table 1: Table of spherical Hermitian parameters for E_6

$\mathcal{O}$	$\mathbf{s}$	$\mathfrak{z}(\mathcal{O})$	$\mathbf{A}(\mathcal{O})$
E_6	$(0,1,2,3,4,-4,-4,4)$	1	1
$E_6(a_1)$	$(0,1,1,2,3,-3,-3,3)$	1	1
D_5	$(\frac{1}{2},\frac{1}{2},\frac{3}{2},\frac{3}{2},\frac{5}{2},-\frac{5}{2},-\frac{5}{2},\frac{5}{2})$	T_1	1
$E_6(a_3)$	$(0,0,1,1,2,-2,-2,2)$	1	S_2
$D_5(a_1)$	$(\frac{1}{4},\frac{3}{4},\frac{3}{4},\frac{5}{4},-\frac{7}{4},-\frac{7}{4},\frac{7}{4})$	T_1	1
A_5	$(-\frac{11}{4},-\frac{7}{4},-\frac{3}{4},\frac{1}{4},\frac{5}{4},-\frac{5}{4},-\frac{5}{4},\frac{5}{4})+\nu(\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},-\frac{1}{2},-\frac{1}{2},\frac{1}{2})$	A_1	1
A_4+A_1	$(0,\frac{1}{2},\frac{1}{2},1,\frac{3}{2},-\frac{3}{2},-\frac{3}{2},\frac{3}{2})$	T_1	1
D_4	$(0,1,2,3,\nu,-\nu,-\nu,\nu)$	A_2	1
A_4	$(-2,-1,0,1,2,0,0,0)+\nu(\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},-\frac{1}{2},-\frac{1}{2},\frac{1}{2})$	A_1T_1	1
$D_4(a_1)$	$(0,0,1,1,1,-1,-1,1)$	T_2	S_3
A_3+A_1	$(-\frac{5}{4},-\frac{1}{4},\frac{3}{4},-\frac{5}{4},-\frac{1}{4},-\frac{3}{4},-\frac{3}{4},\frac{3}{4})+\nu(\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},-\frac{1}{2},-\frac{1}{2},\frac{1}{2})$	A_1T_1	1
$2A_2+A_1$	$(0,1,-\frac{3}{2},-\frac{1}{2},\frac{1}{2},-\frac{1}{2},-\frac{1}{2},\frac{1}{2})+\nu(0,0,1,1,1,-1,-1,1)$	A_1	1
A_3	$(-\frac{3}{2},-\frac{1}{2},\frac{1}{2},\frac{3}{2},0,0,0,0)+(\frac{\nu_1}{2},\frac{\nu_1}{2},\frac{\nu_1}{2},\frac{\nu_1}{2},\frac{\nu_2}{2},-\frac{\nu_2}{2},-\frac{\nu_2}{2},\frac{\nu_2}{2})$	B_2T_1	1
A_2+2A_1	$(\frac{5}{4},-\frac{1}{4},\frac{3}{4},-\frac{3}{4},\frac{1}{4},-\frac{1}{4},-\frac{1}{4},\frac{1}{4})+\nu(-\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{3}{2},\frac{3}{2},-\frac{3}{2},-\frac{3}{2},\frac{3}{2})$	A_1T_1	1
$2A_2$	$(-\frac{1}{2},\frac{1}{2},-\frac{3}{2},-\frac{1}{2},\frac{1}{2},-\frac{1}{2},-\frac{1}{2},\frac{1}{2})+$ $(\frac{\nu_2}{2},\frac{\nu_2}{2},\frac{2\nu_1+\nu_2}{2},\frac{2\nu_1+\nu_2}{2},\frac{2\nu_1+\nu_2}{2},-\frac{2\nu_1+\nu_2}{2},-\frac{2\nu_1+\nu_2}{2},\frac{2\nu_1+\nu_2}{2})$	G_2	1
A_2+A_1	$(-\frac{1}{2},\frac{1}{2},-1,0,-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},\frac{1}{2})+\nu(\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},-\frac{1}{2},-\frac{1}{2},\frac{1}{2})$	A_2T_1	1
A_2	$(0,-1,0,1,0,0,0,0)+(\frac{-\nu_1+\nu_2}{2},\frac{-\nu_1+\nu_2}{2},$ $\frac{-\nu_1+\nu_2}{2},\frac{-\nu_1+\nu_2}{2},\frac{\nu_1+\nu_2}{2},\frac{-\nu_1+\nu_2}{2},\frac{-\nu_1+\nu_2}{2},\frac{\nu_1+\nu_2}{2})$	$2A_2$	S_2
$*3A_1$	$(0,1,-\frac{1}{2},\frac{1}{2},0,0,0,0)+(0,0,\nu_2,\nu_2,\nu_1,-\nu_1,-\nu_1,\nu_1)$	A_2A_1	1
$2A_1$	$(-\frac{1}{2},\frac{1}{2},-\frac{1}{2},\frac{1}{2},0,0,0,0)+$ $(\frac{-\nu_1+\nu_2}{2},\frac{-\nu_1+\nu_2}{2},\frac{\nu_1+\nu_2}{2},\frac{\nu_1+\nu_2}{2},\frac{\nu_1}{2},-\frac{\nu_1}{2},-\frac{\nu_1}{2},\frac{\nu_1}{2})$	B_3T_1	1
A_1	$(\frac{1}{2},\frac{1}{2},0,0,0,0,0,0)+(\frac{-\nu_1+\nu_2}{2},\frac{\nu_1-\nu_2}{2},$ $\frac{-\nu_1+\nu_2}{2}+\nu_3,\frac{\nu_1-\nu_2}{2}+\nu_3,\frac{\nu_1+\nu_2}{2},-\frac{\nu_1+\nu_2}{2},-\frac{\nu_1+\nu_2}{2},\frac{\nu_1+\nu_2}{2})$	A_5	1
1	$(\frac{\nu_1-\nu_2}{2}-\nu_3,\frac{\nu_1-\nu_2}{2}-\nu_4,\frac{\nu_1-\nu_2}{2}+\nu_4,\frac{\nu_1-\nu_2}{2}+\nu_3,$ $\frac{\nu_1+\nu_2}{2},-\frac{\nu_1+\nu_2}{2},-\frac{\nu_1+\nu_2}{2},\frac{\nu_1+\nu_2}{2})$	E_6	1

For the exception $3A_1$, the unitary set is: $\{0 \le \nu_1 \le \frac{1}{2},\ 0 \le \nu_2 \le \frac{1}{2}\} \cup \{\nu_1 = 1, 0 \le \nu_2 \le \frac{1}{2}\} \cup \{\nu_2 = \frac{1}{2}, \frac{1}{2} < \nu_1 < 1\} \cup \{(2, \frac{1}{2})\}$.

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Department of Mathematics, Massachusetts Institute of Technology, Cambridge, Massachusetts 02139
E-mail address: ciubo@math.mit.edu

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