

## CENTER OF INFINITESIMAL CHEREDNIK ALGEBRAS OF $\mathfrak{gl}_n$

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ABSTRACT. We show that the center of an infinitesimal Cherednik algebra of  $\mathfrak{gl}_n$  is isomorphic to the polynomial algebra of  $n$  variables. As consequences of this fact, we show that an analog of Duflo's theorem holds and all objects in the category  $\mathcal{O}$  have finite length.

### 1. INTRODUCTION

This paper is concerned with infinitesimal Cherednik algebras. These algebras are subalgebras of continuous Cherednik algebras introduced by Etingof, Gan and Ginzburg [EGG], as natural continuous analogs of the widely studied rational Cherednik algebras. Let us recall their definition.

Throughout, we will often abbreviate the Lie algebra  $\mathfrak{gl}_n$  as  $\mathfrak{g}$ , and denote by  $\mathfrak{Z}(A)$  the center of any algebra  $A$ . We also fix an algebraically closed ground field  $k$  (which will be assumed characteristic 0 unless explicitly mentioned otherwise). Let  $V$  denote the standard  $n$ -dimensional representation of  $\mathfrak{g}$  (the vector space of column vectors), and let  $V^*$  denote its dual representation.

The tensor algebra  $T(V \oplus V^*)$  is a representation of  $\mathfrak{g}$ ; thus we may form the semi-direct product algebra  $\mathfrak{U}\mathfrak{g} \ltimes T(V \oplus V^*)$  (where  $\mathfrak{U}\mathfrak{g}$  denotes the universal enveloping algebra of  $\mathfrak{g}$ ). Let  $c : V \times V^* \rightarrow \mathfrak{U}\mathfrak{g}$  be a  $\mathfrak{g}$ -invariant pairing; then we will associate to it an algebra  $H_c$  defined as the quotient of  $\mathfrak{U}\mathfrak{g} \ltimes T(V \oplus V^*)$  by the following relations:

$$[v, w^*] = c(v, w^*), [v, v_1] = 0 = [w^*, w_1^*],$$

for all  $v, v_1 \in V, w^*, w_1^* \in V^*$ .

It is clear that if  $c = 0$ , then  $H_0$  (which we will denote by  $H$ ) is just the enveloping algebra of the semi-direct product Lie algebra  $\mathfrak{g} \ltimes (V \oplus V^*)$ . Let us introduce an algebra filtration on  $H_c$  by setting  $\deg(v), \deg(w^*) = 1$  for  $v \in V, w^* \in V^*$ , and  $\deg(\alpha) = 0$  for  $\alpha \in \mathfrak{U}\mathfrak{g}$ . If we pass to the associated graded algebra, we will get a surjective homomorphism  $\mathfrak{U}\mathfrak{g} \ltimes \text{Sym}(V \oplus V^*) \rightarrow \text{gr } H_c$  and if this map is an isomorphism, then one says that the PBW property is satisfied and  $H_c$  is an infinitesimal Cherednik algebra (of  $\mathfrak{gl}_n$ ). The set of pairings  $c : V \times V^* \rightarrow \mathfrak{U}\mathfrak{g}$  for which  $H_c$  satisfies the PBW property is described in [EGG], and it will be recalled later.

As the name suggests, the algebras  $H_c$  are infinitesimal analogs of rational Cherednik algebras [EG]. It is an interesting problem to develop representation theory of such algebras. The first natural step would be to determine their center, which is the goal of this paper. Namely, we prove that the center of an infinitesimal

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Cherednik algebra of  $\mathfrak{gl}_n$  is isomorphic to the polynomial algebra in  $n$  variables. We also briefly discuss the application of this to the analog of the BGG category  $\mathcal{O}$  and primitive ideals of  $H_c$ . The main result of this paper is the following (this is the generalization of our earlier result for  $n = 2$  [T]).

**Theorem 1.1.** *Let  $H_c$  as above be an infinitesimal Cherednik algebra. Then its center is isomorphic to the polynomial algebra in  $n$  variables, and  $\mathfrak{Z}(H_c) = \mathfrak{Z}(\text{gr } H_c)$ .*

The proof of this theorem consists of two parts. The first part is the computation of the center in the undeformed case, i.e., the computation of the center of  $H_{c=0} = H = \mathfrak{U}(\mathfrak{g} \ltimes (V \oplus V^*))$ . The content of this theorem was communicated to us by M. Rais and in fact it is a special case of a more general theorem due to D. Panyushev [P]. The second part is about lifting central elements of  $H = \text{gr } H_c$  to the center of  $H_c$ . This will be proved by establishing the nonexistence of certain outer derivations.

## 2. CENTER OF $\mathfrak{U}(\mathfrak{g} \ltimes (V \oplus V^*))$

We will make use of an anti-involution  $j : H_c \rightarrow H_c$  defined as follows:  $j(A) = A^t$  (transpose) for  $A \in \mathfrak{g}$ , and similarly  $j(v) = v^t, v \in V$ . It follows easily from the explicit formula for  $c$  given below that  $j$  is well defined. There is also another way of seeing this: It follows from [EGG], that if we fix a basis  $y_i$  of  $V$  and its dual basis  $x_i \in V^*$ , there is an element  $c_1 \in \mathfrak{Z}(\mathfrak{U}\mathfrak{g})$  such that  $\sum_i y_i x_i - c_1$  belongs to the center of  $H_c$ , and we may redefine  $H_c$  as a quotient of  $\mathfrak{U}\mathfrak{g} \ltimes T(V \oplus V^*)$  by the relations  $[\sum_i y_i x_i - c_1, a] = 0$  for all  $a \in H_c$ . Now it is clear that this relation is invariant under the anti-involution  $j$ .

To formulate the result about the center of  $\mathfrak{U}\mathfrak{g} \ltimes (V \oplus V^*)$ , we will have to introduce some terminology. We will write elements of  $V$  as column vectors, and elements of  $V^*$  as row vectors. By  $y_i, i = 1, \dots, n$ , we will denote a column vector with 1 in the  $i$ -th place and 0 everywhere else. Similarly,  $x_i$  will denote the transpose (also dual) of  $y_i$ . Let  $Q_1, \dots, Q_n \in k[\mathfrak{g}]$  be defined as follows:

$$\det(t \text{Id} - X) = \sum_{j=0}^n (-1)^j t^{n-j} Q_j(X).$$

(Note that  $Q_0 = 1$ .) For  $0 \leq k < n$ , let  $B_k : \mathfrak{g} \rightarrow \mathfrak{g}$  be the polynomial function ( $B_k$  is the gradient of  $Q_{k+1}$ ) defined by

$$B_k(X) = X^k - Q_1 X^{k-1} + \dots + (-1)^k Q_k.$$

Now let  $L$  denote the Lie algebra  $\mathfrak{g} \ltimes (V \oplus V^*)$ , and let  $S$  denote its Lie group. Then we may identify  $L^*$  with  $\mathfrak{g} \ltimes (V^* \oplus V)$  via the trace pairing. We will denote by  $\beta_i \in \mathfrak{Z}(\mathfrak{U}\mathfrak{g})$  the image of  $Q_i$  under the symmetrization map from  $k[\mathfrak{g}] = k[\mathfrak{g}^*]$  to  $\mathfrak{U}\mathfrak{g}$  (thus  $\beta_1, \dots, \beta_n$  are the standard generators of  $\mathfrak{Z}(\mathfrak{U}(\mathfrak{g}))$ ). We will introduce elements  $t_i = \sum_j [\beta_i, y_j] x_j$ . Now consider the functions  $f_k : L^* \rightarrow k$  ( $0 \leq k \leq n-1$ ) defined as follows:

$$f_k(x, \lambda, v) = \langle \lambda, B_k(x)v \rangle = \lambda B_k(x)v,$$

where  $x \in \mathfrak{g}, \lambda \in V^*, v \in V$ . Now we have the following theorem whose proof is included for the sake of completeness (the proof is also given in a recent preprint by Rais [R2], and a more general result is proved by Panyushev [P]).

**Theorem 2.1** (Panyushev and Rais). *The center of  $\mathfrak{U}L$  is  $k[t_1, \dots, t_n]$ , which is also isomorphic, via the symmetrization, map to  $k[L^*]^L = k[L^*]^S = k[f_0, \dots, f_{n-1}]$ .*

*Proof.* First, let us prove that every element in  $k[L^*]^S$  lies in  $k[f_0, \dots, f_{n-1}]$ . The key observation is that under the coadjoint action of  $S$  on  $L^*$ , the orbit of  $(x_n^*, Y, V)$  (we will denote this affine space by  $M$ ) is dense [R1], where  $Y$  is the matrix with 1's on the subdiagonal and 0's everywhere else, and  $x_n^* = (0, 0, \dots, 1)$ . Thus, it would suffice to prove that  $k[f_0, \dots, f_n]|_M = k[M]$ , but this is immediate.

Now we establish the other direction. It will be more convenient to prove it in  $\mathfrak{UL}$  itself. It is easy to see that the image of  $f_k$  under the symmetrization is  $\sum_{i=1}^n [\beta_k, y_i] x_i$ . Now we claim that for any  $\alpha \in \mathfrak{Z}(\mathfrak{U}\mathfrak{g})$ , the anti-involution  $j$  fixes the following element:  $b = \sum_{i=1}^n [\alpha, y_i] x_i$  (this is true in  $H_c$  for any  $c$ ). Indeed, we have  $b = \sum y_i \alpha x_i - \alpha \sum y_i x_i$ , and  $j$  clearly fixes the first summand. Since  $\sum y_i x_i$  commutes with  $\mathfrak{g}$ , the second summand is fixed too. So it suffices to check that the above element commutes with  $\mathfrak{g}$  and  $V$  (by using the anti-involution). The first part follows from the following easy lemma.

**Lemma 2.1.** *For any  $\alpha \in \mathfrak{Z}(\mathfrak{U}\mathfrak{g})$ , the element  $\sum_i [\alpha, y_i] x_i$  commutes with  $\mathfrak{g}$ .*

So now we need to show that  $[[\beta_k, y_i], v] = 0$  for all  $i$  and  $k$ , and  $v \in V$ . We will check the latter equality in  $\text{Sym } \mathfrak{g} \otimes \text{Sym } V$  via the symmetrization map. It is easy to see that  $[\alpha, y_i]$  maps to  $\sum_j \frac{\partial \bar{\alpha}}{\partial e_{ij}} y_j$ , where  $\bar{\alpha}$  is the symmetrization of  $\alpha$ . Thus our desired equality becomes

$$\sum_{t,p} \frac{\partial^2 Q_k}{\partial e_{ip} \partial e_{jt}} y_t y_p = 0$$

for all  $i, j, k$ . But this is just a consequence of properties of determinants. This concludes the proof of Rais's theorem.  $\square$

### 3. CENTER OF $H_c$

To finish the proof of Theorem 1.1, we need to prove that for any  $1 \leq i \leq n$  there exists  $c_i \in \mathfrak{Z}(\mathfrak{U}\mathfrak{g})$  such that  $\eta_i = t_i - c_i$  is in the center of  $H_c$ . Notice that since such an element will necessarily commute with  $\mathfrak{g}$  and is fixed by the anti-involution  $j$ , it will be sufficient to prove that this element commutes with  $V$ . Notice that the endomorphism of  $H_c$  defined as  $\alpha \mapsto D(\alpha) = [t_i, \alpha]$  preserves  $\mathfrak{U}\mathfrak{g} \rtimes \text{Sym } V = \mathfrak{U}(\mathfrak{g} \rtimes V)$ . Indeed,

$$D(v) = \sum_j [\beta_i, y_j] [x_j, v] + \sum_j [[\beta_i, y_j], v] x_j,$$

but  $\sum [[\beta_i, y_j], v] = 0$  and  $[x_j, v] \in \mathfrak{U}\mathfrak{g}$  so  $D(v) \in \mathfrak{U}(\mathfrak{g} \rtimes V)$  and since  $t_i$  commutes with  $\mathfrak{g}$  we obtain that  $D(\mathfrak{U}(\mathfrak{g} \rtimes V)) \subset \mathfrak{U}(\mathfrak{g} \rtimes V)$ . In particular, using the PBW property of  $H_c$ , we see that  $D$  is a  $\mathfrak{g}$ -invariant derivation of  $\mathfrak{U}(\mathfrak{g} \rtimes V)$ . Thus by Proposition 3.1, it must be an inner derivation, so there exists  $c_i \in H_c$  such that  $D = \text{ad}(c_i)$ . Therefore,  $t_i - c_i$  belongs to the center of  $H_c$ . Since the filtration degree of  $c_i$  is less than the filtration degree of  $t_i$  (which is equal to 2),  $c_i \in \mathfrak{U}\mathfrak{g} \oplus \mathfrak{U}(V \oplus V^*)$ . Also, since  $t_i$  commutes with  $\mathfrak{U}\mathfrak{g}$  and  $\mathfrak{U}\mathfrak{g}V$  and  $\mathfrak{U}\mathfrak{g}V^*$  have weights 1, -1 with respect to the adjoint action by  $\text{Id} \in \mathfrak{g}$ , we conclude that  $c_i \in \mathfrak{Z}(\mathfrak{U}\mathfrak{g})$ .

Let  $\phi : \mathfrak{U}(\mathfrak{g} \rtimes V) \rightarrow \mathfrak{U}(\mathfrak{g} \rtimes V)$  be a derivation defined as follows:  $\phi(\mathfrak{sl}_n \rtimes \text{Sym } V) = 0$  and  $\phi(\text{Id}) = 1$ , where by  $\text{Id}$  we denote the identity matrix. Now we have the following key proposition.

**Proposition 3.1.**  *$\phi$  is a generating outer derivation of  $\mathfrak{U}(\mathfrak{g} \rtimes V)$  over  $k$ .*

The proof below is due to Pavel Etingof, and it is shorter and nicer than our original proof.

*Proof.* Let  $G_1 = G \ltimes V$  be the affine group ( $G = Gl_n(k)$ ), and  $\mathfrak{g}_1$  its Lie algebra, so  $\mathfrak{g}_1 = \mathfrak{g} \ltimes V$ . Let us identify  $\mathfrak{g}_1^*$  with  $V^* \oplus \mathfrak{g}$  via the trace pairing. The coadjoint representation of  $\mathfrak{g}_1$  has a dense orbit [R1]. This orbit  $X$  is a principal homogeneous  $G_1$ -space and consists of pairs  $(A, f)$ ,  $f \in V^*$ ,  $A \in \mathfrak{g}$ , such that  $f, fA, \dots, fA^{n-1}$  are a basis of  $V^*$ . So the complement of the dense orbit  $X$  is the hypersurface defined by the equation  $P(A, f) = 0$ , where  $P \in O(\mathfrak{g}_1^*)$  is the determinant of the above vectors. The function  $P$  is a semi-invariant function with respect to the coadjoint action of  $G_1$ , and any other such function is a multiple of a power of  $P$ , [R1]. This implies that the complement of  $X$ , which will be denoted by  $Z$ , is an irreducible variety.

Our goal is to show that  $H^1(\mathfrak{U}\mathfrak{g}_1, \mathfrak{U}\mathfrak{g}_1)$  is 1-dimensional. This is of course the same as  $H^1(\mathfrak{g}_1, k[\mathfrak{g}_1^*])$ . Now, consider instead  $H^1(\mathfrak{g}_1, k[\mathfrak{g}_1^*][1/P])$ . This is the same as  $H^1(\mathfrak{g}_1, O(X)) = H^1(\mathfrak{g}_1, O(G_1))$ . But for any affine algebraic group  $R$ ,  $H^*(\mathfrak{r}, O(R))$  (where  $\mathfrak{r}$  is the Lie algebra of  $R$ , and  $R$  acts by left translations) is isomorphic to  $H_{DR}^*(G)$ , the de Rham cohomology of  $R$ . So  $H^1(\mathfrak{g}_1, O(G_1))$  is 1-dimensional in our case, and spanned by the derivation  $\phi$  sending  $\mathfrak{sl}(V)$  and  $V$  to zero and  $\text{Id}$  to 1.

Thus, all we need to show is that the map  $H^1(\mathfrak{g}_1, k[\mathfrak{g}_1^*]) \rightarrow H^1(\mathfrak{g}_1, k[\mathfrak{g}_1^*][1/P])$  is injective. Using the long exact sequence, for this it suffices to show that  $H^0(\mathfrak{g}_1, k[\mathfrak{g}_1^*][1/P]/k[\mathfrak{g}_1^*]) = 0$ .

Let  $U_m$  be the space of elements of  $k[\mathfrak{g}_1^*][1/P]$  having a pole of degree at most  $m$  at the surface  $Z$  given by the equation  $P = 0$ . Then it suffices to show that  $H^0(\mathfrak{g}_1, U_m/U_{m-1}) = 0$  for  $i \geq 1$  (as  $k[\mathfrak{g}_1^*][1/P]$  is the direct limit of  $U_m/U_0$ ). Now,  $U_m/U_{m-1} = O(Z) \otimes K^{-m}$ , where  $K$  is a 1-dimensional representation of  $\mathfrak{g}_1$  defined by its action on  $P$ . So it suffices to show that  $\text{Hom}_G(L^m, O(Z)) = 0$  for  $m \neq 0$ .

To do so, let  $h \in \text{Hom}_G(L^m, O(Z))$ . So  $h \in O(Z)$  is a function with the property that  $h(gz) = \det(g)^m h(z)$  for all  $z \in Z, g \in G$ . Pick a generic point  $z = (A, f)$  of  $Z$ . In some basis,  $A = \text{diag}(a_1, \dots, a_n)$  (with distinct  $a_i$ ), and  $f$  is a generic row vector with last coordinate 0. Now consider  $g_t$  a diagonal matrix,  $g_t = \text{diag}(1, \dots, 1, t)$ . Since  $A$  commutes with  $g_t$  and  $fg_t^{-1} = f$  we get that  $h(z) = h(g_t z) = t^m h(z)$ . Thus,  $h(z) = 0$  for generic  $z \in Z$ , hence  $h = 0$ .  $\square$

Thus we conclude that there exist elements  $c_i \in \mathfrak{Z}(\mathfrak{U}\mathfrak{g}), i = 1, \dots, n$  such that elements  $\eta_i = \sum_{j=1}^n [\beta_i, y_j] x_j - c_i, j = 1, \dots, n$  freely generate the center of  $H_c$ . Since  $\beta_1 = \text{Id}$  (the identity matrix) we have  $[c_1, v] = \sum_j y_j [x_j, v]$ , now since  $[x_j, v] \in \mathfrak{U}\mathfrak{g}$ , we get that

$$[\beta_i, [c_1, v]] = [[\beta_i, y_j], [x_j, v]] = \left[ \sum [\beta_i, y_j] x_j, v \right];$$

thus,  $c_i$  is a unique element of  $\mathfrak{Z}(\mathfrak{U}\mathfrak{g})$  such that  $[c_i, v] = [\beta_i, [c_1, v]]$  for all  $v \in V$ .

#### 4. CATEGORY $\mathcal{O}$

We will denote by  $\mathfrak{h}$  the standard Cartan subalgebra of  $\mathfrak{g}$  consisting of the diagonal matrices.

We have the following natural analog of the Bernstein-Gelfand-Gelfand category  $\mathcal{O}$  defined for semi-simple Lie algebras. Recall the definition from [EGG].

**Definition 4.1.** Category  $\mathcal{O}$  is defined as the full subcategory of finitely generated left  $H_c$ -modules on which  $\mathfrak{h}$  acts diagonalizably and  $H_+ = \mathfrak{U}(\mathfrak{b}_+ \ltimes V^*)$  acts

locally finitely, where  $\mathfrak{b}_+$  denotes the standard Borel subalgebra of upper triangular matrices.

This category decomposes into a direct sum of blocks with respect to the center of  $H_c$  in the standard way. We also have the standard definition for Verma modules [EGG].

**Definition 4.2.** Let  $\lambda \in \mathfrak{h}^*$  be a weight. The Verma module with highest weight  $\lambda$  is defined as  $M(\lambda) = H_c \otimes_{H_+} kv_\lambda$ , where  $kv_\lambda$  is the one-dimensional representation of  $H_+$  corresponding to the weight  $\lambda$ .

By  $L(\lambda)$  we denote the unique irreducible quotient of  $M(\lambda)$ , and it easily follows that every simple object in the category  $\mathcal{O}$  is obtained in this manner.

Let  $M(\lambda)$  be a Verma module of highest weight  $\lambda \in \mathfrak{h}^*$ . Then  $\eta_i m = \psi(c_i)(\lambda)m$  for all  $m \in M(\lambda)$ , where  $\psi : \mathfrak{Z}(\mathfrak{U}\mathfrak{g}) \rightarrow k[\mathfrak{h}^*]$  is the usual Harish-Chandra homomorphism of  $\mathfrak{g}$ . Let us denote by  $k[\mathfrak{h}^*]_c$  the image of  $k[c_1, c_2, \dots, c_n]$  under  $\psi$ . Let us denote by  $\Psi_c : \mathfrak{h}^* = \text{Spec } k[\mathfrak{h}^*] \rightarrow \text{Spec } k[\mathfrak{h}^*]_c$  the map corresponding to the restriction of  $\psi$  on  $k[c_1, \dots, c_n] \subset \mathfrak{Z}(\mathfrak{U}\mathfrak{g})$ . To summarize, we have the following.

**Proposition 4.1.** *Two Verma modules  $M(\lambda), M(\mu)$  (or irreducible modules  $L(\lambda), L(\mu)$ ) belong to the same block of the category  $\mathcal{O}$  if and only if  $\Psi_c(\lambda) = \Psi_c(\mu)$ .*

**Proposition 4.2.** *For  $c \neq 0$ , the Harish-Chandra map is finite.*

*Proof.* We need to show that  $\mathfrak{Z}(\mathfrak{U}\mathfrak{g}) = k[\beta_1, \dots, \beta_n]$  is finite over  $k[c_1, \dots, c_n]$ . For this we will show that the ring of symmetric polynomials in  $x_1, \dots, x_n$  is finite over the ring generated by the top terms of images of  $c_i$  under the identification  $k[\mathfrak{g}]^G = \mathfrak{Z}(\mathfrak{U}\mathfrak{g})$ .

Next, we will need to recall the precise computation of the pairing  $c$ . According to the computation from [EGG], for the given deformation parameter  $b = b_0 + b_1\tau + \dots + \tau^m \in k[\tau], m > 0$  one has

$$[y, x] = c(y, x) = b_0 r_0(x, y) + b_1 r_1(x, y) + \dots + r_m(x, y)$$

where  $r_i \in \mathfrak{U}\mathfrak{g}$  are the symmetrizations of the following functions on  $\mathfrak{g}$  :

$$(x, (1 - tA)^{-1}y) \det(1 - tA)^{-1} = r_0(x, y)(A) + r_1(x, y)(A)t + r_2(x, y)(A)t^2 + \dots$$

Recall that if we set  $\sum \beta_j t^{n-j}$  to be the symmetrization of  $\det(t \text{Id} - A) = \sum Q_{n-j}(A)t^j \in k[\mathfrak{g}][t]$  (where  $\mathfrak{g} = \mathfrak{gl}_n$  and  $\beta_j \in \mathfrak{Z}(\mathfrak{U}\mathfrak{g})$ , then  $\sum_i [\beta_j, y_i]x_i$  is in the center of  $H$  (undeformed case). To lift these elements to the center of  $H_c$ , we want to find  $c_i \in \mathfrak{Z}(\mathfrak{U}\mathfrak{g})$  such that

$$\sum_j [\beta_j, y_j][x_j, y] = [c_i, y], i = 1, \dots, n,$$

for all  $y$ . Of course, this is enough to do for  $y = y_1$ . Below we will compute the top symbol of  $c_i$ . We will write these equalities in  $k[\mathfrak{g}] \otimes V$  via the symmetrization map. In this case if  $f \in k[\mathfrak{g}]$ , then  $[f, y_i] = \sum_j \frac{\partial f}{\partial e_{ij}}$ . We will consider diagonal matrices  $A = \text{diag}(\lambda_1, \dots, \lambda_n)$ . For functions appearing in the above equalities we have that  $\frac{\partial}{\partial e_{ij}} = 0$  if  $i \neq j$  and  $\frac{\partial}{\partial e_{ij}} = \frac{\partial}{\partial \lambda_i}$  for  $i = j$ . Thus, we get

$$\sum_i \sum_t \frac{\partial b_j}{\partial e_{it}} y_t[x_i, y_1] = \sum_t \frac{\partial c_j}{\partial e_{1t}} y_t.$$

This gives

$$\frac{\partial c_j}{\partial \lambda_1} = \det(1 - \tau A)^{-1}(x_1, (1 - \tau A)^{-1}y_1) \frac{\partial Q_j}{\partial \lambda_1}.$$

Now multiply this equation by  $t^{n-j}$  and add. Then we want to find  $c' = \sum_{j=1}^n t^{n-j} c_j$ , which is a symmetric function in  $\lambda_1, \dots, \lambda_n$  with values in  $k[t][[\tau]]$ , such that

$$\frac{\partial c'}{\partial \lambda_1} = - \frac{\prod_{i=2}^n (t - \lambda_i)}{\prod_{i=1}^n (1 - \tau \lambda_i)(1 - \tau \lambda_1)}.$$

This element is given by the following formula:

$$c' = \frac{\det(t - A)}{(t\tau - 1) \det(1 - \tau A)} = \frac{\prod_{i=1}^n (t - \lambda_i)}{(t\tau - 1) \prod_{i=1}^n (1 - \tau \lambda_i)}.$$

This follows from

$$\frac{t - \lambda_i}{(1 - \lambda_i \tau)(t\tau - 1)} = \frac{\tau^{-1}(1 - \tau \lambda_i) + t - \tau^{-1}}{(t\tau - 1)(1 - \tau \lambda_i)} = \frac{\tau^{-1}}{t\tau - 1} + \frac{\tau^{-1}}{1 - \tau \lambda_i}.$$

To summarize, if we write

$$\frac{\prod_{i=1}^n (t - \lambda_i)}{(t\tau - 1) \prod_{i=1}^n (1 - \tau \lambda_i)} = \sum_i \sum_j \eta_i^j t^j \tau^i$$

with  $\eta_i^j \in k[\lambda_1, \dots, \lambda_n]$  a symmetric polynomial, then the top symbol of  $c_j$  is equal to the top symbol of the symmetrization of  $\eta_m^{n-j}$ .

It is now easy to see that  $k[\lambda_1, \dots, \lambda_n]$  is finite over  $k[c_1, \dots, c_n]$ . Indeed, from the formulas above we see that  $\sum_j \lambda_1^j \eta_m^j = 0$  and  $\eta_m^{n-i} = c_i$  for  $i = 1, \dots, n$ , and  $\eta_j \in k$  for  $j > n - 1$ , so we are done.  $\square$

For example, if  $c : V \times V^* \rightarrow k$  is the standard pairing (in which case  $H_c = \mathfrak{U}\mathfrak{g} \rtimes \text{Weyl}(V)$ ), then the Harish-Chandra map  $\Psi_c$  is an isomorphism.

We have the following analog of Duflo's theorem (there is a similar result for infinitesimal Hecke algebras of  $\mathfrak{st}_2$  [KT]).

**Theorem 4.1.** *If  $c \neq 0$ , then all objects in the category  $\mathcal{O}$  have finite length, and any primitive ideal of  $H_c$  is equal to  $\text{Ann } L(\lambda)$  for some weight  $\lambda \in \mathfrak{h}^*$ .*

*Proof.* This is just an easy consequence of finiteness of the analog of the Harish-Chandra map  $\Psi_c$  and of Ginzburg's generalization of Duflo's theorem [G]. Indeed, if  $M$  is a simple  $H_c$ -module, then Schur's lemma implies that  $M$  is annihilated by some maximal ideal  $m \subset \mathfrak{Z}(H_c)$ . Now let us consider the following (nonunital) subalgebras of  $H_c$ : let  $A_+$  denote the subalgebra of  $H_c$  generated by  $\mathfrak{n}_+ \rtimes V^*$ , and let  $A_-$  denote the subalgebra generated by  $\mathfrak{n}_- \rtimes V$ . Also, let  $\delta$  be equal to  $-a\tau + h$  where  $a$  is a sufficiently big positive number and  $h \in \mathfrak{h}$  is a sufficiently generic element such that  $\text{ad}(h)$  has strictly positive eigenvalues on  $\mathfrak{n}_+$ . Thus  $\text{ad}(\delta)$  has strictly positive (respectively negative) integer eigenvalues on  $A_+$  (respectively  $A_-$ ). Now the triple  $(A_+, A_-, \delta)$  gives what is called a noncommutative triangular structure on  $H_c$ , and since  $H_c/mH_c$  is finitely generated as  $A_+ - A_-$  bimodule (this follows from finiteness of  $\Psi_c$ ), Ginzburg's theorem implies that  $\text{Ann}(M)$  is the annihilator of a simple object in the category  $\mathcal{O}$ .  $\square$

Let us briefly discuss the characteristic  $p$  case. We have

**Proposition 4.3.** *If  $\text{char}(k) \gg 0$ , then the  $p$ -th powers of  $V, V^*$  and  $a^p - a^{[p]}$ ,  $a \in \mathfrak{g}$  (restricted powers) belong to the center of  $H_c$ .*

*Proof.* It is clear that for any  $a \in \mathfrak{g}$ , the element  $a^p - a^{[p]}$  belongs to the center of  $H_c$  when  $p > \dim V$ . Let  $v \in V$ . Clearly,  $v^p$  commutes with  $V$  and  $\mathfrak{U}\mathfrak{g}$ . Thus it remains to show that  $[v^p, w] = 0$  for any  $w \in V^*$ . Recall the well-known identity in any algebra of characteristic  $p$ :  $\text{ad}(a)^p = \text{ad}(a^p)$ . We have  $[v^p, w] = \text{ad}(v)^{p-1}([v, w])$ ; but  $[v, w]$  is an element of  $\mathfrak{U}\mathfrak{g}$  whose filtration degree with respect to the standard filtration on  $\mathfrak{U}\mathfrak{g}$  is less than  $p - 1$ . Therefore  $\text{ad}(v)^{p-1}([v, w]) = 0$ , and we are done.  $\square$

Let us denote by  $Z_0$  the subalgebra of  $H_c$  generated by the elements in the proposition. Then it follows from the above proposition and the PBW property of  $H_c$  that  $Z_0 \subset Z(H_c)$  and  $H_c$  is a free  $Z_0$ -module of rank  $p^{\dim(H_c)} = p^{n^2+2n}$ . There is a similar result for rational Cherednik algebras, in which case  $\text{Sym}(V^p)^\Gamma, \text{Sym}((V^*)^p)^\Gamma \subset \mathfrak{Z}(H_c)$ , where  $\Gamma$  is the corresponding reflection group.

It is known that for Cherednik algebras in positive characteristic (as long as the characteristic does not divide the order of the reflection group), the smooth and Azumaya loci coincide [BC]. This leads to the following.

**Conjecture 4.1.** *For  $p \gg 0$ , we have that  $\text{gr } \mathfrak{Z}(H_c) = \mathfrak{Z}(\text{gr } H_c)$  and the smooth and the Azumaya loci of  $\mathfrak{Z}(H_c)$  coincide.*

Positive evidence for this conjecture is given by  $\mathfrak{U}\mathfrak{sl}_{n+1}$ , which can be realized as an infinitesimal Cherednik algebra of  $\mathfrak{gl}_n$ . Next, we will verify the above conjecture for  $n = 1$ .

Let us fix the notation:  $H_c$  denotes an algebra generated by elements  $h, e, f$  with relations  $[h, e] = e, [h, f] = -f, [e, f] = c(h)$ , where  $c$  is a polynomial in one variable over the ground field  $k$ , which will be assumed to be algebraically closed of characteristic  $p$ . Also, we will assume that  $\deg c < p - 1$ . This is an infinitesimal Cherednik algebra for  $\mathfrak{gl}_1$ . This algebra was first considered by Smith [S]. The following result also follows from [BC].

**Proposition 4.4.** *We have  $\text{gr } \mathfrak{Z}(H_c) = \mathfrak{Z}(\text{gr}(H_c))$ . Moreover, the smooth and Azumaya loci of  $H_c$  coincide, and the PI-degree of  $H_c$  is equal to  $p$ .*

*Proof.* We have the analog of the Casimir  $\Delta_c = ef + z(h)$  for some  $z(h) \in k[h]$ , and it is a central element in  $H_c$  [S]. It is easy to check that the elements  $e^p, f^p, h^p - h, \Delta_c$  are central. It is clear that the center of  $\text{gr } H_c = H_0$  is generated by  $e^p, f^p, h^p - h, ef$ . Thus we see that  $\text{gr } \mathfrak{Z}(H_c) = \mathfrak{Z}(\text{gr}(H_c))$ . It is also clear that the degree of  $\Delta_c$  over  $Z_0 = k[f^p, e^p, h^p - h]$  is equal to  $p$ ; thus the PI-degree of  $H_c$  is  $p$ .

Now we claim that if for a given character of the center  $\chi : \mathfrak{Z}(H_c) \rightarrow k$  we have  $\chi(f^p) \neq 0$ , or  $\chi(e^p) \neq 0$ , then any irreducible module  $M$  corresponding to  $\chi$  has dimension  $\geq p$ . Indeed, let us assume that  $\chi(f^p) \neq 0$ . Let us pick an arbitrary nonzero weight vector  $v \in V$  with respect to  $h$ ; thus  $hv = \lambda v$  for some  $\lambda \in k$ . Then the elements  $v, fv, \dots, f^{p-1}v$  are nonzero and have different weights with respect to  $h$ ; thus they are linearly independent and  $\dim V \geq p$ . But since  $p$  is equal to the largest dimension of a simple  $H_c$ -module,  $\dim V = p$ . Thus characters of the above type lie in the Azumaya locus. This implies that the codimension of the complement of the Azumaya locus has codimension  $\geq 2$ . By a result of Brown and Goodearl [BG], this implies that the smooth and Azumaya loci coincide.  $\square$

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