

TRANSLATION FOR FINITE W -ALGEBRAS

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ABSTRACT. A finite W -algebra $U(\mathfrak{g}, e)$ is a certain finitely generated algebra that can be viewed as the enveloping algebra of the Slodowy slice to the adjoint orbit of a nilpotent element e of a complex reductive Lie algebra \mathfrak{g} . It is possible to give the tensor product of a $U(\mathfrak{g}, e)$ -module with a finite dimensional $U(\mathfrak{g})$ -module the structure of a $U(\mathfrak{g}, e)$ -module; we refer to such tensor products as translations. In this paper, we present a number of fundamental properties of these translations, which are expected to be of importance in understanding the representation theory of $U(\mathfrak{g}, e)$.

1. INTRODUCTION

Let \mathfrak{g} be a reductive Lie algebra over \mathbb{C} and let $e \in \mathfrak{g}$ be nilpotent. The finite W -algebra $U(\mathfrak{g}, e)$ associated to the pair (\mathfrak{g}, e) is a finitely generated algebra obtained from $U(\mathfrak{g})$ by a certain quantum Hamiltonian reduction; for a definition of $U(\mathfrak{g}, e)$, we refer the reader to Section 3. Finite W -algebras were introduced to the mathematical literature by Premet in 2002; see [Pr1]. A special case of the definition, when there is an even good grading for e , first appeared in the PhD thesis of Lynch [Ly], extending work of Kostant for the case where e is regular nilpotent [Ko]. Since [Pr1], there has been a great deal of research interest in finite W -algebras and their representation theory; see for example [Br, BGK, BK1, BK2, BK3, Gi, GRU, Lo1, Lo2, Lo3, Lo4, Pr2, Pr3, Pr4]. This is largely due to close connections between the representation theory of $U(\mathfrak{g}, e)$ and that of $U(\mathfrak{g})$, which are principally through Skryabin's equivalence; see [Sk]. This is discussed below and provides an important connection between the primitive ideals of $U(\mathfrak{g})$ whose associated variety contains the adjoint orbit of e , and the primitive ideals of $U(\mathfrak{g}, e)$; see [Pr2, Thm. 3.1], [Lo1, Thm. 1.2.2], [Gi, Thm. 4.5.2] and [Lo2, Thm. 1.2.2].

In mathematical physics, finite W -algebras and their affine counterparts have attracted a lot of attention under a slightly different guise; see for example [BT, DK, VD]. It is proved in [D³HK] that the definition in the mathematical physics literature via BRST cohomology agrees with Premet's definition, [Pr1]. The equivalence of the definitions is of great importance in [BGK], and also plays a significant role here.

For the remainder of the introduction M is a finitely generated $U(\mathfrak{g}, e)$ -module and V is a finite dimensional $U(\mathfrak{g})$ -module. We define the *translation* $M \otimes V$ of M by V by transporting the tensor product on $U(\mathfrak{g})$ -modules through Skryabin's

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equivalence; we refer the reader to Section 4 for a precise definition. Such translations are expected to be of importance in understanding the representation theory of $U(\mathfrak{g}, e)$.

In this paper we prove a number of properties of translations. A number of our results are generalizations of results from [BK2, Ch. 8] for the case $\mathfrak{g} = \mathfrak{gl}_n(\mathbb{C})$, though we require different methods in general. We outline our main results and the structure of the paper below.

After giving some preliminaries in Section 2, we consider both the Whittaker model definition of $U(\mathfrak{g}, e)$ and its definition via nonlinear Lie algebras in Section 3; the latter is our preferred definition in the rest of the paper. We recall the equivalence of these two definitions and also present some structure theory of $U(\mathfrak{g}, e)$.

In Section 4, we give the definition of translation for both the Whittaker model definition and the definition via nonlinear Lie algebras. In Lemma 4.5, we show that these two definitions of translation are equivalent.

The principal goal of Section 5 is to prove that there is an isomorphism of vector spaces

$$(1.1) \quad M \circledast V \cong M \otimes V,$$

so that translations lead to the structure of a $U(\mathfrak{g}, e)$ -module on $M \otimes V$. This isomorphism is a consequence of Theorem 5.1, which considers a certain (Kazhdan) filtration on $M \circledast V$ and the associated graded module. Although Theorem 5.1 implies existence of an isomorphism as in (1.1), it does not give an explicit isomorphism. This is remedied in §5.2, where we discuss explicit isomorphisms, which are natural in both M and V ; we note, however, that these isomorphisms are not canonical. In particular, we point the reader to *lift matrices* in Definition 5.15, which are remarkable matrices that allow one to describe the isomorphisms, and are of great importance in the rest of the paper.

In §5.3, we recall the loop filtration on $U(\mathfrak{g}, e)$ and define a loop filtration on M and $M \circledast V$. The associated graded algebra $\text{gr}' U(\mathfrak{g}, e)$ for the loop filtration is $U(\mathfrak{g}^e)$, where \mathfrak{g}^e is the centralizer of e in \mathfrak{g} . In Proposition 5.21, we prove that $\text{gr}'(M \circledast V)$ and $\text{gr}' M \otimes V$ are isomorphic as $U(\mathfrak{g}^e)$ -modules.

In Section 6, we present some elementary properties of translation. In Proposition 6.1, we give a tensor identity for translations of certain $U(\mathfrak{g}, e)$ -modules that occur as restrictions. Then in Lemmas 6.2 and 6.3, we show that translation is “associative”, and that $? \circledast V^*$ is biadjoint to $? \circledast V$, where V^* denotes the dual $U(\mathfrak{g})$ -module.

In Section 7, we consider the relationship between translations and the highest weight theory from [BGK, §4]. Our first main result of this section is Proposition 7.11, which says that the category $\mathcal{O}(e)$ of $U(\mathfrak{g}, e)$ -modules from [BGK, §4.4] is stable under translation; the category $\mathcal{O}(e)$ is an analogue of the usual BGG category \mathcal{O} of $U(\mathfrak{g})$ -modules. Recently, in [Lo3], Losev has proved that the $\mathcal{O}(e)$ is equivalent to a certain category of Whittaker modules for $U(\mathfrak{g})$, which, in particular, verified [BGK, Conj. 5.3]. We note that this equivalence of categories enables an alternative definition of translation for $M \in \mathcal{O}(e)$, which seems likely to be equivalent. The second main result in Section 7 concerns translations of Verma modules for $U(\mathfrak{g}, e)$ as defined in [BGK, §4.2]. In Theorem 7.14, we show that the translation of a Verma module is filtered by Verma modules.

The BRST definition of $U(\mathfrak{g}, e)$ is recalled in Section 8. The definition of translation in the BRST setting is given in Definition 8.7, and shown to be equivalent to

the previous definition in Proposition 8.9. This equivalence, and the explicit form of it given in Theorem 8.12 is of importance in Section 10 as mentioned below.

Section 9 is a short technical section of the paper. Throughout the paper, we work with “left-handed” definitions, but in Section 10 it is necessary to consider “right-handed” versions of certain objects. In Section 9, we present the required definitions and right-handed analogues of results.

The main result of Section 10 says that translation commutes with duality in a certain sense. The exact statement is given in Theorem 10.9; in essence it says that

$$\overline{M} \otimes \overline{V} \cong \overline{M \otimes V},$$

where bars denote *restricted duals*. This result is a consequence of Theorem 10.7, which provides a striking relationship between lift matrices for V and \overline{V} .

The final section of this paper contains a slightly technical result. The Whittaker model definition of $U(\mathfrak{g}, e)$ depends on two choices: of a good grading $\mathfrak{g} = \bigoplus_{j \in \mathbb{R}} \mathfrak{g}(j)$ for e and an isotropic subspace $\mathfrak{l} \subseteq \mathfrak{g}(-1)$. Thanks to a construction of Gan and Ginzburg [GG, Thm. 4.1] it is known that the definition does not depend on the choice of \mathfrak{l} up to isomorphism. In [BG, Thm. 1], it was shown that the definition of $U(\mathfrak{g}, e)$ does not depend on the choice of good grading up to isomorphism; an alternative proof of this result is contained in [Lo1]. In Proposition 11.1 and Theorem 11.2, we show that in the appropriate sense the translation of M by V does not depend on the choice of \mathfrak{l} and of the good grading. This result justifies the fact that, throughout most of the paper, we only consider translations for the definition of $U(\mathfrak{g}, e)$ via nonlinear Lie algebras and a fixed good grading.

We end the introduction with some remarks. First, we note that the definition of translations leads to a definition of translation functors, in analogy to those in other settings; for the case of reductive algebraic groups see [Ja, II.2]. We do not consider these functors in this paper, but remark here that an alternative approach to translation functors for finite W -algebras using the theory of Whittaker \mathcal{D} -modules is given in [Gi, §5]. The two approaches are expected to be related.

Second we comment on Losev’s definition of $U(\mathfrak{g}, e)$ via Fedosov quantization; see [Lo1, §3]. It is possible to define translation with this definition of $U(\mathfrak{g}, e)$ as in [Lo1, §4] and it is expected that this definition of translation is equivalent to those considered in this paper.

Finally, we note that some results in this paper are similar to results proved by Wallach in [Wa].

2. PRELIMINARIES

Throughout this paper we work over the field of complex numbers \mathbb{C} ; though all of our results remain valid over an algebraically closed field of characteristic 0. As a convention throughout this paper, by a “module” we mean a finitely generated left module; we state explicitly when we are considering right modules, which are also always finitely generated.

2.1. Notation. Let G be connected reductive algebraic group over \mathbb{C} , let \mathfrak{g} be the Lie algebra of G . Let $(\cdot | \cdot)$ be a nondegenerate symmetric invariant bilinear form on \mathfrak{g} . For $x \in \mathfrak{g}$, we write \mathfrak{g}^x for the centralizer of x in \mathfrak{g} . We write $\mathfrak{z}(\mathfrak{g})$ for the center of \mathfrak{g} .

Let e be a nilpotent element of \mathfrak{g} and fix an \mathfrak{sl}_2 -triple (e, h, f) . Define the linear functional

$$\chi : \mathfrak{g} \rightarrow \mathbb{C}, \quad x \mapsto (e|x).$$

Let \mathfrak{t}^e be a maximal toral subalgebra of $\mathfrak{g}^e \cap \mathfrak{g}^h$, and let \mathfrak{t} be a maximal toral subalgebra of \mathfrak{g} containing \mathfrak{t}^e and h . The root system of \mathfrak{g} with respect to \mathfrak{t} is denoted by Φ .

We recall that a \mathbb{Z} -grading

$$\mathfrak{g} = \bigoplus_{j \in \mathbb{Z}} \mathfrak{g}(j)$$

of \mathfrak{g} is called a *good grading* for e if $e \in \mathfrak{g}(2)$, $\mathfrak{g}^e \subseteq \bigoplus_{j \geq 0} \mathfrak{g}(j)$ and $\mathfrak{z}(\mathfrak{g}) \subseteq \mathfrak{g}(0)$; see [EK]. The standard example of a good grading for e is the Dynkin grading obtained by taking the $\text{ad } h$ -eigenspace decomposition of \mathfrak{g} . We fix a good grading for the remainder of the paper, which we may assume satisfies $f \in \mathfrak{g}(-2)$ and $\mathfrak{t} \subseteq \mathfrak{g}(0)$. In Section 11, we allow the grading to vary and consider the more general notion of good \mathbb{R} -gradings. One can easily show that there exists $c \in \mathfrak{t}$ such that the good grading is the $\text{ad } c$ -eigenspace decomposition, i.e., $\mathfrak{g}(j) = \{x \in \mathfrak{g} \mid [c, x] = jx\}$.

The vector space $\mathfrak{g}(-1)$ is denoted by \mathfrak{k} . Let $\omega = \langle \cdot | \cdot \rangle$ be the nondegenerate alternating form on \mathfrak{k} defined by

$$\langle x|y \rangle = \chi([y, x]).$$

Let $\mathfrak{n} = \bigoplus_{j \geq 0} \mathfrak{g}(j)$; this is a parabolic subalgebra of \mathfrak{g} . We abbreviate $\mathfrak{n} = \bigoplus_{j < 0} \mathfrak{g}(j)$, which is a nilpotent subalgebra of \mathfrak{g} , and fix a basis b_1, \dots, b_r for \mathfrak{n} such that b_i is the weight vector for \mathfrak{t} with weight $\beta_i \in \Phi$, and $b_i \in \mathfrak{g}(-d_i)$ with $d_i \in \mathbb{Z}_{\geq 1}$. We write f_1, \dots, f_r for the dual basis of \mathfrak{n}^* .

In what follows we require “copies” of \mathfrak{n} and \mathfrak{k} given by $\mathfrak{n}^{\text{ch}} = \{x^{\text{ch}} \mid x \in \mathfrak{n}\}$ and $\mathfrak{k}^{\text{ne}} = \{x^{\text{ne}} \mid x \in \mathfrak{k}\}$, respectively. Given $x \in \mathfrak{g}$, we may write $x = \sum_{j \in \mathbb{Z}} x(j)$ for the decomposition of x with respect to the good grading, i.e., $x(j) \in \mathfrak{g}(j)$ for each j . In some situations we wish to make sense of x^{ch} , when $x \in \mathfrak{g}$ but $x \notin \mathfrak{n}$, by convention we set $x^{\text{ch}} = x(< 0)^{\text{ch}}$, where $x(< 0) = \sum_{j < 0} x(j)$. There is an analogous convention for x^{ne} when $x \in \mathfrak{g}$ but $x \notin \mathfrak{k}$, i.e., $x^{\text{ne}} = x(-1)^{\text{ne}}$.

2.2. Recollection on nonlinear Lie algebras. In this article we use an easy special case of the notion of a nonlinear Lie superalgebra from [DK, Defn. 3.1].

In this article, a *nonlinear Lie superalgebra* means a vector superspace $\mathfrak{a} = \mathfrak{a}_0 \oplus \mathfrak{a}_1$ equipped with a *nonlinear Lie bracket* $[\cdot, \cdot]$; that is, a parity preserving linear map $\mathfrak{a} \otimes \mathfrak{a} \rightarrow T(\mathfrak{a})$ (= the tensor algebra on \mathfrak{a}) satisfying the following conditions for all homogeneous $a, b, c \in \mathfrak{a}$:

- (i) $[a, b] \in \mathbb{C} \oplus \mathfrak{a}$;
- (ii) $[a, b] = (-1)^{p(a)p(b)}[b, a]$, where $p(a) \in \mathbb{Z}_2$ denotes parity; and
- (iii) $[a, [b, c]] = [[a, b], c] + (-1)^{p(a)p(b)}[b, [a, c]]$ (interpreted using the convention that any bracket with a scalar is zero).

This definition agrees with the general notion of nonlinear Lie superalgebra from [DK, Defn. 3.1] when the grading on \mathfrak{a} in the general setup is concentrated in degree 1.

The *universal enveloping superalgebra* of a nonlinear Lie superalgebra \mathfrak{a} is defined to be $U(\mathfrak{a}) = T(\mathfrak{a})/M(\mathfrak{a})$, where $M(\mathfrak{a})$ is the two-sided ideal of $T(\mathfrak{a})$ generated by the elements $a \otimes b - (-1)^{p(a)p(b)}b \otimes a - [a, b]$ for all homogeneous $a, b \in \mathfrak{a}$. By a

special case of [DK, Theorem 3.3], $U(\mathfrak{a})$ is *PBW generated* by \mathfrak{a} in the sense that if $\{x_1, \dots, x_m\}$ is any homogeneous ordered basis of \mathfrak{a} , then the ordered monomials

$$\{x_1^{a_1} \cdots x_m^{a_m} \mid a_i \in \mathbb{Z}_{\geq 0} \text{ if } p(x_i) = \bar{0} \text{ and } a_i \in \{0, 1\} \text{ if } p(x_i) = \bar{1}\}$$

give a basis for $U(\mathfrak{a})$.

By a *subalgebra* of a nonlinear Lie superalgebra \mathfrak{a} we mean a \mathbb{Z}_2 -graded subspace \mathfrak{b} of \mathfrak{a} such that $[\mathfrak{b}, \mathfrak{b}] \subseteq \mathbb{C} \oplus \mathfrak{b}$. In that case \mathfrak{b} is itself a nonlinear Lie superalgebra and $U(\mathfrak{b})$ is identified with the subalgebra of $U(\mathfrak{a})$ generated by \mathfrak{b} .

We call \mathfrak{a} a *nonlinear Lie algebra* if it is purely even.

3. FINITE W -ALGEBRAS

In this section we give both the Whittaker model definition of the finite W -algebra associated to e denoted $W_{\mathfrak{l}}$, and the definition via nonlinear Lie algebras denoted $U(\mathfrak{g}, e)$. Then we recall the equivalence of these definitions (for the case $\mathfrak{l} = 0$) from [BGK, §2]. The definition via nonlinear Lie algebras is the preferred formulation in most of the paper, but the Whittaker model definition is required for Theorem 5.1, which is a fundamental result in this paper. We also present some results on finite W -algebras that are required in the sequel; these are contained in §3.3 and §3.4.

3.1. Whittaker model definition. Before we define $W_{\mathfrak{l}}$ we introduce some notation. We choose an isotropic subspace \mathfrak{l} of $\mathfrak{g}(-1)$ with respect to the alternating form $\omega = \langle \cdot | \cdot \rangle$ on $\mathfrak{k} = \mathfrak{g}(-1)$. The annihilator of \mathfrak{l} with respect to ω is $\mathfrak{l}^{\perp\omega} = \{x \in \mathfrak{k} \mid \langle x | y \rangle = 0 \text{ for all } y \in \mathfrak{l}\}$. Define the nilpotent subalgebras

$$\mathfrak{m}_{\mathfrak{l}} = \mathfrak{l} \oplus \bigoplus_{j < -1} \mathfrak{g}(j) \quad \text{and} \quad \mathfrak{n}_{\mathfrak{l}} = \mathfrak{l}^{\perp\omega} \oplus \bigoplus_{j < -1} \mathfrak{g}(j).$$

Let $I_{\mathfrak{l}}$ be the left ideal of $U(\mathfrak{g})$ generated by $\{x - \chi(x) \mid x \in \mathfrak{m}_{\mathfrak{l}}\}$, and define the left $U(\mathfrak{g})$ -module $Q_{\mathfrak{l}} = U(\mathfrak{g})/I_{\mathfrak{l}}$. The adjoint action of $\mathfrak{n}_{\mathfrak{l}}$ on $U(\mathfrak{g})$ induces a (well-defined) adjoint action on $Q_{\mathfrak{l}}$. The *Whittaker model definition of the finite W -algebra* associated to \mathfrak{g} and e is

$$W_{\mathfrak{l}} = H^0(\mathfrak{n}_{\mathfrak{l}}, Q_{\mathfrak{l}}) = Q_{\mathfrak{l}}^{\mathfrak{n}_{\mathfrak{l}}},$$

where the Lie algebra cohomology is taken with respect to adjoint action of $\mathfrak{n}_{\mathfrak{l}}$ on $Q_{\mathfrak{l}}$. More explicitly, $W_{\mathfrak{l}}$ is the space of *twisted $\mathfrak{n}_{\mathfrak{l}}$ -invariants*:

$$W_{\mathfrak{l}} = \{u + I_{\mathfrak{l}} \in Q_{\mathfrak{l}} \mid [x, u] \in I_{\mathfrak{l}} \text{ for all } x \in \mathfrak{n}_{\mathfrak{l}}\}.$$

It is easy to check that multiplication in $U(\mathfrak{g})$ gives rise to a well-defined multiplication on $W_{\mathfrak{l}}$. We note that the definition of $W_{\mathfrak{l}}$ depends on the choice of \mathfrak{l} and on the choice of good grading; this is discussed in Section 11 where we recall the proof from [GG, §5.5] that $W_{\mathfrak{l}}$ is independent of \mathfrak{l} up to isomorphism, and the proof of independence of good grading from [BG, Thm. 1].

In the sequel, we denote $\mathbb{1}_{\mathfrak{l}} = 1 + I_{\mathfrak{l}} \in Q_{\mathfrak{l}}$, so then we have $u\mathbb{1}_{\mathfrak{l}} = u + I_{\mathfrak{l}}$ for $u \in U(\mathfrak{g})$. There is the $U(\mathfrak{g})$ - $W_{\mathfrak{l}}$ -bimodule structure on $Q_{\mathfrak{l}}$: the right action of $W_{\mathfrak{l}}$ being given by $(u\mathbb{1}_{\mathfrak{l}})(v\mathbb{1}_{\mathfrak{l}}) = uv\mathbb{1}_{\mathfrak{l}}$, it is straightforward to check that this is well-defined. For the case $\mathfrak{l} = 0$, we abbreviate notation and write $I = I_{\mathfrak{l}}$, $Q = Q_{\mathfrak{l}}$, $W = W_{\mathfrak{l}}$ and $\mathbb{1} = \mathbb{1}_{\mathfrak{l}}$.

We now introduce some notation so that we can discuss the main structure theorem for $W_{\mathfrak{l}}$; this is required for the proof of Theorem 5.1, which is a fundamental result for this paper. The *Kazhdan filtration* of $U(\mathfrak{g})$ is defined by declaring that

$x \in \mathfrak{g}(j)$ has Kazhdan degree $j + 2$. The Kazhdan filtration induces filtrations on both Q_Γ and W_Γ . As is shown in [GG, §4], the associated graded module $\text{gr } Q_\Gamma$ of Q_Γ can be identified with the coordinate algebra $\mathbb{C}[e + \mathfrak{m}_\Gamma^\perp]$, where $\mathfrak{m}_\Gamma^\perp$ denotes the annihilator of \mathfrak{m}_Γ in \mathfrak{g} with respect to the form $(\cdot|\cdot)$.

Let N_Γ be the unipotent subgroup of G corresponding to \mathfrak{n}_Γ . The affine space $\mathcal{S} = e + \mathfrak{g}^f$ is called the *Slodowy slice* to the nilpotent orbit of e ; it is a transverse slice to the G -orbit of e . By [GG, Lem. 2.2], there is an isomorphism of varieties $N_\Gamma \times (e + \mathfrak{g}^f) \xrightarrow{\sim} e + \mathfrak{m}_\Gamma^\perp$, given by the adjoint action map. As a consequence, we obtain the identification $\text{gr } Q_\Gamma \cong \mathbb{C}[N_\Gamma] \otimes \mathbb{C}[\mathcal{S}]$. The main structure theorem for W_Γ says that

$$(3.1) \quad \text{gr } W_\Gamma \cong \mathbb{C}[e + \mathfrak{m}_\Gamma^\perp]^{N_\Gamma} \cong \mathbb{C}[\mathcal{S}],$$

where $\text{gr } W_\Gamma$ is the associated graded algebra of W_Γ with respect to the Kazhdan filtration. This was first proved by Premet in [Pr1, Thm. 4.6]; the approach followed here is that given by Gan and Ginzburg, [GG, Thm. 4.1]

3.2. Definition via nonlinear Lie algebras. We begin by defining the nonlinear Lie algebra $\tilde{\mathfrak{g}}$. Recall that \mathfrak{k}^{ne} is a “copy” of \mathfrak{k} . We give \mathfrak{k}^{ne} the structure of a nonlinear Lie algebra with bracket defined by $[x^{\text{ne}}, y^{\text{ne}}] = \langle x|y \rangle$. The nonlinear Lie algebra $\tilde{\mathfrak{g}} = \mathfrak{g} \oplus \mathfrak{k}^{\text{ne}}$ is defined by extending the bracket on \mathfrak{g} and \mathfrak{k}^{ne} and declaring that $[x, y^{\text{ne}}] = 0$ for $x \in \mathfrak{g}$ and $y \in \mathfrak{k}$. We define the subalgebra $\tilde{\mathfrak{p}} = \mathfrak{p} \oplus \mathfrak{k}^{\text{ne}}$ of $\tilde{\mathfrak{g}}$. Note that as \mathfrak{g} commutes with \mathfrak{k}^{ne} , we have tensor decompositions $U(\tilde{\mathfrak{g}}) \cong U(\mathfrak{g}) \otimes U(\mathfrak{k}^{\text{ne}})$ and $U(\tilde{\mathfrak{p}}) \cong U(\mathfrak{p}) \otimes U(\mathfrak{k}^{\text{ne}})$. Further, $U(\mathfrak{k}^{\text{ne}})$ is isomorphic to the Weyl algebra associated to \mathfrak{k} and the form ω .

We define \tilde{I} to be the left ideal of $U(\tilde{\mathfrak{g}})$ generated by $x - x^{\text{ne}} - \chi(x)$ for $x \in \mathfrak{n}$. We define \tilde{Q} to be the $U(\tilde{\mathfrak{g}})$ -module $U(\tilde{\mathfrak{g}})/\tilde{I}$, and denote $\tilde{\mathbb{1}} = 1 + \tilde{I} \in \tilde{Q}$. By the PBW theorem for $U(\tilde{\mathfrak{g}})$ we have a direct sum decomposition $U(\tilde{\mathfrak{g}}) = U(\tilde{\mathfrak{p}}) \oplus \tilde{I}$, so we can identify $\tilde{Q} \cong U(\tilde{\mathfrak{p}})$ as vector spaces. We write $\text{Pr} : U(\tilde{\mathfrak{g}}) \rightarrow U(\tilde{\mathfrak{p}})$ for the projection along the above direct sum decomposition. There is an action of \mathfrak{n} on $U(\tilde{\mathfrak{p}})$ by

$$(3.2) \quad x \cdot u = \text{Pr}((x - x^{\text{ne}} - \chi(x))u),$$

which gives $U(\tilde{\mathfrak{p}})$ the structure of an \mathfrak{n} -module; under the identification of vector spaces $\tilde{Q} \cong U(\tilde{\mathfrak{p}})$ this coincides with the action of \mathfrak{n} on \tilde{Q} by $x - x^{\text{ne}} - \chi(x)$. We note that this action is the same as the *twisted adjoint action* of \mathfrak{n} on $U(\tilde{\mathfrak{p}})$ given by

$$x \cdot u = \text{Pr}([x - x^{\text{ne}}, u]).$$

We define the *finite W-algebra*

$$U(\mathfrak{g}, e) = H^0(\mathfrak{n}, U(\tilde{\mathfrak{p}})) = U(\tilde{\mathfrak{p}})^{\mathfrak{n}},$$

where the cohomology is taken with respect to the action of \mathfrak{n} given in (3.2). More explicitly, we have

$$U(\mathfrak{g}, e) = \{u \in U(\tilde{\mathfrak{p}}) \mid \text{Pr}([x - x^{\text{ne}}, u]) = 0 \text{ for all } x \in \mathfrak{n}\}$$

is the space of twisted \mathfrak{n} -invariants in $U(\tilde{\mathfrak{p}})$. It is a subalgebra of $U(\tilde{\mathfrak{p}})$; see [BGK, Thm. 2.4].

There is a right action of $U(\mathfrak{g}, e)$ on \tilde{Q} making \tilde{Q} into a $U(\tilde{\mathfrak{g}})$ - $U(\mathfrak{g}, e)$ -bimodule. This action is given by $(u\tilde{\mathbb{1}})v = (uv)\tilde{\mathbb{1}}$ for $u \in U(\tilde{\mathfrak{p}})$ and $v \in U(\mathfrak{g}, e)$, i.e., it is given by multiplication in $U(\tilde{\mathfrak{p}})$ under the identification $U(\tilde{\mathfrak{p}}) \cong \tilde{Q}$.

We now recall the isomorphism between $U(\mathfrak{g}, e)$ and W given in [BGK, Lem. 2.3]; see also [Pr2, §2.4]. There is a well-defined action of $U(\tilde{\mathfrak{g}})$ on Q given by extending the regular action of $U(\mathfrak{g})$ and defining $x^{\text{ne}} \cdot (u+I) = ux+I$ for $x \in \mathfrak{k}$, and $u \in U(\mathfrak{g})$. By [BGK, Lem. 2.3], the natural map $U(\tilde{\mathfrak{g}}) \rightarrow Q$ given by $u \mapsto u \cdot (1+I)$ intertwines the twisted adjoint action of \mathfrak{n} on $U(\tilde{\mathfrak{p}})$ with the adjoint action of \mathfrak{n} on Q . This is used to prove [BGK, Thm. 2.4], which we state below for convenience of reference.

Lemma 3.3. *The natural map $U(\tilde{\mathfrak{g}}) \rightarrow Q$ given by $u \mapsto u \cdot \mathbb{1}$ restricts to an isomorphism of vector spaces $U(\tilde{\mathfrak{p}}) \xrightarrow{\sim} Q$ and an isomorphism of algebras $U(\mathfrak{g}, e) \xrightarrow{\sim} W$.*

We next discuss the Kazhdan filtration on $U(\mathfrak{g}, e)$ recalling the required parts of the discussion in [BGK, §3.2]. The Kazhdan filtration is extended to $U(\tilde{\mathfrak{g}})$ by saying that $x^{\text{ne}} \in \mathfrak{k}^{\text{ne}}$ has degree 1. Then $U(\tilde{\mathfrak{p}})$ inherits a nonnegative filtration such that the associated graded algebra $\text{gr } U(\tilde{\mathfrak{p}}) \cong S(\tilde{\mathfrak{p}})$, where the symmetric algebra $S(\tilde{\mathfrak{p}})$ has the Kazhdan grading. The twisted adjoint action of \mathfrak{n} on $U(\tilde{\mathfrak{p}})$ induces a graded action of \mathfrak{n} on $S(\tilde{\mathfrak{p}})$, and through the isomorphism in Lemma 3.3 and (3.1), we get $\text{gr } U(\mathfrak{g}, e) \cong H^0(\mathfrak{n}, S(\tilde{\mathfrak{p}})) = S(\tilde{\mathfrak{p}})^{\mathfrak{n}}$, where the cohomology is taken with respect to this action. By [BGK, Lem. 2.2], there is a direct sum decomposition

$$(3.4) \quad \tilde{\mathfrak{p}} = \mathfrak{g}^e \oplus \bigoplus_{j \geq 2} [f, \mathfrak{g}(j)] \oplus \mathfrak{k}^{\text{ne}}.$$

The projection $\tilde{\mathfrak{p}} \rightarrow \mathfrak{g}^e$ along this decomposition induces a homomorphism $\zeta : S(\tilde{\mathfrak{p}}) \rightarrow S(\mathfrak{g}^e)$. Then [BGK, Lem. 3.5] says that ζ restricts to an isomorphism

$$(3.5) \quad \zeta : S(\tilde{\mathfrak{p}})^{\mathfrak{n}} \xrightarrow{\sim} S(\mathfrak{g}^e).$$

3.3. Some structure theory of $U(\mathfrak{g}, e)$. In this subsection we present some results regarding the structure of $U(\mathfrak{g}, e)$, which are required in the sequel.

First we apply the discussion of the Kazhdan filtration in the previous subsection to show, in Lemma 3.6, that \tilde{Q} is free as a right $U(\mathfrak{g}, e)$ -module; this is a slight generalization of part (3) of the theorem in [Sk]. We set $\mathfrak{r} = \bigoplus_{j \geq 2} [f, \mathfrak{g}(j)] \oplus \mathfrak{k}^{\text{ne}}$

Lemma 3.6.

- (i) \tilde{Q} is free as a right $U(\mathfrak{g}, e)$ -module.
- (ii) A basis of \tilde{Q} as a free right $U(\mathfrak{g}, e)$ -module can be constructed as follows. Choose a basis $(x_1^{\text{ne}}, \dots, (x_{2s}^{\text{ne}}), x_{2s+1}, \dots, x_r)$ of \mathfrak{r} . Then the monomials

$$(x_1^{\text{ne}})^{a_1} \dots (x_{2s}^{\text{ne}})^{a_{2s}} x_{2s+1}^{a_{2s+1}} \dots x_r^{a_r} \tilde{\mathbb{1}} \in \tilde{Q}$$

with $a_i \in \mathbb{Z}_{\geq 0}$ form a basis for \tilde{Q} as a free right $U(\mathfrak{g}, e)$ -module.

Proof. From (3.4) we get $\text{gr } \tilde{Q} \cong S(\mathfrak{r}) \otimes S(\mathfrak{g}^e)$. We also have the isomorphism $\text{gr } U(\mathfrak{g}, e) \xrightarrow{\sim} S(\mathfrak{g}^e)$ given by the restriction of ζ from (3.5). Now an induction on Kazhdan degree shows that $\text{gr } \tilde{Q} \cong S(\mathfrak{r}) \otimes \text{gr } U(\mathfrak{g}, e)$, so that $\text{gr } \tilde{Q}$ is a free right $\text{gr } U(\mathfrak{g}, e)$ -module. As the Kazhdan filtration on \tilde{Q} is nonnegative, a standard filtration argument shows that \tilde{Q} is free as a right $U(\mathfrak{g}, e)$ -module; and moreover, if $\{\text{gr } u_i \tilde{\mathbb{1}} \mid i \in J\}$, where J is some indexing set, is a basis of $\text{gr } \tilde{Q}$ as a free right $\text{gr } U(\mathfrak{g}, e)$ -module, then $\{u_i \tilde{\mathbb{1}} \mid i \in J\}$ is a basis of \tilde{Q} as a free right $U(\mathfrak{g}, e)$ -module. Now the above tensor decomposition $\text{gr } \tilde{Q} \cong S(\mathfrak{r}) \otimes \text{gr } U(\mathfrak{g}, e)$ tells us that if $(x_1^{\text{ne}}, \dots, (x_{2s}^{\text{ne}}), x_{2s+1}, \dots, x_r)$ is a basis of \mathfrak{r} , then the monomials $\text{gr}((x_1^{\text{ne}})^{a_1} \dots (x_{2s}^{\text{ne}})^{a_{2s}} x_{2s+1}^{a_{2s+1}} \dots x_r^{a_r} \tilde{\mathbb{1}}) \in \text{gr } \tilde{Q}$ form a basis for $\text{gr } \tilde{Q}$ as a free right $\text{gr } U(\mathfrak{g}, e)$ -module. \square

We now explain one way to construct a basis $(x_1^{\text{ne}}, \dots, (x_{2s}^{\text{ne}}), x_{2s+1}, \dots, x_r$ of \mathfrak{t} as in the above lemma. Take the basis b_1, \dots, b_r of \mathfrak{n} , which we recall consists of \mathfrak{t} -weight vectors with $b_i \in \mathfrak{g}(-d_i)$ and $d_i \in \mathbb{Z}_{\geq 1}$; then an easy consequence of \mathfrak{sl}_2 -representation theory is that for each i there exists unique $x_i \in \mathfrak{g}(d_i - 2)$ with $(x_i|[b_j, e]) = \delta_{ij}$ and $(x_i|y) = 0$ for all $y \in \mathfrak{g}^f$ (we assume that b_1, \dots, b_r is ordered so that $x_1, \dots, x_{2s} \in \mathfrak{k}$ and $x_{2s+1}, \dots, x_r \in \mathfrak{p}$).

Using this basis of \mathfrak{t} , we may define the projection

$$(3.7) \quad \varphi : \tilde{Q} \rightarrow U(\mathfrak{g}, e)$$

by $\varphi((x_1^{\text{ne}})^{a_1} \dots (x_{2s}^{\text{ne}})^{a_{2s}} x_{2s+1}^{a_{2s+1}} \dots x_r^{a_r} \tilde{\mathbb{1}}) = 0$ if $a_i \neq 0$ for some i , and $\varphi(\tilde{\mathbb{1}}) = 1$. We identify the associated graded module of \tilde{Q} with $S(\tilde{\mathfrak{p}})$, and write

$$(3.8) \quad \eta = \text{gr } \varphi : S(\tilde{\mathfrak{p}}) \rightarrow S(\tilde{\mathfrak{p}})^{\text{n}}$$

for the associated graded map. We record the following technical lemma that we require in Section 5; it is a consequence of [BGK, Lem. 3.7].

Lemma 3.9. *Let ζ be as in (3.5) and η as in (3.8). Then the following diagram commutes:*

$$\begin{array}{ccc} S(\tilde{\mathfrak{p}}) & \xrightarrow{\eta} & S(\tilde{\mathfrak{p}})^{\text{n}} \\ & \searrow \zeta & \downarrow \zeta \\ & & S(\mathfrak{g}^e) \end{array}$$

Next we recall another filtration of $U(\mathfrak{g}, e)$ defined in [BGK, §3.3]; this filtration is called the good filtration in *loc. cit.*, but we choose to use the terminology loop filtration here, as in [BK1, §2]. The good grading on \mathfrak{g} induces a grading of $U(\mathfrak{p})$, which we extend to a grading of $U(\tilde{\mathfrak{p}})$ by declaring that elements of \mathfrak{k}^{ne} have degree 0. Then $U(\mathfrak{g}, e)$ is not in general a graded subalgebra of $U(\tilde{\mathfrak{p}})$, but there is an induced filtration $(F_j^l U(\mathfrak{g}, e))_{j \in \mathbb{Z}_{\geq 0}}$ of $U(\mathfrak{g}, e)$ called the *loop filtration*. The associated graded algebra $\text{gr}^l U(\mathfrak{g}, e)$ is identified with a subalgebra of $U(\tilde{\mathfrak{p}})$; in order to explicitly describe this subalgebra we need to give some notation.

Let z_1, \dots, z_{2s} be a symplectic basis for \mathfrak{k} , so that $\langle z_i | z_j^* \rangle = \delta_{i,j}$ for all $1 \leq i, j \leq 2s$ where

$$z_j^* := \begin{cases} z_{j+s} & \text{for } j = 1, \dots, s, \\ -z_{j-s} & \text{for } j = s + 1, \dots, 2s. \end{cases}$$

The Lie algebra homomorphism $\theta : \mathfrak{g}^e \hookrightarrow U(\tilde{\mathfrak{p}})$ is defined in [BGK, Thm. 3.3] by

$$(3.10) \quad \theta(x) = \begin{cases} x + \frac{1}{2} \sum_{i=1}^{2s} [x, z_i^*]^{\text{ne}} z_i^{\text{ne}} & \text{if } x \in \mathfrak{g}^e(0), \\ x & \text{otherwise,} \end{cases}$$

which restricts to a Lie algebra homomorphism $\mathfrak{g}^e(0) \hookrightarrow U(\mathfrak{g}, e)$. We can extend θ to an algebra homomorphism $\theta : U(\mathfrak{g}^e) \rightarrow U(\tilde{\mathfrak{p}})$, and note that $U(\mathfrak{g}^e)$ is graded from the good grading on \mathfrak{g} . Then [BGK, Thm. 3.8] says that θ gives a \mathfrak{t}^e -equivariant graded algebra isomorphism

$$(3.11) \quad \theta : U(\mathfrak{g}^e) \xrightarrow{\sim} \text{gr}^l U(\mathfrak{g}, e).$$

Finally, in this subsection, we summarize [BGK, Thm. 3.6 and Lem. 3.7], which gives an explicit description of the structure of $U(\mathfrak{g}, e)$; we note that [BGK, Thm. 3.6] is essentially [Pr1, Theorem 4.6].

To begin, we note that (3.5) implies that there exists a (nonunique) linear map

$$(3.12) \quad \Theta : \mathfrak{g}^e \hookrightarrow U(\mathfrak{g}, e)$$

such that $\Theta(x) \in F_{j+2}U(\mathfrak{g}, e)$ and $\zeta(\text{gr}_{j+2}\Theta(x)) = x$ for each $x \in \mathfrak{g}^e(j)$. We can choose Θ so that it is \mathfrak{t}^e -equivariant with respect to the embedding of \mathfrak{t}^e in $U(\mathfrak{g}, e)$ given by θ ; and such that $\text{gr}'\Theta(x) = \theta(x)$ for each $x \in \mathfrak{g}^e(j)$ and $\Theta(t) = \theta(t)$ for each $t \in \mathfrak{g}^e(0)$.

Now let x_1, \dots, x_t be a basis of \mathfrak{g}^e that is homogeneous with respect to the good grading and consists of \mathfrak{t}^e -weight vectors; say $x_i \in \mathfrak{g}^e(n_i)$ with \mathfrak{t}^e -weight γ_i . Then for $j \geq 0$ the monomials

$$\{\Theta(x_1)^{a_1} \dots \Theta(x_t)^{a_t} \mid a_i \in \mathbb{Z}_{\geq 0}, \sum_{i=1}^t a_i(n_i + 2) \leq j\}$$

form a basis for $F_jU(\mathfrak{g}, e)$.

3.4. A version of Skryabin's equivalence. We give a version of Skryabin's equivalence to obtain an equivalence from the category $U(\mathfrak{g}, e)\text{-mod}$ of $U(\mathfrak{g}, e)$ -modules to the category $\text{Wh}(\tilde{\mathfrak{g}}, e)$ of *generalized Whittaker $U(\tilde{\mathfrak{g}})$ -modules* defined below. This equivalence is required in the sequel; more precisely, it is needed for the definition of translation in §4.2 and we require Corollary 3.15 in §8.2. The proof of the equivalence follows the same lines as that given for Skryabin's equivalence in [GG, Thm. 6.1].

Given a $U(\tilde{\mathfrak{g}})$ -module E , there is an action of \mathfrak{n} on E defined by

$$(3.13) \quad x \cdot m = (x - x^{\text{ne}} - \chi(x))m$$

and henceforth referred to as the *dot action*. We call E a *generalized Whittaker module* (with respect to e and \mathfrak{n}) if $x - x^{\text{ne}} - \chi(x)$ acts locally nilpotently on E for all $x \in \mathfrak{n}$, i.e., each $x \in \mathfrak{n}$ acts locally nilpotently in the dot action. We write $\text{Wh}(\tilde{\mathfrak{g}}, e)$ for the category of *generalized Whittaker $U(\tilde{\mathfrak{g}})$ -modules*.

Recall the $U(\tilde{\mathfrak{g}})$ module \tilde{Q} from §3.2; it is clear that this is a generalized Whittaker module. It is also a right $U(\mathfrak{g}, e)$ -module as explained in §3.2. Thus there is a functor

$$\tilde{Q} \otimes_{U(\mathfrak{g}, e)} ? : U(\mathfrak{g}, e)\text{-mod} \rightarrow \text{Wh}(\tilde{\mathfrak{g}}, e).$$

There is also a functor

$$H^0(\mathfrak{n}, ?) : \text{Wh}(\tilde{\mathfrak{g}}, e) \rightarrow \tilde{U}(\mathfrak{g}, e)\text{-mod},$$

where the cohomology is taken with respect to the dot action of \mathfrak{n} ; so that we have

$$H^0(\mathfrak{n}, E) = \{m \in E \mid (x - x^{\text{ne}} - \chi(x))m = 0 \text{ for all } x \in \mathfrak{n}\}.$$

It is straightforward to check that $H^0(\mathfrak{n}, E)$ is a well-defined $U(\mathfrak{g}, e)$ -module where the action is given by restricting the $U(\tilde{\mathfrak{g}})$ action on E .

We are now in a position to state our version of Skryabin's equivalence. We only show how one can apply the proof of [GG, Thm. 6.1].

Theorem 3.14. *The functors $\tilde{Q} \otimes_{U(\mathfrak{g}, e)} ?$ and $H^0(\mathfrak{n}, ?)$ are quasi-inverse equivalences of categories.*

Proof. For $M \in U(\mathfrak{g}, e)\text{-mod}$, we have a natural map $\phi : M \rightarrow H^0(\mathfrak{n}, \tilde{Q} \otimes_{U(\mathfrak{g}, e)} M)$ given by $\phi(m) = \tilde{\mathbb{1}} \otimes m$. Using Lemma 3.3 we may identify \tilde{Q} with Q as a filtered vector space and $U(\mathfrak{g}, e)$ with W . The dot action of \mathfrak{n} on $\tilde{Q} \otimes_{U(\mathfrak{g}, e)} M$ is the same as the twisted adjoint action given by $x \cdot (u\tilde{\mathbb{1}} \otimes m) = [x - x^{\text{ne}}, u]\tilde{\mathbb{1}} \otimes m$, for $x \in \mathfrak{n}$. Under the identification of modules $\tilde{Q} \otimes_{U(\mathfrak{g}, e)} M \cong Q \otimes_W M$, the discussion before

Lemma 3.3 tells us that the dot action of \mathfrak{n} on $U(\tilde{\mathfrak{p}}) \otimes_{U(\mathfrak{g},e)} M$ is identified with the adjoint action of \mathfrak{n} on $Q \otimes_W M$. Now one can prove that ϕ is an isomorphism as in [GG, Thm. 6.1].

For $E \in \text{Wh}(\tilde{\mathfrak{g}})$, there is a natural map $f : \tilde{Q} \otimes_{U(\mathfrak{g},e)} H^0(\mathfrak{n}_l, E) \rightarrow E$ given by $f(u\tilde{1} \otimes v) = uv$. The arguments in the proof of [GG, Thm. 6.1] apply in our situation to prove that f is an isomorphism. \square

Once we have the identifications $\tilde{Q} \cong Q$ and $U(\mathfrak{g}, e) \cong W$ used in the proof of Theorem 3.14, one can deduce that $H^i(\mathfrak{n}, \tilde{Q} \otimes_{U(\mathfrak{g},e)} M) = 0$ for $i > 0$ from the proof of [GG, Thm. 6.1], where the cohomology is taken with respect to the dot action of \mathfrak{n} . Therefore, we obtain the following corollary, which we require in §8.2; it is a slight generalization of part (4) of the theorem in [Sk].

Corollary 3.15. *Let $E \in \text{Wh}(\tilde{\mathfrak{g}}, e)$. Then $H^i(\mathfrak{n}, E) = 0$ for $i > 0$, where the cohomology is taken with respect to the dot action of \mathfrak{n} .*

4. DEFINITION OF TRANSLATION

In this section we give the definition of translation in both the Whittaker model definition and in the definition via nonlinear Lie algebras. In Lemma 4.5, we prove that these definitions are equivalent (for $l = 0$) through the isomorphism in Lemma 3.3. We note that the definition of translation of W_l -modules depends on the choice of l , and on the choice of good grading. In Section 11, we show that in fact the definition is independent up to isomorphism in the appropriate sense, which justifies considering only translations of $U(\mathfrak{g}, e)$ -modules in the rest of the paper.

4.1. Translation of W_l -modules. The definition of translation for W_l -modules given below is a generalization of the definition given in [BK2, §8.2] for the case when $\mathfrak{g} = \mathfrak{gl}_n$ and the good grading for e is even.

Let M be a W_l -module and let V be a finite dimensional $U(\mathfrak{g})$ -module. As explained in §3.1, Q_l has the structure of a right W_l -module and the tensor product $Q_l \otimes_{W_l} M$ is a $U(\mathfrak{g})$ -module. Further, there is an (adjoint) action of \mathfrak{n}_l on $Q_l \otimes_{W_l} M$ given by

$$x \cdot (u\mathbb{1}_l \otimes m) = [x, u]\mathbb{1}_l \otimes m,$$

for $x \in \mathfrak{n}$, $u \in U(\mathfrak{g})$ and $m \in M$; it is straightforward to check that this action is well defined and gives $Q_l \otimes_{W_l} M$ the structure of an \mathfrak{n}_l -module. Therefore, the tensor product $(Q_l \otimes_{W_l} M) \otimes V$ is an \mathfrak{n}_l -module, where \mathfrak{n}_l is acting on V by restriction of the \mathfrak{g} -action and on the tensor product through the comultiplication in $U(\mathfrak{n}_l)$. We refer to this action of \mathfrak{n}_l as the *dot action*; it is given explicitly by

$$(4.1) \quad x \cdot ((u\mathbb{1}_l \otimes m) \otimes v) = ([x, u]\mathbb{1}_l \otimes m) \otimes v + (u\mathbb{1}_l \otimes m) \otimes xv.$$

Definition 4.2. We define the *translation of M by V*

$$M \otimes_l V = H^0(\mathfrak{n}_l, (Q_l \otimes_{W_l} M) \otimes V),$$

where the cohomology is taken with respect to the dot action of \mathfrak{n}_l , i.e., $M \otimes_l V$ is the space of invariants of $(Q_l \otimes_{W_l} M) \otimes V$ with respect to the dot action of \mathfrak{n}_l .

We have that $(Q_l \otimes_{W_l} M) \otimes V$ is a $U(\mathfrak{g})$ -module through the comultiplication in $U(\mathfrak{g})$. It is straightforward to check that this gives rise to a well-defined W_l -module structure on $M \otimes_l V$ with action defined by $u\mathbb{1}_l z = uz$ for $u\mathbb{1}_l \in W_l$ and $z \in (Q_l \otimes_{W_l} M) \otimes V$.

4.2. Translation of $U(\mathfrak{g}, e)$ -modules. In this subsection we define translation for $U(\mathfrak{g}, e)$ -modules. We show, in Lemma 4.5 that this definition agrees with that for W -modules through the isomorphism given in Lemma 3.3.

First we define a “comultiplication” $\tilde{\Delta} : U(\tilde{\mathfrak{g}}) \rightarrow U(\tilde{\mathfrak{g}}) \otimes U(\mathfrak{g})$ by

$$\tilde{\Delta}(x) = x \otimes 1 + 1 \otimes x \quad \text{and} \quad \tilde{\Delta}(y^{ne}) = y^{ne} \otimes 1$$

for $x \in \mathfrak{g}$ and $y \in \mathfrak{k}$. It is trivial to check that $\tilde{\Delta}([x, y]) = \tilde{\Delta}(x)\tilde{\Delta}(y) - \tilde{\Delta}(y)\tilde{\Delta}(x)$ for all $x, y \in \tilde{\mathfrak{g}}$. Therefore, given a $U(\tilde{\mathfrak{g}})$ -module E and a $U(\mathfrak{g})$ -module V , we can give $E \otimes V$ the structure of a $U(\tilde{\mathfrak{g}})$, through $\tilde{\Delta}$. Moreover, if V is finite dimensional, then $? \otimes V$ defines an exact endofunctor of $U(\tilde{\mathfrak{g}})$ -mod. It is clear that if $E \in \text{Wh}(\tilde{\mathfrak{g}}, e)$, then $E \otimes V \in \text{Wh}(\tilde{\mathfrak{g}}, e)$. Therefore, we may transport the functor $? \otimes V$ through the version of Skryabin’s equivalence in Theorem 3.14 to obtain an exact endofunctor of $U(\mathfrak{g}, e)$ -mod, as defined below.

For the rest of this subsection let M be a $U(\mathfrak{g}, e)$ -module and V a finite dimensional $U(\mathfrak{g})$ -module.

Definition 4.3. The *translation of M by V* is defined to be

$$M \circledast V = H^0(\mathfrak{n}, (\tilde{Q} \otimes_{U(\mathfrak{g}, e)} M) \otimes V),$$

where the cohomology is taken with respect to the dot action of \mathfrak{n} given in (3.13).

To be more explicit we note that the dot action of \mathfrak{n} on $(\tilde{Q} \otimes_{U(\mathfrak{g}, e)} M) \otimes V$ is given by

$$(4.4) \quad x \cdot (u\tilde{\mathbb{1}} \otimes m \otimes v) = [x - x^{ne}, u]\tilde{\mathbb{1}} \otimes m \otimes v + u\tilde{\mathbb{1}} \otimes m \otimes xv,$$

for $x \in \mathfrak{n}$, $u \in U(\tilde{\mathfrak{p}})$, $m \in M$ and $v \in V$, and $M \circledast V$ is the space of invariants for this action.

The next lemma says that our definitions of translation are equivalent, it follows easily from Lemma 3.3 using (4.4) and (4.1).

Lemma 4.5. *View M as a W -module through the isomorphism given in Lemma 3.3. Then the natural map $U(\tilde{\mathfrak{g}}) \rightarrow Q$ given by $u \mapsto u \cdot \mathbb{1}$ induces an isomorphism*

$$(\tilde{Q} \otimes_{U(\mathfrak{g}, e)} M) \otimes V \xrightarrow{\sim} (Q \otimes_W M) \otimes V,$$

which intertwines the dot actions of \mathfrak{n} on $(\tilde{Q} \otimes_{U(\mathfrak{g}, e)} M) \otimes V$ and $(Q \otimes_W M) \otimes V$. Therefore, we have

$$H^0(\mathfrak{n}, (\tilde{Q} \otimes_{U(\mathfrak{g}, e)} M) \otimes V) \cong H^0(\mathfrak{n}, (Q \otimes_W M) \otimes V).$$

5. VECTOR SPACE ISOMORPHISM AND LIFT MATRICES

In this section we show that the translation $M \circledast V$ of a $U(\mathfrak{g}, e)$ -module M by a $U(\mathfrak{g})$ -module V is isomorphic as a vector space to $M \otimes V$. We initially do this for the Whittaker model definition by considering suitable Kazhdan filtrations in §5.1. Then we discuss explicit isomorphisms using *lift matrices* in §5.2. Finally, we consider the relationship between translations and the loop filtration in §5.3.

5.1. The Kazhdan filtration. We use the notation from §3.1. Let M be a W_Γ -module and V a finite dimensional $U(\mathfrak{g})$ -module. Below we consider a Kazhdan filtration of $M \otimes_\Gamma V$ and show that the associated graded module $\text{gr}(M \otimes_\Gamma V)$ is a $\text{gr } W_\Gamma$ -module isomorphic as a vector space to $\text{gr } M \otimes V$.

Choose a finite dimensional subspace M_0 of M that generates M and define $F_j M = (F_j W_\Gamma)M_0$, where $F_j W_\Gamma$ denotes the j -part of the Kazhdan filtration on W_Γ . This defines a (Kazhdan) filtration on M , so that M is a filtered W_Γ -module; thus $\text{gr } M$ is a $\text{gr } W_\Gamma$ -module. The semisimple element $c \in \mathfrak{g}$ (that defines the good grading for e) acts diagonally on V , and its eigenspace decomposition gives a grading and thus also a filtration of V . These filtrations on M and V along with the Kazhdan filtration on Q_Γ determine a filtration of $(Q_\Gamma \otimes_{W_\Gamma} M) \otimes V$. It is clear that $(Q_\Gamma \otimes_{W_\Gamma} M) \otimes V$ is a filtered $U(\mathfrak{g})$ -module for the Kazhdan filtration; therefore, $M \otimes_\Gamma V$ is a filtered W_Γ -module and the associated graded module $\text{gr}(M \otimes_\Gamma V)$ is a module for $\text{gr } W_\Gamma$. Also, we can give $\text{gr } M \otimes V$ the structure of a $\text{gr } W_\Gamma$ -module with trivial action on V , i.e., $u(m \otimes v) = um \otimes v$ for $u \in \text{gr } W_\Gamma$, $m \in \text{gr } M$ and $v \in V$.

We are now in a position to state and prove the main theorem of this subsection, which, in particular, implies that there is an isomorphism $M \otimes_\Gamma V \cong M \otimes V$ of vector spaces.

Theorem 5.1. *There is an isomorphism of $\text{gr } W_\Gamma$ -modules*

$$\text{gr}(M \otimes_\Gamma V) \cong \text{gr } M \otimes V.$$

Proof. The good grading of \mathfrak{g} gives a grading, and so is a filtration, of \mathfrak{n}_Γ . One can check that $(Q_\Gamma \otimes_{W_\Gamma} M) \otimes V$ is a filtered \mathfrak{n}_Γ -module for the dot action of \mathfrak{n}_Γ given by (4.1), therefore, $\text{gr}((Q_\Gamma \otimes_{W_\Gamma} M) \otimes V)$ is a module for $\text{gr } \mathfrak{n}_\Gamma = \mathfrak{n}_\Gamma$. We have

$$\begin{aligned} \text{gr}((Q_\Gamma \otimes_{W_\Gamma} M) \otimes V) &\cong (\text{gr } Q_\Gamma \otimes_{\text{gr } W_\Gamma} \text{gr } M) \otimes \text{gr } V \\ &\cong (\mathbb{C}[N_\Gamma] \otimes \mathbb{C}[\mathcal{S}]) \otimes_{\mathbb{C}[\mathcal{S}]} \text{gr } M \otimes V \\ &\cong \mathbb{C}[N_\Gamma] \otimes \text{gr } M \otimes V. \end{aligned}$$

In the above, we use the isomorphisms $\text{gr } W_\Gamma \cong \mathbb{C}[\mathcal{S}]$ and $\text{gr } Q_\Gamma \cong \mathbb{C}[N_\Gamma] \otimes \mathbb{C}[\mathcal{S}]$ from §3.1, and the fact that $\text{gr } V \cong V$, because V is already graded. We explain the action of \mathfrak{n}_Γ on $\mathbb{C}[N_\Gamma] \otimes \text{gr } M \otimes V$. The action of N_Γ on itself by left translations induces a locally finite representation of N_Γ on $\mathbb{C}[N_\Gamma]$ by $(x \cdot f)(n) = f(x^{-1}n)$. The action of \mathfrak{n}_Γ on V can be exponentiated to give an action of N_Γ on V . Then $\mathbb{C}[N_\Gamma] \otimes V$ is a locally finite N_Γ -module with the diagonal action of N_Γ , and $\mathbb{C}[N_\Gamma] \otimes \text{gr } M \otimes V$ is a locally finite N_Γ -module with trivial N_Γ action on $\text{gr } M$. This N_Γ -module structure differentiates to give the \mathfrak{n}_Γ -module structure on $\mathbb{C}[N_\Gamma] \otimes \text{gr } M \otimes V$. Taking \mathfrak{n}_Γ -invariants in $\mathbb{C}[N_\Gamma] \otimes \text{gr } M \otimes V$ is the same as taking N_Γ -invariants so we get

$$\begin{aligned} H^0(\mathfrak{n}_\Gamma, \text{gr}((Q_\Gamma \otimes_{W_\Gamma} M) \otimes V)) &\cong H^0(\mathfrak{n}_\Gamma, \mathbb{C}[N_\Gamma] \otimes \text{gr } M \otimes V) \\ &\cong \text{gr } M \otimes (\mathbb{C}[N_\Gamma] \otimes V)^{N_\Gamma}. \end{aligned}$$

It is a standard result that $(\mathbb{C}[N_\Gamma] \otimes V)^{N_\Gamma} \cong V$ (see for example [Ja, I.3.7(6)]); in fact, the map $\mathbb{C}[N_\Gamma] \rightarrow \mathbb{C}$ given by evaluation at 1 gives a map $\mathbb{C}[N_\Gamma] \otimes V \rightarrow V$, which restricts to the above isomorphism. In turn this gives an isomorphism

$$(5.2) \quad \epsilon : H^0(\mathfrak{n}_\Gamma, \text{gr}((Q_\Gamma \otimes_{W_\Gamma} M) \otimes V)) \cong \mathbb{C}[N_\Gamma] \otimes \text{gr } M \otimes V \xrightarrow{\sim} \text{gr } M \otimes V.$$

Next we want to show that the natural map

$$(5.3) \quad \text{gr } H^0(\mathfrak{n}_\Gamma, (Q_\Gamma \otimes_{W_\Gamma} M) \otimes V) \rightarrow H^0(\mathfrak{n}_\Gamma, \text{gr}((Q_\Gamma \otimes_{W_\Gamma} M) \otimes V))$$

is an isomorphism. To do this we use the standard spectral sequence for calculating the cohomology of the filtered module $(Q_{\mathfrak{t}} \otimes_{W_{\mathfrak{t}}} M) \otimes V$, which we denote by (E_r) , i.e., the standard complex for calculating the $\mathfrak{n}_{\mathfrak{t}}$ -cohomology of $(Q_{\mathfrak{t}} \otimes_{W_{\mathfrak{t}}} M) \otimes V$ is filtered, and one takes (E_r) to be the corresponding spectral sequence. It is a standard result that $H^i(\mathfrak{n}_{\mathfrak{t}}, \mathbb{C}[N_{\mathfrak{t}}] \otimes V) = 0$ for $i > 0$: this can be proved by choosing a filtration $F'_j V$ of V as an $\mathfrak{n}_{\mathfrak{t}}$ -module, such that the action of $\mathfrak{n}_{\mathfrak{t}}$ on the associated graded module $\text{gr}' V$ is trivial; then one can apply a spectral sequence argument along with the fact that $H^i(\mathfrak{n}_{\mathfrak{t}}, \mathbb{C}[N_{\mathfrak{t}}]) = 0$ for $i > 0$, as this is de Rham cohomology of the affine space $N_{\mathfrak{t}}$; see for example [GG, §4.3]. It follows that $H^i(\mathfrak{n}_{\mathfrak{t}}, \text{gr}((Q_{\mathfrak{t}} \otimes_{W_{\mathfrak{t}}} M) \otimes V)) = 0$ for all $i > 0$. This implies that E_1 is concentrated in degree 0, so the spectral sequence (E_r) stabilizes at $r = 2$, namely $E_2 = E_{\infty}$. Therefore, the map in (5.3) is an isomorphism, and in turn we obtain an isomorphism

$$(5.4) \quad \bar{\epsilon} : \text{gr}(M \otimes_{\mathfrak{t}} V) \xrightarrow{\sim} \text{gr } M \otimes V.$$

from ϵ in (5.3).

We are left to show that $\bar{\epsilon}$ is an isomorphism of $\text{gr } W_{\mathfrak{t}}$ -modules. First we consider V as a filtered $U(\mathfrak{g})$ -module for the Kazhdan filtration. It is clear that the action of $\text{gr } U(\mathfrak{g}) \cong S(\mathfrak{g})$ on the associated graded module $\text{gr } V \cong V$ is trivial. Therefore, the action of $\text{gr } U(\mathfrak{g})$ on $\text{gr}((Q_{\mathfrak{t}} \otimes_{W_{\mathfrak{t}}} M) \otimes V)$ is given by

$$a(u \otimes m \otimes v) = au \otimes m \otimes v$$

for $a \in \text{gr } U(\mathfrak{g})$, $u \in \text{gr } Q_{\mathfrak{t}}$, $m \in \text{gr } M$ and $v \in V$. Using the fact that $\text{gr } U(\mathfrak{g})$ and $\text{gr } W_{\mathfrak{t}}$ are commutative, we see that the action of $\text{gr } W_{\mathfrak{t}}$ on $H^0(\mathfrak{n}_{\mathfrak{t}}, (\text{gr } Q_{\mathfrak{t}} \otimes_{\text{gr } W_{\mathfrak{t}}} \text{gr } M) \otimes V)$ is determined by restricting the formula

$$a(u \otimes m \otimes v) = u \otimes am \otimes v,$$

for $a \in \text{gr } W_{\mathfrak{t}}$, $u \in \text{gr } Q_{\mathfrak{t}}$, $m \in \text{gr } M$ and $v \in V$. From these expressions it is straightforward to see that $\bar{\epsilon}$ does indeed restrict to the required isomorphism of $\text{gr } W_{\mathfrak{t}}$ -modules. \square

We now translate Theorem 5.1 into the setting of translations of $U(\mathfrak{g}, e)$ -modules through Lemma 4.5. The discussion below is analogous to that in [BGK, §3.2].

First we explain the commutative diagram below, which regards the translation $M \otimes V$ for the case where $M = U(\mathfrak{g}, e)$ is the regular module.

$$(5.5) \quad \begin{array}{ccccc} (S(\tilde{\mathfrak{p}}) \otimes V)^{\mathfrak{n}} & \xrightarrow{\sim} & S(\tilde{\mathfrak{p}}) \otimes V & \xrightarrow{\sim} & \mathbb{C}[e + \mathfrak{m}^{\perp}] \otimes V & \xleftarrow{\sim} & (\mathbb{C}[e + \mathfrak{m}^{\perp}] \otimes V)^N \\ & \searrow \sim & \downarrow \zeta_V & & \downarrow \text{res} & & \swarrow \sim \\ & & S(\mathfrak{g}^e) \otimes V & \xrightarrow{\sim} & \mathbb{C}[S] \otimes V & & \end{array}$$

We identify the $U(\tilde{\mathfrak{g}})$ -module $(\tilde{Q} \otimes_{U(\mathfrak{g}, e)} U(\mathfrak{g}, e)) \otimes V$ with $U(\tilde{\mathfrak{p}}) \otimes V$, and $U(\mathfrak{g}, e) \otimes V$ with the subspace of \mathfrak{n} -invariants for the dot action. We have a Kazhdan filtration on $U(\tilde{\mathfrak{p}}) \otimes V$, where as before V is graded, and so filtered, by the c -eigenspace decomposition. The associated graded module is isomorphic to $S(\tilde{\mathfrak{p}}) \otimes V$. The dot action from (3.13) is filtered and so gives an action of \mathfrak{n} on $S(\tilde{\mathfrak{p}}) \otimes V$. The map of the left of the diagram is the inclusion of the invariants for this action $(S(\tilde{\mathfrak{p}}) \otimes V)^{\mathfrak{n}} = H^0(\mathfrak{n}, S(\tilde{\mathfrak{p}}) \otimes V)$.

From Lemma 3.3, we obtain an isomorphism $U(\tilde{\mathfrak{p}}) \otimes V \xrightarrow{\sim} Q \otimes V$, which gives the isomorphism $S(\tilde{\mathfrak{p}}) \otimes V \xrightarrow{\sim} \mathbb{C}[e + \mathfrak{m}^{\perp}] \otimes V$ in the diagram through the identification

$\text{gr } Q \cong \mathbb{C}[e + \mathfrak{m}^\perp]$. This isomorphism can also be described as follows: we can view $\mathbb{C}[e + \mathfrak{m}^\perp] \otimes V$ as the space of regular functions from $e + \mathfrak{m}^\perp$ to V ; then $x \otimes v \in \mathfrak{p} \otimes V$ is sent to the function $z \mapsto (x|z)v$ and $y^{\text{ne}} \otimes v \in \mathfrak{k}^{\text{ne}} \otimes V$ to the function $z \mapsto (y|z)v$.

The inclusion of the N -invariants (equivalently \mathfrak{n} -invariants) in $\mathbb{C}[e + \mathfrak{m}^\perp] \otimes V$ is on the right of the diagram. From the proof of Theorem 5.1, it follows that there is an isomorphism $\text{gr } H^0(\mathfrak{n}, Q \otimes V) \cong (\mathbb{C}[e + \mathfrak{m}^\perp] \otimes V)^N$. Thus through Lemma 4.5 we obtain an isomorphism $\text{gr}(U(\mathfrak{g}, e) \otimes V) \cong (S(\tilde{\mathfrak{p}}) \otimes V)^n$.

We now consider the vertical maps. The map $\zeta_V = \zeta \otimes \text{id}_V$ involves the map $\zeta : S(\tilde{\mathfrak{p}}) \rightarrow S(\mathfrak{g}^e)$ from (3.5). The other vertical map is the restriction map. This square commutes, because $(x|z) = 0$ for any $x \in \mathfrak{r}$ and $z \in e + \mathfrak{g}^f$; see the discussion in [BGK, §3.2].

The diagonal map on the right is an isomorphism, because the restriction map identifies with the map $\epsilon : \mathbb{C}[N] \otimes \mathbb{C}[S] \otimes V \rightarrow \mathbb{C}[S] \otimes V$ from (5.2). Hence, also the diagonal map on the left is an isomorphism.

The upshot of this commutative diagram is that the restriction of $\zeta \otimes \text{id}_V$ gives rise to an isomorphism η_V recorded in the following proposition. This isomorphism is obtained as the composition of the diagonal isomorphism in (5.5) with $\zeta^{-1} \otimes \text{id}_V$, where ζ^{-1} means the inverse of $\zeta : S(\tilde{\mathfrak{p}})^n \rightarrow S(\mathfrak{g}^e)$. The formula given in the proposition is a consequence of Lemma 3.9, in the statement we use η from (3.8).

Proposition 5.6. *The map $\eta_V : S(\tilde{\mathfrak{p}}) \otimes V \rightarrow S(\tilde{\mathfrak{p}})^n \otimes V$, defined by*

$$\eta_V(u \otimes v) = \eta(u) \otimes v$$

for $u \in S(\tilde{\mathfrak{p}})$ and $v \in V$, restricts to an isomorphism of vector spaces

$$(5.7) \quad \eta_V : \text{gr}(U(\mathfrak{g}, e) \otimes V) \xrightarrow{\sim} \text{gr } U(\mathfrak{g}, e) \otimes V.$$

We now consider the situation for any $U(\mathfrak{g}, e)$ -module M . There is an analogue of the commutative diagram (5.5), where the triangle on the left is

$$(5.8) \quad \begin{array}{ccc} ((S(\tilde{\mathfrak{p}}) \otimes_{S(\tilde{\mathfrak{p}})^n} \text{gr } M) \otimes V)^n & \xrightarrow{\quad} & (S(\tilde{\mathfrak{p}}) \otimes_{S(\tilde{\mathfrak{p}})^n} \text{gr } M) \otimes V \\ & \searrow \sim & \downarrow \zeta_{M,V} \\ & & (S(\mathfrak{g}^e) \otimes_{S(\mathfrak{g}^e)} \text{gr } M) \otimes V. \end{array}$$

Identifying $\text{gr } \tilde{Q}$ with $S(\tilde{\mathfrak{p}})$ we have

$$\text{gr}(M \otimes V) \cong ((S(\tilde{\mathfrak{p}}) \otimes_{S(\tilde{\mathfrak{p}})^n} \text{gr } M) \otimes V)^n$$

is the module in the top left of the diagram. The horizontal map is simply the inclusion of invariants. We have that $\text{gr } M$ is a module for $\text{gr } U(\mathfrak{g}, e) \cong S(\mathfrak{p})^n$, and we may also view $\text{gr } M$ as an $S(\mathfrak{g}^e)$ -module through ζ from (3.5). Thus, we can make sense of the module at the bottom of the diagram. The vertical map $\zeta_{M,V}$ is defined by

$$\zeta_{M,V}(u \otimes m \otimes v) = \zeta(u) \otimes m \otimes v$$

for $u \in S(\tilde{\mathfrak{p}})$, $m \in \text{gr } M$ and $v \in V$. Analogous arguments to the case $M = U(\mathfrak{g}, e)$ show that the diagonal map is an isomorphism.

There is the obvious isomorphism

$$(5.9) \quad (S(\mathfrak{g}^e) \otimes_{S(\mathfrak{g}^e)} \text{gr } M) \otimes V \xrightarrow{\sim} \text{gr } M \otimes V.$$

Thanks to Lemma 3.9, this is the same as the isomorphism given by $u \otimes m \otimes v \mapsto \zeta^{-1}(u)m \otimes v$ for $u \in S(\mathfrak{g}^e)$, $m \in \text{gr } M$ and $v \in V$, where on the right-hand side

we are viewing $\text{gr } M$ as a $S(\tilde{\mathfrak{p}})^n$ -module. In turn, this means that the composition of the diagonal isomorphism in (5.8) with the isomorphism in (5.9) is given by restricting the map

$$u \otimes m \otimes v \mapsto \eta(u)m \otimes v$$

to $\text{gr}(M \otimes V)$, for $u \in S(\tilde{\mathfrak{p}})$, $m \in \text{gr } M$ and $v \in V$, where η is defined in (3.8).

The above discussion is summarized in the following theorem.

Theorem 5.10. *The map $\eta_{M,V} : (S(\tilde{\mathfrak{p}}) \otimes_{S(\tilde{\mathfrak{p}})^n} \text{gr } M) \otimes V \rightarrow \text{gr } M \otimes V$, defined by*

$$\eta_{M,V}(u \otimes m \otimes v) = \eta(u)m \otimes v$$

for $u \in S(\tilde{\mathfrak{p}})$, $m \in \text{gr } M$ and $v \in V$, restricts to an isomorphism

$$(5.11) \quad \eta_{M,V} : \text{gr}(M \otimes V) \xrightarrow{\sim} M \otimes V.$$

5.2. Explicit isomorphisms and lift matrices. Throughout this subsection, M is a $U(\mathfrak{g}, e)$ -module and V is a finite dimensional $U(\mathfrak{g})$ -module. From Theorem 5.10, it follows that there is an isomorphism of vector spaces $M \otimes V \cong M \otimes V$. However, there is not in general a canonical isomorphism, in this subsection we consider explicit isomorphisms. In Remark 5.19, we discuss all possible isomorphisms satisfying certain technical conditions. Our approach is based on [BK2, Thm. 8.1], which in turn is attributed as a reformulation of [Ly, Thm. 4.2]. First we need to introduce some notation.

We choose an ordered basis $\mathbf{v} = (v_1, \dots, v_n)$ of V , consisting of \mathfrak{t} -weight vectors. We let c_i be the eigenvalue of c on v_i (we recall the good grading is given by the ad c -eigenspace decomposition), and assume that \mathbf{v} is ordered so that $c_1 \geq \dots \geq c_n$. The coefficient functions $b_{ij} \in U(\mathfrak{g})^*$ of V are defined by $uv_j = \sum_{i=1}^n b_{ij}(u)v_i$. We say that an $n \times n$ -matrix $\mathbf{z} = (z_{ij})$ is V -block lower unitriangular if $z_{ij} = \delta_{ij}$ whenever $c_i \geq c_j$.

Let $(x_1^{\text{ne}}, \dots, (x_{2s}^{\text{ne}}, x_{2s+1}, \dots, x_r$ be the basis of $\mathfrak{r} = \bigoplus_{j \geq 2} [f, \mathfrak{g}(j)] \oplus \mathfrak{k}^{\text{ne}}$ introduced after Lemma 3.6. The right $U(\mathfrak{g}, e)$ -module homomorphism $\varphi : \tilde{Q} \rightarrow U(\mathfrak{g}, e)$ is defined in (3.7). We define $\varphi_{M,V} : (\tilde{Q} \otimes_{U(\mathfrak{g}, e)} M) \otimes V \rightarrow M \otimes V$ by

$$(5.12) \quad \varphi_{M,V}(u\tilde{\mathbb{1}} \otimes m \otimes v) = \varphi(u\tilde{\mathbb{1}})m \otimes v.$$

The following theorem gives the desired vector space isomorphism between $M \otimes V$ and $M \otimes V$.

Theorem 5.13. *The restriction of $\varphi_{M,V}$ to $M \otimes V$ is an isomorphism of vector spaces*

$$\varphi_{M,V} : M \otimes V \rightarrow M \otimes V.$$

Moreover, φ is natural in both M and V .

Proof. First we consider the case $M = U(\mathfrak{g}, e)$ is the regular module. In this case we identify $(\tilde{Q} \otimes_{U(\mathfrak{g}, e)} U(\mathfrak{g}, e)) \otimes V$ with $\tilde{Q} \otimes V$, and write $\varphi_V = \varphi_{M,V}$. Under this identification we have $\varphi_V(u\tilde{\mathbb{1}} \otimes v) = \varphi(u\tilde{\mathbb{1}}) \otimes v$ for $u \in U(\tilde{\mathfrak{p}})$ and $v \in V$. Thus the associated graded map $\text{gr } \varphi_V : S(\tilde{\mathfrak{p}}) \otimes V \rightarrow \text{gr}(U(\mathfrak{g}, e) \otimes V)$ is the same as η_V from (5.7). Thus by Proposition 5.6, we have that $\text{gr } \varphi_V$ is an isomorphism. A standard filtration argument now tells us that φ_V is an isomorphism.

The case for general M is similar. We follow the same arguments identifying $\text{gr } \varphi_{M,V}$ with $\eta_{M,V}$ from (5.11) then appeal to Theorem 5.10.

It is clear that φ is natural in both M and V . □

The following lemma is similar to parts of [BK2, Thm. 8.1], though the proof here is different. We recall that $\text{Pr} : U(\tilde{\mathfrak{g}}) \rightarrow U(\tilde{\mathfrak{p}})$ is the projection along the direct sum decomposition $U(\tilde{\mathfrak{g}}) = U(\tilde{\mathfrak{p}}) \oplus \tilde{I}$.

Lemma 5.14. *Let $\mathbf{x}^0 = (x_{ij}^0)$ be an $n \times n$ matrix with entries in $U(\tilde{\mathfrak{p}})$ satisfying the conditions:*

- (i) $\varphi(x_{ij}^0) = \delta_{ij}$, and
- (ii) $\text{Pr}([x - x^{\text{ne}}, x_{ij}^0]) + \sum_{k=1}^n b_{ik}(x)x_{kj}^0 \in \tilde{I}$, for all $x \in \mathfrak{n}$.

Then the inverse to $\varphi_{M,V} : M \otimes V \rightarrow M \otimes V$ is the map $\psi_{\mathbf{x}^0, M, \mathbf{v}} : M \otimes V \rightarrow M \otimes V$ defined by $\psi_{\mathbf{x}^0, M, \mathbf{v}}(m \otimes v_j) = \sum_{i=1}^n x_{ij}^0 \otimes m \tilde{\mathbb{1}} \otimes v_i$, for $m \in M$.

Moreover, \mathbf{x}^0 is uniquely determined by conditions (i) and (ii) and is V -block lower unitriangular.

Proof. We first consider the case $M = U(\mathfrak{g}, e)$ and identify $(\tilde{Q} \otimes_{U(\mathfrak{g}, e)} U(\mathfrak{g}, e)) \otimes V$ with $\tilde{Q} \otimes V$. We write $\psi_{\mathbf{x}^0, \mathbf{v}}$ for $\psi_{\mathbf{x}^0, M, \mathbf{v}}$ in this case.

It is clear that the inverse of φ_V must have the form $\psi_{\mathbf{x}^0, \mathbf{v}}$ for some matrix \mathbf{x}^0 with entries in $U(\tilde{\mathfrak{p}})$. In particular, $\psi_{\mathbf{x}^0, V}(1 \otimes v_j) = \sum_{i=1}^n x_{ij}^0 \tilde{\mathbb{1}} \otimes v_i$. This forces $\varphi(x_{ij}^0) = \delta_{ij}$ giving (i). Also, $\sum_{i=1}^n x_{ij}^0 \tilde{\mathbb{1}} \otimes v_i$ must lie in $M \otimes V = H^0(\mathfrak{n}, (\tilde{Q} \otimes_{U(\mathfrak{g}, e)} U(\mathfrak{g}, e)) \otimes V)$, which forces (ii) to hold.

Conversely, one can check that conditions (i) and (ii) imply that $\psi_{\mathbf{x}^0, \mathbf{v}}$ is inverse to φ_V , so that they uniquely determine \mathbf{x}^0 . To see that \mathbf{x}^0 is V -block lower unitriangular, one observes that conditions (i) and (ii) would still hold if we replaced x_{ij}^0 with δ_{ij} when $c_i \geq c_j$.

For general M , it is immediate from (i) and (ii) that $\varphi_{M,V} \psi_{\mathbf{x}^0, M, \mathbf{v}} = \text{id}_{M \otimes V}$. Therefore, $\psi_{\mathbf{x}^0, M, \mathbf{v}}$ is inverse to $\varphi_{M,V}$, and the proof is complete. \square

The matrix \mathbf{x}^0 in Lemma 5.14 leads us to the following definition of lift matrices, which is key to a number of results in the sequel.

Definition 5.15. Let $\mathbf{x} = (x_{ij})$ be a V -block lower unitriangular matrix with entries in $U(\tilde{\mathfrak{p}})$ satisfying

$$(5.16) \quad \text{Pr}([x - x^{\text{ne}}, x_{ij}]) + \sum_{k=1}^n b_{ik}(x)x_{kj} = 0$$

for all $x \in \mathfrak{n}$, and $x_{ij} \in F_{c_j - c_i} U(\tilde{\mathfrak{p}})$. Then we call \mathbf{x} a *lift matrix* for the basis \mathbf{v} of V .

We note, in particular, that \mathbf{x}^0 is a lift matrix for \mathbf{v} , the condition on the filtered degree of entries to \mathbf{x}^0 holds because $\varphi_{M,V}$ is clearly a filtered map. Our next proposition shows that all lift matrices give rise to a vector space isomorphism $M \otimes V \cong M \otimes V$. It is proved by adapting arguments from the proof of [BK2, Thm. 8.1].

Proposition 5.17. *Let \mathbf{x}^0 be as in Lemma 5.14.*

- (a) *Let \mathbf{x} be a lift matrix for \mathbf{v} .*
 - (i) *The map $\psi_{\mathbf{x}, M, \mathbf{v}} : M \otimes V \rightarrow M \otimes V$ defined by*

$$\psi_{\mathbf{x}, M, \mathbf{v}}(m \otimes v_j) = \sum_{i=1}^n x_{ij} \tilde{\mathbb{1}} \otimes m \otimes v_i$$

is a vector space isomorphism.

- (ii) We have $\mathbf{x} = \mathbf{x}^0 \mathbf{w}^0$, where \mathbf{w}^0 is a V -block lower unitriangular matrix with entries in $U(\mathfrak{g}, e)$
- (b) Suppose \mathbf{x} is of the form $\mathbf{x} = \mathbf{x}^0 \mathbf{w}^0$, where $\mathbf{w}^0 = (w_{ij}^0)$ is a V -block lower unitriangular matrix with $w_{ij}^0 \in F_{c_j - c_i} U(\mathfrak{g}, e)$. Then \mathbf{x} is a lift matrix for \mathbf{v} .
- (c) Let \mathbf{x} be a lift matrix and \mathbf{y} an $n \times n$ matrix with entries in $U(\tilde{\mathfrak{p}})$. Then \mathbf{y} is a lift matrix if and only if there is a V -block lower unitriangular matrix $\mathbf{w} = (w_{ij})$ with $w_{ij} \in F_{c_j - c_i} U(\mathfrak{g}, e)$, such that $\mathbf{x} = \mathbf{y} \mathbf{w}$.

Proof. Let \mathbf{x} be a lift matrix for \mathbf{v} . Consider the case $M = U(\mathfrak{g}, e)$, and as usual identify $(\tilde{Q} \otimes_{U(\mathfrak{g}, e)} U(\mathfrak{g}, e)) \otimes V$ with $\tilde{Q} \otimes V$. Since \mathbf{x} satisfies (5.16), we have that $\sum_{i=1}^n x_{ij} \tilde{\mathbb{1}} \otimes v_i \in U(\mathfrak{g}, e) \otimes V$. It follows from Lemma 5.14 that there exist $w_{kj}^0 \in U(\mathfrak{g}, e)$ such that

$$\sum_{i=1}^n x_{ij} \otimes v_i = \sum_{i,k=1}^n x_{i,k}^0 w_{k,j}^0 \otimes v_i.$$

Equating coefficients gives $\mathbf{x} = \mathbf{x}^0 \mathbf{w}^0$, where $\mathbf{w}^0 = (w_{ij}^0)$. Since \mathbf{x} and \mathbf{x}^0 are V -block lower unitriangular, so is \mathbf{w}^0 . This proves (a)(ii).

Now consider general M . From the factorization $\mathbf{x} = \mathbf{x}^0 \mathbf{w}^0$, we see that $\psi_{\mathbf{x}, M, \mathbf{v}}$ is the composition of the map $M \otimes V \rightarrow M \otimes V$ given by

$$(5.18) \quad m \otimes v_j \mapsto \sum_{i=1}^n w_{ij}^0 m \otimes v_i$$

with $\psi_{\mathbf{x}^0, M, \mathbf{v}}$. The map in (5.18) is an isomorphism as \mathbf{w} is V -block lower unitriangular and thus invertible, and $\psi_{\mathbf{x}^0, M, \mathbf{v}}$ is an isomorphism by Lemma 5.14. Therefore, $\psi_{\mathbf{x}, M, \mathbf{v}}$ is an isomorphism proving (a)(i).

One can check via a straightforward calculation that if \mathbf{x} is of the form given in (b), then it satisfies the conditions to be a lift matrix in Definition 5.15. Then (c) is a consequence of (a)(ii) and (b); it is straightforward to check that the condition on filtered degrees holds. \square

Remark 5.19. We now describe all the isomorphisms $\psi : U(\mathfrak{g}, e) \otimes V \rightarrow U(\mathfrak{g}, e) \otimes V$ of vector spaces that are filtered with respect to the Kazhdan filtration. Let ψ be such an isomorphism and identify $(\tilde{Q} \otimes_{U(\mathfrak{g}, e)} U(\mathfrak{g}, e)) \otimes V$ with $\tilde{Q} \otimes V$. We have $\psi(\tilde{\mathbb{1}} \otimes v_j) = \sum_{c_i = c_j} a_{ij} \tilde{\mathbb{1}} \otimes v_i + \sum_{c_i < c_j} z_{ij} \tilde{\mathbb{1}} \otimes v_i$, where $a_{ij} \in \mathbb{C}$ and $z_{ij} \in U(\tilde{\mathfrak{p}})$. The isomorphisms in Proposition 5.17 are precisely those for which $a_{ij} = \delta_{ij}$. Thus any such filtered isomorphism ψ can be factored as the composition of an isomorphism $U(\mathfrak{g}, e) \otimes V \xrightarrow{\sim} U(\mathfrak{g}, e) \otimes V$ of the form $u \otimes v_j \mapsto \sum_{c_i = c_j} a_{ij} u \otimes v_i$, with an isomorphism $\psi_{\mathbf{x}, \mathbf{v}}$ from Proposition 5.17.

From such an isomorphism ψ we obtain a filtered isomorphism $\psi_M : M \otimes V \rightarrow M \otimes V$ for a $U(\mathfrak{g}, e)$ -module M , by setting $\psi_M(m \otimes V) = \sum_{c_i = c_j} \tilde{\mathbb{1}} \otimes a_{ij} m \otimes v_i + \sum_{c_i < c_j} z_{ij} \tilde{\mathbb{1}} \otimes m \otimes v_i$. However, there can in general be many other isomorphisms not of this form, for example, if M is decomposable.

Remark 5.20. Another characterization of isomorphisms from Proposition 5.17(a)(i) is given in terms of the loop filtration discussed in the next subsection. Note that $\psi : U(\mathfrak{g}, e) \otimes V \xrightarrow{\sim} U(\mathfrak{g}, e) \otimes V$ being filtered with respect to the Kazhdan filtration implies that it is filtered with respect to the loop filtration. Then the $\psi_{\mathbf{x}, \mathbf{v}}$ are

precisely those ψ such that the associated graded map with respect to the loop filtration identifies with the identity map through Proposition 5.21.

5.3. The loop filtration. In Proposition 5.21 below, we prove a compatibility result regarding the loop filtration and translation. Throughout this subsection let M be a $U(\mathfrak{g}, e)$ -module generated by the finite dimensional subspace M_0 , and let V be a finite dimensional $U(\mathfrak{g})$ -module with ordered basis \mathbf{v} as in §5.2.

Using the loop filtration $(F'_j U(\mathfrak{g}, e))_{j \in \mathbb{Z}_{\geq 0}}$ of $U(\mathfrak{g}, e)$ from §3.2 we can define a loop filtration on M by setting $F'_j M = (F'_j U(\mathfrak{g}, e))M_0$. In this way M becomes a filtered $U(\mathfrak{g}, e)$ -module, so the associated graded module $\text{gr}' M$ is a module for $\text{gr}' U(\mathfrak{g}, e) \cong U(\mathfrak{g}^e)$. Also, V is a $U(\mathfrak{g}^e)$ -module by restriction, so $(\text{gr}' M) \otimes V$ has the structure of a $U(\mathfrak{g}^e)$ -module. There is a loop filtration on \tilde{Q} through the identification of $\tilde{Q} \cong U(\tilde{\mathfrak{p}})$. As before, the c -eigenspace decomposition gives a grading and thus also a filtration of V . Putting this all together we obtain loop filtrations of $(\tilde{Q} \otimes_{U(\mathfrak{g}, e)} M) \otimes V$ and $M \otimes V$. It is easy to check that $M \otimes V$ is a filtered $U(\mathfrak{g}, e)$ -module; therefore, the associated graded module $\text{gr}'(M \otimes V)$ is a module for $\text{gr}' U(\mathfrak{g}, e) \cong U(\mathfrak{g}^e)$.

We can now state and prove the compatibility result of this subsection.

Proposition 5.21. *There is an isomorphism $\text{gr}'(M \otimes V) \cong (\text{gr}' M) \otimes V$ of $U(\mathfrak{g}^e)$ -modules.*

Proof. We just consider the case $M = U(\mathfrak{g}, e)$, the general case can be dealt with similarly, and we leave the details to the reader. As usual we identify $(\tilde{Q} \otimes_{U(\mathfrak{g}, e)} U(\mathfrak{g}, e)) \otimes V$ with $\tilde{Q} \otimes V$.

Let \mathbf{x} be a lift matrix for \mathbf{v} , and let $\psi_{\mathbf{x}, \mathbf{v}} : U(\mathfrak{g}, e) \otimes V \xrightarrow{\sim} U(\mathfrak{g}, e) \otimes V$ be the corresponding isomorphism. Consider $\psi_{\mathbf{x}, \mathbf{v}}(1 \otimes v_j) = \sum_{i=1}^n x_{ij} \tilde{\mathbb{1}} \otimes v_i$. The condition on the filtered degree of entries \mathbf{x} in the Kazhdan filtration in Definition 5.15 means that each of the terms $x_{ij} \tilde{\mathbb{1}} \otimes v_i$ for $i \neq j$ is zero or has strictly lower degree than $\tilde{\mathbb{1}} \otimes v_j$ in the loop filtration. Recalling the isomorphism θ from (3.11), we deduce that $\theta \otimes \text{id}_V : U(\mathfrak{g}^e) \otimes V \rightarrow U(\mathfrak{p}) \otimes V$ maps isomorphically onto $\text{gr}'(U(\mathfrak{g}, e) \otimes V)$. It is clear that this isomorphism respects the $U(\mathfrak{g}^e)$ -module structure, so the proof is complete. \square

As a corollary we obtain the following result, which can be proved using a standard PBW basis argument.

Corollary 5.22. *Suppose $\text{gr}' M$ is generated by $\text{gr}' M_0$ as a $U(\mathfrak{g}^e)$ -module and let \mathbf{x} be a lift matrix for \mathbf{v} . Then $M \otimes V$ is generated by $\psi_{\mathbf{x}, M, \mathbf{v}}(M_0 \otimes V)$.*

6. BASIC PROPERTIES OF TRANSLATION

In this section we record some basic properties of translations; they are generalizations of results from [BK2, §8.2].

6.1. Tensor identity. Proposition 6.1 below is a generalization of [BK2, Cor. 8.2] from the case where $\mathfrak{g} = \mathfrak{gl}_n(\mathbb{C})$ (when there is an even good grading for e); it can be proved using the arguments in *loc. cit.* so we do not include a proof here. For the statement, note that we can view a $U(\tilde{\mathfrak{p}})$ -module M as a $U(\mathfrak{g}, e)$ -module by restriction. Therefore, $M \otimes V$ is defined as a $U(\mathfrak{g}, e)$ -module for a finite dimensional $U(\mathfrak{g})$ -module V . Also, $M \otimes V$ can be viewed as a $U(\tilde{\mathfrak{p}})$ -module through $\tilde{\Delta}$, and thus as a $U(\mathfrak{g}, e)$ -module by restriction.

Proposition 6.1. *Let M be a $U(\tilde{\mathfrak{p}})$ -module and V a finite dimensional $U(\mathfrak{g})$ -module. Then:*

(i) *The restriction of the map $(\tilde{Q} \otimes_{U(\mathfrak{g},e)} M) \otimes V \rightarrow M \otimes V$ defined by*

$$u\tilde{\mathbb{1}} \otimes m \otimes v \mapsto um \otimes v,$$

for $u \in U(\tilde{\mathfrak{p}})$, $m \in M$ and $v \in V$, defines a canonical natural isomorphism

$$\mu_{M,V} : M \otimes V \xrightarrow{\sim} M \otimes V.$$

(ii) *Let $\mathbf{v} = (v_1, \dots, v_n)$ be a basis of V as in §5.2, let \mathbf{x} be a lift matrix for \mathbf{v} and \mathbf{y} the inverse to \mathbf{x} . Then the inverse map to $\mu_{M,V}$ sends $m \otimes v_j$ to $\sum_{i,k=1}^n x_{ik}\tilde{\mathbb{1}} \otimes y_{kj}m \otimes v_i$, for $m \in M$.*

6.2. Associativity and adjunction. In the statement of Lemma 6.2, we use the isomorphisms of the form $\varphi_{M,V}$ from (5.12). The lemma can be proved in exactly the same way as in [BK2, §8.2]; so we omit the details.

Lemma 6.2. *Let M be a $U(\mathfrak{g}, e)$ -module and let V, V' be finite-dimensional $U(\mathfrak{g})$ -modules. Then the linear map*

$$(\tilde{Q} \otimes_{U(\mathfrak{g},e)} ((\tilde{Q} \otimes_{U(\mathfrak{g},e)} M) \otimes V)) \otimes V' \rightarrow (\tilde{Q} \otimes_{U(\mathfrak{g},e)} M) \otimes (V \otimes V')$$

given by

$$(u'\tilde{\mathbb{1}} \otimes ((u\tilde{\mathbb{1}} \otimes m) \otimes v)) \otimes v' \rightarrow (u'(u\tilde{\mathbb{1}} \otimes m) \otimes v) \otimes v',$$

for $u', u \in U(\tilde{\mathfrak{p}})$, $m \in M$, $v \in V$ and $v' \in V'$, restricts to a natural isomorphism

$$a_{M,V,V'} : (M \otimes V) \otimes V' \xrightarrow{\sim} M \otimes (V \otimes V').$$

Moreover, the following diagram commutes:

$$\begin{array}{ccc} (M \otimes V) \otimes V' & \xrightarrow{a_{M,V,V'}} & M \otimes (V \otimes V') \\ \varphi_{M \otimes V, V'} \downarrow & & \downarrow \varphi_{M, V \otimes V'} \\ (M \otimes V) \otimes V' & \xrightarrow{\varphi_{M, V} \otimes \text{id}_{V'}} & M \otimes V \otimes V' \end{array}$$

Next we transport the canonical adjunction as in [BK2, §8.2]. In order to do this, note that for the trivial $U(\mathfrak{g})$ -module \mathbb{C} , and a $U(\mathfrak{g}, e)$ -module M there is a natural map $M \otimes \mathbb{C} \rightarrow M$ determined by $i_M(\tilde{\mathbb{1}} \otimes m \otimes 1) = m$, for $m \in M$. Let v_1, \dots, v_n be a basis of V , and let v^1, \dots, v^n be the dual basis of V^* . Then the unit of the canonical adjunction is given by the composition

$$M \xrightarrow{i_M^{-1}} M \otimes \mathbb{C} \longrightarrow M \otimes (V \otimes V^*) \xrightarrow{a_{M,V,V^*}^{-1}} (M \otimes V) \otimes V^*,$$

where the second map is given by $(\tilde{\mathbb{1}} \otimes m) \otimes 1 \rightarrow (\tilde{\mathbb{1}} \otimes m) \otimes (\sum_{i=1}^n v_i \otimes v^i)$, for $m \in M$, and the third map is from Lemma 6.2. The counit is given by the composition

$$(M \otimes V^*) \otimes V \xrightarrow{a_{M,V^*,V}} M \otimes (V^* \otimes V) \longrightarrow M \otimes \mathbb{C} \xrightarrow{i_M} M,$$

where the second map is the restriction of $(u\tilde{\mathbb{1}} \otimes m) \otimes (f \otimes v) = (u\tilde{\mathbb{1}} \otimes m) \otimes \langle f, v \rangle$, for $u \in U(\tilde{\mathfrak{p}})$, $m \in M$, $f \in V^*$ and $v \in V$, where $\langle f, v \rangle$ denotes the natural pairing.

Summarizing the above discussion we obtain.

Lemma 6.3. *Let V be a finite dimensional $U(\mathfrak{g})$ -module and V^* the dual $U(\mathfrak{g})$ -module. Then $? \otimes V$ and $? \otimes V^*$ are biadjoint functors.*

7. TRANSLATIONS AND HIGHEST WEIGHT THEORY

In this section we consider the relationship between translations and the highest weight theory from [BGK, §4]. First in §7.1, we give a brief recollection of some definitions from highest weight theory. Then in §7.2, we recall the definition of the category $\mathcal{O}(e)$ of $U(\mathfrak{g}, e)$ -modules from [BGK, §4.4], and show that it is stable under translations; the category $\mathcal{O}(e)$ is an analogue of the usual BGG category \mathcal{O} of $U(\mathfrak{g})$ -modules. We consider translations of Verma modules for $U(\mathfrak{g}, e)$ in §7.3, and, in particular, show that the translation of a Verma module has a filtration by Verma modules, in Theorem 7.14.

7.1. Recollection on highest weight theory. The recollection below is as brief as possible for our purposes. Full details can be found in [BGK, §4].

The *restricted root system* $\Phi^e \subseteq (\mathfrak{t}^e)^*$ associated to e is defined by the \mathfrak{t}^e -weight space decomposition

$$\mathfrak{g} = \mathfrak{g}_0 \oplus \bigoplus_{\alpha \in \Phi^e} \mathfrak{g}_\alpha,$$

where $\mathfrak{g}_\alpha = \{x \in \mathfrak{g} \mid [t, x] = \alpha(t)x \text{ for all } t \in \mathfrak{t}^e\}$, i.e., Φ^e consists of the elements of Φ restricted to \mathfrak{t}^e . The reader is referred to [BG, §2 and §3] for information on restricted root systems.

We have $e \in \mathfrak{g}_0$ and the good grading on \mathfrak{g} gives a good grading of \mathfrak{g}_0 . Thus the finite W -algebra $U(\mathfrak{g}_0, e)$ is defined in analogy to $U(\mathfrak{g}, e)$. The good grading of \mathfrak{g}_0 for e must be even as e is distinguished in \mathfrak{g}_0 ; see for example [Ca, 5.7.6]. Therefore, $U(\mathfrak{g}_0, e) \subseteq U(\mathfrak{p}_0)$, where $\mathfrak{p}_0 = \mathfrak{p} \cap \mathfrak{g}_0$. The analogue $\theta_0 : U(\mathfrak{g}_0^e) \rightarrow U(\mathfrak{p}_0)$ of θ from (3.10) is simply the inclusion, so we can view $U(\mathfrak{g}_0^e(0))$ as a subalgebra of $U(\mathfrak{g}_0, e)$.

From θ we obtain an embedding $\mathfrak{t}^e \hookrightarrow U(\mathfrak{g}, e)$, which we use to identify \mathfrak{t}^e with a subalgebra of $U(\mathfrak{g}, e)$. Thus there is an adjoint action of \mathfrak{t}^e on $U(\mathfrak{g}, e)$ giving the restricted root space decomposition

$$U(\mathfrak{g}, e) = \bigoplus_{\alpha \in \mathbb{Z}\Phi^e} U(\mathfrak{g}, e)_\alpha$$

of $U(\mathfrak{g}, e)$, where $U(\mathfrak{g}, e)_\alpha = \{u \in U(\mathfrak{g}, e) \mid [t, u] = \alpha(t)u \text{ for all } t \in \mathfrak{t}^e\}$. We choose a system Φ_+^e of positive roots in the restricted root system Φ^e ; we recall from [BG, §2] that this is equivalent to choosing a parabolic subalgebra \mathfrak{q} of \mathfrak{g} with Levi subalgebra \mathfrak{g}_0 . This choice of positive roots gives rise to a partial order on $(\mathfrak{t}^e)^*$ in the usual way, i.e., $\alpha \leq \beta$ if and only if $\beta - \alpha \in \mathbb{Z}_{\geq 0}\Phi_+^e$.

We define $U(\mathfrak{g}, e)_\#$ to be the left ideal of $U(\mathfrak{g}, e)$ generated by $U(\mathfrak{g}, e)_\alpha$ for $\alpha \in \Phi_+^e$. Then by [BGK, Thm. 4.3], $U(\mathfrak{g}, e)_{0, \#} = U(\mathfrak{g}, e)_0 \cap U(\mathfrak{g}, e)_\#$ is a two-sided ideal of $U(\mathfrak{g}, e)_0$, and the quotient $U(\mathfrak{g}, e)_0/U(\mathfrak{g}, e)_{0, \#}$ is isomorphic to $U(\mathfrak{g}_0, e)$. Next we explain this isomorphism explicitly. We define $U(\tilde{\mathfrak{p}})_0$ and $U(\tilde{\mathfrak{p}})_{0, \#}$ in analogy to $U(\mathfrak{g}, e)_0$ and $U(\mathfrak{g}, e)_{0, \#}$. We have $U(\tilde{\mathfrak{p}})_0 = U(\mathfrak{p}_0) \oplus U(\tilde{\mathfrak{p}})_{0, \#}$. Thus we may define the projection $\pi : U(\tilde{\mathfrak{p}})_0 \twoheadrightarrow U(\mathfrak{p}_0)$ along this decomposition. Recall that b_1, \dots, b_r is a basis for \mathfrak{n} with $b_i \in \mathfrak{g}(-d_i)$ of weight $\beta_i \in \Phi$. By [BGK, Lem. 4.1]

$$\gamma = \sum_{\substack{1 \leq i \leq r \\ \beta_i|_{\mathfrak{t}^e} \in \Phi_-^e}} \beta_i$$

is a character of \mathfrak{p}_0 , so we can define the shift $S_{-\gamma} : U(\mathfrak{p}_0) \rightarrow U(\mathfrak{p}_0)$ by $S_{-\gamma}(x) = x - \gamma(x)$. Now [BGK, Thm. 4.3] says that the composition

$$(7.1) \quad U(\mathfrak{g}, e)_0 \hookrightarrow U(\tilde{\mathfrak{p}})_0 \xrightarrow{\pi} U(\mathfrak{p}_0) \xrightarrow{S_\gamma} U(\mathfrak{p}_0)$$

has image equal to $U(\mathfrak{g}_0, e)$ and kernel equal to $U(\mathfrak{g}, e)_{0, \sharp}$; giving the desired isomorphism $U(\mathfrak{g}, e)_0/U(\mathfrak{g}, e)_{0, \sharp} \xrightarrow{\sim} U(\mathfrak{g}_0, e)$.

Let L be a finite dimensional $U(\mathfrak{g}_0, e)$ -module. Define the induced $U(\mathfrak{g}, e)$ -module

$$M(L) = U(\mathfrak{g}, e)/U(\mathfrak{g}, e)_{\sharp} \otimes_{U(\mathfrak{g}_0, e)} L,$$

where $U(\mathfrak{g}, e)/U(\mathfrak{g}, e)_{\sharp}$ is viewed as a right $U(\mathfrak{g}_0, e)$ -module via the isomorphism $U(\mathfrak{g}_0, e) \cong U(\mathfrak{g}, e)_0/U(\mathfrak{g}, e)_{0, \sharp}$. We call $M(L)$ the *quasi-Verma module* of type L . In case L is irreducible, $M(L)$ is a Verma module as defined in [BGK, §4.2]; we consider the more general situation of quasi-Verma modules in §7.3 below.

Note that we have the natural inclusion $L \hookrightarrow M(L)$ allowing us to view $L \subseteq M(L)$. Also, if L' is a $U(\mathfrak{g}_0, e)$ -submodule of L , then $M(L')$ is a $U(\mathfrak{g}, e)$ -submodule of $M(L)$; and we have an isomorphism $M(L/L') \cong M(L)/M(L')$. Thus we obtain the following elementary lemma.

Lemma 7.2. *Let L_1, \dots, L_m be the composition factors of L . Then the quasi-Verma module $M(L)$ has a filtration with quotients isomorphic to $M(L_i)$ for $i = 1, \dots, m$.*

Let M be a $U(\mathfrak{g}, e)$ -module. Given a weight $\lambda \in (\mathfrak{t}^e)^*$ we define the λ -weight space of M to be $M_\lambda = \{m \in M \mid tm = \lambda(t)m \text{ for all } t \in \mathfrak{t}^e\}$. We note that this labelling of weight spaces differs from that in [BGK, §4.2], where the labelling is shifted by

$$(7.3) \quad \delta = \sum_{\substack{1 \leq i \leq r \\ \beta_i |_{\mathfrak{t}^e} \in \Phi_-^e \\ d_i \geq 2}} \beta_i + \frac{1}{2} \sum_{\substack{1 \leq i \leq r \\ \beta_i |_{\mathfrak{t}^e} \in \Phi_-^e \\ d_i = 1}} \beta_i \in (\mathfrak{t}^e)^*.$$

This shift by δ is due to the fact that inclusion of \mathfrak{t}^e into $U(\mathfrak{g}_0, e)$, and the embedding obtained as the composition θ with the map in (7.1) differ.

We say that a weight space M_λ of M is a *maximal weight space* if $M_\mu = \{0\}$ for all $\mu > \lambda$. In this case, we have $U(\mathfrak{g}, e)_{\sharp} M_\lambda = 0$, so that we obtain an action of $U(\mathfrak{g}_0, e) \cong U(\mathfrak{g}, e)_0/U(\mathfrak{g}, e)_{0, \sharp}$ on M_λ . Suppose M_λ is finite dimensional and consider the induced module $M(M_\lambda)$. The universal property for quasi-Verma modules tells us that there is a unique homomorphism

$$(7.4) \quad M(M_\lambda) \rightarrow M$$

sending $M_\lambda \subseteq M(M_\lambda)$ identically to $M_\lambda \subseteq M$; see [BGK, Thm. 4.5(3)].

The main result in [Go] is a compatibility result between Verma modules and the loop filtration. We require this result in §7.3, so we recall it now. There is a loop filtration of $U(\mathfrak{g}_0, e)$ such that the associated graded algebra $\text{gr}' U(\mathfrak{g}_0, e)$ is naturally isomorphic to $U(\mathfrak{g}_0^e)$. Let L be a finite dimensional $U(\mathfrak{g}_0, e)$ -module. We endow L with the trivial filtration concentrated in degree 0, then L is a filtered module for $U(\mathfrak{g}_0, e)$ and the associated graded module $\text{gr}' L$ is a $U(\mathfrak{g}_0^e)$ -module. Note that $x \in \mathfrak{g}_0^e(j)$ acts as zero on $\text{gr}' L$ for $j > 0$. Thus $\text{gr}' L$ is just the restriction of L to $U(\mathfrak{g}_0^e(0))$. For technical reasons, explained in [Go, §3], we have to consider the *shift* $S_\delta(\text{gr}' L)$ of $\text{gr}' L$, where δ is as in (7.3). We define $S_\delta(\text{gr}' L)$ to be equal to

$\mathrm{gr}' L$ as a vector space, and the action of $x \in \mathfrak{g}_0^e$ is given by “ $xv = S_\delta(x)v$ ”, where $S_\delta : U(\mathfrak{p}_0) \rightarrow U(\mathfrak{p}_0)$ is defined in analogy to $S_{-\gamma}$ above; note that δ is a character of \mathfrak{p}_0 by [BGK, Lem. 4.1].

The choice of positive roots Φ_+^e in Φ^e leads to the triangular decomposition $\mathfrak{g}^e = \mathfrak{g}_-^e \oplus \mathfrak{g}_0^e \oplus \mathfrak{g}_+^e$. Therefore, we can define the *quasi-Verma module*

$$M(S_\delta(\mathrm{gr}' L)) = U(\mathfrak{g}^e) \otimes_{U(\mathfrak{g}_0^e \oplus \mathfrak{g}_+^e)} S_\delta(\mathrm{gr}' L)$$

for $U(\mathfrak{g}, e)$, where $S_\delta(\mathrm{gr}' L)$ is extended to a module for $U(\mathfrak{g}_0^e \oplus \mathfrak{g}_+^e)$ by letting \mathfrak{g}_+^e act trivially. We can define a filtration on $M = M(L)$ as in §5.3 with $M_0 = L$, and the associated graded module $\mathrm{gr}' M(L)$ is a module for $\mathrm{gr}' U(\mathfrak{g}, e) \cong U(\mathfrak{g}^e)$. By [Go, Thm. 3.1] we have an isomorphism

$$(7.5) \quad \xi_L : \mathrm{gr}' M(L) \cong M(S_\delta(\mathrm{gr}' L)).$$

We finish this section by giving a more explicit description of the quasi-Verma module $M(L)$, which is needed to make some identifications in the proof of Theorem 7.14 below. Let x_1, \dots, x_t be a basis of \mathfrak{g}^e as at the end of §3.3. So x_i has \mathfrak{t}^e -weight $\gamma_i \in \Phi^e$ for each $i = 1, \dots, t$. We assume that the basis is ordered so that $\gamma_1, \dots, \gamma_s \in \Phi_-^e = -\Phi_+^e$, $\gamma_{s+1}, \dots, \gamma_{s'} = 0$ and $\gamma_{s'+i} = -\gamma_i$ for $i = 1, \dots, s$. Recall the linear map Θ from (3.12). Let l_1, \dots, l_k be a basis of L , which can also be viewed as a basis for $\mathrm{gr}' L$. Then the PBW theorem for $U(\mathfrak{g}, e)$ explained at the end of §3.3 implies that the vectors

$$(7.6) \quad \Theta(x_1)^{a_1} \dots \Theta(x_s)^{a_s} \otimes l_i$$

for $a_1, \dots, a_s \in \mathbb{Z}_{\geq 0}$ and $i = 1, \dots, k$ form a basis of $M(L)$; see also [BGK, Thm. 4.5(1)]. The proof of [Go, Thm. 3.1], tells us that

$$(7.7) \quad \xi_L(\mathrm{gr}' \Theta(x_1)^{a_1} \dots \Theta(x_s)^{a_s} \otimes l_i) = x_1^{a_1} \dots x_s^{a_s} \otimes l_i$$

where ξ_L is as in (7.5).

7.2. The category $\mathcal{O}(e)$. As in [BGK, §4.4], we define $\mathcal{O}(e) = \mathcal{O}(e; \mathfrak{t}, \mathfrak{q})$ to be the category of all (finitely generated) $U(\mathfrak{g}, e)$ -modules M such that:

- (i) the action of \mathfrak{t}^e on M is semisimple with finite dimensional \mathfrak{t}^e -weight spaces; and
- (ii) the set $\{\lambda \in (\mathfrak{t}^e)^* \mid M_\lambda \neq 0\}$ is contained in a finite union of sets of the form $\{\nu \in (\mathfrak{t}^e)^* \mid \nu \leq \mu\}$ for $\mu \in (\mathfrak{t}^e)^*$.

As explained in [BGK, §4.4] the category $\mathcal{O}(e)$ depends (in an essential way) on the choice of positive roots Φ_+^e .

Let $M \in \mathcal{O}(e)$ and let V be a finite dimensional $U(\mathfrak{g})$ -module. In Proposition 7.11 below, we show that $M \otimes V \in \mathcal{O}(e)$, meaning that $\mathcal{O}(e)$ is stable under translations. It is a consequence of the following lemma regarding the isomorphism $\varphi_{M,V}$ from §5.2. For the lemma we note that there is an action of \mathfrak{t}^e on $M \otimes V$ from the embedding $\mathfrak{t}^e \hookrightarrow U(\mathfrak{g}, e)$ and on $M \otimes V$ through the embedding $\mathfrak{t}^e \hookrightarrow U(\mathfrak{g}, e)$ and the inclusion $\mathfrak{t}^e \hookrightarrow U(\mathfrak{g})$.

Lemma 7.8. *The isomorphism $\varphi_{M,V} : M \otimes V \rightarrow M \otimes V$ is \mathfrak{t}^e -equivariant.*

Proof. Let T be the maximal torus of G with Lie algebra \mathfrak{t} and let T^e be the centralizer of e in T , so $\mathrm{Lie} T^e = \mathfrak{t}^e$. By [Pr2, Lem. 2.4], the adjoint action of \mathfrak{t}^e on $U(\mathfrak{g}, e)$ coincides with the restriction of the differential of the adjoint action of T^e on $U(\hat{\mathfrak{p}})$. Let $\mathbf{v} = (v_1, \dots, v_n)$ be an ordered basis of V as in §5.2 and let $\alpha_i \in (\mathfrak{t}^e)^*$ be the \mathfrak{t}^e -weight of v_i ; we identify α_i with the corresponding character of T^e . We may

exponentiate the action of \mathfrak{t}^e on V to get an action of T^e on V (here we may need G to be simply connected, but this assumption does no harm). We write tv for the image of $v \in V$ under $t \in T^e$. Consider the lift matrix \mathbf{x}^0 from Lemma 5.14. For $t \in T^e$, set $t \cdot \mathbf{x}^0 = (t \cdot x_{ij}^0)$, where $t \cdot x_{ij}^0$ is the image of x_{ij}^0 under the adjoint action of t . Using Lemma 5.14, we see that the condition for $\varphi_{M,V}$ to be \mathfrak{t}^e -equivariant is equivalent to the condition $t \cdot \mathbf{x}^0 = (\alpha_i(t)^{-1} \alpha_j(t) x_{ij}^0)$ for all $t \in T^e$.

We define an action of T^e on $\tilde{Q} \otimes V$ by $t \cdot (u\tilde{\mathfrak{l}} \otimes v) = (t \cdot u)\tilde{\mathfrak{l}} \otimes tv$, for $t \in T^e$, $u \in U(\tilde{\mathfrak{p}})$ and $v \in V$; this action can also be seen by restricting the action of $U(\tilde{\mathfrak{g}})$ to \mathfrak{g} , noting that this action is locally finite so that we can exponentiate to an action of G , and then restricting to T^e . Now it is a straightforward calculation to check that

$$(7.9) \quad x \cdot (t \cdot (u\tilde{\mathfrak{l}} \otimes v)) = t \cdot ((t^{-1} \cdot x) \cdot (u\tilde{\mathfrak{l}} \otimes v))$$

for $t \in T^e$, $x \in \mathfrak{n}$, $u \in U(\tilde{\mathfrak{p}})$ and $v \in V$. We have $\sum_{i=1}^n x_{ij}^0 \otimes v_i \in U(\mathfrak{g}, e) \otimes V$ by Lemma 5.14. Now using (7.9), we see that $\sum_{i=1}^n t \cdot x_{ij}^0 \otimes \alpha_i(t)v_i \in U(\mathfrak{g}, e) \otimes V$ for $t \in T^e$. From this we see that the matrix $(\alpha_i(t)^{-1} \alpha_j(t) (t \cdot x_{ij}^0))$ satisfies conditions (i) and (ii) in Lemma 5.14. Now the uniqueness statement in that lemma completes the proof. \square

Remark 7.10. One can give an alternative proof to Lemma 7.8, by considering the map $\varphi_{M,V} : (\tilde{Q} \otimes_{U(\mathfrak{g},e)} M) \otimes V \rightarrow M \otimes V$ and showing that it is \mathfrak{t}^e -equivariant directly, where the action on the left-hand side is through the embedding $\theta : \mathfrak{t}^e \hookrightarrow U(\tilde{\mathfrak{g}})$. This requires use of Lemma 3.6.

The above lemma means that the \mathfrak{t}^e -weight spaces (shifted by δ) of $M \otimes V$ and $M \otimes V$ are identified via $\varphi_{M,V}$. It is clear that the weight spaces of $M \otimes V$ satisfy conditions (i) and (ii) in the definition of $\mathcal{O}(e)$, from which the proposition below follows.

Proposition 7.11. *We have that $M \otimes V \in \mathcal{O}(e)$, so $? \otimes V$ defines an exact endofunctor of $\mathcal{O}(e)$.*

7.3. Translation of Verma modules. In this subsection we consider translations of quasi-Verma modules for $U(\mathfrak{g}, e)$. Throughout, L is a finite dimensional $U(\mathfrak{g}_0, e)$ -module and V is a finite dimensional $U(\mathfrak{g})$ -module. Our main result is Theorem 7.14, which says that a translation of a quasi-Verma module is filtered by quasi-Verma modules; an immediate consequence is Corollary 7.15, giving the corresponding result for Verma modules.

Our first lemma of this subsection considers the associated graded side of translations of quasi-Verma modules.

Lemma 7.12. *There are isomorphisms*

$$\mathrm{gr}'(M(L) \otimes V) \cong \mathrm{gr}' M(L) \otimes V \cong M(S_\delta(\mathrm{gr}' L)) \otimes V \cong M(S_\delta(\mathrm{gr}' L) \otimes V).$$

So, in particular, there is an isomorphism

$$(7.13) \quad \sigma : \mathrm{gr}'(M(L) \otimes V) \xrightarrow{\sim} M(S_\delta(\mathrm{gr}' L) \otimes V).$$

Proof. The first isomorphism is given by Proposition 5.21. The second is an immediate consequence of (7.5).

The obvious homomorphism $S_\delta(\mathrm{gr}' L) \otimes V \rightarrow M(S_\delta(\mathrm{gr}' L)) \otimes V$ of $U(\mathfrak{g}_0^e \oplus \mathfrak{g}_+^e)$ -modules extends to a homomorphism $M(S_\delta(\mathrm{gr}' L) \otimes V) \rightarrow M(S_\delta(\mathrm{gr}' L)) \otimes V$. Now

a standard argument, using the basis of $M(S_\delta(\text{gr}' L)) \otimes V$ consisting of elements of the form in the right-hand side of (7.7), shows that this homomorphism is in fact an isomorphism. This gives the third isomorphism in the statement of the lemma. \square

We now state and prove the main theorem of this section.

Theorem 7.14. *Let L be a finite dimensional $U(\mathfrak{g}_0, e)$ -module, and V a finite dimensional $U(\mathfrak{g})$ -module. There is a filtration of $M(L) \otimes V$ by quasi-Verma modules $M(L_1), \dots, M(L_m)$, where L_i is a finite dimensional $U(\mathfrak{g}_0, e)$ -module for $i = 1, \dots, m$.*

Proof. Note that \mathfrak{t}^e is in the center of $U(\mathfrak{g}_0, e)$ and $U(\mathfrak{g}_0^e)$. Therefore, when considering $M(L) \otimes V$ we can reduce to the case where $\mathfrak{t}^e \subseteq U(\mathfrak{g}_0, e)$ acts on L by a weight, say, $\lambda_0 \in (\mathfrak{t}^e)^*$; reducing from the case where \mathfrak{t}^e does not act semisimply requires a little argument. This means that $L \subseteq M(L)$ is equal to $M(L)_{\lambda_0}$.

Let x_1, \dots, x_t be a basis of \mathfrak{g}^e as at the end of §7.1; so we have bases of quasi-Verma modules as in (7.6) and (7.7). We decompose $V = \bigoplus_{i=1}^m V_{\lambda_i}$ as a direct sum of \mathfrak{t}^e -weight spaces and assume that $\lambda_1, \dots, \lambda_m \in \mathbb{Z}\Phi^e$ are ordered so that they are nonincreasing with respect to the partial order determined by Φ_+^e . Fix an ordered basis \mathbf{v} of V consisting of \mathfrak{t} -weight vectors; so each V_λ is spanned by a subset of the vectors in \mathbf{v} . Let \mathbf{x}^0 be the lift matrix from Lemma 5.14. Then we have the vector space isomorphism $\psi = \psi_{\mathbf{x}^0, M(L), \mathbf{v}} : M(L) \otimes V \xrightarrow{\sim} M(L) \otimes V$, which is \mathfrak{t}^e -equivariant by Lemma 7.8.

Before constructing the desired filtration of $M(L) \otimes V$, we use the isomorphisms in the Lemma 7.12 to construct a filtration of $\text{gr}'(M(L) \otimes V)$ by modules of the form $M(L'_i) = U(\mathfrak{g}^e) \otimes_{U(\mathfrak{g}_0^e \oplus \mathfrak{g}_+^e)} L'_i$, where L'_i is a finite dimensional $U(\mathfrak{g}_0^e)$ -module. Having done this, we construct a corresponding filtration of $M(L) \otimes V$.

Set $M'_0 = \text{gr}'(M(L) \otimes V)$ and consider the \mathfrak{t}^e -weight space of M'_0 of weight $\lambda_0 + \lambda_1$; we denote $L'_1 = (M'_0)_{\lambda_0 + \lambda_1}$. This is a maximal weight subspace of $\text{gr}'(M(L) \otimes V)$, and we have $\sigma^{-1}(S_\delta(\text{gr}' L) \otimes V_{\lambda_1}) = L'_1$, where $\sigma : M'_0 \xrightarrow{\sim} M(S_\delta(\text{gr}' L) \otimes V)$ is given in (7.13). Now by the universal property of Verma modules for $U(\mathfrak{g}^e)$, there is a unique homomorphism $g'_1 : M(L'_1) \rightarrow M'_0$ sending L'_1 identically to itself. It is clear that g'_1 is injective, so we can identify $M(L'_1)$ with its image in M'_0 and define $M'_1 = M'_0/M(L'_1)$.

Now suppose inductively that we have defined M'_1, \dots, M'_{j-1} , where $M'_i = M'_{i-1}/M(L'_i)$ and $L'_i = (M'_{i-1})_{\lambda_0 + \lambda_i}$, for $i = 1, \dots, j-1$. We note that M'_i is a quotient of M'_0 for $i = 1, \dots, j-1$, and inductively that the image of $\sigma^{-1}(S_\delta(\text{gr}' L) \otimes V_{\lambda_i})$ in M'_i is equal to L'_i , for $i = 1, \dots, j-1$.

Consider $L'_j = (M'_{j-1})_{\lambda_0 + \lambda_j}$. Using the basis of M'_0 given by elements of the form in (7.7), dimension counting of \mathfrak{t}^e -weight spaces, and that the weights $\lambda_1, \dots, \lambda_m$ are ordered so that they are nonincreasing, we see that L'_j is a maximal weight space of M'_{j-1} . Further, we see that L'_j is the image of $\sigma^{-1}(S_\delta(\text{gr}' L) \otimes V_{\lambda_j})$ in M'_{j-1} . We have a unique homomorphism $g'_j : M(L'_j) \rightarrow M'_{j-1}$ sending L'_j identically to itself. A standard PBW argument using the basis elements of M'_0 in (7.7), shows that g'_j is injective. Thus, we can view $M(L'_j)$ as a submodule of M'_{j-1} and define $M'_j = M'_{j-1}/M(L'_j)$.

Thus, inductively, we construct M'_1, \dots, M'_m . Counting dimensions of \mathfrak{t}^e -weights spaces tells us that $g'_m : M(L'_m) \rightarrow M'_{m-1}$ must be an isomorphism. Hence, we have constructed a filtration of M'_0 by the quasi-Verma modules $M(L'_i)$ for $i = 1, \dots, m$.

We move on to construct the desired filtration of $M_0 = M(L) \otimes V$. Inductively we construct M_1, \dots, M_m such that $M_i = M_{i-1}/M(L_i)$ and $L_i = (M_{i-1})_{\lambda_0 + \lambda_i}$, for $i = 1, \dots, m$. This is done so that we have canonical isomorphisms $\text{gr}' M_i \cong M'_i$ and $\text{gr}' L_i \cong L'_i$ for $i = 1, \dots, m$. The construction follows the same lines as above for M'_0 . We include the case $j = 1$ explicitly, to demonstrate the idea, even though this is not needed for the induction; this is also the case for the above construction in M'_0 .

Consider $L_1 = M(L)_{\lambda_0 + \lambda_1}$; this is equal to $\psi(L \otimes V_{\lambda_1})$. Then L_1 is a maximal weight space of $M(L)$ and we clearly have $\text{gr}' L_1 \cong L'_1$; for example, since the loop filtration is invariant under the adjoint action of \mathfrak{t}^e . Thus, by the universal property of Verma modules for $U(\mathfrak{g}, e)$, there is a homomorphism $g_1 : M(L_1) \rightarrow M_0$ as in (7.4). Thanks to (7.5) and (7.7), we see that $\text{gr}' g_1 : \text{gr}' M(L_1) \rightarrow \text{gr}' M_0$ identifies with $g'_1 : M(L'_1) \rightarrow M'_0$. Thus as g'_1 is injective, a standard filtration argument tells us that g_1 is injective. We identify $M(L_1)$ with its image in M_0 and define $M_1 = M_0/M(L_1)$. Then we have a canonical isomorphism $\text{gr}' M_1 \cong M'_1$.

Now suppose, inductively, that we have defined M_1, \dots, M_{j-1} , where $M_i = M_{i-1}/M(L_i)$ and $L_i = (M_{i-1})_{\lambda_0 + \lambda_i}$, for $i = 1, \dots, j-1$. Further, we assume, inductively, that there are canonical isomorphisms $\text{gr}' M_i \cong M'_i$ and $\text{gr}' L_i \cong L'_i$, for $i = 1, \dots, j-1$.

Consider $L_j = (M_{j-1})_{\lambda_0 + \lambda_j}$. Through the isomorphism $\text{gr}' M_{j-1} \cong M_{j-1}$, we get $\text{gr}' L_j \cong L_j$ and thus that L_j is a maximal weight space of M_{j-1} . We note also that L_j is the image of $\psi(L \otimes V_{\lambda_j})$ in M_{j-1} . There is a homomorphism $g_j : M(L_j) \rightarrow M_{j-1}$ and, similarly to the $j = 0$ situation, we see that $\text{gr}' g_j : \text{gr}' M(L_j) \rightarrow \text{gr}' M_{j-1}$ identifies with $g'_j : M(L'_j) \rightarrow M'_{j-1}$; here we use the identifications $\text{gr}' M_{j-1} \cong M'_{j-1}$ and $\text{gr}' M(L_j) \cong M(L'_j)$ from (7.5) and we require (7.7) to see that these maps are equal under these identifications. Thus, as g'_j is injective, so is g_j . Therefore, we may identify $M(L_j)$ with a submodule of M_{j-1} and set $M_j = M_{j-1}/M(L_j)$.

Hence, the induction is complete, and we have a filtration of $M(L) \otimes V$ by quasi-Verma modules. □

The following corollary is an immediate consequence of Theorem 7.14 and Lemma 7.2.

Corollary 7.15. *Let L be a finite dimensional irreducible $U(\mathfrak{g}_0, e)$ -module, and V a finite dimensional $U(\mathfrak{g})$ -module. Then there is a filtration of $M(L) \otimes V$ by Verma modules $M(L_1), \dots, M(L_{m'})$, where L_i is a finite dimensional irreducible $U(\mathfrak{g}_0, e)$ -module for $i = 1, \dots, m'$.*

Remark 7.16. In future work, we hope to be more explicit regarding the isomorphism type of factors that occur in the filtration from Theorem 7.14. In particular, this should be possible in the case where e is regular in \mathfrak{g}_0 . In this case $U(\mathfrak{g}_0, e)$ is isomorphic to $Z(\mathfrak{g}_0)$ (the center of $U(\mathfrak{g}_0)$), by [Ko, §2], so we have an explicit parameterization of the irreducible $U(\mathfrak{g}_0, e)$ -modules.

We note here that the proof of Theorem 7.14 does give the weight of \mathfrak{t}^e on each L_i occurring in the filtration. These are precisely the weights $\lambda_0 + \lambda_i$ for $i = 1, \dots, m$. Further, an explicit description of the L_i as subquotients of $M(L) \otimes V$ is given via the map ψ . Finally, we remark that considering the partial order on the weights $\lambda_1, \dots, \lambda_m$ more carefully allows one to shorten the filtration, so that the quotients are certain direct sums of the $M(L_i)$.

8. BRST DEFINITION

In this section we define translation in terms of the definition of the finite W -algebra via BRST cohomology. This turns out to be key for the proof of Theorem 10.9.

8.1. The finite W -algebra. We recall the definition of the finite W -algebra via BRST cohomology. The BRST definition is shown to be equivalent to the Whittaker model definition in [D³HK]; below we recall some results from [BGK, §2] explaining its equivalence with the definition of $U(\mathfrak{g}, e)$.

We recall that $\tilde{\mathfrak{g}} = \mathfrak{g} \oplus \mathfrak{k}^{\text{ne}}$ and that \mathfrak{n}^{ch} is a copy of \mathfrak{n} . The nonlinear Lie superalgebra $\hat{\mathfrak{g}}$ is defined to be

$$\hat{\mathfrak{g}} = \tilde{\mathfrak{g}} \oplus \mathfrak{n}^* \oplus \mathfrak{n}^{\text{ch}}$$

with even part equal to $\tilde{\mathfrak{g}}$ and odd part equal to $\mathfrak{n}^* \oplus \mathfrak{n}^{\text{ch}}$. The nonlinear Lie bracket $[\cdot, \cdot]$ on $\hat{\mathfrak{g}}$ is defined by: extending the bracket on $\tilde{\mathfrak{g}}$; declaring that the bracket is identically zero on \mathfrak{n}^* , \mathfrak{n}^{ch} and between elements of $\tilde{\mathfrak{g}}$ and $\mathfrak{n}^* \oplus \mathfrak{n}^{\text{ch}}$; and setting $[f, x^{\text{ch}}] = \langle f, x \rangle$ for $f \in \mathfrak{n}^*$ and $x \in \mathfrak{n}$, where $\langle f, x \rangle$ denotes the natural pairing of $f \in \mathfrak{n}^*$ with $x \in \mathfrak{n}$. We have the subalgebra

$$\hat{\mathfrak{p}} = \tilde{\mathfrak{p}} \oplus \mathfrak{n}^* \oplus \mathfrak{n}^{\text{ch}}$$

of $\hat{\mathfrak{g}}$.

Since $\tilde{\mathfrak{g}}$ commutes with $\mathfrak{n}^{\text{ch}} \oplus \mathfrak{n}^*$, we have the tensor product decomposition

$$U(\hat{\mathfrak{g}}) = U(\tilde{\mathfrak{g}}) \otimes U(\mathfrak{n}^{\text{ch}} \oplus \mathfrak{n}^*)$$

of $U(\hat{\mathfrak{g}})$ as a superalgebra. The factor $U(\mathfrak{n}^{\text{ch}} \oplus \mathfrak{n}^*)$ is isomorphic to $\wedge(\mathfrak{n}^{\text{ch}}) \otimes \wedge(\mathfrak{n}^*)$ as a vector space with multiplication making it isomorphic to the Clifford algebra on the space $\mathfrak{n} \oplus \mathfrak{n}^*$; here $\wedge(\mathfrak{n}^{\text{ch}})$ and $\wedge(\mathfrak{n}^*)$ denote the exterior algebras of \mathfrak{n}^{ch} and \mathfrak{n}^* , respectively.

We put the *charge grading* on $\hat{\mathfrak{g}}$, hence also on $\hat{\mathfrak{p}}$, consistent with the \mathbb{Z}_2 -grading, by declaring that elements of $\tilde{\mathfrak{g}}$ are in degree 0, elements of \mathfrak{n}^* are in degree 1, and elements of \mathfrak{n}^{ch} are in degree -1 . This induces gradings

$$U(\hat{\mathfrak{g}}) = \bigoplus_{i \in \mathbb{Z}} U(\hat{\mathfrak{g}})^i \quad \text{and} \quad U(\hat{\mathfrak{p}}) = \bigoplus_{i \in \mathbb{Z}} U(\hat{\mathfrak{p}})^i.$$

We note that the charge grading is called the cohomological grading in [BGK, §2.3].

We recall that b_1, \dots, b_r is a basis of \mathfrak{n} and f_1, \dots, f_r is the dual basis for \mathfrak{n}^* . Define the superderivation $d : U(\hat{\mathfrak{g}}) \rightarrow U(\hat{\mathfrak{g}})$ of charge degree 1 by taking the supercommutator with the degree one element

$$\delta = \sum_{i=1}^r f_i(b_i - \chi(b_i) - b_i^{\text{ne}}) - \frac{1}{2} \sum_{i,j=1}^r f_i f_j [b_i, b_j]^{\text{ch}}.$$

One can check that the supercommutator $[\delta, \delta] = 0$, which means that $d = \text{ad } \delta$ satisfies $d^2 = 0$; we note here that $[\delta, \delta] = 2\delta^2$, so that $\delta^2 = 0$. This can be done by computing the action of d on generators of $U(\hat{\mathfrak{g}})$ explicitly as in [D³HK]; see [BGK, §2.3] for these in the present notation. Therefore, we can take the cohomology $H^0(U(\hat{\mathfrak{g}}), d)$ and we obtain an algebra; this is the *BRST definition of the finite W -algebra*.

We now recall the relationship between $H^0(U(\hat{\mathfrak{g}}), d)$ and $U(\mathfrak{g}, e)$ from [BGK, §2.3]. First, we recall based on [D³HK], the quasi-isomorphism between $U(\hat{\mathfrak{g}})$ and

the standard complex $\tilde{Q} \otimes \wedge(\mathfrak{n}^*)$ for computing the \mathfrak{n} -cohomology for the dot action of \mathfrak{n} on \tilde{Q} . By the PBW theorem for $U(\hat{\mathfrak{g}})$ we have $U(\hat{\mathfrak{g}}) = (U(\tilde{\mathfrak{p}}) \otimes \wedge(\mathfrak{n}^*)) \oplus \hat{I}$, where \hat{I} is the left ideal of $U(\hat{\mathfrak{g}})$ generated by $\tilde{I} \subseteq U(\tilde{\mathfrak{g}})$ and \mathfrak{n}^{ch} . The map

$$(8.1) \quad p : U(\hat{\mathfrak{g}}) \rightarrow \tilde{Q} \otimes \wedge(\mathfrak{n}^*)$$

is defined to be the projection along this decomposition composed with the isomorphism $U(\tilde{\mathfrak{p}}) \xrightarrow{\sim} \tilde{Q}$. It is a consequence of results in [D³HK] that p is a quasi-isomorphism of complexes. We write $q : U(\hat{\mathfrak{p}})^0 \rightarrow U(\tilde{\mathfrak{p}})$ for the projection restricted to $U(\hat{\mathfrak{p}})^0$.

The definition of the function ϕ in the next lemma is based on a construction of Arakawa [Ar, §4.8] in the case that e is regular nilpotent. The following lemma, which says that ϕ is a right-inverse of q is part of [BGK, Lem. 2.6].

Lemma 8.2. *There is a well-defined algebra homomorphism*

$$(8.3) \quad \phi : U(\tilde{\mathfrak{p}}) \hookrightarrow U(\hat{\mathfrak{p}})^0$$

such that

$$\phi(x) = x + \sum_{i=1}^r f_i [b_i, x]^{\text{ch}} \quad \text{and} \quad \phi(y^{\text{ne}}) = y^{\text{ne}},$$

for $x \in \mathfrak{p}$ and $y \in \mathfrak{k}$. Moreover, $q\phi = \text{id}_{U(\tilde{\mathfrak{p}})}$.

Next we recall [BGK, Lem. 2.7].

Lemma 8.4. *For $u \in U(\tilde{\mathfrak{p}})$, we have that*

$$d(\phi(u)) = \sum_i f_i \phi(\text{Pr}([b_i - b_i^{\text{ne}}, u]))$$

The above lemma is important for proving [BGK, Thm. 2.8], which says that $U(\mathfrak{g}, e)$ is isomorphic to a certain subalgebra of $U(\hat{\mathfrak{g}})^0$ isomorphic to $H^0(U(\hat{\mathfrak{g}}), d)$; we recall this below.

Theorem 8.5. *We have that*

$$U(\mathfrak{g}, e) = \{u \in U(\tilde{\mathfrak{p}}) \mid d(\phi(u)) = 0\}.$$

Moreover, we have that $\ker d = \phi(U(\mathfrak{g}, e)) \oplus \text{im } d$.

8.2. Translation. In this subsection, we define translation of $U(\mathfrak{g}, e)$ -modules in the BRST definition using the map ϕ from (8.3); see Definition 8.7 below. Before giving this definition we need to recall and introduce some terminology. When we speak of graded $U(\hat{\mathfrak{g}})$ -modules below, we always mean with respect to the charge grading. Throughout this subsection, M is a $U(\mathfrak{g}, e)$ -module and V is a finite dimensional $U(\mathfrak{g})$ -module.

We recall that a *differential graded module* for $U(\hat{\mathfrak{g}})$ is a graded $U(\hat{\mathfrak{g}})$ -module $N = \bigoplus_{j \in \mathbb{Z}} N^j$ for $U(\hat{\mathfrak{g}})$ with a *differential* $d_N : N \rightarrow N$ such that, $d_N : N^j \rightarrow N^{j+1}$ for each j , $d_N^2 = 0$ and

$$d_N(un) = d(u)n + (-1)^{p(u)}ud_N(n)$$

for all homogeneous $u \in U(\hat{\mathfrak{g}})$ and $n \in N$. For a differential graded module N , each of the cohomology groups $H^i(N, d_N)$ is a module for $H^0(U(\hat{\mathfrak{g}}), d)$ with the obvious action.

Example 8.6. Given a graded $U(\hat{\mathfrak{g}})$ -module N . One can define a differential $d_N : N \rightarrow N$ by $d_N(n) = \delta n$: one requires the fact that $\delta^2 = 0$ in $U(\hat{\mathfrak{g}})$. However, this is not the differential that we shall use for the modules that we consider below.

We define a ‘‘comultiplication’’ $\hat{\Delta} : U(\hat{\mathfrak{g}}) \rightarrow U(\hat{\mathfrak{g}}) \otimes U(\mathfrak{g})$ by extending $\tilde{\Delta}$ from §4.2 and setting:

$$\hat{\Delta}(f) = f \otimes 1 \quad \text{and} \quad \hat{\Delta}(x^{\text{ch}}) = x^{\text{ch}} \otimes 1,$$

for $f \in \mathfrak{n}^*$ and $x \in \mathfrak{n}$. It is trivial to verify that $\hat{\Delta}$ is a superalgebra homomorphism. Thus given a $U(\hat{\mathfrak{g}})$ -module N , we can define the structure of a $U(\hat{\mathfrak{g}})$ -module on $N \otimes V$.

We can consider $U(\mathfrak{g}, e)$ as a subalgebra of $U(\hat{\mathfrak{g}})$ through the injective homomorphism $\phi : U(\hat{\mathfrak{p}}) \rightarrow U(\hat{\mathfrak{g}})$ from Lemma 8.2. Therefore, we may define the induced module

$$\hat{M} = U(\hat{\mathfrak{g}}) \otimes_{U(\mathfrak{g}, e)} M.$$

The tensor product $\hat{M} \otimes V$ is then a graded $U(\hat{\mathfrak{g}})$ -module, where the grading is defined by declaring that M and V have degree 0. We can define a right action of δ on $\hat{M} \otimes V$ by $(u \otimes m \otimes v)\delta = u\delta \otimes m \otimes v$; this is well defined since $\phi(U(\mathfrak{g}, e)) \subseteq \ker d$. It is straightforward to check that this right action of δ commutes with the left action of $U(\hat{\mathfrak{g}})$. Thus, we define $d_{M,V} : \hat{M} \otimes V \rightarrow \hat{M} \otimes V$ by taking the supercommutator with δ :

$$d_{M,V}(u \otimes m \otimes v) = \delta(u \otimes m \otimes v) - (-1)^{p(u)}(u \otimes m \otimes v)\delta.$$

An elementary commutator calculation gives

$$d_{M,V}(a(u \otimes m \otimes v)) = d(a)(u \otimes m \otimes v) + (-1)^{p(a)}a \cdot d_{M,V}(u \otimes m \otimes v),$$

for $a, u \in U(\hat{\mathfrak{g}})$, $m \in M$ and $v \in V$. To see that $d_{M,V}^2 = 0$, one uses the fact that $\delta^2 = 0$ in $U(\hat{\mathfrak{g}})$. Therefore, $\hat{M} \otimes V$ is a differential graded $U(\hat{\mathfrak{g}})$ -module, which leads to our definition of translation in this setting.

Definition 8.7. The BRST definition of the translation of M by V is

$$H^0(\hat{M} \otimes V, d_{M,V}).$$

We view this as a $U(\mathfrak{g}, e)$ -module through the map ϕ from Lemma 8.2.

We would like a more explicit description of the action of $d_{M,V}$, which is given by the following lemma.

Lemma 8.8. The action of $d_{M,V}$ is given by

$$d_{M,V}(u \otimes m \otimes v) = d(u) \otimes m \otimes v + \sum_{i=1}^r f_i u \otimes m \otimes b_i v,$$

for $u \in U(\hat{\mathfrak{g}})$, $m \in M$ and $v \in V$.

Proof. This follows from the fact that

$$\delta \cdot (u \otimes m \otimes v) = \delta u \otimes m \otimes v + \sum_{i=1}^r f_i u \otimes m \otimes b_i v,$$

which is easily seen from the definitions. \square

Next we relate this BRST definition of translation to that given in Definition 4.3. To abbreviate notation we write

$$\tilde{M} = \tilde{Q} \otimes_{U(\mathfrak{g}, e)} M.$$

We show that the $U(\mathfrak{g}, e)$ -modules $H^0(\hat{M} \otimes V, d_{M,V})$ and $M \otimes V = H^0(\mathfrak{n}, \tilde{M} \otimes V)$ are isomorphic by arguing that $\hat{M} \otimes V$ is quasi-isomorphic to the standard complex $\tilde{M} \otimes V \otimes \wedge(\mathfrak{n}^*)$ for calculating \mathfrak{n} -cohomology for the dot action.

We define

$$p_{M,V} : \hat{M} \otimes V \rightarrow \tilde{M} \otimes V \otimes \wedge(\mathfrak{n}^*)$$

by

$$p_{M,V}(u \otimes m \otimes v) = p(u)\tilde{\mathbb{1}} \otimes m \otimes v,$$

for $u \in U(\hat{\mathfrak{g}})$, $m \in M$ and $v \in V$, where p is as in (8.1). Now emulating the arguments in [D³HK], one can show that $p_{M,V}$ is a quasi-isomorphism. For this we require that $H^i(\mathfrak{n}, \tilde{M} \otimes V) = 0$ for $i \neq 0$, which is a consequence of Corollary 3.15. Thus we obtain the desired result.

Proposition 8.9. *We have that $H^0(\hat{M} \otimes V, d_{M,V}) \cong H^0(\mathfrak{n}, \tilde{M} \otimes V)$.*

We would like to have a “quasi-inverse” to $p_{M,V}$. We define

$$\phi_{M,V} : \tilde{M} \otimes V \rightarrow \hat{M}^0 \otimes V$$

by

$$\phi_{M,V}(u\tilde{\mathbb{1}} \otimes m \otimes v) = \phi(u) \otimes m \otimes v,$$

for $u \in U(\tilde{\mathfrak{p}})$, $m \in M$ and $v \in V$. We have $p_{M,V}\phi_{M,V} = \text{id}_{\tilde{M} \otimes V}$ so $\phi_{M,V}$ is injective.

The following lemma says that the maps ϕ and $\phi_{M,V}$ preserve module structure.

Lemma 8.10.

$$\phi_{M,V}(a(u\tilde{\mathbb{1}} \otimes m \otimes v)) = \phi(a)\phi_{M,V}(u\tilde{\mathbb{1}} \otimes m \otimes v),$$

for $a, u \in U(\tilde{\mathfrak{p}})$, $m \in M$ and $v \in V$.

Proof. We work by induction on the length of a in a PBW basis for $U(\tilde{\mathfrak{p}})$. First consider the case that $a = x \in \mathfrak{p}$. We have

$$\begin{aligned} \phi_{M,V}(x(u\tilde{\mathbb{1}} \otimes m \otimes v)) &= \phi_{M,V}(xu\tilde{\mathbb{1}} \otimes m \otimes v + u\tilde{\mathbb{1}} \otimes m \otimes xv) \\ &= \phi(xu) \otimes m \otimes v + \phi(u) \otimes m \otimes xv \\ &= \phi(x)(\phi(u) \otimes m \otimes v) \\ &= \phi(x)\phi_{M,V}(u\tilde{\mathbb{1}} \otimes m \otimes v). \end{aligned}$$

The second to last equality follows from the fact that ϕ is an algebra homomorphism and the definitions of ϕ and $\hat{\Delta}$. The case $a = x^{\text{ne}} \in \mathfrak{k}^{\text{ne}}$ is trivial.

The induction step is straightforward and we omit the details. \square

We now prove the following analogue of Lemma 8.4.

Lemma 8.11.

$$d_{M,V}(\phi_{M,V}(u\tilde{\mathbb{1}} \otimes m \otimes v)) = \sum_{i=1}^r f_i \phi_{M,V}([b_i - b_i^{\text{ne}}, u]\tilde{\mathbb{1}} \otimes m \otimes v + u\tilde{\mathbb{1}} \otimes m \otimes b_i v),$$

for $u \in U(\tilde{\mathfrak{p}})$, $m \in M$ and $v \in V$.

Proof.

$$\begin{aligned}
 d_{M,V}(\phi_{M,V}(u\tilde{\mathbb{1}} \otimes m \otimes v)) &= d_{M,V}(\phi(u) \otimes m \otimes v) \\
 &= d(\phi(u)) \otimes m \otimes v + \sum_{i=1}^r f_i \phi(u) \otimes m \otimes b_i v \\
 &= \sum_{i=1}^r f_i \phi(\text{Pr}([b_i - b_i^{\text{ne}}, u])) \otimes m \otimes v + \sum_{i=1}^r f_i \phi(u) \otimes m \otimes b_i v \\
 &= \sum_{i=1}^r f_i \phi_{M,V}([b_i - b_i^{\text{ne}}, u] \tilde{\mathbb{1}} \otimes m \otimes v + u \otimes m \otimes b_i v).
 \end{aligned}$$

The second equality above is given by Lemma 8.8, and the third equality follows from Lemma 8.4. □

Lemma 8.11 easily implies the following analogue of Theorem 8.5.

Theorem 8.12. *We have that*

$$M \otimes V = \{z \in \tilde{M} \otimes V \mid d_{M,V}(\phi_{M,V}(z)) = 0\}$$

and $\ker d_{M,V} = \phi_{M,V}(M \otimes V) \oplus \text{im } d_{M,V}$.

Remark 8.13. In the proof of Theorem 10.7, we consider the case where $M = U(\mathfrak{g}, e)$, where we can identify $\tilde{M} \otimes V$ with $\tilde{Q} \otimes V$ and $\hat{M} \otimes V$ with $U(\hat{\mathfrak{g}}) \otimes V$. We write d_V for the differential on $U(\hat{\mathfrak{g}}) \otimes V$, and apply the above results with this notational convention.

9. RIGHT-HANDED VERSIONS

In Section 10, we require a right-handed versions of certain definitions and results from earlier in the paper. The required material is presented below.

9.1. Right-handed version of $U(\mathfrak{g}, e)$ and translation. There is a right-handed analogue of $U(\mathfrak{g}, e)$, which we denote by $U(\mathfrak{g}, e)'$, note that this notation differs from that used in [BGK, §2]. An isomorphism between $U(\mathfrak{g}, e)$ and $U(\mathfrak{g}, e)'$ is given in [BGK, Cor. 2.9]. We give the definition of $U(\mathfrak{g}, e)'$ below and recall the isomorphism with $U(\mathfrak{g}, e)$, before discussing the right-handed version of translation.

Let \tilde{I}' be the right ideal of $U(\tilde{\mathfrak{g}})$ generated by $x - x^{\text{ne}} - \chi(x)$ for $x \in \mathfrak{n}$, and define $\tilde{Q}' = U(\tilde{\mathfrak{g}})/\tilde{I}'$; we denote $\tilde{\mathbb{1}}' = 1 + \tilde{I}'$. We have a direct sum decomposition $U(\tilde{\mathfrak{g}}) = U(\tilde{\mathfrak{p}}) \oplus \tilde{I}'$; we write $\text{Pr}' : U(\tilde{\mathfrak{g}}) \rightarrow U(\tilde{\mathfrak{p}})$ for the projection along this direct sum decomposition. The right twisted adjoint action of \mathfrak{n} on $U(\tilde{\mathfrak{p}})$ is given by $x \cdot u = \text{Pr}'([x - x^{\text{ne}}, u])$ and we define the *right-handed finite W -algebra*

$$U(\mathfrak{g}, e)' = H^0(\mathfrak{n}, U(\tilde{\mathfrak{p}}))$$

where the cohomology is taken with respect to the right twisted adjoint action.

We define $\beta \in \mathfrak{t}^*$ by $\beta = \sum_{i=1}^r \beta_i \in \mathfrak{t}^*$; recall that β_i is the \mathfrak{t} -weight of b_i , and b_1, \dots, b_r is a basis of \mathfrak{n} . Then β extends to a character of \mathfrak{p}^* , by [BGK, Lemma 2.5], and we can define the shift automorphism $S_\beta : U(\tilde{\mathfrak{p}}) \rightarrow U(\tilde{\mathfrak{p}})$ by $S_\beta(x) = x + \beta(x)$ for $x \in \mathfrak{p}$ and $S_\beta(y^{\text{ne}}) = y^{\text{ne}}$ for $y \in \mathfrak{k}$. By [BGK, Cor. 2.9], S_β restricts to an isomorphism $U(\mathfrak{g}, e)' \xrightarrow{\sim} U(\mathfrak{g}, e)$.

We can define a right-handed version of translation as follows. Let M' be a right $U(\mathfrak{g}, e)'$ -module and let V' be a finite dimensional $U(\mathfrak{g})$ -module. There is an

obvious structure of a left $U(\mathfrak{g}, e)'$ -module on \tilde{Q} , so we can form the $U(\tilde{\mathfrak{g}})$ -module $M' \otimes_{U(\mathfrak{g}, e)'} \tilde{Q}'$. Now we can give $(M' \otimes_{U(\mathfrak{g}, e)'} \tilde{Q}') \otimes V'$ the structure of $U(\tilde{\mathfrak{g}})$ -module through $\tilde{\Delta}$. Given any right $U(\tilde{\mathfrak{g}})$ -module E , we can define a *right dot action* of \mathfrak{n} on E by

$$v \cdot x = v(x - \chi(x) - x^{\text{ne}})$$

for $v \in E$ and $x \in \mathfrak{n}$. Thus we may define

$$M' \circledast V' = H^0(\mathfrak{n}, (M' \otimes_{U(\mathfrak{g}, e)'} \tilde{Q}') \otimes V'),$$

where cohomology is taken with respect to the right dot action. Then $M' \circledast V'$ is a right $U(\mathfrak{g}, e)'$ -module.

We also have a right-handed definition of lift matrices analogous to Definition 5.15 as follows. Let V' be a finite dimensional right $U(\mathfrak{g})$ -module with ordered basis $\mathbf{v}' = (v'_1, \dots, v'_n)$ of \mathfrak{t} -weight vectors. Denote the c -eigenvalue on v_i by c'_i , and assume that $c'_1 \geq \dots \geq c'_n$. We define V' -block lower unitriangular matrices in the analogous way to how V -block lower unitriangular matrices are defined. The coefficient functions $b'_{ij} \in U(\mathfrak{g})^*$ of V' are defined by $v_i u = \sum_{j=1}^n b'_{ij}(u) v_j$ for $u \in U(\mathfrak{g})$. Then we say that a V' -block lower unitriangular matrix $\mathbf{x}' = (x'_{ij})$ with entries in $U(\tilde{\mathfrak{p}})$ is a lift matrix for \mathbf{v}' if

$$\text{Pr}'([x - x^{\text{ne}}, x'_{ij}]) + \sum_{k=1}^n b'_{kj}(x) x'_{ik} = 0,$$

for all $x \in \mathfrak{n}$, and $x'_{ij} \in F_{c_j - c_i} U(\tilde{\mathfrak{p}})$.

There is a right-handed version of Proposition 5.17, which says that if \mathbf{x}' is a lift matrix for V' , and M' is a right $U(\mathfrak{g}, e)'$ -module, then the map $\psi_{\mathbf{x}', M', V'} : M' \otimes V' \rightarrow M' \circledast V'$ defined by $\psi_{\mathbf{x}', M', V'}(m \otimes v'_j) = \sum_{i=1}^n m \otimes x'_{ij} \tilde{\mathbf{1}}' \otimes v'_i$, for $m \in M'$, is a vector space isomorphism. Further, given any two lift matrices \mathbf{x}' and \mathbf{y}' for \mathbf{v}' , there is a V' -block lower unitriangular matrix \mathbf{w}' with entries in $U(\mathfrak{g}, e)'$ such that $\mathbf{y}' = \mathbf{x}' \mathbf{w}'$.

9.2. Right-handed version of BRST definition. Below we outline the right-handed analogues of the results from §8.2 that we require in Section 10; we only consider translations of the regular module, since that is all that is required. We continue to use the notation from the previous subsection.

There is a right-handed version $\phi' : U(\tilde{\mathfrak{p}}) \rightarrow U(\hat{\mathfrak{g}})$ of ϕ from (8.3) defined by $\phi'(x) = x - \sum_{i=1}^r [b_i, x]^{\text{ch}} f_i$, giving a version of Theorem 8.5. By [BGK, Lem. 2.6], we have $\phi' = \phi S_\beta$. We can view $U(\hat{\mathfrak{g}}) \otimes V'$ as a differential graded right $U(\hat{\mathfrak{g}})$ -module, with differential $d_{V'}$ given by taking the right supercommutator with δ . The cohomology $H^0(U(\hat{\mathfrak{g}}) \otimes V', d_{V'})$ gives the right-handed version of the BRST definition of translation of the right regular $U(\mathfrak{g}, e)'$ -module and the analogue of Proposition 8.9 says that this is isomorphic as a $U(\mathfrak{g}, e)'$ -module to $U(\mathfrak{g}, e)' \circledast V'$. Finally, we have a right-handed version of Theorem 8.12, which, in particular, says that for $z \in \tilde{Q}' \otimes V'$, we have that $d_{V'}(\phi'_{V'}(z)) = 0$ if and only if $z \in M \otimes V$, where $\phi'_{V'} : \tilde{Q}' \otimes V' \rightarrow U(\hat{\mathfrak{g}}) \otimes V$ is defined by $\phi'_{V'}(u \tilde{\mathbf{1}}' \otimes v) = \phi'(u) \otimes v$, for $u \in U(\tilde{\mathfrak{p}})$ and $v \in V$.

10. TRANSLATION COMMUTES WITH DUALITY

In this section we prove Theorem 10.9 saying that translation commutes with duality for modules in $\mathcal{O}(e)$; it is proved for the case $\mathfrak{g} = \mathfrak{gl}_n(\mathbb{C})$ in [BK2, Thm.

8.10]. Before we can state and prove this theorem we need to consider the notions of restricted duals and dualizable modules.

10.1. Restricted duals. A special case of following definition is given in [BK2, (5.2)].

Definition 10.1. Let $M \in \mathcal{O}(e)$. We define the *restricted dual* \overline{M} of M to be the vector space

$$\overline{M} = \bigoplus_{\alpha \in (\mathfrak{t}^e)^*} M_\alpha^*,$$

where M_α^* is the normal dual of the finite dimensional space M_α . Then \overline{M} is a subspace of the full dual M^* , so we can define a right action of $U(\mathfrak{g}, e)$ on \overline{M} by $(fu)(m) = f(um)$ for $f \in \overline{M}$, $u \in U(\mathfrak{g}, e)$ and $m \in M$ making \overline{M} into a right $U(\mathfrak{g}, e)$ -module. Further, we view \overline{M} as a right $U(\mathfrak{g}, e)'$ -module through the isomorphism S_β explained in §9.1.

We define the category $\mathcal{O}(e)' = \mathcal{O}(e; \mathfrak{t}, \mathfrak{q})$ of right $U(\mathfrak{g}, e)'$ -modules in analogy to the category $\mathcal{O}(e)$. The following lemma is immediate.

Lemma 10.2. *Let $M \in \mathcal{O}(e)$. Then $\overline{M} \in \mathcal{O}(e)'$.*

10.2. Dualizable modules. The notion of a dualizable finite dimensional $U(\mathfrak{g})$ -module V is defined in [BK2, §8.4] for the case $\mathfrak{g} = \mathfrak{gl}_n(\mathbb{C})$. In [BK2, Thm. 8.10] it is shown that translation commutes with duality for dualizable modules, and in [BK2, Thm. 8.12] all finite dimensional $U(\mathfrak{g})$ -modules are proved to be dualizable. We follow the same approach in our proof of Theorem 10.9 below.

Let V be a finite dimensional $U(\mathfrak{g})$ -module. We use the notation \overline{V} to mean V^* viewed as a right $U(\mathfrak{g})$ -module in the usual way. In the definition below we require lift matrices as defined in Definition 5.15, and §9.1 for the right-handed version. We fix an ordered basis $\mathbf{v} = (v_1, \dots, v_n)$ of V as in §5.2 and let $\mathbf{v}^* = (v^1, \dots, v^n)$ be the dual basis of V^* .

Definition 10.3. We say that V is *dualizable* if it is possible to find lift matrices \mathbf{x} for \mathbf{v} and \mathbf{y} for \mathbf{v}^* such that $S_\beta(\mathbf{y}) = (S_\beta(y_{ij})) = \mathbf{x}^{-1}$.

For our proof that all finite dimensional $U(\mathfrak{g})$ -modules are dualizable in Theorem 10.7, we require the structure of a differential graded superalgebra on $U(\hat{\mathfrak{g}}) \otimes \text{End}(V)$, which is verified in Lemma 10.5.

The endomorphism algebra $\text{End}(V) \cong V \otimes \overline{V}$ of V is a $U(\mathfrak{g})$ -bimodule in the usual way, i.e., $(uau')(v) = u(a(u'v))$ for $u, u' \in U(\mathfrak{g})$, $a \in \text{End}(V)$ and $v \in V$. We set $v_i^j = v_i \otimes v^j$, so that $\{v_i^j \mid i, j = 1, \dots, n\}$ is a basis of $\text{End}(V)$.

We give $U(\hat{\mathfrak{g}}) \otimes \text{End}(V)$ the tensor product structure of an algebra, so that it is isomorphic to the algebra of $n \times n$ matrices with entries in $U(\hat{\mathfrak{g}})$. The charge grading on $U(\hat{\mathfrak{g}})$ is extended to $U(\hat{\mathfrak{g}}) \otimes \text{End}(V)$ by declaring that $\text{End}(V)$ is in degree 0. We note that $U(\hat{\mathfrak{g}}) \otimes \text{End}(V)$ has the structure of a $U(\hat{\mathfrak{g}})$ -bimodule using the comultiplication $\hat{\Delta}$ given in §8.2. Therefore, we can define a differential $d_{\text{End}(V)}$ on $U(\hat{\mathfrak{g}}) \otimes \text{End}(V)$ by taking the supercommutator with δ , i.e.,

$$d_{\text{End}(V)}(u \otimes a) = \delta(u \otimes a) - (-1)^{p(u)}(u \otimes a)\delta$$

for $u \in U(\hat{\mathfrak{g}})$ and $a \in \text{End}(V)$. Now arguing as for Lemma 8.8, we obtain the formula

$$(10.4) \quad d_{\text{End}(V)}(u \otimes a) = d(u) \otimes a + \sum_{i=1}^n f_i u \otimes b_i a - (-1)^{p(u)} u f_i \otimes a b_i.$$

Lemma 10.5. *With the above definitions $U(\hat{\mathfrak{g}}) \otimes \text{End}(V)$ is a differential graded superalgebra. Therefore, $H^0(U(\hat{\mathfrak{g}}) \otimes \text{End}(V), d_{\text{End}(V)})$ is an algebra.*

Proof. We just need to check that $d_{\text{End}(V)}$ is a superderivation of $U(\hat{\mathfrak{g}}) \otimes \text{End}(V)$, i.e.,

$$d_{\text{End}(V)}((u \otimes a)(u' \otimes a')) = d_{\text{End}(V)}(u \otimes a)(u' \otimes a') + (-1)^{p(u)}(u \otimes a)d_{\text{End}(V)}(u' \otimes a').$$

The left-hand side is equal to

$$(d(u)u' + (-1)^{p(u)}ud(u')) \otimes aa' + \sum_{i=1}^n f_i uu' \otimes b_i aa' - (-1)^{p(uu')} \sum_{i=1}^n uu' f_i \otimes aa' b_i$$

and the right-hand side is equal to

$$\begin{aligned} & d(u)u' \otimes aa' + \sum_{i=1}^n f_i uu' \otimes b_i aa' - (-1)^{p(u)} \sum_{i=1}^n u f_i u' \otimes a b_i a' \\ & + (-1)^{p(u)}(ud(u')) \otimes aa' + \sum_{i=1}^n u f_i u' \otimes a b_i a' - (-1)^{p(u')} \sum_{i=1}^n uu' f_i \otimes aa' b_i. \end{aligned}$$

After a cancellation we see that these expressions are equal. □

Remark 10.6. The algebras $H^0(U(\hat{\mathfrak{g}}) \otimes \text{End}_V, d_{\text{End}(V)})$ may be of independent interest. For example, one can show that $H^0(U(\hat{\mathfrak{g}}) \otimes \text{End}_V, d_{\text{End}(V)})$ is a deformation of $U(\mathfrak{g}, e) \otimes \text{End}(V)$. We do not require this here, so we omit the details.

We are now in a position to prove that all finite dimensional $U(\mathfrak{g})$ -modules are dualizable. In the proof we use the notational convention given in Remark 8.13 and its right-handed analogue, as well as the notation given above. This theorem was proved in the case where \mathfrak{g} is of type A in [BK2, Thm. 8.13] by directly computing enough lift matrices.

Theorem 10.7. *Let V be a finite dimensional $U(\mathfrak{g})$ -module. Then V is dualizable.*

Proof. In the right action of \mathfrak{n} on \overline{V} , we have that v^1 is killed by \mathfrak{n} . This implies that the space V^1 spanned by v_1^1, \dots, v_n^1 is a $U(\mathfrak{n})$ -sub-bimodule of $\text{End}(V)$ isomorphic to V as a left module and trivial as a right module. This means that $U(\hat{\mathfrak{g}}) \otimes V^1$ is stable under the action of $d_{\text{End}(V)}$. Moreover, $U(\hat{\mathfrak{g}}) \otimes V^1$ is a sub-left-module of $U(\hat{\mathfrak{g}}) \otimes \text{End}(V)$ and as such is isomorphic to $U(\hat{\mathfrak{g}}) \otimes V$ viewed as a $U(\hat{\mathfrak{g}})$ -module as in §8.2. Further, the differential d_V on $U(\hat{\mathfrak{g}}) \otimes V$ identifies with the restriction of $d_{\text{End}(V)}$ through this isomorphism, by Lemma 8.8 and (10.4).

Similarly, v_n is killed by \mathfrak{n} (in the left action of \mathfrak{n} on V), and this means the space V_n spanned by v_n^1, \dots, v_n^n is a $U(\mathfrak{n})$ -sub-bimodule of $\text{End}(V)$ isomorphic to \overline{V} as a right module and trivial as a left module. This means that $U(\hat{\mathfrak{g}}) \otimes V_n$ is stable under the action of $d_{\text{End}(V)}$ and is isomorphic to $U(\hat{\mathfrak{g}}) \otimes \overline{V}$ viewed as a right $U(\hat{\mathfrak{g}})$ -module as in §9.2. Also, through this isomorphism the differential $d_{\overline{V}}$ identifies with the restriction of $d_{\text{End}(V)}$ on even parts of the grading of $U(\hat{\mathfrak{g}}) \otimes V$ and with $-d_{\text{End}(V)}$ on odd parts.

Now let \mathbf{x} and \mathbf{y} be lift matrices for \mathbf{v} and \mathbf{v}^* , respectively. Then $\sum_{i=1}^n x_{ij} \tilde{\mathbf{1}} \otimes v_i \in \tilde{Q} \otimes V$ is an invariant for the dot action of \mathfrak{n} for each j , by Proposition 5.17; and $\sum_{j=1}^n y_{ij} \tilde{\mathbf{1}}' \otimes v^j$ is an invariant for the right dot action of \mathfrak{n} for each i , by the right-handed version of Proposition 5.17. Now by Theorem 8.12 we have

$$d_V \left(\sum_{k=1}^n \phi(x_{kj}) \otimes v_k \right) = 0,$$

for each j ; and by the right-handed version of Theorem 8.12 discussed in §9.2 we have

$$d_{\bar{V}} \left(\sum_{l=1}^n \phi'(y_{il}) \otimes v^l \right) = 0,$$

for each i .

Putting this altogether and recalling that $\phi' = \phi S_\beta$, by [BGK, Lem. 2.6], we have

$$d_{\text{End}(V)} \left(\sum_{k=1}^n \phi(x_{kj}) \otimes v_k^1 \right) = 0,$$

for each j , and

$$d_{\text{End}(V)} \left(\sum_{l=1}^n \phi(S_\beta(y_{il})) \otimes v_n^l \right) = 0,$$

for each i .

Now if $u \otimes a, u' \otimes a' \in \ker d_{\text{End}(V)}$, then $(u \otimes a)(u' \otimes a') = uu' \otimes aa' \in \ker d_{\text{End}(V)}$, by Lemma 10.5. Thus, the product

$$\left(\sum_{l=1}^n \phi'(y_{il}) \otimes v_n^l \right) \left(\sum_{k=1}^n \phi(x_{kj}) \otimes v_k^1 \right)$$

is in the kernel of $d_{\text{End}(V)}$. Since \mathbf{x} and \mathbf{y} are V -block lower unitriangular, this product is precisely

$$\sum_{k=1}^n \phi(S_\beta(y_{ik})) \phi(x_{kj}) \otimes v_n^1,$$

i.e., the (i, j) entry of $\phi(S_\beta(\mathbf{y})\mathbf{x})$ tensored with v_n^1 . Now since v_n^1 is killed by \mathfrak{n} on both sides, we must have that $d(\phi(S_\beta(\mathbf{y})\mathbf{x})_{ij}) = 0$ for each i and j . Therefore, by Theorem 8.5, we have that $(S_\beta(\mathbf{y})\mathbf{x})_{ij} \in U(\mathfrak{g}, e)$. Thus, $\mathbf{x}S_\beta(\mathbf{y}) = \mathbf{w}$ is a V -block lower unitriangular matrix with entries in $U(\mathfrak{g}, e)$. Now replacing \mathbf{x} with $\mathbf{w}^{-1}\mathbf{x}$ and using Proposition 5.17(c) we see that V is dualizable. \square

The next lemma, which generalizes [BK2, Lem. 8.8], follows from Proposition 5.17(c) and Theorem 10.7.

Lemma 10.8. *Let \mathbf{x} be a lift matrix for \mathbf{v} , then $S_{-\beta}(\mathbf{x})^{-1}$ is a lift matrix for \mathbf{v}^* .*

We now give some notation that we use in the next subsection. Let M' be a right $U(\mathfrak{g}, e)$ -module. Recall the lift matrix \mathbf{x}^0 from Lemma 5.14. Letting $\mathbf{y}^0 = S_{-\beta}(\mathbf{x}^0)^{-1}$, the discussion in §9.1 and Lemma 10.8 imply that

$$\psi'_{M', \bar{V}, \mathbf{y}^0} : M' \otimes' \bar{V} \rightarrow M' \otimes \bar{V}$$

is a vector space isomorphism. We denote its inverse by

$$\varphi_{M', \bar{V}} : M' \otimes \bar{V} \rightarrow M' \otimes' \bar{V}.$$

10.3. Main theorem. We now state and prove the main theorem of this section. The proof is almost identical to that of [BK2, Thm. 8.10], we include the details for completeness. In the proof we use the bases of V and \bar{V} from the previous subsection, and to simplify notation we write $\mathbf{x} = \mathbf{x}^0$ and $\mathbf{y} = \mathbf{y}^0$ for the lift matrices.

Theorem 10.9. *Let $M \in \mathcal{O}_e$ and V be a finite dimensional $U(\mathfrak{g})$ -module. Then there is an isomorphism of $U(\mathfrak{g}, e)'$ -modules*

$$\overline{M} \otimes' \bar{V} \cong \overline{M \otimes V}.$$

Proof. We define $\omega_{M,V} : \overline{M} \otimes' \bar{V} \rightarrow \overline{M \otimes V}$ to be the composite

$$\overline{M} \otimes' \bar{V} \xrightarrow{\varphi_{\overline{M}, \bar{V}}} \overline{M} \otimes \bar{V} \cong \overline{M \otimes V} \xrightarrow{\overline{\varphi_{M,V}}} \overline{M \otimes V},$$

where $\overline{\varphi_{M,V}}$ is the dual map of $\varphi_{M,V}$. Then $\omega_{M,V}$ is a vector space isomorphism.

Define $\delta : U(\tilde{\mathfrak{p}}) \rightarrow U(\tilde{\mathfrak{p}}) \otimes \text{End}(V)$ to be the composite $(\text{id}_{U(\tilde{\mathfrak{p}})} \otimes \rho) \tilde{\Delta}$, where $\rho : U(\mathfrak{g}) \rightarrow \text{End}(V)$ is the representation of $U(\mathfrak{g})$ on V .

So for $u \in U(\mathfrak{g}, e)$ and $m \in M$, we have

$$u \left(\sum_{i=1}^n x_{ij} \tilde{\mathbb{1}} \otimes m \otimes v_i \right) = \sum_{k,i=1}^n u_{ik}^* x_{kj} \tilde{\mathbb{1}} \otimes m \otimes v_i \in M \otimes V,$$

where $\delta(u) = \sum_{i,j=1}^n u_{ij}^* \otimes v_i^j$. Suppose for a moment that $M = U(\mathfrak{g}, e)$, (it does not matter for this part of the argument that $U(\mathfrak{g}, e)$ does not lie in $\mathcal{O}(e)$). By Proposition 5.17, we must have $\sum_{i,k=1}^n u_{ik}^* x_{kj} \tilde{\mathbb{1}} \otimes 1 \otimes v_i = \sum_{i,k=1}^n x_{ik} \tilde{\mathbb{1}} \otimes u_{kj} \otimes v_i$, where $u_{kj} \in U(\mathfrak{g}, e)$; so we have $\sum_{k=1}^n x_{ik} u_{kj} = \sum_{k=1}^n u_{ik}^* x_{kj}$. This means that $\mathbf{u} = S_\beta(\mathbf{y}) \mathbf{u}^* \mathbf{x}$, where $\mathbf{u} = (u_{ij})$ and $\mathbf{u}^* = (u_{ij}^*)$.

Therefore, for general M we have

$$u \left(\sum_{i=1}^n x_{ij} \otimes m \otimes v_i \right) = \sum_{i,k=1}^n x_{ik} \otimes u_{kj} m \otimes v_i.$$

This means that through the isomorphism $\varphi_{M,V}$ the action of $U(\mathfrak{g}, e)$ on $M \otimes V$ is given by

$$(10.10) \quad u(m \otimes v_j) = \sum_{i=1}^n u_{ij} m \otimes v_i,$$

and the u_{ij} are defined from $\mathbf{u} = S_\beta(\mathbf{y}) \mathbf{u}^* \mathbf{x}$.

Now an analogous argument gives the action of $U(\mathfrak{g}, e)'$ on $\overline{M} \otimes \bar{V}$ through the isomorphism $\varphi_{\overline{M}, \bar{V}}$. We define $\delta' = (\text{id}_{U(\tilde{\mathfrak{p}})} \otimes \bar{\rho}) \tilde{\Delta} : U(\tilde{\mathfrak{p}}) \rightarrow U(\tilde{\mathfrak{p}}) \otimes \text{End}'(\bar{V})$, where $\bar{\rho}$ is the right representation of $U(\mathfrak{g})$ mapping into the space of right endomorphisms $\text{End}'(\bar{V})$ of \bar{V} . Given $u \in U(\mathfrak{g}, e)'$ we define the matrix \mathbf{u}'^* by $\delta'(u) = \sum_{i,j=1}^n u'_{ij}^* \otimes v_i^j$. Then the action of $U(\mathfrak{g}, e)'$ is given by

$$(10.11) \quad (f \otimes v^i) u = \sum_{j=1}^n f u'_{ij} \otimes v^j,$$

where $\mathbf{u}' = (u'_{ij})$ is defined by $\mathbf{u}' = \mathbf{y} \mathbf{u}'^* S_{-\beta}(\mathbf{x})$, so that $\mathbf{u}' = S_\beta(\mathbf{u})$.

We view $\overline{M \otimes V}$ as a $U(\mathfrak{g}, e)'$ -module as in Definition 10.1. Then (10.10) and (10.11) imply that $\omega_{M,V}$ is an isomorphism of $U(\mathfrak{g}, e)'$ -modules. \square

Remark 10.12. We note that we can weaken the hypothesis that $M \in \mathcal{O}(e)$: we just require that M is the direct sum of finite dimensional generalized \mathfrak{k}^e -weight spaces.

The first part of the following corollary is immediate from the proof of Theorem 10.9. The second part can be verified by direct calculation.

Corollary 10.13. *Let M be a $U(\mathfrak{g}, e)$ -module and let V, V' be finite dimensional $U(\mathfrak{g})$ -modules. Then the following diagrams commute:*

(i)

$$\begin{array}{ccc} \overline{M} \otimes' \overline{V} & \xrightarrow{\omega_{M,V}} & \overline{M \otimes V} \\ \varphi_{\overline{M}, \overline{V}} \downarrow & & \uparrow \varphi_{M,V} \\ \overline{M} \otimes \overline{V} & \xrightarrow{\sim} & \overline{M \otimes V}, \end{array}$$

where the bottom map is the canonical isomorphism;

(ii)

$$\begin{array}{ccccc} (\overline{M} \otimes' \overline{V}) \otimes' \overline{V'} & \xrightarrow{\omega_{M,V} \otimes \text{id}_{\overline{V'}}} & (\overline{M \otimes V}) \otimes' \overline{V'} & \xrightarrow{\omega_{M \otimes V, V'}} & \overline{(M \otimes V) \otimes V'} \\ a_{\overline{M}, \overline{V}, \overline{V'}} \downarrow & & & & \uparrow a_{M, V, V'} \\ \overline{M} \otimes' (\overline{V} \otimes \overline{V'}) & \xrightarrow{\sim} & \overline{M} \otimes' (\overline{V \otimes V'}) & \xrightarrow{\omega_{M, V \otimes V'}} & \overline{M \otimes (V \otimes V')}, \end{array}$$

where the bottom left map is the canonical isomorphism, $\overline{a_{M, V, V'}}$ is the dual of the isomorphism from Lemma 6.2, and $a_{\overline{M}, \overline{V}, \overline{V'}}$ is the right-handed version defined in analogy.

11. INDEPENDENCE OF \mathfrak{l} AND GOOD GRADING

As mentioned in §3.1, the definition of $W_{\mathfrak{l}}$ depends on the choice of good grading $\mathfrak{g} = \bigoplus_{j \in \mathbb{R}} \mathfrak{g}(j)$ for e and choice of isotropic subspace \mathfrak{l} of $\mathfrak{g}(-1)$. In this subsection we briefly recall the arguments from [GG] and [BG] showing that $W_{\mathfrak{l}}$ is independent up to isomorphism of these choices. Then we show that through these isomorphisms, the definition of translation does not depend on the choices of good grading and \mathfrak{l} in the appropriate sense.

11.1. Independence of choice of \mathfrak{l} . We use the notation from §3.1 and §4.1. In particular, \mathfrak{l} is an isotropic subspace of \mathfrak{k} used in the definition of $W_{\mathfrak{l}}$.

Suppose \mathfrak{l}' is another isotropic subspace of $\mathfrak{g}(-1)$ with $\mathfrak{l}' \subseteq \mathfrak{l}$. Then we have the surjection $Q_{\mathfrak{l}'} \rightarrow Q_{\mathfrak{l}}$. This restricts to a map $\psi_{\mathfrak{l}', \mathfrak{l}} : W_{\mathfrak{l}'} \rightarrow W_{\mathfrak{l}}$, which is shown in [GG, §5.5] to be an isomorphism; this is proved by arguing that the associated graded map is an isomorphism.

Now suppose that \mathfrak{l}' is any isotropic subspace of $\mathfrak{g}(-1)$. Take $\mathfrak{l}'' = 0$, then we have isomorphisms $\psi_{\mathfrak{l}', \mathfrak{l}} : W_{\mathfrak{l}'} \xrightarrow{\sim} W_{\mathfrak{l}}$ and $\psi_{\mathfrak{l}', \mathfrak{l}''} : W_{\mathfrak{l}'} \xrightarrow{\sim} W_{\mathfrak{l}'}$. Therefore, we obtain a canonical isomorphism $\psi_{\mathfrak{l}', \mathfrak{l}} = \psi_{\mathfrak{l}', \mathfrak{l}''} \psi_{\mathfrak{l}'', \mathfrak{l}}^{-1} : W_{\mathfrak{l}'} \rightarrow W_{\mathfrak{l}}$; note that there is no ambiguity in our notation.

Let M be a $W_{\mathfrak{l}}$ -module and V a finite dimensional $U(\mathfrak{g})$ -module. In Proposition 11.1 we show that $M \otimes_{\mathfrak{l}} V$ is isomorphic to $M \otimes_{\mathfrak{l}'} V$ in the appropriate sense. First we note that we may view M as a $W_{\mathfrak{l}'}$ -module via “ $um = \psi_{\mathfrak{l}', \mathfrak{l}}^{-1}(u)m$ ” for $u \in W_{\mathfrak{l}'}$ and $m \in M$; thus we can define $M \otimes_{\mathfrak{l}'} V$.

Proposition 11.1. *There is an canonical isomorphism of vector spaces*

$$\xi_{\ell', \ell} : M \otimes_{W_{\ell'}} V \xrightarrow{\sim} M \otimes_{W_{\ell}} V$$

such that

$$\xi_{\ell', \ell}(uz) = \psi_{\ell', \ell}(u)\xi_{\ell', \ell}(z)$$

for all $u \in W_{\ell'}$, $z \in M \otimes_{W_{\ell'}} V$.

Proof. It suffices to consider the case $\ell' \subseteq \ell$; the general case can be dealt with by taking a composition as for $\psi_{\ell', \ell}$.

The surjection $Q_{\ell'} \rightarrow Q_{\ell}$ gives a map $Q_{\ell'} \otimes_{W_{\ell'}} M \rightarrow Q_{\ell} \otimes_{W_{\ell}} M$. This gives rise to a homomorphism of $U(\mathfrak{g})$ -modules $(Q_{\ell'} \otimes_{W_{\ell'}} M) \otimes V \rightarrow (Q_{\ell} \otimes_{W_{\ell}} M) \otimes V$, which in turn restricts to a map $\xi_{\ell', \ell} : M \otimes_{W_{\ell'}} V \rightarrow M \otimes_{W_{\ell}} V$.

There are Kazhdan filtrations on Q_{ℓ} , $Q_{\ell'}$, M and V as defined in §3.1 and §5.1; we take the same filtration of M considered as a module for W_{ℓ} and $W_{\ell'}$. Therefore, we have filtrations on $M \otimes_{W_{\ell'}} V$ and $M \otimes_{W_{\ell}} V$. To show that $\xi_{\ell', \ell}$ is an isomorphism, it suffices, by a standard filtration argument, to show that the associated graded map $\text{gr } \xi_{\ell', \ell} : \text{gr}(M \otimes_{W_{\ell'}} V) \rightarrow \text{gr}(M \otimes_{W_{\ell}} V)$ is an isomorphism.

As in the proof of Theorem 5.1, we can identify

$$\text{gr}((Q_{\ell} \otimes_{W_{\ell}} M) \otimes V) \cong \text{gr } M \otimes \mathbb{C}[N_{\ell}] \otimes V$$

and $\text{gr}(M \otimes_{W_{\ell}} V)$ with the N_{ℓ} -invariants $(\text{gr } M \otimes \mathbb{C}[N_{\ell}] \otimes V)^{N_{\ell}}$. Under these identifications the map $\text{gr } M \otimes \mathbb{C}[N_{\ell'}] \otimes V \rightarrow \text{gr } M \otimes \mathbb{C}[N_{\ell}] \otimes V$ induced from evaluation at 1 restricts to an isomorphism $\bar{\epsilon} : \text{gr}(M \otimes_{W_{\ell'}} V) \xrightarrow{\sim} \text{gr } M \otimes V$ as in (5.4). Also,

$$\text{gr } \xi_{\ell', \ell} : (\text{gr } M \otimes \mathbb{C}[N_{\ell'}] \otimes V)^{N_{\ell'}} \rightarrow (\text{gr } M \otimes \mathbb{C}[N_{\ell}] \otimes V)^{N_{\ell}}$$

is induced from the restriction map $\mathbb{C}[N_{\ell'}] \rightarrow \mathbb{C}[N_{\ell}]$. Thus, it is straightforward to see that the diagram below commutes, which implies that $\text{gr } \xi_{\ell', \ell}$ is an isomorphism as required.

$$\begin{array}{ccc} (\text{gr } M \otimes \mathbb{C}[N_{\ell'}] \otimes V)^{N_{\ell'}} & \xrightarrow{\text{gr } \xi_{\ell', \ell}} & (\text{gr } M \otimes \mathbb{C}[N_{\ell}] \otimes V)^{N_{\ell}} \\ \bar{\epsilon}' \downarrow & & \downarrow \bar{\epsilon} \\ \text{gr } M \otimes V & \xrightarrow{\text{id}} & \text{gr } M \otimes V \end{array}$$

It is clear from construction that $\xi_{\ell', \ell}$ satisfies the condition in the proposition. \square

11.2. Independence of good grading. We now introduce some notation and terminology required to show that the definition of W_{ℓ} does not depend on the choice of good grading up to isomorphism.

In this section we allow ourselves to consider the more general notion of good \mathbb{R} -gradings: we recall that an \mathbb{R} -grading $\mathfrak{g} = \bigoplus_{j \in \mathbb{R}} \mathfrak{g}(j)$ for e is good for e if $e \in \mathfrak{g}(2)$, $\mathfrak{g}^e \subseteq \bigoplus_{j \geq 0} \mathfrak{g}(j)$ and $\mathfrak{z}(\mathfrak{g}) \subseteq \mathfrak{g}(0)$. Then the alternating form $\langle \cdot | \cdot \rangle$ can be defined on $\mathfrak{k} = \mathfrak{g}(-1)$ in exactly the same way as for case of \mathbb{Z} -gradings. For an isotropic subspace ℓ of \mathfrak{k} we can define \mathfrak{m}_{ℓ} , \mathfrak{n}_{ℓ} , Q_{ℓ} and W_{ℓ} using the same process as in §3.1. In this subsection we only consider the case where ℓ is a Lagrangian subspace of $\mathfrak{g}(-1)$ so that $\mathfrak{m}_{\ell} = \mathfrak{n}_{\ell}$.

We use the notation $\Gamma : \mathfrak{g} = \bigoplus_{j \in \mathbb{R}} \mathfrak{g}(j)$ to denote the good grading for e , and let $\Gamma' : \mathfrak{g} = \bigoplus_{j \in \mathbb{R}} \mathfrak{g}'(j)$ be another good grading for e . Let l' be a Lagrangian subspace of $\mathfrak{g}'(-1)$. Then $\mathfrak{m}'_{l'}$, $Q'_{l'}$ and $W'_{l'}$ are defined in analogy to \mathfrak{m}_l , Q_l and W_l .

The proof, of [BG, Thm. 1], that the definition of W_l does not depend on the choice of good grading is based on the notion of adjacency. We recall from [BG] that the two good gradings $\mathfrak{g} = \bigoplus_{j \in \mathbb{R}} \mathfrak{g}(j)$ and $\mathfrak{g} = \bigoplus_{j \in \mathbb{R}} \mathfrak{g}'(j)$ are called *adjacent* if it is possible to choose Lagrangian subspaces l of $\mathfrak{g}(-1)$ and l' of $\mathfrak{g}'(-1)$ such that $\mathfrak{m}_l = \mathfrak{m}'_{l'}$. In this case we simply have $W_l = W'_{l'}$: equality as algebras and therefore certainly isomorphic.

Next suppose that the two good gradings Γ and Γ' for e are conjugate by $g \in G^e$, where G^e is the centralizer of e in G . Let l' be the image of l under the adjoint action of g . Then it is clear that g induces an isomorphism $\gamma_g : W_l \xrightarrow{\sim} W'_{l'}$.

Let Γ and Γ' be two good gradings for e . The key ingredient for the proof that the definition of W_l does not depend on the choice of good grading is [BG, Thm. 2], which says: there exists a chain $\Gamma_1, \dots, \Gamma_n$ of good gradings for e such that Γ is conjugate to Γ_1 and $\Gamma' = \Gamma_n$, and Γ_i is adjacent to Γ_{i+1} for each $i = 1, \dots, n - 1$. So we obtain an isomorphism $\phi : W_l \xrightarrow{\sim} W'_{l'}$, by composing an isomorphism of the form γ_g (to move from Γ to Γ_1) with isomorphisms of the form $\psi_{l_i, l_{i+1}}$ (to move between Γ_i and Γ_{i+1}).

Let M be a W_l -module and V a finite dimensional $U(\mathfrak{g})$ -module. Theorem 11.2 below, which says that “translation does not depend on the choice of good grading”. In the statement we write M' for M viewed as an $W'_{l'}$ -module through ϕ ; and we write $M' \otimes_{l'} V$ for translation of the $W'_{l'}$ -module M' by V . It is proved by taking a chain $\Gamma_1, \dots, \Gamma_n$ of good gradings as above, and constructing η as composition of isomorphisms. First one takes an isomorphism δ_g that is determined by conjugation by g (in a similar way to how γ_g is defined), then composes with isomorphisms of the form $\xi_{l_i, l_{i+1}}$ from Proposition 11.1.

We have to observe that the arguments required for the proof Proposition 11.1 go through in the more general setting where we are considering a good \mathbb{R} -grading. In particular, the filtrations of algebras and modules considered are not necessarily integral, but indexed by a countable subset I of \mathbb{R} . This subset I is closed under addition and every subset of I has a greatest and least element with respect to $<$. These conditions mean that one can make sense of all the usual notions associated to filtrations, e.g. filtered algebras and associated graded modules.

Theorem 11.2. *There is an isomorphism of vector spaces $\eta : M \otimes_l V \xrightarrow{\sim} M' \otimes_{l'} V$ such that*

$$\eta(uz) = \phi(u)\eta(z)$$

for all $u \in W_{l'}$ and $z \in M \otimes_{l'} V$.

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REFERENCES

- [Ar] T. Arakawa, *Representation theory of W -algebras*, Invent. Math. **169** (2007), 219–320. MR2318558 (2009d:17039)
- [BT] J. de Boer and T. Tjin, *Quantization and representation theory of finite W -algebras*, Comm. Math. Phys. **158** (1993), 485–516. MR1255424 (95c:17040)
- [Br] J. Brown, *Twisted Yangians and finite W -algebras*, Transform. Groups **14** (2009), no. 1, 87–114. MR2480853 (2010c:17015)
- [BG] J. Brundan and S. M. Goodwin, *Good grading polytopes*, Proc. London Math. Soc. **94** (2007), 155–180. MR2293468 (2008g:17031)
- [BGK] J. Brundan, S. M. Goodwin and A. Kleshchev, *Highest weight theory for finite W -algebras*, Internat. Math. Res. Notices, **15** (2008), Art. ID rnn051. MR2438067 (2009f:17011)
- [BK1] J. Brundan and A. Kleshchev, *Shifted Yangians and finite W -algebras*, Adv. Math. **200** (2006), 136–195. MR2199632 (2006m:17010)
- [BK2] ———, *Representations of shifted Yangians and finite W -algebras*, Mem. Amer. Math. Soc. **196** (2008). MR2456464 (2009i:17020)
- [BK3] ———, *Schur-Weyl duality for higher levels*, Selecta Math **14** (2008), 1–57 (2008). MR2480709
- [Ca] R. Carter, *Finite Groups of Lie Type*, Wiley, NY, 1985. MR794307 (87d:20060)
- [D³HK] A. D’Andrea, C. De Concini, A. De Sole, R. Heluani and V. Kac, *Three equivalent definitions of finite W -algebras*, appendix to [DK].
- [DK] A. De Sole and V. Kac, *Finite vs affine W -algebras*, Jpn. J. Math. **1** (2006), 137–261. MR2261064 (2008b:17044)
- [EK] A. Elashvili and V. Kac, *Classification of good gradings of simple Lie algebras*, Lie groups and invariant theory, (E. B. Vinberg ed.), pp. 85–104, Amer. Math. Soc. Transl. **213**, AMS, 2005. MR2140715 (2006c:17013)
- [GG] W. L. Gan and V. Ginzburg, *Quantization of Slodowy slices*, Internat. Math. Res. Notices **5** (2002), 243–255. MR1876934 (2002m:53129)
- [Gi] V. Ginzburg, *Harish-Chandra bimodules for quantized Slodowy slices*, Represent. Theory **13** (2009), 236–271. MR2515934 (2010g:17009)
- [Go] S. M. Goodwin, *A note on Verma modules for finite W -algebras*, J. Algebra **324** (2010), no. 8, 2058–2063. MR2678838
- [GRU] S. M. Goodwin, G. Röhrle and G. Uably, *On 1-dimensional representations of finite W -algebras associated to simple Lie algebras of exceptional type*, LMS J. Comput. Math. **13** (2010), 357–369. MR2685130
- [Ja] J. C. Jantzen, *Representations of Algebraic Groups*, Second Edition, Mathematical Surveys and Monographs, vol. 107, 2003. MR2015057 (2004h:20061)
- [Ko] B. Kostant, *On Whittaker modules and representation theory*, Invent. Math. **48** (1978), 101–184. MR507800 (80b:22020)
- [Lo1] I. Losev, *Quantized symplectic actions and W -algebras*, J. Amer. Math. Soc. **23** (2010), no. 1, 35–59. MR2552248
- [Lo2] ———, *Finite dimensional representations of W -algebras*, preprint, arXiv:0807.1023 (2008).
- [Lo3] ———, *On the structure of the category \mathcal{O} for W -algebras*, preprint, arXiv:0812.1584 (2008).
- [Lo4] ———, *1-dimensional representations and parabolic induction for W -algebras*, preprint, arXiv:0906.0157, (2009).
- [Ly] T. E. Lynch, *Generalized Whittaker vectors and representation theory*, Ph.D. thesis, M.I.T., 1979.
- [Pr1] A. Premet, *Special transverse slices and their enveloping algebras*, Adv. in Math. **170** (2002), 1–55. MR1929302 (2003k:17014)
- [Pr2] ———, *Enveloping algebras of Slodowy slices and the Joseph ideal*, J. Eur. Math. Soc. **9** (2007), 487–543. MR2314105 (2008c:17006)
- [Pr3] ———, *Primitive ideals, non-restricted representations and finite W -algebras*, Mosc. Math. J. **7** (2007), 743–762. MR2372212 (2008k:17012)
- [Pr4] ———, *Commutative quotients of finite W -algebras*, Adv. Math. **225** (2010), no. 1, 269–306. MR2669353

- [Sk] S. Skryabin, *A category equivalence*, appendix to [Pr1].
- [VD] K. de Vos and P. van Driel, *The Kazhdan–Lusztig conjecture for finite W -algebras*, Lett. Math. Phys. **35** (1995), 333–344. MR1358298 (96i:81110)
- [Wa] N. R. Wallach, *Lie algebra cohomology and holomorphic continuation of generalized Jacquet integrals*, Representations of Lie groups, Kyoto, Hiroshima, 1986, 123–151, Adv. Stud. Pure Math. **14**, Academic Press, Boston, MA, 1988. MR1039836 (91d:22014)

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